Bernoulli Convolutions and 1D Dynamics

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August 18, 2015

Abstract

We describe a family ϕ_λ of dynamical systems on the unit interval which preserve Bernoulli convolutions. We show that if there are parameter ranges for which these systems are piecewise convex, then the corresponding Bernoulli convolution will be absolutely continuous with bounded density. We study the systems ϕ_λ and give some numerical evidence to suggest values of λ for which ϕ_λ may be piecewise convex.

1 Introduction

In the study of self similar measures corresponding to non-overlapping iterated function systems, there is a natural way of defining an expanding dynamical system which preserves the measure and which allows one to study various properties of the measure such as dimension. The case of self similar measures with overlaps is much more involved, and it is not clear how best to study them using dynamical systems.

Bernoulli convolutions are a particularly well studied family of self-similar measures. For each $\lambda \in (0,1)$ we define the corresponding Bernoulli convolution ν_λ to be the distribution of the series

$$(\lambda^{-1} - 1)\sum_{i=1}^{\infty} a_i \lambda^i$$

where the digits a_i are picked independently from digit set $\{0, 1\}$ with probability $\frac{1}{2}$. Equivalently, Bernoulli convolutions are the unique probability measures satisfying the self similarity relation

$$\nu_\lambda = \frac{1}{2}(\nu_\lambda \circ T_0 + \nu_\lambda \circ T_1),$$

where the maps $T_i : \mathbb{R} \to \mathbb{R}$ are defined by $T_i(x) = \frac{x}{\lambda} - (\lambda^{-1} - 1)i$. For $\lambda \in (0, \frac{1}{2})$, the self similar measures are generated by a non-overlapping iterated function system consisting of the contractions T_0^{-1} and T_1^{-1}, and are invariant under the interval maps ϕ_λ given by

$$\phi_\lambda(x) = \begin{cases} \lambda^{-1}x & x \in [0, \lambda] \\ 1 & x \in (\lambda, 1-\lambda) \\ 1 - \lambda^{-1}(x - (1-\lambda)) & x \in (1-\lambda, 1) \end{cases}.$$

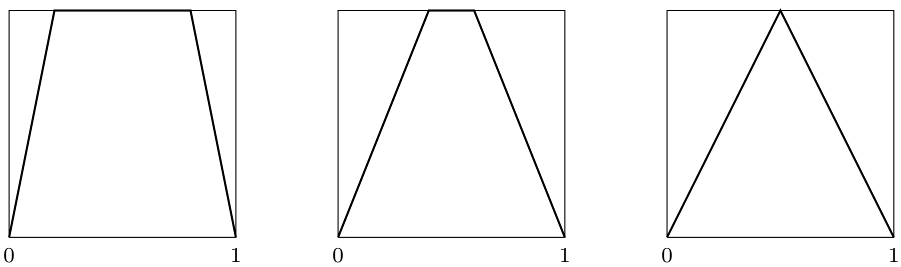

Figure 1: The maps ϕ_λ for λ equal to 0.2, 0.4 and 0.5 respectively

The main aim of this article is to extend the definition of ϕ_λ to the overlapping case, when $\lambda \in (\frac{1}{2}, 1)$, and to study ν_λ using these interval maps.

There are a number of long standing open questions about Bernoulli convolutions, chief among which is the question of for which parameters λ the corresponding measure ν_λ is absolutely continuous. It is known that each Bernoulli convolution is either purely singular or absolutely continuous, see [9]. If λ is the inverse of a Pisot number then ν_λ is singular, see [5], and in fact has Hausdorff dimension less than one, [10]. (A Pisot number is a real algebraic integer, larger than 1 and such that all its conjugates have absolute value smaller than 1.) In [7] Garsia gave a small, explicitly defined class of algebraic integers for which ν_λ is known to be absolutely continuous, and in Solomyak proved in [17] that ν_λ is absolutely continuous for almost every $\lambda \in (\frac{1}{2}, 1)$, see also [12] and [13] which looks at the smoothness of the Bernoulli convolution and the dimension of exceptions.

More recently, Hochman [8] proved that the set of parameters for which the Hausdorff dimension of ν_λ is less than one has Hausdorff dimension 0. Shmerkin [16] built on this result to prove that the set of $\lambda \in (\frac{1}{2}, 1)$ admitting singular Bernoulli convolutions has Hausdorff dimension 0. The question of determining the parameters λ which admit absolutely continuous Bernoulli convolutions remains open. For a good review of progress on Bernoulli convolutions up to the year 2000, see [14].

There are other interesting open questions regarding Bernoulli convolutions. For example, is it the case that any singular Bernoulli convolution must have Hausdorff dimension less than one? Do there exist intervals in the parameter space for which every Bernoulli convolution is absolutely continuous (and even has continuous density)? Does the density evolve continuously with λ?

Similar questions exist in the study of invariant measures associated to various one parameter families of interval maps, and in this area a good deal of progress has been made [4, 6, 11]. With this in mind, we extend the definition of the generalised tent maps ϕ_λ to the overlapping case. These tent maps preserve the corresponding Bernoulli convolutions ν_λ. They are described implicitly in terms of the distribution F_λ of ν_λ, and while we are able to write down explicit formulae for the ϕ_λ only in some special cases, we are able to prove some general properties.

In particular, we prove that if ϕ_λ is piecewise convex for all λ in some interval (a, b) then the corresponding Bernoulli convolution is absolutely continuous with bounded density. For each $x \in [0, 1]$ the map $x \mapsto \phi_\lambda(x)$ is continuous in λ, and convexity is preserved by passing to limits in a continuous family of functions. Thus, piecewise convexity of the functions ϕ_λ seems like an appropriate vehicle for passing from almost everywhere absolute continuity to everywhere absolute continuity for parameters in certain ranges. We can show that ϕ_λ is piecewise convex for certain special cases, and remain optimistic that one may be able to prove analytically that the map ϕ_λ is piecewise convex in certain parameter ranges. For the moment however, our results on piecewise convexity are restricted to some special values of λ, although we are able to run numerical approximations for any λ. There have been previous numerical investigations into Bernoulli convolutions, we mention in particular the work of Benjamini and Solomyak [1] and of Calkin et al [2, 3].

In the next section we define the maps ϕ_λ in which we are interested and prove that they preserve Bernoulli convolutions. We prove some elementary properties of the maps ϕ_λ and give the maps explicitly in some special cases. In section 3 we prove various properties of ν_λ that would follow from ϕ_λ being piecewise convex, and in section 4 we give some numerical evidence on the piecewise convexity of ϕ_λ. Finally in section 5 we state some further questions and conjectures.

2 Generalised Tent Maps

Let $F_\lambda : [0,1] \to [0,1]$ be the distribution of ν_λ, i.e. $F_\lambda(x) := \nu_\lambda[0,x]$. F_λ is strictly increasing because ν_λ is fully supported. We define a map $\phi_\lambda : [0,1] \to [0,1]$ by

$$\phi_\lambda(x) = \begin{cases} F_\lambda^{-1}(2F_\lambda(x)) & x \in [0, \frac{1}{2}] \\ F_\lambda^{-1}(2F_\lambda(1-x)) & x \in [\frac{1}{2}, 1] \end{cases} .$$

Since F_λ is strictly increasing on $[0,1]$, the map ϕ_λ is well defined. We will see later that ϕ_λ preserves ν_λ.

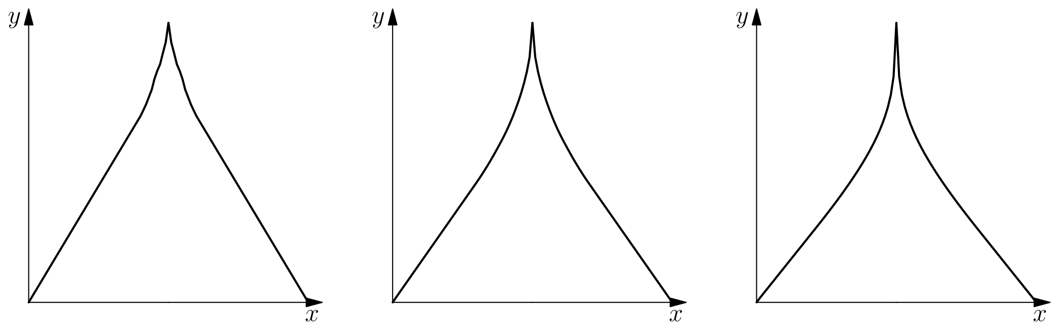

Figure 2: Graphs of ϕ_λ for $\lambda = 0.6, 0.7$ and 0.8.

The map ϕ_λ will be the chief object of study for this article. Since F_λ can be well approximated numerically, to known levels of accuracy, one can gain good numerical approximations to the maps ϕ_λ. Three such approximations are displayed for different values of λ in Figure 2.

We begin by observing some simple properties of ϕ_λ.

Lemma 2.1. *The map ϕ_λ has the following properties for $\lambda \in (\frac{1}{2}, 1)$.*

1. $\phi_\lambda(0) = \phi_\lambda(1) = 0$.

2. $\phi_\lambda\left(\frac{1}{2}\right) = 1$.

3. ϕ_λ *is strictly increasing on* $\left[0, \frac{1}{2}\right]$ *and strictly decreasing on* $\left[\frac{1}{2}, 1\right]$.

4. $\phi_\lambda(x) = \phi_\lambda(1-x)$.

5. ϕ_λ *is continuous.*

Proof. We have that $F_\lambda(0) = 0, F_\lambda(\frac{1}{2}) = \frac{1}{2}$ and $F_\lambda(1) = 1$ because ν_λ is supported on $[0,1]$ and symmetric about the point $\frac{1}{2}$. Then points 1 and 2 follow immediately.

Part 4 can be seen to be true by looking at the piecewise definition of ϕ_λ. Because $\nu_\lambda[a,b] > 0$ for each $0 \leq a < b \leq 1$ we have that F_λ is strictly increasing. Consequently ϕ_λ is strictly increasing on $\left[0, \frac{1}{2}\right]$ (and strictly decreasing on $[\frac{1}{2}, 1]$).

Finally, we observe that continuity of ϕ_λ follows from the fact that ν_λ is non-atomic and that $\nu_\lambda[a,b] > 0$ for each $0 \leq a < b \leq 1$. Then both F_λ and F_λ^{-1} are uniformly continuous, and so ϕ_λ is continuous in x. □

We call maps satisfying the above properties generalised tent maps. Our first theorem is the following

Theorem 2.1. *Let* $\lambda \in (\frac{1}{2}, 1)$*. Then* ν_λ *is invariant under* ϕ_λ*.*

Proof. It is enough to show that for each $a \in [0,1]$ we have that

$$\nu_\lambda[a,1] = \nu_\lambda(\phi_\lambda^{-1}[a,1]).$$

To prove this, we note that

$$\phi_\lambda^{-1}[a,1] = [b, 1-b]$$

where $b \in [0, \frac{1}{2}]$ satisfies $\phi_\lambda(b) = a$. But then

$$\begin{aligned}
\nu_\lambda[b, 1-b] &= 2\nu_\lambda[b, \frac{1}{2}] \\
&= 2(F_\lambda(\frac{1}{2}) - F_\lambda(b)) \\
&= 1 - 2F_\lambda(b) \\
&= F_\lambda(1) - F_\lambda(\phi_\lambda(b)) \\
&= F_\lambda(1) - F_\lambda(a) \\
&= \nu_\lambda[a, 1],
\end{aligned}$$

as required. □

Thus, if ν_λ is absolutely continuous, then it is an absolutely continuous invariant measure of ϕ_λ. We have not been able to prove the converse statement, that ϕ_λ does not have an absolutely continuous invariant measure in the case that ν_λ is singular, this would be a useful statement which would make the relationship between the study of ϕ_λ and the measures ν_λ a little more straightforward.

The following theorem shows that the maps ϕ_λ evolve continuously in λ.

Theorem 2.2. *For each $x \in [0, 1], \lambda_0 \in (\frac{1}{2}, 1)$ we have that $\phi_\lambda(x) \to \phi_{\lambda_0}(x)$ as $\lambda \to \lambda_0$.*

Proof. Fix $\lambda_0 \in (\frac{1}{2}, 1)$. We rely on three facts for this proof.

Firstly we use that the function F_λ^{-1} is continuous in x: for all $\epsilon_2 > 0$ there exists $\epsilon_1 > 0$ such that

$$|x - y| < 2\epsilon_1 \implies |F_\lambda^{-1}(x) - F_\lambda^{-1}(y)| < \epsilon_2. \tag{1}$$

Secondly we use that for each $x \in [0, 1]$ the function $F_\lambda(x)$ is continuous in λ: for all $\epsilon_1 > 0$ there exists $\delta_1 > 0$ such that

$$|\lambda - \lambda_0| < \delta_1 \implies |F_\lambda(x) - F_{\lambda_0}(x)| < \epsilon_1. \tag{2}$$

Finally we use that for each $x \in [0, 1]$ the function $F_\lambda^{-1}(x)$ is continuous in λ. For all $\epsilon_3 > 0$ there exists a $\delta_2 > 0$ such that

$$|\lambda - \lambda_0| < \delta_2 \implies |F_\lambda^{-1}(x) - F_{\lambda_0}^{-1}(x)| < \epsilon_3. \tag{3}$$

We fix x and let $\delta = \min\{\delta_1, \delta_2\}$ and $|\lambda - \lambda_0| < \delta$. Then

$$\begin{aligned}
|\phi_\lambda(x) - \phi_{\lambda_0}(x)| &= |F_\lambda^{-1}(2F_\lambda(x)) - F_{\lambda_0}^{-1}(2F_{\lambda_0}(x))| \\
&\leq \sup_{2F_\lambda(x)-2\epsilon_1 \leq y \leq 2F_\lambda(x)+2\epsilon_1} |F_\lambda^{-1}(2F_\lambda(x)) - F_{\lambda_0}^{-1}(y)| \\
&\leq |F_\lambda^{-1}(2F_\lambda(x)) - F_{\lambda_0}^{-1}(2F_\lambda(x))| + \epsilon_2 \\
&\leq \epsilon_3 + \epsilon_2,
\end{aligned}$$

Here the second line holds since $2F_{\lambda_0}(x) \in (2F_\lambda(x) - 2\epsilon, 2F_\lambda(x) + 2\epsilon)$ by equation 2. Then the third and fourth line follows from equations 1 and 3 respectively. Since ϵ_2, ϵ_3 were arbitrary, we are done. □

In the case that one knows the distribution F_λ, one can write down the map ϕ_λ explicitly. In particular, for the cases $\lambda = 2^{-\frac{1}{n}}$, which are quite well understood, it is not difficult to write down ϕ_λ.

Example 2.1. *In the case $\lambda = \frac{1}{\sqrt{2}}$, F_λ is given by*

$$F_\lambda(x) = \begin{cases} (\frac{3}{4}\sqrt{2}+1)x^2 & x \in \left[0, \frac{1}{1+\sqrt{2}}\right] \\ (1+\frac{1}{\sqrt{2}})x - \frac{\sqrt{2}}{4} & x \in [\frac{1}{1+\sqrt{2}}, \frac{\sqrt{2}}{1+\sqrt{2}}] \\ 1-(1+\frac{3}{4}\sqrt{2})(1-x)^2 & x \in [\frac{\sqrt{2}}{1+\sqrt{2}}, 1] \end{cases} .$$

Consequently ϕ_λ is given by

$$\phi_\lambda(x) = \begin{cases} \sqrt{2}x & x \in [0, \frac{1}{2+\sqrt{2}}] \\ (1+\sqrt{2})x^2 + \frac{1}{2+2\sqrt{2}} & x \in \left[\frac{1}{2+\sqrt{2}}, \frac{1}{1+\sqrt{2}}\right] \\ 1-2\left(\frac{1/2-x}{1+\sqrt{2}}\right)^{1/2} & x \in \left[\frac{1}{1+\sqrt{2}}, \frac{1}{2}\right] \end{cases}$$

which is extended to the whole interval I_λ using the symmetry around $\frac{1}{2}$. We have drawn the graphs of F_λ and ϕ_λ in Figure 3.

2.1 Further properties of ϕ_λ

While we cannot write down ϕ_λ explicitly, we can describe the behaviour near $x = 0$ and the rate of the blowup at $x = \frac{1}{2}$. The following lemma describes ϕ_λ near 0, and hence also the behaviour near 1.

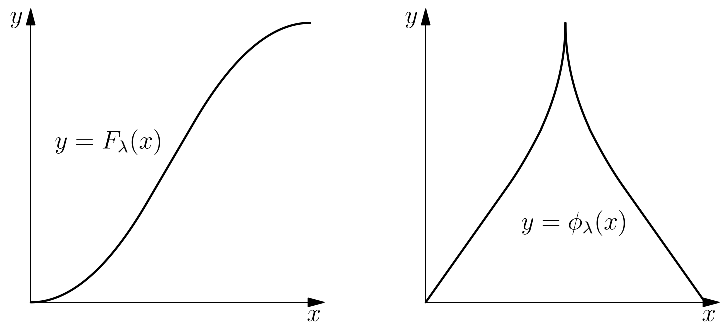

Figure 3: Graphs of F_λ and ϕ_λ for $\lambda = 1/\sqrt{2}$.

Lemma 2.2. *We have that*

$$\phi_\lambda(x) = \lambda^{-1}x$$

for $x \in [0, 1-\lambda]$.

Proof. Self similarity of the measures ν_λ give that

$$F_\lambda(x) = \frac{1}{2}\left(F_\lambda\left(\lambda^{-1}x\right) + F_\lambda\left(\lambda^{-1}x - (\lambda^{-1}-1)\right)\right) \tag{4}$$

Then

$$F_\lambda(\phi_\lambda(x)) = 2F_\lambda(x) = F_\lambda(\lambda^{-1}x) + F_\lambda(\lambda^{-1}x - (\lambda^{-1}-1)).$$

But because $F_\lambda(x) = 0$ for $x \leq 0$, we have

$$F_\lambda(\lambda^{-1}x - (\lambda^{-1}-1)) = 0$$

for $x \leq 1-\lambda$. Then

$$F_\lambda(\phi_\lambda(x)) = 2F_\lambda(x) = F_\lambda(\lambda^{-1}x),$$

for $x \in [0, 1-\lambda]$, which completes the proof. □

It remains to find $\phi_\lambda(x)$ for $x \in \left[1-\lambda, \frac{1}{2}\right]$, and then by symmetry to define ϕ_λ on $[\frac{1}{2}, 1]$. We can also describe the nature of ϕ_λ around $x = \frac{1}{2}$ for typical λ.

Lemma 2.3. *We have that*

$$\phi\left(\frac{1}{2}-x\right) \approx 1 - cx^{-\frac{\log\lambda}{\log 2}}$$

for small x, where c is a constant that depend continuously on λ.

Proof. We start by noting that, since $\phi_\lambda(x)$ evolves continuously in λ, it is enough to describe the nature of the blowup for values of λ corresponding to absolutely continuous ν_λ, as by passing to limits we get the result for all λ.

We consider the behaviour of $F_\lambda(x)$ close to $x = \frac{1}{2}$ and to $x = 1$. Assuming that $h_\lambda(\frac{1}{2})$ exists and is positive, we have that

$$F_\lambda\left(\frac{1}{2}-\epsilon\right) \approx F_\lambda\left(\frac{1}{2}\right) - h_\lambda\epsilon = \frac{1}{2} - h_\lambda\epsilon.$$

Thus we have that

$$\left|\frac{1}{2} - F_\lambda(\frac{1}{2}-\frac{1}{2}\epsilon)\right| \approx \frac{1}{2}\left|\frac{1}{2} - F_\lambda(\frac{1}{2}-\epsilon)\right|. \tag{5}$$

Conversely, equation 4 gives that for small δ

$$\begin{aligned} F_\lambda(1-\delta) &= \frac{1}{2}(1 + F_\lambda(\lambda^{-1}(1-\delta) - (\lambda^{-1}-1))) \\ &= \frac{1}{2} + \frac{1}{2}F_\lambda(1-\lambda^{-1}\delta) \end{aligned}$$

giving

$$1 - F_\lambda(1-\delta) = \frac{1}{2}(1 - F_\lambda(1-\lambda^{-1}\delta)). \tag{6}$$

Now suppose that $\phi_\lambda\left(\frac{1}{2}-\epsilon\right) = 1-\delta$ for some fixed ϵ and δ. Then by equations 5 and 6 we have that

$$\phi_\lambda(\frac{1}{2}-\frac{1}{2}\epsilon) \approx 1 - \lambda\delta.$$

Iterating, we have

$$\phi_\lambda\left(\frac{1}{2} - \left(\frac{1}{2}\right)^n\epsilon\right) \approx 1 - \lambda^n\delta,$$

and we see that we have a blow up of the form

$$\phi_\lambda\left(\frac{1}{2}-x\right) \approx 1 - \frac{\delta}{\varepsilon^{-\log\lambda/\log 2}}x^{-\frac{\log\lambda}{\log 2}}$$

with $c = \delta\varepsilon^{\log\lambda/\log 2}$ depending continuously on λ. □

3 Piecewise Convexity

Numerical approximations of the maps ϕ_λ suggest that there are ranges of λ close to 1 in which the maps ϕ_λ are piecewise convex. For the value $\lambda = \frac{1}{\sqrt{2}}$, one can see directly from the calculation of the previous section that ϕ_λ is piecewise convex, although of course this value of λ is rather special and Bernoulli convolutions are already well understood for $\lambda = 1/\sqrt[n]{2}$.

In this section we prove various properties of ν_λ that would follow from $\phi_\lambda|_{[0,\frac{1}{2}]}$ being convex. We use the term 'piecewise convex' as shorthand for the statement that ϕ_λ is convex on each of the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. The following theorem shows the relevance of the piecewise convexity of ϕ_λ to the study of Bernoulli convolutions.

Theorem 3.1. *Suppose that there exists an interval $(a, b) \subset (\frac{1}{2}, 1)$ such that ϕ_λ is piecewise convex for each λ in (a, b). Then for each $\lambda \in (a, b)$ the Bernoulli convolution ν_λ is absolutely continuous with bounded density.*

We stress that if ϕ_λ is piecewise convex for almost every λ in (a, b), then it is piecewise convex for all λ in (a, b), since the maps ϕ_λ are continuous in λ and convexity is preserved by passing to continuous limits.

Proof. This theorem relies on results of Rychlik [15].

Given a function $g : [0, 1] \to \mathbb{R}$, we define the total variation of g by

$$\operatorname{var} g := \sup_{0=x_0<x_1<\cdots<x_n=1} \sum_{i=1}^{n} |g(x_i) - g(x_{i-1})|.$$

The function g is said to have bounded variation if $\operatorname{var} g < \infty$. Suppose that $T : [0, 1] \to [0, 1]$ is a piecewise continuous map, such that there exists a function g of bounded variation satisfying $g = 1/|T'|$ almost everywhere. We consider the transfer operator L defined on functions of bounded variation by

$$Lf(x) = \sum_{T(y)=x} g(y)f(y).$$

We put $g_n = g \cdot (g \circ T) \cdots (g \circ T^{n-1})$. Then

$$L^n f(x) = \sum_{T^n(y)=x} g_n(y)f(y).$$

Let C_n denote the $(n-1)$th refinement of the partition $\{[0, \frac{1}{2}], (\frac{1}{2}, 1]\}$ by T. In [15, Corollary 3], Rychlik proved that

$$\operatorname{var} L^n f \leq \kappa \operatorname{var} f + D\|f\|_1, \tag{7}$$

where $\kappa = \sup g_n + \max_{C_n} \operatorname{var}_{C_n} g_n$ and $D = \max_{C_n} \operatorname{var}_{C_n} g_n / |C_n|$.

We can apply this to our tent maps, replacing T with ϕ_λ. Suppose that the tent map is convex on each of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Then ϕ_λ is differentiable everywhere except for at most countably many points, and this derivative is increasing on $[0, \frac{1}{2})$ and on $(\frac{1}{2}, 1]$. So there exists a function g which is of bounded variation, which satisfies the assumptions of [15], and which satisfies $g = \frac{1}{|\phi'_\lambda|}$ almost everywhere. We have

$$\sup g = g(0) = g(1) = \lambda \qquad \text{and} \qquad \operatorname{var}_{[0,\frac{1}{2}]} g = \operatorname{var}_{[\frac{1}{2},1]} g \leq \lambda,$$

with equality if and only if $|\phi'_\lambda(x)| \to \infty$ when $x \to \frac{1}{2}$ (which is the case by Lemma 2.3). From this we get that

$$\sup g_n = \lambda^n \qquad \text{and} \qquad \operatorname{var}_{C_n} g_n \leq 2^{n-1}\lambda^n,$$

Combining this with (7) we get that

$$\operatorname{var} L^n f \leq 2\lambda^n \operatorname{var} f + \frac{2^{n-1}\lambda^n}{\min |C_n|}\|f\|_1. \tag{8}$$

In the setting of our tent map, C_n corresponds to the cylinders of generation n, and so C_n depends continuously on n. In particular, for each $n \in \mathbb{N}$ the value of $\min |C_n|$ corresponding to ϕ_λ is continuous in λ.

By Rychlik [15], there is a unique non-negative function h_λ of bounded variation, such that $\|h_\lambda\|_1 = 1$ and $Lh_\lambda = h_\lambda$. The existence follows by Theorem 1 in Rychlik's paper, and the uniqueness is clear since ϕ_λ maps each of the intervals $[0, 1/2]$ and $[1/2, 1]$ onto $[0, 1]$. The function h_λ is the density of the unique absolutely continuous invariant measure of ϕ_λ. If we pick n such that $2\lambda^n < 1$, then (8) implies that

$$\operatorname{var} h_\lambda \leq 2\lambda^n \operatorname{var} h_\lambda + \frac{2^{n-1}\lambda^n}{\min |C_n|},$$

giving

$$\operatorname{var} h_\lambda \leq \frac{2^{n-1}\lambda^n}{\min |C_n|}\frac{1}{1 - 2\lambda^n}.$$

Hence we have that

$$\sup h_\lambda \leq 1 + \frac{2^{n-1}\lambda^n}{\min |C_n|} \frac{1}{1-2\lambda^n},$$

and so h_λ is bounded. Furthermore, since all of the quantities involved are continuous in λ, there is a uniform bound on $\sup h_\lambda$ across all of (a,b).

Now we recall that h_λ was the density of the absolutely continuous invariant measure of ϕ_λ. But for almost every $\lambda \in (a,b)$, the Bernoulli convolution ν_λ is absolutely continuous and is preserved by ϕ_λ, and so h_λ is the density of ν_λ. But now any weak* limit point of a family of measures which is absolutely continuous with uniformly bounded density must also be absolutely continuous with the same bound on the density. Therefore, since the family ν_λ evolves continuously (in the weak* topology), we see that ν_λ is absolutely continuous for all $\lambda \in (a,b)$ and has bounded density h_λ. □

4 Computational Techniques

We have seen in the previous section that showing that ϕ_λ is piecewise convex for all λ in an interval would have significant consequences for Bernoulli convolutions. Analytically, we have been able to show piecewise convexity only for some special values of λ for which the distribution F_λ is already known. We remain optimistic that some further progress could be made here, see the comments section. In this section we show how numerical information on ϕ_λ can show convexity up to a certain scale.

4.1 Showing convexity up to a certain scale for fixed λ

First we choose a natural number M and let x_i denote the point $\frac{i}{M}$ for $i \in \{0, \ldots, M\}$. We wish to show that

$$\phi_\lambda(x_i) \leq \frac{1}{2}(\phi_\lambda(x_{i-1}) + \phi_\lambda(x_{i+1})), \tag{9}$$

for $i < \frac{M}{2}$ and λ in a certain parameter range. It will then follow that

$$\phi_\lambda(x_j) \leq \left(\frac{j-i}{k-i}\right) \phi_\lambda(x_k) + \left(\frac{k-j}{k-i}\right) \phi_\lambda(x_i)$$

for $0 \leq i \leq j \leq k \leq \frac{M}{2}$. This is what we call 'convexity up to scale $\frac{1}{M}$'. It corresponds to the usual definition of convexity restricted to the set of points $\{x_0, \ldots, x_{\frac{M}{2}}\}$, using the fact that

$$x_j = \left(\frac{j-i}{k-i}\right) x_k + \left(\frac{k-j}{k-i}\right) x_i.$$

Because Lemma 2.2 that tells us that $\phi_\lambda(x) = \lambda^{-1}x$ for $0 \leq x \leq 1-\lambda$, we only need to check (9) for i with $1-\lambda < x_i < \frac{M}{2}$.

For large L we estimate $F_\lambda(x)$ by noting that

$$F_\lambda(x) \leq F^+_{\lambda,L}(x) := 2^{-L} \left| \{a_1 \cdots a_L \in \{0,1\}^L : (\lambda^{-1}-1) \sum_{i=1}^{L} a_i \lambda^i \leq x\} \right|$$

and

$$\begin{aligned} F_\lambda(x) \geq F^-_{\lambda,L}(x) &:= 2^{-L} \left| \{a_1 \cdots a_L \in \{0,1\}^L : (\lambda^{-1}-1) \sum_{i=1}^{L} a_i \lambda^i \leq x - \lambda^L\} \right| \\ &= F^+_{\lambda,L}(x - \lambda^{-L}). \end{aligned}$$

Then given the values of $F^+_{\lambda,L}(x_i)$ and $F^-_{\lambda,L}(x_i)$ for $i \in \{1, \ldots, M\}$, we can bound ϕ_λ from below and above by

$$\phi^-_{\lambda,L}(x_i) \leq \phi_\lambda(x_i) \leq \phi^+_{\lambda,L}(x_i),$$

where $\phi^-_{\lambda,L}$ and $\phi^+_{\lambda,L}$ are defined for $0 \leq x_i \leq 1/2$ by

$$\begin{aligned} \phi^-_{\lambda,L}(x_i) = y &\quad \text{where } y \text{ is the largest } y \text{ such that} \quad F^+_{\lambda,L}(y) \leq 2F^-_{\lambda,L}(x_i), \\ \phi^+_{\lambda,L}(x_i) = y &\quad \text{where } y \text{ is the smallest } y \text{ such that} \quad F^-_{\lambda,L}(y) \geq 2F^+_{\lambda,L}(x_i). \end{aligned}$$

Hence, if we have

$$\phi^+_\lambda(x_i) \leq \frac{1}{2}(\phi^-_\lambda(x_{i-1}) + \phi^-_\lambda(x_{i+1})), \tag{10}$$

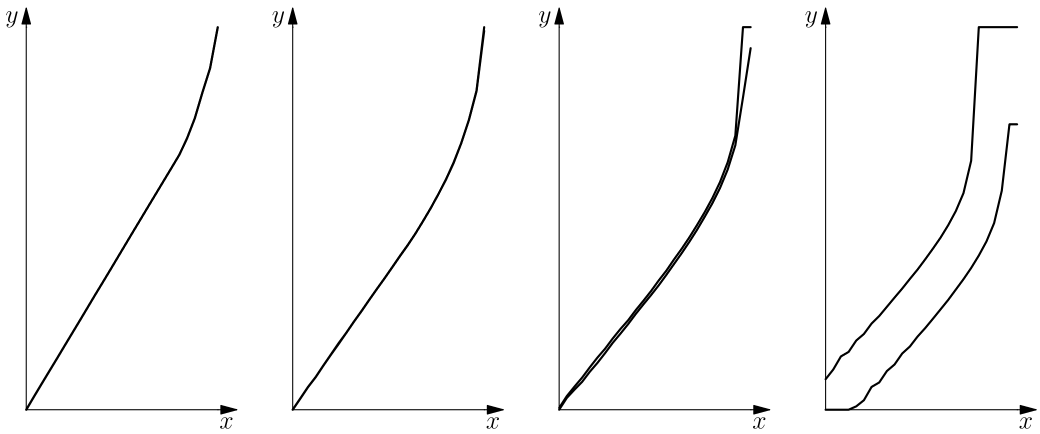

Figure 4: Plot of $\phi^-_{\lambda,L}$ and $\phi^+_{\lambda,L}$ with $L = 24$, $M = 50$, and $\lambda = 0.6$ (left), $\lambda = 0.7$, $\lambda = 0.8$ and $\lambda = 0.9$ (right).

for all i with $1 - \lambda < x_i < 1/2$, then (9) holds for all $i < \frac{M}{2}$.

The inequalities (10) can be checked with a computer, at least up to errors in the floating point arithmetics. We have written a program in C[1], that calculates the approximations $\phi^-_{\lambda,L}$ and $\phi^+_{\lambda,L}$ of $\phi_{\lambda,L}$. Figure 4 shows four plots of the approximations, obtained from the mentioned program.

4.2 Techniques for all λ in an interval

To apply Theorem 3.1 we would like to show that ϕ_λ is piecewise convex for all λ in an interval. We cannot do this computationally, but instead consider how to show ϕ_λ is piecewise convex up to a certain scale for all λ in an interval.

We consider a small interval $I_\epsilon = [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$. The map

$$I_\epsilon \ni \lambda \mapsto (\lambda^{-1} - 1) \sum_{i=1}^{\infty} a_i \lambda^i$$

[1]The source code for this program is available on the homepage of the second author, http://www.maths.lth.se/matematiklth/personal/tomasp/

is differentiable, and if we put $D := \frac{1}{(\lambda_0+\epsilon)(1-\lambda_0+\epsilon)}$, then

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda^{-1}-1)\sum_{i=1}^{\infty} a_i\lambda^i\right| &= \left|-\frac{1}{\lambda^2}\sum_{i=1}^{\infty} a_i\lambda^i + (\lambda^{-1}-1)\sum_{i=1}^{\infty} a_i i\lambda^{i-1}\right| \\ &\le \max\left\{\frac{1}{\lambda^2}\sum_{i=1}^{\infty} a_i\lambda^i, (\lambda^{-1}-1)\sum_{i=1}^{\infty} a_i i\lambda^{i-1}\right\} \\ &\le \max\left\{\frac{1}{\lambda^2}\sum_{i=1}^{\infty} \lambda^i, (\lambda^{-1}-1)\sum_{i=1}^{\infty} i\lambda^{i-1}\right\} \\ &= \max\left\{\frac{1}{\lambda(1-\lambda)}, \frac{1}{\lambda(1-\lambda)}\right\} = \frac{1}{\lambda(1-\lambda)} \le D,\end{aligned}$$

holds for all $\lambda \in I_\epsilon$ and all sequences with $a_i \in \{0,1\}$. We conclude that

$$(\lambda_0^{-1}-1)\sum_{i=1}^{L} a_i\lambda_0^i \le x \qquad \Longrightarrow \qquad (\lambda^{-1}-1)\sum_{i=1}^{L} a_i\lambda^i \le x + |\lambda-\lambda_0|D$$

for any $\lambda \in I_\epsilon$. Similarly, we have

$$(\lambda_0^{-1}-1)\sum_{i=1}^{L} a_i\lambda_0^i \ge x \qquad \Longrightarrow \qquad (\lambda^{-1}-1)\sum_{i=1}^{L} a_i\lambda^i \ge x - |\lambda-\lambda_0|D$$

for any $\lambda \in I_\epsilon$. Using these two estimates we can use $F^{\pm}_{\lambda_0,L}$ to estimate $F^{\pm}_{\lambda,L}$. We get

$$F^{-}_{\lambda,L}(x) \ge F^{-}_{\lambda_0,L}(x-\epsilon D) \qquad \text{and} \qquad F^{+}_{\lambda,L}(x) \le F^{+}_{\lambda_0,L}(x+\epsilon D).$$

Hence, the estimates $F^{\pm}_{\lambda_0,L}$ of F_{λ_0} gives us estimates on $F^{\pm}_{\lambda,L}$ that we can use to estimate $\phi^{-}_{\lambda,L}$ from below and $\phi^{+}_{\lambda,L}$ from above. It is then possible to check with a computer if the inequalities in (9) are satisfied for all $\lambda \in I_\epsilon$. This has been implemented in our program.

Table 1 shows some result of our program. It displays some values for which we have been able to show nummericaly convexity to a certain scale.

A convolution argument shows that h_λ is differentiable for almost all $\lambda \in (2^{-\frac{1}{3}}, 1)$, see [17]. One might suspect that using this information it would be possible to show that ϕ_λ is piecewise convex for all $\lambda \in [2^{-\frac{1}{3}}, 1)$. However, this does not seem to be true, since just as we can sometimes show convexity

λ_0	ϵ	convexity to scale
0.65	0.000001	0.02
0.7	0	0.02
$2^{-1/2} \approx 0.707106781186548$	0.00001	0.125
0.75	0	0.02
0.75	0.00001	0.125
$2^{-1/3} \approx 0.793700525984100$	0.00001	0.125
0.8	0	0.02
0.8	0.00001	0.125
0.85	0.000001	0.125

Table 1: Numerical observations of piecewise convexity to a scale

to a scale using numerics, we are sometimes also capable of observing non-convexity at a certain scale.

Using our program we have observed that ϕ_λ is not piecewise convex when λ is the inverse of the root of $x^5 + x^4 - x^2 - x - 1$ that is larger than 1. We then have $\lambda \approx 0.8501\ldots$, and since $2^{-\frac{1}{3}} \approx 0.7937\ldots$, we do not have piecewise convexity of ϕ_λ for all $\lambda \in [2^{-\frac{1}{3}}, 1)$. In this case, $1/\lambda$ is a Salem number. (A Salem number is a real algebraic integer, larger than 1, such that all its conjugates have absolute value smaller than or equal to 1, and at least one of the conjugates has absolute value equal to 1.)

Similarly, the program can be used to show that ϕ_λ is not convex for $\lambda = \frac{\sqrt{5}-1}{2}$. Since ν_λ is known not to be absolutely continuous for this value of λ, this is not too surprising. We also see a lack of convexity for $1/\lambda$ equal to certain other Pisot numbers. For instance when $1/\lambda$ is the root of $x^4 - x^3 - 1$ or $x^3 - x - 1$, then ϕ_λ is not convex to scale 0.005.

Let us mention some of the computational difficulties associated with trying to prove convexity to a scale for the entire interval I_ϵ. Suppose L is even. Our program calculates all the $2^{L/2}$ sums $\sum_{i=1}^{L/2} a_i \lambda^i$ and stores them in an ordered list. This requires quite a lot of memory even for L as small as 60, but the time required to preform the calculations is rather short. The sums in the list are then combined to get the sums $\sum_{i=1}^{L} a_i \lambda^i$ with double as many terms, when needed. This method of storing only the sums of length $L/2$ instead of storing the sums of length L, saves memory but increases computation time.

However we found that doing so yields a better balance between the use of memory and the computation time.

When λ is close to 1, large values of L are needed to get a good accuracy in the estimates, requiring an unrealistic amount of memory. This is clearly illustrated in Figure 4, where for $\lambda = 0.9$, the two maps $\phi^+_{\lambda,L}$ and $\phi^-_{\lambda,L}$ differ quite a lot, while for $\lambda = 0.6$ they are indistinguishable.

With a computer with 64 GB of memory we are able to run our program for $L \leq 60$. Table 1 was obtained from running the program with $56 \leq L \leq 60$.

5 Further Questions

There are a number of natural questions that follow on from our work.

Question 1: Can one show that for all λ sufficiently close to 1 we have that ν_λ is absolutely continuous? For almost all $\lambda \in (2^{-1/n}, 1)$ one has that the density h_λ is $(n-1)$-times differentiable, does this extra regularity of the density give rise to extra regularity in the functions ϕ_λ?

Question 2: Can one show that there is an interval $J \subset \mathbb{R}$ containing $1/\sqrt{2}$ such that ν_λ is absolutely continuous for all $\lambda \in J$. Perhaps this would involve showing that the map ϕ_λ evolves smoothly in a neighbourhood of $1/\sqrt{2}$.

Question 3: Are there any other properties of ν_λ (besides the question of absolute continuity) which could be studied using ϕ_λ? In particular, can one forbid the possibility of singular Bernoulli convolutions which have Hausdorff dimension 1 by proving a similar result for invariant measures of ϕ_λ? Such results do exist in the literature for one-dimensional dynamics, see e.g. [11] and [4], but at present are not in such a form that they would apply to ϕ_λ.

Such a result would be extremely interesting given recent results of Hochman [8] giving necessary conditions for 'dimension drop' in overlapping iterated functions systems. In the special case of Bernoulli convolutions, Shmerkin [16] was able to use [8] to prove that Bernoulli convolutions are absolutely continuous for all parameters outside of a set of Hausdorff dimension zero, but Shmerkin's techniques were heavily reliant on the convolution structure of Bernoulli convolutions. In this light, an alternative approach for demonstrating that certain fractal measures of Hausdorff dimension one are in fact absolutely continuous would be very interesting.

Acknowldgements

Many thanks to Evgeny Verbitskiy for suggesting to us that piecewise convexity of ϕ_λ would be an interesting property to study. The first author was supported by the EPSRC, grant number EP/K029061/1.

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