STABLE AND MULTISTABLE PROCESSES AND LOCALISABILITY

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Abstract

We first review recent work on stable and multistable random processes and their localisability. Then most of the thesis concerns a new approach to these topics based on characteristic functions.

Our aim is to construct processes on $\mathbb{R}$, which are $\alpha(x)$-multistable, where the stability index $\alpha(x)$ varies with x. To do this we first use characteristic functions to define $\alpha(x)$-multistable random integrals and measures and examine their properties. We show that an $\alpha(x)$-multistable random measure may be obtained as the limit of a sequence of measures made up of α-stable random measures restricted to small intervals with α constant on each interval.

We then use the multistable random integrals to define multistable random processes on $\mathbb{R}$ and study the localisability of these processes. Thus we find conditions that ensure that a process locally 'looks like' a given stochastic process under enlargement and appropriate scaling. We give many examples of multistable random processes and examine their local forms.

Finally, we examine the dimensions of graphs of α-stable random functions defined by series with α-stable random variables as coefficients.

I, Lining LIU, hereby certify that this thesis, which is approximately 23,500 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

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I was admitted as a research student in September, 2006 and as a candidate for the degree of Doctor of Philosophy in September, 2007; the higher study for which this is a record was carried out in the University of St Andrews between 2006 and 2009.

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Contents

Preface

The main part of this thesis studies stable and multistable random measures and integrals leading to the construction of multistable random processes. We then consider the local form of these processes. We give many examples of stable processes and end by examining the dimensions of graphs of α-stable random functions.

In Chapter 1, we give some general background to aspects of probability theory including notation and terminology that we will refer to throughout the thesis. We discuss α-stable random variables and vectors, α-stable random integrals and measures, α-stable random processes and their properties. We review characteristic functions which characterise random variables and vectors and consider the characteristic functions of α-stable random variables and vectors.

In Chapter 2, we review recent developments on stable and multistable random processes and their localisability. We first recall notions of localisability and the local forms of a process. If a process $X(t)$ is enlarged about a point u by scaling by some h and the enlarged process converges to a limiting process, we say X is localisable at u. Thus if the limit (in an appropriate sense)

$$\lim_{r\to 0}\frac{X(u+rt)-X(u)}{r^h}=Y(t)$$

for some process Y, we say that X is h-localisable at u with local form Y. A multistable process is a process such that the local form at x is an $\alpha(x)$-stable random process, so it is a process whose local stability index $\alpha(x)$ varies with x. We review some existing constructions of multistable random processes and conditions for their localisability.

By defining stochastic integrals in term of characteristic functions in Chapter 3, we give a new construction of an $\alpha(x)$-multistable 'random measure' which we denote by $M_{\alpha(x)}$. We examine the properties of this 'random measure'. In particular we show that a multistable random measure $M_{\alpha(x)}$ may be obtained as the limit of a sequence of random measures M_n, each of which is an independent sum of the restriction of α-stable random measures to a large number of small intervals with α constant on each interval. The local forms of $\alpha(x)$-multistable random measures are considered at the end of this chapter.

Our construction of $\alpha(x)$-multistable random measures leads to a new definition of $\alpha(x)$-multistable random processes in term of characteristic functions and stochastic integrals. In Chapter 4, we define and study the localisability of these processes. The first half of this chapter considers conditions for localisability of general processes. The second half contains many examples of $\alpha(x)$-multistable random processes defined by stochastic integrals. We apply our theorems to prove the localisability of these examples under certain conditions.

We look at α-stable processes from a different viewpoint in Chapter 5. We let

$$F_A(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-2)k} \sin(\lambda^k t),$$

where $A_1, A_2, \ldots$ are a sequence of identically independently distributed α-stable random variables, when $0 < h, D < 2$ and $\lambda > 1$. Then $\{F_A(t), t \in \mathbb{R}\}$ is an α-stable random function. The function F_A has a 'fractal' graph and we consider the box-counting dimension and Hausdoff dimension of this graph. We use a potential-theoretic method to get an almost sure lower bound for the Hausdoff dimension and obtain a Hölder inequality which leads to an almost sure upper bound for the box-counting dimension. Under certain conditions, the Hausdoff dimension and box-counting dimension of graph(F_A) equal $\min\{1, D\}$ almost surely.

Chapter 1

General background to probability

1.1 Introduction

In this chapter, we give some general background to aspects of probability theory that we will refer to throughout the thesis. In particular, this will set up the terminology and notation to be used later.

First we review basic properties of probability theory and random processes. We then discuss α-stable random variables and vectors leading on to α-stable integrals, measures and processes. This is in preparation for the multistable processes discussed in the later chapters. The last part of this chapter gives some examples of stable processes. We will develop these examples by introducing multistable versions in Chapters 2 and 4.

1.2 Basic probability theory

We first discuss random variables, random vectors and their distribution and characteristic functions. This leads to random processes which play a major part in this thesis. The final subsection recalls some standard definitions and properties of convergence. Much of this material may be found in [10].

1.2.1 Random variables and random vectors

Let Ω be a non-empty set called a *sample space*. Let F be a *sigma-field* of subsets of Ω, that is F is closed under the operations of taking countable intersections, countable unions and complementation. The sets of F are called *events*.

A *probability measure* $\mathbb{P}$ on (Ω, F) is a function $\mathbb{P} : F \to [0,1]$, such that

1) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$;

2) If $A_1, A_2, \ldots$ is a collection of disjoint members of F, so that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \ldots,$$

that is $\mathbb{P}$ is countably additive on the sets of F.

Definition 1.2.1. *A triple* $(\Omega, F, \mathbb{P})$*, comprising a set* Ω*, a* σ*-field* F *of subsets of* Ω *and a probability measure* $\mathbb{P}$ *on* (Ω, F)*, is called a* probability space.

We assume throughout this thesis that we are working on some appropriate underlying probability space $(\Omega, F, \mathbb{P})$ that will not always be mentioned specifically.

Definition 1.2.2. *A* random variable *is a function* $X : \Omega \to \mathbb{R}$ *such that* $\{\omega \in \Omega : X(\omega) \leq x\} \in F$ *for each* $x \in \mathbb{R}$*. Equivalently,* $\{\omega \in \Omega : X(\omega) \in B\} \in F$ *for every Borel set* $B \subseteq \mathbb{R}$*.*

The higher dimensional analogue of a random variable is a random vector.

Definition 1.2.3. *An n-dimensional* random vector *is a n-dimensional vector of random variables, that is a function* $X : \Omega \to \mathbb{R}^n$ *such that* $\{\omega \in \Omega : X(\omega) \in B\} \in F$ *for every Borel set* $B \subseteq \mathbb{R}^n$*.*

From now on, we will use $\{X \leq x\}$ to denote the event $\{\omega \in \Omega : X(\omega) \leq x\}$ where X is a random variable, and $\{X \in A\}$ to denote the event $\{\omega \in \Omega : X(\omega) \in A\}$ where X is a random variable or vector.

The distribution function of a random variable or random vector X describes the probability distribution of the values taken by X.

Definition 1.2.4. *The* distribution function *of a random variable* X *is the function* $F \equiv F_X \colon \mathbb{R} \to [0,1]$ *given by*

$$F(x) \equiv F_X(x) = \mathbb{P}(X \leq x).$$

If we have two random variables, their joint distribution indicates the dependence between them.

Definition 1.2.5. *Let* X *and* Y *be random variables. The* joint distribution function *of* X *and* Y *is the function* $F \equiv F_{X,Y} \colon \mathbb{R}^2 \to [0,1]$ *given by*

$$F(x,y) \equiv F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y).$$

The distribution function of a random vector can be considered as the joint distribution of its component random variables.

Definition 1.2.6. *The* distribution function *of a n-dimensional random vector* $X = (X_1, X_2, \ldots, X_n)$ *is the function* $F \equiv F_{X_1,\ldots,X_n} \colon \mathbb{R}^n \to [0,1]$ *defined by*

$$F(x) \equiv F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \mathbb{P}\{X_i \leq x_i, i = 1,\ldots,n\},$$

where $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$*. Thus* F *may be regarded as the joint distribution function of* $X_1, X_2, \ldots, X_n$*.*

There are several different senses of equality of two random variables or random vectors. The following definitions are given in increasing order of strength.

Definition 1.2.7. *Two random variables* ($random\ vectors$) X *and* Y*, not necessarily on the same probability space, are* equal in distribution *if they have the same distribution functions. Equivalently, they are* equal in distribution *if*

$$\mathbb{P}(X \in A) = \mathbb{P}(Y \in A)$$

for all Borel sets $A \subseteq \mathbb{R}$ ($A \subseteq \mathbb{R}^n$)*. We write* $X \stackrel{d}{=} Y$ *to mean that* X *and* Y *are equal in distribution.*

Definition 1.2.8. *Two random variables* ($random\ vectors$) X *and* Y *on the same probability space are* equal almost surely *if and only if*

$$\mathbb{P}(X \neq Y) = 0.$$

We write $X \stackrel{a.s}{=} Y$ *to mean that* X *and* Y *are equal almost surely.*

The distribution of a random variable can often be expressed in terms of a density function.

Definition 1.2.9. *If the distribution function of a random variable* X *may be expressed as*

$$F(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(u)du$$

for some integrable $f : \mathbb{R} \to [0,\infty)$*, we call* f *the* (probability) density function *of* X.

We often want to consider the average or mean of a random variable.

Definition 1.2.10. *The* expectation *or* mean *of a random variable* X *on a probability space* $(\Omega, F, \mathbb{P})$ *is*

$$\mathbb{E}(X) = \int_{\omega\in\Omega} X(\omega)d\mathbb{P}(\omega)$$

provided this integral exists.

If X has a probability density function f, then

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx,$$

provided the integral exists. If two random variables have the same probability distribution they will have the same expectation if it is defined.

Let A be an event and let $\mathbf{1}_A : \Omega \to \mathbb{R}$ be the *indicator function* of A; that is

$$\mathbf{1}_A(x) = 1 \text{ if } x \in A;$$

$$\mathbf{1}_A(x) = 0 \text{ if } x \in A^c.$$

Note that

$$\mathbb{E}(\mathbf{1}_A) = \mathbb{P}(A).$$

Note also that

$$\mathbb{E}\left(\sum_{i=1}^{m} X_i\right) = \sum_{i=1}^{m} \mathbb{E}(X_i)$$

for random variables $X_1, X_2, \ldots, X_m$, provided these expectations exist.

Probably the most important distribution is the normal or Gaussian distribution.

Definition 1.2.11. *The* normal *or* Gaussian distribution *with mean μ and standard deviation $\sigma > 0$, denoted by $N(\mu, \sigma^2)$, is defined by the density function*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\},$$

where $-\infty < x < \infty$. If $\mu = 0$ and $\sigma^2 = 1$ then

$$f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right)$$

is the density function of the standard *normal distribution.*

Intuitively, two events are independent if the occurrence of one event does not affect the occurrence of the other.

Definition 1.2.12. *Two events A and B are* independent *if and only if*

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

More generally, an arbitrary collection of events are independent *if and only if for any finite subcollection $A_1, A_2, \ldots, A_n$, we have*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} \mathbb{P}(A_i).$$

This leads to the definition of independence of random variables.

Definition 1.2.13. *Two random variables* (*random vectors*) *X and Y are called* independent *if the events $\{X \in A\}$ and $\{Y \in B\}$ are independent events for all Borel sets $A, B \subseteq \mathbb{R}$ $(A, B \subseteq \mathbb{R}^n)$.*

More generally, an arbitrary collection of random variables (*random vectors*) *are* independent *if and only if for any finite subcollection $X_1, X_2, \ldots, X_n$, and any Borel sets $A_1, \ldots, A_n$ we have that $\{X_1 \in A_1\}, \ldots, \{X_n \in A_n\}$ are independent events.*

Note that if $X_1, \dots, X_n$ are independent and $f_1, \dots, f_n$ are continuous, then $f_1(X_1), \dots, f_n(X_n)$ are independent, since $\{f_i(X_i) \in A_i\} = \{X_i \in f_i^{-1}(A_i)\}$ for all $i = 1, \dots, n$.

Lemma 1.2.14. *If two random variables X and Y are independent, then*

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

provide these expectations exist.

Proof. See [10, Lemma 9, Section 3.3]. □

We now recall the definitions of the characteristic functions of random variables and random vectors, which play a central part in this thesis.

Definition 1.2.15. *The* characteristic function *of a random variable X is the function $\phi : \mathbb{R} \to \mathbb{C}$ defined by*

$$\phi_X(\theta) \equiv \phi(\theta) = \mathbb{E}(e^{i\theta X}) = \mathbb{E}(\cos\theta X) + i\mathbb{E}(\sin\theta X),$$

for all $\theta \in \mathbb{R}$, where $i = \sqrt{-1}$.

Characteristic functions are essentially Fourier transforms, since

$$\phi_X(\theta) = \int e^{i\theta X(\omega)} d\mathbb{P}(\omega) = \int e^{i\theta x} f(x) dx,$$

if X has a density function f.

Characteristic functions have several very useful properties, including that they factorise for independent random variables.

Theorem 1.2.16. *Every characteristic function ϕ satisfies*

1) $\phi(0) = 1$, $|\phi(\theta)| \leq 1$ for all θ;

2) ϕ is uniformly continuous on $\mathbb{R}$.

Proof. See [10, Theorem 3, Section 5.7]. □

Theorem 1.2.17. *$X_1, X_2, \dots, X_m$ are independent random variables if and only if*

$$\phi_{X_1 + \dots + X_m}(\theta) = \phi_{X_1}(\theta)\phi_{X_2}(\theta)\dots\phi_{X_m}(\theta),$$

for all $\theta \in \mathbb{R}$.

Proof. See [10, Theorem 5, Section 5.7]. □

Characteristic functions of random vectors may be defined similarly, using a scalar product.

Definition 1.2.18. *The* characteristic function *of a random vector* $X = (X_1, X_2, \ldots, X_n)$ *is a function* $\phi : \mathbb{R}^n \to \mathbb{C}$ *defined by*

$$\begin{aligned} \phi_X(\theta) &\equiv \phi(\theta) \\ &\equiv \phi(\theta_1, \ldots, \theta_n) \\ &= \mathbb{E}\left(\exp\left\{i \sum_{j=1}^{n} \theta_j X_j\right\}\right) \end{aligned}$$

for all $\theta = (\theta_1, \theta_2, \ldots, \theta_n) \in \mathbb{R}^n$.

Properties similar to those for random variables hold.

Theorem 1.2.19. *Every characteristic function* ϕ *satisfies*
1) $\phi(0, \ldots, 0) = 1$, $|\phi(\theta)| \leq 1$ *for all* $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$*;*
2) ϕ *is uniformly continuous on* $\mathbb{R}^n$.

Proof. See [10, Theorem 3, Section 5.7]. □

Theorem 1.2.20. $X_1, X_2, \ldots, X_m$ *are independent random vectors if and only if*

$$\phi_{X_1 + \cdots + X_m}(\theta) = \phi_{X_1}(\theta)\phi_{X_2}(\theta) \ldots \phi_{X_m}(\theta),$$

for all $\theta \in \mathbb{R}^n$.

Proof. See [10, Theorem 5, Section 5.7]. □

Corollary 1.2.21. *If* $X_1, \ldots, X_n$ *are independent random variables, then the characteristic function of the random vector* $X = (X_1, X_2, \ldots, X_n)$ *can be written as*

$$\begin{aligned} \phi_X(\theta) &= \phi_{X_1 \ldots X_n}(\theta_1, \ldots, \theta_n) \\ &= \mathbb{E}\left(e^{i \sum_{j=1}^{n} \theta_j X_j}\right) \\ &= \prod_{j=1}^{n} \mathbb{E}\left(e^{i \theta_j X_j}\right), \end{aligned}$$

for $\theta = (\theta_1, \ldots, \theta_j) \in \mathbb{R}^n$.

Proof. Since $X_1, \ldots, X_n$ are independent random variables, by Lemma 1.2.14 we have

$$\begin{aligned} \mathbb{E}\left(e^{i \sum_{j=1}^{n} \theta_j X_j}\right) &= \mathbb{E}\left(\prod_{j=1}^{n} e^{i \theta_j X_j}\right) \\ &= \prod_{j=1}^{n} \mathbb{E}\left(e^{i \theta_j X_j}\right). \end{aligned}$$

□

An important property of characteristic functions is that they uniquely determine the distribution of a random variable or random vector.

Proposition 1.2.22. *Let X, Y be random variables (random vectors). Then*

$$\phi_X(\theta) = \phi_Y(\theta)$$

for all $\theta \in \mathbb{R}$ $(\theta \in \mathbb{R}^n)$ if and only if $X \stackrel{d}{=} Y$.

Proof. See [10, Corollary 3, Section 5.9] for details. □

1.2.2 Random processes

Most of this thesis is concerned with random processes or stochastic processes on $\mathbb{R}$. In this section we review some notions relating to general processes.

Definition 1.2.23. *Let T be a set (usually T is $\mathbb{R}$, $\mathbb{R}^+$,$\mathbb{R}^n$ or a set of functions). A* stochastic process *or* random process *on T is a collection of random variables indexed by T on some probability space, that is $\{X(t), t \in T\}$, where $X(t)$ is a random variable for each t.*

When the meaning is clear, we may refer to a process X or $X(t)$. Very often we consider random processes $\{X(t), t \in \mathbb{R}\}$ on the reals and think of t as 'time'.

There are many kinds of processes, we concentrate on specific types which are relevant to later chapters of this thesis.

Definition 1.2.24. *Let $\{X(t), t \in T\}$ be a stochastic process. The* finite-dimensional distributions *of X are the family of joint distributions of the vectors $(X(t_1), X(t_2), \ldots, X(t_n))$, where $t_1, t_2, \ldots, t_n \in T$. Thus, the vector $(X(t_1), X(t_2), \ldots, X(t_n))$ has distribution function $F_t : \mathbb{R}^n \to [0,1]$ given by*

$$F_{t_1,t_2,\ldots,t_n}(x) = \mathbb{P}(X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_n) \le x_n)$$

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

Definition 1.2.25. *We say that two stochastic processes $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$, which may be defined on different probability spaces, are* equal in finite dimensional distributions *if for all $t_1, t_2, \ldots, t_n \in T$,*

$$(X(t_1), X(t_2), \ldots, X(t_n)) \stackrel{d}{=} (Y(t_1), Y(t_2), \ldots, Y(t_n)).$$

We denote this by $\{X(t), t \in T\} \stackrel{fdd}{=} \{Y(t), t \in T\}$ or just $X(t) \stackrel{fdd}{=} Y(t)$.

It is useful to express this in terms of equality of characteristic functions.

Corollary 1.2.26. *Let $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ be stochastic processes. Then*

$$X(t) \stackrel{fdd}{=} Y(t)$$

if and only if

$$\phi_{X(t_1),\ldots,X(t_n)}(\theta_1,\ldots,\theta_n) = \phi_{Y(t_1),\ldots,Y(t_n)}(\theta_1,\ldots,\theta_n),$$

for all $\theta_1,\ldots,\theta_n \in \mathbb{R}$ and all $t_1,t_2,\ldots,t_n \in T$ for all $n \in \mathbb{N}$.

Proof. The result follows directly from Proposition 1.2.22. □

Many interesting processes on $\mathbb{R}$ are stationary, that is their finite-dimensional distributions are invariant under time shifts.

Definition 1.2.27. *A process $\{X(t), t \in \mathbb{R}\}$ is called* stationary *if*

$$X(t) \stackrel{fdd}{=} X(t+h),$$

for all $h \in \mathbb{R}$, that is the vectors

$$(X(t_1), X(t_2), \ldots, X(t_n)) \stackrel{d}{=} (X(t_1+h), X(t_2+h), \ldots, X(t_n+h)),$$

for all $n \in \mathbb{N}$, $t_1,t_2,\ldots,t_n \in \mathbb{R}$ and $h \in \mathbb{R}$.

Another frequent property is that of the increments being invariant under time shifts.

Definition 1.2.28. *A process $\{X(t), t \in \mathbb{R}\}$ has* stationary increments *if for all $h, a \in \mathbb{R}$,*

$$X(t+h) - X(t) \stackrel{fdd}{=} X(t+a+h) - X(t+a),$$

that is the vectors

$$\begin{aligned}(X(t_1+h) - X(t_1), \ldots, X(t_n+h) - X(t_n))& \\ \stackrel{d}{=} (X(t_1+a+h) - X(t_1+a), \ldots, X(t_n+a+h) - X(t_n+a)),&\end{aligned}$$

for all $n \in \mathbb{N}$, $t_1,t_2,\ldots,t_n \in \mathbb{R}$ and $a, h \in \mathbb{R}$.

Definition 1.2.29. *A process $\{X(t), t \in \mathbb{R}\}$ on $\mathbb{R}$ is said to have* independent increments *if for all $n \in \mathbb{N}$ and $t_1 < t_2 \leq t_3 < t_4 \leq \ldots \leq t_{2n-1} < t_{2n}$ the increments $X(t_2) - X(t_1), X(t_4) - X(t_3), \ldots, X(t_{2n}) - X(t_{2n-1})$ are independent.*

Note that a process with stationary independent increments that is continuous on the right is called a *Lévy process*.

Self-similar processes will play an important role in our development.

Definition 1.2.30. *A process* $\{X(t), t \in \mathbb{R}\}$ *on* $\mathbb{R}$ *is* self-similar with index $h > 0$ *if, for all* $r > 0$*,*

$$X(rt) \stackrel{fdd}{=} r^h X(t),$$

that is for all $n \geq 1$*,* $t_1, t_2, \ldots, t_n \in T$ *and all* $r > 0$*,*

$$(X(rt_1), X(rt_2), \ldots, X(rt_n)) \stackrel{d}{=} (r^h X(t_1), r^h X(t_2), \ldots, r^h X(t_n)). \tag{1.1}$$

Note that there are obvious analogues to Definitions 1.2.27 to 1.2.30 for processes $\{X(t), t \in \mathbb{R}^+\}$ on the non-negative reals.

The best known example of a self-similar process with stationary independent increments is Brownian motion, sometimes called the Wiener process.

Definition 1.2.31. Brownian motion *or the* Wiener process*,* $B = \{B(t), t \geq 0\}$ *on* $\mathbb{R}^+$ *is a stochastic process characterised by:*

1) $B(t)$ *is continuous with* $B(0) = 0$ *almost surely;*

2) B *has independent increments;*

3) $B(t+h) - B(t)$ *is* $N(0,h)$ *for all* $t, h \geq 0$*.*

Since the distribution $B(t+h) - B(t)$ depends only on h, it may be shown that $B(t+h) - B(t) \stackrel{fdd}{=} B(t+a+h) - B(t+a)$, for all $t, h, a \geq 0$. Thus B has stationary independent increments. It may also be shown using (1.1) that B is a 1/2-self-similar process.

An alternative approach will be considered in Section 1.4.3 where we consider Brownian motion as a stochastic integral from which these properties may be derived.

Finally in this section we state the *Kolmogorov Existence Theorem* or *Kolmogorov Extension Theorem* which will enable us to assert the existence of random processes that have given families of finite-dimensional distributions. This theorem may be expressed in a number of equivalent forms. Here we give a version in terms of the distributions of random vectors.

Theorem 1.2.32. *Let* T *be a set (usually* $\mathbb{R}, \mathbb{R}^+, \mathbb{R}^n$ *or a set of functions). For each* $k \in \mathbb{N}$ *and* $t_1, t_2, \ldots, t_k \in T$ *let* $X_{t_1,\ldots,t_k}$ *be a random vector in* $\mathbb{R}^k$*. Suppose that these random vectors satisfy the consistency conditions:*

1) for any $k \in \mathbb{N}$ *and permutation* π *of* $\{1, 2, \ldots, k\}$*, with* Q *be the* $k \times k$ *matrix corresponding to the permutation* π*, that is* $[Q]_{i,j} = 1$ *if* $j = \pi(i)$ *and* 0 *otherwise, we have*

$$X_{t_1,\ldots,t_k} \stackrel{d}{=} Q^{-1}(X_{t_{\pi(1)},\ldots,t_{\pi(k)}});$$

2) for all $k, m \in \mathbb{N}$*, writing* $P : \mathbb{R}^{k+m} \to \mathbb{R}^k$ *for the projection* $P(x_1, \ldots, x_k, x_{k+1}, \ldots, x_{k+m}) = (x_1, \ldots, x_k)$*, we have*

$$X_{t_1,\ldots,t_k} \stackrel{d}{=} P(X_{t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}).$$

Then there exists a probability space $(\Omega, F, \mathbb{R})$ *and a stochastic process* X *defined on this probability space, such that for all* $k \in \mathbb{N}$ *and* $t_1, \ldots, t_k \in T$,

$$(X(t_1), \ldots, X(t_k)) \stackrel{d}{=} X_{t_1,\ldots,t_k}.$$

Proof. See [2, Theorem 36.2] for the details of the proof. □

Often the finite-dimensional distributions of a process are specified in terms of characteristic functions, so it is useful to express Theorem 1.2.32 in terms of characteristic functions.

Corollary 1.2.33. *Let* T *be a set (usually* $\mathbb{R}, \mathbb{R}^+, \mathbb{R}^n$ *or a set of functions). For each* $k \in \mathbb{N}$ *and* $t_1, t_2, \ldots, t_k \in T$ *let* $\phi_{t_1,\ldots,t_k}(\theta_1, \ldots, \theta_k)$ *be the characteristic function of a* k*-dimensional random vector. Suppose that these characteristic functions satisfy the consistency conditions:*

1) for every permutation π *of* $\{1, 2, \ldots, k\}$,

$$\phi_{t_{\pi(1)},\ldots,t_{\pi(k)}}(\theta_{\pi(1)}, \ldots, \theta_{\pi(k)}) = \phi_{t_1,\ldots,t_k}(\theta_1, \ldots, \theta_k)$$

for all $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k$;

2) for all $k, m \in \mathbb{N}$ *and* $t_1, t_2, \ldots, t_{k+m} \in T$,

$$\phi_{t_1,\ldots,t_k}(\theta_1, \ldots, \theta_k) = \phi_{t_1,\ldots,t_k,t_{k+1},\ldots,t_{k+m}}(\theta_1, \ldots, \theta_k, 0, \ldots, 0)$$

(where '0' is taken m *times) for all* $(\theta_1, \theta_2, \ldots, \theta_k) \in \mathbb{R}^k$.

Then there exists a probability space $(\Omega, F, \mathbb{P})$ *and a stochastic process* X *defined on this probability space, such that, for all* $k \in \mathbb{N}$ *and* $t_1, \ldots, t_k \in T$,

$$\phi_{t_1,\ldots,t_k}(\theta_1, \ldots, \theta_k) = \phi_{X(t_1),\ldots,X(t_k)}(\theta_1, \ldots, \theta_k),$$

for all $\theta_1, \ldots, \theta_k \in \mathbb{R}$*, where* $\phi_{X(t_1),\ldots,X(t_k)}$ *is the characteristic function of the random vector* $(X(t_1), \ldots, X(t_k))$*. In other words, there is a probability space on which there is a stochastic process* X *such that the characteristic functions of the finite-dimensional distributions of* X *are given by* $\phi_{t_1,\ldots,t_k}$.

Proof. For each $t_1, \ldots, t_k$ let $X_{t_1,\ldots,t_k}$ be a random vector with characteristic function $\phi_{t_1,\ldots,t_k}$. Write $\cdot$ for the dot product and $\theta = (\theta_1, \ldots, \theta_k)$. Then with Q the matrix of the permutation π, from condition 1)

$$\begin{aligned}\mathbb{E}(\exp(i\theta \cdot X_{t_1,\ldots,t_k})) &= \mathbb{E}(\exp(iQ(\theta) \cdot X_{t_{\pi(1)},\ldots,t_{\pi(k)}})) \\ &= \mathbb{E}(\exp(i\theta \cdot Q^{-1}(X_{t_{\pi(1)},\ldots,t_{\pi(k)}}))).\end{aligned}$$

By the uniqueness of characteristic functions, $X_{t_1,\ldots,t_k} \stackrel{d}{=} Q^{-1}(X_{t_{\pi(1)},\ldots,t_{\pi(k)}})$.

Let $P : \mathbb{R}^{k+m} \to \mathbb{R}^k$ be the projection onto the first k coordinates, then condition 2) implies

$$\mathbb{E}(\exp(i\theta \cdot X_{t_1,\ldots,t_k})) = \mathbb{E}(\exp(i\theta \cdot P(X_{t_1,\ldots,t_{k+m}}))),$$

for $\theta = (\theta_1, \dots, \theta_k)$. By the uniqueness of characteristic functions, $X_{t_1,\dots,t_k} \stackrel{d}{=} P(X_{t_1,\dots,t_k,t_{k+1},\dots,t_{k+m}})$.

Thus the Kolmogorov Existence Theorem (Theorem 1.2.32) implies that there is a probability space on which there is a stochastic process X such that $\phi_{t_1,\dots,t_k}$ is the characteristic function of its finite-dimensional distributions. $\square$

1.2.3 Convergence properties

There are several senses in which a sequence of random variables or random vectors may be convergent.

Definition 1.2.34. *Let $X, X_1, X_2, \dots$ be random variables (random vectors) with respective distribution functions $F, F_1, F_2, \dots$. We say that X_m* converges to X in distribution *or* in law*, written $X_m \xrightarrow{\text{d}} X$, if $F_m(x) \to F(x)$ at all $x \in \mathbb{R}$ $(\mathbb{R}^n)$ at which F is continuous.*

Equivalently, $X_m \xrightarrow{\text{d}} X$ if

$$\lim_{m\to\infty} \int_{x\in\mathbb{R}^n} g(x) dF_m(x) = \int_{x\in\mathbb{R}^n} g(x) dF(x).$$

for all continuous bounded $g : \mathbb{R}^n \to \mathbb{R}$,

Equivalently, $X_m \xrightarrow{\text{d}} X$ if

$$\lim_{m\to\infty} \mathbb{P}(X_m \in A) = \mathbb{P}(X \in A)$$

for all continuity sets $A \subseteq \mathbb{R}^n$, that is Borel sets A such that the boundary of A is an event of probability 0*.*

Definition 1.2.35. *Let $X, X_1, X_2, \dots$ be random variables (random vectors) on the same probability space. We say that X_m* converges *to X* in probability *or* stochastically*, written $X_m \xrightarrow{\text{p}} X$, if*

$$\lim_{m\to\infty} \mathbb{P}(|X - X_m| \geq \varepsilon) = 0,$$

for all $\varepsilon > 0$.

Definition 1.2.36. *Let $X, X_1, X_2, \dots$ be be random variables (random vectors) on the same probability space. We say that X_m converges to X* almost surely*, written $X_m \xrightarrow{\text{a.s}} X$, if*

$$\mathbb{P}(\{\omega \in \Omega : X_m(\omega) \to X(\omega)\}) = 1.$$

There are a number of inter-relationships between these definitions of convergence.

Proposition 1.2.37. *Let $X, X_1, X_2, \ldots$ be random variables (random vectors) on the same probability space.*

1) If $X_m \xrightarrow{\text{a.s}} X$ then $X_m \xrightarrow{\text{p}} X$;

2) If $X_m \xrightarrow{\text{p}} X$ then $X_m \xrightarrow{\text{d}} X$;

3) If $X_m \xrightarrow{\text{d}} 0$ then $X_m \xrightarrow{\text{p}} 0$.

Proof. See [18, Theorem 5.16]. $\square$

We will also need the following result of Kolmogorov.

Proposition 1.2.38. *Let $X, X_1, X_2, \ldots$ be a sequence of independent random variables on the same probability space. If $\sum_{j=1}^{\infty} X_j \xrightarrow{\text{p}} X$ then $\sum_{j=1}^{\infty} X_j \xrightarrow{\text{a.s}} X$.*

Proof. See [13, Theorem 6.1]. $\square$

We have already noted that characteristic functions determine the distribution of random variables or random vectors. The next theorem, Lévy's Continuity Theorem, says that pointwise convergence of characteristic functions essentially determines convergence in distribution.

Theorem 1.2.39. *Lévy's Continuity Theorem*

Suppose that $X_1, X_2, \ldots$ is a sequence of random variables (random vectors) with corresponding characteristic functions $\phi_1, \phi_2, \ldots$.

1) If $X_n \xrightarrow{\text{d}} X$ for some random variable (random vector) X with characteristic function ϕ, then $\phi_n(\theta) \to \phi(\theta)$ for all θ;

2) Conversely, if $\phi(\theta) = \lim_{n\to\infty} \phi_n(\theta)$ exists for all θ and is continuous at $\theta = 0$, then ϕ is the characteristic function of some random variable (random vector) X and $X_n \xrightarrow{\text{d}} X$.

Proof. See [10, Theorem 5, Section 5.9]. $\square$

1.3 Stable random variables and stable random vectors

In this section, we introduce stable random variables and stable random vectors which underlie stable and multistable integrals and processes.

Definition 1.3.1. *A random variable X is said to have* stable distribution *if for every pair of positive numbers A and B, there is a positive number C and a real number D such that*

$$AX_1 + BX_2 \stackrel{d}{=} CX + D, \tag{1.2}$$

where X_1 and X_2 are independent copies of X. We call X strictly stable *if (1.2) holds with $D = 0$. If X and $-X$ have the same distribution, then X is* symmetric.

Theorem 1.3.2. *For any stable random variable X, there is a number α, $0 < \alpha \leq 2$ such that the number C in (1.2) satisfies*

$$C^{\alpha} = A^{\alpha} + B^{\alpha}. \tag{1.3}$$

Proof. See [8, Section VI.1]. □

The number α is called the *index of stability* and we say that X is α-*stable*.

A simple example to illustrate Theorem 1.3.2 is a Gaussian random variable.

Example 1.3.3. *If $X \sim N(\mu, \nu)$, that is a normal random variable with mean μ and variance ν^2, then X has a stable distribution with $\alpha = 2$.*

Proof. Let X_1 and X_2 be independent copies of X, then

$$AX_1 + BX_2 \sim N((A+B)\mu, (A^2+B^2)^{1/2}\nu),$$

which implies (1.2) holds with $C = (A^2+B^2)^{1/2}$ and $D = (A+B-C)\mu$. Thus the normal distribution is a 2-stable distribution. □

The next proposition is a useful consequence of Definition 1.3.1, and indeed provides an equivalent definition.

Proposition 1.3.4. *If a random variable X has an α-stable distribution then for any $n \geq 2$, there is a real number D_n such that*

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} n^{1/\alpha} X + D_n, \tag{1.4}$$

where X_1, X_2,...,X_n are independent copies of X.

Proof. This follows from Definition 1.3.1 by induction. □

The following characterisation of stable random variables in terms of characteristic functions is fundamental to the theory.

Recall that $\mathrm{sign}(x) = x/|x|$ $(x \neq 0)$ and $\mathrm{sign}(0) = 0$.

Proposition 1.3.5. *A random variable X has stable distribution with index of stability $0 < \alpha \leq 2$ iff there are $\sigma \geq 0$, $-1 \leq \beta \leq 1$, and $\mu \in \mathbb{R}$ such that the characteristic function of X has the following form:*

$$\begin{aligned}
\mathbb{E}(\exp i\theta X) &= \exp\{-\sigma^{\alpha}|\theta|^{\alpha}(1 - i\beta(\mathrm{sign}\tan(\pi\alpha/2)) + i\mu\theta\}, \alpha \neq 1, \\
\mathbb{E}(\exp i\theta X) &= \exp\{-\sigma|\theta|(1 + i\beta(2/\pi)(\mathrm{sign}\theta)\ln|\theta|) + i\mu\theta\}, \alpha = 1. \quad (1.5)
\end{aligned}$$

The parameters σ, β and μ are unique.

Proof. See [9, Section 34]. □

Notice that (1.5) gives a characterisation of a stable random variable by four parameters, the *stability index* $0 < \alpha \leq 2$, the *scale* $\sigma \geq 0$, the *skewness* $-1 \leq \beta \leq 1$ and the *shift* $\mu \in \mathbb{R}$. We denote the corresponding stable distribution by

$$S_\alpha(\sigma, \beta, \mu).$$

Lemma 1.3.6. *If X_1, X_2,...,X_n have independent identical distributions $S_\alpha(\sigma, 0, 0)$, then*

$$X_1 + X_2 + \ldots + X_n \stackrel{d}{=} n^{1/\alpha} X_1, \tag{1.6}$$

that is it has the distribution $S_\alpha(n^{1/\alpha}\sigma, 0, 0)$

Proof. This follows immediately from (1.4). □

We now recall some properties of the stable distributions.

Lemma 1.3.7. *Let X be an α-stable random distribution with $0 < \alpha < 2$. For all $\beta > 1$*

$$\mathbb{P}(X > \beta) \leq c_\alpha \beta^{-\alpha},$$

where c_α depends only on α.

Proof. See [17, Property 1.2.15]. □

Lemma 1.3.8. *Let $X \sim S_\alpha(\sigma, \beta, \mu)$ with $0 < \alpha < 2$. Then*

$$\mathbb{E}|X|^p < \infty,$$

for $0 < p < \alpha$, and

$$\mathbb{E}|X|^p = \infty,$$

for $p \geq \alpha$.

Proof. See [17, Property 1.2.16]. □

From Lemma 1.3.8, we can see that if $\alpha \leq 1$, X does not have finite expectation.

Lemma 1.3.9. *Let $X \sim S_\alpha(\sigma, \beta, \mu)$ with $\alpha \neq 1$. Then X is strictly stable iff $\mu = 0$. Let $X \sim S_1(\sigma, \beta, \mu)$. Then X is strictly stable iff $\beta = 0$.*

Proof. See [17, Property 1.2.6 and Property 1.2.8]. □

Lemma 1.3.10. *A distribution $X \sim S_\alpha(\sigma, \beta, \mu)$ is symmetric iff $\beta = 0$ and $\mu = 0$.*

Proof. See [17, Property 1.2.5]. □

Note that every symmetric stable random variable is strictly stable. In this thesis, we will generally only require the case of symmetric, and thus strictly stable distributions when $\mu = \beta = 0$, so (1.5) becomes

$$\mathbb{E}(\exp i\theta X) = \exp\{-\sigma^\alpha|\theta|^\alpha\}, \tag{1.7}$$

and in particular that characteristic function is real.

Example 1.3.11. *The characteristic function of a normal distribution $X \sim N(0,1) \sim S_2(2^{-1/2},0,0)$ is*

$$\mathbb{E}(\exp i\theta X) = \exp\left\{-\frac{1}{2}\theta^2\right\}. \tag{1.8}$$

The next lemma shows the behaviour of the scale of the sum of two independent stable random variables.

Lemma 1.3.12. *Let X_1 and X_2 be independent random variables with $X_i \sim S_\alpha(\sigma_i,0,0)$, $i = 1,2$. Then $X_1 + X_2 \sim S_\alpha(\sigma,0,0)$, with $\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}$.*

Proof. See [17, Property 1.2.1]. □

We now introduce stable random vectors, which extend stable random variables to $\mathbb{R}^d$.

Definition 1.3.13. *A random vector $X = (X_1, X_2, ..., X_d)$ is said to be a* stable random vector *in $\mathbb{R}^d$ if for any positive numbers A and B there is a positive number C and a vector $D \in \mathbb{R}^d$ such that*

$$AX^{(1)} + BX^{(2)} \overset{d}{=} CX + D, \tag{1.9}$$

where $X^{(1)}$ and $X^{(2)}$ are independent copies of X. If $X = (X_1, X_2, ..., X_d)$ and $-X = (-X_1, -X_2, ..., -X_d)$ have the same distribution then X is symmetric. *If $D = 0$, then X is a* strictly stable *random vector.*

We generally will work with symmetric, and thus strictly stable random vectors throughout this thesis.

The following theorem is an analogue of Theorem 1.3.2.

Theorem 1.3.14. *Let $X = (X_1, X_2, ..., X_d)$ be a stable random vector in $\mathbb{R}^d$. Then there is a constant $\alpha \in (0,2]$ such that, in (1.9), $C = (A^\alpha + B^\alpha)^{1/\alpha}$. Moreover, any linear combination of the components of X of the type $Y = \sum_{k=1}^d b_k X_k$ is an α-stable random variable.*

Proof. See [17, Theorem 2.1.2]. □

Definition 1.3.15. *A random vector X in $\mathbb{R}^d$ is called* α-stable *if (1.9) holds with $C = (A^\alpha + B^\alpha)^{1/\alpha}$.*

As with α-stable random variables, we can characterise α-stable random vectors by their characteristic functions. As we mentioned before, we only consider the symmetric strictly stable case.

There is a characterisation of characteristic functions of α-stable random vectors with independent components.

Proposition 1.3.16. *Let $X = (X_1, X_2, ..., X_d)$ be a symmetric α-stable random vector in $\mathbb{R}^d$ with $X_j \sim S_\alpha(\sigma_j, 0, 0)$ independently for $j = 1, \ldots, d$. Then the characteristic function of X is*

$$\phi(\theta) \equiv \phi(\theta_1, \theta_2, ..., \theta_d) = \exp\left\{ -\sum_{j=1}^{d} \sigma_j^\alpha |\theta_j|^\alpha \right\}, \tag{1.10}$$

for $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, d$.

Proof. Since $X_1, \ldots, X_d$ are independent, we have

$$\begin{aligned} \mathbb{E}\exp\left\{ i\sum_{j=1}^{d} \theta_j X_j \right\} &= \mathbb{E}\prod_{j=1}^{d} \exp\{i\theta_j X_j\} \\ &= \prod_{j=1}^{d} \mathbb{E}\exp\{i\theta_j X_j\} \\ &= \prod_{j=1}^{d} \exp\{-\sigma_j^\alpha |\theta_j|^\alpha\} \\ &= \exp\left\{ -\sum_{j=1}^{d} \sigma_j^\alpha |\theta_j|^\alpha \right\}. \end{aligned}$$

□

A basic example of a symmetric α-stable random vector is a multivariate normal distribution.

Example 1.3.17. *If $X = (X_1, X_2, ..., X_d)$ with $X_j \sim N(0,1)$ independently for $j = 1, \ldots, d$, then the characteristic function of X is:*

$$\mathbb{E}\exp\left\{ i\sum_{j=1}^{d} \theta_j X_k \right\} = \exp\left\{ -\frac{1}{2}\sum_{j=1}^{d} \theta_j^2 \right\}.$$

Proof. This follows directly from Proposition 1.3.16 and Example 1.3.11. □

1.4 Stable random processes

Let $\{X(t), t \in T\}$ be a stochastic process, where T is a set. For our purposes, T will be $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^n$ or a set of functions. Recall the finite-dimensional distributions of $\{X(t), t \in T\}$ are the joint distributions of the vector $(X(t_1), X(t_2), \ldots, X(t_n))$, where $t_1, t_2, \ldots, t_n \in T$.

We now can introduce the definition of an α-stable stochastic process.

Definition 1.4.1. *An* α-stable stochastic process $\{X(t), t \in T\}$ *is a process such that every vector* $(X(t_1), X(t_2), \ldots, X(t_d))$ *with* $d \geq 1$ *and* $t_1, t_2, \ldots, t_d \in \mathbb{R}$ *is* α-*stable.* X *is* strictly stable *or* symmetric *if all these vectors are strictly stable or symmetric respectively.*

Example 1.4.2. *Brownian motion is a 2-stable process.*

Proof. This may be deduced from Definition 1.2.31 by considering the multivariate Gaussian vector of increments. □

Since we will mainly concentrate on symmetric stable processes in this thesis, we will write "stable" to mean "symmetric stable", unless otherwise stated.

1.4.1 Stable stochastic integrals

We will be particularly concerned with processes on $\mathbb{R}$ that are defined by stochastic integrals. We first define families of functions that will be α-stable integrable.

Definition 1.4.3. *For* $0 < \alpha \leq 2$ *let*

$$\mathcal{F}_\alpha = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is Lebesgue measurable, } \int |f(x)|^\alpha dx < \infty\}. \tag{1.11}$$

(Note that for the purpose of this thesis the control measure, that is the measure of integration in (1.11), will always be Lebesgue measure, see [17, Chapter 3].)

For $a, b \in \mathbb{R}$, and $f_1, f_2 \in \mathcal{F}_\alpha$ we have

$$\begin{aligned} \int |af_1(x) + bf_2(x)|^\alpha dx &\leq c \int |af_1(x)|^\alpha dx + c \int |bf_2(x)|^\alpha dx \\ &= c|a|^\alpha \int |f_1(x)|^\alpha dx + c|b|^\alpha \int |f_2(x)|^\alpha dx \\ &< \infty \end{aligned}$$

where $c = \max\left\{2^{\alpha-1}, 1\right\}$ (see Lemma 3.2.2 for a generalisation). Thus $af_1 + bf_2 \in \mathcal{F}_\alpha$, which implies $\mathcal{F}_\alpha$ is a linear space.

We define a stable integral to be a stochastic process $\{I(f), f \in \mathcal{F}_\alpha\}$ indexed by $\mathcal{F}_\alpha$, by using characteristic functions to define the finite-dimensional distributions and then applying the corollary of Kolmogorov's Existence Theorem (Corollary 1.2.33) to show this defines a process on $\{I(f), f \in \mathcal{F}_\alpha\}$.

Definition 1.4.4. *Given $d \geq 1$ and $f_1, f_2, ..., f_d \in \mathcal{F}_\alpha$, define a probability measure $\mathbb{P}_{f_1,...,f_d}$ on the random vector $(I(f_1), I(f_2), \ldots, I(f_d))$ by the characteristic function*

$$\phi_{I(f_1),...,I(f_d)}(\theta_1,...,\theta_d) \equiv \phi_{f_1,...,f_d}(\theta_1,...,\theta_d) = \exp\left\{-\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^\alpha dx\right\}, \tag{1.12}$$

for $(\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$.

The next Proposition shows that (1.12) defines a stochastic process on $\mathcal{F}_\alpha$.

Proposition 1.4.5. *There exists a stochastic process $\{I(f), f \in \mathcal{F}_\alpha\}$ whose finite-dimensional distributions are given by* (1.12).

Proof. First we note that for each $f_1, \ldots, f_d$, (1.12) is indeed the characteristic function of a random vector, see [17, Section 3.2]. We now apply Kolmogorov's Existence Theorem to the space of functions $\mathcal{F}_\alpha$ to show the consistency of the distribution given by these characteristic functions. Note that for any permutation $(\pi(1), \pi(2), ..., \pi(d))$ of $(1, 2, ..., d)$, we have

$$\begin{aligned}
\phi_{f_{\pi(1),...,\pi(d)}}(\theta_{\pi(1)},...,\theta_{\pi(d)}) &= \exp\left\{-\int \left|\sum_{j=1}^{d} \theta_{\pi(j)} f_{\pi(j)}(x)\right|^\alpha dx\right\} \\
&= \exp\left\{-\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^\alpha dx\right\} \\
&= \phi_{f_1,...,f_d}(\theta_1,...,\theta_d),
\end{aligned}$$

and for any $n \leq d$,

$$\begin{aligned}
\phi_{f_1,...,f_n}(\theta_1,...,\theta_n) &= \exp\left\{-\int \left|\sum_{j=1}^{n} \theta_j f_j(x)\right|^\alpha dx\right\} \\
&= \exp\left\{-\int \left|\sum_{j=1}^{n} \theta_j f_j(x) + \sum_{i=n+1}^{d} 0 f_i(x)\right|^\alpha dx\right\} \\
&= \phi_{f_1,...,f_n,...,f_d}(\theta_1,...,\theta_n,0,...,0).
\end{aligned}$$

By the corollary to Kolmogorov's Existence Theorem (Corollary 1.2.33), there is a stochastic process defined on $\mathcal{F}_\alpha$ which we denote by $\{I(f), f \in \mathcal{F}_\alpha\}$ with finite-dimensional distributions given by (1.12). □

Definition 1.4.4 also ensures that $\{I(f), f \in \mathcal{F}_\alpha\}$ is a linear functional.

Proposition 1.4.6. *If f_1, $f_2 \in \mathcal{F}_\alpha$, then for all real numbers a_1 and a_2,*

$$I(a_1 f_1 + a_2 f_2) = a_1 I(f_1) + a_2 I(f_2)$$

almost surely.

Proof. See [17, Property 3.2.3]; this is shown in a more general setting in Proposition 3.2.6 of Chapter 3. □

Proposition 1.4.7. *For $f \in \mathcal{F}_\alpha$, the random variable $I(f)$ has distribution $I(f) \sim S_\alpha\left((\int |f(x)|^\alpha dx)^{1/\alpha}, 0, 0\right)$ and characteristic function*

$$\mathbb{E}\exp\{i\theta I(f)\} = \exp\left\{-\int |\theta f(x)|^\alpha dx\right\}. \tag{1.13}$$

Proof. This follows from (1.12) taking $d = 1$ and $f_1 = f$, and (1.7). □

From (1.12), if $f_1, f_2, \ldots, f_d \in \mathcal{F}_\alpha$, the characteristic function of random vector $(I(f_1), I(f_2), ..., I(f_d))$ is given by

$$\mathbb{E}\exp\{i\sum_{j=1}^{d}\theta_j I(f_j)\} = \exp\left\{-\int \left|\sum_{j=1}^{d}\theta_j f_j(x)\right|^\alpha dx\right\}. \tag{1.14}$$

In particular, this implies that the random vector $(I(f_1), I(f_2), ..., I(f_d))$ is symmetric α-stable, by Proposition 1.3.5.

1.4.2 α-stable random measure

Let $(\mathbb{R}, \mathcal{E}, \mathcal{L})$ be the Lebesgue measure space, where $\mathcal{L}$ is Lebesgue measure and $\mathcal{E}$ is the σ-field of Lebesgue measurable sets. Let

$$\mathcal{E}_0 = \{A \in \mathcal{E} : \mathcal{L}(A) < \infty\}$$

be the measurable subsets of $\mathbb{R}$ of finite Lebesgue measure.

Let $(\Omega, F, \mathbb{P})$ be the probability space underlying the process $\{I(f), f \in \mathcal{F}_\alpha\}$ and $L^0(\Omega)$ be the set of all real random variables on $(\Omega, F, \mathbb{P})$.

We define α-stable random measure to be a family of random variables on the sets of $\mathcal{E}_0$ that have measure-like properties.

Definition 1.4.8. *The set function*

$$M_\alpha : \mathcal{E}_0 \to L^0(\Omega)$$

such that for $A \in \mathcal{E}_0$

$$M_\alpha(A) = I(\mathbf{1}_A),$$

where $I(.)$ is the α-stable integral, is called α-stable random measure.

Given an α-stable random measure M_α, it is natural to write

$$\int_{-\infty}^{\infty} f(x) dM_\alpha(x) = \int f(x) dM_\alpha(x) = I(f), \tag{1.15}$$

for $f \in \mathcal{F}_\alpha$.

Proposition 1.4.9. *If M_α is an α-stable random measure then*

$$M_\alpha(A) \sim S_\alpha\left((\mathcal{L}(A))^{1/\alpha}, 0, 0\right),$$

for $A \in \mathcal{E}_0$ with characteristic function

$$\mathbb{E}(\exp i\theta M_\alpha(A)) = \exp\left\{-\int_A |\theta|^\alpha dx\right\}. \tag{1.16}$$

Proof. This follows from Proposition 1.4.7 and Definition 1.4.8. □

In the notation of (1.15), the characteristic function (1.13) becomes

$$\mathbb{E}\left(\exp i\theta \int f(x) dM_\alpha(x)\right) = \exp\left\{-\int |\theta f(x)|^\alpha dx\right\}, \tag{1.17}$$

for $f \in \mathcal{F}_\alpha$.

Standard measures are countably additive on a σ-field. We give an analogous definition for random measures.

Definition 1.4.10. *We say that a set function $M : \mathcal{E}_0 \to L^0$ is* σ-additive *or* countably additive *if whenever $A_1, A_2, \ldots, A_k \in \mathcal{E}_0$ are disjoint and $\bigcup_{j=1}^\infty A_j \in \mathcal{E}_0$, then*

$$M\left(\bigcup_{j=1}^\infty A_j\right) = \sum_{j=1}^\infty M(A_j)$$

almost surely.

Independence on disjoint sets is also a useful property and we call this being *independent scattered*.

Definition 1.4.11. *We say that a set function $M : \mathcal{E}_0 \to L^0$ is* independent scattered *if whenever $A_1, A_2, \ldots, A_k \in \mathcal{E}_0$ are disjoint, then the random variables $M(A_1), M(A_2), \ldots, M(A_k)$ are independent.*

It may be shown from Definitions 1.4.4 and 1.4.8 that the α-stable measure M_α is a 'random measure' that is independent scattered and σ-additive.

Proposition 1.4.12. *M_α is independent scattered.*

Proposition 1.4.13. *M_α is σ-additive.*

The proof of these propositions can be found in [17, Section 3.3]. Proofs in a more general setting are given in Chapter 3.

Note that a 2-stable measure is just a *Wiener measure* or *Brownian noise*.

The next lemma on translation and scaling of random measures will be used in our discussion of various examples.

Lemma 1.4.14. *Let M_α be α-stable measure, $\tau \in \mathbb{R}$ and $c > 0$. For $f_1, f_2, \ldots, f_d \in \mathcal{F}_\alpha$, we have*

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} f_1(x-\tau)dM_\alpha(x), \int_{-\infty}^{\infty} f_2(x-\tau)dM_\alpha(x), \ldots, \int_{-\infty}^{\infty} f_d(x-\tau)dM_\alpha(x)\right) \\
&\stackrel{d}{=} \left(\int_{-\infty}^{\infty} f_1(x)dM_\alpha(x), \int_{-\infty}^{\infty} f_2(x)dM_\alpha(x), \ldots, \int_{-\infty}^{\infty} f_d(x)dM_\alpha(x)\right), \qquad (1.18)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{-\infty}^{\infty} f_1(cx)dM_\alpha(x), \ldots, \int_{-\infty}^{\infty} f_d(cx)dM_\alpha(x)\right) \\
&\stackrel{d}{=} \left(|c|^{-1/\alpha}\int_{-\infty}^{\infty} f_1(x)dM_\alpha(x), \ldots, |c|^{-1/\alpha}\int_{-\infty}^{\infty} f_d(x)dM_\alpha(x)\right). \quad (1.19)
\end{aligned}$$

Proof. For $f_j \in \mathcal{F}_\alpha$, $\theta_j \in \mathbb{R}$ and $j = 1, 2, \ldots, d$, we consider the characteristic function of the vector

$$\left(\int_{-\infty}^{\infty} f_1(x-\tau)dM_\alpha(x), \int_{-\infty}^{\infty} f_2(x-\tau)dM_\alpha(x), \ldots, \int_{-\infty}^{\infty} f_d(x-\tau)dM_\alpha(x)\right).$$

We have

$$\begin{aligned}
\mathbb{E}\left(\exp i\left\{\sum_{j=1}^{d}\theta_j\int_{-\infty}^{\infty} f_j(x-\tau)dM_\alpha(x)\right\}\right) &= \exp\left\{-\int_{-\infty}^{\infty}\left|\sum_{j=1}^{d}\theta_j f_j(x-\tau)\right|^{\alpha} dx\right\} \\
&= \exp\left\{-\int_{-\infty}^{\infty}\left|\sum_{j=1}^{d}\theta_j f_j(y)\right|^{\alpha} dy\right\} \\
&= \mathbb{E}\left(\exp i\left\{\sum_{j=1}^{d}\theta_j\int_{-\infty}^{\infty} f_j(y)dM_\alpha(y)\right\}\right),
\end{aligned}$$

after setting $y = x - \tau$, which is the joint characteristic function of

$$\left(\int_{-\infty}^{\infty} f_1(x)dM_\alpha(x), \int_{-\infty}^{\infty} f_2(x)dM_\alpha(x), \ldots, \int_{-\infty}^{\infty} f_d(x)dM_\alpha(x)\right).$$

Thus (1.18) is true.

For (1.19), we consider the joint characteristic function of

$$\left(\int_{-\infty}^{\infty} f_1(cx)dM_\alpha(x), \int_{-\infty}^{\infty} f_2(cx)dM_\alpha(x), \ldots, \int_{-\infty}^{\infty} f_d(cx)dM_\alpha(x)\right).$$

Then

$$\begin{aligned}
\mathbb{E}\left(\exp i\left\{\sum_{j=1}^{d}\theta_j\int_{-\infty}^{\infty}f_j(cx)dM_\alpha(x)\right\}\right) &= \exp\left\{-\int_{-\infty}^{\infty}\left|\sum_{j=1}^{d}\theta_j f_j(cx)\right|^{\alpha}dx\right\}\\
&= \exp\left\{-\int_{-\infty}^{\infty}\left|\sum_{j=1}^{d}\theta_j|c|^{-1/\alpha}f_j(y)\right|^{\alpha}dy\right\}\\
&= \mathbb{E}\left(\exp i\left\{\sum_{j=1}^{d}\theta_j|c|^{-1/\alpha}\int_{-\infty}^{\infty}f_j(y)dM_\alpha(y)\right\}\right),
\end{aligned}$$

after setting $y = cx$, which is the joint characteristic function of

$$\left(|c|^{-1/\alpha}\int_{-\infty}^{\infty}f_1(x)dM_\alpha(x), |c|^{-1/\alpha}\int_{-\infty}^{\infty}f_2(x)dM_\alpha(x), \dots, |c|^{-1/\alpha}\int_{-\infty}^{\infty}f_d(x)dM_\alpha(x)\right).$$

□

1.4.3 Examples of α-stable processes

We now introduce some examples of α-stable processes which are discussed in [17]. We will meet generalisations of many of these in Chapters 2 and 4.

Example 1.4.15. *Symmetric α-stable Lévy motion.*

Let

$$\begin{aligned}
L_\alpha(t) &= \int \mathbf{1}_{[0,t]}dM_\alpha(x)\\
&= \int_0^t dM_\alpha(x)\\
&= M_\alpha[0,t] \qquad (1.20)
\end{aligned}$$

for $t \geq 0$, where M_α is symmetric α-stable measure on $[0,\infty)$.

Then $\{L_\alpha(t), t \geq 0\}$ is an α-stable process. Thus,

$$L_\alpha(0) = 0 \text{ a.s.}$$

$$L_\alpha(t) - L_\alpha(s) = \int_s^t dM_\alpha(x) = M_\alpha([s,t]) \sim S_\alpha(|t-s|^{1/\alpha}, 0, 0),$$

so if $0 \leq t_1 < t_2 \leq \cdots \leq t_{n-1} < t_n$, then

$$\begin{aligned}
&(L_\alpha(t_2) - L_\alpha(t_1), L_\alpha(t_3) - L_\alpha(t_2), \dots, L_\alpha(t_n) - L_\alpha(t_{n-1}))\\
&\qquad \stackrel{d}{=} (M_\alpha[t_1,t_2], M_\alpha[t_2,t_3], \dots, M_\alpha[t_{n-1},t_n]),
\end{aligned}$$

is a vector with independent components since M_α is independent scattered. Thus symmetric L_α has independent increments. It may be checked from Lemma 1.4.14 that it is $1/\alpha$-self-similar, which means the process, $L_\alpha(t)$, with a parameter $1/\alpha$ has the identical probability distribution as a properly rescaled process, $|c|^{1/\alpha}L_\alpha(ct)$. The symmetric α-stable Lévy motion has stationary increments.

Example 1.4.16. *Moving average processes.*

Let $g \in \mathcal{F}_\alpha$ where $0 < \alpha \leq 2$, and define

$$X(t) = \int_{-\infty}^{\infty} g(t-x)\,dM_\alpha(x) \tag{1.21}$$

for $t \in \mathbb{R}$.

Then $\{X(t), t \in \mathbb{R}\}$ is an α-stable process that is stationary. To see this note that, by Lemma 1.4.14, for $\tau \geq 0$, we have the joint distribution

$$\begin{aligned} X(t+\tau) &= \int_{-\infty}^{\infty} g(t+\tau-x)\,dM_\alpha(x) \\ &\stackrel{fdd}{=} \int_{-\infty}^{\infty} g(t-x)\,dM_\alpha(x) \\ &= X(t). \end{aligned}$$

using (1.18).

A specific example of a moving average process is the Reverse Ornstein-Uhlenbeck process.

Example 1.4.17. *Reverse Ornstein-Uhlenbeck process.*

Let $\lambda > 0$ and $0 < \alpha \leq 2$. The process

$$X(t) = \int_{t}^{\infty} e^{-\lambda(x-t)}\,dM_\alpha(x) \tag{1.22}$$

for $t \in \mathbb{R}$ is well defined, since $\int_t^\infty e^{-\lambda(x-t)\alpha}dx = \frac{1}{\alpha\lambda} < \infty$.

For $s > t$,

$$X(t) - e^{-\lambda(s-t)}X(s) = \int_t^s e^{-\lambda(x-t)}\,dM_\alpha(x), \tag{1.23}$$

so

$$e^{-\lambda t}X(t) - e^{-\lambda s}X(s) = \int_t^s e^{-\lambda x}\,dM_\alpha(x),$$

and the reverse Ornstein-Uhlenbeck process has independent increments. When $\alpha = 2$, the reverse Ornstein-Uhlenbeck process is the stationary independent increments Gaussian process.

Example 1.4.18. *Log-fractional stable motion.*

Let $1 < \alpha < 2$. The process

$$\Lambda_\alpha(t) = \int_{-\infty}^{\infty} (\ln|t-x| - \ln|x|)\,dM_\alpha(x), \tag{1.24}$$

for $t \in \mathbb{R}$ is called log-fractional stable motion.

Log-fractional stable motion is $1/\alpha$-self-similar and has stationary increments.

Example 1.4.19. *Well-balanced linear fractional stable motion.*

Let $0 < \alpha \leq 2$, $0 < h < 1$ *and* $h \neq 1/\alpha$. *Then*

$$L_{\alpha,h}(t) = \int_{-\infty}^{\infty} ((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}) dM_\alpha(x), \tag{1.25}$$

for $t \in \mathbb{R}$ *is an* α*-stable process, that is h-self-similar with stationary increments.*

We verify these properties, since they are relevant to our later work.

For well-balanced linear fractional stable motion to be well-defined, we must have

$$\int_{-\infty}^{\infty} \left| (t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right|^\alpha dx < \infty.$$

We need to check the integral does not diverge at $x = 0$, $x = t$ and $x = -\infty$.

If $h > 1/\alpha$, the integrand is bounded near 0 and t. If $h < 1/\alpha$, as $x \nearrow 0$

$$\begin{aligned} \left| (t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right|^\alpha &\approx \left| (-x)_+^{h-1/\alpha} \right|^\alpha \\ &= |(-x)_+|^{h\alpha - 1}, \end{aligned}$$

which is integrable near 0 since $h\alpha > 0$. Similarly the integral converges near $x = t$. As $x \to -\infty$,

$$\begin{aligned} \left| (t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right|^\alpha &= \left| (-x)^{h-1/\alpha} \left(1 + \frac{t}{-x} \right)^{h-1/\alpha} - (-x)^{h-1/\alpha} \right|^\alpha \\ &\approx \left| (-x)^{h-1/\alpha} \left(1 + \frac{(h-1/\alpha)t}{x} \right) - (-x)^{h-1/\alpha} \right|^\alpha \\ &\approx c \left| (-x)^{h-1/\alpha-1} \right|^\alpha \\ &= c(-x)^{h\alpha - 1 - \alpha}, \end{aligned}$$

where c is independent of x, so as $h < 1$ the integral converges at $-\infty$.

Thus

$$\int_{-\infty}^{\infty} \left| (t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right|^\alpha dx < \infty,$$

which implies well-balanced linear fractional stable motion is well-defined.

We can see the self-similarity by considering finite-dimensional distributions. By Lemma 1.4.14, for $t \in \mathbb{R}$ and $c > 0$, we have the joint distribution

$$\begin{aligned} X(ct) &\overset{fdd}{=} \int_{-\infty}^{\infty} \left((ct-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right) dM_\alpha(x) \\ &\overset{fdd}{=} \int_{-\infty}^{\infty} c^{h-1/\alpha} \left((t-x/c)_+^{h-1/\alpha} - (-x/c)_+^{h-1/\alpha} \right) dM_\alpha(x) \\ &\overset{fdd}{=} c^{h-1/\alpha} (1/c)^{-1/\alpha} \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right) dM_\alpha(x) \\ &\overset{fdd}{=} c^h \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right) dM_\alpha(x) \\ &\overset{fdd}{=} c^h X(t). \end{aligned}$$

To show that well-balanced linear fractional stable motion has stationary increments, for $\tau \in \mathbb{R}$, we consider the distribution of $(X(t)-X(0), -\infty < t < \infty)$ and the distribution of $(X(\tau+t)-X(\tau), -\infty < t < \infty)$. From (1.25) and using Proposition 1.4.6 and Lemma 1.4.14, we have

$$
\begin{aligned}
&X(\tau+t)-X(\tau) \\
&\stackrel{fdd}{=} \int_{-\infty}^{\infty} \left((\tau+t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}\right) dM_\alpha(x) - \int_{-\infty}^{\infty} \left((\tau-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}\right) dM_\alpha(x) \\
&\stackrel{fdd}{=} \int_{-\infty}^{\infty} \left((\tau+t-x)_+^{h-1/\alpha} - (\tau-x)_+^{h-1/\alpha}\right) dM_\alpha(x) \\
&\stackrel{fdd}{=} \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}\right) dM_\alpha(x) \\
&\stackrel{fdd}{=} X(t)-X(0),
\end{aligned}
$$

taking $f_j(x) = (t_j - x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}$, for $j = 1,2,\ldots,d$ in Lemma 1.4.14.

A particular instance of Example 1.4.19 is fractional Brownian motion.

Definition 1.4.20. *Let* $0 < h < 1$ *and* $h \neq 1/2$. Standard fractional Brownian motion $\{B_h(t), t \in \mathbb{R}\}$ *can be defined by*

$$
B_h(t) = C \int_{-\infty}^{\infty} \left((t-x)_+^{h-1/2} - (-x)_+^{h-1/2}\right) dM_2(x), \tag{1.26}
$$

where C is a constant chosen so that the variance $\mathbb{E}\left(B_h(1)^2\right) = 1$.

It may be shown that B_h is a Gaussian process, such that for $t_1, t_2 \in \mathbb{R}$

$$
\mathbb{E}(B_h(t_1) - B_h(t_2))^2 = |t_1 - t_2|^{2h}. \tag{1.27}
$$

From Example 1.4.19, fractional Brownian motion exists when $0 < h < 1$, $h \neq 1/2$ and it is a 2-stable self-similar process with stationary increments. For $h = 1/2$ we have the 2-stable Lévy motion

$$
B_{1/2}(t) = \int_0^t dM_2(x) = M_2[0,t] \tag{1.28}
$$

for $t \geq 0$ and

$$
B_{1/2}(t) = -\int_t^0 dM_2(x) = M_2[t,0] \tag{1.29}
$$

for $t < 0$, which is *standard Brownian motion* or *the Wiener process*.

Chapter 2

Review of localisable processes and multistable processes

2.1 Introduction

In this chapter we recall the notion of localisability of a process, that is when a process $X(t)$ is enlarged by scaling about a point $t = u$ and the enlarged process approaches a limiting process called the local form of the process. We then review recent work on the construction of processes with given local form, and we review some possible constructions of multistable processes.

2.2 Localisable random processes

We first define convergence of a sequence of processes in finite-dimensional distributions and then define localisability.

Definition 2.2.1. *Let Y, Y_1, $Y_2, \ldots$ be random processes on $\mathbb{R}$. We say that Y_n* converges to *Y* in finite-dimensional distributions*, written $Y_n \stackrel{\text{fdd}}{\to} Y$, if for all $t_1, t_2, \ldots, t_k \in \mathbb{R}$,*

$$(Y_n(t_1), Y_n(t_2), \ldots, Y_n(t_k)) \stackrel{d}{\to} (Y(t_1), Y(t_2), \ldots, Y(t_k)),$$

where $\stackrel{d}{\to}$ denotes convergence in distribution of a sequence of k-dimensional vectors.

Intuitively, a process $Y(t)$ is h-localisable at u if scaling the process by a factor r^h whilst time is scaled by a factor r converges to a unique process as $r \searrow 0$.

Definition 2.2.2. *A process $Y(t)$ defined on $\mathbb{R}$ is h-*localisable at u *for some $h > 0$ if*

$$\frac{Y(u+rt) - Y(u)}{r^h} \tag{2.1}$$

converges in finite-dimensional distributions to a non-trivial process in t as $r \searrow 0$. We denote the limiting process by $Y'_u = \{Y'_u(t), t \in \mathbb{R}\}$ called the local form of Y at *u. Thus*

$$\frac{Y(u+rt) - Y(u)}{r^h} \overset{\text{fdd}}{\to} Y'_u(t), \tag{2.2}$$

as $r \searrow 0$, for $t \in \mathbb{R}$.

There are many processes which are localisable, perhaps the most well-known one is index-h fractional Brownian motion with local form itself.

Example 2.2.3. *Fractional Brownian motion*

Let B_h denote index-h fractional Brownian motion. For all $u \in \mathbb{R}$, B_h is h-localisable at u with

$$\frac{B_h(u+rt) - B_h(u)}{r^h} \overset{\text{fdd}}{\to} B_h(t) = (B_h)'_u(t), \tag{2.3}$$

for $t \in \mathbb{R}$ as $r \searrow 0$.

Proof. As we have noticed in (1.26), fractional Brownian motion B_h is self-similar and has stationary increments. Thus for $r > 0$, and $u, t \in \mathbb{R}$,

$$\begin{aligned} B_h(u+rt) - B_h(u) &\overset{fdd}{=} B_h(rt) - B_h(0) \\ &\overset{fdd}{=} B_h(rt) \\ &\overset{fdd}{=} r^h B_h(t). \end{aligned}$$

Thus

$$\frac{B_h(u+rt) - B_h(u)}{r^h} \overset{fdd}{=} B_h(t), \tag{2.4}$$

so

$$\frac{B_h(u+rt) - B_h(u)}{r^h} \overset{\text{fdd}}{\to} B_h(t), \tag{2.5}$$

as $r \searrow 0$. □

This example generalises to any self-similar stationary increment processes.

Proposition 2.2.4. *An h-self-similar process $\{Y(t), t \in \mathbb{R}\}$ with stationary increments is h-localisable at all $u \in \mathbb{R}$ with $Y'_u = Y$.*

Proof. Since Y is an h-self-similar process with stationary increments, for $r > 0$, and $u, t \in \mathbb{R}$, noting that $Y(0) = 0$ almost surely,

$$\begin{aligned} Y(u+rt) - Y(u) &\overset{fdd}{=} Y(rt) - Y(0) \\ &\overset{fdd}{=} Y(rt) \\ &\overset{fdd}{=} r^h Y(t). \end{aligned}$$

Hence

$$\frac{Y(u+rt)-Y(u)}{r^h} \overset{fdd}{=} Y(t), \tag{2.6}$$

so

$$\frac{Y(u+rt)-Y(u)}{r^h} \overset{\text{fdd}}{\to} Y(t), \tag{2.7}$$

as $r \searrow 0$, so Y is localisable at u with local form $Y'_u = Y$. □

A generalization of fractional Brownian motion is multifractional Brownian motion, where the similarity index h of fractional Brownian motion is replaced by $h(t)$, so that h varies with time t.

Definition 2.2.5. *A function $f : \mathbb{R} \to \mathbb{R}$ is called a* Hölder function of exponent $\beta > 0$*, if there exists $c > 0$, such that for each $x, y \in \mathbb{R}$ and $|x-y| \le 1$, we have*

$$|f(x) - f(y)| \le c|x-y|^{\beta}.$$

A definition of multfractional Brownian motion can be given as follows.

Definition 2.2.6. *Multifractional Brownian motion.*

Let $h : \mathbb{R} \to (0,1)$ be a Hölder function of exponent $\beta > 0$. For $t \in \mathbb{R}$, the following random process, denoted by $B_{h(t)}$, is called multifractional Brownian motion *with functional parameter $h(t)$,*

$$B_{h(t)}(t) = C\int_{-\infty}^{\infty} \left((t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2} \right) dM_2(x), \tag{2.8}$$

where C is a constant and M_2 is 2-stable measure or Wiener measure.

Note that in Definition 2.2.6, we make the convention that

$$(t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2} = \mathbf{1}_{[0,t]}(x),$$

for $h(t) = 1/2$, $t \ge 0$ and

$$(t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2} = -\mathbf{1}_{[t,0]}(x),$$

for $h(t) = 1/2$, $t < 0$, to allow $B_{h(t)}$ to be defined when $h(t) = 1/2$.

The theorem below states that multifractional Brownian motion is localisable.

Theorem 2.2.7. *Let $h : \mathbb{R} \to (0,1)$ be a Hölder function of exponent $0 < \beta \le 1$ and suppose $h(t) < \beta$ for all $t \in \mathbb{R}$. Then Multifractional Brownian motion*

$$B_{h(t)}(t) = C\int_{-\infty}^{\infty} \left((t-x)_+^{h(t)-1/2} - (-x)_+^{h(t)-1/2} \right) dM_2(x), \tag{2.9}$$

is $h(u)$-localisable at all $u \in \mathbb{R}$ with the local form

$$(B_{h(t)})'_u(t) \overset{fdd}{=} B_{h(u)}(t),$$

where $B_{h(u)}$ is index $h(u)$ fractional Brownian motion.

Proof. Different approaches to this can be found in [15, Proposition 5], [1, Theorem 1.7] or [7, Theorem 3.3], see also Proposition 4.3.7. □

2.3 Localisable stable processes

In 2008, Falconer, Le Guével and Lévy Véhel [6] studied moving average stable processes which provide many examples of localisable processes.

Recall the space of functions

$$\mathcal{F}_\alpha = \{f : f \text{ is measurable and } \int |f(x)|^\alpha dx < \infty\}. \tag{2.10}$$

Moving average processes are stationary processes defined by moving average stochastic integrals, see Example 1.4.16.

A sufficient condition for localisability of moving average processes is as follows.

Proposition 2.3.1. *Let* $0 < \alpha \le 2$ *and let* M_α *be a symmetric* α*-stable measure on* $\mathbb{R}$*. Let* $g \in \mathcal{F}_\alpha$ *and let* X *be the moving average process*

$$X(t) = \int g(t-x)dM_\alpha(x), \tag{2.11}$$

$t \in \mathbb{R}$*. Suppose that there exists a jointly measurable function* $h(t,z)$*, such that* $h(t,.) \in \mathcal{F}_\alpha$*, and*

$$\lim_{r\to 0} \int \left| \frac{g(r(t+z)) - g(rz)}{r^\gamma} - h(t,z) \right|^\alpha dz = 0 \tag{2.12}$$

for all $t \in \mathbb{R}$*, where* $\gamma + 1/\alpha > 0$*. Then* X *is* $(\gamma + 1/\alpha)$*-localisable with local form* $X'_u = \{\int h(t,z)dM_\alpha(z) : t \in \mathbb{R}\}$ *at all* $u \in \mathbb{R}$*.*

Proof. See [6, Proposition 2.1]. □

For convenience, we make the convention that

$$\mathbf{1}_{[u,v]} = -\mathbf{1}_{[v,u]}. \tag{2.13}$$

A particular example of (2.11) is the reverse Ornstein-Uhlenbeck process.

Example 2.3.2. *Reverse Ornstein-Uhlenbeck process*

Let $\lambda > 0$ *and* $1 < \alpha < 2$ *and let* M_α *be* α*-stable measure on* $\mathbb{R}$*. The stationary process*

$$X(t) = \int_t^\infty \exp(-\lambda(x-t))dM_\alpha(x),$$

$t \in \mathbb{R}$ *is* $1/\alpha$*-localisable with* $X'_u(t) = \int_0^t dM_\alpha(z) = M_\alpha[0,t]$ *at all* $u \in \mathbb{R}$*.*

To see this, it is easily checked that (2.12) holds with $g(x) = \exp(\lambda x)$ and $h(t,z) = \mathbf{1}_{[0,t]}(z)$, when $t > 0$. With our convention (2.13), the case $t < 0$ is similar.

We define *asymmetric linear fractional* α*-stable motion* for $0 < \alpha \leq 2$ by

$$L_{\alpha,h,b^+,b^-}(t) = \int \rho_{\alpha,h}(b^+,b^-,t,x)dM_\alpha(x)$$

where $t \in \mathbb{R}$, $b^+, b^- \in \mathbb{R}$, $h \neq 1/\alpha$ and

$$\begin{aligned} \rho_{\alpha,h}(b^+,b^-,t,x) &= b^+\left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha}\right) \\ &\quad + b^-\left((t-x)_-^{h-1/\alpha} - (-x)_-^{h-1/\alpha}\right), \end{aligned} \tag{2.14}$$

where M_α is symmetric α-stable measure see [6]. If $b^+ = 1$ and $b^- = 0$, this is the well-balanced linear fractional stable motion of Example 1.4.19.

Proposition 2.3.1 may be used to show that, with appropriate conditions on $g(x)$ near $x = 0$, the process X has asymmetric linear fractional α-stable motion as its local form.

Proposition 2.3.3. *Let* $0 < \alpha \leq 2$, $g \in \mathcal{F}_\alpha$ *and* M_α *be an* α*-stable symmetric random measure on* $\mathbb{R}$*. Let* Y *be the moving average process*

$$Y(t) = \int g(t-x)dM_\alpha(x),$$

$t \in \mathbb{R}$*. If there exist* $c_0^+, c_0^-, \gamma, a, c, \eta \in \mathbb{R}$ *with* $c > 0$, $\eta > 0$ *and* $0 < \gamma + 1/\alpha < a \leq 1$ *such that*

$$\frac{g(r)}{r^\gamma} \to c_0^+ \tag{2.15}$$

and

$$\frac{g(-r)}{r^\gamma} \to c_0^- \tag{2.16}$$

as $r \searrow 0$ *and*

$$|g(u+\kappa) - g(u)| \leq c|\kappa|^a|u|^{\gamma-a} \tag{2.17}$$

for all $u \in \mathbb{R}$ *and* $|\kappa| < \eta$*, then* Y *is* $(\gamma+1/\alpha)$*-localisable at all* $u \in \mathbb{R}$*, with local form*

$$(a) \quad Y'_u = L_{\alpha,\gamma+1/\alpha,c_0^+,c_0^-}$$

if $\gamma \neq 0$*, and*

$$(b) \quad Y'_u = (c_0^+ - c_0^-)L_\alpha$$

if $\gamma = 0$*, where* L_α *is the* α*-stable Lévy motion as defined in Example 1.4.15.*

Proof. It may be shown that conditions (2.15) and (2.16) imply that (2.12) holds, taking $h(t,z) = \rho_{\alpha,\gamma+1/\alpha}(c_0^+,c_0^-,t,z)$ when $h \neq 1/2$, and when $h = 1/2$, we make the same convention as Definition 2.2.6, so that conclusion follows from Proposition 2.3.1. See [6, Proposition 3.1] for details. □

2.4 Processes with prescribed local form and multistable processes

Many phenomena from physics, medicine, geography and finance have a highly irregular form. We seek stochastic processes which model such phenomena. The local regularity (or 'volatility' in finance language) may vary with time, and this can be allowed for by varying the local scaling factor, that is by using a process with varying local form. The best known example is multifractional Brownian motion where the scaling exponent varies, see Definition 2.2.6. Another possibility is to vary the local stability index. Thus we get multistable processes, that is localisable processes with α-stable local form, but where the index of stability $\alpha(t)$ varies with time.

Falconer and Lévy-Véhel [7] presented a general method for constructing stochastic processes with prescribed local form. In particular they not only constructed multistable processes where the local stability index α varies, but also multifractional multistable processes, where both the local stability index α and the local scaling factor h vary. We review some approaches to constructing multistable processes, and we present a new approach in Chapters 3 and 4.

Let U be an interval with u an interior point. Let $\{X(t,u) : t \in U\}$ be localisable with local form $X'_u(.,u)$ over a range of u. It is useful to have conditions that ensure that the diagonal process $\{X(t,t) : t \in U\}$ looks locally like $\{X(t,u) : t \in U\}$ when t is close to u, in the sense of having the same local forms.

Let $\{X(t,v) : (t,v) \in U \times U\}$ be a random field and let Y be the diagonal process $Y = \{X(t,t) : t \in U\}$. The following theorem gives a sufficient condition for Y to be h-localisable at u with local form $Y'_u(.) = X'_u(.,u)$ where $X'_u(.,u)$ is the local form of $X_u(.,u)$ at u.

Theorem 2.4.1. *Let U be an interval with u an interior point. Suppose that for some $0 < h < \eta \leq 1$ the process $\{X(t,u) : t \in U\}$ is h-localisable at $u \in U$ with local form $X'_u(.,u)$ and that*

$$\mathbb{P}(|X(v,v) - X(v,u)| \geq |v-u|^{\eta}) \to 0 \tag{2.18}$$

as $v \to u$. Then $Y = \{X(t,t) : t \in U\}$ is h-localisable at u with $Y'_u(.) = X'_u(.,u)$. In particular, this conclusion holds if for some $p > 0$ and $\eta > h$

$$\mathbb{E}(|X(v,v) - X(v,u)|^p) = O(|v-u|^{\eta p}) \tag{2.19}$$

as $v \to u$.

Proof. See [7, Theorem 2.3] for the proof of localisability. If (2.19) holds, Markov's inequality implies (2.18). □

This theorem can be used to prove the localisability of multifractional Brownian motion (Theorem 2.2.7). With (2.19), this can be shown by checking that, there exist $c > 0$, $p > 0$, $\eta \leq 1$ such that $h(.) < \eta$ and such that for $u, v \in \mathbb{R}$,

$$\mathbb{E}(|B_{h(v)}(v,v) - B_{h(v)}(v,u)|^p) \leq c|v-u|^{\eta p}.$$

This diagonal approach allows the construction of "multistable processes", that is processes with each local form an α-stable process, but with the stability index α dependent on t.

It is natural to call a stochastic process $\{X(t), t \in \mathbb{R}\}$ *multistable* if for almost all u, X is localisable at u with X'_u an α-stable process for some $\alpha = \alpha(u)$ where $0 < \alpha(u) \leq 2$ (see Chapter 4).

In the next two subsections, we review some existing constructions of multistable processes, and in Chapters 3 and 4 we present an alternative approach to constructing multistable processes.

2.4.1 Poisson representation of multistable processes

As indicated above, one way to set up a multistable process is to use a random field $X(t,v)$. Such a random field may be given by a sum over a suitable Poisson point process, as was done in [7].

We use a Poisson point process on $\mathbb{R}^2$, denoted by Π. Thus Π is a random countable subset of $\mathbb{R}^2$ such that, writing $N(A)$ for the number of points in a measurable $A \subset \mathbb{R}^2$, the random variable $N(A)$ has a Poisson distribution with mean $\mathcal{L}^2(A)$, where $\mathcal{L}^2$ is plane Lebesgue measure on $\mathbb{R}^2$, and with $N(A_1), N(A_2), \ldots, N(A_n)$ independent for disjoint $A_1, A_2, \ldots, A_n \subset \mathbb{R}^2$.

Thus N is an independent scattered σ-additive random set function on $\mathbb{R}^2$ such that, for each set A in $\mathbb{R}^2$, the random variable $N(A)$ has a Poisson distribution with mean $\mathcal{L}^2(A)$, i.e.,

$$\mathbb{P}(N(A) = k) = e^{-\mathcal{L}^2(A)} \frac{(\mathcal{L}^2(A))^k}{k!},$$

where $k = 0, 1, 2, \ldots$. See [12] for details of Poisson point processes.

Define the space $\mathcal{F}_{a,b}$ of measurable functions on $\mathbb{R}$, for $0 < a < b \leq 2$ by

$$\mathcal{F}_{a,b} = \{f : f \text{ is measurable with } \int_{-\infty}^{\infty} |f(x)|^{a,b} dx < \infty\}, \tag{2.20}$$

where

$$|f(x)|^{a,b} = \max\left(|f(x)|^a, |f(x)|^b\right). \tag{2.21}$$

If $0 < \alpha < 2$ is fixed with M_α symmetric α-stable random measure on $\mathbb{R}$, it may be shown that the stochastic integral (1.15) can be written as a Poisson process sum

$$I(f) = \int f(x) dM_\alpha(x) = c(\alpha) \sum_{(X,Y)\in\Pi} f(X) Y^{<-1/\alpha>} \tag{2.22}$$

where $Y^{<-1/\alpha>} = \mathrm{sign}(Y)|Y|^{-1/\alpha}$ and

$$c(\alpha) = \left(2\alpha^{-1}\Gamma(1-\alpha)\cos(\frac{1}{2}\pi\alpha)\right)^{-1/\alpha}, \tag{2.23}$$

for $\alpha \neq 1$, and

$$c(1) = \frac{1}{\pi}, \tag{2.24}$$

see [17, Section 3.12].

Crucially with this representation, the underlying Poisson process Π does not depend on α, so by varying α in (2.22), we may vary the stability index α. Thus for suitable f, we may define a random field

$$X(t,v) = \sum_{(X,Y)\in\Pi} f(t,v,X)Y^{<-1/\alpha(v)>}, \tag{2.25}$$

with diagonal section

$$Y(t) \equiv X(t,t) = \sum_{(X,Y)\in\Pi} f(t,t,X)Y^{<-1/\alpha(t)>}. \tag{2.26}$$

Note that f must satisfy certain conditions to ensure almost everywhere convergence of (2.25), see [7, Sections 8 and 9].

If we choose f so that $X_u'(.,u)$ takes a given form at $u \in \mathbb{R}$, then under certain conditions results such as Theorem 2.4.1 will give that $Y_u'(.)$ has the same local form. We give some examples of this, starting with the following multistable version of Proposition 2.3.1 given in [6].

Proposition 2.4.2. *Let U be a closed interval with u an interior point. Let $\alpha : U \to (a,b) \subset (0,2)$ satisfy*

$$|\alpha(v) - \alpha(u)| \leq k_1|v-u|^{\eta},$$

where $v \in U$ and $0 < \eta \leq 1$. Let $g \in \mathcal{F}_{a,b}$ and define

$$Y(t) = \sum_{(X,Y)\in\Pi} g(X-t)Y^{<-1/\alpha(t)>}, \tag{2.27}$$

where $t \in \mathbb{R}$. Assume that g satisfies

$$\lim_{r\to 0}\int \left|\frac{g(r(t+z)) - g(rz)}{r^{\gamma}} - h(t,z)\right|^{\alpha(u)} dz = 0 \tag{2.28}$$

for jointly measurable functions with $h(t,.) \in \mathcal{F}_{a,b}$ for all t, where $0 < \gamma + 1/\alpha(u) < \eta \leq 1$. Then Y is $(\gamma + 1/\alpha(u))$-localisable at u with local form

$$Y_u'(t) = \int h(t,z)dM_{\alpha(u)}(z), \tag{2.29}$$

where $t \in \mathbb{R}$ and $M_{\alpha(u)}$ is $\alpha(u)$-stable measure.

Proof. See [6, Theorem 4.1]. □

This proposition can be applied to give a multistable version of the reverse Ornstein-Uhlenbeck process.

Example 2.4.3. *Multistable reverse Ornstein-Uhlenbeck process.*
Let $\lambda > 0$ and $\alpha : \mathbb{R} \to (1,2)$ be continuously differentiable, Let

$$Y(t) = \sum_{(X,Y)\in\Pi, X\geq t} \exp(-\lambda(X-t))Y^{<-1/\alpha(t)>}, \tag{2.30}$$

where $t \in \mathbb{R}$. Then Y is $1/\alpha(u)$-localisable at all $u \in \mathbb{R}$ with

$$Y'_u(t) = c(\alpha(u))^{-1}\int_0^t dM_{\alpha(u)}(z) = M_{\alpha(u)}[0,t], \tag{2.31}$$

where $t \in \mathbb{R}$.

Proof. With our convention in (2.13), this follows by taking $g(x) = \exp(\lambda x)$ and $h(t,z) = \mathbf{1}_{[0,t]}(z)$. See [6, Proposition 4.3] for details. □

For more general (i.e. not moving average) processes, we quote the following result that gives conditions for a diagonal process to be localisable using the Poisson process representation. This is typical of a number of results of this type, see [7].

Theorem 2.4.4. *Let U be a closed interval with u an interior point and let $0 < a < b < 2$. Let X be the random field*

$$X(t,v) = \sum_{(X,Y)\in\Pi} f(t,v,X)Y^{<-1/\alpha(v)>} \tag{2.32}$$

where $t,v \in U$, $f(t,v,x)$ is jointly measurable with $f(t,v,.) \in \mathcal{F}_{a,b}$ for all $t,v \in \mathbb{R}$ and $\alpha : U \to (a,b)$. Suppose $X(.,u)$ is h-localisable at u for $h > 0$. Suppose that $\sup_{t\in U}\int|f(t,u,x)|^{a,b}dx < \infty$, and for some $\eta > h$

$$|\alpha(v) - \alpha(u)| \leq k_1|v-u|^{\eta} \tag{2.33}$$

for all $v \in U$, and

$$\int |f(t,v,x) - f(t,u,x)|^{a,b}dx \leq k_2|v-u|^{\eta} \tag{2.34}$$

for all $t,v \in U$. Then $Y = \{X(t,t) : t \in U\}$ is h-localisable at u with local form $Y'_u(.) = X'_u(.,u)$.

Proof. See [7, Theorem 5.2]. □

Theorem 2.4.4 may be used to construct some specific multistable processes.

Example 2.4.5. *Multistable Lévy motion*
Let $\alpha : \mathbb{R} \to (0,2)$*, and define*

$$Y(t) = \sum_{(X,Y)\in\Pi} \mathbf{1}_{[0,t]}(X) Y^{<-1/\alpha(t)>}, \tag{2.35}$$

for $t \in \mathbb{R}$*. Then* Y *is* $1/\alpha(u)$*-localisable at* u *with*
$Y'_u = \left\{ \int \mathbf{1}_{[0,t]}(z) dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \right\} = L_{\alpha(u)}$*, where* $L_{\alpha(u)}$ *is* $\alpha(u)$*-stable Lévy motion.*

Proof. With our convention in (2.13), take $f(t,v,x) = \mathbf{1}_{[0,t]}(x)$ in Theorem 2.4.4, see [7, Theorem 5.4] for details. □

We now indicate multistable analogues of Example 1.4.19 and Example 1.4.18.

Example 2.4.6. *Linear fractional multistable motion.*
Let $\alpha : \mathbb{R} \to (0,2)$*. Define*

$$L_{\alpha(t),h}(t) = \sum_{(X,Y)\in\Pi} (|t-X|^{h-1/\alpha(t)} - |X|^{h-1/\alpha(t)}) Y^{<-1/\alpha(t)>}, \tag{2.36}$$

for $t \in \mathbb{R}$*. Then* $L_{\alpha(t),h}$ *is* h*-localisable at all* $u \in \mathbb{R}$*, where* $h \neq 1/\alpha(u)$*, with* $(L_{\alpha(t),h})'_u = a(u) L_{\alpha(u),h}$*, where* $L_{\alpha,h}$ *is linear* $\alpha(u)$*-stable motion, see Example 1.4.19.*

Proof. Take $f(t,v,x) = |t-x|^{h-1/\alpha(v)} - |x|^{h-1/\alpha(v)}$ in Theorem 2.4.4, see [7, Theorem 5.3] for details. □

Example 2.4.7. *Log-fractional multistable motion*
Let $\alpha : \mathbb{R} \to (1,2)$ *and be continuously differentiable. Define*

$$\Lambda_{\alpha(t)}(t) = \sum_{(X,Y)\in\Pi} (\ln|t-X| - \ln|X|) Y^{<-1/\alpha(t)>}, \tag{2.37}$$

$t \in \mathbb{R}$*. Then* $\Lambda_{\alpha(t)}$ *is* $1/\alpha(u)$*-localisable at all* $u \in \mathbb{R}$*, with* $(\Lambda_{\alpha(t)})'_u = \Lambda_{\alpha(u)}$*, where* $\Lambda_{\alpha(u)}$ *is log-fractional* $\alpha(u)$*-stable motion, see Example 1.4.18.*

Proof. Take $f(t,v,x) = \log|t-x| - \log|x|$ in Theorem 2.4.4, see [7, Theorem 5.5] for details. □

2.4.2 Series representation of multistable processes

Recently Le Guével and Lévy Véhel [14] constructed multistable processes using a series representation which generalises the construction of [17, Section 3.10] for stable stochastic integrals, and we indicate this very briefly here.

Let $\alpha \in (0,2)$ and $(E,\mathcal{E},m)$ be a finite measure space. Let $(\Gamma_i)_{i\geq 1}$ be a sequence of arrival times of a Poisson process, let $(V_i)_{i\geq 1}$ be a sequence of identically independently distributed random variables distributed on E with distribution function $\hat{m} = m/m(E)$, and let $(\gamma_i)_{i\geq 1}$ be a sequence of identically independently distributed random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$. Assume that the three sequences $(\Gamma_i)_{i\geq 1}$, $(V_i)_{i\geq 1}$ and $(\gamma_i)_{i\geq 1}$ are independent. Then for $f \in \mathcal{F}_\alpha$,

$$\int_E f(x)dM_\alpha(x) \stackrel{d}{=} C\sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} f(V_i), \tag{2.38}$$

where C is a constant that is independent of f, where $\int f(x)dM_\alpha(x)$ is as defined in (1.15). This leads to an alternative series definition of an α-stable process,

$$X(t) = C\sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha} f(V_i,t),$$

for all $t \in \mathbb{R}$, where C is a constant, see [17, Section 3.10.1] for details.

In [14], this is extended by varying α to give multistable processes, as in (2.43) below. The following theorem gives conditions for the diagonal section of a random field expressed in this way to have a desired local form.

Theorem 2.4.8. *Let $\alpha : \mathbb{R} \to [a,b] \subset (0,2)$ be C^1. Let $f(t,u,.) \in \mathcal{F}_\alpha$ for $(t,u) \in \mathbb{R}^2$. Consider the random field*

$$X(t,u) = C\sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(u)} f(t,u,V_i), \tag{2.39}$$

where C is a constant and V_i, γ_i and Γ_i are as above. Assume that $X(t,u)$ as a process in t is h-localisable at u with local form $X'_u(.,u)$. Let U be an interval with u an interior point. Assume that

1) The family of functions $v \to f(t,v,x)$ is differentiable for all $v,t \in U$ and all x in E,

2) and

$$\sup_{t\in U} \int_E \sup_{w\in U} \left(|f(t,w,x)|^{\alpha(w)} \right) d\hat{m}(x) < \infty, \tag{2.40}$$

3) and

$$\sup_{t\in U} \int_E \sup_{w\in U} \left(|f'(t,w,x)|^{\alpha(w)} \right) d\hat{m}(x) < \infty, \tag{2.41}$$

where $f'(t,w,x)$ is the derivative of f with respect to w.

4) and

$$\sup_{t\in U} \int_E \sup_{w\in U} \left[|f(t,w,x)\log|f(t,w,x)||^{\alpha(w)} \right] d\hat{m}(x) < \infty. \tag{2.42}$$

Then

$$Y(t) \equiv X(t,t) = C\sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} f(t,t,V_i) \tag{2.43}$$

is h-localisable at u with local form $Y'_u(.) = X'_u(.,u)$.

Proof. See [14, Theorem 3.3]. □

In the above, m needs to be a finite measure on E to specify the distribution of the V_i, whereas we might wish to work, say, with Lebesgue measure on $\mathbb{R}$. However, if m is a σ-finite measure one can transform m to a finite measure and multiplying the terms of (2.39) by a function of V_i and u, see [14].

2.4.3 General remarks

Poisson representation and series representation provide two different ways to construct multistable processes.

Using the representation by sums over Poisson processes, the stochastic integral $I(f)$ with respect to M_α can be expressed conveniently as (2.22). Notice that the stability index α occurs only as an exponent of Y, and the underlying Poisson process does not depend on α, so varying the exponent gives a natural approach to set up a multistable process. However, f must satisfy certain conditions to ensure almost sure convergence of (2.25) and (2.26). Whilst for $0 < \alpha(t) \leq 1$, we get absolute convergence almost surely, for $1 < \alpha(t) \leq 2$, the proof of convergence is more delicate.

Using the series representation, the stochastic integral $I(f)$ can be written as (2.38). This representation is quite complicated to manipulate, and a disadvantage is that results are obtained for finite measures m and then extended. Nevertheless, series representation is convenient for computational purposes, see [14].

We will give another construction of multistable processes in Chapters 3 and 4 by defining the stochastic integrals in terms of characteristic functions. In some ways this is more fundamental, in that it relates to the characteristic function definition of stable random variables and vectors. Localisability of processes constructed in this way will be discussed in Chapter 4.

Chapter 3

Multistable random measures

3.1 Introduction

In this chapter we use Lévy's Continuity Theorem and Kolmogorov's Extension Theorem to construct multistable integrals and measures. Thus, given $0 < \alpha(x) \leq 2$, we show that there is a random measure $M_{\alpha(x)}$ that 'looks like' an $\alpha(x)$-stable measure near x. We investigate convergence properties and show that an $\alpha(x)$-multistable measure may be approximated by independent sums of α-stable measures (α constant) restricted to small intervals.

3.2 Definition of $\alpha(x)$-multistable measure and integral

Recall that for $0 < a \leq b \leq 2$ we write

$$|f(x)|^{a,b} = \max\left\{|f(x)|^a, |f(x)|^b\right\},$$

and define the space of the functions

$$\mathcal{F}_{a,b} = \{f : f \text{ is measurable with } \int |f(x)|^{a,b} dx < \infty\}. \tag{3.1}$$

Lemma 3.2.1. *For any constant c*

$$\int |cg(x)|^{a,b} dx \leq \max\{|c|^a, |c|^b\} \int |g(x)|^{a,b} dx.$$

Proof. For any constant c,

$$\begin{aligned}
\int |cg(x)|^{a,b} dx &= \int \max\{|cg(x)|^a, |cg(x)|^b\} dx \\
&\leq \max\{|c|^a, |c|^b\} \int \max\{|g(x)|^a, |g(x)|^b\} dx \\
&= \max\{|c|^a, |c|^b\} \int |g(x)|^{a,b} dx,
\end{aligned}$$

as required. $\square$

Lemma 3.2.2. *There is a constant c_1 depending only on a and b such that for all $g,h \in \mathcal{F}_{a,b}$*

$$\int |g(z)+h(z)|^{a,b} dz \leq c_1 \int |g(z)|^{a,b} dz + c_1 \int |h(z)|^{a,b} dz. \tag{3.2}$$

Proof. Let $g,h \in \mathcal{F}_{a,b}$. For $p \geq 1$, by Minkowski's inequality, we have

$$\begin{aligned} \int |g(x)+h(x)|^p dx &\leq \left[\left(\int |g(x)|^p dx \right)^{1/p} + \left(\int |h(x)|^p dx \right)^{1/p} \right]^p \\ &\leq 2^{p-1} \left(\int |g(x)|^p dx + \int |h(x)|^p dx \right), \end{aligned}$$

using that for $a,b \geq 0$ and $p \geq 1$, we have $\left(\frac{a+b}{2}\right)^p \leq \frac{a^p+b^p}{2}$ by Jensen's inequality.

For $0 < p < 1$,

$$\int |g(x)+h(x)|^p dx \leq \int |g(x)|^p dx + \int |h(x)|^p dx.$$

So,

$$\int |g(x)+h(x)|^a dx \leq \max\{2^{a-1},1\} \left(\int |g(x)|^a dx + \int |h(x)|^a dx \right),$$

$$\int |g(x)+h(x)|^b dx \leq \max\{2^{b-1},1\} \left(\int |g(x)|^b dx + \int |h(x)|^b dx \right),$$

so

$$\begin{aligned} \int |g(x)+h(x)|^{a,b} dx &\leq \int |g(x)+h(x)|^a dx + \int |g(x)+h(x)|^b dx \\ &\leq \frac{1}{2} c_1 \left(\int |g(x)|^a dx + \int |h(x)|^a dx \right) + \frac{1}{2} c_1 \left(\int |g(x)|^b dx + \int |h(x)|^b dx \right) \\ &\leq c_1 \left(\int |g(x)|^{a,b} dx + \int |h(x)|^{a,b} dx \right) \end{aligned}$$

where $c_1 = 2\max\{2^{a-1}, 2^{b-1}, 1\}$. $\square$

Lemma 3.2.3. *$\mathcal{F}_{a,b}$ is a linear space.*

Proof. If $f,g \in \mathcal{F}_{a,b}$ and $c \in \mathbb{R}$, then $f+g$, $cf \in \mathcal{F}_{a,b}$ by Lemma 3.2.1 and Lemma 3.2.2. $\square$

We define the multistable stochastic integral $I(f)$ of a function $f \in \mathcal{F}_{a,b}$ by specifying the finite-dimensional distributions of I as a stochastic process on the space of functions $\mathcal{F}_{a,b}$ and then using the Kolmogorov Extension Theorem to show that the process is well-defined. Let $\alpha : \mathbb{R} \to [a,b]$ be Lebesgue measurable, and assume $0 < a \leq b \leq 2$.

Given $f_1, f_2, ..., f_d \in \mathcal{F}_{a,b}$, the following proposition shows that we can define a probability distribution on the vector $(I(f_1), I(f_2), ..., I(f_d)) \in \mathbb{R}^d$ by the characteristic function $\phi_{f_1,...f_d}$ defined by (3.3). The crucial point about this definition is that $\alpha(x)$ varies with x, unlike in Proposition 1.4.5.

Proposition 3.2.4. *Let $d \in \mathbb{N}$ and $f_1, f_2, ..., f_d \in \mathcal{F}_{a,b}$, then*

$$\begin{aligned} \phi_{f_1,...f_d}(\theta_1,...,\theta_d) &= \mathbb{E}\left(\exp\left\{ i \sum_{j=1}^{d} \theta_j I(f_j)\right\}\right) \\ &= \exp\left\{ -\int \left| \sum_{j=1}^{d} \theta_j f_j(x) \right|^{\alpha(x)} dx \right\} \end{aligned} \tag{3.3}$$

for $(\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$, is the characteristic function of a probability distribution on the random vector $(I(f_1), I(f_2), ..., I(f_d))$.

Proof. First, we consider $\alpha(x)$ given by the simple function

$$\alpha(x) = \sum_{k=1}^{m} \alpha_k \mathbf{1}_{A_k}(x), \tag{3.4}$$

where $a \leq \alpha_k \leq b$ and A_k are disjoint Lebesgue measurable sets with $\cup_{k=1}^{m} A_k = \mathbb{R}$.

For $\theta_1, \ldots, \theta_d \in \mathbb{R}$

$$\begin{aligned} \exp\left\{ -\int \left| \sum_{j=1}^{d} \theta_j f_j(x) \right|^{\alpha(x)} dx \right\} &= \exp\left\{ -\sum_{k=1}^{m} \int_{A_k} \left| \sum_{j=1}^{d} \theta_j f_j(x) \right|^{\alpha(x)} dx \right\} \\ &= \exp\left\{ -\sum_{k=1}^{m} \int_{A_k} \left| \sum_{j=1}^{d} \theta_j f_j(x) \right|^{\alpha_k} dx \right\} \\ &= \prod_{k=1}^{m} \exp\left\{ -\int_{A_k} \left| \sum_{j=1}^{d} \theta_j f_j(x) \right|^{\alpha_k} dx \right\} \\ &= \prod_{k=1}^{m} \exp\left\{ -\int \left| \sum_{j=1}^{d} \theta_j f_j(x) \mathbf{1}_{A_k}(x) \right|^{\alpha_k} dx \right\}. \end{aligned} \tag{3.5}$$

Now, $\exp\left\{ -\int \left| \sum_{j=1}^{d} \theta_j f_j(x) \mathbf{1}_{A_k}(x) \right|^{\alpha_k} dx \right\}$ is the characteristic function of the α_k-stable random vector $(I(f_1 \mathbf{1}_{A_k}), \ldots, I(f_d \mathbf{1}_{A_k}))$ given by (1.12). Hence (3.5) is the product of the characteristic functions of m α_k-stable random vectors. Thus it is the characteristic function of a random vector given by the independent sum of α_k-stable random vectors, see Theorem 1.2.20. Hence (3.3) is a valid characteristic function of a random vector $(I(f_1),\dots,I(f_n))$ in the case when $\alpha(x)$ is a simple function (3.4).

Now let $a \le \alpha(x) \le b$ be measurable, and take a sequence of simple functions $\alpha_p(x)$ $(p = 1,2,\dots)$ with $a \le \alpha_p(x) \le b$ such that $\alpha_p(x) \to \alpha(x)$ pointwise almost everywhere. Then

$$\left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha_p(x)} \to \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha(x)}$$

pointwise almost everywhere, for all $\theta_1,\dots,\theta_d \in \mathbb{R}$.

Since $f_1,\dots f_d \in \mathcal{F}_{a,b}$, the linear combination $\sum_{j=1}^{d}\theta_j f_j \in \mathcal{F}_{a,b}$, so

$$\left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha_p(x)} \le \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{a,b}$$

with $\int \left|\sum_{j=1}^{d}\theta_j f_j(x)\right|^{a,b} dx < \infty$, using Lemma 3.2.3. By the dominated convergence theorem,

$$\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha_p(x)} dx \to \int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha(x)} dx, \tag{3.6}$$

and so

$$\exp\left\{\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha_p(x)} dx\right\} \to \exp\left\{\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha(x)} dx\right\}, \tag{3.7}$$

as $p \to \infty$, for all $\theta_1,\dots,\theta_d \in \mathbb{R}$.

For $f_1,\dots,f_d \in \mathcal{F}_{a,b}$, using that $a \le \alpha(x) \le b$ and Lemma 3.2.2,

$$\begin{aligned}
\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha(x)} dx &\le \int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{a,b} dx \\
&\le c_1 \sum_{j=1}^{d} \int |\theta_j|^{a,b} |f_j(x)|^{a,b} dx \\
&\le c_1 \sum_{j=1}^{d} \max\{|\theta_j|^a, |\theta_j|^b\} \int |f_j(x)|^{a,b} dx \\
&\le c \sum_{j=1}^{d} \max\{|\theta_j|^a, |\theta_j|^b\} \\
&\to 0
\end{aligned}$$

as $\theta_j \to 0$, where c_2 and c are independent of θ_j. Thus (3.3) is continuous at 0. From (3.5) $\exp\left\{-\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha_p(x)} dx\right\}$ is a valid characteristic function for all p. Applying Lévy's continuity theorem (Theorem 1.2.39) to (3.7), there is a probability distribution on the random vector $(I(f_1), I(f_2), ..., I(f_d))$, whose characteristic function is given by (3.3). $\square$

Theorem 3.2.5. *There exists a stochastic process $\{I(f), f \in \mathcal{F}_{a,b}\}$ whose finite-dimensional distributions are given by (3.3), that is with $\phi_{I(f_1),...,I(f_d)} = \phi_{f_1,...,f_d}$ for all $f_1, \ldots, f_d \in \mathcal{F}_{a,b}$.*

Proof. We know that (3.3) is a valid characteristic function for all $f_1, \ldots, f_d \in \mathcal{F}_{a,b}$. We now apply Kolmogorov's Extension Theorem and Corollary 1.2.33 to the *space of functions* $\mathcal{F}_{a,b}$ to show (3.3) defines a stochastic process on $\mathcal{F}_{a,b}$. Note that for any permutation $(\pi(1), \pi(2), ..., \pi(d))$ of $(1, 2, ..., d)$, we have

$$\begin{aligned}
\phi_{f_{\pi(1),...,\pi(d)}}(\theta_{\pi(1)}, ..., \theta_{\pi(d)}) &= \exp\left\{-\int \left|\sum_{j=1}^{d} \theta_{\pi(j)} f_{\pi(j)}(x)\right|^{\alpha(x)} dx\right\} \\
&= \exp\left\{-\int \left|\sum_{j=1}^{d} \theta_j f_j(x)\right|^{\alpha(x)} dx\right\} \\
&= \phi_{f_1,...,f_d}(\theta_1, ..., \theta_d),
\end{aligned}$$

and for any $n \leq d$,

$$\begin{aligned}
\phi_{f_1,...,f_n}(\theta_1, ..., \theta_n) &= \exp\left\{-\int \left|\sum_{j=1}^{n} \theta_j f_j(x)\right|^{\alpha(x)} dx\right\} \\
&= \exp\left\{-\int \left|\sum_{j=1}^{n} \theta_j f_j(x) + \sum_{i=n+1}^{d} 0 f_i(x)\right|^{\alpha(x)} dx\right\} \\
&= \phi_{f_1,...,f_n,...,f_d}(\theta_1, ..., \theta_n, 0, ..., 0).
\end{aligned}$$

This shows the consistency of the probability distributions given by (3.3). By the corollary of Kolmogorov's Extension Theorem (Corollary 1.2.33), there is a stochastic process on $\mathcal{F}_{a,b}$ which we denote by $\{I(f), f \in \mathcal{F}_{a,b}\}$, whose finite-dimensional distributions are given by the characteristic function (3.3). $\square$

We call $I(f)$ the $\alpha(x)$-*multistable integral* of f. We now check the linearity of the integral.

Proposition 3.2.6. *If $f_1, f_2 \in \mathcal{F}_{a,b}$ and $a_1, a_2 \in \mathbb{R}$,*

$$I(a_1 f_1 + a_2 f_2) = a_1 I(f_1) + a_2 I(f_2) \quad a.s. \tag{3.8}$$

Proof. We show that

$$I(a_1 f_1 + a_2 f_2) - a_1 I(f_1) + a_2 I(f_2) = 0 \quad \text{a.s.}$$

For all real θ we have

$$\begin{aligned}
&\mathbb{E}\left(\exp\{i\theta\left[I(a_1 f_1 + a_2 f_2) - a_1 I(f_1) - a_2 I(f_2)\right]\}\right) \\
&= \mathbb{E}\left(\exp\{i\left[\theta I(a_1 f_1 + a_2 f_2) - (a_1\theta) I(f_1) - (a_2\theta) I(f_2)\right]\}\right) \\
&= \exp\left\{-\int |\theta(a_1 f_1 + a_2 f_2) + (-a_1\theta) f_1 + (-a_2\theta) f_2|^{\alpha(x)} dx\right\} \\
&= \exp\{0\} \\
&= 1,
\end{aligned}$$

where we have used (3.3) with functions $(a_1 f_1 + a_2 f_2)$, f_1, f_2 and variables θ, $-a_1\theta$, $-a_2\theta$, so $I(a_1 f_1 + a_2 f_2) - (a_1) I(f_1) - (a_2) I(f_2) = 0$ almost surely by the uniqueness of characteristic functions. $\square$

Let $\alpha : \mathbb{R} \to [a,b]$ be measurable where $0 < a \le b \le 2$. Analogously to [17, Section 3.3] for α-stable measures, we define the $\alpha(x)$-multistable random measure M in terms of $\alpha(x)$-multistable integrals of indicator functions. With$(\Omega, F, \mathbb{P})$ the underlying probability space, we write $\mathcal{L}^0(\Omega)$ for the set of all real random variables defined on Ω. Let $\mathcal{L}$ be Lebesgue measure on $\mathbb{R}$ and $\mathcal{E}$ the Lebesgue measurable sets. Let

$$\mathcal{E}_0 = \{A \in \mathcal{E} : \mathcal{L}(A) < \infty\}$$

be the sets of finite Lebesgue measure. As usual, $\mathbf{1}_A$ is the indicator function of the set A.

Definition 3.2.7. *We define the* $\alpha(x)$-multistable random measure $M : \mathcal{E}_0 \to \mathcal{L}^0(\Omega)$ *by*

$$M(A) = I(\mathbf{1}_A), \tag{3.9}$$

where I is the process of Proposition 3.2.5, noting that $\int |\mathbf{1}_A|^{a,b} < \infty$, if $A \in \mathcal{E}_0$, so $\mathbf{1}_A \in \mathcal{F}_{a,b}$.

It is natural to write

$$\int f(x) dM(x) = I(f), \quad f \in \mathcal{F}_{a,b}, \tag{3.10}$$

since there are many analogues to usual integration with respect to a measure. Clearly

$$\int \mathbf{1}_A(x) dM(x) = M(A). \tag{3.11}$$

From (3.8), $\int f(x) dM(x)$ is linear, i.e.

$$\int (a_1 f_1(x) + a_2 f_2(x))\, dM(x) = a_1 \int f_1(x) dM(x) + a_2 \int f_2(x) dM(x). \tag{3.12}$$

With this notation the characteristic function (3.3) may be written

$$\begin{aligned}\mathbb{E}\left(\exp i\left(\sum_{j=1}^{d}\theta_j\int f_j(x)dM(x)\right)\right) &= \mathbb{E}\left(\exp i\left(\sum_{j=1}^{d}\theta_j I(f_j)\right)\right)\\ &= \exp\left\{-\int\left|\sum_{j=1}^{d}\theta_j f_j(x)\right|^{\alpha(x)}dx\right\} \quad (3.13)\end{aligned}$$

for $f_j \in \mathcal{F}_{a,b}$. Taking $f_j = \mathbf{1}_{A_j}$ in (3.13) with $A_j \in \mathcal{E}_0$,

$$\mathbb{E}\left(\exp\left\{i\sum_{j=1}^{d}\theta_j M(A_j)\right\}\right) = \exp\left\{-\int\left|\sum_{j=1}^{d}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx\right\}. \quad (3.14)$$

In particular, for $A \in \mathcal{E}_0$

$$\mathbb{E}\left(\exp i\theta M(A)\right) = \exp\left\{-\int_A|\theta|^{\alpha(x)}dx\right\}. \quad (3.15)$$

We need to show that the multistable measure M is independent scattered and σ-additive, recall Definitions 1.4.10 and 1.4.11. We modify the proofs in [17, Section 3.3].

Theorem 3.2.8. *M is independent scattered.*

Proof. Let $A_1,A_2,...,A_d \in \mathcal{E}_0$ with $A_i \cap A_j = \emptyset$ if $i \neq j$. Then

$$\begin{aligned}\mathbb{E}\left(\exp\left\{i\sum_{j=1}^{d}\theta_j M(A_j)\right\}\right) &= \exp\left\{-\int\left|\sum_{j=1}^{d}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx\right\} \quad \text{by (3.14)}\\ &= \exp\left\{-\sum_{j=1}^{d}\int\left|\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx\right\}\\ &\quad \text{since } A_j \text{ are disjoint}\\ &= \prod_{j=1}^{d}\exp\left\{-\int\left|\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx\right\}\\ &= \prod_{j=1}^{d}\exp\left\{-\int_{A_j}\left|\theta_j\right|^{\alpha(x)}dx\right\}\\ &= \prod_{j=1}^{d}\mathbb{E}(\exp\{i\theta_j M(A_j)\}).\end{aligned}$$

By Theorem 1.2.17 $M(A_1), M(A_2), ..., M(A_d)$ are independent, so M is independent scattered. □

Theorem 3.2.9. *M is σ-additive, that is if $A_1,A_2,... \in \mathcal{E}_0$ are disjoint and $\bigcup_{j=1}^{\infty} A_j \in \mathcal{E}_0$ then*

$$M(\bigcup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} M(A_j)$$

a.s.

Proof. Let $A_1,A_2,...,A_k \in \mathcal{E}_0$ be a finite collection of disjoint sets. Then

$$\begin{aligned} M(\bigcup_{j=1}^{k} A_j) &= I(\mathbf{1}_{\cup_{j=1}^{k} A_j}) \quad \text{by (3.9)} \\ &= I(\sum_{j=1}^{k} \mathbf{1}_{A_j}) \\ &= \sum_{j=1}^{k} I(\mathbf{1}_{A_j}) \quad \text{by (3.8)} \\ &= \sum_{j=1}^{k} M(A_j) \quad \text{by (3.9)} . \end{aligned}$$

Now let

$$B = \bigcup_{j=1}^{\infty} A_j \in \mathcal{E}_0$$

where $A_1,A_2,... \in \mathcal{E}_0$ is a countable family of disjoint sets. Then

$$B = \bigcup_{j=1}^{k} A_j \cup \left(\bigcup_{j=k+1}^{\infty} A_j \right),$$

so since $\bigcup_{j=1}^{k} A_j$ and $\bigcup_{j=k+1}^{\infty} A_j$ are disjoint and in $\mathcal{E}_0$, it follows from above that

$$\begin{aligned} M(B) &= M\left(\bigcup_{j=1}^{k} A_j\right) + M\left(\bigcup_{j=k+1}^{\infty} A_j\right) \\ &= \sum_{j=1}^{k} M(A_j) + M\left(\bigcup_{j=k+1}^{\infty} A_j\right). \end{aligned} \tag{3.16}$$

Now we consider the characteristic function of $M\left(\bigcup_{j=k+1}^{\infty} A_j\right)$. As $\alpha(x) \in [a,b]$, where $0 < a < b \leq 2$, we get

$$0 \leq \int |\theta \mathbf{1}_{\cup_{j=k+1}^{\infty} A_j}(x)|^{\alpha(x)} dx \leq \max\{|\theta|^a, |\theta|^b\} \mathcal{L}(\bigcup_{j=k+1}^{\infty} A_j).$$

Since $\lim_{k\to\infty} \mathcal{L}(\bigcup_{j=k+1}^{\infty} A_j) = 0$,

$$\lim_{k\to\infty} \int |\theta \mathbf{1}_{\cup_{j=k+1}^{\infty} A_j}(x)|^{\alpha(x)} dx = 0.$$

Thus

$$\begin{aligned}
\lim_{k\to\infty} \mathbb{E}\left(\exp\left\{i\theta M(\bigcup_{j=k+1}^{\infty} A_j)\right\}\right) &= \lim_{k\to\infty} \exp\left\{-\int |\theta \mathbf{1}_{\cup_{j=k+1}^{\infty} A_j}|^{\alpha(x)}\right\} \quad \text{by (3.15)} \\
&= \exp\left\{-\lim_{k\to\infty} \int |\theta \mathbf{1}_{\cup_{j=k+1}^{\infty} A_j}|^{\alpha(x)}\right\} \\
&= \exp(0) \\
&= 1,
\end{aligned}$$

which is the characteristic function of the random variable 0, so $M(\bigcup_{j=k+1}^{\infty} A_j) \xrightarrow{\mathrm{d}} 0$ as $k \to \infty$ by Lévy's continuity theorem (Theorem 1.2.39).

By (3.16) we get $M(B) - \sum_{j=1}^{k} M(A_j) \xrightarrow{\mathrm{d}} 0$ and so $M(B) - \sum_{j=1}^{k} M(A_j) \xrightarrow{\mathrm{p}} 0$ as $k \to \infty$ by Proposition 1.2.37. Thus $\lim_{k\to\infty} \sum_{j=1}^{k} M(A_j) \overset{p}{=} M(B)$, and since the summands $M(A_j)$ are independent, this implies convergence a.s. See Proposition 1.2.38.

Thus M is σ-additive. $\square$

3.3 Convergence of sequences of multistable measures

In this section, we will obtain conditions for convergence of a sequence of multistable measures with different multistable indexes.

Theorem 3.3.1. *Let $\alpha_n(x)$, $\alpha(x)$ be Lebesgue measurable with $0 < a \le \alpha_n(x), \alpha(x) \le b \le 2$ for all $x \in \mathbb{R}$. Let M_n, M be the associated α_n-multistable and α-multistable measures as in Definition 3.2.7. Suppose $\alpha_n(x) \to \alpha(x)$ for almost all $x \in \mathbb{R}$. Then $M_n \xrightarrow{\mathrm{fdd}} M$ as $n \to \infty$, that is for all $m \in \mathbb{N}$ and $A_1, A_2, \ldots, A_m \in \mathcal{E}_0$,*

$$(M_n(A_1), M_n(A_2), \ldots, M_n(A_m)) \xrightarrow{\mathrm{d}} (M(A_1), M(A_2), \ldots, M(A_m))$$

as $n \to \infty$.

Proof. Let $A_1, A_2, \ldots, A_m \in \mathcal{E}_0$, $\theta_1, \theta_2, \ldots, \theta_m \in \mathbb{R}$, and $\theta_j \neq 0$, and consider the joint characteristic function of M_n,

$$\mathbb{E}\left(\exp i\left\{\sum_{j=1}^{m} \theta_j M_n(A_j)\right\}\right) = \exp\left(-\int \left|\sum_{j=1}^{m} \theta_j \mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)} dx\right) \quad \text{by (3.14)}. \tag{3.17}$$

For the measure M,

$$\mathbb{E}\left(\exp i\left\{\sum_{j=1}^{m}\theta_j M(A_j)\right\}\right)=\exp\left(-\int\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx\right)\quad\text{by (3.14)}\ . \tag{3.18}$$

Since $\alpha_n(x)\to\alpha(x)$ for almost all x, we have

$$\lim_{n\to\infty}\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)}=\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)},$$

for almost all $x\in\mathbb{R}$.

Write

$$C=\max\left\{\left(\sum_{j=1}^{m}|\theta_j|\right)^{a},\left(\sum_{j=1}^{m}|\theta_j|\right)^{b}\right\}.$$

Then for all n and all $x\in\mathbb{R}$

$$\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)}\leq C\mathbf{1}_A(x)$$

where $A=\bigcup_{j=1}^{m}A_j\in\mathcal{E}_0$. Then

$$\begin{aligned}\int\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)}dx &\leq C\int\mathbf{1}_A(x)dx\\ &= C\mathcal{L}(A)\\ &< \infty.\end{aligned}$$

Now we can apply the dominated convergence theorem to get

$$\lim_{n\to\infty}\int\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)}dx=\int\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx.$$

This implies

$$\lim_{n\to\infty}\exp\left(-\int\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)}dx\right)=\exp\left(-\int\left|\sum_{j=1}^{m}\theta_j\mathbf{1}_{A_j}(x)\right|^{\alpha(x)}dx\right).$$

By Lévy's continuity theorem (Theorem 1.2.39), since (3.18) is already the characteristic function of a random vector and therefore is continuous at 0,

$$(M_n(A_1),M_n(A_2),\ldots,M_n(A_m))\xrightarrow{\mathrm{d}}(M(A_1),M(A_2),\ldots,M(A_m))$$

as $n\to\infty$. □

We will use this result in the next section on approximation of $\alpha(x)$-multistable measures.

3.4 Approximation by sums of stable measures

To get a feel for $\alpha(x)$-mutistable measures, we show that, for a continuous $\alpha(x)$, the $\alpha(x)$-multistable measure M defined in Section 3.2 may be approximated by random measures that are the sum of many independent α-stable measures defined on short intervals.

Assume in this section that $\alpha : \mathbb{R} \to [a,b] \subset (0,2]$ is continuous. We may define an $\alpha(x)$-multistable measure M on the sets $\mathcal{E}_0$ as in Section 3.2. We now use the same procedure but with piecewise constant functions $\alpha_n(x)$ to obtain approximating measures M_n. We will arrange for the restriction of M_n to each interval $[r2^{-n},(r+1)2^{-n})$ to be an $\alpha(r2^{-n})$-stable measure independently for all r. We will show that when n is large M_n is close in distribution to M.

Definition 3.4.1. *For each n let $\alpha_n : \mathbb{R} \to [a,b] \subset (0,2)$ be given by*

$$\alpha_n(x) = \alpha(r2^{-n}) \text{ if } x \in [r2^{-n},(r+1)2^{-n}) \text{ for } r \in \mathbb{Z}.$$

Let $M_n \equiv M_{\alpha_n(x)}$ be the $\alpha_n(x)$-multistable measure obtained from this $\alpha_n(x)$ as in Section 3.3. In particular M_n has finite-dimensional distributions given by the characteristic function

$$\mathbb{E}\left(\exp i\left\{\sum_{j=1}^{d}\theta_j M_n(A_j)\right\}\right) = \exp\left\{-\int\left|\sum_{j=1}^{d}\theta_j \mathbf{1}_{A_j}(x)\right|^{\alpha_n(x)} dx\right\}. \tag{3.19}$$

That M_n is independent scattered and σ-additive, follows from Theorem 3.2.8 and Theorem 3.2.9 with $\alpha(x)$ replaced by $\alpha_n(x)$.

We show how to decompose M_n as an independent sum of measures on the intervals $[r2^{-n},(r+1)2^{-n}))$. Let $M_{n,r}$ denote the restriction of $\alpha(r2^{-n})$-stable measure to the interval $[r2^{-n},(r+1)2^{-n}))$,

$$M_{n,r}(A) = M_{\alpha(r2^{-n})}(A\cap[r2^{-n},(r+1)2^{-n})) = M_{\alpha_n(x)}(A\cap[r2^{-n},(r+1)2^{-n})), \tag{3.20}$$

where $M_{\alpha(r2^{-n})}$ is $\alpha(r2^{-n})$-stable measure.

Theorem 3.4.2. *We have that M_n is a random measure given by the sum of independent random measures*

$$M_n(A) = \sum_{r\in\mathbb{Z}} M_{n,r}(A), \tag{3.21}$$

almost surely for $A \in \mathcal{E}_0$.

Proof. By Theorem 3.2.8, $M_{\alpha_n(x)}$ is independent scattered, so since that the sets $A\cap[r2^{-n},(r+1)2^{-n})$ are disjoint for distinct r, we have that

$$M_{\alpha_n(x)}(A\cap[r2^{-n},(r+1)2^{-n})) = M_{n,r}(A)$$

are independent for distinct r from the definition of independent scattered.

Let $A \in \mathcal{E}_0$, then by Theorem 3.2.9 and (3.20)

$$\begin{aligned} M_n(A) &= M_{\alpha_n(x)}(A) \\ &= M_{\alpha_n(x)}\left(\bigcup_{r\in\mathbb{Z}} A\cap[r2^{-n},(r+1)2^{-n})\right) \\ &\overset{a.s}{=} \sum_{r\in\mathbb{Z}} M_{\alpha_n(x)}(A\cap[r2^{-n},(r+1)2^{-n})) \\ &\overset{a.s}{=} \sum_{r\in\mathbb{Z}} M_{n.r}(A), \end{aligned}$$

as required. □

The fact that the random measures M_n approximate M follows from the result of Section 3.3.

Theorem 3.4.3. *Let* $\alpha : \mathbb{R} \to [a,b]$, $0 < a \leq b \leq 2$ *be continuous. Then* $M_n \overset{\text{fdd}}{\to} M$ *as* $n \to \infty$.

Proof. For each n we have $\alpha_n(x) = \alpha(r2^{-n})$ for all $x \in [r2^{-n},(r+1)2^{-n})$. Since $\alpha(x)$ is assumed continuous we have

$$\lim_{n\to\infty} \alpha_n(x) = \alpha(x)$$

for all x. Thus by Theorem 3.3.1, it follows that $M_n \overset{\text{fdd}}{\to} M$ as $n \to \infty$. □

3.5 Local behaviour of $\alpha(x)$-multistable measure

One would expect, partly because of the results of the previous section, that an $\alpha(x)$-multistable measure looks like an $\alpha(u)$-stable measure in a very small interval around u. In this section, we make this idea precise.

Definition 3.5.1. *For* $u \in \mathbb{R}$, $r > 0$, *let* $T_{u,r} : \mathbb{R} \to \mathbb{R}$ *be*

$$T_{u,r}(x) = \frac{x-u}{r}.$$

This induces a mapping $T^{\#}_{u,r}$ *on random measures, given by*

$$\begin{aligned} \int f(x)d(T^{\#}_{u,r}M)(x) &= \int f\left(\frac{x-u}{r}\right)dM(x) \\ &\equiv I\left(f\left(\frac{\cdot-u}{r}\right)\right). \end{aligned} \tag{3.22}$$

In particular (3.9) implies that

$$(T^{\#}_{u,r}M)(A) := M(T^{-1}_{u,r}(A)) = I(\mathbf{1}_{T^{-1}_{u,r}(A)})$$

for $A \in \mathcal{E}_0$.

We now show that the $\alpha(x)$-multistable random measure $M \equiv M_{\alpha(x)}$ "looks like" the $\alpha(u)$-stable measure $M_{\alpha(u)}$ near u.

We firstly introduce a lemma which will be useful in the proof of our next theorem.

Lemma 3.5.2. *Let α: $\mathbb{R} \to \mathbb{R}^+$, let $u \in \mathbb{R}$ and suppose*

$$|\alpha(u) - \alpha(v)| = o\left(\frac{1}{|\log|u-v||}\right) \tag{3.23}$$

as $v \to u$. Let $[z_1, z_2] \subseteq \mathbb{R}$ be a bounded interval. Then

$$\lim_{r\to 0} r^{1/\alpha(u)-1/\alpha(u+rz)} = 1 \tag{3.24}$$

uniformly for $z \in [z_1, z_2]$ and

$$\lim_{r\to 0} r^{1-\alpha(u+rz)/\alpha(u)} = 1 \tag{3.25}$$

uniformly for $z \in [z_1, z_2]$.

Proof. Note that since α is continuous so bounded away from 0 on compact intervals, there is a constant c_1 such that for r sufficiently small,

$$\begin{aligned}
|\log r^{1/\alpha(u)-1/\alpha(u+rz)}| &= \frac{|\alpha(u+rz)-\alpha(u)|}{|\alpha(u)\alpha(u+rz)|}|\log r| \\
&\leq c_1|\log r| o\left(\frac{1}{|\log|rz||}\right) \\
&= o(1)\frac{|\log r|}{|\log|r|+\log|z||} \\
&\to 0
\end{aligned}$$

as $r \to 0$ uniformly for $z \in [z_1, z_2]$. The proof of (3.25) is similar. $\square$

Here is our result on the local form of multistable measures.

Theorem 3.5.3. *Let $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2]$ be continuous with $\alpha(x+r) - \alpha(x) = o(\frac{1}{\log r})$ uniformly on bounded intervals and let $u \in \mathbb{R}$. Then for all functions $f_1, \ldots, f_d \in \mathcal{F}_{a,b}$ with compact support, the vectors*

$$\begin{aligned}
&\left(r^{-1/\alpha(u)}\int f_1(x)d(T^{\#}_{u,r}M)(x), \ldots, r^{-1/\alpha(u)}\int f_d(x)d(T^{\#}_{u,r}M)(x)\right) \\
&\qquad\xrightarrow{\mathrm{d}} \left(\int f_1(x)dM_{\alpha(u)}(x), \ldots, \int f_d(x)dM_{\alpha(u)}(x)\right)
\end{aligned} \tag{3.26}$$

as $r \to 0$. In particular,

$$r^{-\frac{1}{\alpha(u)}}\left((T^{\#}_{u,r}M)(A_1), \ldots, (T^{\#}_{u,r}M)(A_d)\right) \xrightarrow{\mathrm{d}} (M_{\alpha(u)}(A_1), \ldots, M_{\alpha(u)}(A_d)) \tag{3.27}$$

as $r \to 0$, for all bounded sets $A_1, \ldots, A_d \in \mathcal{E}_0$.

Proof. Let $f_1, f_2, \ldots, f_m \in \mathcal{F}_{a,b}$ be functions with compact support, say in $[-z_0, z_0]$. Let $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, m$, and consider the characteristic functions.

$$\begin{aligned}
&\mathbb{E}\left(\exp i \sum_{j=1}^{m} \theta_j r^{-\frac{1}{\alpha(u)}} \int f_j(x) d(T^{\#}_{u,r} M)(x)\right) \\
&\quad = \mathbb{E}\left(\exp i \sum_{j=1}^{m} \theta_j r^{-\frac{1}{\alpha(u)}} \int f_j(\frac{x-u}{r}) dM(x)\right) \quad \text{by (3.22)} \qquad (3.28) \\
&\quad = \exp\left(-\int \left|\sum_{j=1}^{m} \theta_j r^{-\frac{1}{\alpha(u)}} f_j(\frac{x-u}{r})\right|^{\alpha(x)} dx\right) \quad \text{by (3.13)} \\
&\quad = \exp\left(-\int \left|\sum_{j=1}^{m} \theta_j r^{-\frac{1}{\alpha(u)}} f_j(z)\right|^{\alpha(rz+u)} r dz\right) \\
&\quad = \exp\left(-\int \left|\sum_{j=1}^{m} \theta_j f_j(z)\right|^{\alpha(rz+u)} r^{1-\frac{\alpha(rz+u)}{\alpha(u)}} dz\right), \qquad (3.29)
\end{aligned}$$

on writing $\frac{x-u}{r} = z$.

By Lemma 3.5.2 we have

$$\lim_{r \to 0} r^{1-\frac{\alpha(rz+u)}{\alpha(u)}} = 1$$

uniformly for all $z \in [-z_0, z_0]$, and $\lim_{r\to 0} \alpha(rz+u) = \alpha(u)$ also uniformly for all $z \in [-z_0, z_0]$ since α is continuous.

We note that

$$\left|\sum_{j=1}^{m} \theta_j f_j(z)\right|^{\alpha(rz+u)} \leq \sum_{j=1}^{m} (|\theta_j||f_j(z)|)^{a,b} \leq c \sum_{j=1}^{m} |f_j(z)|^{a,b}$$

where c is independent of r. Thus for r sufficiently small,

$$\left|\sum_{j=1}^{m} \theta_j f_j(z)\right|^{\alpha(rz+u)} r^{1-\frac{\alpha(rz+u)}{\alpha(u)}} \leq 2c \sum_{j=1}^{m} |f_j(z)|^{a,b},$$

for $z \in [-z_0, z_0]$, so as $f_j \in \mathcal{F}_{a,b}$, we can apply the dominated convergence theorem to get

$$\lim_{r\to 0} \int \left|\sum_{j=1}^{m} \theta_j f_j(z)\right|^{\alpha(rz+u)} r^{1-\frac{\alpha(rz+u)}{\alpha(u)}} dz = \int \left|\sum_{j=1}^{m} \theta_j f_j(z)\right|^{\alpha(u)} dz.$$

Thus from (3.29)

$$\begin{aligned}
\lim_{r\to 0}\mathbb{E}\left(\exp i\sum_{j=1}^{m}\theta_j r^{-\frac{1}{\alpha(u)}}\int f_j(x)d(T^{\#}_{u,r}M)(x)\right) &= \exp\left(-\int\left|\sum_{j=1}^{m}\theta_j f_j(x)\right|^{\alpha(u)}dx\right)\\
&= \mathbb{E}\left(\exp i\sum_{j=1}^{m}\theta_j\int f_j(x)dM_{\alpha(u)}(x)\right).
\end{aligned}$$

We conclude by Lévy's continuity theorem (Theorem 1.2.39) that (3.26) holds, and then letting $f(x)=\mathbf{1}_A$, that (3.27) holds. □

Chapter 4

Multistable processes and localisability

4.1 Introduction

In this chapter, we study the localisability of the multistable processes. The first part of the chapter considers sufficient conditions for the general processes defined by stochastic integrals to be localisable. The second part contains a number of examples of localisable processes.

4.2 The localisability of multistable processes

In this section, we first state our main theorem on localisability. We then derive a number of lemmas concerning the spaces $\mathcal{F}_{a,b}$, and then prove the theorem.

4.2.1 Localisability theorem

Recall from Definition 2.2.2 that a stochastic process Y is localisable at u if Y has a unique non-trivial scaling limit at u, that is $Y = \{Y(t) : t \in \mathbb{R}\}$ is h-localisable at u with local form $Y'_u = \{Y'_u(t) : t \in \mathbb{R}\}$ if

$$\frac{Y(u+rt) - Y(u)}{r^h} \stackrel{\text{fdd}}{\to} Y'_u(t)$$

as $r \to 0$.

Definition 4.2.1. *A stochastic process* $\{Y(t), t \in \mathbb{R}\}$ *is* multistable *if for almost all u, Y is localisable at u with* Y'_u *an* α*-stable process for some* $\alpha = \alpha(u)$ *where* $0 < \alpha(u) \leq 2$.

In this chapter, we consider multistable processes defined by stochastic integrals, that is for $t \in \mathbb{R}$ and $f \in \mathcal{F}_{a,b}$

$$Y(t) = \int f(t,x)dM(x), \tag{4.1}$$

where M is $\alpha(x)$-multistable measure as in Definition 3.2.7. Thus by (3.13) and (4.1), for $(t_1,t_2,\dots,t_d) \in \mathbb{R}^d$ and $(\theta_1,\theta_2,\dots,\theta_d) \in \mathbb{R}^d$, the characteristic function of the random vector $(Y(t_1),Y(t_2),\dots,Y(t_d))$ is

$$\begin{aligned}\mathbb{E}\left(\exp i\sum_{j=1}^{d}\theta_j Y(t_j)\right) &= \mathbb{E}\left(\exp i\sum_{j=1}^{d}\int f(t_j,x)dM(x)\right)\\ &= \exp\left\{-\int\left|\sum_{j=1}^{d}\theta_j f(t_j,x)\right|^{\alpha(x)}dx\right\}.\end{aligned}$$

For a stochastic process Y, it is natural to ask what conditions are sufficient for Y to be localisable. The following theorem, which is a multistable analogue of Proposition 2.3.1, gives some conditions.

Theorem 4.2.2. *Let*

$$Y(t) = \int f(t,x)dM(x),$$

where $M \equiv M_{\alpha(x)}$ is $\alpha(x)$-multistable measure and $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2)$ is continuous. Assume that $f(t,.) \in \mathcal{F}_{a,b}$ for all t and

$$\lim_{r\to 0}\int\left|\frac{f(u+rt,u+rz)-f(u,u+rz)}{r^{h-1/\alpha(u+rz)}}-h(t,z)\right|^{a,b}dz = 0 \tag{4.2}$$

for a jointly measurable function $h(t,z)$ with $h(t,.) \in \mathcal{F}_{a,b}$ for all t. Then Y is h-localisable at u with local form

$$Y'_u = \left\{\int h(t,z)dM_{\alpha(u)}(z) : t \in \mathbb{R}\right\} \tag{4.3}$$

where $M_{\alpha(u)}$ is $\alpha(u)$-stable measure.

4.2.2 Lemmas

In order to prove Theorem 4.2.2, we need several lemmas.

Lemma 4.2.3. *If $\beta : \mathbb{R} \to \mathbb{R}$ is a measurable function with $0 < a \le \beta(z) \le b$ and $g(z) \in \mathcal{F}_{a,b}$, then*

$$\int |g(z)|^{\beta(z)}dz \le \int |g(z)|^{a,b}dz < \infty. \tag{4.4}$$

Proof. We have

$$\begin{aligned}
\int |g(z)|^{\beta(z)}dz &= \int_{|g(z)|\le 1} |g(z)|^{\beta(z)}dz + \int_{|g(z)|>1} |g(z)|^{\beta(z)}dz \\
&\le \int_{|g(z)|\le 1} |g(z)|^{a}dz + \int_{|g(z)|>1} |g(z)|^{b}dz \\
&= \int |g(z)|^{a,b}dz \\
&< \infty,
\end{aligned}$$

since $g(z) \in \mathcal{F}_{a,b}$ □

Lemma 4.2.4. *Let g, k be Lebesgue measurable functions. If* $0 < p \le 1$*, then*

$$\left| \int |g(z)|^p dz - \int |k(z)|^p dz \right| \le \int |g(z) - k(z)|^p dz, \tag{4.5}$$

and if $p > 1$

$$\left| \left(\int |g(z)|^p \right)^{1/p} - \left(\int |k(z)|^p \right)^{1/p} \right|^p \le \int |g(z) - k(z)|^p dz. \tag{4.6}$$

Proof. If $0 \le p \le 1$, since $(x_1 + x_2)^p \le x_1^p + x_2^p$ for all $x_1, x_2 \ge 0$, we have

$$\begin{aligned}
\int |g(z)|^p dz &= \int |g(z) - k(z) + k(z)|^p dz \\
&\le \int |g(z) - k(z)|^p dz + \int |k(z)|^p dz,
\end{aligned}$$

which implies

$$\int |g(z)|^p dz - \int |k(z)|^p dz \le \int |g(z) - k(z)|^p dz.$$

In the same way,

$$\int |k(z)|^p dz - \int |g(z)|^p dz \le \int |g(z) - k(z)|^p dz,$$

so

$$\left| \int |g(z)|^p dz - \int |k(z)|^p dz \right| \le \int |g(z) - k(z)|^p dz.$$

For $p \ge 1$, (4.6) immediately follows from the reverse triangle inequality for the norm $||f|| = |\int f(z)^p dz|^{1/p}$. □

Lemma 4.2.5. *If* $0 < c \le d$ *and* $\rho, \tau : \mathbb{R} \to [c, d]$ *are continuous, then*

$$\left| \int |g(z)|^{\rho(z)} dz - \int |g(z)|^{\tau(z)} dz \right| \le |c - d| \int |g(z)|^{c,d} |\log |g(x)|| dz, \tag{4.7}$$

for all measurable $g : \mathbb{R} \to \mathbb{R}$*.*

Proof. By the mean value theorem, for all $x>0$, $|x^c - x^d| \le |c-d||\log|x||\max\{x^c, x^d\}$,so

$$\begin{aligned}
& \left|\int |g(z)|^{\rho(z)}dz - \int |g(z)|^{\tau(z)}dz\right| \\
\le & \left|\int \max\{|g(z)|^c, |g(z)|^d\}dz - \int \min\{|g(z)|^c, |g(z)|^d\}dz\right| \\
= & \int \left||g(z)|^c - |g(z)|^d\right| dz \\
\le & \int |c-d||\log|g(z)|||g(z)|^{c,d}dz.
\end{aligned}$$

□

Lemma 4.2.6. *Let $0<a<b$ and $0<\delta<1/2(b-a)$. Then there is a number M_1 such that for all c,d with $a+\delta \le c \le d \le b-\delta$ and all $f \in \mathcal{F}_{a,b}$,*

$$\int |f(z)|^{c,d}|\log|f(z)||dz \le M_1 \int |f(z)|^{a,b}dz. \tag{4.8}$$

Proof. We can find M_1 such that for all x

$$\begin{aligned}
M_1 \max\{|x|^a, |x|^b\} & \ge \max\{|x|^{a+\delta}, |x|^{b-\delta}\}|\log|x|| \\
& \ge \max\{|x|^c, |x|^d\}|\log|x||,
\end{aligned}$$

if $a+\delta \le c \le d \le b-\delta$. Taking $x = f(z)$ and integrating we get inequality (4.8).

□

The following proposition will lead us to the proof of Theorem 4.2.2.

Proposition 4.2.7. *Let $[a,b] \subset (0,2]$, and $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ with $g(r,.) \in \mathcal{F}_{a,b}$ for all $r>0$. Let $k \in \mathcal{F}_{a,b}$ and let $\beta : \mathbb{R} \to [a,b]$ be continuous with $\beta(0) \in (a,b)$. If*

$$\lim_{r\to 0} \int |g(r,z) - k(z)|^{a,b}dz = 0, \tag{4.9}$$

then

$$\lim_{r\to 0} \int |g(r,z)|^{\beta(rz)}dz = \int |k(z)|^{\beta(0)}dz. \tag{4.10}$$

Proof. We may assume $k(z)$ is not almost everywhere 0, otherwise the result follows immediately from Lemma 4.2.3. Let ε be given. Choose M such that

$$\int_{|z|\ge M} |k(z)|^{a,b}dz < \varepsilon, \tag{4.11}$$

so in particular,

$$\int_{|z|\ge M} |k(z)|^{\beta(0)}dz < \varepsilon. \tag{4.12}$$

By (4.9) we may take r_1 such that for all $0 < r \leq r_1$

$$\int_{|z|\geq M} |g(r,z)-k(z)|^{a,b} dz < \varepsilon. \tag{4.13}$$

By Lemma 4.2.3 and Lemma 3.2.2, if $0 < r \leq r_1$,

$$\begin{aligned}
\int_{|z|\geq M} |g(r,z)|^{\beta(rz)} dz &\leq \int_{|z|\geq M} |g(r,z)|^{a,b} dz \\
&\leq c_1 \int_{|z|\geq M} |g(r,z)-k(z)|^{a,b} dz + c_1 \int_{|z|\geq M} |k(z)|^{a,b} dz \\
&\leq 2c_1\varepsilon, \qquad (4.14)
\end{aligned}$$

by (4.11) and (4.13) where c_1 depends only on a and b.

For the other part of the integrals,

$$\begin{aligned}
&\left| \int_{|z|<M} |g(r,z)|^{\beta(rz)} dz - \int_{|z|<M} |k(z)|^{\beta(0)} dz \right| \\
\leq &\left| \int_{|z|<M} |g(r,z)|^{\beta(rz)} dz - \int_{|z|<M} |g(r,z)|^{\beta(0)} dz \right| + \left| \int_{|z|<M} |g(r,z)|^{\beta(0)} dz - \int_{|z|<M} |k(z)|^{\beta(0)} dz \right| \\
\equiv &\ I_1(r) + I_2(r).
\end{aligned}$$

We show that $I_1(r), I_2(r) \to 0$ as $r \to 0$. Let $\varepsilon > 0$ be given. We first estimate I_1. Since $\beta(0) \in (a,b)$, we may find δ such that $a+\delta < \beta(0) < b-\delta$. Let M_1 be given by Lemma 4.2.6 for this a, b and δ. Let c_1 be given by Lemma 3.2.2 for this a and b. Choose c, d such that

$$a+\delta \leq c < \beta(0) < d \leq b-\delta$$

and

$$|c-d| c_1 M_1 \int |k(z)|^{a,b} dz \leq \varepsilon \tag{4.15}$$

Using the continuity of β, let $0 < r_2 \leq r_1$ be such that if $0 < r \leq r_2$ and $|z| \leq M$ then $\beta(rz) \in [c,d]$. By Lemma 4.2.5, Lemma 4.2.6 and Lemma 3.2.2 we get

$$\begin{aligned}
I_1(r) &\leq |c-d| \int |g(r,z)|^{c,d} |\log|g(r,z)|| dz \\
&\leq |c-d| M_1 \int |g(r,z)|^{a,b} dz \\
&= |c-d| M_1 \int |(g(r,z)-k(z)) + k(z)|^{a,b} dz \\
&\leq |c-d| c_1 M_1 \left(\int |(g(r,z)-k(z))|^{a,b} dz + \int |k(z)|^{a,b} dz \right) \\
&\leq \varepsilon + \varepsilon \\
&= 2\varepsilon, \qquad (4.16)
\end{aligned}$$

provided $0 < r \leq r_3$ for some $0 < r_3 \leq r_2$, using (4.9) and (4.15).

For I_2, by (4.9) and Lemma 4.2.3 with $\beta(z)$ replaced by $\beta(0)$, $\lim_{r\to 0}\int_{|z|<M}|g(r,z)-k(z)|^{\beta(0)}dz = 0$, so by Lemma 4.2.4

$$\int_{|z|<M}|g(r,z)|^{\beta(0)}dz \to \int_{|z|<M}|k(z)|^{\beta(0)}dz, \tag{4.17}$$

as $r \to 0$, so $I_2(r) < \varepsilon$ if $0 < r \leq r_4$ for some $0 < r_4 \leq r_3$.

Now if $0 < r \leq r_4$, then

$$\begin{aligned}
& \left|\int |g(r,z)|^{\beta(rz)}dz - \int |k(z)|^{\beta(0)}dz\right| \\
\leq\ & \left|\int_{|z|\geq M}|g(r,z)|^{\beta(rz)}dz - \int_{|z|\geq M}|k(z)|^{\beta(0)}dz\right| + \left|\int_{|z|<M}|g(r,z)|^{\beta(rz)}dz - \int_{|z|<M}|k(z)|^{\beta(0)}dz\right| \\
\leq\ & 2c_1\varepsilon + \varepsilon + 2\varepsilon + \varepsilon \\
=\ & (2c_1+4)\varepsilon,
\end{aligned}$$

using (4.12), (4.14), (4.16) and (4.17). Since ε is arbitrary, (4.9) implies (4.10). □

4.2.3 Proof of the localisability theorem

Now we come to the proof of Theorem 4.2.2.

Proof. Fix $u \in \mathbb{R}$. We consider the characteristic function of the finite-dimensional distributions of $r^{-h}(Y(u+rt)-Y(u))$. Let $\theta_j \in \mathbb{R}$ and $t_j \in \mathbb{R}$ for $j = 1,2,\ldots,m$. Then, using (3.13),

$$\begin{aligned}
& \mathbb{E}\left(\exp i\sum_{j=1}^{m}\theta_j r^{-h}(Y(u+rt_j)-Y(u))\right) && (4.18)\\
=\ & \mathbb{E}\left(\exp i\sum_{j=1}^{m}\theta_j r^{-h}\int(f(u+rt_j,x)-f(u,x))dM(x)\right) \\
=\ & \exp\left\{-\int\left|\sum_{j=1}^{m}\theta_j r^{-h}(f(u+rt_j,x)-f(u,x))\right|^{\alpha(x)}dx\right\} \\
=\ & \exp\left\{-\int\left|\sum_{j=1}^{m}\theta_j r^{-h}(f(u+rt_j,rz+u)-f(u,rz+u))\right|^{\alpha(rz+u)}rdz\right\} \\
=\ & \exp\left\{-\int\left|\sum_{j=1}^{m}\theta_j r^{-h+1/\alpha(rz+u)}(f(u+rt_j,rz+u)-f(u,rz+u))\right|^{\alpha(rz+u)}dz\right\},
\end{aligned}$$

(4.19)

after setting $x = rz + u$.

Defining

$$Z(t) = \int h(t,z) dM_{\alpha(u)}(z), \tag{4.20}$$

its finite-dimensional distributions are given by the characteristic function

$$\mathbb{E}\left(\exp i \sum_{j=1}^{m} \theta_j Z(t_j)\right) = \exp\left\{-\int \left|\sum_{j=1}^{m} \theta_j h(t_j,z)\right|^{\alpha(u)} dz\right\}. \tag{4.21}$$

We now use Proposition 4.2.7, taking

$$g(r,z) = \sum_{j=1}^{m} \theta_j \frac{f(u+rt_j, rz+u) - f(u, rz+u)}{r^{h-1/\alpha(rz+u)}}, \tag{4.22}$$

$$k(z) = \sum_{j=1}^{m} \theta_j h(t_j,z), \tag{4.23}$$

and

$$\beta(x) = \alpha(u+x). \tag{4.24}$$

Observe that

$$\int |g(r,z) - k(z)|^{a,b} dz \to 0, \tag{4.25}$$

as $r \to 0$, by (4.2) and Lemma 3.2.2. Thus by Proposition 4.2.7, since $\beta(0) = \alpha(u) \in (a,b)$ we have

$$\begin{aligned} &\int \left|\sum_{j=1}^{m} \theta_j r^{-h+1/\alpha(rz+u)} (f(u+rt_j, rz+u) - f(u, rz+u))\right|^{\alpha(rz+u)} dz \\ \to \quad &\int \left|\sum_{j=1}^{m} \theta_j h(t_j,z)\right|^{\alpha(u)} dz, \end{aligned}$$

as $r \to 0$.

Since the exponential function is continuous, (4.19), and hence (4.18), is pointwise convergent to (4.21) as $r \to 0$. Thus by the Lévy's Theorem (Theorem 1.2.39), $r^{-h}(Y(u+rt) - Y(u)) \stackrel{\text{fdd}}{\to} Z(t)$ as $r \to 0$, since (4.21) is a characteristic function, it is continuous at 0.

Thus Y is h-localisable with local form Y_u' given in (4.3). $\square$

4.3 Examples

We give a number of examples to illustrate Theorem 4.2.2. Some of these are considered in [7, Section 5] using the Poisson process definition of multistable processes.

Remember that in (2.13) we made the convention that

$$\mathbf{1}_{[u,v]} = -\mathbf{1}_{[v,u]}, \tag{4.26}$$

if $v < u$ which we will also use in the following examples.

Example 4.3.1. *Multistable Lévy motion.*
Let

$$Y(t) = \int \mathbf{1}_{[0,t]}(x) dM_{\alpha(x)}(x) = M_{\alpha(x)}[0,t], \tag{4.27}$$

where α*:* $\mathbb{R} \to (0,2)$ *is continuous. Let* $u \in \mathbb{R}$ *and suppose that as* $v \to u$*,*

$$|\alpha(u) - \alpha(v)| = o\left(\frac{1}{|\log|u-v||}\right). \tag{4.28}$$

Then Y *is* $1/\alpha(u)$*-localisable at* u *with local form*

$$Y'_u = \left\{ \int \mathbf{1}_{[0,t]}(z) dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \right\} = L_{\alpha(u)}, \tag{4.29}$$

where $L_{\alpha(u)}$ *is a* $\alpha(u)$*-stable Lévy motion.*

Proof. Let $f(t,x) = \mathbf{1}_{[0,t]}(x)$. For $t > 0$ and $u > 0$, we have

$$\begin{aligned} f(u+rt, u+rz) - f(u, u+rz) &= \mathbf{1}_{[0,u+rt]}(u+rz) - \mathbf{1}_{[0,u]}(u+rz) \\ &= \mathbf{1}_{[u,u+rt]}(u+rz) \\ &= \mathbf{1}_{[0,t]}(z). \end{aligned}$$

When $t > 0$, $u < 0$; $t < 0$, $u > 0$ and $t < 0$, $u < 0$, applying (4.26) the same argument gives $f(u+rt, u+rz) - f(u, u+rz) = \mathbf{1}_{[0,t]}(z)$.

Then taking $h(t,z) = \mathbf{1}_{[0,t]}(z)$ in Theorem 4.2.2,

$$\begin{aligned} & \overline{\lim}_{r\to 0} \int \left| \frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{1/\alpha(u) - 1/\alpha(u+rz)}} - \mathbf{1}_{[0,t]}(z) \right|^{a,b} dz \\ = \; & \overline{\lim}_{r\to 0} \int \left| \frac{\mathbf{1}_{[0,t]}(z)}{r^{1/\alpha(u) - 1/\alpha(u+rz)}} - \mathbf{1}_{[0,t]}(z) \right|^{a,b} dz \\ = \; & 0, \end{aligned}$$

since $r^{1/\alpha(u) - 1/\alpha(u+rz)} \to 1$ as $r \to 0$ uniformly for $z \in [0,t]$ by Lemma 3.5.2, so the integrand converges to 0 uniformly. By Theorem 4.2.2, we conclude that Y is $1/\alpha(u)$-localisable at u with local form (4.29). $\square$

Here is a weighted version of the previous example.

Example 4.3.2. *Weighted multistable Lévy motion.*

Let

$$Y(t) = \int w(x)\mathbf{1}_{[0,t]}(x)dM_{\alpha(x)}(x), \tag{4.30}$$

where α*:* $\mathbb{R} \to (0,2)$ *is continuous and* $w : \mathbb{R} \to \mathbb{R}$ *is continuous. Let* $u \in \mathbb{R}$, $\alpha(u) \in (a,b)$ *and suppose that as* $v \to u$*,*

$$|\alpha(u) - \alpha(v)| = o\left(\frac{1}{|\log|u-v||}\right). \tag{4.31}$$

Then Y *is* $1/\alpha(u)$*-localisable at* u *such that* $w(u) \neq 0$ *with local form*

$$Y'_u = \left\{\int w(u)\mathbf{1}_{[0,t]}(z)dM_{\alpha(u)}(z), \quad t \in \mathbb{R}\right\} = w(u)L_{\alpha(u)}, \tag{4.32}$$

where $L_{\alpha(u)}$ *is a* $\alpha(u)$*-stable Lévy motion*

Proof. Let $f(t,x) = w(x)\mathbf{1}_{[0,t]}(x)$. For $t > 0$ and $u > 0$, we have

$$\begin{aligned} f(u+rt,u+rz) - f(u,u+rz) &= w(u+rz)\mathbf{1}_{[0,u+rt]}(u+rz) - w(u+rz)\mathbf{1}_{[0,u]}(u+rz) \\ &= w(u+rz)\mathbf{1}_{[u,u+rt]}(u+rz) \\ &= w(u+rz)\mathbf{1}_{[0,t]}(z). \end{aligned}$$

With the convention (4.26), the other three cases give the same answer.

Then

$$\begin{aligned} &\overline{\lim}_{r\to 0} \int \left| \frac{f(u+rt,u+rz) - f(u,u+rz)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} - w(u)\mathbf{1}_{[0,t]}(z)\right|^{a,b} dz \\ = &\ \overline{\lim}_{r\to 0} \int \left| \frac{w(u+rz)\mathbf{1}_{[0,t]}(z)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} - w(u)\mathbf{1}_{[0,t]}(z)\right|^{a,b} dz. \end{aligned} \tag{4.33}$$

By Lemma 3.5.2 we have $r^{1/\alpha(u)-1/\alpha(u+rz)} \to 1$ as $r \to 0$ uniformly for $z \in [0,t]$, and since w is a continuous function, $w(u+rz) \to w(u)$ as $r \to 0$ uniformly for $z \in [0,t]$, so (4.33) equals 0. By Theorem 4.2.2, we conclude that Y is $1/\alpha(u)$-localisable at u with local form (4.32). □

Note that the multistable process (4.30) has independent increments $Y(t_2) - Y(t_1) = \int w(x)\mathbf{1}_{[t_1,t_2]}(x)dM_{\alpha(u)}(x)$.

We now consider multistable reverse Ornstein-Uhlenbeck motion. Notice that in the multistable cases, we get a curious restriction on the range of $\alpha(x)$.

Example 4.3.3. *Multistable reverse Ornstein-Uhlenbeck motion.*

Let

$$Y(t) = \int_t^\infty \exp(-\lambda(x-t))dM_{\alpha(x)}(x), \tag{4.34}$$

where α*:* $\mathbb{R} \to [a,b] \subseteq (1,2)$ *is continuous and* $1 < \sqrt{b} < a \le b < 2$*. Let* $u \in \mathbb{R}$ *and suppose that as* $v \to u$*,*

$$|\alpha(u) - \alpha(v)| = o\left(\frac{1}{|\log|u-v||}\right). \tag{4.35}$$

Then Y *is* $1/\alpha(u)$*-localisable at* u *with local form*

$$Y'_u = \left\{ \int -\mathbf{1}_{(0,t)}(z) dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \right\}. \tag{4.36}$$

Proof. Let $f(t,x) = \exp(-\lambda(x-t))\mathbf{1}_{[t,\infty)}(x)$. Then for $t > 0$,

$$\begin{aligned}
& f(u+rt, u+rz) - f(u, u+rz) \\
&= \exp(-\lambda(rz-rt))\mathbf{1}_{[u+rt,\infty)}(u+rz) - \exp(-\lambda rz)\mathbf{1}_{[u,\infty)}(u+rz) \\
&= \exp(-\lambda r(z-t))\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[0,\infty)}(z) \\
&= \exp(-\lambda r(z-t))\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[0,t)}(z) \\
&= \exp(-\lambda rz)(\exp(\lambda rt) - 1)\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[0,t)}(z).
\end{aligned} \tag{4.37}$$

With the convention (4.26), the same argument implies that (4.37) holds for $t \le 0$.

We take $h(t,z) = -\mathbf{1}_{[0,t)}(z)$ in (4.2). Consider

$$\begin{aligned}
& \int \left| \frac{f(u+rt,u+rz) - f(u,u+rz)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} + \mathbf{1}_{[0,t)}(z) \right|^{a,b} dz \\
= & \int \left| \frac{\exp(-\lambda rz)(\exp(\lambda rt)-1)\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[0,t)}(z)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} + \mathbf{1}_{[0,t)}(z) \right|^{a,b} dz \\
= & \int_{-|t|}^{|t|} \left| \frac{\exp(-\lambda rz)(\exp(\lambda rt)-1)\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[0,t)}(z)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} + \mathbf{1}_{[0,t)}(z) \right|^{a,b} dz \\
& + \int_{|t|}^{\infty} \left| \frac{\exp(-\lambda rz)(\exp(\lambda rt)-1)\mathbf{1}_{[t,\infty)}(z) - \exp(-\lambda rz)\mathbf{1}_{[0,t)}(z)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} + \mathbf{1}_{[0,t)}(z) \right|^{a,b} dz \\
\equiv & \ \phi_{r,1}(t) + \phi_{r,2}(t),
\end{aligned}$$

say.

For $\phi_{r,1}(t)$, by Lemma 3.5.2 we have

$$r^{1/\alpha(u)-1/\alpha(u+rz)} \to 1,$$

$$\exp(-\lambda rz)(\exp(\lambda rt) - 1) \to 0$$

and

$$\exp(-\lambda rz)\mathbf{1}_{[0,t)}(z) \to \mathbf{1}_{[0,t)}(z)$$

all uniformly as $r \to 0$ for $z \in [-|t|, |t|]$. Thus $\phi_{r,1}(t) \to 0$ as $r \to 0$.

For $\phi_{r,2}(t)$, we have

$$\begin{aligned}
\phi_{r,2}(t) &= \int_{|t|}^{\infty} \left| \frac{\exp(-\lambda rz)(\exp(\lambda rt)-1)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} \right|^{a,b} dz \\
&\leq \int_{|t|}^{\infty} \left| r^{-1/a+1/b} \exp(-\lambda rz)(\exp(\lambda rt)-1) \right|^{a,b} dz \\
&\leq r^{1-b/a} \int_{|t|}^{\infty} |\exp(-\lambda rz)(\exp(\lambda rt)-1)|^{a,b} dz \\
&\leq c_1 r^{1-b/a} \int_{|t|}^{\infty} |\exp(-\lambda rz)(\exp(\lambda rt)-1)|^{a} dz \\
&= c_1 r^{1-b/a} |\exp(\lambda rt)-1|^a \int_{|t|}^{\infty} |\exp(-\lambda arz)| dz \\
&\leq c_2 r^{1-b/a} (\lambda r|t|)^a \exp(-\lambda ar|t|)(\lambda ra)^{-1} \\
&\leq c_3 r^{a-b/a},
\end{aligned}$$

for fixed t, where c_1, c_2 and c_3 are independent of $r < 1$, say. Notice that $a - b/a > 0$ since $a > \sqrt{b}$, so

$$\phi_{r,2}(t) \to 0$$

as $r \to 0$.

Thus

$$\lim_{r\to 0} \int \left| \frac{f(u+rt,u+rz) - f(u,u+rz)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} + \mathbf{1}_{(0,t)}(z) \right|^{a,b} dz = 0.$$

By Theorem 4.2.2, we conclude that Y is $1/\alpha(u)$-localisable at u with local form (4.36). □

The next example is linear fractional multistable motion. Recall from (2.14) that asymmetric linear fractional α-stable motion is given by

$$L_{\alpha,h,b^+,b^-}(t) = \int_{-\infty}^{\infty} \rho_{\alpha,h}(b^+,b^-,t,x) dM_\alpha(x) \tag{4.38}$$

where $t \in \mathbb{R}$, $b^+, b^- \in \mathbb{R}$, and

$$\rho_{\alpha,h}(b^+,b^-,t,x) = b^+ \left((t-x)_+^{h-1/\alpha} - (-x)_+^{h-1/\alpha} \right) + b^- \left((t-x)_-^{h-1/\alpha} - (-x)_-^{h-1/\alpha} \right), \tag{4.39}$$

where M_α is a symmetric α-stable random measure $(0 < \alpha < 2)$. Note that by the convention after Definition 2.2.6, if $h - 1/\alpha = 0$ then

$$\rho_{\alpha,h}(b^+,b^-,t,x) = (b^+ - b^-)\mathbf{1}_{[0,t]}(x)$$

if $t \geq 0$, and

$$\rho_{\alpha,h}(b^+,b^-,t,x) = -(b^+ - b^-)\mathbf{1}_{[t,0]}(x)$$

if $t < 0$. Then (4.38) is an α-stable process, see Example 1.4.19 and (2.14). We introduce a multistable analogue of this.

Definition 4.3.4. *Let* $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2)$. *We define linear fractional* $\alpha(x)$*-multistable motion by*

$$L_{\alpha(x),h,b^+,b^-}(t) = \int_{-\infty}^{\infty} \rho_{\alpha(x),h}(b^+,b^-,t,x)dM_{\alpha(x)}(x) \tag{4.40}$$

where $t \in \mathbb{R}$, $b^+, b^- \in \mathbb{R}$, *and*

$$\rho_{\alpha(x),h}(b^+,b^-,t,x) = b^+\left((t-x)_+^{h-1/\alpha(x)} - (-x)_+^{h-1/\alpha(x)}\right) + b^-\left((t-x)_-^{h-1/\alpha(x)} - (-x)_-^{h-1/\alpha(x)}\right), \tag{4.41}$$

where $M_\alpha(x)$ *is an* $\alpha(x)$*-multistable random measure.*

To ensure that (4.40) is well-defined, we need to show that $\rho_{\alpha(x),h}(b^+,b^-,t,.) \in \mathcal{F}_{a,b}$. This may be done as in Example 1.4.19. However, we separate out the convergence at ∞ as a lemma.

Lemma 4.3.5. *Let* α*:* $\mathbb{R} \to [a,b] \subseteq (0,2)$ *be continuous,* $t \in \mathbb{R}$ *and* $0 < h < 1 + 1/b - 1/a$. *For any given* $t \in \mathbb{R}$ *and* $\varepsilon > 0$*, we can choose* M *sufficiently large such that*

$$\int_{|z|\geq M} \left|(t-z)_+^{h-1/\alpha(rz+u)} - (-z)_+^{h-1/\alpha(rz+u)}\right|^{a,b} dz \leq \varepsilon \tag{4.42}$$

for all $r > 0$.

Proof. Given t, for $M \geq \max\{|t|+1, 2|t|\}$, we get, for $z \leq -M$, using the mean value theorem,

$$\begin{aligned}(t-z)_+^{h-1/\alpha(rz+u)} - (-z)_+^{h-1/\alpha(rz+u)} &= (t-z)^{h-1/\alpha(rz+u)} - (-z)^{h-1/\alpha(rz+u)} \\ &= (h-1/\alpha(rz+u))t(w-z)^{h-1/\alpha(rz+u)-1}\end{aligned}$$

where $w \equiv w(z) \in (0,t)$ or $(t,0)$ depending on the sign of t. Hence there is a constant c_2, such that for all $z \leq -M$,

$$\begin{aligned}\left|(t-z)_+^{h-1/\alpha(rz+u)} - (-z)_+^{h-1/\alpha(rz+u)}\right| &\leq c_2|w-z|^{h-1/\alpha(rz+u)-1} \\ &\leq c_2|w-z|^{h-1/b-1} \\ &\leq c_3|t-z|^{h-1/b-1}.\end{aligned}$$

Thus

$$\begin{aligned}&\int_{|z|\geq M} \left|(t-z)_+^{h-1/\alpha(rz+u)} - (-z)_+^{h-1/\alpha(rz+u)}\right|^{a,b} dz \\ \leq\ & c_4 \int_{|z|\geq M} |t-z|^{(h-1/b-1)a} dz \\ <\ & \infty,\end{aligned}$$

since $(h-1/b-1)a < -1$. Thus this integral is convergent at ∞ uniformly in r, as required. $\square$

Corollary 4.3.6. *Let* $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2)$ *be continuous,* $t \in \mathbb{R}$ *and* $0 < h < 1 + 1/b - 1/a$*. Then* $\rho_{\alpha(x),h}(b^+, b^-, t, .) \in \mathcal{F}_{a,b}$*.*

Proof. By Lemma 4.3.5, we have $\int |\rho_{\alpha(x),h}(b^+, b^-, t, x)|^{a,b} dx$ convergent at ∞, and it is clearly convergent at $x = 0$ and $x = t$. Thus $\rho_{\alpha(x),h}(b^+, b^-, t, .) \in \mathcal{F}_{a,b}$. □

We now show that linear fractional multistable motion has linear stable motion as its local form. We consider the case when $b^+ = 1$ and $b^- = 0$, the argument is similar for other b^+ and b^-.

Proposition 4.3.7. *Linear fractional multistable motion.*

Let

$$\begin{aligned} Y(t) &= \int \left[(t-x)_+^{h-1/\alpha(x)} - (-x)_+^{h-1/\alpha(x)} \right] dM_{\alpha(x)}(x) \\ &= \int \rho_{\alpha(x),h}(1,0,t,x) dM_{\alpha(x)}(x) \\ &= L_{\alpha(x),h,1,0}(t), \end{aligned} \tag{4.43}$$

where α*:* $\mathbb{R} \to [a,b] \subseteq (0,2)$ *is continuous. If*

$$1/a - 1/b < h < 1 + 1/b - 1/a, \tag{4.44}$$

then Y *is h-localisable at each* $u \in \mathbb{R}$ *with local form*

$$\begin{aligned} Y'_u(t) &= \left\{ \int (t-z)_+^{h-1/\alpha(u)} - (-z)_+^{h-1/\alpha(u)} dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \right\} \\ &= L_{\alpha(u),h,1,0}(t). \end{aligned} \tag{4.45}$$

Proof. Let $f(t,x) = (t-x)_+^{h-1/\alpha(x)} - (-x)_+^{h-1/\alpha(x)}$, then

$$\begin{aligned} &\frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{h-1/\alpha(u+rz)}} \\ &= \frac{(rt-rz)_+^{h-1/\alpha(u+rz)} - (-rz)_+^{h-1/\alpha(u+rz)}}{r^{h-1/\alpha(u+rz)}} \\ &= \frac{r^{h-1/\alpha(u+rz)}((t-z)_+^{h-1/\alpha(u+rz)} - (-z)_+^{h-1/\alpha(u+rz)})}{r^{h-1/\alpha(u+rz)}} \\ &= (t-z)_+^{h-1/\alpha(u+rz)} - (-z)_+^{h-1/\alpha(u+rz)}. \end{aligned}$$

We apply Theorem 4.2.2 with $h(t,z) = (t-z)_+^{h-1/\alpha(u)} - (-z)_+^{h-1/\alpha(u)}$. Thus we need to consider

$$\begin{aligned} &\int \left| \frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{h-1/\alpha(u+rz)}} - ((t-z)_+^{h-1/\alpha(u)} - (-z)_+^{h-1/\alpha(u)}) \right|^{a,b} dz \\ = &\int \left| (t-z)_+^{h-1/\alpha(u+rz)} - (-z)_+^{h-1/\alpha(u+rz)} - ((t-z)_+^{h-1/\alpha(u)} - (-z)_+^{h-1/\alpha(u)}) \right|^{a,b} dz \\ \equiv &\int |\phi_r(z)|^{a,b} dz, \end{aligned} \tag{4.46}$$

say, given t and u. We consider the domain of the integration in several parts. Let $\varepsilon > 0$.

First using Lemma 3.2.2 and Lemma 4.3.5, we can choose M sufficiently large, so that

$$\int_{|z|>M} |\phi_r(z)|^{a,b} dz < \varepsilon, \tag{4.47}$$

for all $0 < r \leq r_0$, for some $r_0 > 0$.

If $h < 1/\alpha(u+rz)$,

$$(h - 1/\alpha(u+rz))a \geq (h - 1/\alpha(u+rz))b \geq (h - 1/a)b > -1.$$

Hence, for $|t-z| \leq 1$, $|(t-z)_+^{h-1/\alpha(u+rz)}|^{a,b} \leq \max\{1, (t-z)_+^{(h-1/a)b}\}$, so we may find δ such that $\int_{|z-t|<\delta} |(t-z)_+^{h-1/\alpha(u+rz)}|^{a,b} dz < \varepsilon$ for all r. Similarly $\int_{|z|<\delta} (-z)_+^{h-1/\alpha(u+rz)} < \varepsilon$ for all r and $\int_{|z-t|<\delta} (t-z)_+^{h-1/\alpha(u)} < \varepsilon$ and $\int_{|z|<\delta} (-z)_+^{h-1/\alpha(u)} < \varepsilon$ if δ is small enough. Thus we can take δ sufficiently small such that

$$\int_{(|z|<\delta)\cup(|z-t|<\delta)} |\phi_r(z)|^{a,b} dz < 4c_1\varepsilon, \tag{4.48}$$

using Lemma 3.2.2, for all $0 < r \leq r_0$.

Now let $A = \{z : \delta \leq |z| \leq M \text{ and } \delta \leq |z-t|\}$. Then, uniformly for $z \in A$,

$$\alpha(u+rz) \to \alpha(u)$$

and so

$$(-z)_+^{h-1/\alpha(u+rz)} \to (-z)_+^{h-1/\alpha(u)}$$

and

$$(t-z)_+^{h-1/\alpha(u+rz)} \to (t-z)_+^{h-1/\alpha(u)}.$$

Thus $\phi_r(z)$ is bounded and continuous on A with

$$\phi_r(z) \to 0$$

uniformly.

By the bounded convergence theorem,

$$\int_A |\phi_r(z)|^{a,b} dz \to 0 \tag{4.49}$$

as $r \to 0$.

Combining (4.47), (4.48) and (4.49), we conclude that $\int |\phi_r(z)|^{a,b} dz \to 0$ as $r \to 0$, so applying Theorem 4.2.2 to (4.46), we conclude that Y is h-localisable at u with local form (4.45). $\square$

We now generalise Proposition 2.3.3 to multistable processes.

Proposition 4.3.8. *Let $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2)$ be continuous, and let*

$$Y(t) = \int f(t,x) dM_{\alpha(x)}(x), \tag{4.50}$$

for $t \in \mathbb{R}$, where $f(t,.) \in \mathcal{F}_{a,b}$ for $t \in \mathbb{R}$. Let p and h satisfy $1/a - 1/b < h < p + 1/b - 1/a$ with $p < 1$. Let $u \in \mathbb{R}$ and suppose there are numbers c_0^+, c_0^- such that

$$\frac{f(t,x)}{(t-x)^{h-1/\alpha(x)}} \to c_0^+, \quad \text{as } t,x \to u \text{ with } t > x; \tag{4.51}$$

$$\frac{f(t,x)}{(x-t)^{h-1/\alpha(x)}} \to c_0^-, \quad \text{as } t,x \to u \text{ with } t < x. \tag{4.52}$$

Suppose also that there is $c > 0$, such that

$$|f(w,x) - f(v,x)| \le c|w-v|^p |x-v|^{h-1/\alpha(x)-p}, \tag{4.53}$$

for all $x, v.w$ such that $|x-v| \ge 2|w-v|$.

Then Y is h-localisable at $u \in \mathbb{R}$ with local form

$$\begin{aligned} Y'_u &= \left\{ \int \rho_{\alpha(u),h}(c_0^+, c_0^-, t, z) dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \right\} \\ &= L_{\alpha(u),h,c_0^+,c_0^-}. \end{aligned} \tag{4.54}$$

Proof. To use Theorem 4.2.2 we want to show (4.2) is satisfied with $h(t,z) = \rho_{\alpha(u),h}(c_0^+, c_0^-, t, z)$. This is clearly true when $t = 0$, so assume $t \neq 0$. As in [6, Proposition 3.1], we decompose

$$\begin{aligned} \frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{h-1/\alpha(u+rz)}} &= \frac{f(u+rt, u+rz)}{(r(t-z))^{h-1/\alpha(u+rz)}} |t-z|^{h-1/\alpha(u+rz)} \mathbf{1}_{\{t \ge z\}} \\ &\quad + \frac{f(u+rt, u+rz)}{(r(z-t))^{h-1/\alpha(u+rz)}} |t-z|^{h-1/\alpha(u+rz)} \mathbf{1}_{\{t < z\}} \\ &\quad - \frac{f(u, u+rz)}{(rz)^{h-1/\alpha(u+rz)}} |z|^{h-1/\alpha(u+rz)} \mathbf{1}_{\{z \ge 0\}} \\ &\quad - \frac{f(u, u+rz)}{(-rz)^{h-1/\alpha(u+rz)}} |z|^{h-1/\alpha(u+rz)} \mathbf{1}_{\{z < 0\}}. \end{aligned} \tag{4.55}$$

For fixed t and z, since α is continuous, as $r \to 0$,

$$\begin{aligned} \frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{h-1/\alpha(u+rz)}} &\to c_0^+ |t-z|^{h-1/\alpha(u)} \mathbf{1}_{\{t \ge z\}} \\ &\quad + c_0^- |t-z|^{h-1/\alpha(u)} \mathbf{1}_{\{t < z\}} \\ &\quad - c_0^+ |z|^{h-1/\alpha(u)} \mathbf{1}_{\{z \ge 0\}} \\ &\quad - c_0^- |z|^{h-1/\alpha(u)} \mathbf{1}_{\{z < 0\}} \\ &= \rho_{\alpha(u),h}(c_0^+, c_0^-, t, z), \end{aligned} \tag{4.56}$$

so

$$\left|\frac{f(u+rt,u+rz)-f(u,u+rz)}{r^{h-1/\alpha(u+rz)}}-\rho_{\alpha(u),h}(c_0^+,c_0^-,t,z)\right|^{a,b}\to 0$$

pointwise.

We now apply the dominated convergence theorem. For fixed $t\neq 0$ let $|z|\le 2|t|$, we consider the first term of (4.55).

$$\begin{aligned}\left|\frac{f(u+rt,u+rz)}{r(t-z)^{h-1/\alpha(u+rz)}}|t-z|^{h-1/\alpha(u+rz)}\mathbf{1}_{t\ge z}\right| &\le (|c_0|^+ +1)|t-z|^{h-1/\alpha(u+rz)}\mathbf{1}_{t\ge z}\\ &\le c_2|t-z|^{h-1/a}\mathbf{1}_{t\ge z}, \qquad (4.57)\end{aligned}$$

for all $0<r\le r_0$, for some c_2 and $r_0>0$. Note that $\int_{|z|\le 2|t|}|t-z|^{(h-1/a)b}<\infty$, since $h<1/a-1/b$.

Similarly the other three terms in (4.55) raised to the power b are dominated by integrable functions for $|z|\le 2|t|$ for all $0<r\le r_1$, say.

If $|z|\ge 2|t|$, then letting $w=u+rt, v=u$ and $x=u+rz$ in (4.53), we have $|x-v|\ge 2|w-v|$, so

$$\begin{aligned}\left|\frac{f(u+rt,u+rz)-f(u,u+rz)}{r^{h-1/\alpha(u+rz)}}\right|^{a,b} &\le c_3\left|\frac{|rt|^p|rz|^{h-1/\alpha(u+rz)-p}}{r^{h-1/\alpha(u+rz)}}\right|^{a,b}\\ &= c_3\,\big||t|^p\big|^{a,b}\left||z|^{h-1/\alpha(u+rz)-p}\right|^{a,b}\\ &\le c_4\left||z|^{h-1/\alpha(u+rz)-p}\right|^{a,b}\\ &\le c_5|z|^{(h-1/b-p)a}, \qquad (4.58)\end{aligned}$$

where c_3, c_4 and c_5 depend only on t. Since $(h-1/b-p)a<-1$, this has a finite integral on $|z|\ge 2|t|$. Thus by Lemma 3.2.2

$$\begin{aligned}\left|\frac{f(u+rt,u+rz)-f(u,u+rz)}{r^{h-1/\alpha(u+rz)}}\right|^{a,b} &\le c_1c_6|t-z|^{(h-1/a)b}\mathbf{1}_{|z|\le 2|t|}+c_1c_7|z|^{(h-1/a)b}\mathbf{1}_{|z|\le 2|t|}\\ &\quad +c_5|z|^{(h-1/b-p)a}\mathbf{1}_{|z|>2|t|},\end{aligned}$$

where c_6 and c_7 only depend on t, and this is integrable.

Since $\int|\rho_{\alpha(u),h}(c_0^+,c_0^-,t,z)|^{a,b}dz<\infty$, the dominated convergence theorem gives

$$\int\left|\frac{f(u+rt,u+rz)-f(u,u+rz)}{r^{h-1/\alpha(u+rz)}}-\rho_{\alpha(u),h}(c_0^+,c_0^-,t,z)\right|^{a,b}dz\to 0,$$

so the result follows from Theorem 4.2.2. □

Note that Proposition 2.3.3 is a special case of this, taking $\alpha(x)=\alpha$, $f(t,x)=g(t-x)$ and letting $v=x+u$ and $w=x+u+h$ in Proposition 4.3.8.

Chapter 5

The dimensions of graphs of α-stable Weierstrass functions

5.1 Introduction

The Weierstrass function $F : [0,1] \to \mathbb{R}$ is defined as

$$F(t) = \sum_{k=1}^{\infty} \lambda^{(D-2)k} \sin(\lambda^k t), \tag{5.1}$$

where $\lambda > 1$ and $D < 2$. It is a continuous function and when $1 < D < 2$ is nowhere differentiable. As we can see, the Weierstrass function is constructed from a series of rapidly increasing frequency terms and it is known to have a fractal graph. A natural question to ask is what is the dimension of the graph. The graph of the Weierstrass function has box-counting dimension D if $1 < D < 2$ and is believed to have Hausdoff dimension D but finding a lower bound of the Hausdorff dimension has proved difficult. See [5] and [11].

In this chapter, we introduce α-stable Weierstrass functions, that is a random version of the Weierstrass function, by including an α-stable random amplitude with each term. We prove that the box-counting and Hausdoff dimensions of the graphs of α-stable Weierstrass functions equal D almost surely under certain conditions. These random functions provide specific examples of the α-stable random processes discussed earlier.

5.2 Background to fractal dimensions

We first recall a series of standard definitions and lemmas relating to dimensions of a set. See [4, 5, 16] for more details.

We write $|U| \equiv \sup\{|x-y| : x, y \in A\}$ for the diameter of a set $A \subseteq \mathbb{R}^n$, and take the diameter of the empty set to be zero.

Definition 5.2.1. *For F a subset of $\mathbb{R}^n$, a $\delta-$*cover *$\{U_i\}_{i=1}^{\infty}$ of F is a finite or countable collection of sets such that*

$$F \subseteq \bigcup_{i=1}^{\infty} U_i$$

with $|U_i| \leq \delta$ for all i.

Definition 5.2.2. *Let $s \geq 0$. For any $\delta > 0$, we define the* δ-premeasure *$\mathcal{H}^s_\delta(F)$ of $F \subseteq \mathbb{R}^n$ as*

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \textit{ is a } \delta\textit{-cover of } F\right\}, \tag{5.2}$$

where the infimum is taken over all $\delta-$covers of F.

From this definition, we can see as δ decreases, the class of permissible covers of F in (5.2) is reduced. Therefore, the infimum $\mathcal{H}^s_\delta(F)$ increases as $\delta \to 0$, and so it approaches to a limit, which may be zero or infinity.

Definition 5.2.3. *For $s \geq 0$, we define $s-$*dimensional Hausdorff measure *of $F \subseteq \mathbb{R}^n$ to be*

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F). \tag{5.3}$$

There is a critical value at which $\mathcal{H}^s(F)$ jumps from zero to infinity as s increases. We define this value to be the *Hausdorff dimension* of F.

Definition 5.2.4. *The* Hausdorff dimension *of a set $F \subseteq \mathbb{R}^n$ is*

$$\dim_{\mathrm{H}} F = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}. \tag{5.4}$$

Thus

$$\mathcal{H}^s(F) = \begin{cases} \infty & \text{if } 0 \leq s < \dim_{\mathrm{H}} F \\ 0 & \text{if } s > \dim_{\mathrm{H}} F. \end{cases} \tag{5.5}$$

When $s = \dim_{\mathrm{H}} F$, $\mathcal{H}^s(F)$ may equal zero, infinity or be a positive finite real number.

For a nonempty bounded set $F \subseteq \mathbb{R}^n$, let $N_\delta(F)$ be the smallest number of sets with diameter at most δ which cover F.

Definition 5.2.5. *The* lower box-counting dimension *of F is*

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \tag{5.6}$$

and the upper box-counting dimension *of F is*

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}. \tag{5.7}$$

If the two values are equal, the common value is called the box-counting dimension *or* box dimension *of F and we write*

$$\dim_{\mathrm{B}} F = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}. \tag{5.8}$$

Lemma 5.2.6. *For any bounded, nonempty set $F \subseteq \mathbb{R}^n$*

$$0 \leq \dim_{\mathrm{H}}(F) \leq \underline{\dim}_{\mathrm{B}}(F) \leq \overline{\dim}_{\mathrm{B}}(F) \leq n$$

Proof. See [5]. □

The potential theoretic method is one of the main techniques used to find lower bounds for Hausdoff dimensions of sets. We say μ is a *mass distribution on F* if μ is a Borel measure with $0 < \mu(\mathbb{R}^n) < \infty$ and $\mu(\mathbb{R}^n \backslash F) = 0$.

Definition 5.2.7. *For $s \geq 0$, we define the $s-$*energy *$I_s(\mu)$ of μ by*

$$I_s(\mu) = \int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s}. \tag{5.9}$$

The following proposition relates the existence of mass distributions with finite energy to Hausdoff measures and dimensions.

Proposition 5.2.8. *Let F be a subset of $\mathbb{R}^n$.*

1) If there is a mass distribution μ on F with $I_s(\mu) < \infty$, then $\mathcal{H}^s(F) = \infty$ and $\dim_{\mathrm{H}} F \geq s$.

2) If F is a Borel set with $\mathcal{H}^s(F) > 0$, then there exists a mass distribution μ on F with $I_t(\mu) < \infty$ for all $0 < t < s$.

Proof. See [5]. □

5.3 Dimensions of Weierstrass-type graphs

The Weierstrass function

$$F(t) = \sum_{k=1}^{\infty} \lambda^{(D-2)k} \sin(\lambda^k t), \tag{5.10}$$

where $\lambda > 1$ and $1 < D < 2$ has a fractal graph, graph(F), and it is a long-standing problem to find its Hausdorff dimension, $\dim_{\mathrm{H}}(\mathrm{graph}F)$. It is known that

$$\dim_{\mathrm{H}}(\mathrm{graph}F) \leq \underline{\dim}_{\mathrm{B}}(\mathrm{graph}F) = \overline{\dim}_{\mathrm{B}}(\mathrm{graph}F) = D,$$

but finding a lower bound for the Hausdoff dimension has proved difficult. However, it is possible to introduce random versions of the Weierstrass functions whose Hausdoff dimension can be found almost surely.

The Weierstrass function F can be randomized in several ways including the following:

with a random phase added to each term

$$F_\theta(t) = \sum_{k=1}^{\infty} \lambda^{(D-2)k} \sin(\lambda^k(t+\theta_k)),$$

where $\theta_1, \theta_2, \ldots$ are independent uniformly distributed random variables on $[0, 2\pi]$;

with a random amplitude for each term

$$F_A(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-2)k} \sin(\lambda^k t),$$

where $A_1, A_2, \ldots$ are independent identically distributed random variables.

Hunt proved that the graph of F_θ has Hausdorff dimension D almost surely in [11]. Here we study the dimensions of the graph of F_A where the A_k are independent symmetric α-stable random variables. The function F_A provides a realisation of an α-stable process, since $(F_A(t_1), \ldots, F_A(t_m))$ will then be an α-stable random vector for all $t_1, \ldots, t_m$.

The next theorem is the main theorem of this chapter. We will use the rest of the chapter to prove it.

Theorem 5.3.1. *Let $0 < \alpha \leq 2$ and A_k be independent identically distributed symmetric α-stable random variables. Let $\lambda > 1$. Then the random function $F_A : [0,1] \to \mathbb{R}$*

$$F_A(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-2)k} \sin(\lambda^k t) \tag{5.11}$$

is continuous, and has

$$\dim_{\mathrm{B}}(\mathrm{graph} F_A) = \dim_{\mathrm{H}}(\mathrm{graph} F_A) = 1, \tag{5.12}$$

almost surely when $D < 1$, and

$$\dim_{\mathrm{B}}(\mathrm{graph} F_A) = \dim_{\mathrm{H}}(\mathrm{graph} F_A) = D \tag{5.13}$$

almost surely when $1 \leq D < 2$.

The lower bound of the Haudorff dimension comes from a potential-theoretic approach, and the upper bound of the box-counting dimension depends on a random Hölder condition.

5.3.1 Continuity and upper bounds for dimensions

We use the Weierstrass M-test to show that F_A is almost surely continuous.

Lemma 5.3.2. *Weierstrass M-test*

Let $f_k : [a,b] \to \mathbb{R}$ be a sequence of functions, and suppose that there exists a sequence $a_k \geq 0$ with the property that $|f_k(t)| \leq a_k$ for all $k \geq 1$ and $t \in [a,b]$. If $\sum_{k=1}^{\infty} a_k$ is convergent, then $\sum_{k=1}^{\infty} f_k(t)$ is uniformly convergent for $t \in [a,b]$.

Proof. See [3, Theorem 3-4c]. □

Lemma 5.3.3. *With probability one, the random function F_A defined in (5.11) is continuous.*

Proof. Consider

$$F_A(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-2)k} \sin(\lambda^k t) \tag{5.14}$$

where the A_k are independent identically distributed α-stable random variables with $\alpha > 1$ for all k.

Since A_k is an α-stable random variable, by Lemma 1.3.7,

$$\mathbb{P}(|A_k| \geq \beta) \leq c\beta^{-\alpha},$$

for all $\beta > 1$ with c independent of k and β. Take $0 < \delta < 2 - D$, then

$$\mathbb{P}(|A_k| \geq \lambda^{k\delta}) \leq c\lambda^{-k\delta\alpha}.$$

Since $\lambda^{-\delta\alpha} < 1$, we have $\sum_{k=1}^{\infty} \lambda^{-k\delta\alpha} < \infty$. By the Borel–Cantelli lemma, with probability 1, there is a random K_0 such that

$$\mathbb{P}(|A_k| \geq \lambda^{k\delta}) = 0,$$

for all $k \geq K_0$, that is

$$|A_k| \leq \lambda^{k\delta},$$

for $k \geq K_0$.

Let

$$f_k(t) = A_k \lambda^{(D-2)k} \sin(\lambda^k t)$$

and

$$C_k = |A_k| \lambda^{(D-2)k},$$

so $|f_k(t)| \leq C_k$ for all $t \in \mathbb{R}$. But

$$\begin{aligned}
\sum_{k=1}^{\infty} C_k &\leq \sum_{k=1}^{K_0} |A_k| \lambda^{(D-2)k} + \sum_{k=K_0+1}^{\infty} |A_k| \lambda^{(D-2)k} \\
&\leq \max_{1,\ldots,K_0} |A_k| \sum_{k=1}^{K_0} \lambda^{(D-2)k} + \sum_{k=K_0+1}^{\infty} (\lambda^{\delta})^k \lambda^{(D-2)k} \\
&= \max_{1,\ldots,K_0} |A_k| \sum_{k=1}^{K_0} \lambda^{(D-2)k} + \sum_{k=K_0+1}^{\infty} \lambda^{(D-2+\delta)k} \\
&< \infty,
\end{aligned}$$

almost surely.

By the Weierstrass M-test, we have that, almost surely, (5.14) is uniformly convergent on $\mathbb{R}$, so $F_A(t)$ is continuous. $\square$

The case of $D < 1$ is less interesting in that F_A is smooth.

Lemma 5.3.4. *When $D < 1$, F_A is differentiable almost surely with*

$$F_A'(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-1)k} \cos(\lambda^k t),$$

and

$$\dim_{\mathrm{H}}(\mathrm{graph} F_A) = \dim_{\mathrm{B}}(\mathrm{graph} F_A) = 1$$

almost surely.

Proof. Let

$$(F_A)_n(t) = \sum_{k=1}^{n} A_k \lambda^{(D-2)k} \sin(\lambda^k t),$$

then $(F_A)_n$ is differentiable with

$$(F_A)_n'(t) = \sum_{k=1}^{n} A_k \lambda^{(D-1)k} \cos(\lambda^k t),$$

and

$$\lim_{n\to\infty} (F_A)_n(t) = F_A(t).$$

Since A_k is an α-stable random variable, by Lemma 1.3.7,

$$\mathbb{P}(|A_k| \geq \beta) \leq c\beta^{-\alpha},$$

for all $\beta > 1$ with c independent of k and β. Take $0 < \delta < 1 - D$, then

$$\mathbb{P}(|A_k| \geq \lambda^{k\delta}) \leq c\lambda^{-k\delta\alpha}.$$

Since $\lambda^{-\delta\alpha} < 1$, we have $\sum_{k=1}^{\infty} \lambda^{-k\delta\alpha} < \infty$. By the Borel–Cantelli lemma, with probability 1, there is a random K_0 such that

$$\mathbb{P}(|A_k| \geq \lambda^{k\delta}) = 0,$$

for all $k \geq K_0$, that is

$$|A_k| \leq \lambda^{k\delta},$$

for $k \geq K_0$.

Let

$$f_k(t) = A_k \lambda^{(D-1)k} \cos(\lambda^k t)$$

and

$$C_k = |A_k|\lambda^{(D-1)k},$$

so $|f_k(t)| \le C_k$ for all $t \in \mathbb{R}$. But

$$\begin{aligned}
\sum_{k=1}^{\infty} C_k &\le \sum_{k=1}^{K_0} |A_k|\lambda^{(D-1)k} + \sum_{k=K_0+1}^{\infty} |A_k|\lambda^{(D-1)k} \\
&\le \max_{1,\dots,K_0} |A_k| \sum_{k=1}^{K_0} \lambda^{(D-1)k} + \sum_{k=K_0+1}^{\infty} (\lambda^{\delta})^k \lambda^{(D-1)k} \\
&= \max_{1,\dots,K_0} |A_k| \sum_{k=1}^{K_0} \lambda^{(D-1)k} + \sum_{k=K_0+1}^{\infty} \lambda^{(D-1+\delta)k} \\
&< \infty,
\end{aligned}$$

almost surely.

By the Weierstrass M-test, we get $\lim_{n\to\infty}(F_A)_n'(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-1)k}\cos(\lambda^k t)$ converging uniformly almost surely. Thus $F_A(t)$ is differentiable with

$$F_A'(t) = \sum_{k=1}^{\infty} A_k \lambda^{(D-1)k}\cos(\lambda^k t).$$

Finally, we note that the graph of a differentiable function has dimension 1. □

Next we show that F_A satisfies an almost-sure Hölder condition.

Lemma 5.3.5. *Hölder condition*

Let $1 \le D < 2$ *and let* $0 < \eta < 2 - D$. *Then with probability one,* $F_A : [0,1] \to \mathbb{R}$ *satisfies*

$$|F_A(t+h) - F_A(t)| \le B|h|^{\eta} \tag{5.15}$$

if $|h| \le H_0$ *for some random* $H_0 > 0$ *and* $B > 0$.

Proof. Since A_k are identically distributed α-stable random variables, by Lemma 1.3.7,

$$\mathbb{P}(|A_k| \ge \beta) \le c\beta^{-\alpha},$$

for all $\beta > 1$ where c is a constant. Take $\beta = (\lambda^{\xi})^k$ for $0 < \xi < 2 - D$, then

$$\mathbb{P}(|A_k| \ge \lambda^{\xi k}) \le c(\lambda^{\xi k})^{-\alpha}.$$

Since $\lambda^{-\alpha\xi} < 1$, we have $\sum_{k=1}^{\infty} \lambda^{-k\alpha\xi} < \infty$. By the Borel–Cantelli lemma, with probability one there exists K_0 such that,

$$\mathbb{P}(|A_k| \ge \lambda^{\xi k}) = 0,$$

for all $k \geq K_0$. This implies

$$|A_k| \leq \lambda^{\xi k},$$

for all $k \geq K_0$, so

$$\lambda^{(D-2)k}|A_k| \leq \lambda^{(D-2+\xi)k},$$

for all $k \geq K_0$.

For such K_0, suppose $0 < h \leq H_0 \equiv \lambda^{-K_0}$ and let k_1 be the integer such that $\lambda^{-(k_1+1)} < h \leq \lambda^{-k_1}$. Then

$$\begin{aligned}
|F_A(t+h) - F_A(t)| &= \left|\sum_{k=1}^{\infty} A_k \lambda^{(D-2)k}(\sin(\lambda^k(t+h)) - \sin(\lambda^k t))\right| \\
&\leq \sum_{k=1}^{\infty} \lambda^{(D-2)k}|\sin(\lambda^k(t+h)) - \sin(\lambda^k t)||A_k| \\
&\leq \sum_{k=1}^{k_1} \lambda^{(D-2)k}\lambda^k|h||A_k| + 2\sum_{k=k_1+1}^{\infty} \lambda^{(D-2)k}|A_k| \\
&\leq \max_{1,\dots,k_1}|A_k| \sum_{k=1}^{k_1} \lambda^{(D-1)k}|h| + 2\sum_{k=k_1+1}^{\infty} \lambda^{(D-2+\xi)k} \\
&\leq \max_{1,\dots,k_1}|A_k||h| \frac{\lambda^{(D-1)k_1}}{1-\lambda^{1-D}} + 2\frac{\lambda^{(k_1+1)(D-2+\xi)}}{1-\lambda^{(D-2+\xi)}} \\
&\leq C_1|h||h|^{1-D} + c_2|h|^{(2-D-\xi)} \\
&\leq C_3|h|^{(2-D-\xi)},
\end{aligned}$$

where C_1, C_3 are random constants, c_2 is a constant and this valid for $0 < h \leq \lambda^{-K_0} \equiv H_0$. By choosing ξ sufficiently small, we have for all $0 < \eta < 2-D$

$$|F_A(t+h) - F_A(t)| \leq B|h|^{\eta}$$

as required. □

A standard property on the dimension of graphs leads to our upper bound on the box-counting dimension.

Proposition 5.3.6. *Let $F_A : [0,1] \to \mathbb{R}$ be a continuous function. Suppose that for some $h_0 > 0$*

$$|F_A(t) - F_A(u)| \leq c|t-u|^{2-s} \qquad (0 \leq t,u \leq 1 \ \text{ and } \ |t-u| \leq h_0)$$

where $c > 0$ and $1 \leq s \leq 2$. Then $\overline{\dim}_{\mathrm{B}}(\mathrm{graph} F_A) \leq s$.

Proof. See [5, Corollary 11.2]. □

Corollary 5.3.7. *Let F_A be the function (5.11) with $\lambda > 1$ and $1 \leq D < 2$. Then with probability one,* $\dim_{\mathrm{H}}(\mathrm{graph} F) \leq \underline{\dim}_{\mathrm{B}}(\mathrm{graph} F) \leq \overline{\dim}_{\mathrm{B}}(\mathrm{graph} F_A) \leq D$.

Proof. Combining the almost-sure Hölder condition of Lemma 5.3.5 and Proposition 5.3.6, we get

$$\overline{\dim}_{\mathrm{B}}(\mathrm{graph}F_A) \leq D,$$

almost surely, with the other inequalities from Lemma 5.2.6. □

5.3.2 Lower bounds for dimensions

There are several difficulties in estimating the lower bound of the dimension of graph(F_A):

1) The α-stable random variables A_k have infinite variance when $\alpha < 2$ and infinite expectation when $\alpha \leq 1$.

2) When we estimate $F_A(t+h) - F_A(t)$, we get terms of the form

$$\sin(\lambda^k(t+h)) - \sin(\lambda^k t) = 2\sin\left(\lambda^k \frac{h}{2}\right)\cos\left(\lambda^k\left(t+\frac{h}{2}\right)\right)$$

and $\cos\left(\lambda^k\left(t+\frac{h}{2}\right)\right)$ can be very small, giving 'flat' parts of the graph which have to be removed.

Lemma 5.3.8. *For $k = 1, 2, \ldots$, let A_k be independent identically distributed symmetric α-stable random variables, and let a_k be real numbers such that $Z = \sum_{k=1}^{\infty} a_k A_k$ is convergent in distribution. Then for all $r > 0$,*

$$\mathbb{P}(|Z| \leq r) \leq \frac{cr}{|a_k|},$$

for all k, where $c > 0$ is independent of r.

Proof. Note that Z is symmetric α-stable with the distribution of $\left(\sum_{k=1}^{\infty}|a_k|^{\alpha}\right)^{1/\alpha} A_1$. Then

$$\begin{aligned}
\mathbb{P}(Z \leq r) &= \mathbb{P}\left(|A_1| \leq r\left(\sum_{k=1}^{\infty}|a_k|^{\alpha}\right)^{-1/\alpha}\right)\\
&\leq \mathbb{P}(|A_1| \leq r/|a_k|)\\
&\leq \frac{cr}{|a_k|},
\end{aligned}$$

for all k, since the α-stable random variable A_1 has bounded density. (See [17, Section 1.6]) □

We will need the following estimate.

Lemma 5.3.9. *Let $s > 1$ and Z be a random variable satisfying*

$$\mathbb{P}(|Z| \leq r) \leq ar$$

for all $r \geq 0$*, for some* $a > 0$*. Then*

$$\mathbb{E}\left((Z^2+h^2)^{-s/2}\right) \leq ca|h|^{1-s},$$

with c independent of h and a.

Proof. Let F be the distribution function of $|Z|$, so $F(z) \leq az$ for all $z \geq 0$. Then for an appropriate constant c independent of h and a, on integrating by parts

$$\begin{aligned}
\mathbb{E}\left((Z^2+h^2)^{-s/2}\right) &= \int_0^\infty (z^2+h^2)^{-s/2} dF(z) \\
&\leq \int_0^{|h|} |h|^{-s} dF(z) + \int_{|h|}^\infty z^{-s} dF(z) \\
&\leq |h|^{-s} F(|h|) + [z^{-s}F(z)]_{z=|h|}^\infty + s\int_{|h|}^\infty z^{-s-1} F(z) dz \\
&\leq 2a|h||h|^{-s} + s\int_{|h|}^\infty z^{-s-1} az dz \\
&\leq 2a|h|^{1-s} + as\int_{|h|}^\infty z^{-s} dz \\
&\leq ca|h|^{1-s}
\end{aligned}$$

as required. □

We will let $Z = F_A(t+h) - F_A(t)$, and bound the probability $\mathbb{P}(|Z| \leq h)$ for $h > 0$ in order to bound $\mathbb{E}\left((Z^2+h^2)^{-s/2}\right)$ using Lemma 5.3.9. Then we show that there is a random mass distribution on the graph F_A, which has finite s-energy almost surely. By the potential theoretic criterion (Lemma 5.2.8), F_A will have dimension at least s almost surely.

Proposition 5.3.10. *With the notation above, for* $0 < \alpha \leq 2$ *and* $1 \leq D < 2$

$$\dim_{\mathrm{H}}(\mathrm{graph} F_A) \geq D$$

almost surely.

Proof. Fix t and h for the time being and write

$$Z = F_A(t+h) - F_A(t) = \sum_{k=1}^\infty Z_k,$$

where $Z_k = A_k \lambda^{(D-2)k}(\sin(\lambda^k(t+h)) - \sin(\lambda^k t))$.

Since $A_1, A_2, \ldots$ are independent α-stable random variables with $\alpha > 1$, then by Lemma 5.3.8, for $r > 0$ and all k

$$\mathbb{P}(|Z| \leq r) \leq c_1 \frac{r}{\left|\lambda^{(D-2)k}(\sin(\lambda^k(t+h)) - \sin(\lambda^k t))\right|}, \tag{5.16}$$

where c_1 is a constant.

Let $\varepsilon > 0$ be a small number and choose a constant b such that $0 < b\lambda^{-\varepsilon} < \pi/4$. Given $0 < h \le b\lambda^{-(1+\varepsilon)}$, let k be the integer such that,

$$b\lambda^{-(k+1)(1+\varepsilon)} < h \le b\lambda^{-k(1+\varepsilon)}. \tag{5.17}$$

Using that $\sin x \ge (2\sqrt{2}x)/\pi$ if $0 < x < \pi/4$, we get,

$$\begin{aligned} w \equiv w(k,t,h) &\equiv \left|\lambda^{(D-2)k}(\sin(\lambda^k(t+h)) - \sin(\lambda^k t))\right| \\ &= \left|2\lambda^{(D-2)k}\sin\left(\lambda^k\frac{h}{2}\right)\cos\left(\lambda^k\left(t+\frac{h}{2}\right)\right)\right| \\ &\ge \left|2\lambda^{(D-2)k}\frac{\sqrt{2}\lambda^k h}{\pi}\cos\left(\lambda^k\left(t+\frac{h}{2}\right)\right)\right| \\ &\ge c_2\left|\lambda^{(D-1)k}h\cos\left(\lambda^k\left(t+\frac{h}{2}\right)\right)\right|, \end{aligned} \tag{5.18}$$

where c_2 does not depend on k, t and h.

Let S_k be the union of intervals

$$S_k = \bigcup_{n=-\infty}^{\infty}\frac{1}{\lambda^k}[n\pi - (\frac{\pi}{2} - b\lambda^{-k\varepsilon}), n\pi + (\frac{\pi}{2} - b\lambda^{-k\varepsilon})].$$

For $k \ge 1$ suppose $t \in S_k$ and $b\lambda^{-(k+1)(1+\varepsilon)} \le h \le b\lambda^{-k(1+\varepsilon)}$. Then for some integer n

$$n\pi - (\frac{\pi}{2} - b\lambda^{-k\varepsilon}) \le \lambda^k t \le n\pi + (\frac{\pi}{2} - b\lambda^{-k\varepsilon})$$

so

$$n\pi - (\frac{\pi}{2} - \frac{b}{2}\lambda^{-k\varepsilon}) \le \lambda^k(t+\frac{h}{2}) \le n\pi + (\frac{\pi}{2} - \frac{b}{2}\lambda^{-k\varepsilon}).$$

This implies

$$\left|\cos(\lambda^k(t+\frac{h}{2}))\right| \ge \left|\cos(\frac{\pi}{2} - \frac{b}{2}\lambda^{-k\varepsilon})\right| = \left|\sin(\frac{b}{2}\lambda^{-k\varepsilon})\right| \ge \frac{\sqrt{2}b}{\pi}\lambda^{-k\varepsilon} > 0.$$

For some integer k_0 to be defined later, if $t \in \bigcap_{k=k_0}^{\infty} S_k$ and $b\lambda^{-(k+1)(1+\varepsilon)} \le h < b\lambda^{-k(1+\varepsilon)}$ where $k \ge k_0$, by (5.18) and (5.17) we have

$$\begin{aligned} w(k,t,h) &\ge c_2\lambda^{(D-1)k}h\frac{\sqrt{2}b}{\pi}\lambda^{-k\varepsilon} \\ &= c_3(\lambda^{-k})^{(1-D+\varepsilon)}h \\ &\ge c_4 h^{(1-D+\varepsilon)/(1+\varepsilon)+1}, \end{aligned} \tag{5.19}$$

for some constants c_3 and c_4.

Let $t \in \bigcap_{k=k_0}^{\infty} S_k$ and $0 < h \le h_0 \equiv b\lambda^{-k_0(1+\varepsilon)}$, then by (5.16) and (5.19)

$$\begin{aligned} \mathbb{P}(|Z| \le r) &\le \frac{c_1 r}{c_4 h^{(1-D+\varepsilon)/(1+\varepsilon)+1}} \\ &= c_5 r h^{(D-1-\varepsilon)/(1+\varepsilon)-1}, \end{aligned} \tag{5.20}$$

where c_5 is independent of h and r. Thus if $t \in S \equiv (\bigcap_{k=k_0}^{\infty} S_k) \bigcap [0,1]$ and $0 < h \le h_0$, by Lemma 5.3.9, taking $a = c_5 h^{(D-1-\varepsilon)/(1+\varepsilon)-1}$, we have

$$\begin{aligned} \mathbb{E}\left((|F_A(t+h) - F_A(t)|^2 + h^2)^{-s/2}\right) &= \mathbb{E}\left((Z^2 + h^2)^{-s/2}\right) \\ &\le c c_5 h^{(D-1-\varepsilon)/(1+\varepsilon)-1} h^{1-s} \\ &= c_6 h^{(D-1-\varepsilon)/(1+\varepsilon)-s}, \end{aligned} \tag{5.21}$$

where c_6 is independent of h.

Consider the set $S = (\bigcap_{k=k_0}^{\infty} S_k) \bigcap [0,1]$. Then

$$\mathcal{L}(S) \ge 1 - c_7 \sum_{k=k_0}^{\infty} (\lambda^{-k})^{\varepsilon} = 1 - \frac{c_7 \lambda^{-k_0 \varepsilon}}{1 - \lambda^{-\varepsilon}} > \frac{1}{2},$$

where $c_7 > 0$ is a constant and $\mathcal{L}$ is Lebesgue measure, by choosing k_0 large enough.

We may lift Lebesgue measure restricted to S onto the graph of F_A to get a random mass distribution μ_{F_A} on F_A, that is

$$\mu_{F_A}(B) = \mathcal{L}\{t : t \in S \text{ and } (t, F_A(t)) \in B\}$$

for any Borel $B \subseteq \mathbb{R}^2$, with $1/2 \le \mu_{F_A}(\text{graph} F_A) = \mathcal{L}(S) \le 1$. By (5.21),

$$\begin{aligned} &\mathbb{E} \int_{t \in S} \int_{u \in [0,1], |t-u| \le h_0} (|F_A(t) - F_A(u)|^2 + |t-u|^2)^{-s/2} dt du \\ &= \mathbb{E} \int_{t \in S} \int_{t+h \in [0,1], |h| \le h_0} (|F_A(t+h) - F_A(t)|^2 + h^2)^{-s/2} dh dt \\ &= \int_{t \in S} \int_{t+h \in [0,1], |h| \le h_0} \mathbb{E}\left((|F_A(t+h) - F_A(t)|^2 + h^2)^{-s/2}\right) dh dt \\ &\le 2c_6 \int_{t \in S} \int_{t+h \in [0,1], |h| \le h_0} |h|^{-s+(D-1-\varepsilon)/(1+\varepsilon)} dh dt \\ &\le c_8 h_0^{1-s+(D-1-\varepsilon)/(1+\varepsilon)} \\ &< \infty \end{aligned} \tag{5.22}$$

for some $c_8 > 0$, if $1 - s + (D-1-\varepsilon)/(1+\varepsilon) > 0$.

For all $s < D$ we may choose ε small enough, so that $1 - s + (D-1-\varepsilon)/(1+\varepsilon) > 0$, so (5.22) is valid. Thus

$$\int_{t \in S} \int_{u \in [0,1], |t-u| \le h_0} (|F_A(t) - F_A(u)|^2 + |t-u|^2)^{-s/2} dt du < \infty$$

almost surely.

Since

$$\int_{t\in S}\int_{u\in S,|t-u|>h_0}(|F_A(t)-F_A(u)|^2+|t-u|^2)^{-s/2}dtdu<\infty$$

always, we get

$$\int\int|\mathbf{t}-\mathbf{u}|^{-s}d\mu_{F_A}(t)d\mu_{F_A}(u)=\int_{t\in S}\int_{u\in S}(|F_A(t)-F_A(u)|^2+|t-u|^2)^{-s/2}dtdu<\infty$$

almost surely.

Thus the finite measure μ_{F_A} is supported by graph(F_A) and has finite s-energy almost surely, so by the energy criterion (Lemma 5.2.8), we get

$$\dim_{\mathrm{H}}(\mathrm{graph}F_A)\geq s$$

for all $s<D$, so $\dim_{\mathrm{H}}(\mathrm{graph}F_A)\geq D$ almost surely. □

5.4 Proof of the theorem

We now get Theorem 5.3.1 by combining the earlier results.

Proposition 5.4.1. *With probability one,*

$$\dim_{\mathrm{H}}(\mathrm{graph}F_A)=\dim_{\mathrm{B}}(\mathrm{graph}F_A)=1$$

when $0<D<1$ *and* $0<\alpha\leq 2$.

Proof. By Lemma 5.3.3 and Lemma 5.3.4, when $D<1$, the random function F_A is a continuous and differentiable, so

$$\dim_{\mathrm{H}}(\mathrm{graph}F_A)=\dim_{\mathrm{B}}(\mathrm{graph}F_A)=1.$$

□

Proposition 5.4.2. *With probability one, when* $1\leq D<2$ *and* $0<\alpha\leq 2$*, we have*

$$\dim_{\mathrm{H}}(\mathrm{graph}F_A)=\dim_{\mathrm{B}}(\mathrm{graph}F_A)=D.$$

Proof. By Lemma 5.3.6 and Proposition 5.3.10,

$$D\leq\dim_{\mathrm{H}}(\mathrm{graph}F_A)\leq\underline{\dim}_{\mathrm{B}}(\mathrm{graph}F_A)\leq\overline{\dim}_{\mathrm{B}}(\mathrm{graph}F_A)\leq D,$$

almost surely, so

$$\dim_{\mathrm{H}}(\mathrm{graph}F_A)=\dim_{\mathrm{B}}(\mathrm{graph}F_A)=D$$

almost surely. □

Together these two propositions give Theorem 5.3.1.

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