FREE MONOIDS ARE COHERENT

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Abstract. A monoid S is said to be *right coherent* if every finitely generated subact of every finitely presented right S-act is finitely presented. *Left coherency* is defined dually and S is *coherent* if it is both right and left coherent. These notions are analogous to those for a ring R (where, of course, S-acts are replaced by R-modules). Choo, Lam and Luft have shown that free rings are coherent. In this note we prove that, correspondingly, any free monoid is coherent, thus answering a question posed by the first author in 1992.

1. Introduction and preliminaries

The notion of right coherency for a monoid S is defined in terms of finitary properties of right S-acts, corresponding to the way in which right coherency is defined for a ring R via properties of right R-modules. Namely, S is said to be *right (left) coherent* if every finitely generated subact of every finitely presented right (left) S-act is finitely presented. If S is both right and left coherent then we say that S is *coherent*. Chase [1] gave equivalent internal conditions for right coherency of a ring R. The analogous result for monoids states that a monoid S is right coherent if and only if for any finitely generated right congruence ρ on S, and for any $a, b \in S$, the right annihilator congruence

$$r(a\rho) = \{(u, v) \in S \times S : au \, \rho \, av\}$$

is finitely generated, and the subact $(a\rho)S \cap (b\rho)S$ of the right S-act S/ρ is finitely generated (if non-empty) [4]. *Left coherency* is defined for monoids and rings in a dual manner; a monoid or ring is *coherent* if it is both right and left coherent. Coherency is a rather weak finitary condition on rings and monoids and as demonstrated by Wheeler [7], it is intimately related to the model theory of R-modules and S-acts.

A natural question arises as to which of the important classes of infinite monoids are (right) coherent? This study was initiated in [4], where it is shown that the free commutative monoid on any set Ω is coherent. For a (right) noetherian ring R, the free monoid ring $R[\Omega^*]$ over R is (right) coherent [2, Corollary 2.2]. Since the free ring on Ω is the monoid ring $\mathbb{Z}[\Omega^*]$ [6], it follows immediately that free rings are coherent. The question of whether the free monoid Ω^* itself is coherent was left open in [4]. The purpose of this note is to provide a positive answer to that question:

Theorem 1. *For any set Ω the free monoid Ω^* is coherent.*

Date: March 3, 2015.

2010 *Mathematics Subject Classification.* 20M05, 20M30.

Key words and phrases. free monoids, S-acts, coherency.

Our proof of Theorem 1, given in Section 2, provides a blueprint for the proof in [5] that free left ample monoids are right coherent. Further comments are provided in Section 3.

A few words on notation and technicalities follow. If H is a set of pairs of elements of a monoid S, then we denote by $\langle H\rangle$ the right congruence on S generated by H. It is easy to see that if $a, b \in S$, then $a \,\langle H\rangle\, b$ if and only if $a = b$ or there is an $n \geq 1$ and a sequence

$$(c_1, d_1, t_1; c_2, d_2, t_2; \ldots ; c_n, d_n, t_n)$$

of elements of S, with $(c_i, d_i) \in H$ or $(d_i, c_i) \in H$, such that the following equalities hold:

$$a = c_1t_1,\ d_1t_1 = c_2t_2,\ \ldots\ , d_nt_n = b.$$

Such a sequence will be referred to as an *H-sequence* (of length n) *connecting* a and b. It is convenient to allow $n = 0$ in the above sequence; the empty sequence is interpreted as asserting equality $a = b$. Where convenient we will use the fact that Ω^* is a submonoid of the free group $\mathrm{FG}(\Omega)$ on Ω, in order to give the natural meaning to expressions such as yx^{-1}, where $x, y \in \Omega^*$ and x is a suffix of y.

2. Proof of Theorem 1

Let Ω be a set; it is clearly enough to show that Ω^* is right coherent. To this end let ρ be the right congruence on Ω^* generated by a finite subset H of $\Omega^* \times \Omega^*$, which without loss of generality we assume to be symmetric.

Definition 2. A quadruple $(a, u; b, v)$ of elements of S is said to be *irreducible* if $(au, bv) \in \rho$ and for any common non-empty suffix x of u and v we have that $(aux^{-1}, bvx^{-1}) \notin \rho$.

Definition 3. An H-sequence $(c_1, d_1, t_1; \ldots ; c_n, d_n, t_n)$ with

$$au = c_1t_1, d_1t_1 = c_2t_2, \ldots , d_nt_n = bv$$

is *irreducible* with respect to $(a, u; b, v)$ if $u, t_1, \ldots , t_n, v \in \Omega^*$ do not have a common non-empty suffix. Clearly, this is equivalent to one of $u, t_1, \ldots , t_n, v$ being ϵ.

Throughout this note for an H-sequence as above we define $a = d_0, u = t_0, c_{n+1} = b$ and $t_{n+1} = v$. It is clear that if the quadruple $(a, u; b, v)$ is irreducible then any H-sequence connecting au and bv must be irreducible with respect to $(a, u; b, v)$.

We define

$$K = \max\{|p| : (p, q) \in H\}.$$

Lemma 4. *Let the H-sequence $(c_1, d_1, t_1; \ldots ; c_n, d_n, t_n)$ with*

$$au = c_1t_1, d_1t_1 = c_2t_2, \ldots , d_nt_n = bv$$

be irreducible with respect to $(a, u; b, v)$. Then either the empty H-sequence is irreducible with respect to $(a, u; c_1, t_1)$ (in which case $|u| \leq max\ (|b|, K)$ and $u = \epsilon$ or $t_1 = \epsilon$) or there exist an index $1 \leq i \leq n$ such that $t_{i+1} = \epsilon$ (so that $au \,\rho\, c_{i+1}$) and $x \in \Omega^+$ such that $|x| \leq max\ (|b|\,, K)$, the sequence

$$(c_1, d_1, t_1x^{-1}; \ldots ; c_{i-1}, d_{i-1}, t_{i-1}x^{-1})$$

satisfies

$$aux^{-1} = c_1t_1x^{-1}, d_1t_1x^{-1} = c_2t_2x^{-1}, \ldots, d_{i-1}t_{i-1}x^{-1} = c_it_ix^{-1},$$

and is an irreducible H-sequence with respect to $(a, ux^{-1}; c_i, t_ix^{-1})$.

Proof. If the empty sequence is irreducible with respect to $(a, u; c_1, t_1)$ then either $u = \epsilon$ or $t_1 = \epsilon$. In both cases we have that $|u| \leq \max(|b|, K)$. Suppose therefore that the empty sequence is not irreducible with respect to $(a, u; c_1, t_1)$. Let $i \in \{1, \ldots, n\}$ be the smallest index such that $t_{i+1} = \epsilon$ (such an index exists, because our original sequence is irreducible), and let x be the longest common non-empty suffix of $u = t_0, t_1, \ldots, t_i$. Then the sequence

$$(c_1, d_1, t_1x^{-1}; \ldots; c_{i-1}, d_{i-1}, t_{i-1}x^{-1})$$

clearly satisfies

$$aux^{-1} = c_1t_1x^{-1}, d_1t_1x^{-1} = c_2t_2x^{-1}, \ldots, d_{i-1}t_{i-1}x^{-1} = c_it_ix^{-1}$$

and is irreducible with respect to $(a, ux^{-1}; c_i, t_ix^{-1})$. Furthermore, since $t_{i+1} = \epsilon$, we have that $d_it_i = c_{i+1}$, so x is a suffix of c_{i+1}. If $i < n$ then $(c_{i+1}, d_{i+1}) \in H$, while if $i = n$ we have $c_{i+1} = b$. In either case $|x| \leq |c_{i+1}| \leq \max(|b|, K)$. □

We deduce immediately that one condition for coherency of Ω^* is fulfilled.

Corollary 5. *Let $a, b \in S$. Then $(a\rho)S \cap (b\rho)S$ is empty or finitely generated.*

Proof. Let us suppose that $(a\rho)S \cap (b\rho)S \neq \emptyset$ and let

$$X = \{a\rho, b\rho, c\rho : (c, d) \in H\} \cap (a\rho)S \cap (b\rho)S.$$

We claim that X generates $(a\rho)S \cap (b\rho)S$. It is enough to show that for every irreducible quadruple $(a, u; b, v)$ we have that $(au)\rho \in X$. For this, let $(c_1, d_1, t_1; \ldots; c_n, d_n, t_n)$ be an H-sequence with

$$au = c_1t_1, \ldots, d_nt_n = bv.$$

Note that this sequence is necessarily irreducible with respect to $(a, u; b, v)$. Then by Lemma 4, either $u = \epsilon$, or $t_i = \epsilon$ for some $i \in \{1, \ldots, n\}$, or $v = t_{n+1} = \epsilon$. In each of these cases we see that $(au)\rho \in X$. □

It remains to show that for any $a \in \Omega^*$, the right congruence $r(a\rho)$ is finitely generated. To this end we first present a technical result.

Lemma 6. *Let $(c_1, d_1, t_1; \ldots; c_n, d_n, t_n)$ with*

$$au = c_1t_1, \ldots, d_nt_n = bv$$

be an irreducible H-sequence with respect to $(a, u; b, v)$. Then either $u = \epsilon$, or there exist a factorisation $u = x_k \ldots x_1$ and indices $n + 1 \geq \ell_1 > \ell_2 > \ldots > \ell_k \geq 1$ such that for all $1 \leq j \leq k$:

(i) $0 < |x_j| \leq \max(|b|, K)$ and

(ii) $aux_1^{-1} \ldots x_{j-1}^{-1} \,\rho\, c_{\ell_j}$ (note that for $j = 1$ we have $au \,\rho\, c_{\ell_1}$).

Proof. We proceed by induction on $|u|$: if $|u| = 0$ the result is clear. Suppose that $|u| > 0$ and the result is true for all shorter words. If the empty sequence is irreducible with respect to $(a, u; c_1, t_1)$, then $t_1 = \epsilon$ and the factorisation $u = x_1$ satisfies the required conditions, with $k = 1$ and $\ell_1 = 1$. Otherwise, by Lemma 4, there exist an index $1 \leq i \leq n$ such that $t_{i+1} = \epsilon$, so that $au \, \rho \, c_{i+1}$, and $x_1 \in \Omega^+$ such that $|x_1| \leq \max(|b|, K)$ and the sequence

$$(c_1, d_1, t_1 x_1^{-1}; \dots ; c_{i-1}, d_{i-1}, t_{i-1} x_1^{-1})$$

satisfies

$$aux_1^{-1} = c_1 t_1 x_1^{-1}, d_1 t_1 x_1^{-1} = c_2 t_2 x_1^{-1}, \dots, d_{i-1} t_{i-1} x_1^{-1} = c_i t_i x_1^{-1}$$

and is an irreducible H-sequence with respect to $(a, ux_1^{-1}; c_i, t_i x_1^{-1})$. Put $\ell_1 = i + 1$. Since $|ux_1^{-1}| < |u|$, the result follows by induction. □

Lemma 7. *Let $a \in \Omega^*$. Then $r(a\rho)$ is finitely generated.*

Proof. Let $K' = \max(K, |a|) + 1, L = 2|H| + 2, N = K'L$ and define

$$X = \{(u, v) : |u| + |v| \leq 3N\} \cap r(a\rho).$$

We claim that X generates $r(a\rho)$. It is clear that $\langle X \rangle \subseteq r(a\rho)$.

Let $(u, v) \in r(a\rho)$. We show by induction on $|u| + |v|$ that $(u, v) \in \langle X \rangle$. Clearly, if $|u| + |v| \leq 3N$, then $(u, v) \in X$. We suppose therefore that $|u| + |v| > 3N$ and make the inductive assumption that if $(u', v') \in r(a\rho)$ and $|u'| + |v'| < |u| + |v|$, then $(u', v') \in \langle X \rangle$. If the quadruple $(a, u; a, v)$ is not irreducible, it is immediate that $(u, v) \in \langle X \rangle$. Without loss of generality we therefore suppose that the quadruple $(a, u; a, v)$ is irreducible and $|v| \leq |u|$, so that $|u| > N$. Let $(c_1, d_1, t_1; \dots ; c_n, d_n, t_n)$ with

$$au = c_1 t_1, \dots, d_n t_n = av$$

be an irreducible H-sequence with respect to $(a, u; a, v)$. We apply Lemma 6, noting here that $a = b$. Clearly $u \neq \epsilon$, so by Lemma 6, there exists a factorisation $u = x_k \dots x_1$ such that for all $1 \leq j \leq k$ we have $0 < |x_j| \leq K'$ and $aux_1^{-1} \dots x_{j-1}^{-1} \; \rho \; c_{\ell_j}$ for some $1 \leq \ell_j \leq n + 1$. Since $|u| > K'L$ we have that $k > L$. Note that the number of distinct elements among $c_1, \dots, c_n$ is less than $L - 1$. This in turn implies that there exist two indices $1 \leq k - L < j < i \leq k$ such that $c_{\ell_i} = c_{\ell_j}$, so that

$$aux_1^{-1} \dots x_{i-1}^{-1} \; \rho \; c_{\ell_i} = c_{\ell_j} \; \rho \; aux_1^{-1} \dots x_{j-1}^{-1}.$$

Since $i, j > k - L$ we have that $k - i + 1 \leq L$, so $\left|ux_1^{-1} \dots x_{i-1}^{-1}\right| = |x_k \dots x_i| \leq K'L$, and similarly $\left|ux_1^{-1} \dots x_{j-1}^{-1}\right| \leq K'L$. As a consequence $(ux_1^{-1} \dots x_{i-1}^{-1}, ux_1^{-1} \dots x_{j-1}^{-1}) \in X$, and letting $u' = ux_1^{-1} \dots x_{i-1}^{-1} x_{j-1} \dots x_k$, we see that

$$(u', u) = (ux_1^{-1} \dots x_{i-1}^{-1}, ux_1^{-1} \dots x_{j-1}^{-1}) x_{j-1} \dots x_1 \in \langle X \rangle.$$

In particular, $au' \; \rho \; au \; \rho \; av$. Note that $|u'| < |u|$, because $j < i$ and $x_j \neq \epsilon$. Thus by the induction hypothesis we have that $(v, u') \in \langle X \rangle$ and so the lemma is proved. □

In view of the characterisation of coherency given in [4] and cited in the Introduction, Corollary 5 and Lemma 7 complete the proof of Theorem 1.

3. Comments

Given that the class of right coherent monoids is closed under retract [5], it follows from the results of that paper that free monoids are coherent. However, as the arguments in [5] for free left ample monoids are burdened with unavoidable technicalities, we prefer to present here the more transparent proof that Ω^* is coherent, by way of motivation for the work of [5]. With free objects in mind, we remark that we also show in [5] that the free inverse monoid on Ω is not coherent if $|\Omega| > 1$.

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