

# FINITE PRESENTABILITY AND ISOMORPHISM OF CAYLEY GRAPHS OF MONOIDS

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ABSTRACT. Two finitely generated monoids are constructed, one finitely presented the other not, whose (directed, unlabelled) Cayley graphs are isomorphic.

## 1. INTRODUCTION

Is the property of being finitely presented a quasi-isometry invariant for the class of all monoids? This question is posed by Gray and Kambites in [7, Question 1]. It is well known that the answer is positive for groups [3, Proposition I.8.24], and Gray and Kambites also establish it for monoids with finitely many left- and right ideals [7, Theorem 4], and for left cancellative monoids [8, Theorem A]. They also show that finite presentability is not invariant when the direction of edges in Cayley graphs is suppressed ([8, Theorem 11]). The purpose of this paper is to answer the general question negatively, by means of a concrete example. In fact, more strikingly, the two monoids  $M$  and  $N$  that are constructed, one finitely presented and the other not, have *isomorphic* (directed, unlabelled) Cayley graphs.

For a monoid  $M$  with a generating set  $A$  the (right) *Cayley graph*  $\Gamma(M, A)$  is a labelled directed graph with vertices  $M$  and, for every  $a \in A$ , edges  $(x, xa)$  ( $x \in M$ ) labelled  $a$ . It encodes the right regular representation of  $M$ , and hence completely determines  $M$  by Cayley's Theorem (for monoids). One can, of course, dually define the *left* Cayley graph of  $M$ , which, unlike for groups, need not be isomorphic to the right one. Cayley graphs are fundamental objects in combinatorial theory of monoids and groups. Considering undirected paths in  $\Gamma(M, A)$  turns  $M$  into a metric space; this transition has an enormous significance in group theory, and is essentially the interface between combinatorial and geometric theories of groups; see for instance the Introduction to [5]. In a series of papers [6, 7, 8] Kambites and Gray argue that the 'correct' analogue in the case of monoids are semimetric spaces, obtained by considering *directed* paths in the Cayley graph. In either case, quasi-isometry is defined as a natural geometrical notion, which identifies spaces which 'look alike' from 'far away'; see [3, Definition 8.14] and [7, Definition 7]. The question then arises which algebraic properties of the original

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monoid  $M$  remain ‘encoded’ in the new geometric setting, or, more precisely, are invariant under quasi-isometries.

For the purposes of the present paper we will not actually require any of the geometric apparatus, only the notion of the *unlabelled* Cayley graph  $\Gamma'(M, A)$ , obtained from  $\Gamma(M, A)$  by simply ‘omitting’ all the edge labels. Note that this allows for loops and multiple edges between pairs of vertices, although they will actually not occur in the examples we are about to exhibit. Our main result is as follows:

**Main Theorem.** *Let  $M$  and  $N$  be the monoids defined by the presentations*

$$\begin{aligned} M &= \langle a, b \mid ab^n a = aba \ (n = 2, 3, 4, \dots) \rangle \\ N &= \langle c, d \mid cdc = cd^2c = cd^4 = cd^3c^2 = cd^3cdc \rangle. \end{aligned}$$

*Then*

- (a)  $M$  is not finitely presented.
- (b)  $N$  is finitely presented.
- (c) *The unlabelled (directed, right) Cayley graphs  $\Gamma'(M, \{a, b\})$  and  $\Gamma'(N, \{c, d\})$  are isomorphic.*

It is obvious that  $N$  is finitely presented, and it is easy to prove that  $M$  is not. Indeed, if it were, then it would be defined by a finite subset of the given presentation, say

$$M = \langle a, b \mid ab^2a = aba, ab^3a = aba, \dots, ab^{n_0}a = aba \rangle;$$

this general algebraic fact is simply a way of saying that congruence generation is an algebraic closure operator [2, Theorem 5.5] and is stated explicitly for semi-groups in [4, Theorem 9.14]. But then no defining relation could be applied to the word  $ab^{n_0+1}a$ , contradicting the fact that  $ab^{n_0+1}a = aba$  in  $M$ .

Thus, the only task is to prove (c). We do this by utilising rewriting systems to establish a set of normal forms for each monoid (Section 2), and then analysing the action of generators on these normal forms. We first present an intuitive graphical descriptions of the Cayley graphs of  $M$  and  $N$  (Section 3), and then write down an explicit isomorphism of unlabelled versions (Section 4). Since the edge labels play no role in determining the (semi)metric structure on a monoid (see [7, Section 2]), the negative answer to [7, Question 1] is an immediate consequence of our theorem.

## 2. NORMAL FORMS

In order to write down normal forms for  $M$  and  $N$  we will use rewriting systems, and we begin with a brief overview; for full details see [1]. A *rewriting system*  $R$  over an alphabet  $A$  is a subset of  $A^* \times A^*$ , where  $A^*$  denotes the free monoid over  $A$ , including the empty word  $\epsilon$ . The *single step rewriting relation*  $\rightarrow$  on  $A^*$  is  $\{(xuy, xvy) : (u, v) \in R, x, y \in A^*\}$ , and the *rewriting relation*  $\rightarrow^*$

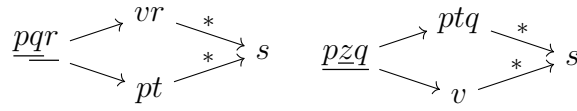
is the reflexive, transitive closure of  $\rightarrow$ . The equivalence relation generated by  $\rightarrow^*$  is denoted by  $\leftrightarrow^*$ ; this is precisely the congruence on  $A^*$  arising from the presentation  $\langle A \mid R \rangle$ .

The rewriting system  $R$  is *noetherian* if there are no infinite sequences of single step rewritings  $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow \dots$ . A word  $w \in A^*$  is *irreducible* if there is no word  $u$  with  $w \rightarrow u$ . If  $R$  is noetherian, for every  $w \in A^*$  there exists an irreducible word  $u$  such that  $w \rightarrow^* u$ .

The rewriting system  $R$  is *confluent* (respectively locally confluent) if for any  $w, u, v \in A^*$  with  $w \rightarrow^* u$ ,  $w \rightarrow^* v$  (respectively  $w \rightarrow u$ ,  $w \rightarrow v$ ) there exists  $z \in A^*$  such that  $u \rightarrow^* z$ ,  $v \rightarrow^* z$ . For a noetherian rewriting system confluence is equivalent to local confluence [1, Theorem 1.1.13]. In turn they are equivalent to the *resolution of overlaps*, namely for for any  $(u, v), (z, t) \in R$  the following being true:

1. If  $u = pq$ ,  $z = qr$  with  $q \neq \epsilon$  then there exists  $s \in A^*$  such that  $vr \rightarrow^* s$  and  $pt \rightarrow^* s$ .
2. If  $u = pqz$  then there exists  $s \in A^*$  such that  $v \rightarrow^* s$  and  $ptq \rightarrow^* s$ ;

see [1, Section 2.3]. Situations described under 1) and 2) can be visualised as follows:



A noetherian and confluent rewriting system is said to be *complete*. For such a system every equivalence class of  $\leftrightarrow^*$  contains a unique irreducible element.

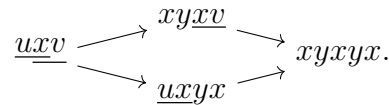
It is now easy to check that the presentations defining  $M$  and  $N$ , if interpreted as rewriting systems

$$R_M : ab^n a \rightarrow aba \quad (n = 2, 3, \dots)$$

$$R_N : cd^2 c \rightarrow cdc, \quad cd^4 \rightarrow cdc, \quad cd^3 c^2 \rightarrow cdc, \quad cd^3 cdc \rightarrow cdc,$$

are complete. Indeed, they are noetherian as all rewriting rules are length-reducing. Furthermore, every overlap in either of the two rewriting systems has the form  $\underline{uxv}$ , where  $x \in \{a, c\}$  and  $ux \rightarrow xyx$  and  $xv \rightarrow xyx$  are rewriting rules

for an appropriate  $y \in \{b, d\}$ , and it resolves as follows:



It follows that sets of normal forms are provided by the irreducible words with respect to the two rewriting systems. In the case of  $R_M$  these are precisely the words which do not contain subwords  $ab^n a$  for  $n \geq 2$ . For future use we will record them as follows:

**Lemma 1.** *Let*

$$U_M = \{a^{i_0} b a^{i_1} b \dots b a^{i_{k-1}} b a^{i_k} : k \geq 0, i_0, \dots, i_k \geq 1\}.$$

*The monoid  $M$  admits the following set of normal forms*

$$\text{NF}_M = \{b^s, b^s u, b^s u b^t : u \in U_M, s \geq 0, t > 0\}.$$

The irreducible normal forms for  $N$  are just a little bit more complicated. Clearly, all  $c^m, d^m$  ( $m \geq 0$ ) are irreducible. Let us consider an arbitrary irreducible word  $w$  which contains occurrences of both  $c$  and  $d$ , and write  $w$  in the following form:

$$w = d^{j_0} c^{i_1} d^{j_1} \dots c^{i_{k-1}} d^{j_{k-1}} c^{i_k} d^{j_k}$$

$$(k \geq 1; i_1, \dots, i_k \geq 1; j_0, j_k \geq 0; j_1, \dots, j_{k-1} \geq 1).$$

Bearing in mind the rewriting rules from  $R_N$  we see that the following hold:

- None of  $j_1, \dots, j_k$  can exceed 3 (because of  $cd^4 \rightarrow cdc$ ).
- None of  $j_1, \dots, j_{k-1}$  equal 2 (because of  $cd^2c \rightarrow cdc$ ).
- If  $j_l = 3$  for some  $l = 1, \dots, k-1$ , then for the subsequent indices we must have  $i_{l+1} = \dots = i_k = 1, j_{l+1} = \dots = j_{k-1} = 3$  (because of  $cd^3c^2 \rightarrow cdc$  and  $cd^3cdc \rightarrow cdcdc$ ).

We can summarise our findings as follows:

**Lemma 2.** *Let*

$$U_N = \{c^{i_0} d c^{i_1} d \dots d c^{i_{k-1}} d c^{i_k} : k \geq 0, i_0, \dots, i_k \geq 1\}.$$

*The monoid  $N$  admits the following set of normal forms*

$$\text{NF}_N = \{d^p, d^p u, d^p u (d^3 c)^q d^r : u \in U_N, p, q \geq 0, 0 \leq r \leq 3, q + r > 0\}.$$

### 3. CONSTRUCTION OF CAYLEY GRAPHS

In this section we describe what the Cayley graphs for  $M$  and  $N$  look like. Each construction will be in three steps. Since these steps turn out to be identical, as are the starting ingredients, provided we ignore the edge labels, this provides an intuitive explanation of the isomorphism between  $\Gamma'(M, \{a, b\})$  and  $\Gamma'(N, \{c, d\})$ . The actual formal proof will follow in Section 4.

We begin with the description of  $\Gamma(M, \{a, b\})$ . The basic building block  $\Gamma_M^{(1)}$  for this graph is shown in Figure 1. In the view of the relations  $ab^n a = aba$  a copy of  $\Gamma_M^{(1)}$  will be found at every vertex of  $\Gamma(M, \{a, b\})$  which receives an edge labelled  $a$ . This process can be captured recursively as follows:

- Start with a single edge labelled  $a$ .
- At each subsequent step, for every vertex  $w$  introduced in the previous step with an in-edge labelled  $a$  attach a new out-edge labelled  $a$  and a copy of  $\Gamma_M^{(1)}$  based at  $w$ .

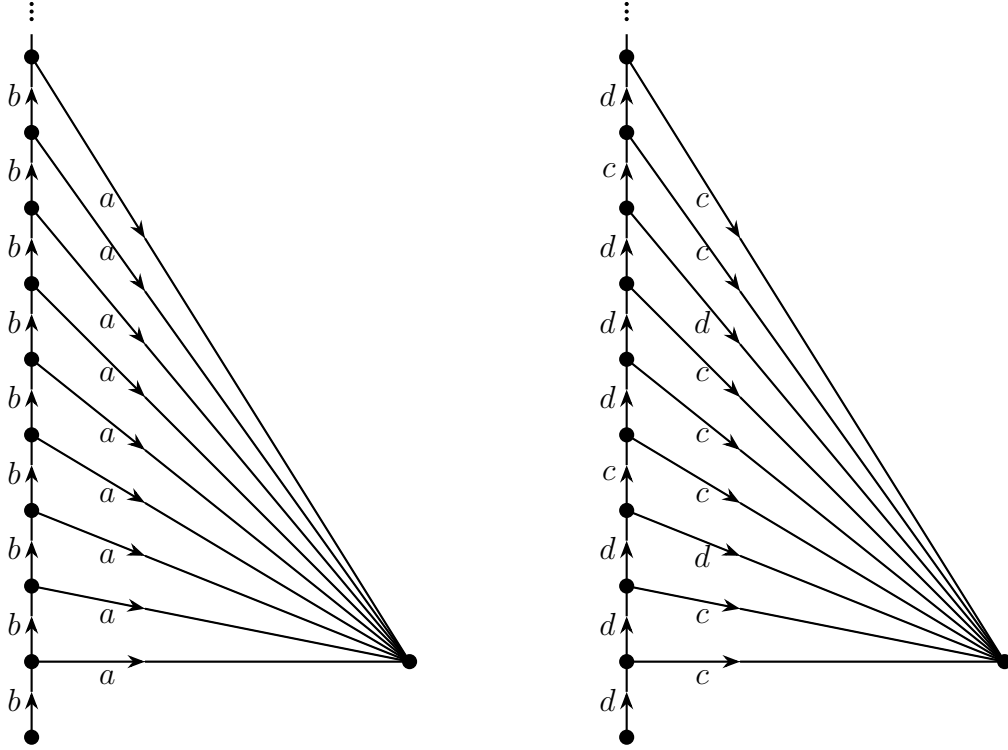


FIGURE 1. The graphs  $\Gamma_M^{(1)}$  (on the left) and  $\Gamma_N^{(1)}$  (on the right).

The resulting graph  $\Gamma_M^{(2)}$  is sketched in Figure 2. This is the portion of  $\Gamma(M, \{a, b\})$  corresponding to the normal forms beginning with  $a$ . All the normal forms are obtained by prepending these by an arbitrary power of  $b$ , yielding the graph  $\Gamma_M^{(3)} = \Gamma(M, \{a, b\})$  shown in Figure 3.

Let us now turn our attention to  $\Gamma(N, \{c, d\})$ . The basic building block  $\Gamma_N^{(1)}$ , depicted in Figure 1, this time reflects the normal forms  $(d^3c)^qd^r$ , and the fact that if they are pre- and post-multiplied by  $c$  the resulting word ‘collapses’ to  $cdc$ . Since there will be a copy of this graph emanating from every vertex which has an in-edge labelled  $c$ , we can set up the following recursive process:

- Start with a single edge labelled  $c$ .
- At each subsequent step, for every vertex  $w$  introduced in the previous step with an in-edge labelled  $c$  attach a new out-edge labelled  $c$  and a copy of  $\Gamma_N^{(1)}$  based at  $w$ .

This results in the graph  $\Gamma_N^{(2)}$  depicted in Figure 2. It is important to notice that in this graph all the words of the form  $u(d^3c)^qd^r$  ( $u \in U_N$ ,  $0 \leq r \leq 3$ ) can be traced from the initial vertex, and that they all lead to distinct vertices. Finally, by Lemma 2, each of these normal forms can be preceded by an arbitrary power of  $d$ , and so we obtain the graph  $\Gamma_N^{(3)} = \Gamma(N, \{c, d\})$  shown in Figure 3.

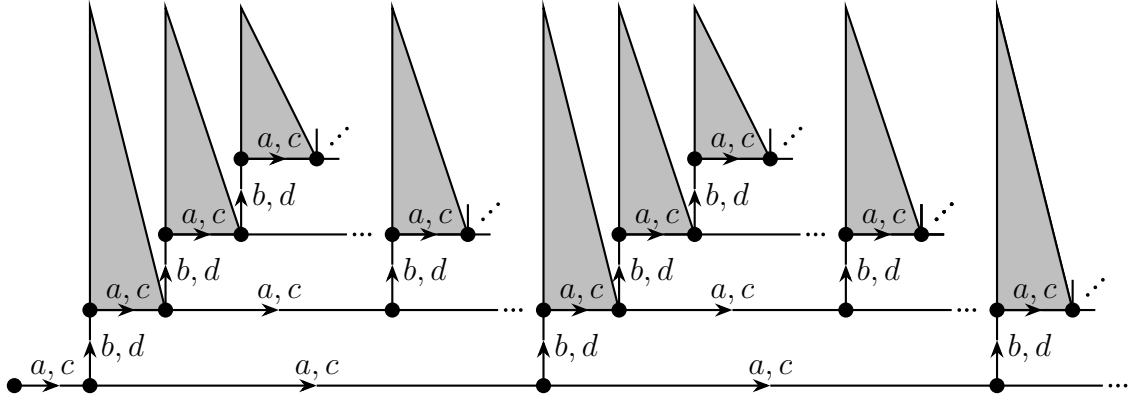


FIGURE 2. The graphs  $\Gamma_M^{(2)}$  (reading the  $a, b$  labels) and  $\Gamma_N^{(2)}$  (reading the  $c, d$  labels). Each constituent copy of  $\Gamma_M^{(1)}$  (resp.  $\Gamma_N^{(1)}$ ) is indicated by its initial  $ba$  (resp.  $dc$ ) path and a shaded triangle.

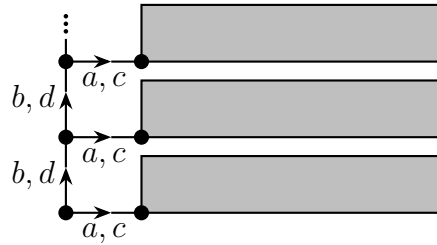


FIGURE 3. The graphs  $\Gamma_M^{(3)} = \Gamma(M, \{a, b\})$  and  $\Gamma_N^{(3)} = \Gamma(N, \{c, d\})$ . Each copy of  $\Gamma_M^{(2)}$  (resp.  $\Gamma_N^{(2)}$ ) is indicated by its initial  $a$  (resp.  $c$ ) edge and a shaded rectangle.

The above discussion and Figures 1, 2, 3 show that when the edge labels are ignored the underlying graphs are isomorphic. We remark that the left Cayley graphs of  $M$  and  $N$  are not isomorphic, as is easily verified.

#### 4. ISOMORPHISM

We now provide a rigorous proof of our Main Theorem, by exhibiting an explicit isomorphism  $f$  between  $\Gamma(M, \{a, b\})$  and  $\Gamma(N, \{c, d\})$ . Essentially, the mapping  $f$  acts on a typical normal form  $b^s u b^t$  ( $u \in U_M$ ) by simple replacement  $a \mapsto c$ ,  $b \mapsto d$  within the  $b^s u$  prefix, and transforms the final  $b^t$  into  $(d^3 c)^q d^r$  where  $t = 4q + r$ ,  $0 \leq r \leq 3$ . Since there are actually several types of normal forms, and since how a generator acts on a normal form depends on its type, the actual definition and the subsequent verification splits into cases. For the sake of conciseness we have organised all the information in tabular form.

Label	Normal Form	Parameters
NFM1	$b^s$	$s \geq 0$
NFM2	$b^s u$	$s \geq 0, u \in U_M$
NFM3	$b^s u b^t$	$s \geq 0, u \in U_M, t > 0$

TABLE 1. Normal forms for  $M$ 

Label	Edge	Vertex Types
NFM1a	$(b^s, b^s a)$	(NFM1, NFM2)
NFM1b	$(b^s, b^{s+1})$	(NFM1, NFM1)
NFM2a	$(b^s u, b^s u a)$	(NFM2, NFM2)
NFM2b	$(b^s u, b^s u b)$	(NFM2, NFM3)
NFM3a	$(b^s u b^t, b^s u b a)$	(NFM3, NFM2)
NFM3b	$(b^s u b^t, b^s u b^{t+1})$	(NFM3, NFM3)

TABLE 2. The edges in  $\Gamma'(M, \{a, b\})$ 

First of all, the normal forms for  $M$  given by Lemma 1 are listed in Table 1 alongside the permitted values of relevant parameters ( $s, t, u$ ). These values will subsequently be taken as read, without explicitly writing them down. We take these normal forms for the vertices of  $\Gamma'(M, \{a, b\})$ . For each vertex there are two edges coming out of it, corresponding to multiplications by  $a$  and  $b$  respectively. For instance, multiplying the normal form  $b^s u b^t$  of type NFM3 by  $a$  and  $b$  yields

$$b^s u b^t \cdot a \rightarrow^* b^s u b a, \quad b^s u b^t \cdot b = b^s u b^{t+1},$$

normal forms of types NFM2 and NFM3 respectively, resulting in two edges

$$(1) \quad (b^s u b^t, b^s u b a), \quad (b^s u b^t, b^s u b^{t+1}).$$

The complete information on the edges of  $\Gamma'(M, \{a, b\})$  is given in Table 2. The only instance where the normal form differs from the simple concatenation with the generator is the row labelled NFM3a, where the rewriting rule  $ab^t a \rightarrow aba$  is applied.

The corresponding information for  $\Gamma'(N, \{c, d\})$  is given in Tables 3, 4. It is worth noting that a further split in cases occurs for edges from vertices  $d^p v (d^3 c)^q d^r$  of type NFN3. Indeed, bearing in mind that  $v$  is non-empty and ends with  $c$ , depending on the value of  $r$  we have:

$$\begin{aligned} d^p v (d^3 c)^q \cdot c &= d^p v (d^3 c)^{q-1} d^3 c^2 \rightarrow d^p v (d^3 c)^{q-1} d c \rightarrow^* d^p v d c \\ d^p v (d^3 c)^q d \cdot c &\rightarrow^* d^p v d c \\ d^p v (d^3 c)^q d^2 \cdot c &\rightarrow d^p v (d^3 c)^q d c \rightarrow^* d^p v d c \\ d^p v (d^3 c)^q d^3 \cdot c &= d^p v (d^3 c)^{q+1}. \end{aligned}$$

Label	Normal Form	Parameters
NFN1	$d^p$	$p \geq 0$
NFN2	$d^p v$	$p \geq 0, v \in U_N$
NFN3	$d^p v (d^3 c)^q d^r$	$p, q \geq 0, v \in U_N, 0 \leq r \leq 3, q + r > 0$

TABLE 3. Normal forms for  $N$ 

Label	Edge	Parameters	Vertex Types
NFN1c	$(d^p, d^p c)$		(NFN1, NFN2)
NFN1d	$(d^p, d^{p+1})$		(NFN1, NFN1)
NFN2c	$(d^p v, d^p v c)$		(NFN2, NFN2)
NFN2d	$(d^p v, d^p v d)$		(NFN2, NFN3)
NFN3c	$(d^p v (d^3 c)^q d^r, d^p v d c)$	$0 \leq r \leq 2$	(NFN3, NFN2)
	$(d^p v (d^3 c)^q d^3, d^p v (d^3 c)^{q+1})$		(NFN3, NFN3)
NFN3d	$(d^p v (d^3 c)^q d^r, d^p v (d^3 c)^q d^{r+1})$	$0 \leq r \leq 2$	(NFN3, NFN3)
	$(d^p v (d^3 c)^q d^3, d^p v d c)$		(NFN3, NFN2)

TABLE 4. The edges in  $\Gamma'(N, \{c, d\})$ 

Type of $w$	$w$	Parameters	$f(w)$	Type of $f(w)$
NFM1	$b^s$		$d^s$	NFN1
NFM2	$b^s u$		$d^s \bar{u}$	NFN2
NFM3	$b^s u b^t$	$t = 4q + r$	$d^s \bar{u} (d^3 c)^q d^r$	NFN3

TABLE 5. The definition of  $f : \text{NF}_M \rightarrow \text{NF}_N$ .

Thus, for  $r = 0, 1, 2$  we obtain normal forms of type NFN2, while for  $r = 3$  it is NFN3. Similarly, for  $r = 0, 1, 2$  multiplication by  $d$  increases  $r$  by 1, while for  $r = 3$  have

$$d^p v (d^3 c)^q d^3 \cdot d = d^p v (d^3 c)^q d^4 \rightarrow d^p v (d^3 c)^q d c \rightarrow^* d^p v d c.$$

We now define the mapping  $f : \text{NF}_M \rightarrow \text{NF}_N$  which will turn out to be the desired isomorphism. Note that there is a natural bijection  $u \mapsto \bar{u}$  between  $U_M$  and  $U_N$  induced by the substitution  $a \mapsto c, b \mapsto d$ . With this notation in mind, the mapping  $f$  is defined in Table 5. It is clearly a bijection, and its inverse can be read off by reading the table from right to left.

It remains to show that  $f$  is a graph isomorphism, i.e. that both  $f$  and  $f^{-1}$  map edges to edges. This is achieved by taking each edge type from Tables 2, 4, applying  $f$  or  $f^{-1}$  to its end-points using Table 5, and verifying that the resulting pair of vertices also forms an edge. For example, consider the edge  $(b^s u b^t, b^s u b a)$  of type NFM3a that we encountered in (1). Under  $f$  its endpoints are mapped to  $d^s \bar{u} (d^3 c)^q d^r$ , where  $t = 4q + r$ , and  $d^s \bar{u} d c$ . From Table 4 we see that for  $r = 0, 1, 2$ , the pair  $(d^s \bar{u} (d^3 c)^q d^r, d^s \bar{u} d c)$  is an edge of type NFN3c, while for  $r = 3$  it is still an



Edge Type	$(w, v)$	$(f(w), f(v))$	Parameters	Edge Type
NFM1a	$(b^s, b^s a)$	$(d^s, d^s c)$		NFN1c
NFM1b	$(b^s, b^{s+1})$	$(d^s, d^{s+1})$		NFN1d
NFM2a	$(b^s u, b^s u a)$	$(d^s \bar{u}, d^s \bar{u} c)$		NFN2c
NFM2b	$(b^s u, b^s u b)$	$(d^s \bar{u}, d^s \bar{u} d)$		NFN2d
NFM3a	$(b^s u b^t, b^s u b a)$	$(d^s \bar{u} (d^3 c)^q d^r,$ $d^s \bar{u} d c)$	$r = 0, 1, 2$ $r = 3$	NFN3c NFN3d
NFM3b	$(b^s u b^t, b^s u b^{t+1})$	$(d^s \bar{u} (d^3 c)^q d^r,$ $d^s \bar{u} (d^3 c)^q d^{r+1})$ $(d^s \bar{u} (d^3 c)^q d^r,$ $d^s \bar{u} (d^3 c)^{q+1})$	$r = 0, 1, 2$ $r = 3$	NFN3d NFN3c

TABLE 6.  $f$  maps the edges of  $\Gamma'(M, \{a, b\})$  to edges of  $\Gamma'(N, \{c, d\})$ 

Edge Type	$(w, v)$	$(f^{-1}(w), f^{-1}(v))$	Parameters	Edge Type
NFN1c	$(d^p, d^p c)$	$(b^p, b^p a)$		NFM1a
NFN1d	$(d^p, d^{p+1})$	$(b^p, b^{p+1})$		NFM1b
NFN2c	$(d^p \bar{u}, d^p \bar{u} c)$	$(b^p u, b^p u a)$		NFM2a
NFN2d	$(d^p \bar{u}, d^p \bar{u} d)$	$(b^p u, b^p u b)$		NFM2b
NFN3c	$(d^p \bar{u} (d^3 c)^q d^r,$ $d^p \bar{u} d c)$ $(d^p \bar{u} (d^3 c)^q d^3,$ $d^p \bar{u} (d^3 c)^{q+1})$	$(b^p u b^{4q+r}, b^p u b a)$ $(b^p u b^{4q+3}, b^p u b^{4q+4})$	$r = 0, 1, 2$	NFM3a NFM3b NFM3b
NFN3d	$(d^p \bar{u} (d^3 c)^q d^r,$ $d^p \bar{u} (d^3 c)^q d^{r+1})$ $(d^p \bar{u} (d^3 c)^q d^3,$ $d^p \bar{u} d c)$	$(b^p u b^{4q+3}, b^p u b^{4q+4})$ $(b^p u b^{4q+3}, b^p u b a)$	$r = 0, 1, 2$	NFM3b NFM3a

TABLE 7.  $f^{-1}$  maps the edges of  $\Gamma'(N, \{c, d\})$  to edges of  $\Gamma'(M, \{a, b\})$ 

edge but its type is NFN3d. The other edge that arose in (1) was  $(b^s u b^t, b^s u b^{t+1})$  of type NFM3b. Its endpoints are mapped under  $f$  to  $d^s \bar{u} (d^3 c)^q d^r$ , where  $t = 4q + r$ , and  $d^s \bar{u} (d^3 c)^{q_1} d^{r_1}$ , where  $t + 1 = 4q_1 + r_1$ . Noting that  $q_1 = q$  and  $r_1 = r + 1$  for  $r = 0, 1, 2$ , while  $q_1 = q + 1$  and  $r_1 = 0$  for  $r = 3$ , in the former case we obtain an edge of the type NFN3d, and in the latter NFN3c. All the cases are presented in Tables 6, 7.

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## REFERENCES

- [1] R. V. Book and F. Otto, *String-Rewriting Systems*, Texts and Monographs in Computer Science, Springer-Verlag, New York, 1993.

- [2] S. Burris and H.P. Sankappanavar, *A Course in Universal Algebra*, Springer, New York, 1981.
- [3] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.
- [4] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups Vol II*, American Mathematical Society, Providence RI, 1967.
- [5] P. de la Harpe, *Topics in Geometric Group Theory*, The University of Chicago Press, Chicago, 2000.
- [6] R. Gray and M. Kambites. A Švarc-Milnor lemma for monoids acting by isometric embeddings. *Internat. J. Algebra Comput.*, 21(7):1135–1147, 2011.
- [7] R. Gray and M. Kambites. Groups acting on semimetric spaces and quasi-isometries of monoids. *Trans. Amer. Math. Soc.*, 365(2):555–578, 2013.
- [8] R. Gray and M. Kambites. Quasi-isometry and finite presentations for left cancellative monoids. *Internat. J. Algebra Comput.*, 23(5):1099–1114, 2013.

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