

STRING C-GROUPS AS TRANSITIVE SUBGROUPS OF S_n

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ABSTRACT. If Γ is a string C-group which is isomorphic to a transitive subgroup of the symmetric group S_n (other than S_n and the alternating group A_n), then the rank of Γ is at most $n/2 + 1$, with finitely many exceptions (which are classified). It is conjectured that only the symmetric group has to be excluded.

1. INTRODUCTION

Classifications of string C-groups of high rank from almost simple groups has been a subject of interest for almost a decade now. Some striking results have been obtained, for instance for the symmetric groups. Leemans and Fernandes classified string C-groups of rank $n-1$ and $n-2$ for S_n [5, 8] and more recently, they extended this classification to rank $n-3$ and $n-4$ with Mixer [9]. The key of such a classification was first to show that, from a certain value of n onwards, all the maximal parabolic subgroups of a string C-group of rank $r \geq n-4$ must be intransitive. This was done by bounding the rank of string C-groups constructed from transitive groups of degree n that are not S_n nor A_n . These ideas motivated the present paper where we deal with transitive groups in a more general way than in the papers cited above. Another motivation is to contribute to the proof of the following conjecture.

Conjecture 1.1. [7, Conjecture 9.1] *Let $n \geq 12$. The highest rank of an abstract regular polytope having A_n as automorphism group is $\lfloor (n-1)/2 \rfloor$.*

Suppose that $\rho_0, \dots, \rho_{d-1}$ are involutions which generate a string C-group Γ . This means that

- if $|i-j| > 1$, then ρ_i and ρ_j commute (the *string property*);
- if Γ_I denotes the group generated by $\{\rho_i : i \in I\}$, for $I \subseteq \{0, \dots, r-1\}$, then

$$\Gamma_I \cap \Gamma_J = \Gamma_{I \cap J}$$

(the *intersection property*).

It is known that string C-groups are the same thing as automorphism groups of regular polytopes [13, Section 2E]. The number d is the *rank* of the string C-group (or of the polytope).

Suppose that Γ is a string C-group of rank d which is a subgroup of the symmetric group S_n .

- If $\Gamma = S_n$, then $d \leq n - 1$, with equality if and only if the generators ρ_i are the *Coxeter generators* of S_n (that is, $\rho_i = (i + 1, i + 2)$ for $0 \leq i \leq d - 1$) when $n \neq 4$, or equivalently, the polytope is the regular simplex. When $n = 4$, the hemicube gives another set of generators. Moreover, the polytopes (or string C-groups) of ranks $n - 2$, $n - 3$ and $n - 4$ with $\Gamma = S_n$ have been classified [5, 9].
- Examples of string C-groups of rank $\lfloor (n - 1)/2 \rfloor$ with $\Gamma = A_n$ have been found [6, 7]. It is conjectured that this is the largest possible rank for $n \geq 12$ (see above).
- Ranks arbitrarily close to n can be realised by string C-groups corresponding to intransitive subgroups of S_n . For example, if $n = n_1 + \dots + n_k$, where $n_i > 1$ for all i , then take Coxeter generators for S_{n_1}, \dots, S_{n_k} : we obtain a string C-group $\Gamma = S_{n_1} \times \dots \times S_{n_k}$ of rank $n - k$ by ordering the factors arbitrarily.

The groups not covered by these remarks are the transitive subgroups of S_n other than S_n and A_n . We prove the following theorem:

Theorem 1.2. *Let Γ be a string C-group of rank d which is isomorphic to a transitive subgroup of S_n other than S_n or A_n . Then one of the following holds:*

- (a) $d \leq n/2$;
- (b) $n \equiv 2 \pmod{4}$, $d = n/2 + 1$ and Γ is $C_2 \wr S_{n/2}$. The generators are

$$\begin{aligned} \rho_0 &= (1, n/2 + 1)(2, n/2 + 2) \dots (n/2, n); \\ \rho_1 &= (2, n/2 + 2) \dots (n/2, n); \\ \rho_i &= (i - 1, i)(n/2 + i - 1, n/2 + i) \text{ for } 2 \leq i \leq n/2. \end{aligned}$$

Moreover the Schläfli type is $[2, 3, \dots, 3, 4]$.

- (c) Γ is transitive imprimitive and is one of the examples appearing in Table 1.
- (d) Γ is primitive. In this case, $n = 6$. Moreover Γ is obtained from the permutation representation of degree 6 of $S_5 \cong \text{PGL}_2(5)$ and it is the group of the 4-simplex of Schläfli type $[3, 3, 3]$.

In Table 1, Degree and Number correspond to the numbering of transitive groups in MAGMA [1]. For instance, the group of degree 6 and number 9 may be obtained using the function `TransitiveGroup(6,9)`.

<i>Degree</i>	<i>Number</i>	<i>Structure</i>	<i>Order</i>	<i>Schläfli type</i>
6	9	$S_3 \times S_3$	36	[3, 2, 3]
6	11	$2^3 : S_3$	48	[2, 3, 3]
6	11	$2^3 : S_3$	48	[2, 3, 4]
8	45	$2^4 : S_3 : S_3$	576	[3, 4, 4, 3]

TABLE 1. Examples of transitive imprimitive string C-groups of degree n and rank $n/2 + 1$ for $n \leq 9$.

We will prove this theorem by separating two cases: the case when Γ is transitive but imprimitive (see Proposition 2.1 and Proposition 2.2); and the case when Γ is primitive (see Proposition 3.2).

2. TRANSITIVE IMPRIMITIVE GROUPS

In this section, we assume that Γ is a string C-group of rank d which is a transitive imprimitive subgroup of S_n .

We want to bound d by a function of n . The cross-polytope (corresponding to the Coxeter group C_d) shows that there is an example with $d = n/2$: its automorphism group is transitive on the $2d = n$ vertices and is given by the wreath product $C_2 \wr S_d$. We will show that this is almost best possible.

Computing all transitive imprimitive string C-groups of degree ≤ 9 takes only a few seconds using MAGMA. Those string C-groups of degree $n \leq 9$ and rank $\geq n/2 + 1$ are listed in Table 1.

Proposition 2.1. *With the above hypotheses, if $n \geq 10$, then $d \leq n/2 + 1$, with equality only when $n \equiv 2 \pmod{4}$.*

We make some observations before getting into the proof of this proposition.

We use the fact that the centraliser (in the symmetric group) of a primitive group is trivial, except for the cyclic groups of prime order (which we will ignore for the moment).

Suppose that we have a set of generators for a primitive group, partitioned into two subsets, so that the elements of the first subset commute with those in the second. Then each subset generates a normal subgroup, and their intersection is central, and so is trivial. Also, each normal subgroup is transitive and semiregular, and so regular, and each is the centraliser of the other in the symmetric group, they are minimal normal subgroups and are products of isomorphic non-abelian simple groups (whence the degree is at least 60). Moreover, since the union

of the two subsets generates the group, it is the direct product of the two subgroups, so in fact each is simple.

A set S of elements of a group G is *independent* if for every $g \in S$, $g \notin \langle S \setminus \{g\} \rangle$. By Whiston's theorem [15], a set of independent elements in S_n has cardinality at most $n - 1$. We also use the fact that, when $n \geq 5$, a set of $n - 1$ involutions generating S_n and having a string diagram must be the Coxeter generators (this follows from the classification of independent sets of size $n - 1$ in S_n when $n \geq 7$ see [3], and from MAGMA computations when $n = 5$ or 6 see [11]).

Proof. Suppose that Γ is transitive but imprimitive, with m blocks of imprimitivity each of size k , where we have chosen the block system so that the blocks are maximal. Let L be a subset of the generating set which is an independent generating set for the group induced on the set of blocks. Since the blocks are maximal, the group generated by L has a primitive action on the blocks. By the remarks above, L induces a subdiagram of the string diagram with at most two connected components. If there are two, then L is an independent generating set for $S \times S$, where S is a non-abelian simple group of order m ; thus $|L| \leq 2 \log_2 m$ and in this case, $m \geq 60$. If there is only one, then $|L| \leq m - 1$, with equality only if the group induced on the blocks is the symmetric group S_m .

Let C be the set of generating involutions which commute with all the involutions in L , and R the remaining set of involutions. Since elements of R do not commute with elements of L , they are adjacent to them in the diagram, so $|R| \leq 4$, and if L is connected then $|R| \leq 2$.

The proof is separated into three cases.

Case 1: $k > 2$, $m > 2$. The group induced on the blocks is not cyclic of prime order, so its centraliser is trivial, so the elements of C fix all the blocks. Since the group they generate centralises a transitive permutation group on the blocks, it acts in the same way on each block. Thus $|C| \leq k - 1$.

If L has two components, then $d \leq 2 \log_2 m + (k - 1) + 4$ and $m \geq 60$. This is then always smaller than $n/2$. Hence we can assume that the induced subdiagram on L is connected, and also, that it is an interval in the string. Since any element of R is joined to an element of L , $|R| \leq 2$ and we have $d \leq k + m$. Let us improve this by one, as follows.

Suppose that equality holds. Then the group induced on the blocks is S_m , and the group induced on a block by its stabiliser is S_k ; the induced subdiagram on C is also connected; and the two elements of R are at each end of L , one between L and C , and one (say ρ) at the end, and so commuting with C . Now the group generated by C

induces the symmetric group on every block, and so ρ must permute the blocks non-trivially; but then, since $\langle C \rangle$ and $\langle L \rangle$ centralise each other, and $\langle L \rangle \cong S_m$, and since ρ also centralises $\langle C \rangle$, ρ has to be in $\langle L \rangle$, a contradiction.

Thus $d \leq k + m - 1 \leq 2 + n/3$, which is at most $n/2$ as long as $n \neq 9$ which is the case here since $n \geq 10$.

Case 2: $k = 2$. We may again assume that the induced subdiagram on L is connected as in Case 1. Thus $|R| \leq 2$. As above, an involution which commutes with a group acting primitively on the blocks must fix every block, and so must fix or interchange the two points in each block. Let ρ be such an involution. If ρ fixes the points in some but not all of the blocks, then (by the primitivity of the group induced on the blocks) we can find an element of the group generated by L which moves a block fixed pointwise by ρ to one not fixed pointwise, and hence not commuting with ρ . So ρ must interchange the points in every block. But then ρ lies in the centre of $C_2 \wr S_{n/2}$, the group fixing the block system. This shows that the set C has cardinality at most one, and its element is the fixed-point-free involution that swaps the points inside the blocks. Therefore the bound $d \leq k - 1 + m - 1 + 2 = 2 + n/2$.

Now we can show that $d \leq n/2 + 1$ and that equality holds if and only if $n \equiv 2 \pmod{4}$ in this case. If $d = n/2 + 2$ then the diagram must have the $n/2 - 1$ nodes of L with one node of R at each end and C must be the fixed-point-free involution fixing each block. Each element of R commutes with a $S_{n/2-1}$ permuting the blocks, since the elements of L correspond to standard Coxeter generators of $S_{n/2}$ as $n/2 > 4$. So in fact both elements of R must fix at least $n/2 - 1$ blocks, and so must fix every block.

The subgroup of S_n fixing every block is $C_2^{n/2}$. As $S_{n/2}$ -module it has just four submodules: the zero module, the module generated by the all-1 vector, the even-weight submodule, and the whole module. Since the involution in C corresponds to the all-1 vector, if $n \equiv 0 \pmod{4}$, adding any even vector (and hence any odd vector as well) will make the all-1 vector dependant and break independence of the generators. Therefore $R = \emptyset$ in that case. If $n \equiv 2 \pmod{4}$, adding an odd vector will break independence of the all-1 vector, but adding an even vector will not. There are only two ways to add such a vector in order to satisfy the commuting property, namely by having an involution which swaps all but two points that are the two points moved by one of the generators at the end of the L -string. Assume we take the two. Then, as we have the symmetric group acting on the blocks, we can move the fixed pair of points of the first to the fixed pair of points of

the second. So, if we take both, independence will not be preserved. Hence $|R| = 1$ in that case and the involution of R swaps all but two points. Moreover, $d = n/2 + 1 < n/2 + 2$. This case exists as will be shown in Proposition 2.2, hence it is not possible to improve the bound here.

Case 3: $m = 2$. In this case, $|L| = 1$, and so $|R| \leq 2$. We claim that $|C| \leq k - 1$ in this case too. Let $L = \{\rho\}$. Elements of C either fix or interchange the blocks. We claim that multiplying them by powers of ρ (i.e. by ρ or the identity) so that they fix the blocks preserves independence. For suppose that, for $\rho_i \in C$, $\rho_i \rho^{\epsilon_i}$ fixes the blocks. Suppose that $\rho_i \rho^{\epsilon_i}$ is a product of other elements of this form. Since all elements of C commute with ρ , we find that ρ_i is a corresponding product of elements ρ_j times a power of ρ , contradicting independence of the original set.

Thus we have $d \leq 1 + 2 + (n/2 - 1) = n/2 + 2$.

Assume that equality holds. Then the induced subdiagram on C is connected, and ρ and the elements of C generate $C_2 \times S_k$. So the string consists of C , followed by an element of R , then ρ , then the other element of R (say ρ'). Now ρ' preserves the block system and commutes with C , so it must necessarily be in $\langle C, L \rangle$, a contradiction. Hence $d \leq n/2 + 1$.

Now suppose that $d = n/2 + 1$. That means R consists of one involution ρ' . Assume first that ρ' permutes the two blocks. In that case, we have that $\rho' \rho \in \langle C \rangle$ and therefore, $\rho' \in \langle C, L \rangle$, a contradiction. Assume next that ρ' does not permute the blocks. Then $\rho'^\rho \in \langle C \rangle$ and therefore, again, $\rho' \in \langle C, L \rangle$, a contradiction.

Hence, in this case, $d \leq n/2$. □

Let us now classify the groups attained by the bound. These groups occur only for degree $n \equiv 2 \pmod{4}$ as shown in the proof above. For $n = 6$ and $n = 10$, we use MAGMA to classify all string C-groups of degree n coming from transitive imprimitive groups.

A string C-group consists of a pair (G, S) where G is a group and S is a set of generating involutions of G satisfying the string property and the intersection property. We denote a string C-group by Γ and a generic permutation group that is not necessarily a string C-group by G .

Proposition 2.2. *Let G be a transitive imprimitive group of degree $n \geq 11$ and let S be an independent generating set of involutions satisfying the intersection property. If $|S| = n/2 + 1$ then $n \equiv 2 \pmod{4}$ and, up to isomorphism, the pair (G, S) is unique.*

Proof. We already saw in the proof of Proposition 2.1 that if $|S| = n/2 + 1$, $n \equiv 2 \pmod{4}$ and that in this case, (G, S) has to be in Case 2 of the proof. So we may assume that $n \geq 14$, $k = 2$ and $m = n/2 \geq 7$. Looking at this case closer, we see that no matter whether C is empty or not, R is of size at most 1. There is, up to isomorphism and duality, a unique way to choose R once C and L have been chosen with $|C| = 1$ and $|L| = n/2 - 1$. Generators for this case are as follows:

$$\begin{aligned}\rho_0 &= (1, n/2 + 1)(2, n/2 + 2) \dots (n/2, n); \\ \rho_1 &= (2, n/2 + 2) \dots (n/2, n); \\ \rho_i &= (i - 1, i)(n/2 + i - 1, n/2 + i) \text{ for } 2 \leq i \leq n/2.\end{aligned}$$

Moreover the Schläfli type is $\{2, 4, 3, \dots, 3\}$ and $\Gamma \cong C_2 \wr S_{n/2}$.

Suppose now that $|L|$ is of size $n/2 - 2$ and C is of size 1. Then by [5, Theorem 2], there is a unique independent generating set satisfying the string condition for the block action. In order to have $|S| = n/2 + 1$, we then would need $|R| = 2$. For similar reasons as in the proof of Case 2 of Proposition 2.1, this cannot occur. \square

Observe that the unique example obtained in Proposition 2.2 is example (b) in Theorem 1.2, which is also line 3 of Table 1 for degree 6.

3. PRIMITIVE GROUPS

In this section we assume that Γ is a string C-group of rank d , which is isomorphic to a primitive subgroup of S_n other than S_n or A_n .

Asymptotically, our bounds for this case are much better than required for the main theorem. Rather than give an explicit bound for the rank of a polytope with primitive group, we give a procedure which will give a good bound for the rank, and deduce that $d < n/2$ in all but a few cases.

The basic tool, which follows from the Classification of Finite Simple Groups, is the fact that primitive groups other than S_n and A_n have small order. The best result along these lines is due to Maróti [12], which gives three possibilities for such a group.

Theorem 3.1 (Maróti [12]). *Let G be a primitive group of degree n which is not S_n or A_n . Then one of the following possibilities occurs:*

- (a) *For some integers m, k, l , we have $n = \binom{m}{k}^l$, and G is a subgroup of $S_m \wr S_l$, where S_m is acting on k -subsets of $\{1, \dots, m\}$;*
- (b) *G is M_{11} , M_{12} , M_{23} or M_{24} in its natural 4-transitive action;*
- (c) $|G| \leq n \cdot \prod_{i=0}^{\lfloor \log_2 n \rfloor - 1} (n - 2^i)$.

Degree	Group	Schläfli types	Reference
10	S_6	[3, 3, 3, 3]	[11]
6	A_5	[3, 5], [5, 5]	[11]
	S_5	[3, 3, 3], [4, 5], [4, 6], [5, 6], [6, 6]	[11]

TABLE 2. Primitive string C-groups of degree n and rank $\geq n/2$.

Proposition 3.2. *With the above hypotheses, $d < n/2$ except for the examples appearing in Table 2.*

Proof. Observe first that we may assume the diagram of Γ to be connected if Γ is primitive. Indeed, suppose the diagram is not connected. Then $d \leq 2 \log_2 n < n/2$ when $n \geq 60$. The strategy for dealing with the cases of Theorem 3.1 is as follows.

In case (a) with $l = 1$, the group is a subgroup of S_m or A_m , so its rank is at most $m - 1$; and we have $n = \binom{m}{k}$, so m is roughly $((k!)n)^{1/k}$, much smaller than linear in n .

In case (a) with $l > 1$, we can replace the primitive action of degree $\binom{m}{k}^l$ with an imprimitive action of degree ml , which is much smaller; by our results for imprimitive groups, the rank is at most $(ml/2) + 1$.

In case (b), the maximum rank is known from explicit computation (see [11, 10]). It is respectively 0, 4, 0 and 5 for M_{11} , M_{12} , M_{23} and M_{24} . In case (c), there are various methods that can be used.

- A theorem of Conder [4] shows that a string C-group of rank d has order at least $2^{2^{d-1}}$ when $d \geq 9$. On the other hand, Maróti's bound in this case is at most $n^{\lfloor \log_2 n \rfloor + 1}$. So we have $d \leq (\log_2 n (\log_2 n + 1) + 1)/2$. A simple calculation shows that $d < n/2$ for $n \geq 23$.
- By the Intersection Property, the boolean lattice of rank d is embedded into the subgroup lattice of a string C-group of rank d . In particular, such a group has a chain of subgroups of length d . The length of the longest chain of subgroups in a group has been investigated by Solomon and Turull with various co-authors in a sequence of papers (see, for example, [14]), and their results can be used to bound d for specific groups.

In particular, just by using the bounds and the length of the boolean lattice, we find that a primitive group other than S_n or A_n has rank necessarily smaller than $n/2$, except maybe for the groups listed in Table 3. Degree and Number correspond to the numbering in Sims' list of primitive groups of degree ≤ 50 (see for instance [2]). These groups

Degree	Number	Group	Maximal rank	Reference
22	1	M_{22}	0	[11]
	2	$\text{Aut}(M_{22})$	4	[11]
16	19	$2^4 : A_7$	At most 7	
	20	$2^4 : A_8$	At most 7	
15	4	$\text{PSL}_4(2) \cong A_8$	0	[11]
13	7	$\text{PSL}_3(3)$	0	[11]
12	3	M_{11}	0	[11]
	4	M_{12}	4	[11]
11	5	$\text{PSL}_2(11)$	4	[11]
	6	M_{11}	0	[11]
10	3	$\text{PSL}_2(9) \cong A_6$	0	[11]
	4	S_6	5	[11]
	5	$\text{PGL}_2(9)$	3	[11]
	6	M_{10}	0	[11]
	7	$\text{P}\Gamma\text{L}_2(9)$	3	[11]
9	5	$\text{A}\Gamma\text{L}(1, 9)$	0	
	6	$3^2 : (2 \cdot A_4)$	0	
	7	$\text{A}\Gamma\text{L}(2, 3)$	0	
	8	$\text{PSL}_2(8)$	0	[11]
	9	$\text{P}\Gamma\text{L}(2, 8)$	3	[11]
8	2	$\text{A}\Gamma\text{L}(1, 8)$	0	
	3	$\text{PSL}_2(7)$	0	[11]
	4	$\text{PGL}(2, 7)$	3	[11]
	5	$\text{ASL}_3(2)$	0	
7	4	$\text{A}\Gamma\text{L}(1, 7)$	0	
	5	$\text{PSL}_2(7)$	0	[11]
6	1	A_5	3	[11]
	2	S_5	4	[11]

TABLE 3. Groups not excluded by bounds and lengths of lattices.

can be constructed in MAGMA using the function `PrimitiveGroupSims`. For most of these groups, the highest rank is known as they were fully investigated in [11].

Let G be the group of degree 16, number 19. It cannot be a string C-group of rank 8. Indeed, it has two conjugacy classes of involutions. The smallest one is of size 15 and hence the centraliser of an involution of that class is of order 2688. Suppose G is a string C-group of rank

8. Then, the centraliser of ρ_0 must be of order at least 2×1728 , a contradiction.

Let G be the primitive group of degree 16, number 20. It has 4 conjugacy classes of subgroups. Only one of them is of length small enough to have $|C_G(\rho)| \geq 2 \times 1728$ for ρ in that conjugacy class. Using MAGMA, it is easy to check that no subgroup of order at least 1728 of that centraliser does not contain ρ . Hence the rank of G cannot be more than 7.

The groups of degree $d \leq 9$ that were not listed in [11] are easily dealt with using MAGMA.

So the proposition is proved. \square

4. CONSEQUENCES

Corollary 4.1. *Suppose $\Gamma = A_n$ of degree n is a string C-group generated by the set of involutions $S = \{\rho_0, \dots, \rho_{d-1}\}$. If $\Gamma_i := \langle \rho_j : j \in \{0, \dots, d-1\} \setminus \{i\} \rangle$ is transitive imprimitive for some $i \in \{0, \dots, d-1\}$, then $d \leq n/2 + 1$.*

Proof. Suppose first that $n \leq 9$. In that case, it is easy to check with MAGMA that, unless $n = 8$, the rank of Γ_i is at most $n/2$. In the case where $n = 8$, A_8 is known not to be a string C-group (see for instance [11]). Suppose then that $n > 9$. By Proposition 2.1, we know that $d \leq n/2 + 1 + 1$. If $d = n/2 + 2$, then Γ_i must be as described in Proposition 2.2. But this implies that one of the generators of Γ_i is an odd permutation and G cannot be A_n , a contradiction. Hence $d < n/2 + 2$. \square

Corollary 4.2. *Suppose $\Gamma = A_n$ of degree n is a string C-group generated by the set of involutions $S = \{\rho_0, \dots, \rho_{d-1}\}$. If $\Gamma_i := \langle \rho_j : j \in \{0, \dots, d-1\} \setminus \{i\} \rangle$ is primitive for some $i \in \{0, \dots, d-1\}$, then $d \leq n/2$.*

Proof. By Proposition 3.2, we know that the only primitive groups (distinct from A_n or S_n) that are string C-groups of rank at least $n/2$ are those listed in Table 2. Since Γ_i must be a subgroup of A_n , it must consist only of even permutations, which is not the case of S_6 or S_5 . Then $\Gamma_i = A_5$ and $n = 6$. But A_6 is not a string C-group (see [11]). Hence Γ_i is of rank at most $n/2 - 1$ and $d \leq n/2$. \square

Corollary 4.3. *Suppose $\Gamma = A_n$ of degree n is a string C-group of rank d . If $d \geq n/2 + 2$, all subgroups Γ_i must be intransitive.*

Proof. This is an immediate consequence of the previous two corollaries. \square

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