

ESSENTIALISM, NOMINALISM, AND MODALITY : THE
MODAL THEORIES OF ROBERT KILWARDBY & JOHN
BURIDAN

Spencer C. Johnston

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews



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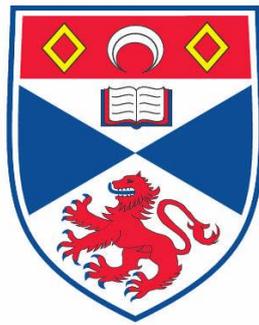
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Essentialism, Nominalism, and Modality:
The Modal Theories of
Robert Kilwardby & John Buridan

Spencer C. Johnston



This thesis is submitted in partial fulfilment for the degree of PhD
at the
University of St Andrews

July 9th 2015

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I, Spencer Johnston, hereby certify that this thesis, which is approximately 90,000 words in length, has been written by me, and that it is the record of work carried out by me, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September, 2010 and as a candidate for the degree of Doctor of Philosophy in September, 2010; the higher study for which this is a record was carried out in the University of St Andrews between 2010 and 2015.

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Abstract

In the last 30 years there has been growing interest in and a greater appreciation of the unique contributions that medieval authors have made to the history of logic. In this thesis, we compare and contrast the modal logics of Robert Kilwardby and John Buridan and explore how their two conceptions of modality relate to and differ from modern notions of modal logic. We develop formal reconstructions of both authors' logics, making use of a number of different formal techniques.

In the case of Robert Kilwardby we show that using his distinction between *per se* and *per accidens* modalities, he is able to provide a consistent interpretation of the apodictic fragment of Aristotle's modal syllogism and that, by generalising this distinction to hypothetical construction, he can develop an account of connexive logic.

In the case of John Buridan we show that his modal logic is a natural extension of the usual Kripke-style possible worlds semantics, and that this modal logic can be shown to be sound and complete relative to a proof-theoretic formalisation of Buridan's treatment of the expository syllogism.

[Logic] is good for this at least, (wherever it is understood,) to make people talk less; by showing them both what is, and what is not, to the point; and how extremely hard it is to prove anything.

John Wesley, *An Address To The Clergy*

Crime is common. Logic is rare. Therefore it is upon the logic rather than upon the crime that you should dwell. You have degraded what should have been a course of lectures into a series of tales.

Sir Arthur Conan Doyle, *The Adventure Of The Copper Beeches*

For consider your calling, brothers [and sisters]: not many of you were wise according to worldly standards, not many were powerful, not many were of noble birth. But God chose what is foolish in the world to shame the wise; God chose what is weak in the world to shame the strong; God chose what is low and despised in the world, even things that are not, to bring to nothing things that are, so that no human being might boast in the presence of God. And because of him you are in Christ Jesus, who became to us wisdom from God, righteousness, and sanctification and redemption. . .

1 Corinthians 1:26-30, English Standard Version Anglicised.

Acknowledgments

Reflecting back on the journey that has brought me to the completion of this thesis, I find myself remembering the many people who have taught, mentored, supported, and otherwise helped me along the way to completing this work. As this period of my studies draws to an end, I find myself thankful to God for the many friends I have made, the many things I have learnt, and the wonderful mentors who have taught me much about philosophy and beyond.

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At the University of Amsterdam, I had the privilege of working under Dr. Sara Uckelman, who was the first person to introduce me to the topic of medieval logic. Dr. Catarina Dutilh Novaes not only served as my M.Sc supervisor during my time at Amsterdam, but has since offered me helpful feedback and suggested that I look at Kilwardby's logic in addition to Buridan's. I am also grateful to Professor Michiel van Lambalgen both for teaching me about Kant's logic and for his suggestion that I be tested for dyslexia.

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Personally, my parents Gordon and Charlene Johnston, deserve immense thanks for their constant support, encouragement, and the wise guidance they have given me, be it in the form of listening to my musings or worrying about where I would end up going next. Both them and my mother-in-law Carol Pierce deserve considerable thanks for kindly reading through this thesis, checking for spelling and grammar. Likewise my wife Esther has supported me throughout this writing process. Not only have I subjected her to many discussions about logic (both medieval and modern), but she has read through multiple drafts of this thesis, correcting the grammar, humorously pointing out my typos, (my personal favourite of which is my proof of the generalised clam for the Lindenbaum lemma. I think the world would be a better place with maximally consistent sea food, however, this thesis will not lead to the development of such things.), and offering many typographical changes. Of course, any errors which remain are wholly my own.

One downside to having met such wonderful people and studied in so many unique environments is that it has taken me far from home and I was unable to be present when my grandmother, Genevieve Meyrick, passed away in 2012. It is to her memory and to my living grandmother, Hilda Johnston, that this thesis is dedicated.

Spencer Johnston
Stanford, California, 2014

Contents

| | |
|--|-----------|
| Bibliography | 1 |
| 1 Introduction | 5 |
| 1.1 Motivation | 5 |
| 2 Kilwardby's Commentary on the Prior Analytics | 11 |
| 2.1 Historical Background | 11 |
| 2.2 Assertoric Syllogistics | 13 |
| 2.2.1 Truth and Signification | 14 |
| 2.3 Consequence & Modality | 18 |
| 2.3.1 Consequence | 18 |
| 2.3.2 Modality | 22 |
| 2.3.3 Syllogisms | 26 |
| 2.3.4 The Syllogism Emerges | 29 |
| 2.3.5 Modal Syllogisms | 31 |
| 2.4 Conclusion | 34 |
| 3 Reconstructing Kilwardby's Logic | 35 |
| 3.1 Introduction | 35 |
| 3.2 Essences, Modality and the Question of Reduction | 36 |
| 3.3 Essences, Syllogisms, and Previous Work | 38 |
| 3.3.1 Reconstructing the Modal Syllogistic | 38 |
| 3.4 Semantics for Kilwardby | 40 |
| 3.4.1 Semantic Reconstructions | 40 |
| 3.4.2 Adequacy of Our Model | 48 |
| 3.5 Connexive Implication and Natural Consequence | 56 |
| 3.5.1 Connexive Logic, Substantivity and Interpretation | 61 |
| 3.6 Conclusions and Future Work | 63 |
| 4 The Modal Syllogism in John Buridan | 65 |
| 4.1 Introduction | 65 |
| 4.2 The Structure of the <i>Treatise on Consequences</i> | 65 |
| 4.3 Buridan's Theory of Consequence | 67 |
| 4.3.1 Medieval Theories of Language: Buridan's Theory of Signification | 71 |
| 4.3.2 Medieval Theories of Language: Buridan's Theory of Supposition | 72 |
| 4.4 Division of Consequence | 75 |
| 4.5 Buridan's Theory of Modality | 84 |

| | | |
|----------|---|------------|
| 5 | A Formal Reconstruction of Buridan’s Modal Logic | 101 |
| 5.1 | Introduction | 101 |
| 5.1.1 | Syllogisms | 102 |
| 5.2 | Formal Theory | 106 |
| 5.2.1 | Preliminaries and Semantics | 106 |
| 5.2.2 | Single Premise Inferences | 109 |
| 5.3 | Buridan’s Syllogistic Theory | 111 |
| 5.4 | The Expository Syllogism | 115 |
| 5.5 | Formalisation | 121 |
| 5.5.1 | Semantics | 121 |
| 5.5.2 | Proof Theory | 122 |
| 5.6 | Soundness and Completeness | 135 |
| 5.6.1 | Completeness | 137 |
| 5.7 | Conclusion | 146 |
| 6 | Comparisons and Philosophical Implications | 147 |
| 6.1 | Introduction | 147 |
| 6.2 | Situation Figures within Modal History | 148 |
| 6.2.1 | Modal Logic in the 12 th & 13 th centuries | 149 |
| 6.2.2 | Modal Logic in the 14 th century | 150 |
| 6.3 | <i>Per Se</i> per Buridan & Kilwardby | 153 |
| 6.3.1 | Buridan’s Essentialist Nominalism | 154 |
| 6.4 | Expositio, Modal and Essential | 158 |
| 6.5 | Buridan & Mere <i>Possibilia</i> | 161 |
| 6.5.1 | Necessitism and Contingentism: The Case of <i>Modal Logic as Meta-physics</i> | 163 |
| 6.5.2 | Quantification in Buridan | 166 |
| 6.5.3 | Predicative and Attributive Readings | 167 |
| 6.5.4 | Barcan & Converse Barcan | 168 |
| 6.6 | The Role of Kripke Semantics in the History of Logic | 171 |
| 6.7 | Buridan, Modality and Kripke Semantics | 172 |
| 6.8 | Kilwardby and Modern Modal Logic | 178 |
| 6.9 | Conclusion | 179 |
| 7 | Conclusions & Further Work | 181 |
| 7.1 | Conclusions | 181 |
| 7.2 | Some Historical Questions | 182 |
| 7.3 | Some Formal and Technical Questions | 183 |
| 8 | Appendix One: Latin References | 185 |
| 8.1 | Chapter Four | 185 |
| 8.2 | Chapter Five | 188 |

| | |
|---|------------|
| 9 Appendix Two: Kilwardby’s Logic | 191 |
| 9.0.1 Single Premise Inferences | 192 |
| 9.1 Inferences Between <i>Per Se</i> , and Assertoric Propositions. | 196 |
| 9.2 Kilwardby’s Validities | 196 |
| 9.2.1 Second and Third Figure Syllogisms | 203 |
| 10 Appendix Three: Buridan’s Modal Syllogism | 207 |
| 11 Appendix Four: Soundness | 237 |
| 11.1 Natural Deduction Rules | 241 |
| 11.1.1 Inferential Rule Structure | 244 |
| 11.2 Soundness | 249 |
| 11.2.1 Interchange Rules | 249 |
| 11.2.2 Introduction Rules | 254 |
| 11.2.3 Elimination Rules | 262 |

1 Introduction

1.1 Motivation

The overarching goal of this dissertation is twofold. On the one hand, this work seeks to better situate our understanding of two very different medieval approaches to the treatment of modality. One tradition places the theory of modality within the study of Aristotle's *Prior* and *Posterior Analytics*. In this tradition, modality is grounded within a broader theory of essences and natures. The other tradition treats modalities as a primitive notion, where the logic is anchored and developed within a systematic exposition of the theory of supposition and the theory of consequence. In both cases, these approaches to modality have broader implications for the logical theories that emerge from these authors. This thesis will look at two authors, one from each tradition. The works of Robert Kilwardby fall within the first tradition and we will discuss his work in the second and third chapters of this thesis. The writings of John Buridan are an example of the second tradition and we will discuss his writings in the fourth and fifth chapters. In both cases, we will situate them within their historical context, offering sustained textual discussions of the theories of the modal syllogism developed by these authors.

The other goal of this dissertation is to explore the connection between these medieval conceptions of modality and modern writings on logic. As has been remarked many times in the history of logic, the medieval period (in particular the 13th and 14th century) was a time of fruitful, technical, thorough, and deep study on the topic of logic. While not carried out within a symbolic language, medieval logicians developed a technical vocabulary and rigorous theories designed to handle inferences within the *Lingua Franca* of the day, Latin. Our aim in this thesis is to contribute to the literature by further exploring the relationship between the modal logics of Kilwardby and Buridan and modern logical systems by illustrating the connections that exist between modern and medieval logic. Our hope is that, historically, we may gain a deeper insight into the medieval theories, and that as modern logicians we may find inspiration for the study of new and interesting logical systems.

It is these two (sometimes complementary, sometimes conflicting) desires that have guided and shaped the present work. Before moving into an outline of the work itself,

we need to introduce the historical figures on whose works we will be focusing.

Biography & Historical Situation

The two figures treated in this work are in many ways very different. One was a Picard (French) secular cleric, who never advanced beyond his arts degree, but whose influence left a lasting mark on medieval thought. The other, an English Dominican, after having taught as an arts master, went on to a successful ecclesiastical career serving as both Archbishop of Canterbury and as a Cardinal. It is to these two figures that we now turn.

Robert Kilwardby

The exact date and location of Robert Kilwardby's birth are unknown, however, given the dates that we do know about him, it seems unlikely that he was born much earlier than 1200.¹ Likewise, the location of Kilwardby's birth is unclear. Kilwardby is clearly English, however none of our historical sources further elaborate on where he is from. The name 'Kilwardby' is a reference to his place of birth, though it appears with many variations in spelling among the various sources. According to Sommer-Seckendorff, by the 15th century, Kilwardby named at least two English villages, one in Leicestershire and the other in Yorkshire, and it seems likely that Kilwardby was from one of these places.[56, p.1]

The early dates of Kilwardby's life are also unclear and rest on some conjecture. We know for certain that Kilwardby was elected as provincial of the English Dominicans in 1261. The exact dates and times of his education are unclear. We know that he completed his Master of Arts degree at the University of Paris and later completed his theology degree at Oxford, however, an exact chronology of this time is not well-known or clear. One theory holds that Kilwardby completed his arts degree in the early 1220's, wrote the majority of his works on Aristotle² and later took the Dominican habit some time between 1240 and 1245. It is during this time that Kilwardby also completed his theological studies at Oxford and stayed on to lecture there. With Kilwardby's election in 1261, his career as an academic came to an end, while his life as an ecclesiastical figure began.³

While Kilwardby's time as provincial of the Dominicans would have placed him in contact with various important political and ecclesiastical figures of his day, it is unclear

1. One source gives his date of birth as 1204, but does not provide reference to the source. See[56, p.47 fn. 7]

2. Including the ones that we will be interested in throughout this dissertation.

3. For further exposition of this position and references see [56] and the references therein.

what views Kilwardby held on many of the political issues of his day. For example, little is known about his views on the Crusade that was initiated during his time as provincial nor about how he sought to rectify the financial issues that faced his predecessor.⁴

Kilwardby went on to be elected Archbishop of Canterbury in 1272. His time as Archbishop was marked by two noteworthy events, the coronation of Edward I on August 19th 1276 and Kilwardby's issuing of a number of condemnations in 1277. A number of important questions in the history of science revolve around Kilwardby's role in issuing these condemnations and how they related to the condemnations issued by Bishop Stephen Tempier in Paris earlier that year. While the reasons why Kilwardby issued these condemnations do not have any bearing on this work, their content does, as some of Kilwardby's condemnations are directly related to logic.⁵ Kilwardby was promoted to Cardinal in 1279 and died later that year.

Our interest in Kilwardby is derived primarily from his writings on logic, in particular, his commentary on Aristotle's *Prior Analytics*, written during his time as an arts master. This will be more fully discussed, along with the reception of the *Prior Analytics* in the Latin West, in the next chapter.

John Buridan

John Buridan was born sometime around 1300 in the Diocese of Arras in Picardy. We know that his early studies were conducted at the Collège Lemoine in Paris and that he was awarded a stipend for needy students. He went on to study for his Masters degree at the University of Paris and was licensed to teach in the mid 1320's. He remained at the University of Paris for the rest of his career and did not go on to read for any higher degrees. He also remained a secular cleric, choosing not to become a member of any of the monastic orders.

Buridan was awarded a number of stipends and benefices over the course of his career. Perhaps most noteworthy was the stipend he was awarded in 1348, where the committee was composed of a theologian, two members of the law faculty, a proctor from each of the four nations at the University of Paris, and the rector of the University.

As we have already remarked, Buridan remained an arts master his entire career. This is unusual, as the usual course of studies for talented students (which Buridan clearly was) was to first read for the arts degree, and then go on to read for a higher degree, often in law or theology. Buridan's reasons for not following this path are unknown, however there have been a number of hypotheses put forward, none of which seem particularly

4. For further discussion of this see [56, pp.41-47].

5. See[63] for further discussions on the relationship between the condemnations and logic.

1 Introduction

conclusive.⁶ Because of Buridan's career path, all of his writings focus on the Arts corpus of his day. His logical works (on which we will focus our attention in this work) are a mixture of commentaries, textbooks, questions, and specific treatises. Buridan's writings covered the entire *Organon*, *Metaphysics* and *Physics*, among other works.

Buridan's decision to remain a secular cleric is also noteworthy. By this point in the Middle Ages, the various mendicant orders had developed distinct and influential positions that had shaped the study of theology at the University of Paris.⁷ By standing outside of these orders, Buridan was free to develop his thoughts and ideas in a way unhindered by commitments to particular figures within a particular order. For example, while Buridan embraces parts of William of Ockham's nominalism, he also was free to criticise Ockham on a number of occasions and may even have contributed to the suspension of the teaching of particular 'Ockhamist' doctrines during his time as Rector at the University of Paris.

Beyond the information sketched above, we know very little about Buridan's life and about the influences that led to the development of his logical theorising. He is believed to have died around 1360. More information about Buridan's life can be found in [69] and the references contained therein.

Buridan's writings covered the traditional arts corpus. His magnum opus is his *Summulae de Dialectica*, a massive textbook based on Peter of Spain's *Summulae* which has been heavily redacted. Buridan provides two main accounts of the modal syllogism in his writings. The first occurs in the final book of the *Treatise on Consequences*. The second occurs in *Book Five* of his *Summulae De Dialectica*.⁸ The two accounts differ on a few points, some of which are relevant for our purposes in this thesis. In particular, the *Summulae* provides counterexamples to QXM Disamis and QXM Baroco.⁹ The semantics that we will present agree with the *Summulae* that these are invalid. The countermodels can be found in the appendix on pages 221 and 222 respectively. Our focal point of discussion will be on the *Treatise on Consequences* as it contains a more compact treatment of the material and because it is situated within a work which discusses Buridan's logic more generally. When it is helpful to understand Buridan's thoughts more generally (for example, his discussion of *per se* terms, see page 155 and onward), we will draw on his material in the *Summulae* as well.

6. This point is discussed or alluded to in a few of the papers in [58].

7. See [69, p.xii]

8. English translations of the two works can be found in [51] and [5] respectively. There is an older translation of the *Treatise*, [25], which will not be referenced. The English translations provided here are due to a recently published translation of the text by Stephen Read. Most of the other translations have been taken from the works that they are cited in.

9. We discuss this terminology on page 106.

The Structure of the Thesis

Having introduced the two main historical figures on which this work is focused and offered some comments explaining the motivation for this thesis, we will now discuss its structure. The thesis is organised into seven chapters which are interspersed with historical, formal and philosophical content. The first and seventh chapters are the Introduction and Conclusion respectively. Chapter Two provides a historical analysis of Kilwardby's commentary on the *Prior Analytics*, and situates his remarks on modality within the broader Medieval Aristotelian tradition. Particular attention will be paid to the reception of the *Prior Analytics* in the Latin West, Kilwardby's analysis of modal terms, his discussion of the modal syllogistic, and his more general theory of inference. This will in turn require that we explicate the distinction between *per se* and *per accidens* modality, and the distinction between *natural* and *accidental* consequences. In Chapter Three we will develop a formal model for Kilwardby's theory of modality. The main aim in this chapter will be to incorporate the essentialist theory that Kilwardby is using within a general modal framework of concepts. The resulting theory will be used to provide a formal reconstruction of Kilwardby's analysis of Aristotle and to develop a formal treatment of Kilwardby's theory of *natural* and *accidental* consequences. In Chapter Four we will discuss Buridan's theory of the modal syllogism as presented in his *Treatise on Consequences*. We will provide an exposition of Buridan's theory of consequence and the results of his analysis. Chapter Five provides a formalisation of Buridan's results within the framework of modal logic. Here we will develop two formal theories, one based on the theory of supposition presented in Chapter Four and another based on Buridan's theory of the expository syllogism, which we will develop in Chapter Five. With this in place we prove that Buridan's treatment of the divided fragment of his modal syllogistic is 1) consistent, and 2) prove that it is complete, relative to our formalisation of Buridan's theory of the expository syllogism. Finally, in Chapter Six, we will bring together the results of chapters Two through Five and conclude with some philosophical reflections on the theories of modality employed by both authors. Our main aims will be to compare Kilwardby's and Buridan's analysis of modal logic. We will pay particular attention to the role that essences play in Buridan's theorising, and to the role of the expository syllogism in Kilwardby. We will then explore some of the ontological questions that Buridan's analysis of modal logic raise. Finally, we will raise a methodological question which will help to draw together the main results of this thesis. Here we will discuss the use of modern formal logic as a tool with which to study such historical theories.

2 Kilwardby's Commentary on the Prior Analytics

The goal of this chapter is to outline Kilwardby's theory of the modal syllogism and situate it within its larger historical context. To that end, the structure of this chapter will be tailored to show how the various elements of Kilwardby's logic come together to give rise to the modal syllogism. The chapter will be divided into three sections. The first section will begin with some historical background and set the stage for the reception of the *Prior Analytics*, discussing the form of the text that Kilwardby was likely commenting from. The second section will discuss Kilwardby's treatment of the assertoric syllogism. In the final section, we will look at the key parts of Kilwardby's theory of modal propositions.

There are a number of important connections that we will need to focus on in this chapter. In our discussion of the assertoric syllogism, we will focus on Kilwardby's theory of signification and how this can be used to give truth conditions for assertoric and tensed categorical propositions. What we will see here is that Kilwardby distinguishes between two ways of giving truth conditions to assertoric propositions, one based on conceptual connections between terms and the other based on classes of objects that fall under terms. After that, we will turn to Kilwardby's analysis of consequence and discuss when various propositions entail other propositions. Here particular attention will be paid to Kilwardby's distinction between natural and accidental consequences, which will be important in our formal reconstruction of Kilwardby's theory. The section will then conclude with Kilwardby's definition of a syllogism. Within the modal portion of this chapter, our main focus will be on Kilwardby's conception of necessity. We will not discuss Kilwardby's analysis of propositions of contingency. Our main focus will be on Kilwardby's distinction between *per se* and *per accidens* modalities. Our main goal will be to clarify what they are, how they are to be understood, and the inferential properties that they possess. We will then conclude by bringing all of these components together and explore how Kilwardby conceives of the modal syllogism.

2.1 Historical Background

Kilwardby's writings covered a wide range of topics including grammatical, philosophical, biblical, and theological writings. In this chapter we will concern ourselves mostly with Kilwardby's commentary on Aristotle's *Prior Analytics*. However we will draw from papers dealing with Kilwardby's comments on the *Logica Vetus* as well. The commentary on the *Prior Analytics* is an exposition of Aristotle's text together with Kilwardby's analysis, and, in some places, attempted defence of the various views Aristotle puts

2 Kilwardby's Commentary on the *Prior Analytics*

forward. While Kilwardby attempts to portray Aristotle in as positive a light as possible, he also strives for an account of logic that is both systematic and consistent.

The reception of the *Prior Analytics* in the Latin West is an interesting story and a brief summary of it helps illuminate the historical importance of Kilwardby's commentary. The *Prior Analytics* is one of six Aristotelian texts that made up the *Organon*.¹ These texts were often 'introduced' by Porphyry's *Isagoge*. [7, p.1] After the collapse of the Roman Empire, only some of these texts were transmitted and closely studied in the Latin West. The *Prior Analytics* was one of the texts that was not transmitted directly. Before the 10th century, those working in the Latin West had to make do with second-hand accounts. It was not until c.980 that a copy of Boethius' *De Syllogismo Categoricalis* was recovered [7, p.2] and it was not until the writings of Abelard that we find explicit references to the *Prior Analytics* again.

However, during the 11th century the influence of the *Prior Analytics* was slight. It was not until the 12th century that the *Prior Analytics* begins to exert some influence on medieval thinkers. For example, Peter Abelard seems to have made some study of the text and his pupil Otto Freisinger is said to have brought a copy of the text to Germany. [12, p.97] The reception of the *Prior Analytics* seems to have been slowed by a number of factors. For one, it seems that many 12th century scholars believed the views of the book were better articulated in other works. For example, John of Salisbury remarks that:

Although we need its doctrine, we do not need the book itself that much. For whatever is contained there is presented in an easier and more reliable manner elsewhere, though nowhere in a truer or more forceful manner.²[12, p.97]

Very few commentaries on the *Prior Analytics* have survived from the 12th century and those that have, only in fragments. [12, p.99] Thus it is Kilwardby's *Commentary on the Prior Analytics* that, so far as we are aware, is the first completely preserved commentary on this work.[12, p.101] As such, it offers us insights into the role of this text and its function in medieval academic practice.

Kilwardby's treatment of Aristotle's work, as we shall see, is sophisticated, interesting, and exerted considerable influence on the logicians and philosophers that followed Kilwardby. The work itself is massive, coming to almost 300,000 words, and does not currently exist in a critical edition, although one is currently being undertaken by Paul Thom. To better understand this work, we will attempt to collect various elements of Kilwardby's views.

1. The other five texts were the *Categories*, *On Interpretation*, the *Posterior Analytics*, the *Topics*, and the *Sophistical Refutations*.

2. According to Ebbesen the work to which John refers is probably Boethius' *On Categorical Syllogisms*. [12, p.97]

2.2 Assertoric Syllogistics

In order to understand Kilwardby's analysis of the modal syllogism, it may prove helpful to look at the following parts of Kilwardby's theory:

1. Kilwardby's discussion of truth and signification.
2. Kilwardby's analysis of logical consequence and syllogistic form.
3. Kilwardby's analysis of modality.

We will treat these parts in succession in separate sections. With each of these pieces of the theory in place, we will then look at how the modal syllogism emerges from these different elements.

Kilwardby's account of logical consequence differs from the modern 'classical' conceptions of logical consequence in a number of important ways. For example, Kilwardby distinguishes two kinds of logical consequence, one that, we will argue, is connexive in nature, and another that is not. Kilwardby's theory counts syllogistic reasoning as valid in connexive logic (and in classical logic as well). For our purposes in this chapter, the term 'connexive logic' will refer to any logic where the principles: 'If A implies B then it is not the case that A implies not B' and 'It is not the case that A implies not A' both hold. It should be observed that, depending on how we interpret the negation in the second principle, there are two ways of conceiving of it. On one view, we would understand these to be saying that for all formulae A 'It is not a theorem of our logic that A implies not A'. The stronger view would assert that, for all A, 'It is not the case that A implies not A' is valid in our system. It should be observed that, regardless of how the second principle is understood, it is not compatible with classical logic, since, '(B & not B) implies not (B & not B)' is a theorem of classical logic.

We will have more to say about some of the formal properties of connexive logics in the next chapter.³ The starting point for the contemporary discussion of connexive logic is [42] and [41]. Following McCall, we will refer to the first principle as Boethius' Thesis and the second as Aristotle's Thesis. Later in this chapter we will also touch on parallels between Abelard and Kilwardby's accounts of implication. In that context we will also discuss something that Martin [40, p.191] calls Abelard's first thesis, which states that 'it is not the case that both A implies B and A implies not B'.

Kilwardby treats several kinds of modals in his work. In this thesis we are going to only focus on those modals that are grounded in the essence of the terms which occur in the syllogism. The other sort of modals, which are used to distinguish two senses of contingency, are based on the disposition of objects to change. As such, for Kilwardby, there are two kinds of contingencies. One kind of contingency is the usual modal definition of contingency. The other kind of contingency is grounded in the tendencies of particular objects to change or develop in particular ways. Both the essentialist modalities and the dispositional modalities are underpinned by an Aristotelian ontology. When Kilwardby discusses modalities of necessity, there are two kinds. Necessity can be divided into

3. See page 56

necessity *per se* or necessity *per accidens*. This distinction turns on the idea that some properties are held in virtue of the essence of a term, and other properties are held necessarily but are not essential to the object. As we shall see, this is at the heart of much of Kilwardby's theory. However, before we turn to this, we need to unpack Kilwardby's treatment of assertoric propositions.

2.2.1 Truth and Signification

As is well-known, for most medieval thinkers the theory of signification served as the backbone and foundation for medieval theories of truth and, by extension, theories of inference.⁴ Kilwardby, like many medieval authors, grounds his discussion of signification in Aristotle's comments in the *De Interpretatione* 16a 3-8:

Now spoken sounds are symbols of affections in the soul, and written marks symbols of spoken sounds. And just as written marks are not the same for all men, neither are spoken sounds. But what these are in the first place signs of—affections of the soul—are the same for all; and what these affections are likenesses of—actual things—are also the same. These matters have been discussed in the work on the soul and do not belong to the present subject.[1, p.25]

According to Conti, Kilwardby starts his discussion of signification in his commentary on *De Interpretatione* by explaining exactly what written and spoken terms signify. [10, p.70-72]⁵ Kilwardby's view is that words primarily signify *passiones animae* and the objects themselves. However, Kilwardby does not think that words signify a multiplicity.[10, p.71]⁶ On Kilwardby's account this is because "concepts imitate the things by which they come to be, taking on their forms" [10, p.72].⁷ It is this similarity between the concept and the thing picked out by the concept that ensures that words do not signify a multiplicity. They signify the concepts, and then the concepts have this similarity with the things in the world. Visually, we can represent this as follows:

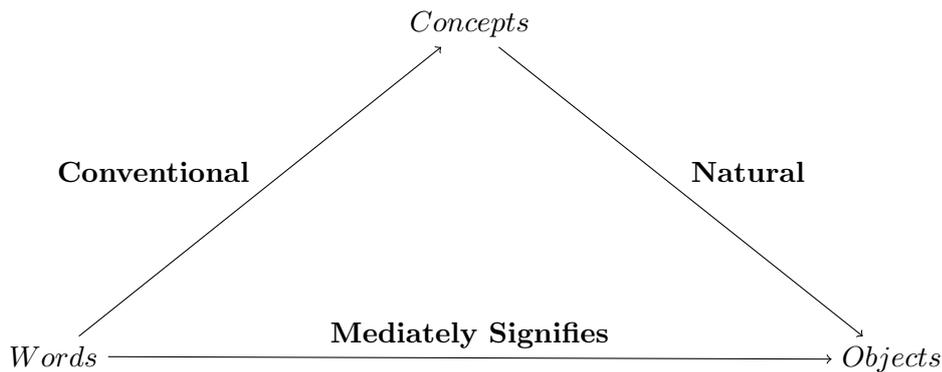
4. Buridan being a noteworthy exception, preferring to use the theory of supposition. We will consider Buridan's views more fully in Chapter Three.

5. It should be noted that in medieval discussions of signification it is either assumed that one is working with terms that do signify or a distinction is drawn between those terms that signify and those which do not. This is to rule out cases where the propositions are grammatically well-formed, but one or both of the terms is meaningless.

6. For Kilwardby, a word is an utterance (*vox*) together with the act of signifying. The utterance functions as the matter, while the act of signifying gives the utterance form.[10, p.71]

7. Conti cites two passages from Kilwardby for this:

"Ex quo patet quod non dicemus ipsam vocem rem et intellectum significantem plura significare" and "Intellectus est similitudo rerum; fit enim ad imitationem rei et ex ea generatur".[10, p.83, f.20].



For many of the medievals the signification of concepts by words was conventional. On Kilwardby's view there is no intrinsic relationship between, for example, the word 'homo' in Latin (or the word 'man' in English) and the concept that the word signifies (in both cases the concept of humanity). However, the relationship between concepts and the objects which fell under those concepts was not a conventional one, but natural. To use the modern phraseology suggested by the diagram, there is a relation from concepts to objects that, in some important sense, maps the structure of the concept onto structure in the objects. Kilwardby's view is a fairly standard medieval realist reading of signification. On this view, written terms immediately signify mental terms. Mental terms, in turn immediately and *naturally* signify things in the world. Hence for Kilwardby, there are two ways of understanding the signification of terms. On the one hand we can think of terms as the names that are given to various things.⁸ This corresponds to the mediate sense of signification. On this account we can think of terms as composed of utterances and which in turn can be combined with other utterances to form propositions. For example, on this reading, the term 'man' can be used to signify the various men in the world (if there are any). On the other hand we can think of the signification of terms as expressing relationships between various kinds of concepts in a mental language. In this case we are interested in the immediate signification of the words. On this view, terms signify thoughts or concepts. Both of these are important for signification, because it means that either a term can signify a class of objects, or it can refer to the concept the term expresses. Both notions of signification will be important for our discussions about logical consequence and for how we formalise Kilwardby's modal logic.

With the theory of signification in place, we can then move on to discuss when propositions are true or false. Kilwardby follows the standard medieval formulation of truth, which states that a proposition is true if and only if it signifies how things are in the world.⁹ One of the concerns that Kilwardby focused on was how to use the theory of signification to account for the truth of complex propositions, e.g. conjunctions, disjunctions, etc.¹⁰ For Kilwardby, truth is to be understood as a kind of relationship between

8. i.e. 'est sermo de termino ut est praedicabilis'

9. [10, p.70] It is worth observing that the signification here would be the mediate kind.

10. For now we will use the term 'proposition' in a loose and imprecise sense to range over both significative utterances and the content expressed by such utterances. Kilwardby's account has some difficulty in separating these two ideas and at this point we will not attempt to disentangle the two either.

concepts and things in the world. However, this does not tell us how we are to account for the truth or falsity of complex expressions. Terms (like 'Socrates' or 'man') cannot be true or false as they only signify things in the world. They do not signify how things are. In contrast, complex expressions generally, and in particular subject–predicate propositions, can be true or false.¹¹ According to Kilwardby this is because complex expressions signify the realities to which the sentence refers.[10, p.76] For example, the expression 'Socrates' may signify the philosopher Socrates¹², but it does not describe or signify anything about how the world is, so it cannot be true or false. Whereas, if we say 'Socrates exists' (and Socrates refers to the ancient Greek philosopher) then this signifies that Socrates exists, which is currently false, because Socrates is dead.

An interesting complication of this analysis emerges when Kilwardby gives truth conditions for past and future tensed propositions. As we just observed, 'Socrates exists' is false (assuming we are referring to the ancient Greek philosopher) because he is dead, and so the name 'Socrates' does not signify an object that exists. What is important here is that the tense of the verb, in this case, 'exists' (Latin: 'est'), determines the time at which the proposition is evaluated. Now, Kilwardby wants the proposition 'Socrates existed' to come out true. But then, how are we to understand the signification of 'Socrates existed'? Kilwardby's answer is that it signifies the present tense proposition referring to the past.¹³ On Kilwardby's reading, 'Caesar fuit' signifies 'praeteritio Caesaris sive memoria est' ([10, p.76]) i.e. 'Caesar existed' signifies that 'Caesar exists in the past or is remembered.' Kilwardby's strategy is a natural one: he tries to paraphrase the difficulty of past tense propositions away by using present tensed propositions that refer to those times. This also brings out some of the subtlety of Kilwardby's theory. First, when we say that 'Caesar exists in the past' what exactly is the signification of this proposition? The name Caesar still signifies the person. The term 'exists in the past' presumably signifies the various things that existed in the past. What is interesting here is that, given what we have already seen, there needs to be some sort of fact that grounds this relationship. Unfortunately, Kilwardby does not tell us what kind of fact this might be. The other disjunct is simpler, and might also be more helpful. According to Kilwardby it is sufficient for the past-tense proposition to be true if there is an individual who is thinking about Caesar. In this case, it is the fact that there is a person who currently exists who remembers Caesar that grounds the truth of the proposition. Regardless, either way the goal is to paraphrase past tense expressions in terms of present facts about those past tense terms and phrases.

As we have already seen, Kilwardby analyses tensed terms disjunctively. For simple propositions like 'Socrates runs', all that is required is that Socrates be running. However, when we turn to complex propositions things become more complicated. In this

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11. At least in a loose way of speaking. Kilwardby would argue that, strictly speaking, truth is a property that is held by things in the world, not by sentences or other linguistic expressions. See [10, p.75].
 12. At this point it is helpful to simply bracket the discussion of how we assign names to the objects that they pick out. In many cases, the medievals used names like Socrates, Aristotle, etc., in a way that is similar to how modern philosophers use the names John, Alice, Bob, etc., not as referring to particular historical figures, but as generic persons for making a particular point.
 13. The follow follows Conti's analysis. See [10, p.76].

context, a complex proposition is one that contains conjunctions, disjunctions, or similar constructions. These operations can occur in a number of places. For example ‘Socrates or Plato is running’ and ‘John runs and Plato sleeps’ are both complex propositions. In the case of complex propositions such as ‘Socrates runs or Plato swims’, it is not obvious if there needs to be a single fact in the world that corresponds to this disjunction, i.e. what we might now call a ‘disjunctive fact’ which exists over and above the disjuncts, or if all that is required is that one of the disjuncts be true. Regardless, categorical propositions can then be given truth conditions in terms of their signification in the usual way. For example, ‘Every A is B’ is true just in case anything that is signified by A is also signified by B. This can then be disambiguated according to Kilwardby’s notion of signification to give two different sets of truth conditions, one based on the concepts expressed by A and B and another for the objects which fall under the extension of A and B. Kilwardby does not discuss when negative propositions are true or false. This can easily be done by defining negative propositions as the contradictory of the affirmative propositions. It should be remarked, however, that this is not a very medieval way handle this. Usually medieval authors treat affirmative and negative categorical propositions equally and as primitive.¹⁴

This distinction between what we might call the ‘mediate’ and ‘immediate’ senses of signification is extremely important for our formal treatment of Kilwardby’s theory and will be required to make sense of a number of things that Kilwardby speaks of. This distinction gives rise to two ways different ways proposition such as e.g. ‘Every A is B’ can be true. For example, the proposition ‘Every man is an animal’ will always be true on the conceptual reading, since the concept of ‘man’ contains or makes use of the concept of ‘animal’. However, if we read ‘Every man is an animal’ based on the objects that fall under the terms, then this will come out false in the cases where there are no objects that fall under the term ‘man’. In [61, p.14] a number of distinctions that Kilwardby desired to preserve are listed. Inspired by the last item in this list, let us refer to the ‘conceptual’ reading as expressing ‘habitual predications’, and the ‘object’ reading as ‘actual predications’. In our formal treatment of Kilwardby, we will separate out these two classes of semantic readings and explore the kinds of logical machinery that can be used to represent them.¹⁵

Before turning to our analysis of other parts of Kilwardby’s theory, we need to take a metaphysical detour and examine his theory of being and essence, as the theory plays an important role in his logical theorising. As we have already remarked, the proposition ‘Every man is running’ can be considered in one of two ways. The crucial difference hinges on what the terms ‘man’ and ‘running’ signify. They could signify one of two things. They could signify men and things running respectively, or they could signify the concepts of ‘man’ and ‘running’. In the first case the proposition would be true if every man were indeed running. In the second case, the proposition would be true if

14. Alternatively, and equivalently, one could simply put a negation in front of the truth conditions that Kilwardby gives. According to the usual semantics of negative categorical propositions, a universal negative is true if everything that supposits for A does not supposit for B, and a particular negative is true if there is some A that does not supposit for B or A does not supposit for anything.

15. See page 46.

the concept 'man' contained or in some sense entailed the concept of 'running'(which presumably it does not). But what exactly is this notion of containment? Its use is critical to Kilwardby's analysis of *natural* and *accidental* consequences to which we now turn.

2.3 Consequence & Modality

2.3.1 Consequence

Kilwardby's analysis of logical consequence takes two notions of consequence as primitive. The first kind of consequence is one of *natural* consequence. These are consequences that follow because the consequent is understood in the antecedent (i.e. they follow because there is an essential relationship between the consequent and the antecedent such that the consequent follows in virtue of the essence of the antecedent). The second kind is an *accidental* consequence and is illustrated by the general idea that the necessary follows from everything.¹⁶

Kilwardby illustrates accidental inferences with the example: "You are an ass, so you are not an ass". Kilwardby tells us that, "in such consequences, when one opposite follows from the other, it is not because one opposite posits the other; but the consequent posits itself on account of its own necessity and not on account of the antecedent. So, in natural consequences the antecedent posits its consequent; but in accidental consequences this is not necessary." [60, pp.55-56]¹⁷

It is this distinction, the idea that in a *natural* consequence the antecedent posits the consequent, whereas in *accidental* consequences this does not have to happen, that differentiates the two classes of logical consequence. But this raises the question as to how we should conceive of the sort 'understanding' that Kilwardby speaks of. How exactly are we supposed to be able to tell if a consequence is *natural* or *accidental*? When Kilwardby says that in *natural* consequences the consequent is understood in its antecedent, this should not be taken in an epistemic way. The containment in question is one of containment in virtue of meaning, or put differently, in virtue of the essence of the thing signified by the term. What we see here is that if the consequent is part of the definition of the antecedent, then the consequence is *natural*.

A consequence is said to be *accidental* (or said to be a material consequence) if there is no situation in which the antecedent can be true and the consequent false. This is the sense of logical consequence that is more familiar to most modern students of logic. Having said how we can identify a *natural* consequence and an *accidental* consequence, we should discuss how these two notions relate to one another. Are these two different

16. It is important to realise here that this principle is only illustrative. It would be possible to illustrate the same idea by using *Ex Impossibile Quodlibet* or another logical principle that violates the impossible containment relationship. What is important is that this kind of logical consequence is not fine grained enough to distinguish between essential and non-essential implications.

17. Quod autem in talibus consequentiis unum oppositorum sequitur ad alterum non est quia unum oppositorum ponit alterum, sed ipsum consequens propter necessitatem sui ponitur et non propter suum antecedens. In consequentiis ergo naturalibus antecedens ponit suum consequens, in consequentiis autem accidentalibus non est hoc necesse.

but equally good notions of logical consequence or is one a subset of the other (in the sense that, e.g., all natural consequences are accidental consequences but not vice versa)? We shall see that these are in fact two different notions of logical consequence, where we can think of *natural* consequences and *accidental* consequences as a subdivision of the kinds of valid logical consequences. We will also see in our formal treatment of Kilwardby that we can think of *natural* consequences as a finer grained notion of logical consequence that does not accept all *accidental* consequences as valid. Likewise, we can treat *accidental* consequences as the logical consequences that are valid, but are not *natural* consequences.

It is interesting to observe that for Kilwardby, natural consequences give rise to a class of what are now referred to as connexive logics. There are two principles that Kilwardby takes to characterise natural consequences. The first is that a disjunction follows from either of its disjuncts.¹⁸ The second is that no proposition follows from its own negation in a natural consequence.

And it is to be said that there are two types of consequences: positive or accidental (and here it is not unacceptable that one of a pair of opposites follows from the other, as has been shown), and natural or essential consequence (and here one of a pair of opposites does not follow from the other and this is what Aristotle means here).¹⁹

What is important to notice about this passage is that it emerges within Kilwardby's discussion of *Prior Analytics* 2.4.²⁰²¹

This is a very complicated passage of Aristotle and it is the place where Aristotle endorses what has come to be known as Aristotle's Thesis. The reconstruction of the passage sketched below follows Smith's notes in his translation of the *Prior Analytics* [2, p.190-191]. In this passage, Aristotle starts his deduction with the assumptions:

1. If A is white, then B is large.
2. If A is not white, then B is large.

From these two pairs, with a bit of work, Aristotle deduces that 'If B is not large then B is large', which Aristotle rejects as an impossibility. The general form of the contradictory of 'If B is not large then B is large' has come to be known as Aristotle's thesis, and can be stated, somewhat convolutedly in English as saying that 'It is impossible that if not

18. Adhuc disiunctiua sequitur ad utrumque sui partem, et hoc naturali consequentia. See [24, ad B4 59a]. It should be noted that the first part of the quoted passage occurs within the context of an objection and the second part is Kilwardby's response, where he accepts the principle.

19. Et dicendum quod consequentia duplex est, scilicet positiva sive accidentalis (et ibi non est inconueniens unum oppositorum sequi ex alio, sicut ostensum est), et consequentia naturalis sive essentialis (et ibi non sequitur unum oppositorum ex alio, et hoc intendit hic Aristoteles) [24, p.ad B4 60a]

20. Much of what follows is inspired by an unpublished book that Professor Thom was very kind to allow me look at.

21. ie. *Prior Analytics* 57A37-57B17. See [3, III, 104 18-19].

A then A'.²² What is important for our purposes here is to note that Aristotle says the proposition needs to be impossible, otherwise the reductio argument does not work. This is a strong claim, and, when working in non-modal connexive propositional systems, the formula $\neg(\neg A \rightarrow A)$ is taken either as an axiom or shown to follow from other axioms.²³ It should also be observed that, in classical logic, $\neg(\neg A \rightarrow A)$ is equivalent to $\neg A$ and is not a theorem of classical logic. We include this digression here because it is important to be clear that this principle, which Kilwardby affirms, is the same principle that some modern commentators view as connexive, and in what follows we will take Kilwardby to be defending the strong form of this thesis.

Further, a disjunction follows from either of its parts – and in a natural consequence. Hence, it follows ‘If you are sitting then you are sitting or you are not sitting; and if you are not sitting then you are sitting or you are not sitting’. And so the same thing follows from the being and the not being of the same thing in a natural consequence and thus of necessity.²⁴

We can also further strengthen the idea that Kilwardby is endorsing a kind of connexive logic by drawing parallels to other medieval connexive logicians. Our focus here will be on Peter Abelard. In the recent literature on Abelard, he has been seen as a defender of connexive logic. As the connections made here should hopefully bring out, there are a number of interesting parallels between Kilwardby's thinking on natural consequences and Abelard's discussion of entailment. By pointing out these parallels we will strengthen the case for Kilwardby as a defender of connexive implication as well. At this point, our intention is to simply point out the parallels between these two thinkers' approaches to logic, not to discuss what kind of relationship might (or might not) exist between Kilwardby and Abelard on logic (e.g. we will not explore the possibility that Kilwardby may be aware of Abelard's logic or the common texts that may have served as the basis for both Kilwardby's and Abelard's views on logic).

Like Kilwardby, Abelard distinguishes between two different kinds of necessity. Martin writes:

The explanation of Abelard's reference to two kinds of necessity is to be found in his theory of the relation of substances to their properties and accidents. In the *Isagoge* Porphyry introduces a distinction between separable accidents such as being seated for humans, and inseparable accidents such as being black for crows, and gives as the general definition of an accident that it is something which may be present or absent without the corruption of its subject. [40, p.183]

22. In the first case by we are taking ‘B is not large’ to be not A, and B is large, to be A.

23. For example, see the treatments of connexive logic in [42], [41].

24. Kilwardby, *Notule libri Priorum* 2.4 dub.1: *Adhuc disiunctiua sequitur ad utrumque sui partem, et hoc naturali consequentia. Quare sequitur si tu sedes, tu sedes vel tu non sedes; et si tu non sedes, tu sedes vel non sedes. Et ita naturali consequentia sequitur idem ad idem esse et non esse et ita ex necessitate.* [24, ad B4 59a] starting on line 36.

We have already seen a very similar form of this distinction in Kilwardby, namely the distinction between *per se* and *per accidens* modalities. Also, like Kilwardby, Abelard observes that there seem to be two kinds of necessity of consecution i.e. two kinds of implications. Abelard writes:

There seem to be two kinds of necessity of consecution. A broader kind, which is found where the antecedent cannot hold without the consequent. Another narrower kind, where not only can the antecedent not be true without the consequent but also of itself requires (exigit) the consequent. This latter necessity is the proper sense of consecution and the guarantee of immutable truth. As, for example, when it is said if something is human, then it is an animal, human is properly antecedent to animal since it of itself requires animal. Because animal is contained in the substance of human, animal is always predicated with human. [40, p.182]

As in the case of Kilwardby, Abelard also distinguishes two kinds of implications, one based on the idea that an implication holds just in case the antecedent cannot hold without the consequent, and another more narrow account. As in the case of Kilwardby, this more narrow account is based on a further criterion, namely that the essence of the antecedent contains in it, as part of the essence, the consequent, as in the case of animal being part of the essence of human.

We quoted Kilwardby earlier as denying that the same thing follows from a proposition and its negation in natural consequences (see pg. 19). We find this principle in Abelard as well. Martin writes:

Two of these principles stand at the center of Abelard's logic and provide, as it were, the rules of proof corresponding to the semantics of containment. The first of them is Abelard's version of Aristotle's Principle, noted above, that the same cannot follow both from something and its opposite. Abelard's version is propositional and properly represented as: "not $\{(p \rightarrow q) \& (\neg p \rightarrow q)\}$ " [40, p.191]

Martin reconstructs Abelard's reasoning for why this form follows from Aristotle's Thesis by means of the following reductio argument:

- (1) $p \rightarrow q$ Hypothesis
- (2) $\neg p \rightarrow q$ Hypothesis
- (3) $\neg q \rightarrow \neg p$ I1, Modus Tollens
- (4) $\neg q \rightarrow q$ 3,2 Transitivity

A similar argument is given to show that $p \rightarrow \neg p$ would follow. Another point of observation is that on line (3) what Martin cites as Modus Tollens is what we might also refer to as contraposition.

That (4) is inconsistent on Abelard's view, is supported by the following passage:

2 Kilwardby's Commentary on the Prior Analytics

No one doubts them to be embarrassing, or inconsistent, because the truth of one of a pair of dividing propositions not only does not require the truth of the other but rather entirely expels and extinguishes it. (Dial. 290.2527)[40, p.191]

As we saw in our discussion of Aristotle, Abelard also endorses the stronger sense of Aristotle's Thesis.²⁵ Given these considerations then, it seems plausible to take Kilwardby as also committed to a strong kind of connexive implication. For reasons for space we will not further explore the possible connections between Abelard and Kilwardby, however, this will be briefly remarked on in the conclusion of this thesis as possible future work.

2.3.2 Modality

The theory of modality has a long history going back to Aristotle. Within this work we will limit our attention to Kilwardby's treatment of the modal syllogism as it is discussed in his *Commentary on the Prior Analytics*. The most natural place to start is with Kilwardby's understanding of modes and the kinds of modal terms that result from them.

Kilwardby distinguishes between two kinds of necessary modal relationships that can hold between terms. Two terms can be related so as to be necessary *per se* or be related so as to be necessary *per accidens*. It should be noted that terms may fail to be related to each other in either way. This distinction between *per se* and *per accidens* is not unique to Kilwardby, but traces its origins to Aristotle. In *Posterior Analytics I 4-6*, Aristotle introduces four different ways that the term *per se* can be taken to function. The passage is known for being difficult in a number of ways. Thankfully, we only need to take away two things from Kilwardby's treatment of Aristotle, which are nicely summed up by the following quotation from Henrik Lagerlund:

Aristotle discusses four different notions of *per se* predication but Kilwardby . . . seems to only have the first two in mind when referring to *per se*. Aristotle says that the first type of *per se* predication (*per se primo modo*) occurs when the definition of the subject includes the predicate. The second type of *per se* predication (*per se secundo modo*) occurs when the definition of the predicate includes the subject. . . to complete the picture of *per se* predication, one should discuss the concepts of genus, species, difference and *proprium*. This is not done by Kilwardby. . . who only very briefly states what [he] means by necessity *per se*. [35, p.30-31]

The first thing to take away is the unfortunate reality that, as far as I am aware, Kilwardby does not provide a sustained discussion of the notions of genus, species,

25. Recall that this is of the form: The stronger view would assert that, for all A, 'It is not the case that A implies not A' is valid in our system.

etc., in the commentary on the *Prior Analytics*. As such, this means that some of our reconstruction of Kilwardby's logic will have to be somewhat speculative on these points.

The second thing to observe is the distinction between *per se primo modo* and *per se secundo modo*. This distinction is very important and will be helpful to understand exactly how Kilwardby understands *per se* modalities. As was stated above, A is said to be *per se primo modo* B if the definition of A includes or makes use of the definition of B, while A is *per se secundo modo* B if the definition of B includes or makes use of A. For example, 'Every man is an animal' is true *per se primo modo* since the definition of 'man' makes use of the definition of 'animal'. While 'Every man is capable of laughter' would be true *per se secundo modo* since 'man' is part of the definition of 'capable of laughter' because, according to the medievals, this a property unique to man alone.

One of the questions that we will need to answer is, by *per se*, does Kilwardby mean *per se primo modo* or *per se secundo modo*? From what we have already seen about *natural* consequences it is clear that what Kilwardby has in mind for these kinds of consequences are *per se primo modo* predications. To see this, recall that in a natural consequence it is the *antecedent* that posits the consequence, i.e. it is in virtue of the nature of the antecedent that the consequent is said to follow. In what follows when we speak of one term being *per se*, we will mean *per se primo modo* unless otherwise stated.

The use of the distinction between being *per se* predicated of another term or *per accidens* predicated was employed to attempt to dissolve some purported counterexamples to conversion. In looking at these counterexamples, the relationship to essentialist modalities comes through clearly. The counterexample to conversion normally ran as follows: 'Every man is necessarily literate' therefore 'Something literate is necessarily a man'. There are two challenges here. The first challenge is to understand why this inference may be problematic, and second, to explain how that problematic inference is to be dissolved.²⁶

To address the first point, observe that the proposition 'Every man is necessarily literate' is true because every man has the potential to be literate.²⁷ However, the converted proposition 'Something literate is necessarily a man' is, according to many authors of this time, not true. The exact reason for why this proposition is false is often unclear, but given an essentialist reading of the modality, we can easily make sense of the problem. It is part of the nature of being a man that they are able to be literate, since they are rational, and being rational includes the capacity for being literate. However, it is not part of the nature of being literate that one be a man.

Now, to explain why this inference does not work, the response by Kilwardby and others is to observe that man is *per se* literate, but that literate is not *per se* man. Again, it should be observed that this reading only makes sense if we read *per se* in the sense of *per se primo modo*. Thus the solution adopted by Kilwardby was to distinguish the two senses of necessity mentioned above.

Kilwardby sums this up by saying:

26. The first point is an important one here. On a normal reading of the modal operators, reading them *de dicto*, the conversion is logically valid and unproblematic.

27. The term 'literate' here needs to be understood as, 'has the ability or capacity to be literate', not as saying that the person necessarily is able to read.

2 Kilwardby's Commentary on the Prior Analytics

For a *per se* necessity-proposition requires the subject to be *per se* some of the predicate itself. But when it is said, “all who are literate are of necessity men”, the subject is not *per se* some of the predicate itself; but it is granted that it is necessary, because the literate are not separate from what is some man. But this is a *per accidens* necessity.[60, p.20]²⁸

It should be observed here that Kilwardby is speaking not only of the relationship that exists between terms, but also of when a proposition (in this case ‘Every A is B’) can be *per se* necessary. As the quote illustrates, for Kilwardby a proposition is said to be *per accidens* necessary if a relationship of necessity *per accidens* holds between the subject and the predicate. Again, we see here that it is whether the *subject* is *per se* the predicate that determines if a given proposition is *per se* or not.

So, it seems that *per se* modalities do not always convert. This raises a very natural problem for Kilwardby's interpretation of Aristotle, since Aristotle makes use of the conversion of modal propositions in various places in the *Prior Analytics*. What is Kilwardby to do about this? The answer is that Kilwardby sees a kind of symmetry within the terms that are essentially predicated.[60, p.21] This is very much in keeping with modern treatments of Aristotle's modal syllogism. For example, in his paper on Aristotle's syllogistic, Malink writes:

By analogy, the particular affirmative necessity $N^i ab$ could be defined as

$$(13) \exists z \Upsilon bz \wedge \hat{E}az$$

However, using this definition, $N^i ab$ fails to convert to $N^i ba$; in order to avoid this problem we follow Thom and Brenner in weakening (13) by disjunction as follows:

$$(14) \exists z ((\Upsilon bz \wedge \hat{E}az) \vee (\Upsilon az \wedge \hat{E}bz))$$

[37, p.109][38, p.179]²⁹

What is important here is that we will need to find a way to weaken the reading of the particular necessary propositions to ensure that modal conversion remains valid for Kilwardby as long as the inference does not confuse *per se* and *per accidens* modalities.

Up to this point, we have seen that the distinction between two kinds of necessity, necessity *per se* and necessity *per accidens*, is essential for understanding Kilwardby's reading of the modal syllogism. This distinction presupposes a difference between the essence of a thing and the properties that are necessarily true of a thing. We have also seen that Kilwardby does not discuss how essences and some of the related notions (such

28. “Propositio enim per se de necessario exigit subiectum esse per se aliquid ipsius praedicati. Cum autem dicitur ‘Omne grammaticum de necessitate est homo’, ipsum subiectum non est aliquid per se ipsius praedicati, sed quia grammaticum non separatur ab eo quod est aliquid ipsius hominis, ideo conceditur esse necessaria. Sed quae sic est de necessario, per accidens est de necessario.”

29. We will make considerable use of Malink's reconstruction of Aristotle in our formal reconstruction of Kilwardby in Chapter Three. As such, there will be a sustained discussion of his system there as well as definitions for all of the logical symbols.

as genus, species, etc.) function within his logical theorising. As such, it may be helpful to look at other authors' views about essences around the time of Kilwardby.

How does this distinction function within the 13th century metaphysics of Kilwardby's day? First, it is helpful to realise that there were two theories about how the term *esse* functions in sentences.

According to the inherence theory, the copula is a sign of the fact that the 'nature' or 'form' signified by the predicate term is present in the individual things (supposita) denoted by the subject term. The identity theory instead claims that the terms should be taken as denoting classes and that the affirmative propositions state that all or some members of the subject class are simultaneously members of the predicate class.[35, p.29]

What is important is to note that Kilwardby held to the inherence theory of the copula and it is this theory that allows him to distinguish the *per se* and *per accidens* senses of the necessary propositions. Given that there can be a distinction between the necessary and the essential, this distinction makes sense within an inherence framework. But how are we to make sense of this distinction? When is it true to say that 'A is essentially B', and when is it false?

There were two main questions that drove the metaphysical debates around the conception of an essence in Kilwardby's time. First, the medievals wanted to develop a theory of essence and existence that could account for the existence of non-physical entities like angels and demons (i.e. creatures that were non-physical, but not simple and whose existence was not necessary.)[68, p.662] Second, the medievals wanted a distinction that would help explain the difference between God, whose existence was necessary and who depended on nothing, and everything, which depended on God for existence.[68, p.662] The development of these positions drew from a number of ancient sources, the most important authors being Aristotle, Boethius and Ibn Sina.

As we have already observed, so far as modern authors are aware, Kilwardby does not provide a sustained discussion of essences. However, what is clear is the following: First, the essence of a thing is to be separated from the existence of a thing, so that one can posit a thing's essence without positing the existence of the thing.³⁰ Second, when we are dealing with necessary *per se* propositions, the universal affirmative lacks existential import. To claim that 'Every man is essentially an animal' does not require that there be some man who instantiates the term. The reasons for this are, at least to some degree, connected to the relationship between the interpretation of Aristotle and Christian theology. According to Christian theology, humans, animals etc. did not always exist and were created by God. Now, if the subject of a proposition of necessity needs to necessarily exist in order for the proposition to be true, many of the propositions that Aristotle says are necessary turn out to be false, such as the 'Every

30. For the medievals there is one obvious counter-example to this, namely God, who due to His simplicity, is the place where existence and essence are identical. Such considerations will not concern us at this point.

man is an animal' example cited above. According to Christian theology, there was some time when men did not exist, and so, it follows that men did not necessarily exist. Hence, 'Every man is an animal' would come out false if the subject had to necessarily exist. In fact, Kilwardby had this proposition, namely, that necessary truth depends on persistence of the subject, condemned in 1277! According to Kilwardby, "That necessary truth depends on persistence of the subject³¹" is to be condemned.³² As an interesting corollary to this, we should note here that if, for Kilwardby, the truth of the universal affirmative proposition requires the existence of the subject, then necessary universal affirmative propositions do not entail the corresponding assertoric proposition. As we shall see later on this is one way of providing an interpretation of Aristotle's modal logic that does not run afoul of theological issues, while also preserving the inferences Aristotle takes to be valid. Another interpretation, consistent with Kilwardby's condemnation, is to widen the class of objects under consideration. This interpretation, which Buridan uses, requires only that there be an object that can fall under the subject, regardless of whether that object exists or not. This approach yields a different theory of the modal syllogism that is weaker than Aristotle's.³³

Now, given the way that Kilwardby employs the term *per se*, it seems that it is best to think of this as concerning essential relationships that hold between the various terms under discussion. What we then need to develop is a logic that can relate the conceptual grounds of various terms to the objects that they pick out, and is fine-grained enough to be able to separate out *per se* from *per accidens* modalities.³⁴ Before looking for such a formal model, we need to discuss how Kilwardby brings these results together in his treatment of the syllogism.

2.3.3 Syllogisms

Kilwardby's discussion of the syllogism looks at three different readings of Aristotle's text. One view appears to be based on that of Boethius. Another one is broadly Aristotelian and based on comments Aristotle makes elsewhere in the *Prior Analytics*. The third and final reading, according to Thom, is advanced in the *Dialectica Monacensis*. [60, p.40]³⁵ The differences between these views can be seen in a number of places, but is perhaps clearest if we start with how the different views look at Aristotle's opening definition of a syllogism in the *Prior Analytics*. Kilwardby quotes Aristotle saying that "A syllogism is a discourse in which, certain things being set out [positis] something else comes about [contingit] of necessity from their being so."³⁶ [60, p.42] The three different readings all try to address the following problem: why does Aristotle give discourse as the genus of the syllogism? [60, p.42] The Boethian reading sees the use of the term 'discourse' as a way to preclude other classes of arguments such as induction, example, and

31. Item quod veritas cum necessitate tantum est cum constancia subjecti. [63, p.217]

32. See [63, p.217] and the reference to Lewry therein.

33. See Chapter Four, section 5 for Buridan's ampliative account of modality.

34. Ideally, the logical system should also help us make sense of the distinction between natural and accidental consequence and so give rise to a connexive logic.

35. Hereafter, the Boethian, the Aristotelian and the Munichian readings.

36. The quote from Aristotle is *Prior Analytics* 24b1820.

enthymeme.³⁷ On this view the syllogism is a kind of complex argument that is used to create a belief in the soul of an individual.[60, pp.43-44] Kilwardby spends much less time on the Aristotelian exposition. On this view, the genus is not intended to exclude induction, enthymeme, or example since these can be reduced to syllogisms.[60, p.45] Instead, the reading focuses on the ‘necessity of the conclusion’ excluding syllogism pairs that do not produce a valid conclusion.³⁸ This reading also excludes *petito principii* and *non causa ut causa* on the same grounds.

On the Munichian reading, the definition of the syllogism is intended to preclude all other kinds of reasoning. Accordingly, they state that:

The definition given can therefore make clear from what has been said that all other species of argumentation are excluded (namely Induction, Example, Enthymeme), and it also excludes the sophistical syllogism no matter what its cause (and this includes fallacies).[60, p.46]³⁹

The first part of this definition is similar to the Boethian account, however the Munichian account goes further, in that it also precludes so called ‘sophistical’ syllogisms. Unfortunately, after mentioning these sophisms, the Munichian author moves onto a discussion of *quid sit dici de omini* and does not bother to tell us what the author means by sophistical syllogisms. However, given Kilwardby’s rejection of this view, it seems that Kilwardby took this to refer to arguments that have the form of a syllogism, but are deficient in matter, where the form of the syllogism refers to how the propositions and terms are arranged in the argument, while (at least part of) the matter of the syllogism is whether the propositions are true or false. Thom connects this view with William of Sherwood’s[60, p.47] [67, pp.57-58]. Kilwardby rejects the view because:

For Syllogism formally (as dealt with in this Book) is merely two propositions and three terms. Form and Figure and Mood can be saved in a sophistical syllogism, as is clear from this “Every dog runs, everything that barks is a dog, therefore etc.” and in many others.⁴⁰[60, p.47]

Kilwardby is dissatisfied with this reading because he thinks it excludes too much. According to a view like the Munichian, it is only sound syllogistic arguments that

37. Thus the particular “certain things” excludes Enthymeme. By “set out” is understood arrangement in Mood and Figure and this excludes the useless premise-pairs and Induction. “Of necessity” excludes example, which possesses mere probability since it is a rhetorical argument... By “something other comes about” *petito principii* is excluded, not as a sophistical ground but as a fault in syllogism simpliciter... By “from their being so” *non causa ut causa* is excluded. [60, p.42]

38. Thom uses the term ‘useless premise pairs’ [60, p.45]

39. “Potest igitur manifestum esse ex predictis per diffinitionem datam, quod excluduntur omnes alie species argumentationis, scilicet inductio, exemplum, entimema, et preterea excluditur sophisticus sillogismus secundum omnes sui causas, et hoc quantum ad tredecim fallacias.” The quote is taken from [11, p.490:12-15]

40. Syllogismus enim formaliter, de quo determinatur in hoc libro, tantum modo sunt duae propositiones et tres termini. Forma autem et figura et modus potest salvari in syllogismo sophistico, sicut patet hic: Omnis canis currit, omne latrabile est canis, ergo etc., et in multis aliis.

count as syllogisms. Phrased slightly differently, this view entails that a syllogism which is faulty in its matter is not a syllogism.[60][p.47] This is a view that Kilwardby explicitly condemns in the condemnation of 1277 and he objects to in his commentary. The argument is of interest to us because it hinges on how the form and the matter of the syllogism are to be understood.

It is to be said that the material principles of the syllogism without qualification are two propositions (and if this is lacking there will be no syllogism); but of the ostensive syllogism [the material principles are] two *true* propositions. So, even though a syllogism with false premises is lacking in matter, it is not lacking in the matter of a syllogism without qualification, but in the matter of an ostensive syllogism; and so, even though it has false premises, it does not follow that it is not a syllogism without qualification, but that it isn't an ostensive syllogism.⁴¹[60][pp.47-48]

At the heart of Kilwardby's point is that while a syllogism with false premises will fail to be an ostensive syllogism, it does not outright fail to be a syllogism. For Kilwardby an ostensive syllogism must have true premises, while a syllogism is only required to have two propositions. It should also be observed that here we see what Kilwardby thinks the matter of a syllogism is. Kilwardby takes the matter of a syllogism to be the two propositions that are used to make up the syllogism. For him it is simply the propositions. The truth value of the propositions is irrelevant, as long as they are not ruled out by the previous kinds of considerations given in the other two readings. For example, the propositions cannot be ambiguous.

Kilwardby goes on to analyse the syllogism in terms of its material, formal, and final causes. Kilwardby tells us that:

And it is to be said that there is an order in materials. For some are remote and unarranged, and some are proximate and arranged. And so it is in forms. Some are material forms, which are in potentiality to an ulterior form, and some are ultimate and completing forms. Thus we find an order in a syllogism's materials and forms. For, in materials, the term is its remote and unarranged material, and the proposition is its proximate and arranged materials; and in forms, Figure is the incomplete form which is in potentiality to an ulterior form, and Mood is the ultimate form completing the syllogism.⁴²[60][p.57]

41. Dicendum quod syllogismo simpliciter sunt principia materialia, quae sunt duae propositiones, et si in hoc sit defectus non erit syllogismus, syllogismi autem ostensivi sunt duae propositiones verae. Quamvis ergo syllogismus ex falsis deficiat in materia, non deficit in materia syllogismi simpliciter sed in materia syllogismi ostensivi, et ideo, quamvis ex falsis sit, non tamen sequitur quod non sit syllogismus sed quod non ostensivus simpliciter.

42. Et dicendum quod sicut ordo est in materiis, quaedam enim est remota et indisposita, quaedam autem propinqua et disposita, sic est in formis. Quaedam est forma materialis et in potentia ad formam

The above passage is one of the clearest summaries of Kilwardby's view of the form and matter of the syllogism. The distinction between proximate and remote matter is a standard medieval distinction and is fairly straightforward. The idea is that the proximate matter is the matter that is closest to what the particular thing is made up of, while the remote matter is conceptually further away. Using modern biology as an example, the remote matter in a human would be the basic atomic components that make them up, while the proximate matter could be their various body parts (heart, arms, legs etc). In the case of the syllogism, the terms are the remote and most 'basic' part of it. The propositions that make up the syllogism are its proximate matter. As is standard, two of the necessary conditions for being a syllogism are that the argument be composed of a pair of propositions, composed of three terms from which a conclusion can be drawn. Since propositions are made up of terms together with the copula, they serve as the remote matter of the syllogism. Then the propositions themselves form the proximate matter of the syllogism.

2.3.4 The Syllogism Emerges

When discussing the validity of the syllogism, Kilwardby discusses two necessary conditions common to all syllogistic arguments that entail a valid conclusion. This is what we mean when we speak of valid syllogisms. The first property is that every valid syllogism must have at least one universal premise. The second property is that every valid syllogism must have one affirmative premise.[60][p.114] In fact, Kilwardby, following Aristotle, takes syllogisms to be valid by definition, i.e. only productive pairs of propositions can form syllogisms. One does not have invalid syllogisms.

Kilwardby is quick to point out that: "And it is to be said that Aristotle does not mean that without a universal there is no sort of syllogism- for a necessary conclusion is drawn from singulars in the Third Figure- but that no syllogism competently related and arranged according to mood is produced without universals".⁴³[60][p.114] It seems likely that Kilwardby is here referring to expository syllogisms, as he is referring to the use of singular terms that occur in third figure syllogisms.⁴⁴ The expository syllogism was often discussed in the context of the third figure, as in this figure the validity of expository syllogisms is particularly clear. Here Kilwardby points out that Aristotle is not ruling out arguments using singular terms with these claims, since inferences like: 'Socrates is mortal' and 'Socrates is running' therefore 'Some mortal is running' are

ulteriorem, quaedam autem est ultima et completiva. Et sic invenimus in syllogismo ordinem in materia et in formis. In materiis quia terminus... est materia eius remota et indisposita, propositio vero est materia propinqua et disposita; in formis etiam quia... figura est forma incompleta et in potentia ad ulteriorem formam. Modus autem est forma ultima syllogismi completiva.

43. Et dicendum est quod non est intentio Aristotelis quod sine universali nullo modo sit syllogismus quia ex singularibus in tertia figura concluditur necessario sed quod syllogismus competenter se habens et secundum modum dispositus non fiat sine universalibus.

44. An expository syllogism is an argument where the middle term is a singular term, instead of a general term. For example, 'Socrates is mortal' and 'Socrates is a human' therefore 'Some mortal is human'. In this case the term 'Socrates' picks out an individual. For more on the Expository syllogism see our discussion of it on page 116.

valid. Instead, Kilwardby is pointing out that while such arguments may be valid, on Kilwardby's reading they are not syllogisms since they are not properly arranged.

The point is that this property holds when the terms are categorical propositions. It should be observed that Kilwardby does not consider such 'syllogisms' to be syllogisms proper. According to him such syllogisms have figure, but lack a determinate mood.⁴⁵ [60][p.114] When discussing the second property, Kilwardby starts by explaining why Aristotle does not explicitly mention it in the *Prior Analytics*. Kilwardby offers three arguments for this. One is based on the definition of the syllogism. The other uses the first property (in a valid syllogism one premise must be universal) to infer the second. The first argument goes as follows:

For when it is said in the definition of syllogism that something follows from their being so, it is signified that the premises are the cause of the conclusion. But no negation is a cause. So there has to be an affirmative.⁴⁶[60][p.115]

The first thing to be aware of is that 'cause' here should be understood in a fairly broad way, in terms similar to Aristotle's use of the four causes.⁴⁷ In contemporary thought we would normally only consider an efficient cause to be a 'cause' proper. It may be better to think of all four causes here as expressing the reasons why something happens or why a particular inference holds. Kilwardby's second argument goes as follows:

Or, it can be said that Aristotle, in showing that some universal is in a syllogism, shows that something in the syllogism stands to something else in it as whole to part. But taking a part under its whole cannot be with a negation, but with an affirmation. Hence in showing that something in the syllogism is universal he has thereby shown that something is affirmative. And I am speaking of perfect syllogisms in the First Figure, from which all others come and to which they reduce.⁴⁸[60][p.115]

At the heart of this argument is an assumption that Kilwardby frequently uses. He assumes that the subject stands to the predicate as the part stands to the whole. Hence, this argument turns on a natural sense of the word 'part'. Let X and Y be terms. Then

45. . . . duae propositiones factae in tribus terminis per situm terminorum ex necessitate determinant figuram, sed non de necessitate determinant modum.

46. Quia cum dicitur in definitione syllogismi quod aliquid sequitur eo quod haec sunt, significatur quod praemissae sint causa conclusionis. Sed negatio nullius est causa. Quare oportet aliquam esse affirmativam.

47. Recall that the 'four causes' are Material, Formal, Efficient, and Final. Aristotle discusses them in *Metaphysics* V, 2 among other places.

48. Vel dici potest quod per hoc quod Aristoteles hic ostendit aliquam esse universalem in syllogismo ostensum. . . . est quod aliquid in syllogismo se habet ad aliud in . . . ipso sicut totum se habet ad partem. Sed acceptio partis sub suo toto non potest esse cum negatione, sed cum affirmatione. Quare per hoc quod hic ostendit aliquam esse universalem in syllogismo relinquitur ostensum esse aliquam esse affirmativam. Et loquor de syllogismo perfecto, scilicet primae figurae, a quo omnes alii extrahuntur et in quem omnes reducuntur.

there will be cases where X is a part of Y and Y is a part of X. These cases occur when the proposition in question can be converted simply. The third argument proceeds as follows:

It is to be said that every syllogism aims to deny something of something or to affirm something of something. Now, one thing can be denied of another only through some difference between them; and this difference is necessarily some of one of them that does not agree with the other. So, since this difference has to be a Middle, it will necessarily be affirmed of one though it is denied of the other. And so if the aim is to deny something of something it is necessary for one at least of the propositions to be affirmative. [60][p.116]

According to this argument, the aim of the syllogism is to end up with a proposition which is either affirmative or negative. This is fairly obvious, as categorical propositions are always equivalent to one that is either affirmative or negative. If the conclusion is a negative proposition, then one of the premises must have described some kind of difference between the subject and predicate term in the conclusion. For this to be the case, one of the premises needs to be negative (otherwise, they would both be affirmative, and it is easy to check that no positive propositions ever yield a negative syllogism.)⁴⁹ Now, because of this, it should be clear that the only way to get this difference to connect the two propositions is that they share a term that expresses this difference. That difference is expressed by the middle terms of the syllogism, and it is exactly this difference which is the middle term of a valid syllogism.

2.3.5 Modal Syllogisms

Kilwardby's discussion of the modal syllogism is divided according to the kinds of propositions that occur within the syllogism. The LLL⁵⁰ syllogisms are treated in P8, the mixed necessary/assertoric proposition in A9-11 respectively. Contingency syllogisms are treated in P14, P17, and P20. The mixed contingency/assertoric propositions are treated in P15, P18, and P21. Finally, the contingency/necessity propositions are treated in P16, P19, and P22. [60][p.147, p.180]

As has already been mentioned, necessary propositions must be *per se* necessary. When considering LL pairs, Kilwardby says that the valid syllogisms in the first figure are the same as the ones that are valid in the assertoric case (i.e. aa, ea, ai, ei all yield valid syllogisms). This uniformity continues in his treatment of the second and third figures.[60][p.150] Kilwardby then goes on to show how the second and third figure can be reduced to the first figure. The reductions are quite straightforward. The only cases worth commenting on are the proofs of LLL Baroco and Bocardo. In both cases,

49. Strictly speaking this needs to be sharpened to rule out cases where one of the terms is implicitly negative or entails a negative proposition.

50. In what follows we use the letter L to denote a premise of necessity, M to denote a premise of one way contingency/possibility, X to denote an assertoric premise and Q to denote a premise of two way contingency.

2 Kilwardby's Commentary on the Prior Analytics

Kilwardby makes use of expository syllogisms.⁵¹ The use of an expository syllogism here could be because Kilwardby is following Aristotle's exposition of the text, and when Aristotle proves that these syllogisms are valid, he makes use of an expository syllogism, because he has not yet treated syllogisms with possibilities.

In the LX case, Kilwardby adds an additional principle to describe the valid syllogisms in this mood. He gives us the following principle:

P8 In First Figure assertoric/necessity syllogisms, the necessity-proposition must be the major.

From this, together with the principles P1-P4 Kilwardby is able to derive the same syllogistic validities as Aristotle, namely LXL Barbara, Celarent, Darii and Ferio. In interpreting this, what is important is to look at Kilwardby's justification for P8. P8 can be used (with a bit of extra work) to rule out the XLL syllogisms that have traditionally been seen as problematic.⁵² Kilwardby justifies P8 as follows:

The conclusion is part of the Major, and mostly in regard to the predicate, which they share. With regard to the subject, it is part of the Minor. And so it follows the Minor in features affecting the subject (such as universality and particularity) and the major in features affecting the predicate (such as affirmative and negative, assertoric and modal).⁵³[60][p.154]

Kilwardby's point pertains to the question of what sorts of properties are transmitted by which parts of the syllogism. Here, Kilwardby claims that it is the major premise that conveys the mode and the quality of the syllogistic conclusion, whereas the minor term determines if the conclusion is universal or particular. Kilwardby then proceeds to sketch how useless premise pairs can be excluded using these principles. Kilwardby's justification for P8, in some sense, seems a little thin. The feature that he points to is clearly true of the LXL syllogisms that Aristotle takes to be valid. However, what is missing is an explanation of why exactly this is the case.

In Thom's presentation of Kilwardby it does not seem that Kilwardby has a principled reason for requiring that P8 be true. Thom says "As I read him, Kilwardby holds that since the assertoric Major in the first Figure XLL moods may be true merely as-of-now, those moods are invalid." [60][p.161] This would be sufficient to generate counterexamples to the various XLL syllogisms, however it is unclear how this could be justified in a way that is not ad hoc. This is particularly problematic given that Kilwardby takes

51. We will discuss the expository syllogism in more detail when we talk about Buridan's theory of the syllogism. At this point it suffices to know that an expository syllogism is a syllogism where the two premises are linked by a common singular term, i.e. a term that refers to an individual. We will discuss how Kilwardby uses the expository syllogism in Chapter Six, pg. 158.

52. The most famous of these being XLL Barbara, but any first figure XLL syllogism will do.

53. Et dicendum quod conclusion est pars maioris et maxime secundum praedicatum in quo communicat cum ipsa et quantum ad subiectum pars minoris. Et ideo sequitur minorem in dispositionibus accidentibus subiecto eius quae sunt universalitas et particularitas, maiorem autem in dispositionibus accidentibus praedicato eius quae sunt affirmativum et negativum de inesse et de modo.

the assertoric propositions in such syllogisms to be unrestricted.⁵⁴ This leads to one natural question: If Kilwardby wants to try and preserve Aristotle’s reading of the modal syllogism, how is he going to be able to justify the rejection of various XLL syllogisms (most importantly XLL Barbara)?

First, observe that if we require that the assertoric premise in an XLL syllogism be unrestricted, then we can generate a counterexample to P8. For example, consider the syllogism:

Every animal is a substance. (true and necessary in Aristotelian ontology.)

Every man is necessarily an animal.

Every man is necessarily a substance.

Kilwardby gives additional principles that explain the validity of the syllogisms in the other figures. As these principles hold if a syllogism is to produce a valid conclusion and as we will be making use of them in the next chapter, we restate them here:

P1 In every syllogism, at least one premise must be universal.

P2 In every syllogism, at least one premise must be affirmative.⁵⁵

P3 In first figure syllogisms, the major must be universal.⁵⁶

P4 In first figure syllogisms, the minor must be affirmative.⁵⁷

P5 In second figure syllogisms, the major must be universal.⁵⁸

P6 In second figure syllogisms, at least one of the premises must be negative.

54. “He[Kilwardby] holds that the Minor in the LXL moods must be an unrestricted assertoric which is the same in reality as a necessary proposition, even if it is not the same in mode and he deals with the issue by stating that despite the syntactic differences, there is no difference ‘in reality’ between the minor premises in the LLL and the LXL case”[60][p.158]

55. See [24, p.ad A4 Part 2 Dub. 8 10^{vb}].

56. Kilwardby writes: “Alternatively, it can also be said that if the Major were particular, the Middle could be more common than the Major Extreme. For, an inferior is predicated of a superior in part, affirmatively and negatively. And if this were so, it could happen similarly that the Major was negative and the Extremes convertible, or exceeding and exceeded. And a negative conclusion couldnt follow unless it was false as it clear from the terms man, animal, ass.”[60, p.120], [24, ad A4 Part 2 dub.11 (IIra)]

57. Kilwardby writes: Now, of the remaining principle, namely that the Minor is affirmative, it is to be said that if the Minor were negative, either the Major would be negative (and then a syllogism would not be produced– for the stated reason), or the Major would be affirmative (and then there would be a fallacy of the consequent), because from the negation of a things inferior there doesnt follow the negation of the same things superior.[60, pp.119-120], [24, ad A4 Part 2 dub.11 (IIra)].

58. Kilwardby affirms P5 and P6 is given in the following passage: “Next, someone will enquire concerning the sufficiency of the moods in this Figure, why when there are premise-pairs, only are useful. This is to be solved by supposing the common principles we had before, and two that are proper to this Figure, of which one is that the Major is universal, and the second is that one of the propositions is negative.”[60, p.131]. [24, ad A5 dub.2 12ra-b]

2 Kilwardby's Commentary on the *Prior Analytics*

P7 In third figure syllogisms, the minor must be affirmative.⁵⁹

P8 In first figure assertoric/necessity syllogisms, the necessity-proposition must be major.⁶⁰

P9 In second figure assertoric/necessity syllogisms, one premise must be a universal negative necessity proposition.⁶¹

P10 In affirmative third figure assertoric/necessity syllogisms, the necessity premise must be a universal affirmative.⁶²

P11 In negative third figure assertoric/necessity syllogisms, the necessity premise must be a universal negative.⁶³

In his book, Thom does not always provide quotations for each of these propositions. In cases where he does, where he provides additional arguments that Kilwardby offers for these propositions, or references to the Renaissance edition, they can be found in the footnotes.

For a full summary of these principles and the citations to the references see [60, pp.145,176—177,238].

2.4 Conclusion

Our main aim in this chapter has been to sketch the important details about Kilwardby's theory of the modal syllogism that are necessary to formalise his logic. To that end we have explored Kilwardby's use of the *per se/per accidens* distinction in modal logic and shown that Kilwardby believes that modal syllogisms are only valid if they contain *per se* terms. We have also explored the kind of essential relationship that was envisioned in *per se* terms by the medievals. With these details in place, we can now develop a formal reconstruction of Kilwardby's logic, connecting his ideas with modern notions from the literature on essences.

59. Kilwardby writes: "And it is to be said that the two common principles are to be supposed, and one principle proper to this Figure, namely that the Minor is affirmative." [60, p.136]. [24, ad A6 dub.2 (13vb)]

60. See [24, ad A9 dub.5 (16vb)].

61. [24, ad A10 dub.1 (17^{va})].

62. See [60, p.170].

63. [24, ad A11 dub.5 (19^{ra})]

3 Reconstructing Kilwardby's Logic

3.1 Introduction

Our goal in this chapter is to provide a formal analysis of Kilwardby's modal logic that is expressive enough to capture Kilwardby's ideas about *per se* and *per accidens* necessity, and to capture the distinction between natural and accidental consequences.

As we saw in the last chapter, there are a number of challenges that face such a construction. Kilwardby's distinction between *per se* and *per accidens* modalities is a fine-grained notion and it looks as though this would be difficult to faithfully represent with a standard possible world semantics. On Kilwardby's own analysis, 'Every person is literate' and 'Someone literate is a person' are both necessary truths, but only the first is necessary *per se* while the second is only necessary *per accidens*. As we already observed, the reason for this is that *per se* necessities preserve essential connections between concepts or objects (depending on the kind of signification), while *per accidens* necessities need not do so.

Our goal in this chapter is to develop a formal theory of Kilwardby's modal syllogism, and his more general theory of inference revolving around accidental and natural consequences. As we shall show, Kilwardby's theory of the syllogism, together with the distinction between *per se* and *per accidens* modality, naturally generalise to a connexive theory of inference.

The distinction between *per se* and *per accidens* is not unlike some contemporary debates about the relationship between essence and modality. In [15], Kit Fine makes a number of observations that, he thinks, tell against various contemporary views that attempt to reduce essences and essential predication to modal predications. Fine argues that such approaches are, in some sense, treating the problem back to front. On his view, essences should be the primitive notion, and it is then through the notion of essences that we can obtain other kinds of necessity.[14] We will argue here that this analysis is not unlike Kilwardby's, and that we can draw inspiration from the logical machinery that has been developed to treat Fine's semantics also when we treat Kilwardby's. In doing this, we will show that Kilwardby's interpretation of the modal syllogism succeeds in validating all of the apodictic syllogisms that Aristotle claimed were valid and refuting all of the apodictic 'syllogisms' that are invalid.

In this chapter, we will sketch the necessary formal and philosophical background to connect Kilwardby's discussion of *per se* terms to modern debates, both philosophical and logical. After a brief exposition of Fine's views on essence, we will highlight some of the logical features of Marko Malink's recent treatment of Aristotle's modal logic. These features will be helpful in our treatment of Kilwardby. With that in place, we will then do two things. First, we will formalise Kilwardby's distinction between *per se* and *per*

accidens necessity in a framework that allows us to see the connections with both modern modal logic and Fine's analysis of essence. Second, we will show that our reconstruction of Kilwardby's logic can be extended to account for Kilwardby's distinction between natural and accidental consequences. In doing so, we will show how this gives rise to a connexive implication relationship.

3.2 Essences, Modality and the Question of Reduction

In [14] and [16], Kit Fine articulates some of the reasons for rejecting the equivalence between essences and modality [14][p. 3]. There are a number of important philosophical ideas that underpin Fine's theory. First and most importantly, for Fine, essences are not to be analysed by means of modal properties. Fine points to three kinds of counterexamples that seem to tell against any possible modal reduction [14][pp.4-5]. In each case, the counterexample picks a particular necessary property or relation that holds of an object or between objects, but does not seem to be essential to the object or objects.¹ Fine's response to these counterexamples is to argue that what has gone wrong in the modal account is that it is not 'fine grained' enough to account for the differences in essences that exist between objects. Instead, he argues that essences need to be treated as the logically primitive notion and that other kinds of modal properties should then be defined in terms of those [15][p.241]. Second, Fine suggests that the relationship between essence and modality is similar to the relationship between meaning and analyticity. On this view, the notion of essence is a very fine-grained relationship that can then be used to express the more coarse-grained notion of necessity. Fine writes:

The concept of metaphysical necessity, on the other hand, is insensitive to source: all objects are treated equally as possible grounds of necessary truth; they are all grist to the necessitarian mill. [14, p. 8]

On Fine's view, an adequate treatment of essences needs to be able to track exactly this distinction. On Fine's view, not all statements that are necessarily true about an object are part of that object's essence, as the quote about Socrates illustrates. This idea, namely that an object's essence is not all of the properties that are necessarily true of the object, is not foreign to Kilwardby. In fact, the distinction between *per se* and *per accidens* modalities seems to require a distinction that is in a similar spirit to this one. Consider again the problematic inference: 'Every man is necessarily literate'

1. To cite one example:

Consider, then, Socrates and the set whose sole member is Socrates. It is then necessary, according to standard views within modal set theory, that Socrates belongs to singleton Socrates if he exists; for, necessarily, the singleton exists if Socrates exists and, necessarily, Socrates belongs to singleton Socrates if both Socrates and the singleton exist. It therefore follows according to the modal criterion that Socrates essentially belongs to singleton Socrates. [14][p. 4]

therefore ‘Something literate is necessarily a man’. Substituting the term ‘essentially’ for ‘necessarily’ makes the issue clearer. For ‘Every man is essentially literate’ is true if we see being literate as something that follows from the rational nature of a man (i.e. because you have a rational nature, it follows that you have the capacity to be literate). However, ‘Something literate is essentially a man’ is false, since there is nothing in the nature of being literate that connects with being a man. One natural way to understand why conversion fails in this case, is in the essentialist terms that Fine suggests.

The failure of conversion leaves us with the challenge of thinking about Kilwardby’s distinction between natural and accidental consequences. Characterising accidental consequences is straightforward,² as they appear to be a classical consequence relation. However, natural consequences are not, and since Fine’s logic is thoroughly classical, they will prove to be ill-suited as a starting point. As such, our challenge in modelling Kilwardby’s logic, then, is to develop a formal system that, unlike the modern literature on the logics of essence, allows for a connexive implication.

It should be recalled that connexive logics are a family of contra-classical logics. Following [21, p.1], a logic is said to be *contra-classical* (in the deep sense) if not everything provable in the logic is provable in classical logic. Contra-classical logics, then, are unusual families of non-classical logics. Nearly all of the non-classical logics are commonly discussed (e.g. relevance and para-consistent logics, intuitionistic logic, and many multi-value logics) are *not* contra-classical. They reject particular tautologies and/or theorems that classical logic claims are valid. However, they do *not* include any validities that are not valid in classical logic. They are proper sub-logics of classical logic. This is not the case for connexive logics. First, they accept as valid some theorems that are invalid in classical logic. Second, for such logics to be non-trivial they also need to reject some theorems of classical logic. The easiest way to see this is to recall that classical logic is Post-complete, i.e. that there are no logical systems that can consistently extend classical logic. Because of this, if we were to add any of the distinctive theorems (i.e. any of the theorems of connexive logic that are not theorems of classical logic) of connexive logic to classical logic, the logic would become trivial.

What are some of the theorems that are taken to characterise connexive systems? The hallmark of connexive logic systems is that:

The definition of connexive implication is transmitted to us by Sextus Empiricus:

And those who introduce the notion of connexion say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedents.

2. To deal with the simplicity of the models, we will choose to represent Kilwardby’s accidental modals in terms of the usual possible worlds semantics. This is just to make more transparent what is going on in the models. If it turns out that Kilwardby’s accidental modals are better thought of in terms of temporal operations (e.g. that ϕ is possible just means either ϕ is, was, or will be the case), the models can be easily changed to reflect this. We will have more to say about the legitimacy of this in 5.6.2.

3 Reconstructing Kilwardby's Logic

It is characteristic of this variety of implication that no proposition connexively implies or is implied by its own negation, since it is never incompatible with its own double negation, nor is its own negation incompatible with itself.[41][p. 415]

Using \rightarrow as the symbol for implication, this approach to logic is often seen to give rise to the following theses:

Aristotle's Thesis: $\neg(\neg A \rightarrow A)$

Boethius' Thesis: $(A \rightarrow B) \rightarrow \neg(A \rightarrow \neg B)$

It can easily be checked that neither of these formulae is valid in classical logic. Likewise, one can also check that they are not valid in any of the logics of essence presented by Fine[16]. Part of our goal in this chapter, then, is to show how we can obtain a connexive implication from the logic of essence. While other semantics for connexive logics have been given in e.g.[54],[48], our semantics will turn out to provide a fairly natural understanding of the operations, and are (comparatively) simpler than semantics found in other literature.

3.3 Essences, Syllogisms, and Previous Work

3.3.1 Reconstructing the Modal Syllogistic

At this point, it will be helpful to look at a reconstruction of Aristotle's modal syllogistic that gives a number of the features which we will want to make use in our formal treatment of Kilwardby and which have been used in previous approaches to Kilwardby's logic. A longstanding problem in the interpretation of the *Prior Analytics* has been to find a consistent interpretation of Aristotle's modal syllogistic. This challenge has been met with varying degrees of success over the past two millennia. One answer, which succeeds in a number of ways in which other interpretations of Aristotle have failed, is given by Malink in [37] and further developed in [38]. The aim of [37] is:

to provide a single formal model that exactly captures Aristotle's claims on (in)validity and inconclusiveness in the whole modal syllogistic. This model is intended to be not without a certain explanatory value for our understanding of why Aristotle's modal syllogistic looks the way it does; but in the following we shall focus on the logical reconstruction and sketch the explanatory background only in a cursory manner. [37][p.96]

Our main focus here will be to discuss the logical reconstruction offered by Malink, with only some attention paid to the historical interpretation. Malink starts by introducing the following three primitive relations Υ , \mathbf{E} and $\tilde{\mathbf{E}}$, where Υ stands for accidental predication, \mathbf{E} for substantial essential predication and $\tilde{\mathbf{E}}$ for non-substantial essential

predication. The symbol $a\hat{E}b$ is a shorthand used by Malink for the disjunction aEb or $a\tilde{E}b$.

Malink's system is governed by the following five axioms:

ax. 1 $\forall a\Upsilon aa$

ax. 2 $\forall a, b, c((a\Upsilon b \wedge b\Upsilon c) \rightarrow a\Upsilon c)$

ax. 3 $\forall a, b(a\hat{E}b \rightarrow a\Upsilon b)$

ax. 4 $\forall a, b, c((aEb \wedge b\Upsilon c) \rightarrow aEc)$

ax. 5 $\forall a, b, c((a\tilde{E}b \wedge b\Upsilon c) \rightarrow a\tilde{E}c)$

Slightly less formally, Υ is a reflexive and transitive relation. E and \tilde{E} are transitive subrelations of Υ that are downwardly closed under Υ .

This also gives us an easy way of thinking about the class of models that this gives rise to. Let A be a non-empty set (of terms), and Υ, E and \tilde{E} be subsets of A^2 . Then, $\mathcal{A} = \langle A, \Upsilon \rangle$ is a preorder on A with E and \tilde{E} as designated down-sets of \mathcal{A} .³

Malink is able to show that axioms 1-5 are sufficient to prove all of the validities that Aristotle claims are valid in the *Prior Analytics* and that the class of models satisfying 1-5 are sufficient to prove counterexamples for all of the syllogisms that Aristotle says are invalid.

What is important for our purposes is the idea that at the heart of Aristotle's modal logic are the various relationships that obtain between terms. Of particular interest in the case of the modal syllogistic are the relationships of being essentially predicated and of being accidentally predicated. It is these two notions that form the basis of how we should understand the predication relationships.

Second, the modal logic that Malink proposes for Aristotle rests on a very close connection between the truth of necessary propositions and the predicate being essentially predicated of the subject if the proposition is affirmative and the predicate being essentially incompatible with the subject, if the proposition is negative. Malink writes:

We have seen that the universal affirmative necessity $\mathbb{N}^a ab$ is not obtained from the corresponding assertoric proposition $\mathbb{X}^a ab$ by adding modal sentential operators, but by replacing the Υ -copula of accidental predication by the \hat{E} -copula of essential predication.[37][p.106]

We may suspect Kilwardby would wholeheartedly agree with this. The basic idea of thinking of *per se* necessities in terms of the essential predication of one term for another, or the incompatibility of one term for another, is very much within the spirit of Kilwardby's project. What we will draw on is the underlying idea that the terms of the modal logic are, in some sense 'ordered' by relationships of accidental containment and essential containment.

3. Recall that, given an order \leq on S and an element $y \in S$ the down-set of y is $\{x : x \in S \text{ and } x \leq y\}$.

3.4 Semantics for Kilwardby

At the heart of our approach to Kilwardby's logic will be an augmentation of the usual possible worlds semantics with additional lattice-theoretic machinery that then imposes constraints on how objects are assigned to predicates. This is in keeping with recent literature on the logics of essence [16],[17]. In particular, given what we have seen in Malink's work [37], we will approach the construction as one that is based on ordering the terms in a given language using a lattice-like construction. What we also saw in Malink's paper is that the order conditions turn out to be fairly weak. The order relations are only preorders on the set of terms. If we help ourselves to the usual meta-logical resources, we only need our essential predications to be preorders on the set of terms, downwardly closed under accidental predication. Unlike Malink, as we shall see below, we will add an additional relationship that captures the idea of an essential incompatibility between terms.

3.4.1 Semantic Reconstructions

As we saw in our discussion of Kilwardby's logic (see page 15) there are two ways that we can think about the signification relationship. One of these corresponds to the mediate sense of signification pertaining to objects in the world, while the other corresponds to the immediate sense of signification, and is concerned with the relationships that exist between various sorts of concepts. The challenge will be to bring these two senses together, since in some cases they yield different truth conditions for some propositions.

First, let us define the language that we will be working in for this section. We define $\mathcal{L}_{\mathfrak{R}}$ in the following way:

A Language for Kilwardby Models 1. Let $\mathcal{L}_{\mathfrak{R}} = \langle TERMS, a, e, i, o, p.s., p.a., \rangle$ where:

TERMS is a countable set of terms.

a, e, i, and o are operators used to form the usual categorical propositions 'Every A is B', 'No A is B', 'Some A is B', and 'Not every A is B', as the formation sequence will make clear.

p.s. and p.a. are the modal operations corresponding to per se necessity and per accidens necessity.

The set of well-formed formulae, $WFF_{\mathcal{L}_K}$ is the least set closed under the following conditions: For any $A, B \in TERMS$:

1. $AaB, AeB, AiB,$ and AoB are in $WFF_{\mathcal{L}_K}$.
2. if $A \times B \in WFF_{\mathcal{L}_K}$ where \times is either $a, e, i,$ or $o,$ then $A \overset{p.s.}{\times} B \in WFF_{\mathcal{L}_K}$

3. if $A \times B \in WFF_{\mathcal{L}_K}$ where \times is either a , e , i , or o , then $A \overset{p.a.}{\times} B \in WFF_{\mathcal{L}_K}$

At the heart of our semantics are the following two ideas. The first is the usual idea from modal logic that, at various points, objects may fall under different predicates. This will be used to capture the idea of accidental necessity and possibility. The second idea is that the relationships between terms determine the essential and accidental relationships that hold between objects. As we explained in Chapter One (see page 22), the idea here is that we interpret ‘Every A is *per se* B’ as *per se primo modo* and so is true if the definition of A is part of the definition of B.

To that end, we will start by introducing the algebraic machinery we will be using.

Let $\mathcal{T} = \langle T, \sqsubseteq, \leq, | \rangle$

where:

T is a non-empty set and \sqsubseteq , \leq , and $|$ are binary relations on T .

We require that \leq be a preorder, \sqsubseteq be a transitive relation on T , and that $|$ is irreflexive and symmetric. We further require that:

1. $\forall x, y$ if $x \sqsubseteq y$ then $x \leq y$.
2. $\forall x, y, z$ if $x \leq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$.
3. $\forall x, y$ if $x \leq y$ then not $x|y$
4. $\forall x, y, z$ if $x \leq y$ and $y|z$ then $x|z$.

In what follows we shall refer to these as order properties 1-4. The basic idea behind these structures is that we use the relations \leq , \sqsubseteq , and $|$ to represent the principles of accidental predication, essential predication, and (definitional) incompatibility respectively. The set T corresponds to the set of concepts that we are evaluating.

In this context, each of our conditions makes good sense. Every concept is contained in itself accidentally, however a term does not need to be defined in terms of itself (and so the relation does not need to be reflexive). Similarly for essential containment. In the case of incompatibility, we require that no concept is incompatible with itself (in essence, a consistency requirement) and that incompatibility is symmetric, (i.e. if A is incompatible with B, then B is also incompatible with A).

The remaining conditions are used to explain how the various operations interact with one another. Our first condition tells us that if x is essentially y , then x is also accidentally y . It should be observed that when we speak of x being accidentally y , this is taken in a broad sense to include those cases where x is also essentially y i.e. we can think of \leq as accidentally or essentially.

The second condition tells us that if x is accidentally y and y is essentially z then x is essentially z as well. This is similar to the principle used by Malink in his reconstruction of Aristotle. The rationale for this principle comes from the idea of treating essences as real definitions. What this principle tells us is that, if x happens to be y , and it is

3 Reconstructing Kilwardby's Logic

part of the definition of y is that it is z , then x also has that property as part of its essence.

The third condition is a further consistency requirement, and tells us that if x is accidentally predicated of y then x cannot be incompatible with y .

The fourth condition is similar. If x is accidentally y and y is incompatible with z , then x cannot be compatible with z either, for otherwise, we would have something that is both compatible and incompatible with itself.

With these conditions in place, we can give truth conditions for the various assertoric and modal propositions that Kilwardby treats as follows:

Let $\mathbb{T} = \langle \mathcal{T}, c \rangle$ where \mathcal{T} was as before, and c is a function from terms to elements of T . Then we can give truth conditions for the immediate sense of signification as follows:

Semantics for Assertoric and *Per Se* Immediate Signification.

| | | |
|---|----------------|--|
| $\mathcal{T} \models AaB$ | if and only if | $c(A) \leq c(B)$ |
| $\mathcal{T} \models AeB$ | if and only if | $\neg \exists D \in T \text{ s.t. } D \leq c(A) \text{ and } D \leq c(B)$ |
| $\mathcal{T} \models AiB$ | if and only if | $\exists D \in T \text{ s.t. } D \leq c(A) \text{ and } D \leq c(B)$ |
| $\mathcal{T} \models AoB$ | if and only if | $c(A) \not\leq c(B)$ |
| $\mathcal{T} \models A \overset{p.s.}{a} B$ | if and only if | $c(A) \sqsubseteq c(B)$ |
| $\mathcal{T} \models A \overset{p.s.}{e} B$ | if and only if | $c(A) c(B)$ |
| $\mathcal{T} \models A \overset{p.s.}{i} B$ | if and only if | $\exists D \in T \text{ s.t. } D \sqsubseteq c(A) \text{ and } D \sqsubseteq c(B)$ |
| $\mathcal{T} \models A \overset{p.s.}{o} B$ | if and only if | $\exists D \in T \text{ s.t. } D c(A) \text{ and } D c(B)$ |

What this account does not provide for us is a natural way to separate *per se* necessity from *per accidens* necessity. For that, we will make use of the normal possible worlds semantics. We will also give semantics for the mediate sense of signification, i.e. the case where the word signifies the objects that fall under a given concept. To do that we will make use of constant domain modal logic:

Let $\mathcal{T}' = \langle D, W, R, v \rangle$ where:

D and W are non-empty sets.

R is a reflexive binary relation on W .

$v : W \times TERM \rightarrow \mathcal{P}(D)$.

We can then give semantic definitions for mediate signification as follows:

Semantics for Assertoric and *Per Accidens* Necessity in the case of Mediate Signification 1.

| | | |
|---|----------------|---|
| $\mathcal{T}', w \models AaB$ | if and only if | $v(w, A) \subseteq v(w, B)$ and $v(w, A) \neq \emptyset$ |
| $\mathcal{T}', w \models AeB$ | if and only if | $v(w, A) \cap v(w, B) = \emptyset$ |
| $\mathcal{T}', w \models AiB$ | if and only if | $v(w, A) \cap v(w, B) \neq \emptyset$ |
| $\mathcal{T}', w \models AoB$ | if and only if | $v(w, A) \not\subseteq v(w, B)$ or $v(w, A) = \emptyset$ |
| $\mathcal{T}', w \models A \overset{p.a.}{a} B$ | if and only if | $\forall x \in W$ if wRx then $\mathcal{T}', x \models AaB$. |
| $\mathcal{T}', w \models A \overset{p.a.}{e} B$ | if and only if | $\forall x \in W$ if wRx then $\mathcal{T}', x \models AeB$. |
| $\mathcal{T}', w \models A \overset{p.a.}{i} B$ | if and only if | $\forall x \in W$ if wRx then $\mathcal{T}', x \models AiB$. |
| $\mathcal{T}', w \models A \overset{p.a.}{o} B$ | if and only if | $\forall x \in W$ if wRx then $\mathcal{T}', x \models AoB$. |

Now, in order to bring *per se* and *per accidens* modalities together into a common system, there are a number of tensions between these two formalisations that need to be resolved. What we will see in the next few paragraphs is that, given how things are currently set up, our system does not correctly capture some key insights that we should have.

First, it should be observed that the definition of AaB using the semantics based on \mathcal{T} and \mathcal{T}' do not always agree with each other. To see this consider the following structure:

| | | |
|-----------------------|---|-----------------|
| $D = \{a\}$ | $W = \{w\}$ | $T = \{A, B\}$ |
| $R = \{(w, w)\}$ | $\leq = \triangleleft = \{(A, A), (A, B), (B, B)\}$ | $ = \emptyset$ |
| $c(A) = A$ | $c(B) = B$ | |
| $v(w, A) = \emptyset$ | $v(w, B) = \{a\}$ | |

Since $A \leq B$ we have $\mathcal{T}, w \models AaB$. However, since $v(w, A) = \emptyset$ it follows that $\mathcal{T}', w \not\models AaB$.

There is a similar problem in the other direction:

| | | |
|-------------------|---|-----------------|
| $D = \{a\}$ | $W = \{w\}$ | $T = \{A, B\}$ |
| $R = \{(w, w)\}$ | $\leq = \triangleleft = \{(A, A), (B, B)\}$ | $ = \emptyset$ |
| $c(A) = A$ | $c(B) = B$ | |
| $v(w, A) = \{a\}$ | $v(w, B) = \{a\}$ | |

In this case we have $\mathcal{T}', w \models AaB$, since $v(w, A) \neq \emptyset$ and $v(w, A) = v(w, B)$, which entails $v(w, A) \subseteq v(w, B)$. However, $\mathcal{T}, w \not\models AaB$ since, $A \not\leq B$.

In one sense, this difference shows us that we have got something correct. This difference captures Kilwardby's idea that while 'Every man is an animal' is true (and necessarily true) of the concepts 'man' and 'animal', it is not always true when we are referring to the classes of objects that fall under this concept. On the other hand, it tells us that there is somewhat of a mismatch between how \leq encodes accidental predication and how objects are assigned to terms.

However, this is not our only problem. Consider the following situation:

| | | |
|-------------------|---|-----------------|
| $D = \{a\}$ | $W = \{w\}$ | $T = \{A, B\}$ |
| $R = \{(w, w)\}$ | $\leq = \triangleleft = \{(A, A), (A, B), (B, B)\}$ | $ = \emptyset$ |
| $c(A) = A$ | $c(B) = B$ | |
| $v(w, A) = \{a\}$ | $v(w, B) = \emptyset$ | |

3 Reconstructing Kilwardby's Logic

Observe that on this structure we have $\mathcal{T} \models A \overset{p.s.}{a} B$ since $A \sqsubseteq B$. However, we also have that $\mathcal{T}', w \not\models AaB$ and $\mathcal{T}', w \not\models A \overset{p.a.}{a} B$. Informally, this would mean that A is essentially B, but that A is not B nor is it accidentally B, even where there is something that is A. Again, this shows that there is a mismatch, but in this case it is between how \sqsubseteq relates terms and how v assigns the extensions of those terms to objects in the domain. This will need to be fixed.

There is a third problem, which is itself more complicated. With what we have sketched above, it appears that *per se* modalities do not entail *per accidens* modalities. The question is, is this correct? There are some remarks that Kilwardby makes later in life that suggest this may not be so. As we have already seen, in the condemnation of 1277, Kilwardby condemns the proposition “that necessary truth depends on persistence of the subject”.⁴ Now, does Kilwardby here mean to include both *per se* and *per accidens* necessity? The condemnation is not clear on this, but this can easily be done in our semantics, especially if we are working with the immediate sense of signification. If we are working with the mediate sense of signification, things will become more complicated.⁵

If we follow this idea, then we can give semantics for the entire modal logic which will rectify these issues. In what follows, we will refer to these as \mathfrak{KM}_M for Kilwardby Models for Mediate Signification and \mathfrak{KM}_I for Kilwardby Models for Immediate Signification. We will start with KM_I .

Kilwardby Models for Immediate Signification 1. Let $\mathfrak{K}_I = \{D, W, T, R, \leq, \sqsubseteq, |, c, v\}$ where

D, W, and T are non-empty sets. (Informally, D is our Domain, W is a set of worlds, and T is a set of interpretations for terms or predicates).

$R \subseteq W^2$.

\leq, \sqsubseteq , and $|$ are subsets of T^2 and satisfy the conditions previously given for them.

$c : \text{Terms} \rightarrow T$.

$v : W \times T \rightarrow \mathcal{P}D$.

In order to bring these two families of structures together we need the following principles:

1. For all terms, A, B $c(A) \leq c(B)$ if and only if for some $w \in W$ $v(w, A) \subseteq v(w, B)$.
2. For all terms, A, B if $c(A) \sqsubseteq c(B)$ then for all $w \in W$ $v(w, A) \subseteq v(w, B)$.
3. For all terms, A, B if $c(A) | c(B)$ then for all $w \in W$ $v(w, A) \cap v(w, B) = \emptyset$

4. Item quod veritas cum necessitate tantum est cum constancia subjecti.[63, p.217]

5. As we shall see later, the ampliative approach that Buridan takes leads him to a different understanding of modal propositions.

The first principle ensures that accidental predication is respected by our valuation relation in some world and that our ordering relation tracks each instance of an accidental containment holding.

The second principle tells us that essential properties are preserved by objects across all worlds (i.e. if A is essentially B then in every world if something is A, it is also B). Likewise, incompatibility is preserved in the same manner.

The reason why 2. and 3. are not biconditionals has to do with the distinction between *per se* and *per accidens* modalities. As the semantics below will make clear, it will follow from both *per se* and *per accidens* modalities that $\forall w \in W v(w, A) \subseteq v(w, B)$. However, the distinction between the two is that, in the case of *per se* modalities, we have the further relationship holding that $c(A) \trianglelefteq c(B)$ while in the case of *per accidens* modalities, this does not need to hold. This is in keeping with Kilwardby's own use of the distinction, where he uses it to distinguish between essential and non-essential, but still necessary properties (someone literate is a man).

It should be observed that most of the interpretive work in this logic is done by the order-theoretic relationships. As such, our logic is much more term focused than modern modal logic, where formulae are usually unstructured and not constrained in such a way. In some sense, we can think of the function v as determining the supposition of a particular term at a particular world, while the relation \leq describes how the various terms relate to each other.

With this in place, we can give our first attempt to treat all of Kilwardby's modal operations together:

| | | |
|---|----------------|---|
| $\mathfrak{K}_J, w \models AaB$ | if and only if | $c(A) \leq c(B)$ |
| $\mathfrak{K}_J, w \models AeB$ | if and only if | $\neg \exists D \in \mathcal{T}$ s.t. $D \leq c(A)$ and $D \leq c(B)$ |
| $\mathfrak{K}_J, w \models AiB$ | if and only if | $\exists D \in \mathcal{T}$ s.t. $D \leq c(A)$ and $D \leq c(B)$ |
| $\mathfrak{K}_J, w \models AoB$ | if and only if | $c(A) \not\leq c(B)$ |
| $\mathfrak{K}_J, w \models A \overset{p.s.}{a} B$ | if and only if | $c(A) \trianglelefteq c(B)$ |
| $\mathfrak{K}_J, w \models A \overset{p.s.}{e} B$ | if and only if | $c(A) c(B)$ |
| $\mathfrak{K}_J, w \models A \overset{p.s.}{i} B$ | if and only if | $\exists D \in \mathcal{T}$ s.t. $D \leq c(A)$ and $D \trianglelefteq c(B)$ or $D \leq c(B)$ and $D \trianglelefteq c(A)$ |
| $\mathfrak{K}_J, w \models A \overset{p.s.}{o} B$ | if and only if | $\exists D \in \mathcal{T}$ s.t. $D c(A)$ and $D c(B)$ |
| $\mathfrak{K}_J, w \models A \overset{p.a.}{a} B$ | if and only if | $\forall x \in W$ if wRx then $v(x, A) \subseteq v(x, B)$. |
| $\mathfrak{K}_J, w \models A \overset{p.a.}{e} B$ | if and only if | $\forall x \in W$ if wRx then $v(x, A) \cap v(x, B) = \emptyset$. |
| $\mathfrak{K}_J, w \models A \overset{p.a.}{i} B$ | if and only if | $\forall x \in W$ if wRx then $v(x, A) \cap v(x, B) \neq \emptyset$. |
| $\mathfrak{K}_J, w \models A \overset{p.a.}{o} B$ | if and only if | $\forall x \in W$ if wRx then $v(x, A) \not\subseteq v(x, B)$. |

With this in place, we can define logical consequence for these models as follows:

Let Γ be a set of well-formed formulae and ϕ a well-formed formula. We say that ϕ is a logical consequence of Γ and write $\Gamma \models \phi$ if:

For all models \mathfrak{K}_J and for all $w \in W$ (where $W \in \mathfrak{K}_J$) if $\mathfrak{K}_J, w \models \gamma$ (for all $\gamma \in \Gamma$) then $\mathfrak{K}_J, w \models \phi$.

This provides a semantic reconstruction of immediate signification. Here, we should

3 Reconstructing Kilwardby's Logic

observe that in all cases *per se* necessary propositions imply *per accidens* necessary ones, in virtue of our three principles. Likewise, both *per se* and *per accidens* necessities entail assertoric propositions. However, it should be observed that on this account, propositions such as 'Every animal is an animal' and 'Some animal is an animal' come out true, even if their extensions are empty. As we have already seen, this is unacceptable when we are concerned with objects. In order to rectify this, let us now consider how to account for such things by giving models for mediate signification and respecting Kilwardby's remarks in the condemnation of 1277.

Kilwardby Models for Immediate Signification 2. Let $\mathfrak{K}_M = \{D, W, T, R, \leq, \sqsubseteq, \mid, c, v\}$ where

$D, W,$ and T are non-empty sets. (Informally, D is our Domain, W is a set of worlds, and T is a set of interpretations of terms or predicates)

$$R \subseteq W^2.$$

$\sqsubseteq, \sqsubseteq,$ and \mid are subsets of T^2 and satisfy the conditions previously given for them.

$$c : Terms \rightarrow T.$$

$$v : W \times T \rightarrow \mathcal{P}D.$$

As before, we impose the following conditions:

1. For all terms, A, B $c(A) \leq c(B)$ iff for some $w \in W$ $v(w, A) \subseteq v(w, B)$.
2. For all terms, A, B if $c(A) \sqsubseteq c(B)$ then for all $w \in W$ $v(w, A) \subseteq v(w, B)$.
3. For all terms, A, B if $c(A) \mid c(B)$ then for all $w \in W$ $v(w, A) \cap v(w, B) = \emptyset$

Truth conditions are given as follows:

| | | |
|---|----------------|---|
| $\mathfrak{K}_M, w \models AaB$ | if and only if | $v(w, A) \subseteq v(w, B)$ and $v(w, A) \neq \emptyset$ |
| $\mathfrak{K}_M, w \models AeB$ | if and only if | $v(w, A) \cap v(w, B) = \emptyset$ |
| $\mathfrak{K}_M, w \models AiB$ | if and only if | $v(w, A) \cap v(w, B) \neq \emptyset$ |
| $\mathfrak{K}_M, w \models AoB$ | if and only if | $v(w, A) \not\subseteq v(w, B)$ or $v(w, A) = \emptyset$ |
| $\mathfrak{K}_M, w \models A \overset{p.s.}{a} B$ | if and only if | $c(A) \sqsubseteq c(B)$ |
| $\mathfrak{K}_M, w \models A \overset{p.s.}{e} B$ | if and only if | $c(A) \mid c(B)$ |
| $\mathfrak{K}_M, w \models A \overset{p.s.}{i} B$ | if and only if | $\exists D \in \mathcal{T}$ s.t. $(D \leq c(A) \text{ and } D \sqsubseteq c(B))$ or $(D \leq c(B) \text{ and } D \sqsubseteq C(A))$ |
| $\mathfrak{K}_M, w \models A \overset{p.s.}{o} B$ | if and only if | $\exists D \in \mathcal{T}$ s.t. $D \leq c(A)$ and $D \mid C(B)$ |
| $\mathfrak{K}_M, w \models A \overset{p.a.}{a} B$ | if and only if | $\forall x \in W$ if wRx then $v(x, A) \subseteq v(x, B)$ |
| $\mathfrak{K}_M, w \models A \overset{p.a.}{e} B$ | if and only if | $\forall x \in W$ if wRx then $\mathfrak{K}_M, x \models AeB$ |
| $\mathfrak{K}_M, w \models A \overset{p.a.}{i} B$ | if and only if | $\forall v \in W$ if wRx then $\mathfrak{K}_M, x \models AiB$ |
| $\mathfrak{K}_M, w \models A \overset{p.a.}{o} B$ | if and only if | $\forall x \in W$ if wRx then $\mathfrak{K}_M, x \models v(x, A) \not\subseteq v(x, B)$ |

As in the case of immediate signification, we require that 2. and 3. are not biconditionals on pain of collapsing the distinction between *per se* and *per accidens* modalities.

There is, of course, one glaring issue with the semantics given here. The problem is that, given the way we have defined $A \overset{p.a.}{a} B$, which is in accordance with Kilwardby's comments in the condemnation of 1277, the condemnation does not tell us what we should do with particular necessity statements. Is it true to say that 'Some man is necessarily an animal' even if no animals exist? The semantics presented here presuppose that this is in fact not the case (i.e. $A \overset{p.a.}{a} B$ does not entail $A \overset{p.a.}{i} B$). This is a problem for our semantic presentation, and at this point, it is not clear how we might be able to reformulate this so as to avoid this problem. Because of these problems, when we move on to discuss Kilwardby's theory of the syllogism and its relationship to Aristotle, we will make use of \mathfrak{K}_3 models.

Logical Consequence

We have already defined a fairly standard notion of logical consequence for \mathfrak{K}_3 models. However, as we have already observed, Kilwardby further distinguishes two notions of logical consequence: *natural* consequences and *accidental* consequences. How shall we capture these two notions of necessity? We propose to do this as follows:

As before let Γ be a set of well-formed formulae and let $A \overset{\nabla}{\times} B$ be a well-formed formula where ∇ is either blank (corresponding to an assertoric proposition), *p.s.* or *p.a.* and \times is one of *a, e, i, o*.

We say that Γ naturally entails $A \overset{\nabla}{\times} B$ and write $\Gamma \models_N A \overset{\nabla}{\times} B$ if:

- 1) $\Gamma \models A \overset{\nabla}{\times} B$.

and

1. $c(A) \trianglelefteq c(B)$ if $A \times B$ is an affirmative proposition.
2. $c(A) | c(B)$ if $A \times B$ is a negative proposition.

Likewise, we say that Γ accidentally entails $A \overset{\nabla}{\times} B$ and write $\Gamma \models_A A \overset{\nabla}{\times} B$ if:
 $\Gamma \models A \overset{\nabla}{\times} B$ and $\Gamma \not\models_N A \overset{\nabla}{\times} B$.

At this point we are in a position to formulate and demonstrate some of the claims Kilwardby makes about the difference between accidental and natural consequences. As we have already seen for Kilwardby 'ex falso' type arguments are accidentally valid, but not naturally/essentially valid. At this point, there is, however, an interpretive point that we need to make. When Kilwardby rejects 'ex falso' style arguments, it seems that what he is objecting to is the lack of an essential connection between the premises and the conclusion. By these sorts of examples, we do not take Kilwardby to mean that all 'ex falso' type arguments will fail to have the requisite connections, but only that some arguments do. Read this way, all we need to do is provide a recipe for constructing a

3 Reconstructing Kilwardby's Logic

counterexample to an 'ex falso' style argument. We do not need to show that there is *no* arrangement of premises and conclusion that results in an essentially valid 'ex falso' style argument.⁶

We can now prove this easily. Let $\Gamma = \{AeA, AiA\}$.⁷

To see that $\{AeA, AiA\} \not\equiv_N BiB$, one can simply construct a countermodel.⁸ We construct the model as follows:

| | | |
|-----------------------------|------------------------------------|------------------|
| $D = \{a\}$ | $W = \{w\}$ | $R = \{(w, w)\}$ |
| $T = \{A, B\}$ | $\leq = \leq = \{(A, A), (B, B)\}$ | $ = \emptyset$ |
| $c(A) = A$ | $c(B) = B$ | |
| $v(w, A) = v(w, B) = \{a\}$ | | |

Since $c(A) \not\subseteq c(B)$, the second condition of natural consequence fails, and the argument is not naturally valid. However, such an argument is clearly accidentally valid as there are no \mathfrak{R}_7 models of $\{AeA, AiA\}$.

3.4.2 Adequacy of Our Model

As we saw on 34 of the previous chapter, Kilwardby listed a number of principles that he gives as rules for understanding when a syllogism is valid. According to Kilwardby the following rules given necessary conditions for validities in the apodictic and assertoric syllogisms, where the apodictic necessities are restricted to *per se* necessities. This is important to note, as we will only focus on *per se* necessary propositions here.:

P1 In every syllogism, one premise must be universal.

P2 In every syllogism, one premise must be affirmative.

P3 In first figure syllogisms, the major must be universal

P4 In first figure syllogisms, the minor must be affirmative

P5 In second figure syllogisms, the major must be universal.

P6 In second figure syllogisms, one of the premises must be negative.

P7 In third figure syllogisms, the minor must be affirmative.

P8 In first figure assertoric/necessity syllogisms, the necessity-proposition must be major.

P9 In second figure assertoric/necessity syllogisms, one premise must be a universal negative necessity proposition.

6. In fact, this latter constraint is not possible in our logic.

7. Since we do not have access to a conjunction operation in this language, we need to use both of these formulae to express a contradiction.

8. It could also be observed that such an argument would not be counted by Kilwardby as a valid syllogism since a valid syllogism with a negative premise must have a negative conclusion.

P10 In affirmative third figure assertoric/necessity syllogisms, the necessity premise must be a universal affirmative.

P11 In negative third figure assertoric/necessity syllogisms, the necessity premise must be a universal negative.

Can our models correctly model these conditions for syllogisms? We contend that they can, when we restrict our attention to *per se* necessary propositions, as Kilwardby does (see page 31). Properties P1-P7 are standard features of assertoric syllogisms and we will prove these in the appendix, starting on page 191. We will focus here on the modal claims. Before we move on to the proof of this, we will often appeal to the following lemma:

Let \times range over the categorical operations a, e, i or o and let ∇ range over $p.s.$, $p.a.$, and $-$. Then:

1. $A \overset{p.s.}{\times} B \models A \overset{p.a.}{\times} B$
2. $A \overset{p.a.}{\times} B \models A \times B$
3. $A \overset{p.s.}{a} B \models A \overset{\nabla}{i} B$
4. $A \overset{p.s.}{e} B \models A \overset{\nabla}{o} B$
5. $A \overset{p.a.}{a} B \models A \overset{p.a.}{i} B$
6. $A \overset{p.a.}{e} B \models A \overset{p.a.}{o} B$
7. $A \overset{p.a.}{a} B \models A i B$
8. $A \overset{p.a.}{e} B \models A o B$
9. $A a B \models A i B$
10. $A e B \models A o B$
11. $A \overset{p.s.}{e} B \models A \overset{\nabla}{e} B$
12. $A \overset{\nabla}{i} B \models A \overset{\nabla}{i} B$

The proof of these claims is routine and follow from what we show on page 194 of the Appendix on Kilwardby's logic.

A triple, S , is a pair of premises, $\langle M, m \rangle$ from which a conclusion C can be drawn. We require that M , m , and C are well-formed formulae and satisfy the following conditions:

1. M, m , and C are all categorical propositions;
2. M, m , and C have exactly three terms;

3 Reconstructing Kilwardby's Logic

3. The predicate of C occurs in M ;
4. The subject of C occurs in m ;
5. M and m share a common term that does not occur in C .⁹

It should be observed that the definition of S given here is entirely syntactic. In what follows, a triple will count as a syllogism even if one or both of the premises in the syllogism are false.

Syllogistic Validity. *A triple S is valid when the following obtains:
if $\mathfrak{K}_3, w \models M$ and $\mathfrak{K}_3, w \models m$ then $\mathfrak{K}_3, w \models C$.*

We will denote this by $\models S$. A valid triple is called a syllogism.

We prove P8-P11 as follows:

P8: If S is a valid assertoric/necessity syllogism in the first figure, then the necessity-proposition must be major.

Proof: We prove the contrapositive of this claim and restrict ourselves to the cases where the syllogism's assertoric counterparts are valid. For XLL syllogisms, we need to verify that the following are not valid:

| | | | |
|-------------------------|-------------------------|-------------------------|-------------------------|
| $B a C$ | $B e C$ | $B a C$ | $B e C$ |
| $A \overset{p.s.}{a} B$ | $A \overset{p.s.}{a} B$ | $A \overset{p.s.}{i} B$ | $A \overset{p.s.}{i} B$ |
| $A \overset{p.s.}{a} C$ | $A \overset{p.s.}{e} C$ | $A \overset{p.s.}{i} C$ | $A \overset{p.s.}{o} C$ |

Consider the following countermodels:

For XLL Barbara

| | | |
|-------------------|---|---|
| $D = \{a, b\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (A, B), (B, C), (A, C)\}$ | $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a, b\}$ | $v(w, B) = \{a\}$ |
| $v(w, C) = \{a\}$ | | |

Call this model \mathfrak{D} . Now, since $B \leq C$ it follows that $\mathfrak{D}, w \models BaC$ and since $A \leq B$ it follows that $\mathfrak{D}, w \models A \overset{p.s.}{a} B$. However, $\mathfrak{D}, w \not\models A \overset{p.s.}{a} C$ since $A \not\leq C$, thus showing XLL Barbara to be invalid.

9. This definition is based on the one found in [65].

For XLL Darii:

| | | |
|----------------------|---|---|
| $D = \{a, b\}$ | $W = \{w\}$ | $T = \{A, B, C, D\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C, c(D) = D$ | $\leq = \{(A, A), (B, B), (C, C), (D, D), (D, A), (D, B), (D, C), (B, C)\}$ | $\leq = \{(A, A), (B, B), (C, C), (D, D), (D, B)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a, b\}$ | $v(w, B) = \{a\}$ |
| $v(w, C) = \{a\}$ | | |

Call this model \mathfrak{D} . Again, observe that $\mathfrak{D}, w \models BaC$ since $B \leq C$. Since $D \leq B$ and $D \leq A$, it follows that $\mathfrak{D}, w \models A \overset{p.s.}{i} B$. To see that $\mathfrak{D}, w \not\models A \overset{p.s.}{i} C$, need to show that there is no $D \in T$ s.t. $D \leq A$ and $D \leq C$ or $D \leq C$ and $D \leq A$. For the left disjunct, observe that only $C \leq C$, and $C \not\leq A$. For the right disjunct, observe that only $A \leq A$ and that $A \not\leq C$. Hence $\mathfrak{D}, w \not\models A \overset{p.s.}{i} C$.

For XLL Celarent and Ferio:

| | | |
|-------------------|---|---|
| $D = \{a\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ | $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a\}$ | $v(w, B) = \{a\}$ |
| $v(w, C) = \{b\}$ | | |

Call this model \mathfrak{D} . For XLL Celarent observe that $v(w, B) \cap v(w, C) = \emptyset$ and so, $\mathfrak{D}, w \models BeC$. Likewise, since $A \leq B$, it follows that $\mathfrak{D}, w \models A \overset{p.s.}{a} B$. However, we do not have $A|C$, and so, $\mathfrak{D}, w \not\models A \overset{p.s.}{e} C$.

For XLL Ferio, again, we have $\mathfrak{D}, w \models BeC$. Likewise, since $A \leq A$ and $A \leq B$ we have $\mathfrak{D}, w \models A \overset{p.s.}{i} B$. However, since only $A \leq A$ and we do not have $A|C$ it follows that $\mathfrak{D}, w \not\models A \overset{p.s.}{o} C$.

That no XXL syllogism is valid in the first figure is fairly obvious. Observe that in each case simply let $| = \emptyset$ and $\leq = (A, A), (B, B), (C, C)$ and ensure that the model verifies the assertoric propositions. Then the \leq condition of *per se* propositions will fail in each case.

Conversely, we show that the following syllogisms are valid:

LXL Barbara, LXL Celarent, LXL Darii, and LXL Ferio.

Proof of LXL Barbara:

Let \mathfrak{K}_3 be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_3, w \models B \overset{p.s.}{a} C$ and that $\mathfrak{K}_3, w \models A a B$. From this it follows that $c(B) \leq c(C)$ and that $c(A) \leq c(B)$. From order property 2 it follows that $c(A) \leq c(C)$ and hence $\mathfrak{K}_3, w \models A \overset{p.s.}{a} C$ as required.

Proof of LXL Celarent:

3 Reconstructing Kilwardby's Logic

Let \mathfrak{K}_J be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_J, w \models B \overset{p.s.}{e} C$ and that $\mathfrak{K}_J, w \models A a B$. Hence, $c(B)|c(C)$ and $c(A) \leq c(B)$ by order property 3 from which it follows that $c(A)|c(C)$, and so $\mathfrak{K}_J, w \models A \overset{p.s.}{e} C$ as desired.

Proof of LXL Darii:

Let \mathfrak{K}_J be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_J, w \models B \overset{p.s.}{a} C$ and that $\mathfrak{K}_J, w \models A i B$. From this it follows that $c(B) \sqsubseteq c(C)$ and that $\exists D \in T \ D \leq c(A)$ and $D \leq c(B)$. From this it follows by order property 2 that $c(D) \sqsubseteq c(C)$ and it follows by basic logic that $\mathfrak{K}_J, w \models A \overset{p.s.}{i} C$ as required.

Proof of LXL Ferio:

Let \mathfrak{K}_J be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_J, w \models B \overset{p.s.}{e} C$ and that $\mathfrak{K}_J, w \models A i B$. Hence, $c(B)|c(C)$ and there is some $D \in T$ such that $D \leq c(A)$ and $D \leq c(B)$. Hence by order property 3 it follows that $D|c(C)$, and so $\mathfrak{K}_J, w \models A \overset{p.s.}{o} C$ as desired.

P9 In second figure assertoric/necessity syllogisms, one premise must be a universal negative necessity proposition.

Based on this we need to check that the following second figure syllogisms are valid:

LXL Cesare:

Let \mathfrak{K}_J be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_J, w \models C \overset{p.s.}{e} B$ and that $\mathfrak{K}_J, w \models A \overset{p.s.}{a} B$. Then it follows that $c(C)|c(B)$ and that $c(A) \sqsubseteq c(B)$. Since $|$ is symmetric, it follows that $c(B)|c(C)$. Likewise, by order property 1 $c(A) \sqsubseteq c(B)$ entails that $c(A) \leq c(B)$. Hence by order property 4 it follows that $c(A)|c(C)$. Hence $\mathfrak{K}_J, w \models A \overset{p.s.}{e} C$ as desired.

XLL Camestres:

Let \mathfrak{K}_J be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_J, w \models C \overset{p.s.}{a} B$ and that $\mathfrak{K}_J, w \models A \overset{p.s.}{e} B$. From this it follows that $c(C) \sqsubseteq c(B)$ and that $c(A)|c(B)$. Since $|$ is symmetric, it follows that $c(B)|c(A)$. By order property 1, $c(C) \sqsubseteq c(B)$ entails $c(C) \leq c(B)$. By order property 4, it follows that $c(C)|c(A)$, which entails, $c(A)|c(C)$. Hence $\mathfrak{K}_J, w \models A \overset{p.s.}{e} C$ as desired.

LXL Festino:

Let \mathfrak{K}_J be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_J, w \models C \overset{p.s.}{e} B$ and that $\mathfrak{K}_J, w \models A \overset{p.s.}{i} B$. Then it follows that $c(C)|c(B)$ and that for some D either $D \leq c(A)$ and $D \sqsubseteq c(B)$ or $D \leq c(B)$ and $D \sqsubseteq c(A)$. We need to show that there is some D such that $D \leq c(A)$ and $D|C$. For the left disjunct, assume that $D \leq c(A)$ and $D \sqsubseteq c(B)$. By order property 1, it follows that $D \leq c(B)$. By symmetry, $c(C)|c(B)$ entails $c(B)|c(C)$. By order property 4 it follows that $D|c(C)$ as desired.

For the right disjunct, assume that $D \leq c(B)$ and $D \triangleleft c(A)$. $D \leq c(B)$ together with $c(B)|c(C)$ entails that $D|(C)$. By order property 1, it follows that $D \leq c(A)$ as desired.

Hence, in both cases we have shown that there is some D such that $D \leq c(A)$ and that $D|(C)$ and so $\mathfrak{K}_3, w \models A \overset{p.s.}{o} C$.

To see that these are the only valid syllogisms in the second figure, we need to show that the following syllogisms are invalid:

LXL Camestres, LXL Baroco, XXL Cesare, and XXL Festino.

Countermodel for LXL Camestres and LXL Baroco:

| | | |
|-------------------|---|--|
| $D = \{a, b, c\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (C, B), (C, A)\}$ | $\triangleleft = \{(A, A), (B, B), (C, C), (C, B)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a\}$ | $v(w, B) = \{b, c\}$ |
| $v(w, C) = \{b\}$ | | |

Call this model \mathfrak{D} . In this model, observe that $C \triangleleft B$. Hence $\mathfrak{D}, w \models C \overset{p.s.}{a} B$. Likewise, observe that for no $D \in T$ do we have $D \leq A$ and $D \leq B$, since $A \not\leq B$, $B \not\leq A$, and $C \not\leq A$. Hence $\mathfrak{M}, w \models A e B$. However, since $|$ is empty, we have $\mathfrak{M}, w \not\models A \overset{p.s.}{e} C$ and $\mathfrak{M}, w \not\models A \overset{p.s.}{o} C$.

For LXL Baroco, observe that $A \not\leq B$ and so $\mathfrak{M}, w \models C o B$. We have already shown that the conclusion does not follow.

Countermodel for XXL Cesare and XXL Festino:

| | | |
|-------------------|---|--|
| $D = \{a, b, c\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ | $\triangleleft = \{(A, A), (B, B), (C, C), (A, B)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a\}$ | $v(w, B) = \{a, b\}$ |
| $v(w, C) = \{c\}$ | | |

In this model observe that for no $D \in T$ is $D \leq B$ and $D \leq C$, since $A \not\leq C$, $B \not\leq C$ and $C \not\leq B$. Hence $\mathfrak{M}, w \models C e B$. Likewise, observe that $A \triangleleft B$. Hence $\mathfrak{M}, w \models A \overset{p.s.}{a} B$. However, $\mathfrak{M}, w \not\models A \overset{p.s.}{e} C$ and $\mathfrak{M}, w \not\models A \overset{p.s.}{o} C$ since $|$ is empty.

For XXL Festino, observe that $A \triangleleft B$ and that $A \triangleleft A$. Hence $\mathfrak{M}, w \models C \overset{p.s.}{i} B$. Likewise, since $|$ is empty, it follows that $\mathfrak{M}, w \not\models C \overset{p.s.}{o} B$. We have already shown that the conclusion does not follow in this model.

Moving to syllogisms in the third figure, we need to prove P10 and P11. Recall that P10 states:

P10 In affirmative third figure assertoric/necessity syllogisms, the necessity premise must be a universal affirmative.

3 Reconstructing Kilwardby's Logic

Based on this we need to show that LXL Datisi and XLL Disamis are valid.

For LXL Datisi: Let \mathfrak{K}_3 be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_3, w \models B \overset{p.s.}{a} C$ and $\mathfrak{K}_3, w \models B i A$. Then we have $c(B) \triangleleft c(C)$ and there is some D such that $D \leq c(B)$ and $D \leq c(A)$ or $D \leq c(A)$ and $D \leq c(B)$. We will show that there is some D such that $D \leq c(A)$ and $D \triangleleft c(C)$. The first conjunct follows from our second assumption. The second conjunct follows from $D \leq c(B)$ and $c(B) \triangleleft c(C)$ by the order property 2. Hence some D such that $D \leq c(A)$ and $D \triangleleft c(C)$ and it follows by basic logic that $\mathfrak{K}_3, w \models A \overset{p.s.}{i} C$.

For XLL Disamis:

Let \mathfrak{K}_3 be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_3, w \models B i C$ and $\mathfrak{K}_3, w \models B \overset{p.s.}{a} A$. Hence 1) there is some D such that either $D \leq c(B)$ and $D \leq c(A)$ or $D \leq c(B)$ and $D \leq c(A)$ and 2) $c(B) \triangleleft c(A)$. By order property 2 $D \leq c(B)$ and $c(B) \triangleleft c(A)$ entail that $D \triangleleft c(C)$. This, together with $D \leq c(A)$ and some basic logic entails that $\mathfrak{K}_3, w \models A \overset{p.s.}{i} C$.

We will also show that LXL Disamis and XLL Datisi are invalid.

For LXL Disamis :

| | | |
|-------------------|---|------------------------------|
| $D = \{a, b, c\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (B, A), (C, B), (C, A)\}$ | $\triangleleft = \{(B, C)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a, b, c\}$ | $v(w, B) = \{b, c\}$ |
| $v(w, C) = \{c\}$ | | |

In this model, observe that $B \leq B$ and $B \triangleleft C$. Hence $\mathfrak{K}_3, w \models B \overset{p.s.}{i} C$. Likewise, since $B \leq A$ it follows that $\mathfrak{K}_3, w \models B a A$. In order to show that $\mathfrak{K}_3, w \not\models A \overset{p.s.}{i} C$ it suffices to observe that: $A \not\leq C$, $B \not\leq A$, $C \not\leq A$, $A \not\leq C$, $B \not\leq C$, and $C \not\leq C$.

For XLL Datisi:

| | | |
|----------------------|---|------------------------------|
| $D = \{a, b, c\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (B, C), (A, B), (A, C)\}$ | $\triangleleft = \{(A, B)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a, b\}$ | $v(w, B) = \{b\}$ |
| $v(w, C) = \{b, c\}$ | | |

In this model, observe that $B \leq C$. Hence $\mathfrak{K}_3, w \models B a C$. Likewise, since $B \leq B$ and $B \triangleleft A$ it follows that $\mathfrak{K}_3, w \models B \overset{p.s.}{i} A$. However, in order to show that $\mathfrak{K}_3, w \not\models A \overset{p.s.}{i} C$ it suffices to observe that: $A \not\leq C$, $B \not\leq C$, $C \not\leq C$, $A \not\leq A$, $B \not\leq A$, and $C \not\leq A$.

P11 In negative third figure assertoric/necessity syllogisms, the necessity premise must be a universal negative.

Based on this we need to show that LXL Ferison is valid.

LXL Ferison:

Let \mathfrak{K}_3 be an arbitrary model and w a point in W . Assume that $\mathfrak{K}_3, w \models B \overset{p.s.}{e} C$ and

$\mathfrak{K}_3, w \vDash B \text{ i } A$. Based on this we have 1) $c(B)|c(C)$ and 2) there is some D such that $D \leq c(B)$ and $D \leq c(A)$. We will show that there is some D such that $D \leq c(A)$ and that $D|c(C)$. The first conjunct follows from 2). For the second conjunct, observe that $D \leq c(B)$ and $c(B)|c(C)$ entails $D|c(C)$ by order property 4. Hence, $\mathfrak{K}_3, w \vDash B \overset{p.s.}{o} C$.

We will also show that LXL Bocardo, XLL Bocardo, XLL Felapton and XLL Ferison are invalid.

For the three XLL syllogisms, consider the following countermodel:

| | | |
|----------------------|---|--------------------------------|
| $D = \{a, b, c\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (A, C), (B, A), \}$ | $\trianglelefteq = \{(B, A)\}$ |
| $ = \emptyset$ | $v(w, A) = \{a, b\}$ | $v(w, B) = \{b\}$ |
| $v(w, C) = \{b, c\}$ | | |

First, observe that in this model, $\mathcal{M}, w \not\vDash A \overset{p.s.}{o} C$ since $|$ is empty.

For XLL Felapton, observe that there is no D such that $D \leq B$ and $D \leq C$, since $A \not\leq B$, $B \not\leq C$ and $C \not\leq B$. For this it follows that $\mathcal{M}, w \vDash BeC$. Likewise, since $B \trianglelefteq A$ it follows that $\mathcal{M}, w \not\vDash B \overset{p.s.}{a} A$.

For XLL Ferison, first observe that $\mathcal{M}, w \vDash BeC$, as we proved in the case of XLL Felapton. Likewise, since $B \trianglelefteq A$ and $B \leq B$, it follows that there is some D such that $D \leq B$ and $D \leq A$. From this it follows that $\mathcal{M}, w \not\vDash B \overset{p.s.}{i} A$.

For XLL Bocardo first observe that $\mathcal{M}, w \vDash B \overset{p.s.}{a} A$, as we proved in the case of XLL Felapton. Likewise, observe that $B \not\leq C$ and so, $\mathcal{M}, w \vDash BoC$.

For LXL Bocardo consider the following countermodel:

| | | |
|------------------------------------|---|--------------------------------|
| $D = \{a, b, c\}$ | $W = \{w\}$ | $T = \{A, B, C\}$ |
| $R = W^2$ | $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | $\leq = \{(A, A), (B, B), (C, C), (B, A), \}$ | $\trianglelefteq = \{(B, A)\}$ |
| $ = \{(B, C), (C, B)\} \emptyset$ | $v(w, A) = \{a, b\}$ | $v(w, B) = \{b\}$ |
| $v(w, C) = \{c\}$ | | |

In this model, observe that since $B \leq A$, it follows that $\mathcal{M}, w \vDash B \overset{p.s.}{a} A$. Likewise, since $B|C$ and $B \leq B$, it follows that $\mathcal{M}, w \vDash B \overset{p.s.}{o} C$. To show that $\mathcal{M}, w \not\vDash A \overset{p.s.}{o} C$, it suffices to observe that $B \not\leq C$, not $A|C$ and not $C|A$.

This completes our treatment of the modal syllogisms that Kilwardby outlines in his treatment of Aristotle's modal logic. There is an interesting implication of our analysis that we should mention at this point. In [37, p.117] Malink provides a standard table that tracks all of the LXL, LLL, XLL, and XXX syllogisms that Aristotle claims are valid as well as references to where those validities are affirmed in the *Prior Analytics*. We reproduce a modified version of the table in Figure 2.1, omitting the numbers in square brackets (which are references to theorems in Malink's paper), stating the syllogisms using their medieval mnemonic names, and using L where Malink uses N. Valid syllogisms are in bold font while invalid syllogisms are in italics. If the square is blank,

3 Reconstructing Kilwardby's Logic

| 1 st figure | XXX | LLL | LXL | XLL |
|------------------------|--------------|--------------|--------------|--------------|
| Barbara | 25b37 | 29b36 | 30a17 | <i>30a23</i> |
| Celarent | 25b40 | 29b36 | 30a17 | <i>30a23</i> |
| Darii | 26a23 | 29b36 | 30a17 | <i>30b2</i> |
| Ferio | 26a25 | 29b36 | 30b1 | <i>30b5</i> |
| 2 nd Figure | | | | |
| Cesare | 27a5 | 29b36 | 30b9 | <i>30b18</i> |
| Camestres | 27a9 | 29b36 | <i>30b20</i> | 30b14 |
| Festino | 27a32 | 29b36 | 31a5 | |
| Baroco | 27a36 | 30a6 | <i>30a10</i> | <i>31a15</i> |
| 3 rd Figure | | | | |
| Darapti | 28a17 | 29b36 | 31a24 | 31a31 |
| Felapton | 28a26 | 29b36 | 31a33 | <i>31a37</i> |
| Disamis | 28b7 | 29b36 | <i>31b31</i> | 31b12 |
| Datisi | 28b11 | 29b36 | 31b19 | <i>31b20</i> |
| Bocardo | 28b15 | 30a7 | <i>32a4</i> | <i>31b40</i> |
| Ferison | 28b31 | 29b36 | 31b33 | <i>32a1</i> |

Figure 3.1: Apodictic Syllogisms in Aristotle

Aristotle does not say if the syllogism is valid or not.

As the reader can verify from what we have shown above, our reconstruction of Kilwardby correctly tracks the validities in Aristotle's modal logic. In some sense this is not surprising, as there are a number of close connections between the formalisation of Kilwardby developed here, and the formalisation that Malink uses for Aristotle's modal logic. The main difference, which we will not dwell on here, is that we make explicit use of an operation to define definitional incomparability, while Malink prefers to handle this in the semantic definitions of his formulae.

3.5 Connexive Implication and Natural Consequence

There is one final question that we will discuss and it also emerges from Kilwardby's discussion of logical consequence. As we saw in the previous chapter¹⁰, Kilwardby provides a characterisation of natural inferences that makes use of logical connectives other than the categorical operations a, e, i and o. In particular, Kilwardby discusses implication and disjunction. The main results of Kilwardby's theory can be summarised as follows:

1. The natural implication relationship validates Aristotle's Thesis and Abelard's Thesis.
2. Disjunction introduction is a natural inference.
3. The natural implication relationship does not validate *ex impossibile quodlibet*.

10. See chapter 1, section 3, subsection 1

3.5 Connexive Implication and Natural Consequence

We treat each of the first two claims in some detail.

In Chapter One, we argued that Kilwardby is committed to Aristotle's Thesis based on textual analysis of Kilwardby's commentary, the passage of the *Prior Analytics* and some analogies between Kilwardby's views and Abelard's. We also gave explicit textual references showing Kilwardby's commitment to Abelard's Thesis.¹¹

In the case of the principle that "Disjunction introduction is a natural inference", the following passage in Kilwardby can be cited:

Moreover, a disjunction follows from each of its parts in a natural consequence. Hence it follows that: 'if you are sitting, then you are sitting or you are not sitting'; and 'if you are not sitting, then you are sitting or not sitting'. And in the same way, in a natural consequence, the same follows from its being and not being and similarly of necessity.[29, p.760]¹²

Given what we have seen in our model, we can motivate this sort of implication as a generalisation of the notion of meaning containment. Just as *per se* necessary propositions capture the idea that the meaning of one term is contained in the other, the more generalised idea is that the meaning of the consequent should be contained in the antecedent. Based on this, the failure of *ex falso* should generalise from the usual syllogistic case. Inferences like disjunction introduction (and dually, as we shall see, conjunction elimination) preserve this notion of meaning containment, since in each case the antecedent 'contains in it' the meaning relationship necessary to ground the truth of the consequent.

This leaves the connexive nature of this implication to be discussed. At the heart of this is the idea that no true proposition contains in its definitions anything that is inconsistent with it. In the framework we are working with here, this idea becomes the notion that the definition of any term does not contain the negation of that term, or taken from the object side, for no term, if an object satisfies the definition of that term, does it then fail to satisfy the definition of that term.¹³

Formally, there are some interesting challenges present. Given the framework that Kilwardby is working in, it is natural to think of his logic as term-based logic augmented with the operations of \wedge (and), \vee (or), \neg (not) and \rightarrow (implies), generalised to allow the modal operations of *per se* necessity and *per accidens* necessity (which we will here denote as \boxplus and \boxminus respectively) to range over any of these propositions. In some sense, we could think of this as a propositional modal logic, where the atomic terms just happen to be categorical propositions. There are some technical challenges with this approach. Instead, we will settle for a more modest proposal. We will have to require that our

11. See 21 and 19.

12. Adhuc disiunctiua sequitur ad utrumque sui partem, et hoc naturali consequentia. Quare sequitur si tu sedes, tu sedes vel tu non sedes; et si tu non sedes, tu sedes vel non sedes. Et ita naturali consequentia sequitur idem ad idem esse et non esse et ita ex necessitate. [24, ad B2 Dub 4 60ra].

13. It should be noted that one very natural class of terms that *do* seem to violate this condition are liar-like properties, e.g., the membership condition of the Russell set.

3 Reconstructing Kilwardby's Logic

well-formed formulae be restricted in particular ways to make this idea precise. As such, we will call this a quasi first-order logic.

The other interesting challenge is to figure out exactly how we should think about meaning containment in some of the propositional connectives. For example, when is $\boxplus(\phi \wedge \psi)$ true? One very natural way forward involves treating \wedge in terms similar to how particular affirmative propositions function. Then we would say that a conjunction is *per se* just in case the truth of each of the conjuncts is guaranteed by a *per se* connection (i.e. we would want to say that either $\phi \sqsubseteq \psi$ or $\psi \sqsubseteq \phi$, and further require that both propositions be true.) As we shall see, \rightarrow will be similar to the notion of conceptual containment and mirror the truth conditions for $\boxplus AaB$.

The interesting challenge here concerns the notion of negation that we want to work with. What we want to do is use the relation $|$ to define a notion of negation that captures the idea of a proposition being *per se* impossible. This is not a notion that Kilwardby treats, but we can think of it as saying that a concept or idea is *per se* impossible if there is something incompatible about the way that the concept expressed by the proposition has been defined.

Quasi First-Order Logic 1.

Let $\mathcal{L}_{QFOL} = \{Term, a, e, i, o, \square, \boxplus, \wedge, \vee, \rightarrow, \neg, \Rightarrow\}$

The definition of well-formed formulae of \mathcal{L}_{FOL} is a natural generalisation of what we have already seen:

Let $A, B \in Term$, and ϕ, ψ be well-formed formulae.

$A \times B$ is well-formed, when \times is one of a, e, i , or o .

$\neg\phi, \phi \wedge \psi, \phi \vee \psi, \phi \rightarrow \psi, \square\phi$, and $\boxplus\phi$ are well-formed formulae.

Nothing else is a well-formed formula.

We shall denote the well-formed formulae of a language, \mathcal{L} as $WFF(\mathcal{L})$. If ϕ is of the form $A \times B$, $\square A \times B$, or $\boxplus A \times B$, then we say that ϕ is atomic.

Our Kilwardby Models for this language will be similar to the previous models we considered. The only thing that will need to change is how we assign truth conditions to newly added operations.

Kilwardby Models for FOL. Let $\mathcal{L} = \{D, W, T, R, \sqsubseteq, |, c, v\}$ where

D, W , and R are non-empty sets. (Informally, D is our Domain, W is a set of worlds, and T is a set of interpretations of terms or predicates)

$R \subseteq W^2$.

$\sqsubseteq \subseteq T^2, | \subseteq T^2, \leq \subseteq T^2$.

$c : Terms \rightarrow T$.¹⁴

14. For clarity, it should be observed that c is *not* the contradictory operator, but an operation that assigns to each term in *Terms* its interpretation in T .

3.5 Connexive Implication and Natural Consequence

$$v : T \times W \rightarrow \mathcal{P}D.$$

Again, we require that the following properties hold:

1. $\forall x, y$ if $x \sqsubseteq y$ then $x \leq y$.
2. $\forall x, y, z$ if $x \leq y$ and $y \sqsubseteq z$ then $x \sqsubseteq z$.
3. $\forall x, y$ if $x \leq y$ then not $x|y$
4. $\forall x, y, z$ if $x \leq y$ and $y|z$ then $x|z$.

We also require that our domains be related in the following way: In order to bring these two families of structures together we need the following principles:

1. For all terms, A, B $c(A) \leq c(B)$ if and only if for some $w \in W$ $v(w, A) \subseteq v(w, B)$.
2. For all terms, A, B if $c(A) \sqsubseteq c(B)$ then for all $w \in W$ $v(w, A) \subseteq v(w, B)$.
3. For all terms, A, B if $c(A)|c(B)$ then for all $w \in W$ $v(w, A) \cap v(w, B) = \emptyset$

With the exception of *per se* necessity, truth conditions for the logical operations are defined as usual:

| | | |
|--|----------------|---|
| $\mathfrak{L}, w \models A \times B$ | if and only if | as in our treatment of the modal syllogism. |
| $\mathfrak{L}, w \models A \overset{p.a.}{\times} B$ | if and only if | as in our treatment of the modal syllogism. |
| $\mathfrak{L}, w \models A \overset{p.s.}{\times} B$ | if and only if | as in our treatment of the modal syllogism. |
| $\mathfrak{L}, w \models \neg\phi$ | if and only if | if $\mathfrak{L}, w \not\models \phi$ |
| $\mathfrak{L}, w \models \phi \wedge \psi$ | if and only if | $\mathfrak{L}, w \models \phi$ and $\mathfrak{L}, w \models \psi$ |
| $\mathfrak{L}, w \models \phi \vee \psi$ | if and only if | $\mathfrak{L}, w \models \phi$ or $\mathfrak{L}, w \models \psi$ |
| $\mathfrak{L}, w \models \phi \rightarrow \psi$ | if and only if | if $\mathfrak{L}, w \models \phi$ then $\mathfrak{L}, w \models \psi$ |
| $\mathfrak{L}, w \models \Box\phi$ | if and only if | for all v if wRv then $\mathfrak{L}, v \models \phi$ |

In order to define how the *per se* operations work, we will make use of the following:

Let $Ref : WFF_{\mathfrak{L}_K} \rightarrow \mathcal{P}(T)$ governed by the following conditions:

1. $Ref(A \overset{\nabla}{\times} B) = \{A, B\}$
2. $Ref(\phi \vee \psi) = Ref(\phi) \cup Ref(\psi)$
3. $Ref(\phi \wedge \psi) = Ref(\phi) \cap Ref(\psi)$
4. $Ref(\phi \rightarrow \psi) = \{A \in Ref(\phi) : \exists B \in Ref(\psi) A \sqsubseteq B\}$
5. $Ref(\neg\phi) = \{A \in T : A|Ref(\phi)\}$
6. $Ref(\boxplus(\phi)) = Ref(\Box\phi) = Ref(\phi)$

3 Reconstructing Kilwardby's Logic

Informally, we can think of Ref as the operation that collects the various terms that occur in formulae.

Finally, we generalise \sqsubseteq and $|$ as follows:

$\phi \sqsubseteq \psi$ if and only if $\forall A \in Ref(\phi) \exists B \in Ref(\psi)$ such that $A \sqsubseteq B$.
 $\phi | \psi$ if and only if $\forall A \in Ref(\phi) \forall B \in Ref(\psi) A | B$.

The concept of ϕ is contained in ψ just in case the content of ϕ jointly expresses the content of ψ . Likewise, ϕ and ψ are incompatible (we might say strongly incompatible) if none of their content is consistent.

The idea in each case, will then be to use the operation \boxplus to change the truth conditions under which the main connective of the formula is evaluated. In the cases with categorical formulae, it transforms them into *per se* formulae if they are not, and leaves them unchanged if they are already *per se*. In the case of the new operations, we use the Ref clauses to give the relevant truth conditions.

$$\begin{array}{ll}
 \mathfrak{L}, w \models \boxplus A \times^{p.s.} B & \text{if and only if} & \mathfrak{L}, w \models A \times^{p.s.} B \\
 \mathfrak{L}, w \models \boxplus A \times^{p.a.} B & \text{if and only if} & \mathfrak{L}, w \models A \times^{p.s.} B \\
 \mathfrak{L}, w \models \boxplus A \times B & \text{if and only if} & \mathfrak{L}, w \models A \times^{p.s.} B \\
 \mathfrak{L}, w \models \boxplus \neg \phi & \text{if and only if} & \neg \exists A \in Ref(\phi). \\
 \mathfrak{L}, w \models \boxplus (\phi \rightarrow \psi) & \text{if and only if} & Ref(\phi) \sqsubseteq Ref(\psi) \\
 \mathfrak{L}, w \models \boxplus (\phi \wedge \psi) & \text{if and only if} & \mathfrak{L}, w \models \boxplus \phi \text{ and } \mathfrak{L}, w \models \boxplus \psi \\
 \mathfrak{L}, w \models \boxplus (\phi \vee \psi) & \text{if and only if} & \mathfrak{L}, w \models \boxplus \phi \text{ or } \mathfrak{L}, w \models \boxplus \psi
 \end{array}$$

The only clause that should require some comment is our definition of $\boxplus \neg \phi$. Here, the idea is that $\boxplus \neg$ tells us that the formula that follows it is '*per se* false'. The key idea to observe is that Ref will be empty just in case there is some formula ϕ such that $A \in Ref(\phi)$ and $A \in Ref(\neg \phi)$, i.e. the term A occurs in both ϕ and its negation. This leads to a *per se impossibility*, because $A | A$ is always false.

With these in place, we now need to explain how we formalise *natural* and *accidental* consequences. As before, we start with the standard account of logical consequence. We say that:

$\Gamma \models \phi$ if for all models, \mathfrak{L} and all $w \in W$, if $\mathfrak{L}, w \models \Gamma$ then $\mathfrak{L}, w \models \phi$.

ϕ is a *natural* consequence of Γ (denoted by $\Gamma \models_N \phi$) if whenever $\mathfrak{L}, w \models \boxplus \gamma_i$ for all $\gamma_i \in \Gamma$, $\mathfrak{L}, w \models \boxplus \phi$.

ϕ is an *accidental* consequence of Γ (denoted by $\Gamma \models_A \phi$) if $\Gamma \models \phi$ but $\Gamma \not\models_N \phi$.

The following lemma will be helpful:

Incompatibility Lemma For no $A \in T$ do we have $A \in Ref(\phi)$ and $A \in Ref(\neg \phi)$

Proof: Assume not for a contradiction, so that for some A , $A \in Ref(\phi)$ and $A \in Ref(\neg \phi)$. This holds if and only if $A \in Ref(\phi)$ and $A | Ref(\phi)$. Then $A | A$, by the requirement of strong incompatibility. But this is impossible as $|$ is irreflexive.

We claim that our account of natural consequences can prove the main features of Kilwardby's connexive logic. It can also prove the following:

Aristotle's Thesis We claim that $\models_N \neg(\neg\phi \rightarrow \phi)$: this holds, if and only if $\models \boxplus \neg(\neg\phi \rightarrow \phi)$. We will show that $\neg\exists A \in Ref(\neg\phi \rightarrow \phi)$. We will show that $Ref(\neg\phi \rightarrow \phi) = \emptyset$. To see this, observe that $Ref(\neg\phi \rightarrow \phi) = \{A \in Ref(\neg\phi) : \exists B \in Ref(\phi), \text{ and } A \sqsubseteq B\}$. Now, assume for a contradiction that $\{A \in Ref(\neg\phi) : \exists B \in Ref(\phi), \text{ and } A \sqsubseteq B\}$ is non-empty and call the witness of this C . Then $C \in Ref(\neg\phi)$, and there is some B (which will call D) such that $D \in Ref(\phi)$ and $C \sqsubseteq D$. It then follows by the definition of Ref that $C|Ref(\phi)$ and so $C|D$. However, since $C \sqsubseteq D$, it follows that $C \leq D$ and so not $C|D$ which is a contradiction. Hence $Ref(\neg\phi \rightarrow \phi) = \emptyset$ and so $\neg\exists A \in Ref(\neg\phi \rightarrow \phi)$.

Abelard's Thesis We claim that $\models_N \neg((\phi \rightarrow \neg\psi) \wedge (\phi \rightarrow \psi))$. It suffices to show that $Ref(\phi \rightarrow \neg\psi) \cap Ref(\phi \rightarrow \psi) = \emptyset$. Assume not. Then $\exists B$ s.t. $B \in Ref(\phi \rightarrow \neg\psi) \cap Ref(\phi \rightarrow \psi)$ i.e. the following both hold:

1. $B \in Ref(\phi)$ and $\exists C$ such that $B \sqsubseteq C$ and $C \in Ref(\psi)$
2. $B \in Ref(\phi)$ and $\exists C$ such that $B \sqsubseteq C$ and $C \in Ref(\neg\psi)$

Let D be a witness to 1 and E be a witness to 2. Then by 1. we have $B \sqsubseteq D$ and $D \in Ref(\psi)$. By 2. we have $B \sqsubseteq E$ and $E \in Ref(\neg\psi)$. By the definition of $Ref(\neg\psi)$ it follows that $E|Ref(\psi)$ and so $E|D$ by the definition of $|$. Since we have $B \sqsubseteq E$ we also have $B \leq E$ by order property 1. and so $B|D$ by order property 4. However, $B \sqsubseteq D$, and so $B \leq D$ by order property 4. and so not $B|D$ by order property 3. This is a contradiction. Hence $Ref(\phi \rightarrow \neg\psi) \cap Ref(\phi \rightarrow \psi) = \emptyset$.

Boethius' Thesis We claim that $\models_N ((\phi \rightarrow \psi) \rightarrow \neg(\phi \rightarrow \neg\psi))$. This holds if and only if For all A , if $A \in Ref(\phi \rightarrow \psi)$ then $\exists B$ such that $B \in Ref(\neg(\phi \rightarrow \neg\psi))$ and $A \sqsubseteq B$.

Take an arbitrary term, C and assume that $C \in Ref(\phi \rightarrow \psi)$. Then $C \in Ref(\phi)$ and $\exists D$ such that $D \in Ref(\psi)$ and $C \sqsubseteq D$. Now, we claim that $D \in Ref(\neg(\phi \rightarrow \neg\psi))$. It suffices to show that $D|Ref(\phi \rightarrow \neg\psi)$. So, take an arbitrary E such that $E \in Ref(\phi \rightarrow \neg\psi)$. It then follows that $\exists B$ such that $E \sqsubseteq B$ and $B|Ref(\psi)$. Call this F . Then $E \sqsubseteq F$ and $F|Ref(\psi)$. From the second conjunct it follows by the definition of $|$ that $F|D$. Hence by order properties 1. and 4. it follows that $D|E$. As E was arbitrary, this holds for all $E \in Ref(\phi \rightarrow \neg\psi)$. Hence by the definition of $|$ we have $D|Ref(\phi \rightarrow \neg\psi)$ as claimed.

It is also easy to see that disjunction introduction is a natural relationship. To see this, observe that $Ref(\phi) \sqsubseteq Ref(\phi \vee \psi)$, since for any $A \in Ref(\phi)$ it follows that $A \in Ref(\phi) \cup Ref(\psi)$ (for any ψ), and hence $A \in Ref(\phi \vee \psi)$.

3.5.1 Connexive Logic, Substantivity and Interpretation

Over the years there have been a number of attempts, both formally and historically, to understand and offer motivation for the connexive account of implication. One sort of motivation, discussed by Graham Priest, comes from a particular reading of negation:

The connexivist principles appear rather odd to the modern eye, and it is not clear what might justify them. The answer is, in fact, simple. They are all

3 Reconstructing Kilwardby's Logic

justified by the null account of negation. . . The connection is also explained by Routley and Routley (1985), p. 205, as follows:

Entailment is inclusion of logical content. So if A were to entail $\neg A$, it would include as part of its content what neutralizes it, $\neg A$, in which event it would entail nothing, having no content.

So it is not the case that A entails $\neg A$, that is, Aristotle's thesis, $\neg(A \rightarrow \neg A)$, holds. [48][p. 144] who cites [53][p.205].

This is one possible interpretation of the connexivist principles generally and one way to justify Aristotle's thesis in particular. On this interpretation, the connexivist principles are to be justified by appealing to a non-classical notion of negation. In [48], Priest refers to this view as a 'null account' of negation. It is null because the idea is that, in a contradiction, the two propositions, in some sense 'destroy' or 'extinguish' each other. [48, p.142] In both cases the use of the terms 'destroy' and 'extinguish' are taken from historical works, one due to Abelard and the other due to Berkeley.

The main idea that Priest tries to capture in his system is the idea that, in a connexive framework, contradictions do not have any logical content, and as such, should not imply anything. Priest writes:

The main problem in formulating a null account of negation, as should be clear, is how to make sense of the idea that a contradiction has no content. We will enforce this in the most simple-minded way. Let us say that: $\Sigma \vDash \alpha$ iff Σ is consistent and $\Sigma \vdash \alpha$ [48, p.142]

In his semantics, Priest uses \vDash for his consequence relation and \vdash for classical consequence. To avoid confusion in what follows, we will use \Vdash for Priest's \vDash and always subscript our account to make clear we are talking about \vDash_N .

There are a few interesting technical details worth noting about Priest's system. First, it is helpful to make the following observation about connexive implication:

As with the semantics of section 1.3, these semantics are not monotonic or closed under uniform substitution, and for exactly the same reason. In particular, none of the inferential principles (i.e., those with something to the left of the turnstile) just cited is valid for arbitrary substitutions (though the logical truths are).[48][p. 146]

The issue in each of these cases is that while inferences such as: $\{p \rightarrow q, q \rightarrow r\} \Vdash p \rightarrow r$ hold, instances like the following entailments do not hold with Priest's semantics.

1. $\{p \wedge \neg p, p \rightarrow q, q \rightarrow r\} \Vdash p \rightarrow r$
2. $\{(p \wedge \neg p) \rightarrow q, q \rightarrow r\} \Vdash p \rightarrow r$

The reason for this is straightforward. In his logic, Priest requires that Σ be consistent, and clearly both 1. and 2. are not. Priest then goes on to observe that it is possible to generalise this idea. If one were to move into a modal framework then we can use the \diamond operator¹⁵ to make this requirement explicit in the object language. Priest observes that:

We may define a connexivist conditional, $\alpha \rightarrow \beta$ as $\diamond\alpha \wedge (\alpha \Rightarrow \beta)[\wedge \diamond \neg\beta]$. For example, \Rightarrow can be any strict conditional or the conditional of many relevant logics. [48, p.146]

The idea that we have taken from Kilwardby is, in some sense different from the account of Priest, although there is clearly some overlap. Our idea is to think of \vDash_N as capturing a notion of meaning containment and to account for the validity of Aristotle's Thesis and Boethius' Thesis as expressing principles about the underlying definitional relationships that exists between terms in our language. On our account, Aristotle's Thesis is valid because no formula is ever compatible with its own negation and Boethius' Thesis is valid because if a formula, ϕ posits ψ because of how the terms in ϕ are defined, it does not, on this account, also imply $\neg\psi$. As such, this approach to connexive logic differs from other attempts to formulate connexive logic in the literature, and it would be interesting to see if this analysis of connexive logic has relationships either to other medieval discussions of connexivity or to more recent ones.

3.6 Conclusions and Future Work

Our goal in this chapter was to provide a formal reconstruction of Kilwardby's modal logic, his theory of validity and his account of the syllogism. As has become clear, there are a number of interesting and unique components that go into Kilwardby's theory of consequence. One observation, which will be useful in the next chapter, is that Kilwardby's modal logic cannot be represented in the usual possible worlds semantics of modal logic. In our final chapter, our aim is to bring the results of the previous four chapters together and explore the differences that exist between Kilwardby's and Buridan's theories of logical consequence.

This work also suggests two interesting technical questions for future research. First, Kilwardby's analysis of non-syllogistic proposition gives rise to a connexive logic. Do the semantics developed here naturally generalise to a connexive framework? In particular, can we use the essential operations to define a notion of implication and/or negation that is 'natural' in the sense that Kilwardby and others intended? If so, then is this perhaps why medieval authors writing before William of Soissons (e.g. Peter of Spain and Peter Abelard) wanted to endorse some kind of connexive hypothetical? Some of these ideas have been developed in the literature, in particular by Chris Martin e.g. [8], [39], and [40] but so far as we are aware, the idea of connecting this ideas with the notion

15. An operation, \diamond where $\neg \diamond (p \wedge \neg p)$ is always true.

3 Reconstructing Kilwardby's Logic

of *per se* as found in the *Posterior Analytics* and *The Metaphysics* has not been deeply explored.

Second, there is a well-known duality that exists between normal modal logics, on the one hand, and Boolean algebras with operations on the other. Such algebraic structures give rise to families of well-studied algebraic structures. Again, two lines are interesting here. First, it should be noted that we need very weak order conditions to be able to define structures that mirror the assertoric syllogism. This may be nothing more than a simple technical observation, but it should be noted. Second, and perhaps more interestingly, can we use the semantics provided here to find a natural framework for viewing the logic of essence as a family of algebraic structures closed under particular operations? It is unclear, but it may be possible, for sufficiently rich algebraic structures.

4 The Modal Syllogism in John Buridan

4.1 Introduction

The main goal of this chapter is to understand Buridan's theory of logical consequence and to situate his modal logic within this theory. To do this we will proceed in two parts. First, we will start with Buridan's general theory of inference and assertoric inferences. Our aim here will be to explore Buridan's theory so that we can 1) understand his distinction between formal and material consequences 2) understand his theory of supposition and see how it can be used to provide 'truth conditions' for the various categorical propositions, and 3) sketch Buridan's understanding of the syllogism and explore some of the unique features of his theory. Then, 4) we will explain how Buridan is able to 'reduce' modal syllogistics to his theory of consequence together with his analysis of modal propositions.¹

With this in place we will then turn to modal inferences. We will start with Buridan's distinction between divided and composite modal propositions. With these in place we will then sketch Buridan's theory of single-premise inferences contained in Book Two of the *Treatise on Consequences*. Finally, we will turn to the modal syllogism and highlight some of the interesting conclusions that Buridan draws.

4.2 The Structure of the *Treatise on Consequences*

Our main focus in this chapter will be on Buridan's *Treatise on Consequences*. From time to time we will supplement this with passages from the *Summulae De Dialectica* when they will help clarify Buridan's meaning. Hubert Hubien, in his introduction to the critical edition of the *Treatise*, dates the work to around 1334[4, p. 9-10] based on allusions made to a 'white cardinal'. However this dating is somewhat questionable. The main concern is that the 'white cardinal' was actually the Pope at the time that this work was written. Given that Buridan makes some unflattering comments about this 'white cardinal' it would be a brave move by Buridan to be insulting the current Pope.

1. Buridan distinguishes between two kinds of propositions, subject–predicate propositions and composite propositions, where composite propositions are composed of several subject–predicate propositions along with an expression such as 'if'. We shall see shortly what it means for a proposition to be modal.

To this end, I say that propositions are divided into subject–predicate and compound propositions. Now a consequence is a compound proposition; for it is constituted from several propositions conjoined by the expression "if" or the expression "therefore" or something equivalent.[51, p.66]

The Latin can be found on page 185.

4 The Modal Syllogism in John Buridan

Buridan's *Treatise on Consequences* is divided into four books. Each of these books follows a general format.

The book opens with a preamble.

A number of definitions are given, remarked on, proved, or agreed to.

A number of conclusions are proved based on the definitions laid down.

The first book outlines Buridan's general theory of consequence. Here 'consequence' is defined and then divided into a number of different kinds. The notions of ampliation and supposition are introduced and defined for various kinds of propositions. Buridan also outlines the various causes of truth for propositions and explains when different kinds of propositions are true and false. Most of the conclusions are fairly standard results about consequences, including Ex Impossibile Quodlibet (this is one of the medieval phrases used to express the principle of explosion²; literally 'from an impossibility anything follows')³ transitivity of consequence⁴ and how consequence relates to the truth and possibility of truth.⁵

Buridan's second book provides an analysis of modal consequences where the antecedent and conclusion are simple expressions.⁶ In this chapter, Buridan provides an analysis of modality using the theory of ampliation and supposition. He then gives a number of definitions about propositions in which modals occur. In particular, he identifies two ways that a modal can occur in a proposition. These give rise to two senses: the composite sense and the divided sense.⁷ From these definitions and the previous results in the first book, Buridan proves a number of interesting results about such propositions. The conclusions are separated into three sections. First he proves conclusions for divided propositions. Second he offers some more remarks about composite modal propositions and proves some conclusions. Finally, he proves conclusions about the relationship between divided and composite modal propositions.

Buridan's third book provides a discussion of assertoric syllogisms. The book is divided into two parts. The first part concerns syllogisms between direct terms.⁸ The second part

2. Formally, the principle of explosion says for any formulae ϕ , ψ from ϕ and $\neg\phi$ one may infer ψ .

3. See Book One Conclusions 1 and 7. See [51, p.75] and [51, p.79] respectively.

4. See Book One Conclusion 4

5. See Book One Conclusion 5. See [p.77]sr:10.

6. We will define simple expressions when we introduce the book in more detail. For now, it is easiest to think of these as terms with nothing modifying them.

7. This is a standard medieval distinction used to cover a number of differences in scope that occur with various kinds of operations.

8. The distinction between direct and oblique terms is an important one. Buridan explains what oblique terms are in the following passage

Accordingly, it will first be explained what an oblique term is when it is used with a direct term that is governed by it as a determination of that direct term, just as an adjective is a determination of a substantive [term]. For just as when saying 'A white horse is running,' the expression 'white' determines the expression 'horse' to supposit only for white ones, so if I say 'Socrates's horse is running,' the expression 'Socrates's' restricts the expression

concerns syllogisms with oblique terms. In this book he discusses the usual Aristotelian moods, however, he displays considerable originality both in his presentation of these moods and in his discussion of additional, non-Aristotelian syllogisms involving oblique terms. He discusses non-standard forms of the various propositions, as well as syllogisms with oblique terms. Oblique terms are terms that are modified by a genitive or an accusative construction. For example, in the proposition the term 'Everything that is a man's ass is running' the term 'man's' is an oblique term since 'man' is in the genitive and in this context implies possession of an ass. Oblique terms are technically interesting (at least) because they can be seen as a medieval attempt to treat relations within the context of syllogisms.

Buridan's fourth and final book brings together the results from the previous three books. Here he develops his theory of the modal syllogism using the resources proven in the other three books. All of the English translations given here are drawn from a recent book by Professor Read, [51].

4.3 Buridan's Theory of Consequence

The cornerstone of the *Treatise on Consequences* is the definition of consequence. After proposing a number of definitions Buridan settles on the following definition:

One proposition is antecedent to another which is such that it is impossible for things to be altogether as it signifies unless they are altogether as the other signifies when they are proposed together[51, p.67]...⁹

First, it should be observed that Buridan's theory of consequence is a 'non-reductive' analysis of consequence. In this definition, the notion of 'impossibility' is left unanalyzed. It is simply taken as a primitive notion. This raises the question of how this definition relates to the analysis of modality provided in the second book. There, as here, the modal notions are taken as conceptually primitive. There is no attempt, as in some other medieval authors¹⁰ to define the modalities in terms of temporal operations or other notions.

Second, Buridan's definition of consequence reflects his philosophical commitment to medieval nominalism. His commitments are most clearly seen in his requirement that the two terms must be "proposed together." In his rejection of one of the previous definitions, Buridan remarks:

'horse' to supposit only for those that are Socrates's.[51, p.128]

The Latin can be found on page 185. The basic idea is that an oblique term is an adjective, possessive etc, that changes the supposition of the term it is modifying and restricts it accordingly. A direct term lacks such restriction. In the case of 'Socrates's horse' the oblique term 'Socrates's' restricts the supposition of 'horse' to only range over those horses owned by Socrates.

9. For the Latin see page 185.

10. For example Lambert of Auxerre. For a formal reconstruction of his system and some discussion of his modal theory, see Chapter Five of [62].

But this definition is defective or incomplete, because ‘Every human is running, so some human is running’ is a good consequence, but it is possible for the first to be true with the second not being true, when the second does not exist at all.[51, p. 67]¹¹

There are some details in this passage that should be teased out to help clarify exactly what Buridan thinks is going wrong here. Before we fully explain what is going on here, there is a distinction, drawn by Arthur Prior which he uses to help explain some remarks that Buridan makes about negative and affirmative propositions in the *Sophismata*.¹² Prior’s explanation of what is going on in Buridan starts with the following supposition:

Suppose we have several sheets of white paper on which there are black marks arranged in lines. We may classify the marks according to their shapes.[49, p.482]

Prior then goes on to distinguish three families of shapes, one for terms (in particular he includes *propositio*, *affirmativa*, and *negativa*), one for signs of quantity (in particular, he gives *omnis*, *quaedam*, and *nulla*), and one for copulae (which include *est* and *non est*).[49, p.482].

Prior then goes on to define a sentence as follows:

A line of marks is called a sentence if and only if it consists of a sign of quantity followed by a term followed by the copula *est* (or by *non est*, if the sign of quantity is *quaedam*) followed by a term. Thus sentences are any lines of marks of these four shapes

omnis A est B

quaedam A est B

nulla A est B

quaedam A non est B,

where A and B are marks of the shapes called “terms”. [49, pp.482-483]

With this in place, Prior gives truth conditions for these propositions, relative to the sheet on which they are written as follows:

Whether sentences are or are not true on their sheets is determined by the shapes of the marks on their sheets. In this sense, the sentences are “about” the shapes of the marks. Each term is associated with a particular group of shapes, which it may be said to connote, though this means no more than that the presence on a sheet of marks of certain shapes will determine, in ways which we shall shortly detail, whether or not sentences containing certain terms are to be counted as “true on their sheets”. The shapes connoted

11. The Latin can be found on page 185.

12. In particular, the eighth chapter of the *Sophismata*. See [49, p.481]

by the term *propositio* are all those which count, by the rules given above, as shapes of sentences. The shapes connoted by the term *negativa* are those of sentences which either begin with *nulla* or have *non est* for their copula, and the shapes connoted by the term *affirmativa* are those of all sentences which do not either begin with *nulla* or have *non est* for their copula.

The rules by which we classify sentences as true or false on their sheets are as follows:

(1) A sentence of the type *Omnis A est B* is true on a sheet if and only if (i) it is written on the sheet, (ii) there is at least one mark on the sheet of a shape *A* (i.e. of a shape connoted by the term *A*) and (iii) there is no mark on the sheet of a shape *A* which is not also of a shape *B*. For example, the sentence:

Omnis propositio est affirmativa

is true on a sheet if and only if (i) it is on the sheet, (ii) there is at least one sentence on the sheet and (iii) there is no sentence on the sheet which begins with *nulla* or has *non est* for its copula. [49, p.483-484]

Similar truth conditions are given for the other three types and a sentence. Likewise, a sentence is said to be false if it is written on the sheet and is not true of that sheet.

With this in place, Prior goes on to define when a sentence on a sheet can be said to be 'possibly true' and can be said to be merely 'possible'. Prior says that:

A sentence on a sheet may be said to be possibly-true on that sheet if and only if there is some sheet (that one or another) on which it is true, and possibly-false on that sheet if and only if there is some sheet (that one or another) on which it is false.

A sentence on a sheet may be said to be possible on that sheet under the following conditions:

(1) If it is of the shape *Omnis A est B*, it is possible on a sheet X if and only if (i) it is on the sheet X, and (ii) there is some sheet Y such that (a) some mark on Y is of a shape connoted by *A*, and (b) no mark on Y is of shape *A* but not of shape *B*. (2) If it is of the shape *Nulla A est B*, it is possible on a sheet X if and only if (i) it is on the sheet X, and (ii) there is some sheet Y such that no mark on Y is at once of shapes *A* and *B*.

The possibility-conditions for the other forms may be worked out similarly. . . [49, p.485-486]

What Prior then goes on to note is that for a sentence, S to be possible on a sheet Y, does *not* require that S be written on Y. This for Prior, is the distinction between the 'possibly true' and the 'possible'. For a sentence to be possible it does not need to be written down, whereas for it to be possibly true, it does need to be.

4 The Modal Syllogism in John Buridan

With the image of the sheets in mind, and the distinction between the possibly true and the possible, we can hopefully get clearer on Buridan's definition of logical consequence and the counterexamples that he puts forward. First, recall Buridan's definition of logical consequence:

One proposition is antecedent to another which is such that it is impossible for things to be altogether as it signifies unless they are altogether as the other signifies when they are proposed together.[51, p.67]

Phrased in terms of sheets, we can express this as saying that one proposition S is antecedent to another proposition, S' just in case there is no sheet X such that S is true of X and S' is false of X .¹³

Observe that if we require S to be true *at* X and S' to be false *at* X , they both need to be written down on the piece of paper. This is what is being required when Buridan says that the propositions need to be proposed together. With this in place we can then see why this condition is important. If we were to drop this requirement and instead say that:

S is antecedent to another proposition, S' just in case there is no sheet X such that S is true at X and S' is false at X .

Then we can immediately rephrase Buridan's counterexample. In the case of 'Every man is running, so some man is running', what Buridan is considering is a situation where the sentence 'Every man is running' is written on some sheet of paper (call it Z), but there is no sheet of paper where the sentence 'Some man is running' has been written down. In such a situation the antecedent is true, since 'Every man is running' is true of Z and true at Z . However, there is no sheet of paper where 'Some man is running' is true, since the sentence has not been written down.¹⁴

In this way, the problem with the defective definition of validity is that it would rule out as invalid cases where the proposition does not exist. Phrased slightly more metaphysically, this failure is caused by the nominalist insistence that the truth of propositions are contingent upon an agent thinking, uttering or writing them.¹⁵ Phrased in terms of Prior's sheets, as a necessary condition for a proposition to be true, it needs to be marked

13. Prior phrases this as:

The most satisfactory definition of validity (on a sheet in a set of sheets) is to say that a sentence on a sheet may be validly inferred from other sentences on this sheet if and only if there is no sheet (in the set) of which all the premises-sentences are true but of which the conclusion-sentence is false.[49, p.489]

Clearly Prior's definition is broader than the one given above. He restricts his definition to single premise inferences later on.[49, p.489]

14. Compare [49, p.489-490].

15. For Buridan, as well as many medieval authors, there is a threefold distinction between mental, spoken and written propositions, which they ground in Aristotle's remarks in *On Interpretation* [16a4-16a9]. The standard view is that there is a priority of signification in such propositions. Written propositions

on the relevant sheet. This is not so for signification. Buridan allows propositions to signify even if they are not written down or in Prior's terms, marked on a sheet. In some ways this is a curious notion, however it appears to be a consequence of Buridan's commitment to nominalism.

Third and finally, Buridan's discussion of consequence frequently alludes to or mentions the concept of signification. Signification is a key component in medieval analysis of language and we will briefly sketch the theory in what follows, so as to better understand Buridan's discussion of logical consequence.

4.3.1 Medieval Theories of Language: Buridan's Theory of Signification

The theory of signification is one of the two related linguistic notions that Buridan employs in his analysis of consequence.¹⁶ The theory of signification analyses the relationship between an utterance in a given language and the concept that the utterance refers to. For Buridan, utterances can be broken down into two kinds.¹⁷ The first kind of utterance is called signficative and the second kind is called non-signficative. Signficative utterances represent something to the hearer of the utterance, whereas non-signficative utterances do not represent anything. As an example of the first, take any meaningful word of English. For an example of the second, take any sound or arrangement of letters that does not mean anything. Signficative utterances can then be distinguished between those that signify by nature and those that signify by convention. Utterances that signify by nature are things like screams of pain, shouts of joy etc.[5, p. 11] For Buridan, these are understood by everyone and do not require additional information about the language in question. In contrast to this are utterances that signify by convention. For Buridan, Latin, as well as other natural languages are composed of such conventional utterances. What Buridan means by 'conventional' is that the meaning of an utterance is something that is determined by the linguistic community that a speaker finds herself in.¹⁸ For example, that the utterance 'homo' signifies the concept 'man', and not 'donkey', is for Buridan a matter of convention. There is nothing in the choice of the particular utterance of those four letters that connects it with the concept in question in any 'natural' way.¹⁹

The important thing to observe for our purposes here, is that, Buridan simple utterances such as 'man' or 'turkey' immediately signify concepts and by extension also signify things in the world.

signify spoken propositions which in turn signify mental propositions, however on Buridan's account, all three count as propositions for the purposes here. See, for example, [5, p.849] for Buridan's discussion of signification.

16. The other is supposition which will be discussed in the next section.

17. All of what follows can be found in[5, pp. 9-10].

18. The Latin term, *ad placitum*, translated in Klima as 'conventional', literally means 'at one's pleasure', but its medieval usage carries connotation of judgement, and so could also be translated 'as one judges' or more colloquially, 'as one sees fit'.

19. It should be pointed out that for Buridan this 'conventionalism' only applies to written and spoken languages. For Buridan, terms in mental language always signify the things they pick out naturally.

... for Buridan, mental expressions are naturally similar to things outside the mind, but they are not signs of them. Rather, spoken expressions are signs of mental expressions and so derivatively signs of external things.[51, p.7]

However, for Buridan the notion of signification is a very ‘coarse grained’ notion:

For example, the written proposition, “A man is white,” signifies the spoken proposition, which immediately signifies the concepts “man” and “white” and their combination, the mental proposition, but it ultimately signifies only men and white things. In particular, “A man is white” and “A man is not white,” and generally any proposition and its contradictory, have the same (ultimate) signification.[51, p.8]

The challenge that then emerges for Buridan is to explain how one can give truth conditions for various complex propositions. This difficulty was further compounded by the tendency of logicians in Buridan’s days to say that ‘things being as it signifies’ was a good definition of truth.[51, p.8]. What Buridan instead does is uses the notion of supposition to provide truth conditions for various propositions.

4.3.2 Medieval Theories of Language: Buridan’s Theory of Supposition

If signification is the relationship that expressions bear to concepts, supposition is the relationship that different terms have to objects. The basic idea behind supposition is that “a term, when it occurs in a proposition ‘stands for’ [supponit pro] each member of a certain class of things and the truth conditions are stated in terms of the relationship between these classes.”[20, pp. 1-2] When discussing supposition Buridan starts by looking at the two classes of propositions, affirmative and negative. Buridan tells us that,

Now in the fifth chapter, I set down further that an affirmative proposition means that the terms supposit for what are the same, or were or will be or can be the same, depending on the kind of proposition. For if I say ‘A is B,’ I mean that A and B are the same, and if I say ‘A was B,’ I mean that A was the same as B, and so on. A negative proposition means the opposite, namely, that what the subject supposits for and what the predicate supposits for are not the same or that they were not the same or will not be, and so on.[51, pp.69-70]²⁰

As Buridan’s examples make clear, when he is speaking of affirmative proposition here is not yet giving us truth conditions of universal propositions (either affirmative or negative), but he is describing when unquantified (and particular) propositions are true.

The idea is that a particular affirmative proposition is true if the terms in question supposit for the same thing and they are false otherwise. Likewise, a negative proposition

20. The Latin can be found on page 185.

is true if the terms do not supposit for the same thing. Buridan has to restrict this definition to rule out a number of possible counterexamples (liar sentences, chimeras etc.) but Buridan is very terse about how this goes, and remarks that a full treatment of such issues would require another treatise.[52, p. 9] He goes on to address these issues in the *Sophismata*.

In the case of universal propositions, Buridan goes on to remark that:

We need to describe different kinds of proposition differently. For depending on the expressions used, we describe universal [propositions] one way and particulars another, for example, that nothing for which the subject supposits is the same as anything for which the predicate supposits, or for a particular, that something for which the subject supposits is not the same as something for which the predicate supposits.[51, p.70]²¹

Here Buridan describes the situation when a universal affirmative proposition is true. The conditions for universal negative propositions can be similarly worked out, although they are not explicitly stated by Buridan here.

At this point it would also be helpful to remark on Buridan's use of the notion of 'causes of truth'. Buridan gives three examples, and in a style reminiscent of many modern logic textbooks, leaves the details to the reader to figure out how causes of truth are to be assigned. These examples are, 'Colin's horse cantered well', 'The Antichrist will preach' and 'Something which never will be can be.' The first example illustrates a past tense proposition, the second illustrates a future tensed example that medieval philosophers believed to be true and the third illustrates a modal proposition.

If Colin's horse, which cantered well, is dead, 'Colin's horse cantered well' is true, but things are not in reality as the proposition signifies. . . [51, p. 63]²²

Buridan's point here is that when we evaluate the truth or falsity of a proposition we must pay due attention to the presence of modal, temporal and other factors that affect when we evaluate the truth of a proposition.

Assigning causes of truth to these propositions is straightforward. The first proposition is true because, in the past things were as the proposition describes (i.e. the subject and predicate supposit for the same thing). Analogously, the second proposition is true just in case things will be as the proposition signifies they will be. The third proposition is more interesting. The proposition contains both modal and tensed verbs.²³ All Buridan says about this is that it is true because things can be as the proposition signifies they can be. Buridan's point is that here the signification does not only deal with things that are, but also with things that can be.

From what Buridan has said, it is clear that the 'cause of truth' for a given proposition, is the set of conditions that are sufficient for the proposition to be true. Buridan himself

21. The Latin can be found on page 185.

22. The Latin can be found on page 185.

23. This example will also be important when we come to discuss Buridan's view on modality below.

defines ‘causes of truth’ as “whatever is enough for the proposition to be true”.²⁴ A present tense proposition will be true if the terms signify as things are.²⁵ A past tense proposition will be true if the terms signify as was the case etc. For example, the proposition ‘Socrates is running’ is true just in case there is something (now) that is Socrates, and that same thing is also running. This formulation has close connections to Buridan’s phrasing of the expository syllogism as we shall see later in Chapter Five.

Buridan defines causes of falsity by exploiting a connection that exists between true and false propositions:

Since it is impossible for the same proposition to be both true and false, and if it is formed it is necessarily true or false, one must assign the cause of truth and the cause of falsity to the one proposition in opposite ways.[51, p.64]²⁶

Buridan takes it for granted that no proposition can be true and false at the same time. Now, if the propositions in question are formed, then the cause of truth for one proposition will be the cause of falsity for its contradictory and vice versa.²⁷ So, the cause of falsity for a present tense assertoric proposition will be that the terms do not signify as is the case. For example, the proposition ‘Socrates runs’ will be false just in case Socrates is not the same as any running thing. This analysis can be extended to other propositions in the natural way.

Given what we have seen about causes of truth, signification, and supposition, we can now explain when the four main kinds of categorical propositions are true or false.

| Proposition | | Truth Conditions |
|--------------------|----------------|---|
| Every A is B | if and only if | everything that A supposits for B also supposits for and there is something that A supposits for. |
| No A is B | if and only if | it is not the case that something is supposit for by both A and B, |
| Some A is B | if and only if | there is something that both A and B supposit for. |
| Some A is not B | if and only if | there is something both A supposits for and B does not or there is nothing that A supposits for. |

There is another complication with this analysis. Some terms ampliate their subject or predicate terms. In essence, what this does is require that we need to consider a broader range of possible supposita than surface formulation would suggest we need. Ampliation

24. The Latin can be found on page 186.

25. Buridan’s analysis of truth is sometimes referred to as an identity theory.[27] [p.3] This is because for a proposition such as ‘Socrates is white’ to be true, it needs to be the case that the object picked out by the name ‘Socrates’ is identical to one of the things signified by the word ‘white’.

26. The Latin can be found on page 186.

27. Again, the formation of the propositions is because of Buridan’s nominalism.

is another feature of the medieval account of language that emerges from the theory of supposition. A term is amplified if it stands for more objects than the term would normally stand for in a given present tense proposition. The standard example of this is the term ‘dead’. In the proposition ‘a horse is dead’ the term ‘dead’ ampliates ‘horse’. Specifically, it ampliates the term to range over horses that used to exist as well as the horses that currently do exist. The reason for this is straightforward. For many of the medievals, if a horse is dead, it is not a horse. The horse is a corpse or may have entirely passed out of existence. So, if the term ‘dead’ did not amplify, then ‘a horse is dead’ could never be true, because nothing will be both dead and a horse at the same time. Buridan’s theory of modality applies ampliation to the subjects in modal propositions where the verb is modalized.²⁸

With all of these concepts in place, we are now able to fully discuss Buridan’s theory of consequence as well as his theory of modality. We now return to Buridan’s discussion of consequence.

4.4 Division of Consequence

Within the *Treatise on Consequences*, after laying down the definition of consequence, Buridan distinguishes various subclasses of consequences. The first distinction that Buridan draws is between formal and material consequences. According to Buridan:

“A consequence is called formal if it is valid in all terms retaining a similar form. Or if you want to put it explicitly, a formal consequence is one where every proposition similar in form that might be formed would be a good consequence.[51, p.68]²⁹

For Buridan a consequence is formal, just in case all similar forms of the argument are also valid, i.e. if we hold everything in the argument constant except (possibly) the terms, and if the argument remains valid no matter how the terms are changed, then the argument is formally valid. A material consequence is one that preserves truth but does not retain validity if the logical form was retained but different terms were substituted, i.e. “A material consequence, however, is one where not every proposition similar in form would be a good consequence”. [51, p. 68]³⁰ This is best illustrated with the following example. The consequence: ‘Some donkey is an animal’ so ‘Some animal is a donkey’ is valid according to Buridan. Moreover, it is valid formally, since it is true regardless of which terms are used. In this case the argument is an instance of simple conversion. In contrast the consequence, ‘Some donkey is a man’ so ‘A stick stands in the corner’ is valid, but is only valid materially. It is valid because it is impossible that something is both a man and a donkey, and so the definition of consequence is satisfied.³¹ However,

28. A verb is modalized if it has a modal term, e.g. ‘possibly’, modifying it adverbially.

29. The Latin can be found on page 186.

30. The Latin can be found on page 186.

31. This is the ‘Ex Impossibili Quodlibet’ principle, or ‘from the impossible anything follows.’

the inference is not valid in all propositions of a similar form. For example, from ‘Some man is running’ it does not follow that ‘A stick stands in the corner.’ This definition of ‘formal’ and ‘material’ also generalises to cases where we have conjunctions of terms in the antecedent or are dealing with multiple antecedent propositions. For example: the inference from ‘Every man is mortal and Socrates is a man’ to ‘Socrates is mortal’ is a valid, formal consequence for Buridan, since every uniform replacement of the terms in the premises and conclusion yield a valid argument. It should be noted here that for Buridan the terms ‘man’, ‘mortal’, and ‘Socrates’ can be ‘replaced’ or abstractly represented using term variables. Everything else will remain constant.³² Because of this it is possible for Buridan to apply his definition of consequences to syllogisms.

In order to do that, we should say a few words about what Buridan takes the form and matter of a proposition to be. He says,

By the matter of a proposition or consequence we mean the purely categoric terms, namely, the subject and predicate, setting aside the syncategoremes attached to them ... [whereas] we say all the rest pertains to the form. [51, p. 74]³³

According to Buridan, the matter of a proposition is simply the categoric terms. These are expressions like ‘Socrates’, ‘human’ ‘animal’ etc. For Buridan everything else in the proposition pertains to its form. In the case of syllogisms this will be important, for Buridan includes the number of distinct terms as part of the form and uses this to avoid considering syllogisms with fewer than three categorical terms.

Using this definition it is easy to distinguish the usual kinds of categorical propositions (affirmative, negative, universal, particular, indefinite, etc.) from each other on the basis of their form. For example, the universal differs from the particular by the presence of the syncategorematic term ‘all’ or some similar term denoting universality.³⁴ Likewise the particular differs from the indefinite by the presence of the term ‘some’ or a similar term. As noted, Buridan adds that the number of categoric terms in the proposition also pertains to the form of the propositions. For example ‘Every A is A’ and ‘Every B is A’ have different logical forms for Buridan. This becomes important when we differentiate syllogistic arguments from non-syllogistic ones.

To determine the form and matter of syllogistic arguments, the situation becomes slightly more complicated. Buridan spends the first few pages of Book Three narrowing down the possible candidates for what a syllogism is. He then settles on the following definition of a syllogism:

We want to understand by ‘syllogism’ ... only a formal consequence to a single

32. The more general form of this would be: ‘Every A is B’ and ‘S is A’, therefore ‘S is B’. It should be observed that for Buridan (as well as Ockham) the copula (in this case ‘is’) is also treated as part of the logical form of the proposition.

33. The Latin can be found on page 186.

34. For example: ‘every’.

simple subject-predicate conclusion by a middle term different from each of the terms in the conclusion.[51, p.115]³⁵

This definition has a few important points. First, Buridan is interested only in arguments that are valid in virtue of form. Arguments that will be valid in virtue of the matter are not considered, presumably because they will be handled by the general definition of consequence. Likewise, the syllogisms that Buridan wants to deal with are those that are composed of a ‘simple’ subject–predicate conclusion where the terms in the premises are linked by a common term. This excludes arguments like disjunctive ‘syllogism’ and modus ponens as syllogistic arguments properly so called. It also excludes arguments where the conclusion is a conjunction or disjunction of terms. On this reading, an argument which were to conclude ‘Every human or donkey is an animal’ would not count as a syllogism. This is because the term ‘human or donkey’ is a disjunction of two terms. Second, it should be observed that Buridan defines a syllogism to contain the premises and is drawn to a conclusion. The terminology here is slightly ambiguous and invites the question: is a syllogism a pair of premises from which a conclusion can be drawn, or is it the premises together with a designated conclusion? According to Buridan, every syllogism has the following property:

We take it that every syllogism links the middle term in the premises with each extreme from the conclusion, so that on account of that linking the linking of the extremes is inferred, either affirmatively or negatively. Then it is clear that every syllogism, as we here intend syllogism, is made up from only three terms, namely, from two extremes which are the terms of the conclusion, and from a middle term with which those extremes are linked in the premises...it follows from this that there are only four figures of this kind of syllogisms. For the relation of the middle to the extremes in the premises as subject and predicate is called the syllogistic figure. This can only happen in four combinations.[51, p.115]³⁶

Because of this, it is clear that there are four possible ways the premises can be arranged. Buridan then goes on to point out that Aristotle does not need to treat syllogisms in the ‘fourth figure’ because they can be immediately reduced to the first figure by transposition of the premises. Buridan also does not provide a detailed discussion of the fourth figure. This suggests that Buridan is thinking of syllogisms along the lines of an ordered pair of propositions together with a conclusion. When a syllogism is thought of along these lines, it is clear why the fourth figure should be included and why sustained discussion of it is unnecessary. Any pair of premises in the fourth figure are equivalent to a different pair in the first figure.³⁷

35. The Latin can be found on page 186.

36. The Latin can be found on page 186.

37. One simply needs to change the order of the major and the minor premise and relabel the terms as required.

Third, given what Buridan says here, it seems that he is speaking of syllogisms simply as a particular subclass of formally valid argument. As we already saw in our discussion of validity, the criterion of validity does not require that the premises of the argument be true, only that it be impossible for the antecedent to be true and the consequent false when the antecedent and consequent are proposed together. This is unusual, since in most Latin authors there is usually an additional requirement that the premises be true.³⁸ However, this seems not to be what Buridan says in Book Three of the *Treatise on Consequences*. This becomes even more surprising when we turn to Buridan's discussion of syllogisms in the *Summulae*.

In this work Buridan discusses syllogisms in a several places. The natural place to focus our attention will be on Book Five, which is dedicated to the treatment of syllogisms. As a word of caution, it needs to be recalled that the *Summulae* is a textbook which is an expanded and revised commentary on Peter of Spain's *Summulae Logicales*. As such, Buridan often cites Peter's text at the start of a section, and then goes on to comment on a passage later on. In many cases this makes it hard to pin down exactly what Buridan thinks. Buridan quotes Peter of Spain as follows (all references are taken from Klima's translation):

A syllogism is an expression in which, after some things have been posited, it is necessary for something else to occur on account of what has been posited [quibusdam positis necesse est aliud accidere per ea quae posita sunt], as in Every animal is a substance; every man is an animal; therefore, every man is a substance; this whole [phrase] is an expression in which after certain things, namely, the two premises, have been posited, it is necessary for something else, namely, the conclusion, to occur, i.e., to follow.[5, p.308]

As a matter of fact, Buridan is very clear about what he thinks of this definition. In responding to one objection to it, Buridan writes:

I reply that although a syllogism is composed of several expressions, it is nevertheless a single hypothetical proposition, connecting the conclusion with the premises through the conjunction 'therefore'. Further it can be relegated to the species of conditional propositions, for just as a conditional is one consequence, so too is a syllogism, whence a syllogism could be formulated as a conditional, in the following manner: 'if every animal is a substance and every man is an animal, then every man is a substance'. Strictly speaking, however, a syllogism has an additional feature in comparison to a conditional in that a syllogism posits the premises assertively, whereas a conditional does not assert them. Therefore it would not be inappropriate to place syllogisms

38. I am grateful to Professor Hodges for raising this point with me.

in a species of hypotheticals different from those that the author enumerated earlier; that species could then be described, as far as its nominal definition is concerned in terms of the following expression: a consequence that asserts the consequent and the antecedent'... [5, pp. 308-309]

As can be seen, the conditions that Buridan is here imposing on a syllogism are much more elaborate than what is stated in the *Treatise on Consequences*. The requirement that the consequent and antecedent need to be asserted is completely absent from the *Treatise on Consequences* and from Peter of Spain's text.³⁹ One may wonder if the use of the expression 'Strictly speaking' is doing some theoretical work here. Perhaps the absence of this requirement from the *Treatise on Consequences* suggests a more general class of arguments that Buridan is there willing to allow to be counted as syllogistic. It is also possible that he may have simply forgotten to add this requirement to the *Treatise*, although this would be uncharacteristically sloppy of Buridan.

The second thing that should be observed here is Buridan's use of the term 'hypothetical' is somewhat different from how modern logicians use the word. In the *Summulae*, Buridan describes a 'hypothetical' proposition as:

The second section describes categorical proposition, stating that its principal parts are subject and predicate. And by this it is clear what distinguishes it from a hypothetical, for the principal and major parts of a hypothetical are not subject and predicate, but several propositions, each of which is composed of a subject and a predicate, as we shall see later in more detail. [5, p.23]

Buridan elaborates this as follows:

Next, we deal with hypothetical propositions. A hypothetical proposition is one that has two categorical propositions joined by some conjunction or by some adverb. And its name derives from 'hypo', i.e., 'sub-' [under] and from 'thesis', i.e., 'positing', i.e., as it were, 'suppositive locution'. [5, p.57]

What sounds odd to modern ears about this use of 'hypothetical' is that, for Buridan, conjunctions, disjunctions, conditionals, as well as propositions joined by some adverbs (such as 'where' or 'when') count as hypothetical propositions. Normally we reserve the term 'hypothetical propositions' for 'if...then' statements and similar.

39. Peter of Spain defines a syllogism in *Tractatus IV.2* as follows:

Sillogismus est oratio in qua quibusdam positus necesse est aliud accidere per ea que posita sunt. Ut 'omne animal est substantia omnis homo est animal ergo omnis homo est substantia'. Hoc totum est quedam oratio in qua quibusdam positus, idest duabus premissis propositionibus, necesse est per illas sequi aliud, idest conclusionem. [44, p. 43]

At this point it should also be observed that when Buridan speaks of *conditional* hypotheticals in the *Summulae*, he means the following:⁴⁰

A conditional is [a hypothetical proposition] in which two categorical propositions are joined by the conjunction ‘if’, as in ‘If a man is, an animal is’. 2) The truth of a conditional requires that the antecedent cannot be true without the consequent, hence every true conditional amounts to one necessary consequence.[5, p.61]

As Buridan goes on to make clear here, he is thinking of these ‘if...then...’ constructions as consequences, and it seems to include both relationships that we would now treat using notation such as $\phi \vdash \chi$ (where ϕ, ψ , and χ are formulae of some language and \vdash is a consequence relation) and $(\phi \wedge \psi) \rightarrow \psi$.

At this point we should clarify Buridan’s terminology. Syllogisms, as he defines them, are two premise arguments where the two premises are linked by a common term. The common term is called the ‘middle term’ while the other two terms are called the ‘extremes’. The term which occurs in the first premise of the syllogism is called the ‘major term’ or ‘major extreme’, the term occurring in the second premise is the ‘minor term’ or ‘minor extreme’. What is interesting is Buridan’s requirement that each term must be distinct from the other two. To illustrate this, observe that by the above definition the following does not have the form of a syllogism: Every A is B, Every B is B, and therefore Every A is B. This fails to have the correct form on two counts. First, the middle term, B, occurs in the conclusion, which is not allowed. Second, there are only two distinct terms that occur in the argument. The reason for this failure goes back to Buridan’s formality constraint. Since the number of terms in an argument pertains to the form of the argument, an argument with only two terms, like the one above, has a different ‘logical form’ than the syllogisms that Buridan is interested in.⁴¹

As we already remarked, from the way Buridan defines a syllogism it is easy to see that there are four possible ‘figures’ or ways that the middle term can be related to the extreme terms so as to produce syllogisms. These can be summarized as follows:

| Figure 1 | Figure 2 | Figure 3 | Figure 4 |
|----------|----------|----------|----------|
| B-A | A-B | B-A | A-B |
| C-B | C-B | B-C | B-C |
| C-A | C-A | C-A | C-A |

It should be noted that the presentation of the figures is ‘backwards’ from Aristotle’s own formulation of the syllogism as well as from some modern presentations. The difference is due to the formulation of the categorical proposition. For Aristotle (as well as some medievals) categorical propositions are of the form: first term, copula, quantifier,

40. the passage is, again, a quote from Peter of Spain, but Buridan says nothing in his comments on it to suggest he disagrees with it, as long as these are not applied to ‘as of now’ consequences.

41. Note that Buridan does not say arguments like the one above are in any sense wrong or bad. They simply do not have the form of syllogisms. He discusses such ‘syllogisms’ in [5][pp.386-387].

second term. For Aristotle, the first figure inference Barbara is of the form:

$$\begin{array}{l} \text{C belongs to every B} \\ \text{B belongs to every A} \\ \hline \text{C belongs to every A} \end{array}$$

In what follows we will follow Buridan's notation and terminology.

At this point we may also wonder what kind of inference a syllogism is. Keeping in mind some recent historical debates about Aristotle's understanding of the syllogism, we may wonder how Buridan thinks of syllogisms.⁴² At the heart of this debate is how we ought to think of the syllogistic and how syllogisms should be represented in modern formalisations. Łukasiewicz thought of syllogisms as hypothetical propositions where the antecedent is composed of two categorical propositions and the conclusion is a single categorical proposition.⁴³ Łukasiewicz argued that the best way to represent Aristotle's syllogistic is by means of an axiomatic theory. In contrast, Corcoran thinks of syllogisms as inferences from two assumed propositions from which another conclusion can be drawn. Corcoran argues that the best way to represent syllogistic reasoning is by using a natural deduction system. This will be important when we come to formalise Buridan's syllogistic theory. In particular this will have bearing on the kind of proof theory we use to represent Buridan's system.

As we have already seen, there are some interesting questions that emerge here, and because of reasons of space, these issues can only be briefly touched on.⁴⁴ As we already observed, the question of where Buridan would fall in this debate seems to depend, to some degree, on which text of Buridan's we are reading. In both cases, Buridan does treat the syllogisms as a hypothetical proposition. However, as we observed, the use of the term 'hypothetical' in Buridan does not naturally map onto the modern usage of the term, and Buridan is somewhat ambiguous as to what level the consequence is functioning on. However, he does seem to be somewhat closer to Łukasiewicz than to Corcoran. Conversely, when we ask the question, do the premises of a syllogism need to be true for it to count as a syllogism, Buridan's own writings do not present an entirely consistent picture. As we saw, in the *Treatise on Consequences*, the definition of a syllogism is simply a particular kind of formal validity, and there is no requirement that the premises be true. However, in the *Summulae* the antecedent and consequent do both need to be asserted. The question that is open here, is do the premises need to be asserted and be true, or do they merely need to be asserted and the conclusion follow from the premises? Buridan does not tell us. The only passages I am aware of where he discusses how truth and falsity relate to asserted propositions is in his discussion of Fallacies in Book 7.⁴⁵ The passage reads as follows:

42. This harks back to the disagreement between John Corcoran and Łukasiewicz over the nature of the syllogism. See [9] and [36] for the start of this debate.

43. See [9, 696-697] for Corcoran's formal reconstruction.

44. I am very grateful to Professor Hodges for raising these points in discussion with him. His observations made me aware of differences between the *Summulae* and the *Treatise* that I was not aware of.

45. Again, this passage was helpfully pointed out to me by Professor Hodges

As to the fourth mode, it is obvious that many poets and philosophers often put forth false propositions in an interrogative manner, and thus, if, because of the authority of these authors, someone assents to these propositions as if they had been put forth assertively, then he will be deceived. For these expressions, which are materially the same, are different in form, since in a question the last word is uttered at a high pitch, whereas in an assertion it is uttered at a low pitch; in writing, however, a question has a question mark after the last word, whereas the assertion has a full stop. [5, p.534]

This passage is part of a discussion of the fallacy of accent and the fourth mode pertains to accents. We are interested in the following characteristic:

The fourth characteristic is accent properly so called, which is the modulation of an utterance with respect to raising or lowering its various parts. And such accent is usually classified as acute, grave, or circumflex.[5, p.532]

What should be observed here is that the ‘deception’ is not due to the person merely asserting a false proposition, nor is the assertion of a false proposition a depiction. The deception occurs when a person has been led to assert a false proposition on the basis of an authority who was putting forward a proposition in an interrogative manner. In such a case, the person has failed to notice the relevant phonetic or morphological features that indicated this difference and thought the authority was asserting the proposition. Buridan says nothing here about whether asserting a false proposition is acceptable or possible, he is only outlining one way in which people can be deceived into asserting false things.

As such, Buridan does not settle this matter for us. His position may be different from both Corcoran and Lukasiewicz, however, it may also be possible to read Buridan’s ideas about assertion as governing how the propositions are treated in the drawing of the conclusion, in which case he would be somewhat closer to Corcoran on this point. i.e. the propositions need to be asserted, but they do not need to be true.

Applying Buridan’s definition of formal and material validity to syllogisms is fairly straightforward. As we have seen, a syllogism is a two premise argument made up of three terms which meet the criteria we discussed above. A syllogistic argument is formally valid if it is valid in *all* terms which have a similar form. In this case, that similar form would be if we hold the structure of the particular syllogism in question fixed, and then change the terms.

It should further be noted that the term *all* in this definition is actually a restricted quantifier. Buridan, taking note of various propositions in medieval theology, observes that there are counter-instances to all of the formally valid syllogisms as well as the principles that support syllogistic validity. This is the rationale for Buridan’s first conclusion in Book Three: ‘No syllogisms are formal in the common and customary way of speaking.’[52][p. 65] His justification is that the following arguments are all invalid:

1. This God is the Father and this same God is the Son, so the Father is the Son.

2. This Father is not the Son and this same Father is God, so God is not the Son.⁴⁶
3. Every God is the Divine Father, every Divine Son is God, so every Divine Son is the Divine Father.⁴⁷
4. No Son is the Divine Father, every God is the Son, so no God is the Divine Father.⁴⁸

Buridan's response to these counterexamples is amusing. Buridan simply says that it is not his place to speak on matters of theology,⁴⁹ and takes 'formal' in this context to mean 'in all terms excluding divine ones'. This is presumably because the standard medieval solutions to explain what has gone wrong in these sorts of cases would require Buridan to discuss theological issues (e.g. how one distinguishes the divine persons and in what sense they are still one.)⁵⁰ This response is amusing because Buridan makes copious use of one divine term (*deus*) throughout Books Two and Four of his work. In Chapter Four of Book Two Buridan uses divine terms as an example to explain the difference between his analysis of divided modal propositions and Pseudo-Scotus'. He also uses divine terms as counterexamples to various modal propositions and syllogisms.⁵¹ It should be pointed out that all Buridan wants to restrict in this case are the presence of divine terms that are problematic for syllogistic validity. In particular, he only needs to remove the divine terms 'Father', 'Son', and 'Holy Spirit'. If Buridan removes these he does not need to remove the term 'God'. Third, given what Buridan has said about consequences, it is quite easy to see how we can classify syllogistic arguments as one subclass of arguments treated by Buridan's theory of logical consequence. Recall that a syllogistic argument is composed of two premises and its conclusion. We can reduce this to a consequence, in the sense that Buridan defines in Book One, by taking the two premises, conjoining them together, and taking the consequent to be the conclusion of the syllogism. Looking at Buridan's definition of validity, we may wonder how we are supposed to be able to check that a particular syllogism is valid. It is quite easy to check for invalidity using this definition, but validity is quite a bit harder. To see which syllogisms are valid, Buridan puts forward two rules:

We lay down that affirmative syllogisms hold in virtue of the principle, 'Whatever are the same as one and the same are the same as each other'. Hence, from the fact that the extremes are said in the premises to be the same as

46. These serve as counterexamples to the validity of expository syllogisms. This is sufficient to undercut the validity of all syllogisms on Buridan's analysis.

47. This is a counterexample to Barbara.

48. This is a counterexample to Celarent.

49. It would be interesting to look at how Buridan's remarks here are picked up in medieval theology after him. In particular we may wonder if any medieval theologians explicitly took up the challenge Buridan gives here, i.e. to develop a syllogistic framework that works when divine terms are allowed. For a start to this question, see Chapter 7 of [64]. However, that question does not concern us here.

50. See [51], Book Three, Conclusion 1.

51. For a taste of this, see [51] Book Two, Conclusions 5, 6 and, 17 as well as Book Four, Conclusions 9, 10, 15, etc.

the one middle it is concluded in the conclusion that they are the same as each other. However, negative syllogisms hold by another principle, ‘Two things are not the same as each other if one is the same as something and the other is not’. Thereby an affirmative conclusion must be concluded from two affirmatives and a negative from an affirmative and a negative, since an affirmative proposition indicates identity and a negative non-identity.⁵²

These principles are the basis of the expository syllogism. One of the unique features of Buridan’s analysis of the syllogism is that he bases the validity of the syllogism on these two rules. In doing this, he departs from Aristotle, who based his syllogistic theory on the reduction of ‘imperfect’ syllogisms to the ‘perfect’ ones. He also differs from various medievals who understood the validity of the syllogism in terms of the ‘dictum de omni et nullo’. We will have quite a bit more to say about these rules and the role of the expository syllogism in Buridan when we formalise Buridan’s theory in Chapter Five. However, at this point, we make two observations. First, the principle holds by taking a singular term and showing that its suppositum either has two properties or that it has one property and lacks the other. As we have already seen for Buridan, an affirmative proposition is true if the subject and predicate supposit for the same thing(s). This principle is a very natural extension of this idea. In the affirmative case, the idea is that if we can show that one and the same thing have two properties we can conclude that there is something that has both of those properties. This principle also can be used in the case of universals in a similar way. Using this Buridan can justify the ‘dictum de omni et nullo’ and by extension show the first figure syllogisms to be valid.

Second, at the heart of this principle is the idea that we take a singular term that has the respective properties in question. At this point it should simply be noted that this principle is not modified or changed when we look at Buridan’s analysis of modal propositions. Having sketched Buridan’s theory of the syllogism and provided some explication of how the form/matter distinction works in Buridan’s theory, we now turn to his analysis of modal propositions.

4.5 Buridan’s Theory of Modality

Having looked at Buridan’s treatment of syllogistics as it pertains to assertoric propositions, we need to look at Buridan’s theory of modality. Two books of the *Treatise on Consequences* are concerned with modality. In the second book, Buridan develops his theory of consequence for single premise inferences. In the fourth book Buridan combines this material with the material in the third book to deal with syllogistic theory. Our goal in this section is to provide a brief summary of Buridan’s theory of modality as presented in Book Two of the *Treatise on Consequences*. This will serve as the basis from which we will develop our formal reconstruction of Buridan’s modal syllogistics in the next chapter, as well as the basis for some of our conjectures about Buridan’s theory of modality. For Buridan, a modal proposition is one that contains a modal term or adjective within the proposition itself:

52. The Latin can be found on page 186.

It should be noted that propositions are not said to be of necessity or of possibility in that they are possible or necessary, rather, from the fact that the modes 'possible' or 'necessary' occur in them.[51, p. 95]⁵³

Buridan's point is a linguistic one. For a proposition to be modal, a modal term must occur in the proposition. For example, the proposition 'Every human is an animal' is necessarily true (at least for Aristotle), but the proposition is assertoric, because no modal term or adjective occurs in the proposition.⁵⁴ Likewise, the proposition 'Every human is necessarily running' is a proposition of necessity and it is false. Modal propositions can be separated into two different groups: divided modal propositions and composite modal propositions. Concerning these, Buridan writes:

They are called 'composite' when a mode is the subject and a dictum is the predicate, or vice versa . . . They are called 'divided' when part of the dictum is the subject and the other part the predicate. The mode attaches to the copula as a determination of it.[51, p. 96]⁵⁵

For Buridan, the standard form of a proposition is: quantifier, subject, verb, predicate.⁵⁶ A composite modal proposition is one where the verb is not modalised,⁵⁷ but either the subject or the predicate is a modal term. The non-modal term is called the dictum of the proposition. In Latin the dictum is designated by using an accusative–infinitive construction. In such a construction the verb of the dictum is placed in the infinitive and both the terms relating to that verb are placed in the accusative. There are some challenges with literally translating this into English and it is standard to use dependent clauses to translate the dictum. For example, 'That every B is A is necessary' and 'It is possible that some B is A' are examples of composite modal propositions.

In contrast, a divided modal proposition does not contain a modal term i.e. the subject or the predicate is not a modal such as 'possible', 'necessary' etc. Instead it is the verb that has been modalized. This is best illustrated in English either by the use of verbs where the modal is an adverb. For example, 'A person is of necessity an animal' is a divided modal proposition for Buridan. In what follows we will often write e.g. 'A is necessarily B' for a divided modal proposition.⁵⁸

53. The Latin can be found on page 187.

54. On Buridan's reading this proposition is also necessarily true, as long as the modality is given the composite reading.

55. The Latin can be found on page 187.

56. This order can be changed in various ways to create non-standard propositions and gives rise to syllogisms with oblique terms. We will not discuss those sorts of propositions or syllogisms in this thesis.

57. A verb is modalised if either the modality is a 'feature' of the verb, e.g. 'can' or if a verb is modified by a modal adverb, e.g. 'is of necessity'. See [51, p.172], footnote 9 of Book Two.

58. This is ambiguous in English between 'A is of necessity B' and 'A is necessarily-B' where the hyphen indicates that the modality goes with the term, not with the verb. In the first case, according to Buridan, A is amplified to the possible, while, in the second case, A is not amplified. Unless we say otherwise, 'A is necessarily B' should be read as a divided proposition.

It should be observed that, in divided modal propositions, the negation operation can occur in three places.⁵⁹ First, the negation can occur in front of the modalized copula as in ‘Some A is not necessarily B.’ The second place the negation can occur is between the modal and the verb as in ‘Some B is necessarily not A’. The third place the negation can occur is in front of one of the terms as in ‘Some B is necessarily non-A.’ The third location requires the use of the hyphen, since the distinction is difficult to represent in English. The main difference between the second and third kind of proposition is that the former is a negative, particular proposition, whereas the later is actually a positive proposition. For Buridan, a proposition is counted as negative when the negation occurs in *front* of either the modal or the verb;^{[52][pp.38-39]} Otherwise it is positive. This construction is much more natural in Latin where it is perfectly grammatical and intelligible to write something like ‘*A non est B*’ whereas in English, ‘A not is B’ is not grammatical. To capture this distinction we will use ‘B is non-A’ for when the negation occurs after the verb and modifies the term. We will use ‘B is not A’ when the negation modifies the copula.

This distinction is important, because the truth conditions for negative and affirmative propositions differ in a number of ways. In assertoric propositions, only affirmative propositions have ‘existential import’. For example, if the proposition ‘Every A is B’ is true then there must exist some singular term whose supposita are not empty (i.e. there is something which is A), and everything that is A must also be B. In contrast, a negative proposition is true if there is nothing that the subject term supposits for or there is nothing that both subject and predicate supposit for. To see the difference, observe that for the proposition, ‘A person is non-running’ to be true, there must be some person who is not running. In contrast the proposition ‘A person is not running’ is true if there are no people in existence at all.

Buridan also points out that if two negations occur in the modal or the verb, or one occurs in each, then the proposition is equivalent to a positive one. Buridan illustrates this with the following example ‘B is not possibly not A.’ Buridan observes that this is clearly equivalent to an affirmative proposition (B is necessarily A), and so Buridan says he will treat those propositions as affirmative.

Before moving on to discuss the truth conditions of these propositions, it should be observed that there does not seem to be any principled reason in Buridan’s theory to prevent us from combining these two kinds of propositions together. For example, ‘It could be necessary that every A is B’ or ‘It is necessary that some B is possibly A’ seem to be perfectly good constructions by Buridan’s own lights. The main issue would be which kind of proposition is this, and if it does not fall into one or the other is this a problem for Buridan’s analysis? Such propositions are not considered by Buridan. And this should not pose a problem to his theory since, given what Buridan says about each kind of proposition, the combination should follow from the analysis of the two kinds of

59. Buridan does include truth and falsity among his list of modals. Because of this, it is possible to use falsity in the composite sense to define something similar to what modern logicians think of as propositional negation. We will briefly discuss this when we discuss the truth conditions for composite modal propositions. However we do not count this as one of the possible locations for negation and we do not intend to treat these modalities in our formalisation of Buridan’s theory.

modals discussed in his theory.

Buridan's analysis of divided modal propositions is one of the unique features of his modal theory. Buridan tells us that

It should be realised that a divided proposition of possibility has a subject amplified by the mode following it to supposit not only for things that exist but also for what can exist even if they do not.[51, p. 97]⁶⁰

According to Buridan, in divided modal propositions of possibility, the subject is amplified to supposit for that which is or can be. At this point, one may wonder why it is only the subject that is amplified and not the predicate. In a sense, both terms are amplified, since both the subject and the predicate are modified by a modal. In the case of the predicate, the modal component is made explicit by the modal that is modifying the verb. For example, in the proposition 'Every man is necessarily an animal' the predicate animal is modified by modal 'necessarily is'. In this way, one could say that the predicate is amplified. However, this is not a way that Buridan speaks. He only speaks of the subject being amplified. One possible reason for this may be that usually when a term is amplified there are usually no grammatical markers to indicate that it needs to be modified to have modal force. For example, in both the proposition 'Buridan's horse is dead' and 'every donkey is possibly an animal' there is nothing in the grammar of the sentence that suggests that either 'Buridan's horse' or 'donkey' needs to range over things that were Buridan's horse and things that can be donkeys respectively.

As Buridan goes on, he will later prove that given the assumption about the ampliation of propositions of possibility, and the equivalences between necessity and possibility, it follows that propositions of necessity also have their subject amplified in the same way.⁶¹ On Buridan's theory, 'Some B is possibly A' is equivalent to 'that which is or can be B is possibly A'. Likewise, 'Some B is necessarily A' is equivalent (via Book Two, Conclusion two) to 'that which is or can be B is necessarily A'. Why might we think this? According to Buridan this is a general feature about the way these kinds of verbs amplify their subjects. As an example, consider the proposition, 'Someone labouring was healthy'. Buridan tells us that this can be true in different ways. This proposition is true if there is currently someone labouring who was healthy at some point in the past. The proposition would also be true if there was some person in the past who was labouring. The case for future tensed propositions is analogous. This seems clear enough, but what is interesting is that Buridan goes on to remark that,

Thus, because possibility is about the future and all that is possible, the verb

60. The Latin can be found on page 187.

61. See [51], Book Two Conclusion 2. Buridan writes:

In every divided proposition of necessity the subject is amplified to supposit for those that can be.[51]

The Latin can be found on page 187.

4 The Modal Syllogism in John Buridan

‘can be’ similarly ampliates the supposition of the subject to everything that can be.⁶²

This could be one of the reasons that Buridan thinks that the divided propositions of possibility and necessity ampliate their subject.⁶³ Buridan’s observation is based on this connection between temporal and modal propositions. Roughly, we could think of this as arguing that,

- 1) Temporal propositions ampliate their subject.
- 2) If something will be the case, then it can be the case.
- 3) Therefore: modal propositions also ampliate their subjects.

To see this, assume that modal propositions do not ampliate their subject. We will show that a counterexample to 2) can then be constructed. Assume for the sake of argument that Socrates does not currently exist, but that he will, and that he will be white. Then the following proposition is true: ‘Socrates will be white’ since temporal propositions ampliate their subjects. By 2) it follows that ‘Socrates can be white’ is also true. But since we assumed that modal propositions do not ampliate their subject, this holds only if ‘that which is Socrates can be white’ is true. But it is not, since we assumed Socrates does not exist.

To muster further support for his theory, Buridan later goes on to contrast this reading of the modal terms with Pseudo-Scotus’ analysis of modal propositions.⁶⁴ On Pseudo-Scotus’ analysis, ‘Some B can be A’ is equivalent to saying ‘That which is B can be A or that which can be B can be A’. Buridan points out that this definition is not equivalent to his own. To illustrate the point he uses the proposition ‘A creating God can fail to be God’. Buridan argues that on his view the proposition comes out false. This is because the contradictory, ‘Every creating God is necessarily God’ is true on Buridan’s analysis.⁶⁵ On Pseudo-Scotus’ reading this must come out false, since the contradictory of the particular distinctions will have to be conjoined universal propositions. (i.e. the contradictory of ‘Everything which can be B is not necessarily A or everything that is B is not necessarily A’ is ‘Everything which can be B is necessarily A and everything that is B is necessarily A’.) The gist of the argument, is that, since God can fail to be creating, a creating God can be nothing. Then, since God is something, the conjunct ‘Everything that is B is necessarily A’ is not true.

Buridan prefers the first reading of the divided proposition over the second because,

If I say ‘Every B can be A’, there is just one subject and just one predicate and a single simply predicative proposition, and the subject is distributed all

62. The Latin can be found on page 187.

63. In modern terms we would express this as the following modal ‘bridge’ principle: $F\phi \rightarrow \diamond\phi$, where F is a future tense operation.

64. See Pseudo-Scotus Questions on the Prior Analytics (included in John Duns Scotus, *Opera Omnia*, ed. Vivs, Paris 1891-5, vol. 2).

65. Observe that everything which is or can be a creating God is necessarily God. Implicit here is the assumption that God exists necessarily, which most medievals accepted.

at once by a single distribution. So it seems better to analyse it by a single predicative proposition with a single subject and a single predicate, even though the ampliation of the subject makes the subject of the analysis and a disjunction of the verb 'can be' with the verb 'is'. [51, p. 98]⁶⁶

This will be important when we propose an analysis of what Buridan's theory of modality is. For our analysis to be successful, it should be able to explain why modal terms amplify the subject in the way Buridan argues they do. It also needs to be consistent with the connections he sees between tense and modality.

We should also observe that Buridan does not push his rejection of Pseudo-Scotus' position too far. He is quite happy to allow people to define divided propositions however they wish, since names and utterances signify by convention, but he prefers this reading because of the reasons sketched above.

Given this conventional caveat, there is also an important theoretical benefit to having uniform ampliation of subject terms. If the modal terms all amplify their subjects uniformly, then we obtain an octagon of opposition between the modal propositions of necessity and possibility. If the ampliation is non-uniform, then such an octagon may not arise. For example, if we were to posit that 'No A is necessarily B' does not amplify its subject, but 'No A is possibly B' does, then it is easy to check that this will block the inference from 'No A is necessarily B' to 'No A is possibly B'. What this means is that the terms need to have a uniform modal ampliation. This leaves only a few possible options:

1. The subject is amplified to supposit for what can be.
2. The subject is amplified to supposit for what necessarily is.
3. The subject is amplified to supposit for what can contingently be.
4. The subject is not modally amplified.

Given what we have already seen about temporal propositions, the possibility reading is the most natural. In practice, the second and third readings will not work very well. The second reading makes the modal propositions very 'narrow' in the sense that they would only range over the things that are necessarily the subject. The third reading is also problematic. It would make it very difficult to talk about things which are necessarily both A and B. The fourth reading seems to be a plausible alternative. The truth conditions for composite modal propositions are, in a sense, more complicated than their divided counterparts. However, there is also a sense in which they are much simpler. In composite modal propositions, we need to understand that the terms 'possible' and 'necessary' need to be read differently than they are in divided propositions. Buridan tells us that,

66. The Latin can be found on page 187.

Here ‘possibility’ is taken not for what can be but for a possible proposition, which is said to be possible in so far as things can be altogether as it signifies.[51, p. 105]⁶⁷

In composite modal propositions the term ‘possible’ ranges over a different class of things than it does in divided propositions. In the divided case, the modality ranges over the objects which a term supposit for, e.g. the things that can be A or must be B. In the composite case the modalities range over different kinds of propositions, not over different supposita. As we have already seen, a proposition is possible for Buridan just in case it signifies what can be the case. Likewise, a proposition is necessary if it signifies what is necessarily the case. Structurally, these propositions are read differently than the divided ones. For example, a universal affirmative proposition with the modal as the subject, would read ‘Every possibility is that B is A’ or slightly more fleshed out, ‘Every possible proposition is that B is A’. It is this difference in reading that prompts Buridan to remind his readers to be very careful to avoid conflating the two senses.⁶⁸ Another thing to observe is that the supposition in the composite case is different than it is in the divided case. In the divided case, the subject and predicate both supposit for objects. In composite modes, the dictum has ‘material supposition’ i.e. the term supposit for the proposition the dictum names.

When we have a composite modal with the dictum as the subject, Buridan’s analysis of the proposition is similar, however expressing these propositions in grammatical English can be somewhat difficult. At this point it is best to simply observe that in such cases we would analyse these propositions as: every proposition which is X is a possibility, where X expresses the dictum of the proposition.

Composite modal propositions are simpler, because they do not cause the terms of the propositions to be amplified. Buridan observes that because of this, there is very little that needs to be proven about such propositions and their corresponding syllogisms. Buridan only has two conclusions discussing syllogisms with composite modals. See [51, pp. 141-142] Book 4, Conclusions 1 and 2.

In a sense, what we have are two distinct sets of modalities, or better: The location of the modality in a sentence determines the class of objects we are dealing with. Divided modals are modals that amplify their subjects and range over objects. Composite modals range over dictums/propositions, interact with the signification of terms, but do not amplify anything in the dictum.⁶⁹ Because of this, when we speak of Buridan’s theory of modality, we need to realize that we are speaking of modality in both of these senses.

What are the implications that Buridan draws from his theory of modality? In Book Two he draws 19 conclusions. We will start with the first eight inferences as they all

67. The Latin can be found on page 187.

68. As an easy example of the difference, consider the proposition ‘A white thing can be black’. If we read this as a divided proposition, then it is true, since there is something which can be white and can (at a later point) be black. However, the composite reading is impossible, since nothing can be white and black (all over) at the same time.

69. The lack of ampliation is not trivial. As Buridan points out, some terms cause other terms to be amplified. The best example of this is the term ‘dead’.

pertain to divided modal consequences which are the main focus of our formalisations in Chapter Five. We will then sketch conclusions 10, 17 and 18, as these conclusions explain how inferences between divided and composite modal propositions function. We will need this information for our discussion of Buridan's ontology in Chapter Six.

Buridan's first conclusion is that:

From any proposition of possibility there follows as an equivalent another of necessity and from any of necessity another of possibility, such that if a negation was attached either to the mode or to the dictum or to both in the one it is not attached to it in the other and if it was not attached in the one it is attached in the other, other things remaining the same.[51, p. 99]⁷⁰

What Buridan means is that it is always possible to convert a proposition of necessity to a proposition of possibility, by correctly replacing the negations in a proposition. For Buridan, a proposition is said to be 'of necessity' (*de necessario*) if it explicitly contains the modal 'necessary' and similarly for possibility. As an example, Buridan observes that 'Every B is necessarily A' is equivalent to the proposition 'Every B is not possibly not A', which then yields 'No B is possibly not A'. This conclusion is important as it gives some insight into how negation works in Buridan's logic, and it is necessary to establish the second conclusion.

The second conclusion shows that divided propositions of necessity amplify their subjects to supposit for what does or can exist. Buridan writes:

In every divided proposition of necessity the subject is amplified to supposit for those that can be.[51, p. 100]⁷¹

Buridan's proof for this conclusion can be reconstructed as follows: First, let us assume that propositions of possibility amplify their subjects. Now the proposition 'Some B is possibly not A' contradicts 'No B is possibly not A'. Observe that both of these are propositions of possibility and so amplify their subjects. That they are contradictory follows from what Buridan assumed about contradictory propositions. Now, from the previous conclusion, we have 'No B is possibly not A' is equivalent to 'Every B is not possibly not A' which (again by the first conclusion) is equivalent to 'Every B is necessarily A'. But for 'Every B is necessarily A' to be equivalent to 'No B is possibly not A', the subjects of both propositions need to be amplified.

These two conclusions provide the foundation for the analysis of modal propositions that Buridan goes on to offer in the rest of Book Two of the *Treatise*. In a number of places, Buridan needs to ensure that the subject is amplified for his conclusions to follow.

In presenting this conclusion Buridan implicitly rejects William of Ockham's analysis of propositions of necessity. Due to lack of space we do not have the time to develop this observation in detail. The full details of this argument can be found in my paper [23].

70. The Latin can be found on page 187.

71. The Latin can be found on page 187.

4 The Modal Syllogism in John Buridan

Buridan's argument for this conclusion functions, in some sense, as an argument against Ockham. Buridan argues that if one wants possibility propositions to amplify their subjects to the possible (as Ockham does), then one must either reject the relationships of contradiction that Buridan lays down in Chapter Five, or accept that necessity propositions also amplify their subjects in the same way. Conclusions three and four show, respectively, how necessity and possibility propositions relate to assertoric propositions. Conclusion three states that:

From no proposition of necessity does there follow an assertoric or vice versa, except that from a universal negative of necessity a universal negative assertoric follows.[51, p. 101]⁷²

While conclusion four propounds that:

From no proposition of possibility does there follow an assertoric or vice versa, except that from every affirmative assertoric proposition there follows an affirmative particular of possibility.[51, p. 102]⁷³

Here Buridan claims that only universal negative necessity propositions entail negative assertoric propositions, and that only particular affirmative assertoric propositions entail particular affirmative possibility propositions. Before going on to sketch the motivation for this, it should be observed that when Buridan speaks of a universal negative of necessity, what he means is a proposition of the form: 'Every B is necessarily not A', as he gives this form in his proof of the proposition.⁷⁴ Given the equivalences we discussed previously, this is equivalent to 'No B is not necessarily not A' which in turn is equivalent to 'No B is possibly A'. The motivation for these conclusions is as follows: Assume that No B is possibly A. For Buridan, this means that the set of things that are possibly B and possibly A are disjoint. Buridan implicitly assumes that everything that is A is also possibly A, and so it follows that what is A and what is B must also be disjoint. The reasoning is similar when looking at the particular affirmative of possibility. As our formal semantics will make perspicuous, the reason that the inference only holds in these cases is because the subject is amplified to supposit for possible objects.⁷⁵

Conclusions five and six address the conversion of modal terms. Conclusion five reads as follows:

From every affirmative of possibility there follows by conversion of the terms a particular affirmative of possibility, but not a universal, and from no negative of possibility does there follow by conversion of the terms another of possibility.[51, p.103]⁷⁶

72. The Latin can be found on page 187.

73. The Latin can be found on page 187.

74. See [4, p.65].

75. For the proofs of these conclusions, it should be noted that, from a modern formal perspective, this amounts to assuming that our frames must be reflexive.

76. The Latin can be found on page 187.

Conclusion six asserts that:

From no proposition of necessity does there follow by conversion of the terms another of necessity, except that from a universal negative there follows a universal negative.[51, p.104]⁷⁷

In these conclusions Buridan shows that the particular affirmative of possibility and the universal negative of necessity convert simply. Again, if we think of this in terms of disjoint sets, the reasoning is clear. If 'no A is necessarily B', then the set of things that are possibly A and the set of things that are possibly B are disjoint. Hence, 'no B is necessarily A' will also be true.

The seventh and eight conclusions discuss the valid inferences that hold between propositions of contingency. The seventh conclusion asserts that:

Every proposition of each-way contingency having an affirmed mode is converted into [one of] the opposite quality with an affirmed mode, but none is converted if the result of conversion or what was converted had a negated mode.[51, p.104]⁷⁸

While the eighth conclusion reads as follows:

No proposition of contingency can be converted in terms into another of contingency, but any having an affirmed mode can be converted into another of possibility.[51, p.105]⁷⁹

Buridan's semantics for contingency propositions are tricky and hinge on the location of the negation operation which Buridan handles by the distinction between 'affirmed modes' and 'negated modes.' As Hughes rightly observes:

The other main point that [Buridan] makes about divided contingency propositions is that 'it is contingent to be' and 'it is contingent not to be' are equivalent. In this way 'contingent' differs strikingly from 'necessary' or 'possible'. So, if I extend my notation for divided modal propositions by writing ' $A \overset{Q}{a} B$ ' for 'A is contingent to be B' etc., we shall find that the two forms AaB and AiB will be equivalent to the corresponding E and O forms, which will therefore not be needed. [20][p.100]

The equivalence rests on observing that if something is contingently B then it both can be B and can fail to be B. If we say that something is contingently not B, this is equivalent

77. The Latin can be found on page 188.

78. The Latin can be found on page 188.

79. The Latin can be found on page 188.

(assuming double negation elimination) to ‘Something can be B and something can fail to be B’, yielding the desired equivalence. If the negation occurs after the modal operation, this means that mode is affirmed. In this case, we read the negation operation as occurring after the mode, modifying the copula. This reading of contingency propositions is required for a number of the syllogisms that Buridan claims are valid, to be valid in our semantics.⁸⁰

In contrast to this is the case where the mode is negative. In this case, we would place the negation in front of the modal operation, as in the proposition ‘Every A is not contingently B’. These create their own contradictory pairings. Hence ‘Every A is not contingently B’ contradicts ‘Some B is contingently not A’ and hence ‘Some B is contingently A’ as we have already seen. Likewise ‘Some A is not contingently B’ contradicts ‘No A is contingently B’ and ‘Every A is contingently B’.

In commenting on Buridan, Hughes also notes that these two modal propositions correspond to the cases where the negation occurs outside the scope of the modal operation. Here we will use \bar{Q} to represent such cases, and present semantics that also capture these propositions. These cases correspond to the cases where the negation occurs in front of the modal. These are needed for some of the conclusions that Buridan discusses in Book Four of the *Treatise*, which we will not treat in detail here.

Buridan’s ninth conclusion highlights some of the unique implications that follow from the material supposition of the dictum. The conclusion states that:

In all composite modals in which the dictum is subject, from a particular there follows a universal, the rest being unchanged.[51, p.106]⁸¹

The proof of this has to do with the relationship between the dictum and signification. As an example, consider the proposition ‘Some proposition ‘B is A’ is possible’. Buridan observes that among all the propositions ‘B is A’ each one of these signifies the same thing as all of the other ones.[4][p.71] Hence, if one of them is true, then all of them will be true, and vice versa. The upshot of this is that Buridan can dispense with the universal and particular quantifiers in the context of composite modal propositions.

Buridan’s seventeenth and eighteenth conclusions deal with the relationship between composite and divided propositions. The seventeenth conclusion shows that the only composite modal claim that entails a divided modal claim is the particular affirmative. Buridan doesn’t really prove the positive part of the proposition. He simply remarks that ‘if “Some B is A” is possible then “Some B can be A” clearly follows.’ This can be clearly seen, since “Some B is A” is possible if and only if “Some B is A” signifies as can be the case’ and from this it seems clear that something which is or can be B can be A. Buridan offers counterexamples to rule out the other possible inferences. In the universal affirmative, he uses the following counterexample:

Although ‘Everything running is a horse’ is possible, it does not follow univer-

80. For example, QMQ Ferison and Bocardo are not valid if the negation is in front of the mode.

81. The Latin can be found on page 188.

sally that everything running can be a horse, because an ass may be running, but it cannot be a horse.[51, p.110]⁸²

Buridan's point is that even though the composite is true (imagine a world in which only horses are running), the divided sense is clearly false, since a horse can be running, but it is impossible that a horse be an ass. That nothing follows from the divided to composite direction is equally clear. Buridan provides counterexamples to these as well. The reason all of the counterexamples work has to do with the fact that divided terms are amplified. For example, the proposition 'Something white can be black' is true, because something which is currently white can be black. However, the proposition 'Something white is black is possible' does not have the dictum amplified. It is only possible if the proposition 'Something white is black' signifies as can be the case. However it cannot signify this because 'Something white is black' signifies that something white is black and that is impossible.

So, having sketched elements of Buridan's modal theory, let us turn to the natural question this raises. What is Buridan's theory of modality? That is to say, what does Buridan think modals actually are? To clarify this question it is helpful to refer to one modern theory of modality. According to a view popular since the introduction of Kripke semantics, and having roots going back at least as far as Leibniz, modalities can be analyzed in terms of possible worlds. On this theory, when we say that ' ϕ is possible' (relative to some world w) what we mean is that 'there is some possible world accessible from w such that ϕ is true at w '. Necessity is defined in a similar manner.

If we were to ask what are the modalities on the Kripke semantics view, the answer would be that it is a reductive account of modality where possibility and necessity are reduced to possible worlds and accessibility relations on those worlds.⁸³ In the modern case, this has led into a number of discussions about the nature and ontology of possible worlds, discussions about 'impossible' worlds, the nature of actuality and many other philosophical debates. When we ask this question of Buridan's theory, we are asking for a few things. First, we want to know what kinds of things these modal operations are. Can possibility talk be defined in terms of tense? Can it be defined in terms of 'ways the world can be'? Does Buridan's account of modality not yield to any kind of explication? What we are asking for is an explanation as to what possibility and necessity are. Second, we are asking for the truth conditions for the various kinds of modal operations. To this question, hopefully the above remarks have made sufficiently clear when Buridan thinks various kinds of modal sentences are true or false. Our main concern is with the first question. What does Buridan take modal talk to be about? Before we present some possible options, we should observe that in no place in the *Treatise on Consequences* does Buridan explicitly attempt to provide a definition or reduction of modality. What we are doing is a bit of historical reconstruction. There is one other piece of information that we need to keep in mind. When we speak of Buridan's theory of modality, it seems that we are actually dealing with *two* sets of modal operations.

82. The Latin can be found on page 188.

83. We say that a theory of modality is reductive if modal terms do not occur in the definitions of possibility and necessity.

As we already observed, composite and divided modals range over different kinds of objects and have different sets of truth conditions. Because of this, when we discuss Buridan's view, we will need to evaluate proposals against both aspects of his modal theory. We have already seen that Buridan draws a close connection between temporal operations like 'will be' and 'was' and modal operations like 'possibility' and 'necessity'. Could Buridan think that modality can just be defined in terms of temporal operations? This view was defended by some in the Middle Ages.⁸⁴ On this view, a proposition is possible just in case the proposition either was true, is true or will be true. Likewise a proposition is necessary if it always was, is and will be true. I think it is safe to say that this view is not Buridan's view for divided propositions. Buridan says nothing to explicitly rule out this reading for composite modals. He makes the following remark which tells against a temporal reading of his divided modalities, "Many are the possible things which never are, will be or have been" [51, p.72].⁸⁵ This is found in a list of propositions that Buridan takes to be uncontroversial and conceded by most people. It may be possible to reject such a reading by pointing out that it doesn't follow that Buridan is one of these people. However, Buridan does not argue against the proposition in this list, and in the absence of evidence to the contrary, this seems to tell against such a reading. It should be fairly clear how this undermines the temporal reading. The reason is that if the proposition "Many are the possible things which never are, will be or have been" is true, then possibility is not exhausted by temporality. While that is not sufficient to rule out the temporal interpretation as applied to divided propositions, I think this is enough to suggest we should be looking elsewhere for Buridan's modal theory. G. Hughes in his paper on Buridan's modal logic makes the following remark:

For a long time I was puzzled about what Buridan could mean by talking about possible but non-actual things of a certain kind...what I want to suggest here, very briefly, is that we might understand what he is saying in terms of modern 'possible world' semantics...It seems to me, in fact that in his modal logic he is implicitly working with a kind of possible worlds semantics throughout.[20][p. 3]

This is a very interesting proposal. If Hughes' remarks are correct, then in Buridan we have a kind of 'proto possible worlds' analysis of modality. In order to properly assess Hughes' remarks we will need to better situate Buridan within his medieval intellectual context. This will be done in Chapter Five, and so a complete discussion of this proposal will have to wait until then. However, two points should be made here. First, it should be remarked that, because of the success possible worlds semantics has enjoyed in the last 30 years or so, for many philosophers, possible worlds semantics has become synonymous with modality. For some, to say ' ϕ is possible' is just to assert that ϕ is true in some possible world. When we turn to historical situations, however, we need to be very cautious about anachronism and projecting our philosophical theories onto those who

84. For example, Lambert of Lagny. See [64] for a discussion of Lambert's logic and a formal reconstruction of it.

85. The Latin can be found on page 188.

have come before us. Just because we have a very popular and successful theory, it does not follow that this is what our philosophical precursors thought.

Second, from what we have already seen in Buridan, there may be some reason to see a 'proto possible worlds' analysis in Buridan. This is best seen when we look at composite modal propositions, which Hughes did not dwell on in his paper. Recall that composite modal propositions have to do with the relationship between the dictum, the signification of the proposition that the dictum expresses, and the way that things can be. 'It is possible that B is A' is true for Buridan, just in case it is possible that what the dictum 'B is A' supposits for is true. The connection with possible worlds semantics seems to come from the requirement that things 'can be such that ...'. When Buridan talks of the ways things can be, a natural thought would be to see in this a sort of 'possible worlds' analysis of modality. The move here that needs to be supported textually is the idea of there having been multiple different ways that the world could have been. Unfortunately, there is little in *The Consequences* that would suggest this beyond the considerations we have already mentioned. It would be helpful at this point to look at Buridan's discussion of modality in the *Summulae* as well as some of his commentaries and questions on *The Physics* to see if Buridan suggests a closer connection between possibility and ways things could be. We will say more about this in Chapter Six, starting on page 153.

Another factor that comes into play here is Buridan's nominalist commitments. As we have remarked, for a proposition to be true, it must be formed. This raises some important issues as it relates to the definition of consequence. One such example is the following argument: 'No proposition is negative, so no ass is running'[51, p. 67].⁸⁶ As Buridan points out, this inference is not a good one. If this argument were valid then its contrapositive would be valid as well. The contrapositive is 'Some ass is running', therefore 'Some proposition is negative'. However, this clearly does not follow, if we allow for propositions to be contingent entities (of either the mental, spoken or written kind). As Buridan points out, God could wipe out all of the negative propositions while the donkey is running.⁸⁷ This requires Buridan to tweak the definition of consequence to rule out such arguments as valid. The way this is done, implicitly, is by drawing a distinction between what is true of a situation or of some time and what is true at something/sometime. This idea was developed by Arthur Prior in his paper *The Possibly True and the Possible*. For Buridan, the idea is that 'No proposition is negative' may be true of a given situation even if it is not true at that situation. Already this kind of distinction seems to point toward a theory of 'ways the world could be.'

The divided case is more interesting and is what leads Hughes to initially propose that Buridan had something along the lines of possible worlds in mind. Hughes remarks:

Did he mean by a 'possible A', I wondered, an actual object which is not in

86. The Latin can be found on page 188.

87. Implicit in this point is a theological assumption that Buridan does not point out, namely that however God conceives of truths, God does not form the relevant type of negative propositions, otherwise, this counterexample would fail.

fact A but might have been or might become A? My house, e.g., is in this sense a possible green thing because although it is not in fact green, it could become green [...]. But this interpretation won't do; for Buridan wants to talk e.g., about possible horses; and it seems quite clear that he does not believe that there are, or even could be things which are not in fact horses but which might become horses. [20, p.97]

Recall that a divided modal proposition, e.g. 'A can be B' is true just in case 'that which is or can be A can be B' is true. Hughes starts with the observation that when Buridan speaks of 'that which can be A' he is not only quantifying over the things that now exist, and could become A. Buridan's temporal analogy makes this point very clear. Recall that, 'B was A' has two causes of truth and in the case of 'that which was B was A', this is true even if the object which was both B and A does not exist. Hughes goes on to point out that when Buridan uses divided modals he wants to quantify over possible objects as well as actual objects. This is very well supported in the text and in Buridan's semantics. But how are we to make sense of this talk about possible objects? Here Hughes draws a very natural connection between possible objects and possible worlds.[20, p. 97] He points out one very natural reading of 'that which is or can be A can be B' is as saying that there is some possible world where something is A and there is another (possibly the same) possible world where the thing which is A is also B.⁸⁸ This provides a very natural reading of Buridan's divided modal propositions.

From what we have sketched above, there is a reading of Buridan where the modalities are understood by a kind of possible worlds semantics. Both divided and composite modal propositions can be understood in terms of possible worlds. Formally, this invites us to take up Hughes' sixth challenge with which he ends for [20]. Here he remarks that,

A much more elaborate project still would be to try to give a Kripke-style possible worlds semantics for Buridan's modal system and then an axiomatic basis for it. [20, p. 108]

Parts of this challenge have been taken up by other authors. For example, in [59] an axiomatic analysis of Buridan's modal syllogistic is developed. As we will point out briefly in the next chapter, this formalisation is not entirely satisfactory. Further, until my paper [22], a semantic analysis of Buridan's theory had not been put forward nor has any formal analysis that takes seriously Buridan's use of the expository syllogism.⁸⁹ Our goal in the next chapter will be to develop a possible worlds semantics that coincides with what Buridan says in the *Consequences*. In a sense this will vindicate the claim that Buridan's modal syllogism is consistent with and can be understood in terms of a

88. Notice that the requirement is not that the object in question be both A and B at some world. This would be to confuse the composite and divided sense of the proposition. Consider the proposition 'Something white can be black'. There will be no world where any object will be both white and black.

89. There are some mismatches between Thom's formalisation of Buridan and the inferences that Buridan says are valid. Specifically, Thom's formalisation predicts some syllogistic figures to be valid, even though Buridan denies they are and provides counterexamples for them.

possible worlds semantics. In our next chapter we will develop our own semantic and syntactic reconstruction of Buridan's logic and show that it is both sound and complete.

In conclusion, we have outlined and developed Buridan's understanding of the form-matter distinction as he applies it to propositions and syllogisms. We have explored some of the implications of this theory as it pertains to the validity of syllogisms. Likewise we have developed Buridan's account of modal propositions to unpack Buridan's use of the terms 'necessary' and 'possible' and pondered exactly what Buridan takes these terms to mean. Having done this we now turn our attention to the formal component of this thesis. In what follows we will use this analysis of Buridan to provide a formal reconstruction of his modal syllogistics.

5 A Formal Reconstruction of Buridan's Modal Logic

5.1 Introduction

G. E. Hughes concluded his paper, *The Modal Logic of John Buridan*[20] with the following challenge:

A much more elaborate project still would be to try to give a Kripke-style possible worlds semantics for Buridan's modal system and then an axiomatic basis for it. I think this could probably be done and would be worth doing; but it would take us well into the twentieth century. [20][p.108]

To that end, this chapter will pursue two tasks related to this project and in doing so will answer Hughes' challenge. Our first aim in this chapter is to develop a Kripke-style possible worlds semantics for the divided fragment of Buridan's modal logic. This theory will be grounded in Buridan's remarks about how ampliation and supposition are used to account for the causes of truth for various modal and non-modal propositions. This will draw strongly from our observations about Buridan's modal logic in the previous chapter.

Up until this point in the literature of Buridan there have been no attempts to present a formal semantics for Buridan's modal logic, nor, so far as I am aware, has there been a syntactic treatment of Buridan's theory of the expository syllogism. Buridan's modal logic has been discussed in a number of places informally (e.g. [25] and [35]). In [59] a syntactic treatment of Buridan's modal logic was developed, the results of which will be discussed below. The informal discussions of Buridan's logic have raised a number of questions about his logic. Our aim in this chapter is to prove that Buridan's discussion of the modal syllogism is consistent, in the sense that we can develop a semantic system that captures all of the inferences Buridan says are valid, and refutes the ones he says are not. With this semantics in place, we will extend the semantic theory to include singular terms, and prove that it is complete with respect to Buridan's views on the expository syllogism.

As is well-known, there are a number of different approaches one can take to establishing the completeness of a logic, which have various virtues. Our aim here will be to develop a natural deduction system for Buridan's modal logic. Our motivation for this system, however, will not be entirely formal in nature and some of the rules, while inspired by Buridan's remarks, will not be explicitly justified by Buridan's remarks in the *Treatise on Consequences*.

In his discussion of the assertoric syllogism, Buridan grounds the syllogism not in the dictum de omni et nullo, but in the expository syllogism. While Buridan does not spend very much time discussing differences between how the expository syllogism works in assertoric cases and how it works in modal cases as we have seen, he does make use of it in the modal syllogism, and as such formalising this theory suggests a very natural way to prove completeness for Buridan's semantics.¹ Our goal will be to provide one representation of Buridan's logic semantically, and the other syntactically, and to then show that the systems are sound and complete relative to each other. This will answer Hughes' challenge, but also show something more. It will show that Buridan's two different ways of treating modal propositions in the *Treatise on Consequences* are compatible with each other and show that Buridan's modal logic can answer to the standards of modern modal logic. In doing this, we hope this will demonstrate just how impressive a logician Buridan was.

To that end, this chapter will start by providing an exposition of Buridan's theory of the modal syllogism and answer the first part of Hughes' challenge: developing a semantics that is able to formalise Buridan's divided modal syllogisms as outlined in Book Four of the *Treatise on Consequences*. After this we will have a brief diversion to discuss the expository syllogism and its function within Buridan's theory. We will then go on to expand the semantics with singular terms and show how we can develop a proof-theoretic account of the expository syllogism. This will be based on Buridan's treatment of singular terms in Books Three and Four of the *Treatise*. We will conclude with a proof of completeness using a natural (although not entirely trivial) modification of the canonical model construction.

We have already laid the foundation for our treatment of single premise inferences in the previous chapter. We recall a few points about syllogisms before we turn to our formal treatment of Buridan's logic.

5.1.1 Syllogisms

In Book Three of Buridan's *Treatise on Consequences* Buridan defines a syllogism in the following way:

In the second chapter, we take it that every syllogism links the middle term in the premises with each extreme from the conclusion, so that on account of that linking the linking of the extremes is inferred, either affirmatively or negatively.[51, p.115]²

From this definition Buridan makes a number of points about syllogisms. First, he takes it that a syllogism only contains three terms; second, he observes that the middle term does not occur in the conclusion, and third; that the two extreme terms must occur in the conclusion.[4][p.82]. Following usual phraseology, Buridan refers to the first

1. In his treatment of the modal syllogism Buridan mentions the use of the expository syllogism in Book Four Conclusions 6, 9, 12, 18, 19, 23, and 28.

2. e Latin can be found on page 188.

premise in the syllogism as the major premise and the second premise as the minor premise. From his definition it also follows that there are only four possible figures. He remarks that,

It should be noted that the fourth figure differs from the first only in the transposition of the premises, and that transposition does not permit inferring another conclusion or prevent that inference, but only affects whether the conclusion inferred is direct when in the first figure and indirect in the fourth and vice versa.[51, p.116]³

When considering valid syllogisms, Buridan observes that the fourth figure only differs from the first figure in terms of the order in which the premises occur, and if the conclusion is direct or indirect. Because of this he does not bother to treat the fourth figure.[51, p.116][4][pp.82-83] Likewise, when we define the syllogism formally and treat Buridan's theory we will only consider the first three figures. The treatment of the fourth figure follows easily from our treatment of the first figure.

When it comes to the modal syllogisms that Buridan takes to be valid, there are a number of interesting differences between Buridan and other writers on the syllogism. Stephen Read has very helpfully summarised the syllogisms that Buridan claims are valid. The tables, which we reproduce below, can be found in[51, pp.41-42, 44] of Read's translation of the *Treatise*. As we are not treating restricted modals in this thesis, we will only include the cases for L, M, and Q. References to the *Treatise* can be found in Read's Tables.

| | L | M | X | Q |
|---|------------------|---------------------|--|---------------------|
| L | L | L, M, Celarent X | Darii, Ferio L, Barbara X, Celarent X | L, M, Celarent X |
| M | M | M | Darii, Ferio, M | M |
| X | M, Celarent X | ∅ | Darii, Ferio M, | ∅ |
| Q | M, Q | M, Q | Darii, Ferio Q, | Q |

Table 5.1: Valid First Figure Syllogisms

3. The Latin can be found on page 188.

5 A Formal Reconstruction of Buridan's Modal Logic

| | L | M | X | Q |
|---|---------------------------------------|----------------------------------|--------------------------------------|----------------------------------|
| L | L | L,M, Cesare X, Camestres X | Festino L, Camestres, Baroco X | L,M, Cesare X, Camestres X |
| M | L,M,Cesare, <i>Camestres</i> X, | ∅ | ∅ | ∅ |
| X | M, Cesare X, Camestres X | ∅ | Festino M | ∅ |
| Q | M, Cesare X, Camestres X | ∅ | ∅ | ∅ |

Table 5.2: Valid Second Figure Syllogisms

| | L | M | X | Q |
|---|-------------------------|-------------------------|--|-------------------------|
| L | L,X | L,M | Darapti X, Felapton X ⁴ , Datisi X, Ferison X Darapti L, Felapton L, Datisi L, Ferison L | L,M |
| M | M | M | Darapti, Felapton, Datisi, Ferison M | M |
| X | Darapti X, Disamis X | Darapti M, Disamis M | Datisi M | Darapti M, Disamis M |
| Q | M, Q | Q | Disamis M, Bocardo M, Datisi Q, Ferison Q | Q |

Table 5.3: Valid Third Figure Syllogisms

In each table, the modal in the leftmost column corresponds to the modality of the major premise. The modal in the top row corresponds to the modality of the minor premise. Each of the remaining boxes lists the valid syllogisms in that figure which

4. The original text reads X. However Professor Read notes in his errata that is not correct. The errata can be found at http://www.st-andrews.ac.uk/~slr/Buridan_errata.html.

follow from that pair.⁵ If the conclusion is simply a modal, then this means that the argument is valid for every syllogism in which the corresponding XXX syllogism is valid.

There is an interesting interpretive question as to whether or not the syllogisms *Barbari* and *Celaront* should be included in the first figure tables and the corresponding weakened modes included for the second and third figure tables. Buridan rarely talks about *Barbari* or *Celaront*. In his formulation of the syllogism Buridan is primarily interested in syllogistic propositions that can generate valid arguments.[5][p.308] (i.e. Buridan's interest is in syllogistic pairs from which some conclusion follows.) He is less interested in which conclusions those happen to be and how many conclusions can be drawn from a given pair. This is why Buridan does not need to include *Barbari* if *Barbara* is valid or include *Celaront* if *Celarent* is valid, nor does he need to treat the fourth figure syllogisms. However, in cases where *Barbara* (or *Celarent*) are invalid, it does not follow that *Barbari* will also be invalid. In any mode for which a modal *Darii* syllogism is valid (e.g. *Darii LXL*) it will follow that *LXL Barbari* is also valid, since universal affirmative propositions entail particular ones. This does occur in a few cases, most notably in the *LXL* mood. Buridan's phrasing of the conclusions may help settle this matter. He writes:

From a major [premise] of necessity and an assertoric minor there is always a valid syllogism in the first figure to a particular conclusion of necessity, but not to a universal.[51, p.151]⁶

Buridan's phrasing certainly allows for *Barbari* and *Celaront* to be counted as valid syllogisms here, since they satisfy the conditions for the conclusion. In addition, if we are only interested in syllogistic tuples that generate productive pairs, then *Barbari* and *Celaront* should be included, since they are both productive tuples.

Another interesting part of Buridan's theory stems from his analysis of *LXL* propositions in the first figure. In the previous chapter we mentioned the so called 'two *Barbara*'s' problem, namely the problem of explaining why, for Aristotle, *LXL Barbara* is valid, while *XLL Barbara* is not. In his sixteenth conclusion (just cited above) Buridan rejects the validity of *LXL Barbara* and *Celarent*, but affirms the validity of any inference to a particular conclusion. This is interesting, first, because Buridan makes no reference to Aristotle here and, second, because Buridan finds a middle way in the debate about the validity of *LXL* syllogisms. In rejecting *LXL Barbara* and *Celarent* Buridan rejects Aristotle's theory of the modal syllogism, to some degree. But Buridan does affirm *LXL Darii*, *Ferio*, *Celaront*, and *Barbari* as valid. By doing this, Buridan agrees with Aristotle that *L-X* pairs with universal premises are productive, but they do not produce a universal conclusion, only a particular one. The reason for these validities goes back to the ampliation of the subject in propositions of necessity. For example, consider the affirmative particular of necessity. For it to be true that 'Some A is necessarily B', there needs to be something that is possibly A and is necessarily B. The validity

5. In these tables we will always list the strongest inferences that follow, and omit reference to weaker inferences. For example, we would not make mention of *Barbari* if *Barbara* follows.

6. The Latin can be found on page 188.

of each of these syllogisms can be seen as follows: First, as we have already remarked (see our discussion of conclusions three and four in the previous chapter 92), we need to assume that everything which exists is also possible. Then in each of the valid first figure syllogisms the minor premise either entails an assertoric particular proposition or is an assertoric particular premise. It then follows by Buridan's fourth conclusion in Book Two of the *Treatise on Consequences*, that this entails a proposition of possibility. In which case, the validity will follow using either LML Darii or Ferio (the validity of which establishes in Bk. 4 Con. 4).

5.2 Formal Theory

In this section our aim will be to present a formal reconstruction of Buridan's divided modal syllogism as contained in the *Treatise on Consequence*.⁷

5.2.1 Preliminaries and Semantics

Buridan Language. *A Buridan Language*

$\mathcal{L} = \langle CONS, PRED, VAR, a, e, i, o, L, M, Q, \bar{Q} \rangle$ where:

CONS is a countable set of singular terms/objects.⁸

PRED is a countable set of (monadic) predicates.⁹

Well-Formed Formulae. *If $A, B \in PRED$ then:*

AaB, AeB, AiB, AoB are well-formed.

If $A \times B$ is a well-formed formula where \times is one of a, e, i, o then:

$\nabla A \times B$ is a well-formed formula where ∇ is one of L, M, Q, \bar{Q} , or $-$.¹⁰

Nothing else is a well-formed formula.

A formula is said to be *categorical* if it is any of the well-formed formulae above. In what follows we will introduce another set of formulae which will be called singular formulae and will make use of CONS. We include the full language here to avoid having to expand the language later in this chapter. A formula is said to be modal if it contains the symbol L, M, Q , or \bar{Q} . In what follows, we will often contract 'well-formed formulae' and speak only of formulae. A brief comment on the notation conventions should be made here. As Buridan remarks, and as will be clear from our semantic definitions below, divided modal propositions are different from both composite propositions (which Buridan treats but will not be treated in this thesis) and from the usual semantics given

7. For a similar treatment of modal propositions by other medieval logicians see chapters 4 and 5 of [62].

8. In what follows we will use terms from the beginning of the alphabet for objects and terms from the end of the alphabet for variables or worlds.

9. To avoid confusion we will not include Q or R among our predicates.

10. To avoid ambiguity between divided and compounded senses of the modality, we will superscript the modal operation over the copula. Following standard convention, we will sometimes write X instead of $-$ for the assertoric proposition.

in modern logic for the operation \square . As such, and to help avoid confusing divided modal propositions with composite ones, we avoid the use of the notation \square , \diamond , and ∇ , which are often (thought not exclusively) used in the modern literature preferring L , M , Q , and \bar{Q} , which are used less frequently, and are also common in the historical literature about modal syllogisms.

Buridan Modal Model. A Buridan Modal Model is a tuple:
 $\mathfrak{M} = \langle D, W, R, O, v \rangle$ such that:

D and W are non-empty sets. D is the domain of objects and W is a set of worlds.

$R \subseteq W^2$.

$O : W \rightarrow \mathcal{P}(D)$ s.t. $O(w) \subseteq D$

$v : W \times PRED \rightarrow \mathcal{P}(D)$

We require that R be an equivalence relationship.

Semantic Abbreviations. Let P be a term. Using the semantics we can define the following operations:

$V(w, P) = O(w) \cap v(w, P)$

$V(w, \neg P) = D \setminus (O(w) \cap v(w, P))$

Informally, we can think of V , (and M and L ; see below) as giving the supposition or extension of a particular term. $V(w, P)$ returns the extension of the predicate for the objects that exist at w . $V(w, \neg P)$ gives the anti-extension of P at w . It should be noted that an object will fall into the anti-extension at a world w if it either fails to exist at w or if it is not in the valuation of P at w . It is for this reason that we use the notation \neg as it is suggestive of the negation operation in a more familiar setting.

Moving into the modal context, we will want to consider situations where an object, say d can fall under a predicate (say P) and cases where an object d does not fall under a predicate. As such we will need to use both $V(w, P)$ and $V(w, \neg P)$ in our definitions of modal operations. As such, we will abuse notation slightly and write, e.g. $V(w, K)$ with the understanding that K ranges over terms and terms with \neg in front of them. To that end we define our modal operations as follows:

$M(w, K) = \{d \in D : \text{there is some } z \text{ s.t. } wRz \text{ and } d \in V(z, K)\}$

$L(w, K) = \{d \in D : \text{for all } z \text{ if } wRz \text{ then } d \in V(z, K)\}$ ¹¹

11. If one finds the use of K unclear, it may be helpful to alternatively think of these as four operations defined as follows:

$M(w, P) = \{d \in D : \text{there is some } z \text{ s.t. } wRz \text{ and } d \in V(z, P)\}$

$M(w, \neg P) = \{d \in D : \text{there is some } z \text{ s.t. } wRz \text{ and } d \in V(z, \neg P)\}$

$L(w, P) = \{d \in D : \text{for all } z \text{ if } wRz \text{ then } d \in V(z, P)\}$

$L(w, \neg P) = \{d \in D : \text{for all } z \text{ if } wRz \text{ then } d \in V(z, \neg P)\}$

When we move to consider the operations $M(w, K)$ and $L(w, K)$, these are being used to encode the set of objects that are possible relative to a world and are necessary relative to a world. The basic idea is that an object (d) is possibly K relative to a world (w) if there is some world v such that wRv and the object d is K at w . From this it should be clear that, as in the usual semantics for modal logic, the operation R is used to encode which worlds are accessible from a given world and which worlds are not. In practice, since we take R to be an equivalence relationship, we can dispense with R in our formulations of many of these principles. We retain mention of R since we will prove some results that do not depend on what kind of relationship R is.

At this point we should mention two important caveats. As may have been clear from a moment's reflection on the language in which we are working in, we do not have the ability to iterate modalities in our well-formed formulae. As such, our language is incapable of distinguishing between T and stronger modal systems where the characteristic formulae involve iterative modal operations (such as 4, 5, or B , to give three familiar examples). As such, the choice to work with a universal accessibility relation is less interesting than it otherwise might be. In some senses, this is, however, as it should be. As we shall see, none of Buridan's conclusions require iterated modalities. In fact, as Read points out in his review of Thom's *Medieval Modal Systems*:

But in fact, this appeal to theses of K4 is unnecessary and misleading. Axioms 1.12 – 1.15 are only ever used [144, 152– 5, 163–4, 180–4] in conjunction with 1.11 (the characteristic T-thesis if $\Box p$ then p), but in each case the appeal to $T + 4$ is unnecessary (for in each case, T simply cancels the L which 4 preserves). . . What little modal labour there is, is carried out by 1.7 – 1.8 (matching the T-theses CLpp and CpMp) and the revised 1.12 – 1.15 (matching the K-theses CLpqCLpLq and CLpqCMpMq—they are also all valid in S2). [50, p.612]

Buridan's modal logic is one of the logics treated there, and in our discussion of Buridan's modal logic in the previous chapter at no point did we observe Buridan making use of iterative modal operations. As such, the choice to work with a universal accessibility relation simplifies some of the formal details.

With this in place, we can simplify the various collections we defined above as follows:

$$M(w, K) = \{d \in D : \text{there is some } z \text{ such that } d \in V(z, K)\}$$

$$L(w, K) = \{d \in D : \text{for all } z \text{ } d \in V(z, K)\}^{12}$$

12. Again, if one finds the use of K unclear, the four operations are defined as follows:

$$M(w, P) = \{d \in D : \text{there is some } z \text{ such that } d \in V(z, P)\}$$

$$M(w, \neg P) = \{d \in D : \text{there is some } z \text{ such that } d \in V(z, \neg P)\}$$

$$L(w, P) = \{d \in D : \text{for all } z \text{ } d \in V(z, P)\}$$

$$L(w, \neg P) = \{d \in D : \text{for all } z \text{ } d \in V(z, \neg P)\}$$

Using these operations we can define the truth for categorical propositions.

Assertoric Categorical Propositions.

- $\mathfrak{M}, w \models AaB$ if and only if $V(w, A) \subseteq V(w, B)$ and $V(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models AeB$ if and only if $V(w, A) \cap V(w, B) = \emptyset$
 $\mathfrak{M}, w \models AiB$ if and only if $V(w, A) \cap V(w, B) \neq \emptyset$
 $\mathfrak{M}, w \models AoB$ if and only if $V(w, A) \not\subseteq V(w, B)$ or $V(w, A) = \emptyset$

Modal Categorical Propositions.

- $\mathfrak{M}, w \models A \overset{L}{a} B$ if and only if $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{L}{e} B$ if and only if $M(w, A) \cap M(w, B) = \emptyset$
 $\mathfrak{M}, w \models A \overset{L}{i} B$ if and only if $M(w, A) \cap L(w, B) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{L}{o} B$ if and only if $M(w, A) \not\subseteq M(w, B)$ or $M(w, A) = \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{a} B$ if and only if $M(w, A) \subseteq M(w, B)$ and $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{e} B$ if and only if $M(w, A) \cap L(w, B) = \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{i} B$ if and only if $M(w, A) \cap M(w, B) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{o} B$ if and only if $M(w, A) \not\subseteq L(w, B)$ or $M(w, A) = \emptyset$
 $\mathfrak{M}, w \models A \overset{Q}{a} B$ if and only if $M(w, A) \subseteq M(w, B) \cap M(w, \neg B)$ and $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{Q}{e} B$ if and only if $\mathfrak{M}, w \models A \overset{Q}{a} B$
 $\mathfrak{M}, w \models A \overset{Q}{i} B$ if and only if $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{Q}{o} B$ if and only if $\mathfrak{M}, w \models A \overset{Q}{i} B$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$ if and only if $M(w, A) \cap M(w, B) \cap M(w, \neg B) = \emptyset$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{e} B$ if and only if $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{i} B$ if and only if $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$ or $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{o} B$ if and only if $\mathfrak{M}, w \models A \overset{\bar{Q}}{i} B$

The \bar{Q} notation is used to indicate that the negation occurs in front of the modal operator instead of after it.

As we have already observed Buridan tells us that modal propositions amplify their subject to supposit for what is or can be the case. Because R is universal, we do not need to mention the union $M(w, P) \cup V(w, P)$. It can be simplified to $M(w, P)$. The reason for this is that since R is reflexive (which follows from R being universal), it is easy to show that $V(w, A) \subseteq M(w, A)$. This is proven on 211.

5.2.2 Single Premise Inferences

Just as it is possible to visualize Aristotle's assertoric propositions as a diagram with four points, it is possible to envision Buridan's modal propositions in diagrammatic form. If

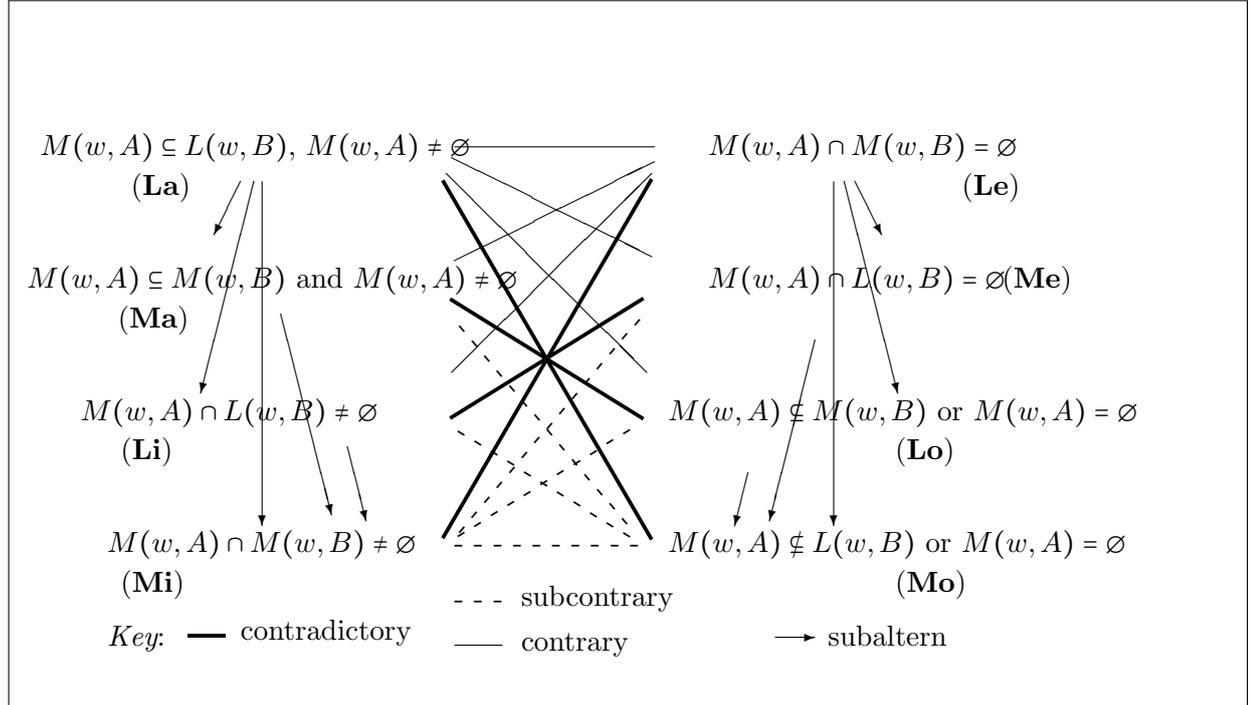


Figure 5.1: Buridan's Modal Octagon of Opposition

we limit ourselves to possibility and necessity, we obtain an octagon of opposition.¹³

Formally, the octagon gives rise to 24 distinct inferences. The proofs of these properties are all obvious semantic consequences of the system. We present two examples.

1. $A \overset{L}{a} B$ contradicts $A \overset{M}{o} B$
2. $A \overset{L}{a} B$ is a contrary of $A \overset{M}{e} B$

Proof of 1: Normally two propositions are said to be contradictory if the truth of one entails the falsehood of the other and vice versa. In this context we say that two well-formed formulae are said to be contradictory if, in every model the truth of one of the formulae entails the falsity of the other, and the falsity of the one entails the truth of the other.

Proof. Assume that $\mathfrak{M}, w \models A \overset{L}{a} B$. Then $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$. We claim that $\mathfrak{M}, w \not\models A \overset{M}{o} B$. If this were not so, then either $M(w, A) \not\subseteq L(w, B)$ or $M(w, A) = \emptyset$ is true. However, it is clear that both of these contradict our assumptions that $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$. The other direction is similar. \square

13. Adding contingency and negated contingency produces a hexadecagon of opposition. Figure 1 (seen below) is due to Stephen Read and can be found in his paper *Non-Contingency Syllogisms in Buridan's Treatise on Consequences* [52, p.450].

Proof of 2: Normally two propositions are said to be contrary if they cannot both be true, but they can both be false. In this case, we interpret this in the following way: Two well-formed formulae are said to be contrary if there are no models in which both of the formulae are true, but there is a model where they are both false. Subcontraries are treated in a similar way. Two well-formed formulae are said to be subcontrary if there is a model where both formulae are true, but there is no model where both formulae are false.

To see that $A \overset{L}{a} B$ and $A \overset{M}{e} B$ cannot both be true:

Proof. Assume that 1) $\mathfrak{M}, w \models A \overset{L}{a} B$ and 2) $\mathfrak{M}, w \models A \overset{M}{e} B$. From 1) we have $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$. From 2) we have $M(w, A) \cap L(w, B) = \emptyset$. Since $M(w, A) \neq \emptyset$ we know that there is some $d \in M(w, A)$. From 1) it follows that $d \in L(w, B)$ and so $M(w, A) \cap L(w, B) \neq \emptyset$, contradicting 2). \square

To see that they can both be false consider the following model:

| | | | |
|-------------------|-------------------|-------------------|--------------|
| $D = \{a, b\}$ | $W = \{w_1\}$ | $R = W^2$ | $O(w_1) = D$ |
| $a \in v(w_1, A)$ | $a \in v(w_1, B)$ | $b \in v(w_1, A)$ | |

Since $a \in v(w_1, A)$ and $a \in v(w_1, B)$ we have $a \in M(w_1, A)$ and $a \in L(w_1, B)$. Hence $\mathfrak{M}, w_1 \not\models A \overset{M}{e} B$. Likewise, $b \in v(w_1, A)$ and $b \notin v(w_1, B)$, we have $a \in M(w_1, A)$ and $a \notin L(w_1, B)$, from which it follows that $\mathfrak{M}, w_1 \not\models A \overset{L}{a} B$.

The results of conclusions three through eight in Book Two of the *Treatise on Consequences* are summarised in Table 4. As in the case of the modal octagon, the proofs and construction of the relevant countermodels are straightforward. The proofs of the contradictories can be found on page 239.

5.3 Buridan's Syllogistic Theory

As we have already seen, according to Buridan a syllogism is an inference from two premises to a single conclusion composed of three distinct terms. All three propositions in a syllogism must be categorical. The terms in a syllogism can be arranged in the following ways:

| Figure 1 | Figure 2 | Figure 3 | Figure 4 |
|----------|----------|----------|----------|
| B-A | A-B | B-A | A-B |
| C-B | C-B | B-C | B-C |
| C-A | C-A | C-A | C-A |

Formally, we define a syllogism, S , to be a triple $\langle M, m, C \rangle$ such that:

1. M , m , and C are all categorical formulae;
2. M , m , and C have exactly three terms;

| Conclusion | Inferences |
|---------------------|--|
| Conclusion Three | If $\mathfrak{M}, w \models A \overset{L}{e} B$ then $\mathfrak{M}, w \models AeB$ $\mathfrak{M}, w \models A \overset{L}{a} B$ does not entail $\mathfrak{M}, w \models AaB$ $\mathfrak{M}, w \models A \overset{L}{i} B$ does not entail $\mathfrak{M}, w \models AiB$ $\mathfrak{M}, w \models A \overset{L}{o} B$ does not entail $\mathfrak{M}, w \models AoB$ No assertoric formula entails an L formula. |
| Conclusion Four | $\mathfrak{M}, w \models AiB$ entails $\mathfrak{M}, w \models A \overset{M}{i} B$ $\mathfrak{M}, w \models AaB$ entails $\mathfrak{M}, w \models A \overset{M}{a} B$ $\mathfrak{M}, w \models AaB$ does not entail $A \overset{M}{a} B$ $\mathfrak{M}, w \models AeB$ does not entail $\mathfrak{M}, w \models A \overset{M}{e} B$ $\mathfrak{M}, w \models AoB$ does not entail $\mathfrak{M}, w \models A \overset{M}{o} B$ No M formulae entails an assertoric formulae |
| Conclusion Five | $\mathfrak{M}, w \models A \overset{M}{i} B$ if and only if $\mathfrak{M}, w \models B \overset{M}{i} A$ If $\mathfrak{M}, w \models A \overset{M}{a} B$ then $\mathfrak{M}, w \models B \overset{M}{a} A$ No other conversions are valid with M formulae. |
| Conclusion Six | if $\mathfrak{M}, w \models A \overset{L}{e} B$ then $\mathfrak{M}, w \models B \overset{L}{e} A$ if $\mathfrak{M}, w \models A \overset{L}{o} B$ then $\mathfrak{M}, w \models B \overset{L}{o} A$ No other conversions are valid with L formulae. |
| Conclusion Seven | If $\mathfrak{M}, w \models A \overset{Q}{a} B$ then $\mathfrak{M}, w \models A \overset{Q}{e} B$ If $\mathfrak{M}, w \models A \overset{Q}{i} B$ then $\mathfrak{M}, w \models A \overset{Q}{o} B$ If $\mathfrak{M}, w \models A \overset{Q}{a} B$ then $\mathfrak{M}, w \not\models A \overset{Q}{a} B$ and $\mathfrak{M}, w \not\models A \overset{Q}{e} B$ if $\mathfrak{M}, w \models A \overset{Q}{i} B$ then $\mathfrak{M}, w \not\models A \overset{Q}{i} B$ and $\mathfrak{M}, w \not\models A \overset{Q}{o} B$ |
| Conclusion Eight | $\mathfrak{M}, w \models A \overset{Q}{e} B$ does not entail $\mathfrak{M}, w \models B \overset{Q}{e} A$ $\mathfrak{M}, w \models A \overset{Q}{o} B$ does not entail $\mathfrak{M}, w \models B \overset{Q}{o} A$ $\mathfrak{M}, w \models A \overset{Q}{a} B$ does not entail $\mathfrak{M}, w \models B \overset{Q}{a} A$ $\mathfrak{M}, w \models A \overset{Q}{i} B$ does not entail $\mathfrak{M}, w \models B \overset{Q}{i} A$ $\mathfrak{M}, w \models A \overset{Q}{a} B$ entails $\mathfrak{M}, w \models B \overset{M}{i} A$ $\mathfrak{M}, w \models A \overset{Q}{i} B$ entails $\mathfrak{M}, w \models B \overset{M}{i} A$ |

 Table 5.4: Summary of Conclusions Three through Eight of Book Two of the *Treatise on Consequences*

3. The predicate of C occurs in M ;
4. The subject of C occurs in m ;
5. M and m share a common term that does not occur in C .¹⁴

Syllogistic Validity. *A syllogism S is valid when the following obtains:*

For all Buridan Modal Models \mathfrak{M} and all worlds $w \in W$ if $\mathfrak{M}, w \models M$ and $\mathfrak{M}, w \models m$ then $\mathfrak{M}, w \models C$.

We will denote this by $\models S$.

There are two strategies that can be adopted in showing the correctness of Buridan's system. One could simply check that all of the syllogisms that Buridan claims to be valid are valid in our semantics. They are, but this process is tedious. Instead the process can be streamlined by proving a number of first figure syllogisms and then using the inferences in Book Two and the Modal Octagon to reduce the second and third figure syllogisms to the first figure. The first figure syllogisms are justified by appeals to the *dictum de omni et nullo* or other relevant principles and the remaining syllogisms are then reduced to the first figure. To perform the reduction we will need the following two principles:

Interchange: Let $S = \langle M, m, C \rangle$ and $S' = \langle M', m', C' \rangle$ be two syllogisms such that $M = m'$, $M' = m$, and $C = C'$ then $\models S$ if and only if $\models S'$.

The proof is trivial.

Proof *per impossibile* [PPI]:

Formally, let $S = \langle M, m, C \rangle$ and $S' = \langle M', m', C' \rangle$ be two syllogisms such that C' is the contradictory of m , m' is the contradictory of C and $M = M'$, then we claim that $\models S$ if and only if $\models S'$.

Proof. We prove the left to right direction. The other direction is similar. Take an arbitrary model \mathfrak{N} and an arbitrary world, $w_n \in \mathfrak{N}$ and assume that $\mathfrak{N}, w_n \not\models S'$. We show that $\mathfrak{N}, w_n \not\models S$

Since $\mathfrak{N}, w_n \not\models S'$ it follows by the definition of a valid syllogism that $\mathfrak{N}, w_n \models M'$ and $\mathfrak{N}, w_n \models m'$ but that $\mathfrak{N}, w_n \not\models C'$. Per our assumption, $M = M'$ so $\mathfrak{N}, w_n \models M$. Since $\mathfrak{N}, w_n \not\models C'$ and every categorical proposition has a contradictory, it follows by the definition of contradictory that \mathfrak{N}, w_n satisfies the contradictory of C' . Per our assumption, this is m . Hence $\mathfrak{N}, w_n \models m$. Analogously since $\mathfrak{N}, w_n \models m'$ it follows that $\mathfrak{N}, w_n \not\models C$. Hence $\mathfrak{N}, w_n \models M$ and $\mathfrak{N}, w_n \models m$ but $\mathfrak{N}, w_n \not\models C$. Hence $\mathfrak{N}, w_n \not\models S$ which suffices to prove the claim. \square

14. This definition is standard and can be found in [65].

Informally what PPI tells us is that if we take a valid syllogism and rearrange it so that the contradictory of the minor premise is the conclusion and the contradictory of the original conclusion is the minor premise, then the resulting syllogism is also valid. Using this we can reduce the validity problem of the third figure to the second, and the second figure to the first.

For example, consider the case of MLL Camestres. By PPI we need to show that there is a valid syllogism $S = \langle M, m, C \rangle$ such that if we replace m with the contradictory of C , and we replace C with the contradictory of m , we obtain Camestres MLL. Clearly, the syllogism in question is MMM Darii, which we can show to be valid. Hence MLL Camestres is also valid.

The proof that our semantics correctly track the validities that Buridan gives in the *Treatise on Consequences* is a fairly long and somewhat tedious exercise in checking syllogistic validities and invalidities. For reasons of space, this result is proven in Appendix 3. We illustrate some of the proofs and countermodels below.

Proof of LML Darii:

Proof. Assume that 1) $\mathfrak{M}, w \models B \overset{L}{a} C$ and that 2) $\mathfrak{M}, w \models A \overset{M}{i} B$. It follows that $M(w, B) \subseteq L(w, C)$ and $M(w, B) \neq \emptyset$. From 2) it follows that $M(w, A) \cap M(w, B) \neq \emptyset$. Since $M(w, A) \cap M(w, B)$ is non-empty, let e be an object such that $e \in M(w, A)$ and $e \in M(w, B)$. From 1) it follows that $e \in L(w, C)$ and so $M(w, A) \cap L(w, C) = \emptyset$. Hence, $\mathfrak{M}, w \models A \overset{L}{i} C$. □

Countermodel for LXL Barbara:

| | |
|--|------------------------|
| $W = \{w, x\}$ | $D = \{a, b\}$ |
| $O(w_1) = O(w_2) = D$ | $R = W^2$ |
| $v(w_1, A) = v(w_1, B) = (w_1, C) = \{a\}$ | |
| $v(w_2, A) = \{a\}$ | $v(w_2, C) = \{a, b\}$ |

By construction we have $M(w_1, B) = \{a\}$ (since $a \in O(w_1)$, $a \in v(w_1, B)$, and wRw) and so $M(w_1, B)$ is non-empty. $M(w_1, B) \subseteq L(w_1, C)$ because, in the case of a , $a \in V(w_1, C)$ and $a \in V(w_2, C)$ (and so $a \in L(w_1, C)$) while in the case of b , $b \notin M(w_1, B)$. Hence, $\mathfrak{M}, w_1 \models B \overset{L}{a} C$. To see that $\mathfrak{M}, w_1 \not\models A \overset{L}{a} B$, observe that $a \in V(w_1, A)$ and $a \in V(w_1, B)$, while $b \notin V(w_1, A)$. Hence, $V(w_1, A) \subseteq V(w_1, B)$. Since $a \in V(w_1, A)$ we have $V(w_1, A) \neq \emptyset$. However, $\mathfrak{M}, w_1 \not\models A \overset{L}{a} C$ since $b \in M(w_1, A)$ and $b \notin L(w_1, C)$, since $b \notin V(w_2, A)$.

In his book *Medieval Modal Systems*, Paul Thom claims that all of the first figure XXM syllogisms are valid.[59][p.178 Thm 9.10a] Only Darii and Ferio are. Consider the following countermodel for XXM Barbara:

| | |
|-----------------------|-------------------|
| $W = \{w_1, w_2\}$ | $D = \{a, b\}$ |
| $O(w_1) = O(w_2) = D$ | $R = W^2$ |
| $a \in v(w_1, B)$ | $a \in v(w_1, C)$ |
| $a \in v(w_1, A)$ | $b \in v(w_2, C)$ |

Call this model \mathfrak{J} . Since $a \in V(w_1, C)$, $a \in V(w_1, B)$, and $a \in V(w_1, A)$ and $b \notin V(w_1, B)$, it follows that $\mathfrak{J}, w_1 \models BaC$ and $\mathfrak{J}, w_1 \models AaB$. However, per construction of the model $b \notin M(w_1, A)$, since $b \notin V(w_1, A)$ and $b \notin V(w_2, A)$ and $b \in M(w_1, C)$. Thus $M(w_1, A) \not\subseteq M(w_1, C)$ and so $\mathfrak{J}, w_1 \not\models A \overset{M}{a} C$.

The reason that Thom gives for this is that *XLX* Baroco is valid. Unfortunately, it is not. The proof in the book is flawed and rests on the invalid inference from ‘No B is A’, it follows that ‘No B is necessarily A’.[59][p.177] To see that the syllogism is invalid, consider the following countermodel:

$$\boxed{\begin{array}{llll} W = \{w_1, w_2\} & D = \{a, b\} & O(w_1)D & R = W^2 \\ a \in v(w_1, A) & a \in v(w_1, B) & a \in v(w_1, C) & b \in v(w_2, C) \end{array}}$$

$\mathfrak{M}, w_1 \models A a B$ since both $a \in V(w_1, B)$ and $b \notin V(w_1, A)$; likewise $\mathfrak{M}, w_1 \models C \overset{L}{o} B$ since $b \in V(w_2, C)$ and $b \notin V(w_2, B)$ and $b \notin V(w_1, B)$. However, $\mathfrak{M}, w \models CaA$ since $b \notin V(w, C)$ while $a \in V(w, C)$ and $a \in V(w, A)$. Hence $\mathfrak{M}, w \not\models CoA$.

One may, at this point, be tempted to suggest that this follows because of Conclusion 17 in Book Four on the *Treatise on Consequences*. Here Buridan writes:

From a negative major [premise] of necessity and an assertoric minor there is always a valid syllogism in the second figure to a particular conclusion of necessity, but not to a universal; but if the major is affirmative of necessity or assertoric, there is no valid syllogism to a conclusion of necessity, but it is valid to an assertoric conclusion.[51, p.152]¹⁵

However, as is clear from what Buridan has written, he is claiming that the *LXX* syllogisms are productive. This should not be taken to suggest that the *XLX* syllogisms are valid. Buridan is quite clear here that he is referring to a major of necessity and an assertoric minor.

Hence when we read on in Conclusion 17 and come to the following passage:

But that the said moods are valid to an assertoric conclusion is clear, for from a [premise] of necessity there always follows an assertoric except in the case where it is true only for those which can be. But this case does not prevent the truth of a negative assertoric conclusion.[51, p.152]¹⁶

The referent of ‘the said moods’ should be seen as referring back to the *LXX* syllogisms, not the *XLX* ones as is noted by Read in footnote 16.[51, p.176 fn16]

5.4 The Expository Syllogism

The inferential process called Ekthesis or ‘setting out’ occupies an unusual place in the history of logic. The inference was used by Aristotle to provide alternative deductions

15. The Latin can be found on page 188.

16. The Latin can be found on page 188.

of various third figure syllogisms in the *Prior Analytics*. For example, Aristotle uses the technique when proving Darapti:

When both P and R belong to every S, it results of necessity that P will belong to some R. For since the positive premise converts, S will belong to some R; consequently, since P belongs to every S and S to some R, it is necessary for P to belong to some R (for a deduction through the first figure comes about). It is also possible to carry out the demonstration through an impossibility or through the setting-out (ekthesis). For if both terms belong to every S, then if some one of the S's is chosen (for instance N), then both P and R will belong to this; consequently, P will belong to some R.[55][p.9,28^a18 – 25]

The key move that is made here, which has raised questions for interpreters afterwards, is 'what exactly is this term N that has been selected?' If it is another general term, then the argument looks circular, but if not, then what kind of term is it?¹⁷ When the medievals took up the study of the *Prior Analytics*, they chose to read the term selected here as a singular term, at least in the assertoric case. They also simplified the inference somewhat and referred to it as the expository syllogism.

Expository syllogisms differ from categorical syllogisms in at least two important ways: First, unlike in the categorical syllogism, the middle term in an expository syllogism needs to be a singular term, where a singular term is a term that refers to a particular object. For example, the following is an expository syllogism:

Socrates is wise

(The same) Socrates is human

Therefore Something wise is a human.

Second, when we turn to the writings of Buridan, the expository syllogism is not justified by means of the *dictum de omni et nullo*. Instead, Buridan provides two rules that justify such syllogisms. The first rule states that

Whatever are the same as one and the same are the same as each other.[51, pp.116-117]¹⁸

It is important to stress that when Buridan speaks of identity here, he does not mean that the terms A and B are identical, but that the singular terms in each of the premises of the syllogism refer to the same object.¹⁹

Buridan distinguishes two kinds of expository syllogisms: affirmative and negative. As an example of each, consider the following:

-
17. For references and a more complete discussion of how to understand Ekthesis see the discussion in e.g.[55],[38]
 18. The Latin can be found on page 188. The principle comes from Aristotle. See *Sophistical Refutations* [168b28-29]
 19. Buridan discusses this in the fifth chapter of the first book of his *Treatise on Consequences*[51, pp.69-71][4, p.25].

- 1) This C is an B and the same C is a A, so some B is A
- 2) This C is an B and the same C is not A so not every B is A [51, pp.116-117][4][pp.84-85]

Obviously, the principle cited above justifies 1). However, the principle is not sufficient to justify 2). Buridan notices this and uses a different principle to justify 2), namely that:

Two things are not the same as each other if one is the same as something and the other is not [51, p.117].²⁰

As in the case of the affirmative rule used above, the principle is used to show that a negative particular proposition is true.²¹ In what follows, I will refer to the first rule as the ‘rule of sameness’ and the second as the ‘rule of difference.’

In both cases, these rules explain how singular terms relate to categorical propositions. Consider one of Buridan’s examples:

The first part is clear by an expository syllogism. For if B can be A, then designate such a B as C. Then this C is or can be B and the same thing can be A; so that which can be A is or can be B.²²

There are two inferences which should be noted. The first inference is the move from an affirmative particular categorical proposition to a singular term which falls under both terms. We might think of this as a form of existential instantiation. If one has shown that ‘Some B can be A’, then this inference allows for a (new) singular term to be selected which is possibly B and possibly A.

The second inference, namely the move from ‘This C is possibly B’ and ‘The same C can be A’ to ‘Some B can be A’, is justified by the rule of sameness. In this context, the use of the words ‘this’ and ‘this same’ are important in designating the singular term. It is the presence of these words that Buridan uses to avoid context shifts which could invalidate the principle.²³ Since the C in ‘This C is possibly B’ and the C in ‘This same C is possibly A’ are the same, we may use the principle to infer that ‘possibly B’ and ‘possibly A’ supposit for the same thing, and so infer that some B is possibly A.

For what comes later, it is helpful to observe that the expository syllogism, in some ways, looks very similar to a kind of existential elimination. The idea is that we start

20. The Latin can be found on page 188.

21. It should be observed that these two principles are *not* Buridan’s formulation of the *dictum de omni et nullo* but are separate principles. Buridan formulates the *dictum de omni et nullo* as “*dici de omni* applies when nothing is taken under the subject of which the predicate is not predicated. . . *Dici de nullo* applies when nothing is taken under the subject of which the predicate is not denied.” [5][p.306]

22. The Latin can be found on page 189.

23. It is important to note that if the term ‘this same’ does not occur, then we can create counterexamples to Buridan’s principle. For example, ‘This person is running’ said pointing to one person ‘This person is not running’ said pointing to another person, therefore ‘Something is running that is not running’, which is impossible.

from two singular terms, 'This C is B' and 'The same C is A' and then we remove the singular term C to conclude that 'Some B is A'. This similarity will be helpful for thinking about formal reconstructions of Buridan's theory.

There is an interesting metaphysical question which we will raise only to set aside. Buridan does not limit himself to only looking at singular terms that actually do or do not fall under a particular term. He also considers objects that can, or must, or only contingently fall under a particular term. On Buridan's reading of modal propositions, 'Some B can be A' is true if something can be B (even if it currently is not or does not exist), that can also be A.[22] Formally, this raises the question of exactly what do the categorical operators and the copulae range over? Do they range over names of terms and names of singular objects? If they do not, what exactly would the nature of these possible objects be and how would this fit with Buridan's commitment to nominalism?

With our understanding of the expository syllogism in place, we now turn to see how Buridan used the theory to analyse the assertoric and modal syllogism. Unlike Aristotle and many medieval authors, Buridan does not ground the assertoric syllogism in the *dictum de omni et nullo*. Instead, Buridan observes that:

Affirmative syllogisms hold in virtue of the principle, 'Whatever are the same as one and the same are the same as each other'. Hence, from the fact that the extremes are said in the premises to be the same as the one middle it is concluded in the conclusion that they are the same as each other. However, negative syllogisms hold by another principle, 'Two things are not the same as each other if one is the same as something and the other is not'.²⁴

Buridan's claim is that all of the syllogisms where the conclusion is affirmative are justified by the principle of sameness while those which are negative are justified by the principle of difference. The other important difference about Buridan's analysis is that, unlike Aristotle, he starts in the third figure, and then reduces the other figures to the third. He writes:

It is obvious that expository syllogisms are self-evident, especially in the third figure; and all six modes of the third figure are easily proved by reducing them to expository syllogisms.²⁵ [5][p.324]

In each case, this is done by using the affirmative term to select a particular object which instantiates the required term, and then use one of the two principles to derive the desired proposition. Buridan proves Darapti as follows:

From the premises of this syllogism it follows that some C is A and the same C is B, whence it is inferred, by an expository syllogism, that some

24. The Latin can be found on page 189.

25. The Latin can be found on page 189.

B is A; since, therefore, this conclusion follows from the premises of the expository syllogism, and those premises followed from the premises of the original syllogism, it follows that the conclusion validly followed from the premises of the original syllogism, by the rule ‘whatever follows from the consequent follows from the antecedent.’[5][p.324]

What is interesting here is how exactly Buridan obtains the propositions ‘Some C is A’ and ‘The same C is B’. To obtain one of these propositions is simple. First, we use subalternation to infer that ‘Some B is C’ from ‘Every B is C’ and then convert ‘Some B is C’ to ‘Some C is B’. Now, in order to get a singular term, we need to apply an exposition principle that allows us to select one of the singular terms that falls under C. This move is similar to a kind of existential introduction, in that we are picking an object that is an instance of the truth of ‘Some C is B’.

What is slightly unclear is that Buridan seems to again use C for the singular term that he has selected. We will use the term D to avoid confusion. So we use an exposition principle to conclude that this D (which is a C) is B. From this point there are two ways that Buridan’s reasoning could go. One way would be to use the assumption that ‘Every B is A’, combined with ‘This D is B’ to infer that ‘The same D is A’. If we use the *dictum de omni*, the inference would follow. But then it seems that the third figure syllogisms require the *dictum de omni* in addition to the rules for expository syllogism, which is not what Buridan claims. The other way would be to take this as a given rule, i.e. if you have shown that ‘Every B is A’ and you have shown ‘This D is B’ you may conclude ‘The same D is A’. This reading ties in naturally with Buridan’s discussion of the causes of truth for a proposition. He writes:

Let me say first, that every proposition with an undistributed general term, or one similar in form to it, has or can have more causes of truth than a proposition with the same general term distributed, other things being equal. . . E.g., if I say ‘A human is running’, this would be true if only Socrates were running, and if only Plato were, and so on, and no less if they all were. But if the term is distributed, it can only have one cause of truth on account of that term, namely, that it holds for all of them, not only for one or two.[51, p.65]²⁶

Buridan’s point here is that while ‘A human is running’ only requires one individual to be both a human and running for the proposition to be true, the proposition ‘Every human is running’ requires that each individual who is a human also be running. Hence, if ‘Every human is running’ is true, then if Socrates is a human, it follows that he is running. This licences the move from ‘This D is B’ and ‘Every B is A’ to the ‘This same D is A’, since whatever is B is also A, and we have already shown that ‘This D is B’, so it must also be A. This will also provide an alternative way of proving the validity of the *dictum de omni et nullo*.

26. The Latin can be found on page 189.

One final point about Buridan's theory concerns the presence of negative propositions. Buridan, like many medieval authors, allows his theory to contain empty terms, i.e. terms where there are no objects that do fall or can fall under the term. In the case of negative particular propositions, for example, 'Not every A is B', this requires there be two ways for the proposition to be true.²⁷ First, there could be some object which is A, but is not B. The other way this proposition could be true is if there are no A's. This is important, because the move from 'Not every A is B' to 'This C is A' and 'This same C is not B' is invalid if we have not already shown that A is non-empty.

It is important to note that while Buridan uses the expository syllogism within his modal logic, he spends very little time describing what has changed when one moves to a modal framework. In fact, his mention of the expository syllogism is somewhat terse. For example, when stating the proof of the 18th conclusion, he simply writes

In the third figure a syllogism is always valid to a conclusion of necessity from a universal major of necessity and an assertoric minor. . . The first part of the conclusion would be evident in all moods by expository syllogisms.[51, p.152]²⁸

Likewise, within the rest of the *Treatise on Consequences*, when Buridan uses the expository syllogism to discuss modal propositions, he simply remarks that ϕ can be shown by an expository syllogism, or something similar.²⁹ Likewise, Buridan says nothing about the use of expository syllogisms in his treatment of the modal syllogism in the *Summulae*. In terms of reconstructing Buridan's thoughts on this matter, this leaves us with scant information to go on, except that we should be able to make modal singular terms (e.g. 'This D can be A', 'The same D is necessarily B' etc.) fit into the same mold as non-modal terms.

What we then need to formalise are the following:

1. The syllogistic framework that Buridan is working in.
2. The principles of sameness and difference.
3. The instantiation principles.
4. Rules that capture the subaltern, contrary and contradictory relationships between the various categorical and modal propositions.

It is to this formalisation that we now turn.

27. We use the formulation 'Not every A is B' to stress that the particular negative proposition is the contradictory of the universal affirmative proposition and because this is closer to Aristotle's formulation. It should be observed that for the universal affirmative, we need to assume that there is some term that falls under the subject.

28. The Latin can be found on page 189.

29. For example, see Book Four Conclusions 6,9, 19, and 23.

5.5 Formalisation

In order to formalise the expository syllogism we will need to expand both our language and the semantics of the system. On the side of the language, we will need to expand the language to allow for formulae that will stand in for singular propositions and we will need to add a conjunction operation. This is added merely as a technical convenience to simplify some of the details of the completeness proof (in particular the construction of the canonical models). On the side of the semantics, we will need to expand the models so as to have the correct semantic clauses to make a completeness proof with singular elements possible. We also have to formalise the expository syllogism. We will start with the language changes.

Buridan Language. Let $\mathcal{L} = \langle CONS, PRED, a, e, i, o, L, M, Q, \bar{Q}, \wedge \rangle$ where:

CONS and PRED are disjoint, countable sets.

We will use CONS for singular terms and will denote them with underlined lowercase letters (to avoid confusion with the categorical operations). We will use PRED for categorical terms, and will denote them with capital letters. To avoid confusion we will avoid using the letters L, M, Q, R, and V for terms.

Well-Formed Formulae. Let $A, B \in PRED$ and $\underline{d} \in CONS$ be well-formed formulae then:

$AaB, AeB, AiB, AoB, \underline{d}aB, \underline{d}eB$ are well-formed.

If $A \times B$ is a well-formed formula where \times is one of a, e, i, o then:

$\nabla A \times B$ is a well-formed formula where ∇ is one of L, M, Q, \bar{Q} , or $-$.

if $\underline{d}aB$ is well-formed, then so is $\nabla \underline{d}aB$.

if $\underline{d}eB$ is well-formed, then so is $\nabla \underline{d}eB$.

if ϕ and ψ are both well-formed formulae then $\phi \wedge \psi$ is well-formed.

Nothing else is a well-formed formula.

The operations a, e, i and o should be understood as before. A well-formed formula is *categorical* if it is only formed from elements of $PRED$. If it contains an element from $CONS$ it is said to be *singular*.

5.5.1 Semantics

In the case of the semantics, we need to add clauses that give truth conditions for singular propositions. To that end, we add a new function to our semantics that assigns elements $CONS$ to elements of the domain. For simplicity, we will assume that this function is surjective. This is done as follows:

Buridan Modal Model (Expanded). A Buridan Modal ModSpruyt's el is a tuple: $\mathfrak{M} = \langle D, W, R, O, c, v \rangle$ such that:

5 A Formal Reconstruction of Buridan's Modal Logic

D and W are non-empty sets. D is the domain of objects and W is a set of worlds.
 $R \subseteq W^2$ which is an equivalence relationship.

$O : W \rightarrow \mathcal{P}(D)$.

$c : CONS \rightarrow D$ such that c is surjective.

$v : W \times PRED \rightarrow \mathcal{P}(D)$.

With this in place, all that we need to add are clauses that deal with the relationship between particular objects and the terms that fall under them as well as clauses for \wedge . To that end, we add the following clauses to our theory:

New Propositions.

| | | |
|---|----------------|---|
| $\mathfrak{M}, w \models \underline{f}aA$ | if and only if | $c(f) \in V(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}eA$ | if and only if | $c(f) \notin V(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^L a A$ | if and only if | $c(f) \in L(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^M e A$ | if and only if | $c(f) \notin L(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^M a A$ | if and only if | $c(f) \in M(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^L e A$ | if and only if | $c(f) \notin M(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^Q a A$ | if and only if | $c(f) \in M(w, A) \cap M(w, \neg A)$ |
| $\mathfrak{M}, w \models \underline{f}^Q e A$ | if and only if | $\mathfrak{M}, w \models \underline{f}^Q a A$ |
| $\mathfrak{M}, w \models \underline{f}^Q a A$ | if and only if | $c(f) \in L(w, A) \cup L(w, \neg A)$ |
| $\mathfrak{M}, w \models \underline{f}^Q e A$ | if and only if | $\mathfrak{M}, w \models \underline{f}^Q a A$ |
| $\mathfrak{M}, w \models \phi \wedge \psi$ | if and only if | $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$ |

When we write well-formed formulae, we will underline singular terms so that they stand out (and, in principle, to distinguish the singular term a from the operation that denotes universal affirmative propositions i.e. the a in AaB . However in practice we will simply avoid writing formulae such as $\underline{a}aA$.)

As in the case of categorical propositions, the singular contingency and non-contingency propositions can be reduced to combinations of the L and M operations along with meta-level negation. The other feature of these propositions that should be observed is that in singular propositions $\underline{L} a$ contradicts $\underline{L} e$, $\underline{M} a$ contradicts $\underline{M} e$, and $\underline{Q} a$ contradicts $\underline{Q} e$. This is intended to directly parallel the cases in categorical propositions. In the context of modern modal logic, these can be thought of as modal operations where the negation occurs after the modal. E.g. $\underline{d}^L e A$ can be thought of as saying that \underline{d} is possibly not A . We will remind the reader of this on occasion, to avoid confusion.

5.5.2 Proof Theory

As we have just remarked above, the principles of the expository syllogism function in a way that is similar to existential instantiation. Recall that in usual first-order logic, existential instantiation is the rule:

From $\exists x\phi$ infer $\phi[x/a]$

Where $[x/a]$ is the result of uniformly replacing all of the occurrences of x with a and the term a is appropriately restricted so as to be a new term in the proof.

The challenge with existential instantiation both in our framework and in usual first-order logic, is that:

Proofs in a system employing this latter rule [existential instantiation] do not obey the principle that each line of a proof is a semantic consequence of all the assumptions that are ‘active’ at that point in the proof. For, even if $\exists xFx$ were semantically implied by whatever active assumptions there are, it is not true that Fy will be implied by those same assumptions, since the rule’s restriction on variables requires that y be new.[45, p.12]

This is a challenge for formulating our completeness proof and for developing our treatment of Buridan’s logic. As such, we will develop the idea of the expository syllogism in a way that avoids these sorts of problems. In what follows, we will treat the expository syllogism as a kind of existential elimination, which simplifies the technical challenges. With the completeness result in place, it would then be much easier to formulate a proof theory where the expository syllogism is treated as a kind of existential instantiation, show that the proof theory is equivalent (in the appropriate sense) to what is developed below, and conclude that it is also complete with respect to our semantic theory. This project will not be done in this thesis, for reasons of time and space.

The proof theory we develop here is a natural deduction system where we will use the notation $\mathcal{C}(\phi)$ to pick out a contradictory of ϕ .³⁰ We state this as an assumption to reduce the number of rules that we need to consider in the completeness proof. In what follows it may be helpful to recall that:

| | | |
|----------------------|-------------|----------------------------|
| $A a B$ | contradicts | $A o B$ |
| $A i B$ | contradicts | $A e B$ |
| $A \overset{L}{a} B$ | contradicts | $A \overset{M}{o} B$ |
| $A \overset{L}{i} B$ | contradicts | $A \overset{M}{e} B$ |
| $A \overset{M}{a} B$ | contradicts | $A \overset{L}{o} B$ |
| $A \overset{M}{i} B$ | contradicts | $A \overset{L}{e} B$ |
| $A \overset{Q}{a} B$ | contradicts | $A \overset{\bar{Q}}{o} B$ |
| $A \overset{Q}{i} B$ | contradicts | $A \overset{\bar{Q}}{e} B$ |

30. If ϕ is assertoric, an L or an M formula, then $\mathcal{C}(\phi)$ is unique. If ϕ is either Q and \bar{Q} then there will be two formula, equivalent to each other, but not the same formula. See Appendix Three for further elaboration of this.

In the case of singular propositions, these are mostly the same, except, as we have already remarked, it is the a and e propositions that contradict in the assertoric, necessity and possibility cases, e.g $\underline{f}^a A$ contradicts $\underline{f}^e A$ and $\underline{f}^L A$ contradicts $\underline{f}^M A$. In the case of contingency, $\underline{f}^Q A$ contradicts $\underline{f}^{\bar{Q}} A$.

The proof system that we will make use of here is a *Gentzen type* natural deduction system. In this system proofs are presented as trees of formulae. Following Prawitz[47, p.22], we say that if Π_1, \dots, Π_n are sequences of well-formed formulae then $(\Pi_1, \dots, \Pi_n/A)$ is the tree obtained by arranging the configuration of Π 's such that the Π 's end on a horizontal line immediately above A .

Instances of inference rules are also defined as in Prawitz[47]. The basic idea behind these rules is that we use the sequence Π_1, \dots, Π_n to list the formulas that occur above a particular designated formula, A . When it comes to expressing inference rules, we will use this notation to express the idea of what formulae are needed to license a particular inference. This will involve nesting the notion in such a way as to make it clear which sets of formulae are used in the derivation of a particular formula. For example, the conditions for \wedge elimination can be given as:

$$\langle\langle\Gamma, \phi\rangle, \langle\Delta, \psi\rangle, \langle\Gamma \cup \Delta, \phi \wedge \psi\rangle\rangle$$

This says that, if we have a derivation of ϕ using Γ and a derivation of ψ using Δ , then we are allowed to derive $\phi \wedge \psi$ from the union of Γ and Δ .

In order to better categorise our rules we introduce three tables, one for elimination rules, one for introduction rules, and one that relates the various propositions to one another. To reduce the number of rules present, we will use the symbol ∇ to range over modal operations M, L, \bar{Q} and Q , in that order. We will use Δ to refer to the operations L, M, Q and \bar{Q} in that order. We use this notation to allow us to express multiple rules as a schema. For example, when we write the rule:

$$\frac{\begin{array}{c} [^M_a fA] \\ \underline{f}^{\nabla} e B \end{array}}{A \overset{\Delta}{e} B} e^{\Delta}\text{-introduction}$$

What this rule tells us is that:

If from assuming $^M_a fA$ we can derive $^L_e fB$ then we may infer $A \overset{L}{e} B$.

If from assuming $^M_a fA$ we can derive $^M_e fB$ then we may infer $A \overset{M}{e} B$.

If from assuming $^M_a fA$ we can derive $^Q_e fB$ then we may infer $A \overset{\bar{Q}}{e} B$.

If from assuming $^M_a fA$ we can derive $^{\bar{Q}}_e fB$ then we may infer $A \overset{Q}{e} B$.

| | |
|---|---|
| $\frac{[faA \wedge faB]}{AiB \quad C} \text{Exposition } +$ $\frac{[f_a^M A \wedge f_a^L B]}{A_i^L B \quad C} \text{Exposition } L^+$ $\frac{[f_a^M A \wedge f_a^M B]}{A_i^M B \quad C} \text{Exposition } M^+$ $\frac{[f_a^M A \wedge f_a^Q B]}{A_i^Q B \quad C} \text{Exposition } Q$ | $\frac{[faA \wedge feB]}{AoB \quad C \quad AiA} \text{Exposition } -$ $\frac{[f_a^M A \wedge f_e^L B]}{A_o^L B \quad C \quad A_i^M A} \text{Exposition } L^-$ $\frac{[f_a^M A \wedge f_e^M B]}{A_o^M B \quad C \quad A_i^M A} \text{Exposition } M^-$ $\frac{[f_a^M A \wedge f_a^{\bar{Q}} B]}{A_i^{\bar{Q}} B \quad C \quad A_i^M A} \text{Exposition } \bar{Q}$ |
| $\frac{AaB \quad daA}{daB} DDO$ | $\frac{AeB \quad daA}{deB} DDN$ |
| $\frac{AeB \quad daB}{deA} DDN$ $\frac{A_a^L B \quad d_a^M A}{d_a^L B} DDO$ | $\frac{A_e^L B \quad d_a^M A}{d_e^L B} DDN$ |
| $\frac{A_e^L B \quad d_a^M B}{d_e^L A} DDN$ $\frac{A_a^M B \quad d_a^M A}{d_a^M B} DDO$ | $\frac{A_e^M B \quad d_a^M A}{d_e^M B} DDN$ |
| $\frac{A_e^M B \quad d_a^L B}{d_e^L A} DDN$ $\frac{A_a^Q B \quad d_a^M A}{d_a^Q B} DDO$ | $\frac{A_a^{\bar{Q}} B \quad d_a^M A}{d_a^{\bar{Q}} B} DDO$ |
| $\frac{d_a^{\bar{Q}} A \quad d_a^M A}{d_a^L A} \bar{Q}\text{-Out}$ | $\frac{d_a^{\bar{Q}} A \quad d_e^M A}{d_e^L A} \bar{Q}\text{-Out}$ |
| $\frac{\phi \wedge \psi}{\phi} \wedge \text{Elimination}$ | $\frac{\phi \wedge \psi}{\psi} \wedge \text{Elimination}$ |
| $\frac{\phi \quad C(\phi)}{\psi} \text{Ex Falso Quodlibet}$ | |

Table 5.5: Elimination Rules

| | |
|---|--|
| $\frac{\underline{d} a A \wedge \underline{d} a B}{A i B} \text{ Expository Syllogism } ^+$ | $\frac{\underline{d} a A \wedge \underline{d} e B}{A o B} \text{ Expository Syllogism } ^-$ |
| $\frac{\underline{d}^M a A \wedge \underline{d}^L a B}{A^L i B} \text{ Expository Syllogism } L^+$ | $\frac{\underline{d}^M a A \wedge \underline{d}^L e B}{A^L o B} \text{ Expository Syllogism } L^-$ |
| $\frac{\underline{d}^M a A \wedge \underline{d}^M a B}{A^M i B} \text{ Expository Syllogism } M^+$ | $\frac{\underline{d}^M a A \wedge \underline{d}^M e B}{A^M o B} \text{ Expository Syllogism } M^-$ |
| $\frac{\underline{d}^M a A \wedge \underline{d}^Q a B}{A^Q i B} \text{ Expository Syllogism } Q$ | $\frac{\underline{d}^M a A \wedge \underline{d}^{\bar{Q}} a B}{A^{\bar{Q}} i B} \text{ Expository Syllogism } \bar{Q}$ |
| $\frac{A i A \quad \frac{[faA] \quad \underline{f} a B}{A a B} a\text{-introduction}}{A a B}$ | $\frac{[faA] \quad \underline{f} e B}{A e B} e\text{-introduction}$ |
| $\frac{A^M i A \quad \frac{[f^M a A] \quad \underline{f}^{\nabla} a B}{A^{\nabla} a B} a^{\nabla}\text{-introduction}}{A^{\nabla} a B}$ | $\frac{[f^M a A] \quad \underline{f}^{\nabla} e B}{A^{\Delta} e B} e^{\Delta}\text{-introduction}$ |
| $\frac{\underline{f} e A}{A o B} \text{ Empty Exposition}$ | $\frac{\underline{f}^L e A}{A^{\nabla} o B} \text{ Empty Exposition}$ |
| $\frac{\underline{d}^L a A}{\underline{d}^{\bar{Q}} a A} \bar{Q}\text{-In}$ | $\frac{\underline{d}^L e A}{\underline{d}^{\bar{Q}} e A} \bar{Q}\text{-In}$ |
| $\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge \text{ Introduction}$ | |
| $\frac{\underline{d} a A}{\underline{d}^L a A} \text{ Necessity Introduction}$ | $\frac{\underline{d} e A}{\underline{d}^L e A} \text{ Necessity Introduction}$ |
| $\frac{\underline{d} a A}{\underline{d}^M a A} \text{ Reflexivity}$ | $\frac{\underline{d}^L e A}{\underline{d} e A} \text{ Reflexivity}$ |
| $\frac{[\phi] \quad \psi}{\mathcal{C}(\phi)} \text{ C-Introduction}$ | |

Table 5.6: Introduction Rules

| | |
|---|--|
| $\frac{A \times^L B}{A \times^M B} \text{Modal Subalternation}^L$ | $\frac{A a/e B}{A \overset{M}{a} B} \text{Modal Subalternation}^Q$ |
| | $\frac{A i/o B}{A \overset{M}{i} B} \text{Modal Subalternation}^Q$ |
| $\frac{A a B}{A i B} \text{Subalternation}^+$ | $\frac{A e B}{A o B} \text{Subalternation}^-$ |
| $\frac{A \overset{L/M}{a} B}{A \overset{L/M}{i} B} \text{Subalternation}^+$ | $\frac{A \overset{L/M}{e} A}{A \overset{L/M}{o} B} \text{Subalternation}^-$ |
| $\frac{A a/e B}{A i/o B} \text{Subalternation}^+$ | $\frac{A a/e A}{A i/o B} \text{Subalternation}^-$ |
| $\frac{\underline{d} \overset{Q}{a} A}{\underline{d} \overset{Q}{e} A} Q \text{ sing Equivalence}$ | $\frac{\underline{d} \overset{\bar{Q}}{a} A}{\underline{d} \overset{\bar{Q}}{e} A} \bar{Q} \text{ sing Equivalence}$ |
| $\frac{A \overset{Q}{a} B}{A \overset{Q}{e} B} Q a - e \text{ Equivalence}$ | $\frac{A \overset{Q}{i} B}{A \overset{Q}{o} B} Q i - o \text{ Equivalence}$ |
| $\frac{A \overset{\bar{Q}}{a} B}{A \overset{\bar{Q}}{e} B} \bar{Q} a - e \text{ Equivalence}$ | $\frac{A \overset{\bar{Q}}{i} B}{A \overset{\bar{Q}}{o} B} \bar{Q} i - o \text{ Equivalence}$ |
| $\frac{\underline{d} \overset{Q}{a} A}{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A} Q - Out$ | $\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A}{\underline{d} \overset{Q}{a} A} Q - In$ |
| $\frac{\mathcal{C}(\mathcal{C}(\phi))}{\phi} \text{CC Elimination}$ | |

Table 5.7: Interaction Rules

Here \underline{f} denotes a new constant that does not occur previously in the proof, while \underline{d} has no such restriction.

The specifications of the proper inference rules are clear from the way the rules are given. For a complete list of the definitions of the proper inference rules see Appendix Four, 244.

As the reader will have observed, a number of the rules used here are *improper inference rules* and we need to stipulate how assumptions are to be discharged. We do this as follows:

5 A Formal Reconstruction of Buridan's Modal Logic

The various forms of *Expository Syllogism* are as follows:

For the assertoric cases:

$$\langle\langle\Gamma_1, A \text{ } i \text{ } B\rangle, \langle\Gamma_2, C\rangle, \langle\Lambda, C\rangle\rangle$$

Where $\Lambda = \Gamma_1 \cup (\Gamma_2 - \Gamma_3)$ where Γ_3 consists of some $\underline{d} a A \wedge \underline{d} a B \in \Gamma_2$ such that \underline{d} does not occur in Λ or C .

For the modal cases:

$$\langle\langle\Gamma, A \overset{\nabla}{i} B\rangle, \langle\Gamma_2, C\rangle, \langle\Lambda, C\rangle\rangle$$

Where $\Lambda = \Gamma \cup (\Gamma_2 - \Gamma_3)$ where Γ_3 consists of some formula $\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{\Delta}{a} B \in \Gamma_2$ such that \underline{d} does not occur in Λ or C

For *Empty Exposition* we have in the assertoric case:

$$\langle\langle\Gamma, \underline{f}eA\rangle, \langle\Gamma, AoB\rangle\rangle$$

Where we require that \underline{f} does not occur in Γ .

In the modal case:

$$\langle\langle\Gamma, \underline{f} \overset{L}{e} A\rangle, \langle\Gamma, A \overset{\nabla}{o} B\rangle\rangle$$

Where we require that \underline{f} does not occur in Γ .

The rule for *Necessity Introduction* is:

$$\langle\langle\Gamma, A \times B\rangle, \langle\Gamma, A \overset{L}{\times} B\rangle\rangle$$

Where Γ is a set of singular formulae of possibility or necessity, i.e. Γ does not contain any propositions of contingency or non-contingency, and it contains no categorical formulae either. A formula is said to be modalised if and only if it is not of the form $A \overset{\nabla}{\times} B$ or $\underline{d} \overset{\nabla}{\times} B$ where ∇ is any one of L, M, Q , or \bar{Q} .

With the exception of the rules using \wedge and *Necessity Introduction*, each of the modal rules is a natural modal analogue of a principle of Buridan's that we have discussed. Buridan discusses *Ex Impossible Quodlibet* in Book One, Conclusion 1.

The idea behind *Necessity Introduction* is not entirely foreign to our discussion of what we have already seen. We can think of the rule for *Necessity Introduction* as a very restricted generalisation of Conclusions 4 and 5 of Book Two, to singular propositions. Conclusions 4 and 5 told us when an assertoric proposition follows from a necessary proposition and when an assertoric proposition entails a proposition of possibility. *Necessity Introduction* is a kind of 'corollary' to this. It tell us that if we have a set of singular modalised formulae and this set entails an assertoric singular proposition, then it must also entailed the corresponding proposition of necessity. While more general forms of *Necessity Introduction* can be formulated for this system (obviously by allow other kinds of modal formulae to occur in Γ), it turns out that for completeness this is form is sufficient.

Each instance of *Expository Syllogism*⁺ and *Expository Syllogism*⁻ corresponds to the principles of sameness and difference respectively, for various assertoric or modal singular terms. The rule for the various forms of exposition are straightforward elimination rules

for the i and o categorical formulae. In the assertoric case, the idea here is that if we have proved AiB from Γ and then can show some formula C by assuming $\underline{f}aA \wedge \underline{f}aB$ (assuming that \underline{f} satisfies the criteria listed) we may conclude C . Slightly less formally, this is a natural modification of the idea of existential elimination for AiB and AoB . The modal cases generalise this to cover the various combinations of modal introduction.

The rules DDO and DDN can either be thought of as the *dictum de omni et de nullo* or as a kind of modus ponens, (e.g. if ‘Every A is B’ and ‘ \underline{c} is A’ then ‘ \underline{c} is B’.) The remaining rules capture the square and octagon of opposition that Buridan endorses, and outline how Q and \bar{Q} function. In the case of rules of *subalternation*, these rules simply capture the usual subalternation inferences as expressed in the square of opposition and the modal octagon. It should be noted that since negative singular propositions should be read as having the negation coming before the modal (e.g. $\underline{f} \overset{Q}{e} A$ should be read as saying the singular term f is not contingently A), the rules for \bar{Q} – *sing* are not entirely parallel with the cases for categorical propositions since, following Buridan, e.g. $A \overset{Q}{e} B$ is translated as ‘Every A is contingently not B’.

With this theory in place, it will follow as a corollary of completeness theorem and our results in Appendix Three that this system can prove all of the conclusions about divided modal propositions in Book Two of the *Treatise on Consequences*. However, the system developed so far does *not* identify or pick out the syllogistic inferences that Buridan considers in Books Three and Four. Thus, our strategy here, following Buridan’s remarks in Book Three, is to identify the particular features that account for syllogistic structures and use these to define syllogisms. As such, we define a syllogistic structure S to be a tuple $\langle M, m \rangle$ along with a designated conclusion C that satisfies the following constraints:

1. M , m , and C are all categorical propositions;
2. M , m , and C have exactly three terms;
3. The predicate of C occurs in M ;
4. The subject of C occurs in m ;
5. M and m share a common term that does not occur in C .³¹

If $\{M, m\} \vdash C$ is provable and M, m , and C are as above, then we call $\{M, m\} \vdash C$. In what follows we will drop the brackets and simply write $M, m \vdash C$.

First, it should be observed that these criteria are used to rule out a number of valid inferences as non-syllogistic. This is in keeping with Buridan’s own language where he distinguishes valid syllogisms from other kinds of valid arguments. For example, inferences such as:

$$\begin{array}{c} B \overset{L}{a} C, A \overset{L}{a} B \text{ entails } B \overset{M}{i} A \\ \text{and} \\ BaC, BaB \text{ entails } BaC \end{array}$$

31. This definition is standard and can be found in e.g.[65].

both fail to constitute syllogistic structures. In the first case, conditions 3 and 4 are violated while in the second conditions 2 and 5 are both violated. It should be observed however that both of these inferences are formally valid for Buridan.

Second, we claim that the proof system is able to prove all of the syllogisms that Buridan claims are valid. This will follow from the completeness proof combined with what we have already established about the semantics for the system. What is interesting, we conjecture, is that the reasoning used in the proof system tracks, to some degree, Buridan's own claims about the expository syllogism. Unfortunately, due to space constraints, we are unable to provide a proof of this result in the present work.

Since both the semantics and proof theory for Buridan's *divided fragment* are based on objects that satisfy particular sets of properties, one natural way to prove completeness would be to use the various propositions to 'force' objects to belong to particular classes of models. This is the approach we will use here. The proof of completeness is fairly routine, being a variation of the usual canonical model construction, modified for singular terms. In this chapter we will prove completeness for the general case where Γ is any set of consistent formulae in our language. The specific case using syllogistic structures as the basis for maximally consistent sets is an easy corollary. The generalised case for when the consistent set is not a syllogistic structure is an easy extension of the results presented here. We begin by recalling the definitions of consistency, inconsistency, and maximality.

Consistency and Inconsistency. Γ is inconsistent if for some ϕ $\Gamma \vdash \phi$ and $\Gamma \vdash \mathcal{C}(\phi)$. Γ is consistent just in case it is not inconsistent.

Maximally Consistent Set. A set Γ is Maximally Consistent if Γ is consistent and there is no consistent set Δ such that Γ is a proper subset of Δ .

Given any set of categorical or singular propositions, we can extend it to a maximally consistent set of propositions using the usual Lindenbaum construction. We assume that we have a well-ordering of the well-formed formulae in our language. Given a consistent set γ we proceed entirely as usual. If we are working with a syllogistic structure $S = \langle M, m \rangle$ consistent with a designated conclusion C , we let $\gamma_0 =$ be the initial three propositions M , m , and C .

We let $\gamma_{n+1} = \gamma_n \cup \{\phi_{n+1}\}$ if $\gamma_n \cup \{\phi_{n+1}\}$ is consistent, and $\gamma_{n+1} = \gamma_n$ if $\gamma_n \cup \{\phi_{n+1}\}$ is inconsistent. We then let $\Gamma = \bigcup_{n < \omega} \gamma_n$.

With such maximally consistent sets in hand, we now want to verify that they preserve all of the inferential relationships that we will need for the construction of our canonical model. The following lemma establishes this.

Maximally Consistent Set Lemma. Let Γ be a maximally consistent set. Then the following hold:

1. $\phi \in \Gamma$ if and only if $\Gamma \vdash \phi$;
2. Exactly one of $\phi, \mathcal{C}(\phi) \in \Gamma$;
3. At most one of A or $B, A \text{ e } B \in \Gamma$;

4. At most one of $A \overset{L}{a} B, A \overset{L}{o} B \in \Gamma$;
5. At most one of $A \overset{L}{a} B, A \overset{M}{e} B \in \Gamma$;
6. At most one of $A \overset{L}{a} B, A \overset{L}{e} B \in \Gamma$;
7. At most one of $A \overset{L}{e} B, A \overset{M}{a} B \in \Gamma$;
8. At most one of $A \overset{L}{e} B, A \overset{L}{i} B \in \Gamma$;
9. If $A a B \in \Gamma$ then $A i B \in \Gamma$;
10. If $A e B \in \Gamma$ then $A o B \in \Gamma$;
11. If $A \overset{\nabla}{a} B \in \Gamma$ then $A \overset{\nabla}{i} B \in \Gamma$;
12. If $A \overset{\nabla}{e} B \in \Gamma$ then $A \overset{\nabla}{o} B \in \Gamma$;
13. If $A \overset{L}{\times} B \in \Gamma$ then $A \overset{M}{\times} B \in \Gamma$;
14. If $A \overset{Q}{\times} B \in \Gamma$ then $A \overset{M}{\times} B \in \Gamma$;
15. $A \overset{Q}{i} B \in \Gamma$ if and only if $A \overset{Q}{o} B \in \Gamma$;
16. $A \overset{Q}{a} B \in \Gamma$ if and only if $A \overset{Q}{e} B \in \Gamma$;
17. $A \overset{\bar{Q}}{i} B \in \Gamma$ if and only if $A \overset{\bar{Q}}{o} B \in \Gamma$;
18. $A \overset{\bar{Q}}{a} B \in \Gamma$ if and only if $A \overset{\bar{Q}}{e} B \in \Gamma$;
19. If $A i B \in \Gamma$ then $A \overset{M}{i} B \in \Gamma$;
20. $A i B \in \Gamma$ if and only if there is some \underline{m} such that $\underline{m} a A \in \Gamma$ and $\underline{m} a B \in \Gamma$;
21. $A \overset{L,M,Q}{i} B \in \Gamma$ if and only if there is some \underline{m} such that $\underline{m} \overset{M}{a} A \in \Gamma$ and $\underline{m} \overset{L,M,Q}{a} B \in \Gamma$;
22. $A o B \in \Gamma$ if and only if either there is some \underline{m} such that $\underline{m} a A \in \Gamma$ and $\underline{m} e B \in \Gamma$ or there is no \underline{n} s.t. $\underline{n} a A \in \Gamma$;
23. $A \overset{L,M,\bar{Q}}{o} B \in \Gamma$ if and only if either there is some \underline{m} such that $\underline{m} \overset{M}{a} A \in \Gamma$ and $\underline{m} \overset{L,M,\bar{Q}}{e} B \in \Gamma$ or there is no \underline{n} s.t. $\underline{n} \overset{M}{a} A \in \Gamma$;
24. $A a B \in \Gamma$ if and only if 1) there is some \underline{m} such that $\underline{m} a A \in \Gamma$ and 2) if $\underline{m} a A \in \Gamma$ then $\underline{m} a B \in \Gamma$;
25. $A e B \in \Gamma$ if and only if for all \underline{m} if $\underline{m} a A \in \Gamma$ then $\underline{m} e B \in \Gamma$;
26. $A \overset{\nabla}{a} B \in \Gamma$ if and only if there is some \underline{m} such that $\underline{m} \overset{M}{a} A \in \Gamma$ and 2) for all \underline{n} if $\underline{n} \overset{M}{a} A \in \Gamma$ then $\underline{n} \overset{\nabla}{a} B \in \Gamma$;
27. $A \overset{\nabla}{e} B \in \Gamma$ if and only if for all \underline{m} if $\underline{m} \overset{M}{a} A \in \Gamma$ then $\underline{m} \overset{\Delta}{e} B \in \Gamma$;
28. $\underline{m} \overset{Q}{a} A \in \Gamma$ if and only if $\underline{m} \overset{M}{a} A \in \Gamma$ and $\underline{m} \overset{M}{e} A \in \Gamma$.
29. if $\underline{m} \overset{\bar{Q}}{a} A \in \Gamma$ and $\underline{m} \overset{M}{a} A \in \Gamma$ then $\underline{m} \overset{L}{a} A \in \Gamma$.

5 A Formal Reconstruction of Buridan's Modal Logic

30. if $\underline{m} \bar{a} A \in \Gamma$ and $\underline{m} \bar{e} A \in \Gamma$ then $\underline{m} \bar{e} A \in \Gamma$.

31. $\phi \wedge \psi \in \Gamma$ if and only if $\phi \in \Gamma$ and $\psi \in \Gamma$

Two observations should be made before proving the MCS Lemma. First, it should be observed that the subcontraries are not included in this lemma because they follow from 2 and 3-7, depending on which subcontrary is in question. For example, assume that neither AiB nor $AoB \in \Gamma$. Then by 2, it follows that AaB and $AeB \in \Gamma$ which contradicts 3. Second, it should be observed that the resulting M.C.S.s are used to provide models for the entire *divided fragment* of Buridan's modal logic, not only for the syllogistic inferences.

The proof of each of these propositions is fairly routine and follows from the construction of Γ and the rules that we have formulated. When a group of cases are similar, but only require the application of a slightly different rule, we will provide an instance and observe that the other cases are similar.

The proofs of 1 and 2 are routine and follow from the properties of a Maximally Consistent Set.

Proof of 3: immediate from *Subalternation*⁺

Proof of 4: immediate from *Subalternation*⁻

Proof of 5: Assume that 5 is false, then we could have $A a B \in \Gamma$ and $A e B \in \Gamma$ where Γ is a maximally consistent set. Then by 4, we have $A o B \in \Gamma$, contradicting the consistency of Γ .

Proof of 6: Assume 6 is false, then we could have $A \bar{a} B \in \Gamma$ and $A \bar{o} B \in \Gamma$ where Γ is a maximally consistent set. Then by *Modal Subalternation*^L we have $A \bar{o} B \vdash A \bar{e} B$. By 1. it follows that $A \bar{e} B \in \Gamma$. But this is the contradictory of $A \bar{a} B$, which means Γ is inconsistent, violating 2.

Proof of 7: Assume that 7 is false, then $A \bar{a} B \in \Gamma$ and $A \bar{e} B \in \Gamma$. By *Modal Subalternation*⁻ it follows that $A \bar{o} B \in \Gamma$, which contradicts 2.

The proofs of 8-10 are similar.

The proofs of 11-14 are immediate from the relevant rules of *Subalternation*.

Proof of 15: immediate from *Modal Subalternation*^L.

Proof of 16: immediate from *Modal Subalternation*^Q.

Proof of 17: immediate from *Q i - o Equivalence*.

Proof of 18: immediate from $Q\ a - e$ Equivalence.

The **Proof of 19** and **20** are analogous using the \bar{Q} Equivalences.

Proof of 21: Assume that $AiB \in \Gamma$. Now, consider the following proof:

$$\frac{\frac{\frac{[faA \wedge faB]}{faA} \wedge - \text{Elimination}}{\underline{f}^M_a A} \text{Reflexivity}}{AiB} \quad \frac{\frac{[faA \wedge faB]}{faB} \wedge - \text{Elimination}}{\underline{f}^M_a B} \text{Reflexivity}}{\underline{f}^M_a A \wedge \underline{f}^M_a B} \wedge - \text{Introduction}}{\frac{A^M_i B}{A^M_i B} \text{ExpositorySyllogism}^{M+}} \text{Exposition}^+$$

Hence $A^M_i B \in \Gamma$ by 1.

Proof of 22: For the left to right direction, assume that $AiB \in \Gamma$ but that for all \underline{n} , we have either $\underline{na}A \notin \Gamma$ or $\underline{na}B \notin \Gamma$. By 2 and some logic we have: for all \underline{n} , either $\underline{ne}A \in \Gamma$ or $\underline{ne}B \in \Gamma$. If either disjunct holds, we show AeB follows. We will prove the case for the first disjunct and note that the case of the second disjunct is similar.

Now, consider the following proof:

$$\frac{\frac{\frac{[\underline{na}A \wedge \underline{na}B]}{\underline{na}A} \wedge - \text{Elimination}}{AiB} \quad \underline{ne}A \text{ Ex Falso Quodlibet}}{AeB} \text{Exposition}^+}{AeB} \text{Exposition}^+$$

Hence, $AeB \in \Gamma$, contradicting 2. Hence there is some \underline{n} such that $\underline{na}A$ and $\underline{na}B \in \Gamma$.

For the other direction, assume that there is some \underline{m} such that $\underline{ma}A \in \Gamma$ and $\underline{ma}B \in \Gamma$. Then, it follows by \wedge Introduction and Expository Syllogism that $AiB \in \Gamma$.

The **Proof of 23** is similar to the proof of 22, only changing the relevant Exposition and Expository Syllogism rules.

For example, assume that $A^M_i B \in \Gamma$ but that for all \underline{n} , either $\underline{ne}^L A \in \Gamma$ or $\underline{ne}^L B \in \Gamma$. Consider the following proof:

$$\frac{\frac{\frac{[\underline{n}^M_a A \wedge \underline{n}^M_a B]}{\underline{n}^M_a A} \wedge - \text{Elimination}}{A^M_i B} \quad \underline{ne}^L A \text{ Ex Falso Quodlibet}}{A^L_e B} \text{Exposition}^{M+}}{A^L_e B} \text{Exposition}^{M+}$$

It would then follow that $A^L_e B \in \Gamma$, contradicting 2. Hence there is some \underline{n} such that $\underline{n}^M_a A$ and $\underline{n}^M_a B$.

The other direction is straightforward. Assume that there is some \underline{n} such that $\underline{n} \overset{M}{a} A \wedge \underline{n} \overset{M}{a} B \in \Gamma$. It then follows by *Exposition*^M that there is some \underline{n} such that $A \overset{M}{i} B \in \Gamma$.

Proof of 24: For the right to left direction, assume that $AoB \in \Gamma$. Assume for a contradiction that 1) there is some \underline{m} such that $\underline{m}aA$ and 2) $\forall \underline{n}$ if $\underline{n}aA$ then $\underline{n}aB$. From 1) it follows that AiA . Taking \underline{l} as a witness for the existential we have:

$$\frac{\frac{\underline{l}aA \quad \underline{l}aA}{\underline{l}aA \wedge \underline{l}aA} \wedge \text{Introduction}}{A i A} \text{Exposition}^+$$

with this and 2) it then follows by a-introduction that $AaB \in \Gamma$, contradicting the consistency of Γ .

For the left to right direction, assume that we have two cases to consider: If $\underline{m}aA \in \Gamma$ and $\underline{m}eB \in \Gamma$, then it follows by *Expository Syllogism*⁻ that $AoB \in \Gamma$. Otherwise, if for no term, \underline{n} is $\underline{n}aA \in \Gamma$ then it follows by *Empty Exposition* that $AoB \in \Gamma$.

The **Proof of 25** is similar, only using the relevant cases of *Expository Syllogism*[∇] and *Empty Exposition*[∇].

Proof of 26: This is the contrapositive of 24.

Proof of 27: This is the contrapositive of 22.

Proof of 28: This is the contrapositive of 25.

Proof of 29: This is the contrapositive of 23.

Proof of 30: This is immediate from *Q – In* and *Q – Out*.

The **Proof of 31** and **32** are immediate from both instances of $\bar{Q} - Out$.

Proof of 33: this is immediate from the introduction and elimination rules for \wedge .

As the reader will have noticed, with the exception of the first rule and the last, each of these rules is a natural extension of results that Buridan discusses. For example, inferences 5–10 together with 13–15 capture the inferences in the modal octagon. The relationships of contradictories are covered by the presence of the operation $\mathcal{C}()$.

We can use this system to prove the various conclusions that Buridan proves in his *Treatise on Consequences*. For example, using *Exposition* and *Reflexivity* we can prove Buridan's conclusion 4 in Book Two of his *Treatise on Consequence*, namely that AiB entails $A \overset{M}{i} B$, as was done in the proof of 21. As an easy corollary of this, we have: $A \overset{L}{e} B$ entails $A e B$.

Alternatively, we can derive the proof as Buridan proves it (and infer 21 as a corollary). To that end we want to show that $A \overset{L}{e} B \vdash A e B$:

$$\begin{array}{c}
 \frac{[\underline{f}aA]}{\underline{f} \overset{M}{a} A} \quad A \overset{L}{e} B}{\underline{f} \overset{L}{e} B} DDN^L \quad \frac{[\underline{f}aB]}{\underline{f} \overset{M}{a} B} \quad A \overset{L}{e} B}{\underline{f} \overset{L}{e} A} DDN^L \\
 \frac{\underline{f} \overset{L}{e} B}{\underline{f} \overset{L}{e} B} Reflexivity \quad \frac{\underline{f} \overset{L}{e} A}{\underline{f} \overset{L}{e} A} Reflexivity \\
 \hline
 AeB \quad e - Introduction
 \end{array}$$

This will turn out to be a very important conclusion, because it textually justifies the inclusion of *Reflexivity* as a rule which in turn ensures that our models are reflexive.

5.6 Soundness and Completeness

The soundness of \vdash is, with one exception, an easy (but tedious) consequence of the semantics presented above. As such, the full details have been relegated to Appendix Four starting on page 237.

The only rules that are not straightforward are the cases for *Necessity Introduction*. To that end, we claim that the rules:

For all \underline{d}

$$\begin{array}{l}
 \text{For all } \underline{d}, \text{ if } \Gamma \vdash \underline{d}aA \text{ then } \Gamma \vdash \underline{d} \overset{L}{a} A \\
 \text{For all } \underline{d}, \text{ if } \Gamma \vdash \underline{d}eA \text{ then } \Gamma \vdash \underline{d} \overset{L}{e} A
 \end{array}$$

both preserve validity, where Γ is a set of singular modalised formulae. In order to prove this, we will prove two lemmas first. The first lemma tells us what singular formulae follow from other singular formulae. The second tells us what categorical formulae follow from singular formulae. As a corollary of the second lemma, we will infer that singular formulae never entail universal categorical formulae.

Lemma 1: Singular Propositions

1. $c \overset{M}{a} A \not\models caA$
2. $c \overset{L}{e} A \not\models ceA$
3. if $c \overset{L}{a} A \models caA$ then $c \overset{L}{a} A \models c \overset{L}{a} A$.
4. if $c \overset{M}{e} A \models ceA$ then $c \overset{M}{e} A \models c \overset{M}{e} A$.

First, observe that 3 and 4 are included for exhaustiveness, but are trivial. For 1) consider the following countermodel:

| | |
|---|----------------|
| $ \begin{array}{l} D = \{c\} \\ R = W^2 \\ O(w) = D \\ v(w, A) = \emptyset \\ v(x, A) = D \end{array} $ | $W = \{w, x\}$ |
|---|----------------|

Likewise, for 2:

| |
|---|
| $ \begin{aligned} D &= \{c\} & W &= \{w, x\} \\ R &= W^2 \\ O(w) &= D \\ v(w, A) &= D \\ v(x, A) &= \emptyset \end{aligned} $ |
|---|

Lemma 2: Let Γ be a set of singular formulae then for all A, B we have:

1. $\Gamma \not\models A \overset{\nabla}{a} B$
2. $\Gamma \not\models A \overset{\nabla}{e} B$
3. $\Gamma \models A \overset{i}{i} B$ if and only if for some \underline{d} $\Gamma \models \underline{d}aA$ and $\Gamma \models \underline{d}aB$
4. $\Gamma \models A \overset{o}{o} B$ if and only if for some \underline{d} $\Gamma \models \underline{d}aA$ and $\Gamma \models \underline{d}oB$
5. $\Gamma \models A \overset{i}{i}^{L,M,Q,\bar{Q}} B$ if and only if either for some \underline{d} $\Gamma \models \underline{d}^M a A$ and $\Gamma \models \underline{d}^{L,M,Q,\bar{Q}} a B$ or for no \underline{d} does $\Gamma \models \underline{d} a A$
6. $\Gamma \models A \overset{o}{o}^{L,M,Q,\bar{Q}} B$ if and only if either for some \underline{d} $\Gamma \models \underline{d}^M a A$ and $\Gamma \models \underline{d}^{M,L,\bar{Q},Q} e B$ or for no \underline{d} does $\Gamma \models \underline{d}^M a A$

The proofs of 3–6 immediately follow from the semantic definitions (and will be appealed to in the MCS Lemma below)

The general strategy for constructing countermodels in the case of 1 and 2 is also straightforward. We sketch the case for AaB and observe that the other cases are similar. In the case of 1) Take an arbitrary model, \mathfrak{M} and world $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma$. Let f be an element such that $c(f)$ does not occur in Γ . Let \mathfrak{M}_f be the same as \mathfrak{M} , except $f \in V(w, A)$ and $f \notin V(w, B)$. Hence $\mathfrak{M}_f \not\models AaB$, however, since f does not occur in Γ we have $\mathfrak{M}_f \models \Gamma$. This follows because in this system, the only valid inferences from singular categorical propositions of the form:

$\underline{d} \overset{\nabla}{a} A$ or $\underline{d} \overset{\nabla}{e} A$ to $\underline{d} \overset{\nabla}{a} B$ or $\underline{d} \overset{\nabla}{e} B$ occur when a universal affirmative or negative categorical proposition holds.³² As such adding $c(f)$ will not provide a counterexample to any of the formulae in Γ .

The soundness of both kinds of necessity introduction follows as a straightforward corollary of Lemmas 1 and 2 together with our observation.

In the case of:

32. The verification of this is a straightforward observation following from the general observation that, $\underline{d} \overset{\nabla}{a} A, A \overset{\nabla}{i} B \not\models \underline{d} \overset{\nabla}{a} B$ and $\underline{d} \overset{\nabla}{a} A, A \overset{\nabla}{o} B \not\models \underline{d} \overset{\nabla}{e} B$ in the affirmative case. The negative case is also similar.

For all \underline{d} , if $\Gamma \vdash \underline{d}aA$ then $\Gamma \vdash \underline{d}^L A$

Take an arbitrary \underline{d} and assume that $\Gamma \vdash \underline{d}aA$. As the proof of $\Gamma \vdash \underline{d}aA$ is shorter than n it follows by our inductive hypothesis that $\Gamma \vDash \underline{d}aA$ also. From Lemma 2. it follows that $\Gamma \not\vDash A \overset{\nabla}{a} B$, $\Gamma \not\vDash A \overset{\nabla}{e} B$, $\Gamma \not\vDash B \overset{\nabla}{a} A$, and $\Gamma \not\vDash B \overset{\nabla}{e} A$. By our observation, it follows that the only way $\Gamma \vDash \underline{d}aA$ is if it follows from a singular categorical proposition. By Lemma 1, this is only possible if $\Gamma \vDash \underline{d}^L A$.

5.6.1 Completeness

Completeness. *If $M, m \vDash C$ then $M, m \vdash C$*

Our proof of completeness will be by the usual canonical model construction, modified for our language. In defining our canonical model, some care needs to be taken to ensure that the relation R is universal.³³

To that end, assume that $M, m \not\vDash C$. Then M, m , and $\mathcal{C}(C)$ is consistent. Hence there exists an M.C.S. Λ containing M, m , and $\mathcal{C}(C)$.

First, we first define our canonical model:

Let $\mathfrak{M}^c = \langle W^C, D^C, R^C, O^C, c^C, v^C \rangle$ where:

$W^C = \{ \Gamma : \Gamma \text{ is a maximally consistent having exactly the same modal singular sentences as } \Lambda. \}$.

$D^C = \{ \underline{n} : \underline{n} \text{ occurs in some maximally consistent set } \Gamma \in W^C \}$.

$\Gamma R^C \Theta$ iff 1) Γ and Θ are in W^C and 2) for all terms singular terms \underline{c} and terms A if $\underline{c} \overset{L}{e} A \in \Gamma$ then $\underline{c} e A \in \Theta$.

$O^C(\Gamma) = \{ \underline{n} : \underline{n} \text{ occurs in } \Gamma \text{ and } \Gamma \text{ in } W^C \}$.

$c^C(\underline{n}) = \underline{n}$.

$v(\Gamma, A) = \{ \underline{n} : \underline{n}aA \in \Gamma \}$

At first glance, it may seem that our requirement that Γ is a maximally consistent set having exactly the same modal singular sentences as Λ is too weak a condition to ensure that Γ and Λ make all of the same modal formulae true. In fact it is not, as the following proof shows:

For all M.C.S. Γ, Δ if Γ and Δ contain exactly same modal singular sentences then for all modal formulae ϕ $\phi \in \Gamma$ if and only if $\phi \in \Delta$.

Take arbitrary Γ and Δ and assume the antecedent. We only need to consider the 16 possible modal categorical formulae. For the cases of $A \overset{\nabla}{i} B$ recall that $A \overset{\nabla}{i} B \in \Gamma$ if and only if $\exists \underline{m}$ such that $\underline{m}^M a A \in \Gamma$ and $\underline{m}^{\nabla} a B \in \Gamma$ by the M.C.S. Lemma 23. We then have $\underline{m}^M a A \in \Gamma$ and $\underline{m}^{\nabla} a B \in \Gamma$ if and only if $\underline{m}^M a A \in \Delta$ and $\underline{m}^{\nabla} a B \in \Delta$ since Γ and Δ contain exactly the same modal singular formulae. An analogous proof holds for the o categorical formulae using M.C.S. lemma 24.

33. I am grateful to Professor Hodges for suggesting how this might be done.

For the various universal propositions, we prove the cases for $A \overset{L}{a} B$ and observe that the rest are similar. Assume that $A \overset{L}{a} B \in \Gamma$ this holds if and only if $\mathcal{C}(A \overset{L}{a} B) \notin \Gamma$ by M.C.S. 2, if and only if $A \overset{M}{o} B \notin \Gamma$ by MCS Lemma 24, this holds if and only if there is some \underline{m} such that $\underline{m} \overset{M}{a} A \in \Gamma$ and for all \underline{m} if $\underline{m} \overset{M}{a} A \in \Gamma$ then $\underline{m} \overset{L}{a} B \in \Gamma$. However, since Γ and Δ agree on all singular modal formulae, both of these properties are preserved in Δ . Hence there is some \underline{m} such that $\underline{m} \overset{M}{a} A \in \Delta$ and for all \underline{m} if $\underline{m} \overset{M}{a} A \in \Delta$ then $\underline{m} \overset{L}{a} B \in \Delta$. Hence by M.C.S. Lemmas 24 and 2, it follows that $A \overset{L}{a} B \in \Delta$.

With these definitions in place, it is routine to verify that we can define the canonical analogues and that they have the following properties:

- i $V^C(\Gamma, A) = v(\Gamma, A) \cap O(\Gamma) = \{\underline{n} : \underline{n}aA \in \Gamma\}$.
- ii $V^C(\Gamma, \neg A) = D \setminus (v(\Gamma, A) \cap O(\Gamma)) = \{\underline{n} : \underline{n}aA \notin \Gamma\}$.
- iii $M^C(\Gamma, A) = \{\underline{n} : \text{There is some } \Delta \text{ such that } \Gamma R \Delta \text{ and } \underline{n}aA \in \Delta\}$
 $= \{\underline{n} : \text{There is some } \Delta \text{ such that } \underline{n}aA \in \Delta\}$
- iv $L^C(\Gamma, A) = \{\underline{n} : \text{For all } \Delta \text{ if } \Gamma R \Delta \text{ then } \underline{n} \overset{L}{a} A \in \Delta\} = \{\underline{n} : \text{For all } \Delta \underline{n} \overset{L}{a} A \in \Delta\}$

In the proof of i., the first equivalence is definitional. The proof of ii follows by basic set theory. In the case of iii. the right to left inclusion is trivial. For the left to right direction, take an arbitrary term $\underline{m} \in \{\underline{n} : \text{There is some } \Delta \text{ such that } \underline{n}aA \in \Delta\}$. We need to show that $\Gamma R \Delta$ this follows immediately by our construction of W . The reason for this is because of how we constructed W^C . To see this, observe that, for any two M.C.S.s Γ and Δ , Γ and Δ contain all of the same modalised singular formulae. (Recall that Γ and Δ are M.C.S.s in W^C). Now, observe that the contrapositive of the second condition on R^C and the maximality of Γ and Θ this is equivalent to:

For all singular terms \underline{c} and terms A if $\underline{c}aA \in \Theta$ then $\underline{c} \overset{M}{a} A \in \Gamma$.

Now, clearly since all the M.C.S. that are in W^C agree on their singular modal formulae, the consequent of the hypothetical is satisfied. This form of the R condition is very useful.

Hence the consequent of the contrapositive of R is satisfied.

The case of iv. is similar.

Semantic clauses for the various formulae are analogous to the ones given before and can be seen below.

As an easy corollary of this and the MCS Lemma, observe that we have the following:

1. if $\Gamma R^C \Theta$ and $A \overset{M}{i} B \in \Theta$ then $A \overset{M}{i} B \in \Gamma$
2. if $\Gamma R^C \Theta$ and $A \overset{L}{e} B \in \Gamma$ then $A \overset{L}{e} B \in \Theta$

Second, observe that we an alternative way to demonstrate that $\Gamma R^C \Gamma$. Take arbitrary \underline{c} and A . From $\underline{c} \stackrel{M}{e} A \in \Gamma$ it follows that $\underline{c} e A \in \Gamma$ by the rule reflexivity.

Before proving each categorical case, the following lemmas will be very useful in what follows.

Existence Lemma

Let Γ be a Maximally Consistent Set in W^C . Then we claim that:

$A \stackrel{M}{i} B \in \Gamma$ if and only if there exists M.C.Ss Δ and Ξ such that, for some \underline{m} ,

1. $\underline{m} \stackrel{M}{a} A \in \Gamma$ and $\underline{m} \stackrel{M}{a} B \in \Gamma$
2. $\underline{m} a A \in \Delta$ and $\underline{m} a B \in \Xi$
3. $\Gamma R^C \Delta$ and $\Gamma R^C \Xi$, and so
4. $M^C(\Gamma, A) \neq \emptyset$ and $M^C(\Gamma, B) \neq \emptyset$.

Left to Right:

Assume that $A \stackrel{M}{i} B \in \Gamma$. Then by the MCS Lemma part 21 for some \underline{m} , $\underline{m} \stackrel{M}{a} A \in \Gamma$ and $\underline{m} \stackrel{M}{a} B \in \Gamma$, proving (1).

We construct Δ and Ξ as follows: let $\delta = \{\underline{b} \stackrel{L}{e} A \in \Gamma : \text{for every singular term } b \text{ and categorical term } A\} \cup \{\underline{b} \stackrel{L}{a} A \in \Gamma : \text{for every singular term } b \text{ and categorical term } A\}$.

By construction $\delta \subseteq \Gamma$ and so δ is consistent. We claim that $\delta \cup \{\underline{m} a A\}$ and $\delta \cup \{\underline{m} a B\}$ are both consistent. Assume not. We prove the case for $\delta \cup \{\underline{m} a A\}$ and observe that the other case is analogous. First, suppose $\delta \cup \{\underline{m} a A\}$ is inconsistent: it follows by C-Intro that $\delta \vdash \underline{m} e A$. As δ is a set of modalised singular formulae, it follows by *Necessity Introduction* that $\delta \vdash \underline{m} \stackrel{L}{e} A$. Since $\delta \subseteq \Gamma$ it follows that $\Gamma \vdash \underline{m} \stackrel{L}{e} A$ contradicting the consistency of Γ .

Let Δ be an MCS extending $\delta \cup \{\underline{m} a A\}$ and let Ξ be an MCS extending $\delta \cup \{\underline{m} a B\}$. First, observe that $\Delta \in W^C$ and $\Xi \in W^C$. This follows since $\Gamma \in W^C$ and Δ and Ξ agree with Γ on all modalised singular formulae. Second, observe that $\underline{m} a A \in \Delta$ and $\underline{m} a B \in \Xi$, proving (2).

Now recall that $\Gamma R^C \Theta$ iff for all singular terms \underline{c} and terms A if $\underline{c} \stackrel{L}{e} A \in \Gamma$ then $\underline{c} e A \in \Theta$. Take an arbitrary singular term \underline{g} and term A such that $\underline{g} \stackrel{L}{e} A \in \Gamma$. It then follows that $\underline{g} \stackrel{L}{e} A \in \delta$ and hence by construction that $\underline{g} \stackrel{L}{e} A \in \Delta$ and $\underline{g} \stackrel{L}{e} A \in \Xi$. That $\underline{g} e A \in \Delta$ and $\underline{g} e A \in \Xi$ follows by the rule *Reflexivity*. Hence $\Gamma R^C \Delta$ and $\Gamma R^C \Xi$, proving (3).

Proof of 4: This follows from 2 and 3.

Right to Left

Immediate from 1. and the MCS Lemma part 21.

In what follows we will mostly be using Existence Lemma 4.

The previous result, together with the results:

1. if $\Gamma R^C \Theta$ and $A i B \in \Theta$ then $A \overset{M}{i} B \in \Gamma$
2. if $\Gamma R^C \Theta$ and $A \overset{L}{e} B \in \Gamma$ then $A e B \in \Theta$

forms the basis of what is normally called the Existence Lemma in standard treatments of modal logic. e.g. [13, p.200]. As will become clear, our existence lemma together with these two observations will have a very similar function.

To establish some other important properties, the following lemmas will be used extensively:

M^C -Lemma

Let Γ be an M.C.S. Then for all singular terms \underline{c} and terms, A :

$\underline{c} \in M^C(\Gamma, A)$ if and only if $\underline{c} \overset{M}{a} A \in \Gamma$.

Proof:

Left to Right direction:

Assume that $\underline{c} \in M^C(\Gamma, A)$. Then by iii on 138 there is some M.C.S. Δ such that 1) $\Gamma R \Delta$ and 2) $\underline{c} a A \in \Delta$. Recall that the contrapositive of 1) states that:

if $\underline{c} a A \in \Delta$ then $\underline{c} \overset{M}{a} A \in \Gamma$. This combined with 2) entails that $\underline{c} \overset{M}{a} A \in \Gamma$ as desired.

Right to Left direction:

Assume that $\underline{c} \overset{M}{a} A \in \Gamma$. We construct an M.C.S. Δ such that $\underline{c} a A \in \Delta$ and $\Gamma R \Delta$.

Let $\delta = \{\underline{d} \overset{L}{e} A : \underline{d} \overset{L}{e} A \in \Gamma\}$. We claim that $\delta \cup \{\underline{c} a A\}$ is consistent. Assume it is not. Then $\delta \vdash \underline{c} e A$ by C-Introduction. Since δ is modalised, it follows by *Necessity Introduction* that $\delta \vdash \underline{c} \overset{L}{e} A$. Now, since $\delta \subseteq \Gamma$ it follows that $\underline{c} \overset{L}{e} A \in \Gamma$, contradicting the consistency of Γ . So $\delta \cup \{\underline{c} a A\}$ is consistent. Let Δ be an M.C.S. based on $\delta \cup \{\underline{c} a A\}$. Clearly $\underline{c} a A \in \Delta$, per construction. Likewise, $\Gamma R \Delta$, since per construction Δ contains every instance of $\underline{d} \overset{L}{e} A$ for any \underline{d} and A that occur in Γ .

As an easy corollary of this construction, observe that if $\Gamma \in W^C$ then $\Delta \in W^C$. This follows since Δ contains every instance of $\underline{d} \overset{M}{a} A$ for any \underline{d} and A that occur in Γ , as we noted above.

L^C - Lemma

Let Γ be an M.C.S. Then for all singular terms \underline{c} and terms, A :

$\underline{c} \in L^C(\Gamma, A)$ if and only if $\underline{c} \overset{L}{a} A \in \Gamma$.

Proof:

Left to Right direction:

Assume that $\underline{c} \in L^C(\Gamma, A)$. Assume that $\underline{c} \overset{L}{a} A \notin \Gamma$. Then $\underline{c} \overset{M}{e} A \in \Gamma$, since Γ is an M.C.S. We claim that if this were the case then there exists an M.C.S. Δ such that $\underline{c} e A \in \Delta$ and $\Gamma R \Delta$, contradicting our assumption that $\underline{c} \in L^C(\Gamma, A)$.

We construct Δ as follows: Let $\delta = \{A \overset{L}{e} B : A \overset{L}{e} B \in \Gamma\}$. We claim that $\delta \cup \{\underline{c} e A\}$ is consistent. This follows by an argument analogous to the one used in the M^C only using the negative singular form of *Necessity Introduction*. As before Δ is an M.C.S. based on $\delta \cup \{\underline{c} e A\}$, and clearly $\underline{c} e A \in \Delta$ and $\Gamma R \Delta$.

Right to Left direction:

Assume that $\underline{c} \overset{L}{a} A \in \Gamma$. Now, assume that there is some Δ such that $\Gamma R \Delta$ and $\underline{c} e A \in \Delta$ i.e. $\underline{c} \notin L^C(w, A)$. Then $\underline{c} a A \in \Delta$ since $\underline{c} \overset{L}{a} A \in \Gamma$ and $\Gamma R \Delta$. This contradicts the consistency of Γ . Hence $\underline{c} \in L^C(w, A)$.

With these two lemmas in place, we now introduce the following notational shorthand:

Similarly to our semantics, we define the following negative operations:

$$M^C(\Gamma, \neg A) = \{\underline{c} : \Gamma R \Delta \text{ and } \underline{c} e A \in \Delta\}$$

$$L^C(\Gamma, \neg A) = \{\underline{c} : \text{if } \Gamma R \Delta \text{ then } \underline{c} e A \in \Delta\}$$

From the M^C and the L^C Lemmas we have the following easy corollaries:

For all MCS Γ

1. $\underline{c} \in M^C(\Gamma, \neg A)$ if and only if $\underline{c} \overset{M}{e} A \in \Gamma$
2. $\underline{c} \in L^C(\Gamma, \neg A)$ if and only if $\underline{c} \overset{L}{e} A \in \Gamma$

To prove 1. observe that $\underline{c} \in M^C(\Pi, \neg A)$ if and only if $\underline{c} \notin L^C(\Pi, A)$. Hence $\underline{c} \overset{L}{a} A \notin \Pi$.

Since Π is an M.C.S. it follows that $\underline{c} \overset{M}{e} A \in \Pi$

The proof of 2. is similar.

In what follows we will refer to 1. as the \bar{M}^C Lemma and 2. as the \bar{L}^C Lemma.

We now prove the main lemma for our completeness proof:

Truth Lemma:

For all formulae ϕ , and all MCS $\Pi \in W^C$ we have:

$\phi \in \Pi$ if and only if $\mathfrak{M}^C, \Pi \models \phi$

As each categorical and singular proposition has a contradictory, by MCS Lemma 1. we will only need to consider the left to right direction. The right to left direction will follow from the observation that if $\phi \notin \Pi$ then $\mathcal{C}(\phi) \in \Pi$ (by the MCS Lemma) and so $\mathfrak{M}^C, \Pi \models \mathcal{C}(\phi)$ (which will follow from the left to right direction) which entails $\mathfrak{M}^C, \Pi \not\models \phi$ since Π is an MCS.

For singular propositions we have the following cases to consider:

1. if $\underline{c} a A \in \Pi$ then $\underline{c} \in V^C(\Pi, A)$.
2. if $\underline{c} e A \in \Pi$ then $\underline{c} \notin V^C(\Pi, A)$.
3. if $\underline{c} \overset{L}{a} A \in \Pi$ then $\underline{c} \in L^C(\Pi, A)$.
4. if $\underline{c} \overset{M}{e} A \in \Pi$ then $\underline{c} \notin L^C(\Pi, A)$.
5. if $\underline{c} \overset{M}{a} A \in \Pi$ then $\underline{c} \in M^C(\Pi, A)$.
6. if $\underline{c} \overset{L}{e} A \in \Pi$ then $\underline{c} \notin M^C(\Pi, A)$.

7. if $\underline{c} \overset{Q}{a} A \in \Pi$ then $\underline{c} \in M(w, A) \cap M(w, \neg A)$.
8. if $\underline{c} \overset{Q}{e} A \in \Pi$ then $\underline{c} \in M(w, A) \cap M(w, \neg A)$.
9. if $\underline{c} \overset{\bar{Q}}{a} A \in \Pi$ then $\underline{c} \notin M(w, A) \cap M(w, \neg A)$.
10. if $\underline{c} \overset{\bar{Q}}{e} A \in \Pi$ then $\underline{c} \notin M(w, A) \cap M(w, \neg A)$.

The cases of 1 and 2 are trivial and follow by construction. 3-6 follow from the M^C Lemma and the L^C Lemma. 7-10 are also easy consequences of the M^C and \bar{M}^C Lemmas.

For categorical formulae we have 16 cases to consider.

Assertoric cases:

Assume that $AaB \in \Pi$. First observe that $V^C(\Pi, A)$ is non-empty. To see this observe that since Π is a maximally consistent set, it follows by the MCS Lemma that $AiB \in \Pi$ and so there is some \underline{m} such that $\underline{m}aA \in \Pi$. Hence $V^C(\Pi, A)$ is non-empty. Next, observe that $V^C(\Pi, A) \subseteq V^C(\Pi, B)$. To see this, assume not. Then there is some \underline{n} such that $\underline{n}aA \in \Pi$ and $\underline{n}eB \in \Pi$. Then it follows by MCS Lemma 22 that $\Pi \vdash AoB$ and so $AoB \in \Pi$, contradicting the consistency of Π . Hence $V^C(\Pi, A) \subseteq V^C(\Pi, B)$ and $V^C(\Pi, A)$ is non-empty. i.e. $\mathfrak{M}^C, \Pi \models AaB$.

Assume that $AiB \in \Pi$. We claim that $V^C(\Pi, A) \cap V^C(\Pi, B) \neq \emptyset$. This holds if and only if there is some \underline{m} such that $\underline{m}aA \in \Gamma$ and $\underline{m}aB \in \Gamma$. This is immediate from MCS Lemma 20. From $V^C(\Pi, A) \cap V^C(\Pi, B) \neq \emptyset$ it follows that $\mathfrak{M}^C, \Pi \models AiB$, as desired.

The proofs for e and o are corollaries.

Modal Propositions:

1. If $A \overset{M}{a} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{M}{a} B$
2. If $A \overset{M}{e} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{M}{e} B$
3. If $A \overset{M}{i} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{M}{i} B$
4. If $A \overset{M}{o} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{M}{o} B$
5. If $A \overset{L}{a} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{L}{a} B$
6. If $A \overset{L}{e} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{L}{e} B$
7. If $A \overset{L}{i} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{L}{i} B$
8. If $A \overset{L}{o} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{L}{o} B$

9. If $A \overset{Q}{a} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{Q}{a} B$
10. If $A \overset{Q}{e} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{Q}{e} B$
11. If $A \overset{Q}{i} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{Q}{i} B$
12. If $A \overset{Q}{o} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{Q}{o} B$
13. If $A \overset{\bar{Q}}{a} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{\bar{Q}}{a} B$
14. If $A \overset{\bar{Q}}{e} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{\bar{Q}}{e} B$
15. If $A \overset{\bar{Q}}{i} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{\bar{Q}}{i} B$
16. If $A \overset{\bar{Q}}{o} B \in \Pi$ then $\mathfrak{M}^C, \Pi \models A \overset{\bar{Q}}{o} B$

Proof of 1:

Assume that $A \overset{M}{a} B \in \Pi$. We need to show two things. First, that $M^C(\Pi, A)$ is non-empty and second, that $M^C(\Pi, A) \subseteq M^C(\Pi, B)$. Observe that by *Subalternation*^M we have $A \overset{M}{i} B \in \Pi$. By the Existence Lemma it follows that $M^C(\Pi, A) \neq \emptyset$.

To prove the second, take an arbitrary term \underline{m} and assume that $\underline{m} \in M^C(\Pi, A)$. Then by the M^C Lemma it follows that $\underline{m} \overset{M}{a} A \in \Pi$. However, it follows from $A \overset{M}{a} B \in \Pi$ and $\underline{m} \overset{M}{a} A \in \Pi$ that $\underline{m} \overset{M}{a} B \in \Pi$, using *DDO*. Hence, $\underline{m} \in M^C(\Pi, B)$ by the M^C Lemma, as required.

Proof of 2:

Assume that $A \overset{M}{e} B \in \Pi$. We need to show that $M^C(\Pi, A) \cap L^C(\Pi, B) = \emptyset$. So, assume that $M^C(\Pi, A) \cap L^C(\Pi, B) \neq \emptyset$.

From our assumption it follows that there is some term \underline{n} such that $\underline{n} \in M^C(\Pi, A) \cap L^C(\Pi, B)$. Then by the M^C Lemma it follows that $\underline{n} \overset{M}{a} A \in \Pi$ and the L^C Lemma entails that $\underline{n} \overset{L}{a} B \in \Pi$. However, $\underline{n} \overset{M}{a} A \in \Pi$ and our assumption that $A \overset{M}{e} B \in \Pi$ entails $\underline{n} \overset{M}{e} B \in \Pi$ by the *DDN*, contradicting the consistency of Π .

Proof of 3:

Assume that $A \overset{M}{i} B \in \Pi$. We need to show that $M^C(\Pi, A) \cap M^C(\Pi, B) \neq \emptyset$. This follows immediately from the Existence Lemma and the M^C Lemma.

Proof of 4:

Assume that $A \overset{M}{o} B \in \Pi$. We need to show that either $M^C(\Pi, A)$ is empty, or that $M^C(\Pi, A) \not\subseteq L^C(\Pi, B)$.

Assume that neither of these are the case, then 1) $M^C(\Pi, A) \neq \emptyset$ and 2) $M^C(\Pi, A) \subseteq L^C(\Pi, B)$. From 1) it follows that there is some M.C.S. Λ and some term \underline{m} such that

$\underline{m}aA \in \Lambda$ and $\Pi R\Lambda$. By an argument analogous to the ones we have used above, it follows that $\underline{m} \overset{M}{a} A \in \Pi$ and so $\Pi \vDash \underline{m} \overset{M}{a} A$.

Now, assume that $\underline{n} \overset{M}{a} A \in \Pi$ for an arbitrary \underline{n} . Then, by the M^C Lemma it follows that $\underline{n} \in M^C(\Pi, A)$. By 2) it follows that $\underline{n} \in L^C(\Pi, B)$. By the L^C Lemma, it follows that $\underline{n} \overset{L}{a} B \in \Pi$. Hence by the MCS Lemma 26, it follows that $A \overset{L}{a} B \in \Pi$, contradicting the consistency of Π .

Necessary Propositions:

Proof of 5: Assume that $A \overset{L}{a} B \in \Pi$. We need to show that 1) $M^C(\Pi, A)$ is non-empty and that 2) $M^C(\Pi, A) \subseteq L^C(\Pi, B)$.

To show 1), observe that $A \overset{L}{a} B \vdash A \overset{M}{i} B$. It then follows by the Existence Lemma and M^C Lemma that $M^C(\Pi, A)$ is non-empty.

To see that $M^C(\Pi, A) \subseteq L^C(\Pi, B)$, take an arbitrary $\underline{n} \in M^C(\Pi, A)$. By the M^C Lemma, it follows that $\underline{n} \overset{M}{a} A \in \Pi$. It then follows by the *DDO* that $\underline{n} \overset{L}{a} B \in \Pi$. By the L^C Lemma it follows that $\underline{n} \in L^C(\Pi, B)$.

Proof of 6:

Assume that $A \overset{L}{e} B \in \Pi$. We need to show that $M^C(\Pi, A) \cap M^C(\Pi, B) = \emptyset$. Assume not, then $M^C(\Pi, A) \cap M^C(\Pi, B) \neq \emptyset$ and so there is some term \underline{m} such that $\underline{m} \in M^C(\Pi, A)$ and $\underline{m} \in M^C(\Pi, B)$. It then follows by the M^C Lemma that $\underline{m} \overset{M}{a} A \in \Pi$ and that $\underline{m} \overset{M}{a} B \in \Pi$. By \wedge Introduction and *Expository Syllogism*^{M+}, it follows that $A \overset{M}{i} B \in \Pi$ contradicting the consistency of Π .

Proof of 7:

Assume that $A \overset{L}{i} B \in \Pi$. We need to show that $M^C(\Pi, A) \cap L^C(\Pi, B) \neq \emptyset$. By MCS Lemma 21 it follows that there is some \underline{n} such that $\underline{n} \overset{M}{a} A \in \Pi$ and $\underline{n} \overset{L}{a} B \in \Pi$. Hence by the M^C Lemma $\underline{n} \in M^C(\Pi, A)$ and by the L^C Lemma it follows that $\underline{n} \in L^C(\Pi, B)$. Hence $M^C(\Pi, A) \cap L^C(\Pi, B) \neq \emptyset$.

Proof of 8:

Assume that $A \overset{L}{o} B \in \Pi$. Assume for a contradiction that 1) $M^C(\Pi, A)$ is non-empty and that 2) $M^C(\Pi, A) \subseteq M^C(\Pi, B)$.

From 1) it follows by the M^C Lemma that there is some term \underline{d} such that $\underline{d} \overset{M}{a} A \in \Pi$. From 2) and the M^C Lemma it follows that, for all terms \underline{n} if $\underline{n} \overset{M}{a} A$ then $\underline{n} \overset{M}{a} B$.

From these it follows by a^M introduction that $A \overset{M}{a} B \in \Pi$, contradicting the consistency of Π .

Contingency Propositions:

Proof of 9 & 10:

For 9, assume that $A \overset{Q}{a} B \in \Pi$. We need to show that 1) $M(w, A) \neq \emptyset$ and that 2) $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$.

To see 1) it follows by subalternation that $A \overset{M}{a} B \in \Pi$ and that $A \overset{M}{i} B \in \Pi$. By the Existence Lemma and M^C Lemma it follows that $M(w, A) \neq \emptyset$.

For 2) take an arbitrary \underline{d} and assume that $\underline{d} \in M(w, A)$. It then follows by the M^C Lemma that $\underline{d} \stackrel{M}{a} A \in \Pi$. Since $A \stackrel{Q}{a} B \in \Pi$, it follows by the DDO that $\underline{d} \stackrel{Q}{a} B$. It then follows by $Q - Out$ and \wedge elimination that $\underline{d} \stackrel{M}{a} B \in \Pi$ and $\underline{d} \stackrel{M}{e} B \in \Pi$. It then follows by the M^C Lemma and the \bar{M}^C Lemma, that $\underline{d} \in (M(w, B) \cap M(w, \neg B))$. Hence $\underline{d} \in (M(w, B) \cap M(w, \neg B))$, completing the proof.

For 10, assume that $A \stackrel{Q}{e} B \in \Pi$. Again, we need to show that 1) $M(w, A) \neq \emptyset$ and that 2) $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$. By $Qa - e$ Equivalence it follows that $A \stackrel{Q}{a} B \in \Pi$, and by the previous proof, we are done.

Proof of 11 & 12:

For 11, assume that $A \stackrel{Q}{i} B \in \Pi$. We need to show that $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$. It suffices to show, by the M^C Lemma and that \bar{M}^C Lemma the following: there exists a \underline{m} such that:

1. $\underline{m} \stackrel{M}{a} A \in \Pi$
2. $\underline{m} \stackrel{M}{a} B \in \Pi$
3. $\underline{m} \stackrel{M}{e} B \in \Pi$

However, by M.C.S. lemma 21. it follows that there is some \underline{m} such that $\underline{m} \stackrel{M}{a} A \in \Pi$ and that $\underline{m} \stackrel{Q}{a} B \in \Pi$. By M.C.S. Lemma 28 it follows that:

there is some \underline{m} such that $\underline{m} \stackrel{M}{a} A \in \Pi$, $\underline{m} \stackrel{M}{a} B \in \Pi$, and $\underline{m} \stackrel{M}{e} B \in \Pi$, which is what we needed to show.

For 12 assume that $A \stackrel{Q}{o} B \in \Pi$. Again, we need to show that $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$. By $Qi - o$ Equivalence it follows that $A \stackrel{Q}{i} B \in \Pi$, which entails the conclusion by our proof of 11.

Proof of 13 & 14: As in the cases of Q the case of \bar{e} will follow from \bar{a} .

Assume that $A \stackrel{\bar{Q}}{a} B \in \Pi$. We need to show that $M(w, A) \cap M(w, B) \cap M(w, \neg B) = \emptyset$. Assume that it is not. Then by the M^C and \bar{M}^C Lemmas it follows that there is some \underline{m} such that 1) $\underline{m} \stackrel{M}{a} A \in \Pi$, 2) $\underline{m} \stackrel{M}{a} B \in \Pi$, and 3) $\underline{m} \stackrel{M}{e} B \in \Pi$. By 2) and 3) it follows by $Q - In$ that $\underline{m} \stackrel{Q}{a} B \in \Pi$. Which together with 1) entails that $A \stackrel{Q}{i} B \in \Pi$ by *Expository Syllogism* Q^+ . This contradicts the consistency of Π .

Proof of 15 & 16: As in the cases of Q the case of \bar{o} will follow from \bar{i}

Assume that $A \stackrel{\bar{Q}}{i} B \in \Pi$. We need to show that either $M(w, A)$ is empty or that $M(w, A) \not\subseteq M(w, B) \cap M(w, \neg B)$. Assume that neither of these is the case. Then 1) $M(w, A)$ is non-empty and 2) $M(w, A) \subseteq M(w, B) \cap M(w, \neg B)$. By the M^C Lemma and 1 it follows that there is some $\underline{m} \stackrel{M}{a} A \in \Pi$.

Now, assume that, for an arbitrary \underline{n} that $\underline{n} \overset{M}{a} A \in \Pi$. By the M^C Lemma it follows that $\underline{n} \in M^C(\Pi, A)$ and by 2) it follows that $\underline{n} \in M^C(\Pi, B)$ and that $\underline{n} \in M^C(\Pi, \neg B)$. Again, by the M^C Lemma and the \bar{M}^C Lemma it follows that $\underline{n} \overset{M}{a} B \in \Pi$ and $\underline{n} \overset{M}{e} B \in \Pi$. It then follows by $Q - In$ that $\underline{n} \overset{Q}{a} B \in \Pi$.

As \underline{n} was arbitrary, it follows that $\forall \underline{n}$ if $\underline{n} \overset{M}{a} A$ then $\underline{n} \overset{Q}{a} B$.

From both of these things, it then follows by the M.C.S. Lemma 26, that $A \overset{Q}{a} B \in \Pi$, contradicting the consistency of Π .

To conclude our proof of completeness recall that we assumed $M, m \neq C$. Then there exists an M.C.S. Ω such that M, m , and $\mathcal{C}(C)$ are all in Ω . We construct \mathfrak{M}^C with Ω as the M.C.S. used to select which M.C.S.s are in W^C . By the Truth Lemma, it follows that $\mathfrak{M}^e, \Lambda \models M$, $\mathfrak{M}^e, \Lambda \models m$ and $\mathfrak{M}^e, \Lambda \models \mathcal{C}(C)$. Hence by the truth lemma and the MCS Lemma $\mathfrak{M}^e, \Lambda \neq C$. Hence $M, m \neq C$ as claimed.

By contra-position it then follows that if $M, m \models C$ then $M, m \vdash C$.

Q.E.D.

5.7 Conclusion

It is difficult to fully appreciate how valuable Buridan's contribution to the history of modal logic is. In presenting Buridan's work on the modal syllogism in light of modern logic some of this value should become clear: First, Buridan's semantic analysis of modal propositions and his treatment of the expository syllogism naturally line up with each other, in the sense that the two systems fit seamlessly together. Because of this it is possible to show that the formal analogues of these two systems are sound and complete. Second, unlike Aristotle, whose remarks about the modal syllogism have confused his interpreters and require elements of his metaphysics to explain the validity of the apodictic modal syllogistic (see e.g. [37] and [38] and it is still unclear if Aristotle's modal logic can be given a plausible reading, either in natural language or formally), Buridan's theory follows clearly and perspicuously from the assumptions that he lays down. Within the *Treatise on Consequences* his reasoning is flawless, the syllogisms that he treats as valid are indeed so, and his counterexamples establish the invalidity of the invalid ones. Third, the semantics that he employs suggest an interesting extension of the usual resources for variable domain first-order modal logic, and the resources he employs point to some interesting and potentially fruitful technical developments.

In answering Hughes' challenge we are able to see just how impressive Buridan's treatment of modal logic was. Beyond being an original reformulation of the modal syllogism, we see that Buridan's logic is able to answer to the highest standards expected of modern logic. With this in mind, it is no wonder that Buridan's work on logic came to be so dominant in the late Middle Ages, nor that contemporary study of his work proves to be enlightening.

6 Comparisons and Philosophical Implications

6.1 Introduction

In our previous chapters we have been exploring two very different accounts of modality and the modal logics that they yield. Our goal in this chapter is to bring together these two theories and explore a number of natural questions that have emerged during this study of these two theories.

The general structure of this chapter will proceed in the following way: we will first start with a fairly brief sketch of the history of modal logic. Our goal here is to situate Kilwardby and Buridan within the logical developments of their times and to see how their views on modality map onto the general spectrum of views on offer at the time. As we shall see, situating Buridan in this history proves to be non-trivial. After situating these two views, we will go on to look at some of the connections, common ground and differences that exist between Buridan and Kilwardby. We will start with the most concrete connection, by comparing Kilwardby's treatment of *per se* modalities with Buridan's discussion of essential properties and essences in a number of works (but with particular attention to the *Summulae de Dialectica*).

The next topic we will treat in this chapter is the ontological underpinnings of Buridan's framework. We will attempt to see if we can obtain a clearer view of how Buridan conceives of modality and the ontology that underpins his modal theory. We will frame this debate in terms of the modern debate between contingentism and necessitism, as this will help us get clearer on the ontological question. We will use these theories to help answer two questions that arise in the context of Buridan's analysis of modality. Of these, one will look at the question of the ontological status of the objects in Buridan's theory and the other will concern the sorts of inferences that Buridan admits in his modal theory. The first question can be broached by asking 'what sorts of objects does Buridan's nominalist ontology commit him to when he talk about non-existent or 'possible objects' in some of the passages in the *Treatise on Consequence*?' The second can be discerned by looking at some of the inferences that Buridan accepts and rejects as valid. In particular, we should enquire as to whether Buridan's views about the inferential relationship between divided and composite modal propositions commit Buridan to a rejection of the Barcan and Converse Barcan formulae.

The Barcan and Converse Barcan formulae, named in honour of Ruth Barcan Marcus, can be formulated as:

$$\text{BF} \quad \diamond \exists x \phi x \rightarrow \exists x \diamond \phi x$$

CBF $\exists x \diamond \phi x \rightarrow \diamond \exists x \phi x$

Let v and w be worlds and D_w and D_v be the set of objects that exist at w and v respectively. The logical significance of the Barcan formula is that it amounts to requiring that if wRv then $D_w \subseteq D_v$. The converse Barcan formula requires that if wRv then $D_v \subseteq D_w$. The philosophical importance of these formulae has been discussed at length in various places, and a number of philosophical positions closely connect the validity of these formulae to necessitism, the view that necessarily, everything is necessarily something. For example, see [66]. We will argue that, because Buridan seems committed to reject both of these propositions, and given some other remarks by Buridan, he is committed to some sort of contingentism. We will discuss what this means for Buridan's ontology and its relationship to the modal syllogism. As we develop this framework, we will be exploring the suitability of understanding Buridan's modal logic in terms of the modern notions of possible worlds.

In the final section of this chapter, we will raise a meta-level question about the project that we have been undertaking. Throughout these logical reconstructions we have been analysing the modalities present in medieval authors by means of Kripke frames, a modern formal apparatus. How valid is this construction? What exactly is the point of doing this, and, more importantly, can we conclude anything substantive about medieval modal theories based on their representations within the lights of modern logic? Here we will argue that this issue relates closely to the interpretive gloss offered on the Kripke semantics used to formalise the theories¹. We will argue that in the case of Buridan, the use of possible worlds semantics is historically appropriate and the results of our modal logic may prove useful for interpreting Buridan's logic. In the case of Kilwardby we will argue that this is not the case and that our formal treatment of Kilwardby should be read as offering a modern take on Kilwardby's logic.

6.2 Situation Figures within Modal History

In order to situate Kilwardby and Buridan's modal theories, it may be helpful to rehearse some of the main details of the development of medieval theories of modality and modal logic. Broadly, there were three main groups whose thinking and writings influenced the development of medieval modal logic. As we have already remarked in a couple of places, the medieval study of modal logic was one that developed and was deeply influenced by the Greek sources that were available to them. The most influential of these, at first, were Boethius's translations of Aristotle's *De interpretatione* and the *Categories* as well as Porphyrius's *Isogoge*.² Later, as more of Aristotle's works became available, the rest of the *Organon* exerted a considerable influence on medieval theories of modality, as we have already seen in the case of Kilwardby. A second influence came through the theological writings of a number of 'Church Fathers' in particular Augustine, who had a considerable impact on the way modality was understood and

1. By gloss we mean, what sort of motivation that is offered to interpret the various parts of the mathematical model (e.g. W, R, O, D, c and v in the case of Buridan)

2. See [33, p.505]

conceived. Most of these writings influenced the development of modal theories focusing around theological issues such as God's power and on the sorts of things that it was possible or impossible for God to do. Third, and finally, there was the influence of medieval Arabic logicians/philosophers, who we will not discuss in detail here.

Our survey in this section will cover a few selected interpretations of modality that are relevant to our discussions of Buridan and Kilwardby. We will start by looking at the statistical interpretation of modality, the *potential* interpretation of modality and (briefly) the temporal interpretation of modality, and discuss the role that essentialism and modal conversion rules played in these 12th and 13th century developments. We will then look at theories of modality in the 14th and in particular some of the contours of Duns Scotus' modal theory.

6.2.1 Modal Logic in the 12th & 13th centuries

In the 12th and 13th centuries, approaches to modality were closely connected with a number of related Aristotelian metaphysical ideas. Our commentary here follows Knuuttila's in [33] observing, the standard view of the time can be well summed up in the following passage from Aquinas' commentary on the *Peri Hermeneias*:

In necessary matter, all affirmative propositions are determinately true; this holds for propositions in the future tense as well as in the past and present tenses; and negative ones are false. In impossible matter the contrary is the case. In contingent matter, however, universal propositions are false and particular propositions are true. This is the case in future tense propositions as well as those in the past and present tenses. In indefinite ones, both are at once true in the future tense propositions as well as those in the past and present tenses. [33, p.507]

There are a number of things here that are interesting, but our main focus will be to observe that the ideas of necessity, impossibility, and contingency are closely tied to the matter of the proposition. The matter of a proposition can be elucidated in the following way:

If the predicate is *per se* in the subject, it will be said to be a proposition in necessary or natural matter, for example, 'Man is an animal' and 'Man is risible'. If the predicate is *per se* repugnant to the subject, as in a way excluding the notion of it, it is said to be a proposition in impossible or remote matter, for example 'Man is an ass'. If the predicate is related to the subject in a way midway between these two, being neither *per se* repugnant to the subject nor *per se* in it, the proposition is said to be in possible or contingent matter. [33, p.508]

It is helpful to observe that by the time we come to Aquinas, we have a very succinct and clear presentation of the ideas of *per se* predication of terms, and the relationship

between them and the matter of a proposition. Aquinas's theory is one example of what is referred to as a *statistical* model for modality. In such a model, the notion of necessity is connected with a notion of containment or of omni-temporal actuality of something within all members of a species, contingency with some members of the species possessing the property and impossibility with no members of the species possessing the property. As we have already seen (see page 13), Kilwardby's theory of modality falls under this framework, specifically under the second disjunct.

Another view that was considered during the middle ages was the view of possibility as potency or the potential interpretation of modality. As in the previous case, the inspiration for this view came from Aristotle, in this case by way of *Metaphysics* V.12 and IX.1. In these passages potency is described as a principle of motion where something is either an activator or a receptor of the relevant influence. Modal notions were then developed based on these ideas, often following Aristotle's observation that this is one legitimate sense of 'can'. Something is necessary on this view if its potency is never unrealised, i.e. the thing in question is always actual. Their nature is such that they are always actual.[33, p.513] Similarly, a thing is impossible if its potency is never actualised. The original idea behind this theory was to account for various kinds of changes in objects.

A third approach to understanding possibility and necessity is to employ various notions of time and tense. The view was motivated by Aristotle's comments in the *Peri Hermeneias* 9.19 A23-24. In this much studied passage, Aristotle draws a distinction between what is necessary without qualification and what is actual. Interpreting this passage and the distinctions that Aristotle was using gave rise to a number of different interpretations.³ However, what eventually developed was a view of modality that sought to unpack modal notions by thinking of them in temporal terms. One very simple view is to read 'possibly ϕ ' as saying that either ϕ is true, was true or will be true. Likewise, 'necessarily ϕ ' is read as saying that ϕ is true, was always true and will always be true. This is the *temporal* interpretation of modality. We note this view, only to set it aside, as it does not play an important role in either Kilwardby or Buridan's modal logic.

6.2.2 Modal Logic in the 14th century

As we move to look at the time when Buridan wrote, one observes that in the intellectual landscape of 14th century, a number of important changes had occurred over the past 200 years. Most important among them for our purposes was that:

In contrast to the twelfth and thirteenth centuries, many Scholastics of the fourteenth and fifteenth viewed the world as radically contingent, depending upon a divine will able to will other than it does will. Belief in God's absolute power to do anything that does not involve a contradiction meant a concomitant belief that only some of the logically possible possibilities could be actual at any one time. . . Moreover, since this world unfolds its events over

3. See [33][p.516] and references given there.

time, it seemed possible that alternative futures still lay open to divine and human choice— that what would come to pass was not ultimately determined or perhaps even determinate.[18][p.2]

It is these two commitments, combined with the idea of God’s omniscience, that led to a number of interesting and striking problems for medieval theologians. The developments are equally important for the analysis of possibility. The impact of Aquinas’ statistical views on modality seems to have been underwhelming. For example, when we look at Dominican discussions of possibility and necessity in the 1320’s and 1330’s we find that

And yet absent from their works are the Thomist analysis of necessity and contingency according to Aristotelian causal theory, the Thomist emphasis on providence and its allied compatibilism, the Principle of Plenitude, and other hallmarks of Aquinas system. [18][p.123]

What we find in place of this view is the final view in the history of modal logic that we are interested in, the view that something is possible just in case it is within (or compatible with) the absolute power of God to bring it about. According to Knuuttila, this view has its origin in the bible[30, p.199] and was present in Christian writers as early as Tertullian. However, it is John Duns Scotus who is usually credited with the development of this view and of seeing the implications this had for logic and metaphysics more generally. In his writings Scotus rejects Aristotle’s thesis that the present is necessary, and develops a conception of synchronic alternatives.[31, p.129] Scotus’s view can be described as follows:

The main lines of Scotus’ theory of modality are easy to understand without any speculative interpretations. This depends on the fact that his starting point is a criticism of the statistical interpretation of modality. An extensive discussion about the theory of modality is to be found in distinction 39 of the first book of Scotus’ commentary on the Sentences. . . According to Scotus a *causatum* can be contingent only if the first cause functions in a contingent way. The contingency of phenomena in a causal chain depends on whether the whole universe (which is ultimately reducible to the first cause) could be different. Scotus take this to be conceptually possible. [30, p.225]

What is important here is the idea that the actual world does not need to be the way it is. Scotus’ ideas cut against the statistical interpretation of modality and

Scotus often treated modal notions in a way which shows similarities to what has been done in the contemporary possible worlds semantics. [31, p.130]

As Knuuttila goes on to observe, there are a number of interpretations of Scotus that make his views appear to be very modern in spirit.

Punch's interpretation of Scotus's modal theory is possibilistic and that of Mastrius is conceptualistic and mind dependent. I have argued that Scotus's theory of purely logical possibilities is not possibilistic in any standard sense and that that it is not mind dependent at this level either.[31, p.142] ⁴

What is important for our purposes here is to observe that, broadly speaking, there are important philosophical parallels that exist between modern theories of modality and the theory of modality advocated by Scotus. In part, this is important in our attempts to situate Buridan's theory, and it is also important when we turn to think about the relationship between the formal systems we have developed for these two figures and the faithfulness of our representations of them.

This leaves us with the question of situating Buridan within these theories. As we have already remarked in a few places, Buridan does not attempt to offer an interpretation or a gloss on what it means for a proposition to be necessary or possible in the *Treatise on Consequences*. As such, this means that any attempt to situate Buridan's analysis must be somewhat speculative in nature. From what we know of the life of Scotus, he was lecturing on the *Sentences* at the University of Paris from 1302, subsequently being expelled from France in June 1303, returning to Paris in April 1304. He remained in Paris, likely around October of 1307. From this, it seems likely that Scotus' views, both theological and philosophical (including his views on modality) would have been in the air while Buridan was a student in Paris.⁵ The difficulties become further complicated when we observe that Buridan's philosophical positions are, generally speaking, not consistent with Scotus. However, there seems to be at least one place where Buridan does make implicit use of Scotus' views on modality.

Buridan seems to show some awareness of Scotus' views on modality in his *Quaestiones super octo Physicorum libros Aristotelis*. According to Knuuttila,

In question 22 of the first book of the questions on Aristotle's *Physics*, Buridan analyzes the terms "potency" and "possibility" as follows. One can take these terms to refer (1) to a proposition which expresses something that is possible, (2) to something which can be actualized by the interplay of an active potency and a passive potency which exist in nature, or (3) to something which can be realized by the supreme active potency which is God's omnipotence. He says that things which are possible in sense (2) are also possible in sense (3) and the same holds of (1) as far as propositions are treated. . . Even though divine omnipotence is the ultimate executive power, the possibilities which can be realised are possible by themselves. As for

4. By 'possibilistic' Knuuttila writes: "In the possibilist theories, possible worlds are treated as having some kind of reality, either all of them as having equal being as Lewis thinks, or the actual world of ours as having the superior sort of being." [31, p.142]

5. Even this is not entirely clear. Scotus would have likely been a master in the English nation and as such, it is unclear how quickly or completely Scotus' views on modality would have been disseminated to the other nations in the university. However, it is not unreasonable to suppose that by the time Buridan was an Art's master (some 15-20 years later) these views would have been known.

the unrealized possible beings (*possibilia*), Buridan states that they have no existence and are not founded on anything.[32, p.71]

As Knuuttila goes on to point out, the view of possibility that Buridan sketches in the *Questions on the Physics* is one that only makes sense in the framework of the modal metaphysics developed by Scotus. What this suggests is that by the time Buridan wrote his *Quaestiones* he was aware of Scotus' position and was willing to make use of it in understanding the distinction between something that is physically possible/impossible and something that is possible *simpliciter*. This offers some further evidence for seeing Buridan's modal theory in the *Summulae* and perhaps the *Treatise on Consequences* as being aware of Scotus' modal theory, albeit developed in a different direction.

One final point, if this sort of theologically informed modal view is what Buridan is working with, then it does make some sense as to why he might not have wanted to offer a gloss or an interpretation of necessity and possibility. As is well-known, the Arts faculty at the University of Paris had, by the time of Buridan, gotten itself into theological troubles with the church a number of times, perhaps most famously in the condemnation of 1277, but other times as well. By Buridan's time Arts masters were normally required to promise to not speak on matters outside of their discipline (i.e. on matters of theology) and at least one point in the *Treatise on Consequences* Buridan makes reference to this.⁶ Because of this, if Buridan wanted to avoid courting theological controversy, it would be prudent of him to avoid mentioning any more theologically relevant ideas concerning modality than he needs to in order to establish his conclusions about modal consequences.

6.3 *Per Se per Buridan & Kilwardby*

There are a few challenges that are present if we want to compare and contrast Buridan and Kilwardby's accounts of modality. First, and perhaps most glaringly, while Kilwardby offers us some explanation and motivation for the different sorts of modal operators that he is using (those that are *per se* and *per accidens*), Buridan does not offer us a similar sort of distinction for the sorts of modal operations that he is using.⁷ This entire chapter is intended to help us better compare these two theories. A second, and related problem comes from both the historical and philosophical distance between Kilwardby and Buridan. Kilwardby's metaphysics and ontology are fundamentally realist in nature, while Buridan is a chief defender of nominalism.

On the metaphysical front, things are not as grim as they appear. As has been pointed out by Klima (see [27],[28]), while seeking to be a nominalist, Buridan also wants to

6. As we have already noted, this was in the context of the validity of the syllogism as it relates to the presence of Trinitarian terms. Buridan remarks that: "Now whether according to another way of speaking syllogisms in divine terms are formally valid and what that form is, I leave to the theologians. And it should be noted and always kept in mind that, because it is not for me, an Arts man, to decide regarding the foregoing beyond what was said. . ." [51]Book Three, Conclusion 1

7. So far as I am aware, the closest we get to this in the *Treatise on Consequences* is the difference between divided and composite modalities. However, in neither case are these modal operations reduced to simpler notions nor are we given explicit motivation for how they should be analysed.

‘affirm’ the ‘existence’ of essences and admit them into his theories. This has been coined Buridan’s “essentialist nominalism”. In order to better discuss the relationship between Kilwardby and Buridan’s analysis of modality, it will prove helpful to start with this natural touchstone. While (so far as I am aware) Buridan does not use *per se* to provide a distinction between different sorts of modalities, Buridan does have quite a bit to say about essences and essential properties, which we will review below.

6.3.1 Buridan’s Essentialist Nominalism

When speaking of an ‘essentialist nominalism’ it is very important that we be clear on exactly what such a position amounts to.⁸

Before we venture into this directly, it may be helpful to set aside a few possible misconceptions. First, the debate between medieval realists and nominalists was not a debate between those who accepted the existence of ‘platonic’ or ‘abstract objects’ (the realists) and those who denied their existence (the nominalists). Both camps took Aristotle to have soundly refuted such a position, to the point that some found the ‘platonic’ view so absurd that they doubted if Plato really subscribed to it.⁹ Hence both parties to this debate took the objects in question to be just as individual and temporal as any other object would be.[26, p.476]

So then, what is the difference between nominalists like Buridan and realists? At this point it is helpful to distinguish between ‘pre-Ockham’ realists, like Giles of Rome and Kilwardby, and ‘post-Ockham’ realists such as Burley, Scotus, etc.¹⁰ The main difference between ‘pre-Ockham’ realists and both nominalists and ‘post-Ockham’ realists was that:

while the former [‘pre-Ockham’ realists] would consider abstract terms in the accidental categories to be essential predicates of their particulars, the latter would reject this assumption. . .

What seems to be at the bottom of the “older realist” commitment, then, in interpreting abstract accidental terms as the genera and species, that is, essential predicates of their particulars. To be sure, even those authors who can justifiably be regarded as “older realists” in the sense of working within the semantic framework outlined above plus endorsing the view that abstract terms in the accidental categories are essential predicates of their supposita. . . were prepared to regard several abstract terms as non-essential predicates of their supposita. [26, p.483, ft 18]

What then makes Buridan an essentialist nominalist? In [27, p.740] Klima suggests that we can draw a distinction between ‘predicate-essentialism’ and ‘realist essentialism’. The view attributed to Buridan here is that Buridan wants to be able to attribute essential predicates to things, but in doing so, he is not committed to positing some

8. As is pointed out in [27, p.739], on many contemporary conceptions, nominalism and essentialism turn out to be in strong tension or are flat out incompatible.

9. See [26, p.476] and the references in ft.7 to Giles of Rome and John Wyclif.

10. The terms ‘pre-Ockham’ and ‘post-Ockham’ are not ideal terms, since, for example, Burley is contemporaneous with Ockham. However, they are better than using terms such as ‘older’ and ‘newer’.

shared common essence. The challenge, of course, is to see if it is actually possible to pull these two notions apart in a way that is coherent.

So then, how does Buridan conceive of essential predication? The following quote offers some insight:

Since something is called a predicable because it is apt to be predicated of many things, it is reasonable to distinguish the species or modes contained under the term ‘predicable’ according to the different modes of predication. Therefore, everything that is predicated of something is either predicated essentially, so that neither term adds some extraneous connotation to the signification of the other; or it is predicated denominatively, so that one term does add some extrinsic connotation to the signification of the other. This division is clearly exhaustive, for it is given in terms of opposites. [5, p.106]

What is interesting here is how Buridan goes about defining essential and denominative predication. The definition of essential predication is cashed out in terms of what is ‘added’ to the connotation of a term. This is, at least at first glance, different from Kilwardby’s conception of a *per se* necessity, where the consequent of the predication is understood in the antecedent. However, things become more interesting when we try to flesh out Buridan’s definition. In order to do this, we need to look at the connotation of a term. So far as I am aware, Buridan does not explicitly define connotation, however, there are numerous examples that make it clear how Buridan sees the relationship working. For example:

The terms ‘white’ and ‘black’ connote qualities of the substances for which they supposit, and it is on account of these [qualities] that [the substances] are said to be such and such; again, the terms ‘two cubits long’ and ‘three cubits long’ [connote] the quantities [of substances] by which they are measurable. But it is in accordance with another mode of pertaining to [adiacentia], or relation [habitudinis], that terms from the category of time [quando When?], such as ‘today’ or ‘tomorrow’, appellate the motion of the heavens around the things of which we say that they are today or will be tomorrow, and so on for the other cases. [5, p.880]

The point is that a term like ‘black’ or ‘two cubits long’ supposits for the particular object in question (the thing that is black, the thing that is two cubits long etc.) but it also connotes additional information about the thing for which it supposits. For example, in the proposition ‘Socrates is black’ both ‘Socrates’ and ‘black’ supposit for the same object, namely Socrates, but in this case the term ‘black’ provides additional information about Socrates. From this definition we can see how predicates get many of the properties they are usually taken to have. For example, on Buridan’s account, accidental (denominative) predicates can cease to be true about an object, even if the

6 Comparisons and Philosophical Implications

object still exists. For example, ‘Socrates is walking’ will be true as long as Socrates is walking, but when he stops walking, the proposition becomes false. Likewise, the only way that an object can cease to have an essential property is if that object ceases to exist. For example, in the proposition ‘Socrates is a human’, both ‘Socrates’ and human supposit for Socrates, and since ‘humanity’ does not connote anything extraneous about Socrates, the only way this proposition could be false, is if Socrates ceased to exist. For this interpretation to work, we need to place quite a bit of emphasis on the connotation of the particular words and the way they constrain when a term can and cannot be truly predicated of a thing.

The role of connotation is also closely connected with Buridan’s analysis of appellative terms, and some discussion of this will help us get clearer on what is going on with connotation and essential terms.

First, translation of the term ‘appellatio’ varies from author to author and situation to situation. For example, in many authors other than Buridan, ‘appellatio’ is best translated by ‘connotative’ or similar expressions. This is because Buridan’s understanding of appellative terms is an idiosyncratic one. According to Lambert of Auxerre (Lagny) there are four possible ways that ‘appellatio’ can be understood:

Because appellation is a kind of supposition, supposition was considered first. Now we must discuss appellation... Now it is essential to know that ‘appellation’ is used in four ways. In one way, proper names, or the proper name of any person, is called appellation. In this connection it is said that someone has the appellation ‘Peter’ or ‘William’. Taken in this way, appellation is nothing other than the establishment of an utterance for signifying some complex or noncomplex thing; and ‘appellation’ is often used this way in obligations, in connection with which it is said that ‘A’ appellates Socrates or appellates that a man is running. ‘Appellation’ used in the second way is a property of names in according with which names are called appellative. In this sense appellation is nothing other than the positing of a common nature containing more than one suppositum under it. (Appellation is something common when it belongs to more than one but something proper when it belongs to one.)

‘Appellation’ used in the third way is the acceptance of a term for a suppositum or {212} for supposita contained under its thing signified, whether or not those supposita are existing things. ‘Appellation’ taken in this way applies to terms having supposita under them either actually or potentially, and also to names of things signified. Used in the fourth way, ‘appellation’ is the acceptance of a term for a suppositum or for suppositia actually existing. [34, pp. 113-114]

As Lambert goes on to observe, every instance of appellation requires an instance of supposition, while the converse does not hold.[34][p.114] The first sense of appellation is the one that is most closely connected with the dictionary definition of ‘appellare’,

meaning to name. This sense is just the usual action of giving something a name. The second sense of appellation is an extension of this, where a particular name supposits for something that is common (here called a common nature) to multiple supposita. This is further subdivided into common appellation if the name supposits for multiple objects, and proper appellation if it supposits for only one name. The third sense of appellation is when a term is taken for a particular suppositum, regardless of whether or not the thing exists. In this sense, the name ‘Socrates’ (referring to the historical philosopher) appellates a philosopher, even though Socrates does not currently exist. The final sense of appellation imposes the condition that the suppositum of the appellated term must actually exist.

Buridan distinguishes appellative and non-appellative terms in the following way:

Now we turn to appellation. Some terms are appellative and others are not. For substantial terms in the nominative case or terms not connoting anything at all beyond the things for which they supposit are not appellative terms properly speaking. But every term connoting something other than what it supposits for is called ‘appellative’ and appellates that which it connotes as pertaining to [adiacens] that which it supposits for, as when ‘white’ [album] appellates whiteness as pertaining to that which the term ‘white’ [album] is apt to supposit for.[5, p.291]

As is clear from this definition, appellation is defined in terms of connotation. As in the case of essential terms, non-appellative terms do not connote anything more than they supposit for. However, what an appellative term does is “always appellates its form, whether it is placed on the side of the subject, i.e., before the verb, or on the side of the predicate, i.e., after the verb” [5, p.291]. Buridan’s take on this classical expression is the following:

So I say that, conventionally, by the ‘matter’ of a term we usually understand that for which the term is apt to supposit. . . . But by the ‘form’ of a term we usually understand whatever the term appellates, whether it is an accident or a substance and whether it is matter or form, a composite of matter and form, or an aggregate of many things. For example, the term ‘wealthy’ supposits for a man, and so the man is called its ‘matter’, and it appellates houses, lands, and money, and other things he possesses as pertaining to him as to their possessor, and so such things, insofar as [ea ratione qua] they are possessed, are called the ‘form’ of the term ‘wealthy’ [5, p.292]

So, on Buridan’s account of essences, a property is essential to an object if that object is among the supposita of the property in question, and the property does not connote anything additional to what the object is.

What are we to make of the difference between Buridan and Kilwardby on Essences? In one sense, there is not much of interest to say here that has not already been said

in connection with other debates between realists and nominalists in the Middle Ages. Buridan's approaches to essences is cast in terms of the individuals and the connotation of terms, while Kilwardby grounds the essences of things in the meaning of the terms, which are things do not simply reduce down to the objects that fall under a particular term. It is, however, helpful to see the two theories in some detail to appreciate just how different they are.

6.4 Expositio, Modal and Essential

In the previous chapter we spoke at length about Buridan's theory of the expository syllogism and how a slightly extended version of it could be used to provide a basis for the modal syllogism. At this point, two natural questions present themselves. First, 'How does Kilwardby conceive of the expository syllogism?' and second, 'In what ways does Kilwardby's conception of the expository syllogism differ from Buridan's?' It is to these questions that we now turn.

In his writing on Aristotle's logic, Kilwardby observes that there are three methods by which a syllogism can be perfected.¹¹ According to Aristotle, first figure syllogisms are perfect and require no further justification to make them evident.¹² Hence, one way to perfect a particular non first figure syllogism is to show that one can transform that syllogism into a valid first figure one. For example, Ferison, ('No B is C', 'Some B is A' therefore 'Not every A is C'), can be reduced to Ferio, by observing that 'Some B is A' is equivalent to 'Some A is B'. Normally such transformations involve either converting one of the premises, switching the major and minor premise, or moving from a universal premise to a particular one.

The second way is to prove the syllogism *per impossibile*. This form of inference involves assuming the contradictory of the conclusion and showing that a contradiction¹³ follows from this assumption together with the major and minor premises. The third way of showing an inference is perfect is by use of an expository syllogism. As we have already seen, the expository syllogism usually involves selecting a singular term and using these singular terms to show that a particular inference is valid. According to Kilwardby, the principle of expository syllogism always yields a valid conclusion and one that is evident to the senses.

Kilwardby writes:

And he [presumably, Aristotle] says that it can shown per impossibile, and

-
11. The notion of perfectibility is connected to our ability to know if a particular syllogism is valid or not. Aristotle remarks that: "I call that a perfect syllogism which needs nothing other than what has been stated to make plain what necessarily follows; a syllogism is imperfect, if it needs either one or more propositions which are indeed the necessary consequences of the terms set down, but have not been expressly stated as premises." *Prior Analytics* 24b23-26.
 12. For Barbara and Celarent see *Prior Analytics* 26a3-12, for Darii and Ferio, see *Prior Analytics* 26a16-30.
 13. Here a contradiction amounts to deriving either (AaB and AoB) or (AiB and AeB) for some terms A and B.

also by exposition. To show it by exposition is to descend to some designated individual and to posit a singular beyond the universal, and so to exhibit what was proposed to the senses. So if some designated object is taken under the middle of which each of the extremes is said it is necessary that one extreme is said particularly of the other extreme.¹⁴[61, p.149]

What is important to note here is that, in the case of the assertoric syllogism, the terms selected for the expository syllogism are always singular terms. When we move to the modal context, this will not be the case. The main reason for this should be obvious. As we have already seen, for Kilwardby the truth of necessary propositions do not require that there be anything that necessarily or actually falls under the subject. As such, the move from ‘Some B is necessarily A’ to ‘This C is A’ and ‘This same C is B’ would be invalid if C refers to some object that does not currently exist. Kilwardby is well aware of this problem and fixes it in a natural way. According to Kilwardby, the term ‘C’ in the expository syllogism can either refer to a particular singular object (presumably we can do this if we have already shown that such an object exists) or we can descend to a term that is less general than the terms in the premise, but of which it is true to say, ‘This C is necessarily A’ and ‘This same C is necessarily B’.¹⁵

Given what we have already seen formally, the idea is that we need to find the following inferences:

| Premise | | Less General Term |
|--|---------|--|
| ‘Every A is <i>per se</i> necessarily B’ | entails | $A \leq B$ |
| ‘Some A is <i>per se</i> necessarily B’ | entails | $\exists C$, such that $C \leq A$ and $C \leq B$ |
| ‘No A is <i>per se</i> necessarily B’ | entails | $\neg \exists C$, such that $C \leq A$ and $C \leq B$ |
| ‘Not every A is <i>per se</i> necessarily B’ | entails | $A \not\leq B$ |

Given what we have set up, our theory does not have any problem tracking this theory. We can always make the required inferences. In the worst case, we will have to select one of the terms, (either A or B) to stand in as our ‘less general term’.¹⁶

We can reconstruct the negative syllogism Thom considers[60, p.152]:

‘Some C is necessarily not A’, ‘Every B is necessarily A’ therefore ‘Some C is necessarily not B’ as follows: First, select some common term D , such that $D \leq C$ and $D \not\leq A$.

14. Et dicit quod per impossibile ostendi potest; similiter etiam per expositionem. Et est ostendere per expositionem descendere ad aliquid individuum signatum et ponere singulare extra suum universale, et sic ad sensum manifestare propositum. Si ergo sub medio accipiatur aliquid signatum de quo dicitur utrumque extremorum necesse est extremum de extremo dici particulariter.

15. Kilwardby writes: “Et dicendum, ut dicant aliqui, quod non fit hic expositio per singularia vere sed per minus universalia, et illa sumi, dicunt, universaliter sic ‘Necesse est omnem hominem esse animal, necesse est quoddam album non esse animal, ergo necesse est album non esse hominem.’ Et exponi debet ‘album’ non per aliquid signatum sensible, sed per aliquid particulare album cuiusmodi est nix. Et ideo sumi debet universaliter, et fiet sillogismus in secundo secunde sic ‘Necesse omnem hominem esse animal, necesse nullam nivem esse animal, ergo necesse est nullam nivem esse hominem.’ Et ita cum nix sit aliquod album, necesse est aliquod album non esse hominem. Consequenter facienda est expositio in quinto tertie, et fiet sillogismus expositivus in secundo tertie. Et ita utrobique fit sillogismus expositivus in eadem figura cum eo qui exponitur, licet non in eodem modo. Sic satis bene dici potest.”

16. This will occur in cases where A is lower bound of a particular sequence in which B occurs and B is immediately above A.

Then $D \not\leq A$, together with ‘Every B is necessarily A’ ($B \leq A$) entails that $D \not\leq B$, which together with $D \leq A$ entails that ‘Some C is necessarily not B’ as desired.

In fact, what this analysis suggests is that Kilwardby could have, if he so desired, restricted the expository syllogism in the modal case to only allow for the selection of less general terms. If singular terms happened to fall under a particular general term, this would be covered by the inclusion of meaning of the singular terms in the general terms.¹⁷ On this reading, it would seem that Kilwardby could have placed a much lower stress on the particular individuals that fall under particular predicates than other medievals (say, Buridan for example) did. What seems to be more important for Kilwardby is the relationships that hold or fail to hold between the particular terms of interest.

As we saw in Buridan, the validity of the expository syllogism was based on two rules, one dealing with the case when one singular term was affirmative and the other negative (the principle of difference) and one when both premises were affirmative (the principle of sameness). What, according to Kilwardby, accounts for the validity of the expository syllogism?

According to Kilwardby, the validity of the expository syllogism is the principle ‘whatever follows from the consequent also follows from the antecedent.’ He writes:

But it may be asked how the syllogism of this figure is perfected by means of an exposition, since it is necessary that what follows from a universal proposition, follows from a singular proposition because a singular contraction is made in the universal proposition to some of these singulars. And it is said that the necessity of this is seen through the principle: what follows from the consequent follows from the antecedent.¹⁸ [61, p.150]

For Kilwardby, the validity of the expository syllogism is grounded in a much simpler way than Buridan’s. His observation is simply that, in each case, the inferences to the singular terms follow from the particular syllogisms, and the inferences from the singular terms/less general terms also meet the request definition of validity.

Unfortunately, while what Kilwardby says is true, it is somewhat uninformative as to why these inferences are valid. Part of this likely stems from Kilwardby’s interest in the syllogism. Unlike Buridan, Kilwardby does not ground his understanding of the syllogism in the expository syllogism. This, combined with Kilwardby’s more textually based approach to the modal syllogism (in contrast with Buridan’s more systematic presentation of a modal syllogism that is weaker than Aristotle’s), suggests that the

17. For this to work, it would be required that singular terms are the ‘lowest’ elements in any sequence of \leq . While this assumption was not made in our formal treatment of Kilwardby, it can be accommodated. For example, we could select a particular subset, S of terms to be singular terms, and impose requirements on \leq and v requiring that each term in S be incomparable with all of the others in S and require that every term not in S is greater than or incomparable with every element in S .

18. Sed queretur de expositione per quam perficit quosdam sillogismos huius figure, qua necessitate sequatur ad propositiones universales quod sequitur ad singulares quando fit contractio subiecti in propositionibus universalibus ad hoc aliquod et singulare. Et dicendum quod necessitas patet per hanc maximam: Quod sequitur ad consequens sequitur ad antecedens.

expository syllogism played a much less important role in Kilwardby's analysis of the syllogism than it did for Buridan.

Likewise, Buridan and Kilwardby's approach to the expository syllogism in the context of modal propositions is very different. Because of Buridan's ampliative reading of the subject, Buridan does not run afoul of Kilwardby's condemnation in 1277 since, in Buridan's case, the subject is amplified to cover all of the objects that do or could ever fall under that proposition. However, Buridan's analysis of the expository syllogism is entirely based on singular terms which is entirely in keeping with the key roles supposition and ampliation play in Buridan's analysis of the syllogism and his nominalism more generally. In contrast, Kilwardby's use of the expository syllogism admits both singular terms and less general categorical terms. To a nominalist, the inclusion of terms of various degrees of generality will be unsatisfactory if such terms cannot ultimately be reduced down to the singular terms that either do or do not fall under those other terms. Given Kilwardby's realist commitments, this is clearly not a problem for his logic.

This reading seems closer to what Aristotle might have had in mind with his use of the expository syllogism, and also allows Kilwardby to preserve the validity of the expository syllogism for modal operations without having to raise any theologically or metaphysically loaded questions. This is not the case for Buridan, as we shall see in the next section.

There is one other tangential question that is worth asking at this point, though Kilwardby does not address the question here: On Kilwardby's theory, is the expository syllogism naturally valid, like the syllogisms or is it only accidentally valid? From what we have seen formally, we would predict that the inference would be a naturally valid inference, as meaning is preserved when we descend to a common term, and then reascend to the terms above it.

6.5 Buridan & Mere Possibilia

A few times in our discussion of Buridan's logic we have flagged some of the interesting metaphysical comments or assumptions that Buridan has been making. These observations are best summed up in the following passage, due to G. Hughes:

A short digression seems in order here. For a long time I was puzzled about what Buridan could mean by talking about possible but non-actual things of a certain kind. Did he mean by a 'possibly A', I wondered, an actual object which is not in fact A, but might have been or might become, A?... But this interpretation will not do; for Buridan wants to talk, e.g., about possible horses; and it seems quite clear that he does not believe that there are, or even could be, things which are not in fact horses but which might become horses. What I want to suggest here, very briefly, is that we might understand what he says in terms of modern 'possible world semantics'. Possible world theorists are quite accustomed to talking about possible worlds in which there are more horses than there are in the actual world. And then, if Buridan

assures us that by ‘Every horse can sleep’ he means ‘Everything that is or can be a horse can sleep’ we could understand this to mean that for everything that is a horse in any possible world, there is a (perhaps other) possible world in which it is asleep. It seems to me, in fact, that in his modal logic he is implicitly working with a kind of possible worlds semantics throughout. [20, p.9]

Hughes’ remarks here nicely summarise a natural set of questions that emerge when we look at Buridan’s modal logic, which we will place under the heading of Buridan’s modal ontology. Buridan talks quite freely about things which can be water even if they are not water, or of stars and planets that can be in a particular position in the sky, even if they currently are not.¹⁹ The first question to ask then, is what is the ontological status of these objects? When Buridan speaks of these objects, what exactly are they? As the example of water makes clear, Buridan is not only speaking of objects that currently do exist but might be in a different configuration, but also of things that do not exist.

A second and related question that flows out of this concerns the permanence of these objects.²⁰ To cast this in even more modern terms, where would Buridan stand on the debate between necessitism and contingentism? Very briefly, we can sum up the two positions with the following quote:

Call the proposition that it is necessary what there is *necessitism* and its negation *contingentism*. In slightly less compressed form, *necessitism* says that necessarily everything is necessarily something; still more long-windedly: it is necessary that everything is such that it is necessary that something is identical with it. [66, p.3]

Our methodology for answering these questions is as follows. In his book *Modal Logic as Metaphysics*[66] Williamson sets out a number of arguments for necessitism and reflects on a number of consequences that follow from adopting necessitism. What we will do is see in which places Buridan goes along with Williamson (or it seems like he would) and in which places he differs from Williamson. The idea is that we will use *Modal Logic as Metaphysics* as giving us a collection of criteria for identifying someone who holds to some flavour of *necessitism*. We use *Modal Logic as Metaphysics* as this is one of the most recent and most thorough defences of *necessitism*.

Our goal here is twofold. First, historically, by comparing Buridan’s views to this modern question we will hopefully gain a somewhat better understanding of exactly what Buridan was or was not up to with his modal theory. Second, we will see how Buridan’s logic and his theory connect to this interesting metaphysical debate and see how Buridan’s modal logic better relates to modern modal logic and modal metaphysics. In doing

19. See *Treatise on Consequence* Book Two Chapter 4 P1.

20. At this point we will limit our attention to what can be gleaned from Buridan’s logic. As the quote from Knuuttila’s treatment of Buridan’s physics suggests, there are other places where this topic comes up. We will not include this in the discussion in part because of scope and in part because I was not able to look at the Renaissance edition of Buridan’s text.

this we will also have the opportunity to focus on a few features of Buridan’s modal logic that will help us better understand Buridan’s position. In order to accomplish this, we will first sketch some of the key features of *necessitism*. After doing this, we will look at the inferences and principles that Buridan accepts and see if they commit Buridan to either necessitism or contingentism or are consistent with both. We will argue that Buridan’s position on a number of features of his modal language are not compatible with necessitism but that his remarks on modality show that he thinks about it in ways that are shared by the necessitist. In particular, Buridan seems to be committed to denying the Barcan and Converse-Barcan formulae. As we shall see, this renders his modal logic inconsistent with necessitism and consistent with contingentism. Next, drawing on some of the larger metaphysical themes within Buridan we will argue that his metaphysical commitment to nominalism, together with his views about propositions, seem to commit Buridan to a kind of contingentism about possible objects.

6.5.1 Necessitism and Contingentism: The Case of Modal Logic as Metaphysics

In his recent book, [66] Williamson offers a spirited, vigorous and insightful defence of necessitism. As we have already remarked, this is the view that “it is necessary that everything is such that it is necessary that something is identical with it”[66][p.3]. Williamson offers a number of arguments for this position within his book and he identifies a number of key principles that either follow from necessitism, imply it, or are required for us to formulate the relevant distinctions between the two positions. We will employ a number of these factors as ways of testing to see where Buridan might have fallen on such a debate.

The first distinction we will need is the distinction between the predicative reading of a modal attribution and the attributive reading of a modal attribution. According to Williamson:

Someone might object that it is absurd to postulate a non-concrete possible stick, because being concrete is necessary for being a stick. But that is to mistake the intended sense of ‘possible stick’. The objector reads ‘x is a possible stick’ as equivalent to something like ‘x is a stick and x could have existed’. Call that the *predicative* reading. On this reading, it is trivially necessary that that all sticks are concrete.

On the relevant alternative reading ‘x is a possible stick’ is simply equivalent to ‘x could have been a stick’. Call that the *attributive reading* of ‘possible stick’. . . it is not necessary that all possible sticks are sticks on the attributive reading. [66][p.10]²¹

21. Throughout this book, Williamson holds that ‘concrete’ and ‘abstract’ are not best thought of as contradictory pairs, i.e. that something is non-concrete if and only if it is abstract, but are better thought of as contraries. See [66][p.7]

As Williamson later points out the necessitist will (unless context or other features of the way the proposition is expressed) want to conceive of modal attributions following the attributive reading. Not only does this help the necessitist avoid being confused with other, less plausible theories (e.g. Meinongianism)[66][pp. 18-21] but it also avoids some trivialising issues with the theory.

The next feature of this debate that is worth highlighting here is that the quantifiers used in the formulation of necessitism and contingentism need to be understood as unrestricted quantifiers that range over absolutely everything.

Both necessitists and contingentists can also use quantifiers with various restrictions, and may always regard such uses as typical of everyday discourse. In particular, necessitists can simulate contingentist discourse by tacitly restricting their quantifiers to the concrete. Then they sound like contingentists, saying ‘Concrete things are only contingently something’. But they just mean that concrete things are only contingently something *concrete*. The restriction makes the words express different claims from those they express when used unrestrictedly. The disagreement is made explicit only when both sides use their quantifiers unrestrictedly. In what follows, our interest is in the unrestricted uses. [66][p.15]

In what follows in our treatment of Buridan, it will be important to establish that he views the quantifiers in his modal theory as sufficiently non-restricted to not run afoul of this issue. To see why this is an interpretive problem, say that we argue for the conclusion that Buridan is a contingentist. One natural response would go, ‘you cite evidence X,Y and Z for showing that what Buridan says requires him to reject necessitism, but it is consistent with what Buridan says that these quantifiers be read in a restricted way, and so he is not required to reject necessitism.’ Interpretively, there is a helpful warning here: so far as I am aware the medievals did not discuss issues related to unrestricted generality, and so it will not be clear what Buridan thinks on the matter. As such, we will need to present some evidence about how Buridan understands his modal propositions.

From a formal perspective, perhaps the most important feature of necessitism is its commitments to the Barcan and Converse Barcan formulae. Williamson writes:

The metaphysical disputes discussed in Chapter 1 between contingentism and necessitism turns out to be intimately connected with some technical issues in quantified modal logic, over two principles usually known as the Barcan formula and its converse. When those principles are interpreted in the relevant way, they are typically accepted by necessitists, and rejected by contingentists. Indeed, in some natural logical settings, each of them is equivalent to the central necessitist claim that necessarily everything is necessarily something. [66][pp. 31]

What is important to observe is that²² from the validity of the Barcan Formula, the Converse Barcan Formula, and necessitation, it is possible to derive the necessitist claim that “necessarily everything is necessarily something.” [66][p.38]

While there are other important features of necessitism that Williamson highlights, the collection of quotes and thumbnail sketches of the view are sufficient for what will follow. Williamson also points out a number of consequences that, he argues, the contingentist is under pressure to adopt. The main one which interests us is the following:

The challenge to contingentists is to identify a fallacy in Barcan Marcus’s proof. . . They have a natural line. Her proof involves the claim that $\neg\exists yx = y$ strictly implies $\exists x\neg\exists yx = y$, in other words:

$$(8) \quad \Box(\neg\exists yx = y \rightarrow \exists x\neg\exists yx = y)$$

. . . As we have seen, contingentists cannot accept (8) as a theorem, where (8) is the necessitation of (11) [$\neg\exists yx = y \rightarrow \exists x\neg\exists yx = y$]. . . thus a contingentist must either reject (11) as a theorem or reject the rule of necessitation. . . First, suppose that the contingentist rejects (11) as a theorem. But (11) is a theorem of standard non-modal first-order logic. It is simply an instance of ‘existential generalisation’, $A \rightarrow \exists xA$. Thus the contingentist is under pressure to adopt some form of ‘free logic’ in which that principle is not unrestrictedly valid. [66][p.39]

Williamson goes on to point out that by duality, the contingentist is also required to deny the principle $\forall vA \rightarrow A$. What is important to observe here is that contingentism is under pressure to work in a sort of free logic, one where particular inferences require that the objects in question already exist.

As a brief foreshadowing of Buridan, it is worth observing that, in the eyes of at least one metaphysician, there are analogues of these principles identified by Buridan. In his book *The Nature of Necessity*, Plantinga observes that:

Jean Buridan once remarked that

(31) Possibly everything is F

does not in general entail:

(32) Everything is possibly F.

That is, he rejected

(33) necessarily, if possibly everything is F, then everything is possibly F.

His counterexample is as follows. God need not have created anything; hence it is possible that (34) Everything is identical with God.

It does not follow from this, he says, that everything is possibly identical with God. You and I, for example are not.

[46][p.58]

22. Assuming that one is working in classical first-order logic.

Plantinga does not include references for this and it is not actually clear what passage in Buridan he has in mind. What seems likely here is that he is extrapolating from a number of Buridan's counterexamples in Book Two where Buridan starts from the assumption that God is the only one creating. We will have quite a bit more to say about this counterexample of Buridan's in what follows.

6.5.2 Quantification in Buridan

Before we turn explicitly to see how Buridan's modal logic relates to necessitism and contingentism, we should pause and think about how quantification works in Buridan's modal logic. As we already saw, there is a natural way that necessitists can express contingentist questions speaking within their logical framework, namely by restricting the quantification of their quantifiers.

We have already seen that Buridan's logic has the resources to express different sorts of restricted quantifiers. Throughout his writing in the *Treatise on Consequences* Buridan uses the phrase 'quod est X' ('that which is X') as a way of making explicit the ampliation of a particular subject term by particular modal operations. For example,

It should be realised that a divided proposition of possibility has a subject amplified by the mode following it to supposit not only for things that exist but also for what can exist even if they do not. Accordingly, it is true that air can be made from water, although this may not be true of any air which exists. So the proposition 'B can be A' is equivalent to 'That which is or can be B can be A'. [51, p.97]

Elsewhere Buridan observes that ampliation is blocked in cases where 'quod est' is used. He remarks that:

'That which is B is A', does not permit the ampliation of the subject, namely, of 'B'; for [B] is contracted and restricted to the present by the verb 'is' in the present tense, which precedes it. [51, p.83]

What is important here is that, in modal propositions, Buridan intends that the subject be amplified in the most general way possible. I.e. that when the subject is amplified, it should range over all of the things that could possibly fall under the subject, including the things that do not exist. How does this relate to quantification? In the following way. We have already seen that, for Buridan, 'Some A is B' is true if there is an object of which we can say, 'This thing is A' and 'This same thing is B'. Thus, if the supposition of the subject term ranges over everything that can fall under the subject, the quantification inherits the range given by the ampliation of the subject and the predicate. Since Buridan seems to be intending his unrestricted modal propositions as ranging over everything that is or can be, it seems a fair extrapolation of Buridan's views that he intended the proposition to range over all of the relevant objects in question.

Because of how Buridan uses his quantifiers, there is a way that Buridan could mimic both necessitist and contingentist readings of various modal propositions. On the contingentist reading (according to Williamson), ‘Every A is necessarily B’ states that ‘Everything that is concretely A is necessarily B’ while the necessitists would hold that ‘Every A is necessarily B’ states that ‘Everything that is concretely A or is non-concretely A is necessarily B’. Notice that all we have done here is made the range of the quantification explicit in both cases. It is also instructive to notice the parallel with how Buridan sets up his modal framework. These sorts of quantifiers give Buridan a way to talk about either sort of quantification, regardless of which reading he would regard as the correct reading of the proposition.

As such, it seems that it would be an unmotivated view of Buridan’s modal logic to argue that he is implicitly restricting his quantification to only range over concrete objects. For such a reading to be plausible, a gloss would need to be offered “although this may not be true of any air which exists” which either restricts the range of the supposita of air in this passage or argue that here Buridan means to only speak of concrete objects. The second disjunct seems to go directly against what is said in the passage²³ while the first disjunct goes against the spirit of Buridan’s unrestricted ampliation of the subject. As such it seem that a fair extrapolation of Buridan’s logic is to see him quantifying over absolutely everything.²⁴

6.5.3 Predicative and Attributive Readings

As we have already seen the distinction between predicate and attributive readings of the modal operations is important for understanding and formulating necessitism. What is interesting to observe here is that, broadly speaking, Buridan’s ways of reading the various terms within his modal logic are either attributive readings or do not fall under either. As we have already seen, Buridan reads divided modal propositions as ranging over the things that ‘can be A’ or ‘are necessarily B’. Formally, we treated these as ranging over classes of object in the domain. For example, $A \overset{L}{a} B$ is true if and only if $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$. Here we defined $M(w, A)$ as $M(w, A) = \{d \in D : \text{there is some } z \text{ s.t. } wRz \text{ and } d \in V'(z, A)\}$. Likewise $L(w, B)$ was defined as $\{d \in D : \text{for all } z \text{ if } wRz \text{ then } d \in V'(z, B)\}$. What is important to see here is that the formal readings offered match Williamson’s gloss on the attributive reading, assuming that by ‘x could have been a stick’ he intends the modality ‘could’ to be read as a diamond and not as a counterfactual. $M(w, A)$ picks out the class of all objects that could have been A. Likewise $L(w, A)$ picks out the class of all objects that are necessarily A. The point here is that Buridan seems to situate his discussion of modal logic within an attributive framework. Given Buridan’s ampliative reading of the subject and his

23. This assumes that ‘if something does not exist then it is not concrete’, a principle which seems to not be ruled out by anything Williamson has said, and is in keeping with the spirit of non-concrete objects.

24. Again, it should be stressed that this is an extrapolation from Buridan’s views as presented in the *Treatise on Consequences*. There may be other passages in Buridan’s works that tell against such a reading. If so, I am currently unaware of them.

views about the expository syllogism, this is not surprising. On Buridan's account, the predicative reading of 'x is a possible stick', namely 'x is a stick and x could have existed' is too narrow in its ampliative force. First, such a reading does not cover the cases where x could have been a stick and x could have existed. Second, such a reading is starting to have a bit of similarity with his rejection of Pseudo-Scotus' reading of the modal operations. Buridan's point was that we should not analyse 'some stick can exist' as 'either there is something that is a stick and it can exist or there is something that can be a stick and can exist'. The possible parallel requires us to observe that the first disjunct of Pseudo-Scotus' reading of the modal operation is the predicative reading that Williamson comments about.

6.5.4 Barcan & Converse Barcan

As we mention in our previous chapter, our primary focus in this work has been Buridan's analysis of divided modal propositions. Buridan distinguishes two kinds of modal propositions, composite and divided modals. The second half of Book Two of the *Treatise on Consequences* addresses inferences between composite modal propositions, and the relationship between composite and divided propositions. Our main interest here will be to sketch enough of Buridan's theory to allow for us to unpack the following conclusions:

from no affirmative composite of possibility does there follow a divided one of possibility with the mode affirmed, or conversely, except that from an affirmative composite with an affirmed dictum there follows a divided particular affirmative. . . . from no composite affirmative of necessity does there follow a divided one of necessity with an affirmed mode, nor conversely, except that from a divided universal negative there does follow a composite universal with a negated dictum. . . . from no proposition, [whether] assertoric, of possibility or of necessity does there follow one of contingency with both modes affirmed; similarly, from none of contingency does there follow an assertoric or one of necessity, but there does follow one of possibility. [51, pp. 55-57] ²⁵

Grammatically, composite modal propositions are ones where the modality occurs as one of the two terms in the proposition.²⁶ In English these kinds of modals are usually translated with the presence of a 'that' clause. The other term in a composite modal proposition is normally a categorical proposition as well. For example, 'that 'Every A is B' is necessary' and 'it is possible that 'No B is A"' are both composite modal propositions. In Latin such propositions are distinguished by the presence of an accusative-infinitive construction. Buridan offers the following two examples: "Hominem currere est possibile et haec: Necessarium est hominem esse animal". [4, Bk2-C2]

25. These are conclusions 17–19 of Book Two.

26. For our purposes here, as well as Buridan's, we will assume that a proposition is either composite or divided, and not both, i.e. we will not consider cases where one of the terms is a modal and the copula is modified by a modal term. For example, we will rule out cases like: that 'Every A is B' can be necessary.

Here the clauses ‘Hominem currere’ (a man runs) and ‘hominem esse animal’ (a man is an animal) are examples of the accusative-infinitive construction.

When referring to composite modal propositions, Buridan refers to the infinitive–accusative construction as the dictum. For example, in ‘it is possible that ‘No B is A’” ‘No B is A’ is the dictum of the proposition. The truth conditions for these propositions are also very different from the ones for divided modal operations. Composite modals do not amplify either the subject or the predicate to supposit for anything. Instead, Buridan discusses the signification of such propositions.

Here ‘possibility’ is taken not for what can be but for a possible proposition, which is said to be possible in so far as things can be altogether as it signifies. So in the examples above, saying ‘Every possibility is that B is A’ is the same as to say ‘Every possible proposition is that B is A’... It should also be noted that in the proposition ‘Every possibility is that B is A’, the predicate ‘that B is A’ supposits materially for the proposition ‘B is A’, and does not supposit for itself, since the phrase ‘That B is A’ is not a proposition.[51, p.49]

There are a number of points to pay attention to here. The main one that concerns us is Buridan’s point that the proposition which falls under the ‘that’ clause supposits for the proposition itself, and then the modal term is assessed based on if the proposition is in fact the way the term describes. For example, in ‘it is necessary that ‘Every man is an animal’” the sentence ‘Every man is an animal’ supposits for that proposition, and it is true just in case it is necessary, which, according to Aristotle, it is.

After this, Buridan goes on to discuss an important grammatical difficulty that occurs because of the Latin in which the propositions are expressed. The main problem here has to do with propositions like: ‘Nullum B esse A est possibile’. Here ‘nullum’ could be either nominative or accusative, and so it is ambiguous as to whether the quantifier is part of the accusative–infinitive construction, and thus part of the dictum, or if it ranges over the whole proposition.

Notice that in some ways this analysis is much closer to the way modern modal logic relates to propositions. The modal operator binds to the truth of the entire proposition, not to the various terms in the proposition ampliating the supposition of various terms. In other ways, this is rather different. The modals here are not functioning as operators (as they do in our standard modal logics), but instead they function as terms that modify various propositions. These term-based operations expand the expressive power of the syllogistic logic in some very interesting ways. For example, if we add the modality ‘false’ to our language (as Buridan does), then we can define an operation that looks very similar to what we now think of as propositional negation.²⁷

When it comes to thinking about truth conditions between these sorts of propositions, it is easiest if we resort to the usual kinds of relationships that we think of in an operator-based modal logic. We have already argued that Buridan’s account of modal logic is in the same spirit as possible worlds semantics, and as such, reading his operations this

27. The idea here is that, we equate *not* – ϕ with ϕ is false (or perhaps even more clearly ‘it is false that ϕ ’). Then observe that ‘ ϕ is false’ is clearly a composite modal of falsity.

way should not do too much damage. Strictly speaking, since Buridan does not offer us a reductive account of modality, it would be safer if we consider truth conditions in terms of the primitive notions of possibility and necessity. Those persuaded that Buridan is working with a kind of possible worlds semantics can then supply the needed truth conditions.

What we have said so far should be enough to help us understand the importance of Buridan's final conclusions in Book Two. When we come to the four conclusions listed above, it will be helpful to take them each in turn. First,

From no affirmative composite of possibility does there follow a divided one of possibility with the mode affirmed, or conversely, except that from an affirmative composite with an affirmed dictum there follows a divided particular affirmative.[51, p.110]

What this tells us that 'it is possible that 'some A is B'' entails 'Some A is possibly B', but that in the other cases there is no valid inference.²⁸ From what we have already seen, this makes sense. Reading, 'it is possible that 'some A is B'' as telling us that there is some world where 'Some A is B' is true, we know from what Buridan has already said, that this is only true if there is some object, say D, such that 'This D is A' and 'The same D is B'. But then, it is possible that 'This D is A' and it is possible that 'This D is B'. As we have already seen, it follows by expository syllogism then that 'Some A is possibly B.' At first glance, this might seem to look like Buridan endorsing the Barcan formula and we can find such a view in the literature.²⁹

However things are somewhat more complicated when it comes to translating Buridan's logic into first-order logic, and $\exists x \diamond (Ax \wedge Bx)$ is not equivalent to $A \overset{M}{i} B$, as the quantifier gets the ampliation of the terms wrong.³⁰ The problem is that the quantification used here does not range over the specific world at which the formula is evaluated, but should range over all of the objects at all of the worlds.

So, at least here, it seems Buridan is not committed to either of the Barcan formulae.

In fact, a counterexample to the Barcan formula is easily seen to follow from Buridan's consideration of the definition of possibility modals. Recall that Buridan said: "Accordingly, it is true that air can be made from water, although this may not be true of any air which exists."³¹ [4][p.58]. Let us assume that this situation does indeed obtain, there is some air that can be made from water. Let us assume further that there is currently

28. This is equally clear since, even if something can be A and can be B, it does not entail that something can be A and B at the same world. i.e. $(\diamond A \wedge \diamond B) \rightarrow \diamond(A \wedge B)$ is not valid.

29. See [35, pp.158,160 fn. 56] In addition to Lagerlund's own concerns about his proof, the comments we make here raise similar problems of Lagerlund's formalisation of 'Every B is necessarily not A \Rightarrow That every B is not A is necessary.'

30. A proper spelling out of this would require a formal reconstruction of Buridan's composite modal propositions and then a proof of the inference in a system that does not validate the Barcan and Converse Barcan formulae. Unfortunately, due to time and space constraints, this will not be attempted here.

31. Unde sic est uerum quod aer potest fieri ex aqua, licet hoc non sit uerum de aliquo aere qui est.

no water but that there will be. Then, we have $A \overset{M}{i} W$ (reading A for air and W for water) is true, and hence so is $\diamond \exists x(Ax \wedge Wx)$ but $\exists x \diamond (Ax \wedge Wx)$ is not true because no water currently exists at the world of evaluation ex hypothesi.

For the Converse Barcan formula, things are a little bit more tricky but Buridan will reject it, given the following sorts of remarks:

As to whether the proposition ‘A horse is an animal’ is necessary, I believe it is not, speaking simply of a necessary proposition, since God can annihilate all horses all at once, and then there would be no horse; so no horse would be an animal, and so ‘A horse is an animal’ would be false, and so it would not be necessary. But such [propositions] can be allowed to be necessary, taking conditional or temporal necessity, analysing them as saying that every human is of necessity an animal if he or she exists, and that every human is of necessity an animal when he or she exists.[51, p.141]

Informally, what Buridan is pointing out here is that it is entirely possible for all objects to cease existing. It is within the power of God to bring it about that no horses exist, or in fact ever existed. More to the point, such objects also lose all of the properties that they might have, upon ceasing to exist concretely. As such, it seems that Buridan allows for objects to pass out of existence.

As is well-known, we can use this to construct a counterexample to the Converse Barcan formula along the usual lines. Let us assume that some horse exists. Then, clearly given what Buridan has said above, it is clearly possible that this horse does not exist and hence, $\exists x \diamond \neg Ex$, where Ex stands for ‘x exists’. However, since Buridan maintains that horses (and objects more generally) lose their properties once they cease to exist, $\diamond \exists x \neg Ex$, will turn out to be impossible on Buridan’s view, as it would require the existence of a non-existent object.

From what we’ve shown here, it seems then, that Buridan would be some sort of contingentist. What he says about his modal logic suggests he would deny the validity of the Barcan and Converse Barcan formulae. As further support for this, we should observe that to formalise the divided fragment of Buridan’s modal logic, we did not need to impose any sort of domain restriction on the models.³² This does raise some interesting questions, as Buridan does opt for a particularly broad reading of the modal operations and his definitions of quantification could be interpreted in nonstandard ways.

6.6 The Role of Kripke Semantics in the History of Logic

Before concluding, we end with one meta-level reflection on the relationship between modern formal logic and the historical logics of Kilwardby and Buridan. In the case

32. At this point this is fairly weak evidence as the most likely inferences to require such domain assumptions will be the conclusions discussed above, which were not formalised.

of both Kilwardby and Buridan we have provided formal reconstructions of their theories that are, at least in some sense, grounded in a Kripke-style semantics for possible worlds.³³

In this section we will proceed as follows. We will first start by considering our formalisation of Buridan. Drawing on what we have already seen concerning Scotus and the development of modalities, it seems natural to think of Buridan's logic in terms of possible worlds and this is no serious anachronism. Hence, the corresponding formal representation of Buridan's logic using Kripke semantics is motivated, and is interpretively useful. As such, the variable domain modal logic together with our formal reconstruction does have value in understanding how Buridan thought about modality, and on how we might read the modal operations in Buridan. In arguing for this conclusion we will discuss one methodology that can be used to ground this interpretation.

We will then move on to consider Kilwardby's modal logic. Here the case is far less clear in both directions. Historically, we will argue that there is good reason to be sceptical about thinking of Kilwardby's modal theory in terms of possible worlds semantics. As we have already seen in our analysis of Kilwardby's views, it makes better sense to view his modal theory as a sort of essentialist (a subspecies of statistical) modality. See page 13. Conversely, while the formal reconstruction of Kilwardby's modal logic is based on Kripke semantics, the combination of it with the lattice-theoretic machinery makes its interpretation as a kind of possible worlds semantics less clear. As such, we will argue that our formalisation is best understood as a way of drawing connections between Kilwardby's ideas about modal logic and modern theories of modality.

6.7 Buridan, Modality and Kripke Semantics

The natural question that arises at this point is to ask, what do these formalisations tell us about the modal theories of the two thinkers we have studied. What can we take away from such regimentations?

Obviously, the answer to this question turns on which figure we look at. In this section we will focus mostly on Buridan. This is for a few reasons. First, on the formal side, our reconstruction of Buridan's modal logic is much closer to modern accounts of modal logic, and does not require any additional logical or conceptual primitive machinery, as in the case of Kilwardby. As such, focusing on Buridan's modal logic introduces fewer complications in discussing possible worlds semantics and it seems that the case of Buridan provides us with a situation in which we could learn more about his logic through formalisation. Also, as we have seen, historically there were ideas that are in

33. In this section we will talk exclusively about possible worlds and Kripke semantics. From a formal perspective we could very well have used any of the other well-known semantic frameworks for developing modal logics, e.g. Topological semantics, co-algebras, Boolean algebras with operations, etc. In what follows we will use Kripke semantics to refer to the formal machinery we used, while we will use 'possible worlds' and variants to cover the usual interpretation of the Kripke semantics which range over the various positions in contemporary, diachronic readings of modality. This is intended to be very generous in its scope and, for example, the account should be indeterminate between actualism and various ersatzisms or contingentism vs. necessitism.

the same spirit as modern discussions of possible worlds semantics and these may have influenced Buridan.

Second, and related to this, is that to unpack Kilwardby's analysis of modality is one that, at least according to our formalisation, requires more than just the usual Kripke semantics for modal logic.³⁴ Given how we are using the term Kripke semantics, there is already a good case to be made for the contention that Kilwardby's modal logic cannot be expressed using the usual resources of Kripke semantics.

Third, there is a bit more literature where people want to make inferences about how the modalities can be conceived of in Buridan. So far as I am aware, there is less of a discussion about this in the literature on Kilwardby's modal logic. We have already cited Hughes' paper where he argues for the conclusion that Buridan's modal logic is best understood in terms of 'possible worlds'.

Turning then to Hughes, he starts with the following remark:

It seems to me, in fact, that in his modal logic he is implicitly working with a kind of possible worlds semantics throughout. [20, p.97]

What are we to make of claims like this? The historically precise person may want to pull away or resist such a claim. First, is not this talk of 'possible worlds' all rather anachronistic? It seems safe to say that Buridan was not working with Kripke semantics, and we have already observed that Buridan does not attempt to explain or reduce modal terms to anything more primitive. Why not simply be content with the observation that there is a reconstruction of Buridan's modal logic using modern logic (in this case, using Kripke semantics) and observe that, in so far as Kripke semantics captures the modern notion of possible worlds, Buridan's modal logic is consistent with that? There are two possible views here. One that argues that any attempt to formalise a historical logical system is bad or anachronistic and should not be done. Another, slightly more positive position would be the one where we can show that this or that logical formalisation is *consistent* with a particular historical figure's logic, or that it is an *adequate representation* of this figure's position, but that the logical system tells us little more than that.

In contrast, there is, of course, another natural line that we could take here. We have a formal reconstruction of Buridan that seems to be faithful to his views³⁵ and formally adequate, both in the sense that it captures all of the validities and invalidities that Buridan claims about his modal system and also in the sense of being sound and complete. So, why should we not use this as a guiding interpretation of Buridan's modal logic and see where this takes us? At the heart of this idea is to view the formal reconstruction of Buridan's modal logic just like any other sort of interpretation of a historical work, only, in this case, it happens to be done in a formalised language

34. Given what we have already said about Fine and Aristotle, this is to be expected, and is probably a good thing. If we are correct in seeing Kilwardby as situating his theory of modality within the 'statistical' tradition, then it makes sense that a model done purely in terms of possible worlds semantics would be incorrect.

35. In a moment we will consider one interesting objection to exactly how faithful our reconstruction is.

6 Comparisons and Philosophical Implications

using mathematical and logical machinery. Charges of anachronism are, on this analysis, simply missing the point rather severely. Just as one would not object to using a different language or vocabulary or framework to articulate a particular historical perspective, perhaps one should not reject the use of a mathematical language to analyse historically interesting logical figures.

Moves like this are not unprecedented in the history of logic. For example, Malink, writing on the assertoric syllogism remarks:³⁶

As such the heterodox *dictum de omni et de nullo* is informative. It states that, for any A and B A is a_X -predicated of B if and only if A is a_X predicated of everything which B is a_X predicated. Given classical propositional and quantifier logic, this implies that the relation of a_X -predication is both reflexive and transitive. In other words, it implies that the following holds for any A,B,C:

Reflexivity: Aa_XA

Transitivity: if Aa_XB and Ba_XC then Aa_XC [38, p.66]

In a footnote, Malink goes on to observe that the reason a_X is reflexive is because $\forall Z(Aa_XZ \supset Aa_XZ)$ is a theorem of first-order logic.³⁷ He also observes that it follows as a corollary that Aa_XA is always valid. This is one of the features that makes his interpretation *heterodox*. The interesting move that is made here is the direction that Malink goes with his interpretative pressure. He does point to passages in Aristotle that also seem to suggest Aa_XA is always valid.³⁸ But he also uses the basis of his reconstruction in classical logic to motivate this interpretation of Aristotle. In fact, Malink makes good uses of the resources of classical logic as the framework for his interpretation of Aristotle, while freely acknowledging that these resources were not available to Aristotle.³⁹

Malink's approach here seems, at least at first glance, to be a reasonable one. If we can offer a faithful interpretation of a historical figure's logic that is textually well-informed, formally adequate, and captures the key notions that an author was working with, then we are allowed to use this as a basis for an interpretation of that figure's logic (or ideas more generally). Our aim here will be to briefly sketch how such a methodology functions.

36. In what follows Malink uses Aa_XB in the same way that we used AaB , to express that 'B is predicated of every A', or equivalently, 'every A is predicated of B'.

37. If this is unclear, simply treat a_X as a binary relation on terms, and have the quantifiers range over terms. This yields the more familiar: $\forall z(R(a, z) \supset R(a, z))$

38. See [38, p.69], *Prior Analytics* 2.15 and *Prior Analytics* A2.22

39. In formulating the *dictum de omni*, Malink remarks that: "The formulae on the right-hand side employ the resources of modern propositional and quantifier logic. Of course, these resources were not available to Aristotle. Nevertheless, the four equivalences give, I think, a sufficiently faithful representation of Aristotle's views on the semantics of assertoric propositions." [38, p.37]

The value of each of these criteria as necessary conditions for ensuring that our logical interpretations remain faithful to the author should be clear.⁴⁰ That our interpretation be textually well-grounded and well-interpreted needs to be the starting and foundational point of any sort of analysis. One might justify this by observing that if we fail to understand the texts that our particular interpretation is based on, then we have failed in attempting to understand our author's logic and views.

The formal adequacy of the theory is the logical flip-side to this. Just as we need to ensure that the interpretation of our author is well-grounded in the text, we need to ensure that the formal machinery we are using accurately captures the inferences and ideas that they are working with. Minimally, this means that the formal system that we are working in should, if the author has been careful, correctly handle all of the valid and invalid inferences that the author claims are valid. The reason for the 'careful' caveat is that our sources are human and as such, may make logical mistakes. Ideally this should be identified in the interpretive stage, if the mistakes are obvious. If they are not obvious, then things become much more difficult.⁴¹

However, in some sense, this is only the starting point for the formal adequacy. As we saw in our reconstruction of Buridan, there is more our logic can do than simply get the validities and invalidities correct. Our system can also systematically recapture the kinds of proofs and inferences that Buridan is making within the *Treatise on Consequences*. Though it is a rather difficult question to say when two proofs are the same, we can make do with the following weaker kind of adequacy. A formal construction of an informal argument is formally adequate in this way, when it:

1. Translates all of the features of the informal argument into the formal framework.
2. Each step that is made in the informal argument is a valid inference in the formal framework.
3. The underlying language of the framework is robust enough to express the relevant propositions that the author is using.

As we already shown in Chapter Five, our reconstruction of Buridan allows us to do this with all of the conclusions in the *Treatise on Consequences*.

In terms of the translation, the language that we use maps fairly naturally onto the parts of Buridan's logic that we are interested in modelling. We are able to follow his use of introducing schematic terms for the features he treats as variable and hold constant most of the things he holds constant. The only feature that is not clearly preserved in our representation is the presence of the copula, which we will discuss below.

Again, consider the following remarks by Calvin Normore concerning William of Ockham's modal logic:

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40. At this point, we can only offer a shallow justification of this approach. A thorough defence of the kind of historiography that is being argued for here would require considerably more space than is allowed in the present work.
 41. The case of Aristotle's modal syllogistic is an excellent example of when one needs to be very careful to distinguish between a logical mistake on the part of the author, and an analysis of modality or predication that is deeply different from our modern way of formalising and representing things. In the case of Aristotle it is entirely possible that both issues are present.

On this picture, then, Ockham does not think that all affirmative assertoric present-tensed sentences commit one to the actual existence of what their terms stand for. Some, for example those involving semantic expressions such as “signifies” or “true”, do not. Such expressions affect the supposition of terms in sentences containing them, so they may stand for things that do not exist. Unlike the usual semantics for late twentieth-century modal and tense logics, which analyze modal and tense locutions in terms of quantification over an expanded domain of past, future, or possible objects, Ockham’s semantics does not attempt to eliminate modal or tense expressions in favor of assertoric ones. . . . Contemporary quantification theory runs together counting and existential commitment. Ockham keeps them separate. For him quantity is the business of quantifiers, but existential commitment is the business of the copula. Thus when we use quantifiers together with non-assertoric copulae there is no commitment to the things under discussion existing in the sense in which present and actual things exist.⁴²[43, p.92]

While the general thrust of this passage is clear, there are a few points that should be developed. First, it should be noted that Ockham and Buridan seem to be following similar ideas about the nature of modality. Comments like Hughes’ notwithstanding, the account offered by Buridan is a non-reductive one, just like Ockham’s is. This point is important and we will come back to it in a moment.

Second, what are we to make of the remark that ‘Contemporary quantification theory runs together counting and existential commitment’? This should probably be viewed as a complaint about the language of first-order logic and the way quantifiers are interpreted within this framework. Recall that the standard semantics for first-order quantification tell us that $\mathfrak{A} \models \exists x\phi x$ if and only if there is some element, a in the domain s.t. $\mathfrak{A} \models \phi[x/a]$. What Normore contends is that Ockham avoids ‘running together’ the conjunction on the right hand side of the biconditional, i.e. the clauses that require there be an object that exists in the domain, and the satisfaction clause for a particular formula. For Ockham, these are separate parts of the logic, where the object’s existence or non-existence in a particular domain is handled by the quantifier, and the status of the object as regards its existence is handled by the copula. The problem with standard accounts of first-order logic is that this distinction is collapsed.

What should we make of this? First, matters may not be quite as straightforward as Normore makes them. While we may grant that counting and existential commitment need to be kept separate in medieval logical theories, there is still the question of exactly what the quantifiers in Ockham’s semantics are counting. Put in a more modern framework, one standard way to think of ($\{1\}$ -type) quantifier is as a function from the domain of objects to elements of the power-set of the domain. In modern semantics, the natural question is to then ask, what is the domain we are applying our quantifiers to? To put the worry more colloquially, if we say that ‘Every dinosaur was an animal’ what is the quantifier ranging over? Things that exist? Things that existed but do not

42. It is worth observing that this concern is not unique to Normore. A similar worry is raised by Henry in [19]. Similar issues arise throughout the book.

now? It seems that, while we may want to keep these issues separated, we will need to fix the range of things we are counting over before we can decide the truth or falsity of the proposition in question. So there does need to be some connection between the copula and the quantifier, at least when it comes to analyzing the truth or falsity of a particular proposition. However, given what we have said about quantifiers, it may be that the copula should actually take priority in such propositions, since on theories like Ockham's, it tells us the range of the domain. Alternatively, we could develop a formal reconstruction of propositions that does exactly this and allows for the various readings of the quantifier.⁴³

Let us take these ideas and develop them into a slightly more radical position. Here is a natural objection to the formalisation project: the formal techniques of modern quantificational logic are inappropriate to be used to analyse medieval theories of logic. The reasons for this are clear from the quotes above. In terms of expressive power, modern logical systems lack the required resources to separate out important *logical* distinctions that are needed to capture the medieval theories. For example, the distinctions Normore highlights in Ockham. Semantically, modern logical analysis of medieval theories such as Buridan's and Ockham's are fundamentally reductionistic in nature, which runs against the spirit of the theories.

What are we to make of such an objection? First, it is very important that we be clear on exactly what kind of systems we are interested in when it comes to logic. The following example may prove useful. It is a well-known result that there are a number of natural frame conditions that are inexpressible within the framework of propositional modal logic. For more information see the discussion in Chapter Seven of [13]. For example, one cannot give modal formulae that force the accessibility relation to be irreflexive, anti-symmetric or asymmetric. So, what are we to do if we want to study modal systems where we can express such axioms? The well-known answer is that we need to expand the expressive resources of our language to be able to capture such things. In this case, such examples are used to motivate various kinds of hybrid logics, where we add literals that allow us to evaluate particular propositions at particular worlds. In the case of something like Normore's concern, this becomes strong motivation for the separation and addition of the necessary formal distinctions required to capture these distinctions. The logics presented in this dissertation do not limit themselves in such a way, and the resulting systems do have the expressive power necessary to capture the relevant class of distinctions. What Normore's argument suggests is that we are limiting ourselves to the well-known systems and languages of logic, which is exactly what makes the study of such logics of potential interest to contemporary logicians as well as historians.

Reading this as an objection to the logical vocabulary is no objection to our project. It should be noted that what Normore observes about Ockham's logic also seems to be the case for Buridan's logic. Buridan spends considerable time discussing the function of the copula in various modal sentences. Syntactically, we have done something similar to what Normore attributes to Ockham's logic. When we represent a proposition as

43. For some modern attempts to do something in this spirit, see e.g. Lesniewski's works on Logic and Ontology. [57]

$A \overset{M}{a} B$, for example, we are tracking two things; the modal on top notes the quality of the copula, while the term underneath tracks the kind of quantifier that is being used in this sentence.⁴⁴ In fact, given some of the difficulties that result with translating our semantics into modern modal predicate logic, this may in some sense offer further support for Normore's reservation about using standard quantificational logic to represent medieval logic. From a modern point of view this also offers another reason for the study of medieval logical theories, they give rise to comparatively weak-looking logics that are not always expressible with the usual resources of first-order logic.⁴⁵

6.8 Kilwardby and Modern Modal Logic

When we move to Kilwardby's modal logic, our formal reconstruction raises a number of more interesting questions.

First, let us recall that according to the statistical definition of modality given above, a categorical proposition is necessary if the subject is *per se* the predicate and contingent if it is neither *per se* nor is it incompatible with the subject.

What is important to observe is that the definition of necessity given here does not need to make any appeal to a synchronic notion of possibility in order to ground the definitions of possibility, necessity or contingency. This view seems to offer one of the best ways of thinking about Aristotle's modal logic, as the following quote makes clear:

In semantics, I follow the path-breaking work of Johnson and Thomason. Johnson showed us how to understand Aristotle's modal sentences in terms of structured sets which makes no appeal to possible worlds. . . . One has to regard this kind of semantical analysis of the modal syllogistic as particularly appropriate if one thing that Aristotelian metaphysics are in any way implicit in that syllogistic. For, Aristotle's metaphysics envisages a single world. [60, p.5]

We have already argued that Kilwardby's theory is best situated in the statistical tradition. Given Kilwardby's desire to follow Aristotle's logic, and seeing the keys to understanding Aristotle's logic in his ontology, it is unsurprising sense that we would encounter these problems in our treatment of Kilwardby as well.

There are two questions that make this difficulty sharp. First, does our interpretation of the modal operators, \square and \diamond require a synchronic reading of the operators to make sense in the context of possibility. If this is so, then what parts of our modal logic crucially turn on this difference and how does it affect our formal treatment of Kilwardby?

44. Recall that in divided modal propositions, the modal term occurs adverbially (or is equivalent to one that is, in the case of words like can) so that $A \overset{M}{a} B$ should be translated into English as saying that 'Every B is possibly B'.

45. It is worth noting that this goes against a slightly generalised interpretation of the syllogism as the monadic fragment of first-order logic. While it is possible to translate Aristotle's assertoric syllogistic into first-order logic, this may not generalise to other modal syllogistic theories, e.g. Buridan's.

On all of the standard readings of the modal operators in modal logic, there does seem to be something fundamentally synchronic about the reading.⁴⁶ For example, the epistemic reading of \Box_a as ‘knows’ in effect says that (an agent a) knows that ϕ just in case ϕ is true in all of the worlds that the agent believes are compatible with the real world. What is clear from each of these cases is that the interpretation offered on the formal models requires a synchronic reading of the modal operations.

So, what does this say about our reconstruction of Kilwardby’s modal logic? First, it is important to note that the main use for the modal semantics is to separate necessary propositions from contingent and possible ones. While the distinction between *per se* and *per accidens* does make use of this modal notion, it does not actually need to.⁴⁷ Likewise, the main interpretive work required to separate these two senses of necessity is given by $\langle T, \leq \rangle$, not by the modal semantics. As such, the additional semantic machinery is necessary for understanding Kilwardby’s logic and does seem to be, in some sense, motivated by his theory.

Unlike in our treatment of Buridan’s modal logic, what we have is a formal reconstruction of Kilwardby’s logic that is an attempt to relate his logic to contemporary possible worlds semantics. It preserves some of the features of Kilwardby’s logic, but does work in a framework that is rather alien to his motivations and thought. It is worth pointing out, however, that the use of possible worlds semantics is anachronistic (and as such our formal model should not be used as an interpretive tool in understanding Kilwardby).

What sort of framework would fare better? One natural way forward would be to approach the syllogism from an entirely tree-based semantics. The idea here, following Malink (and ultimately Thomason and Johnson) would be to view the syllogism as a relationship between ordered terms, where the order is given by underlying assumptions about Aristotelian ontology.

6.9 Conclusion

The aim of this chapter was to bring together a number of philosophical implications for our analysis of Kilwardby and Buridan. The topics covered here are somewhat diverse but serve to bring out some of interesting differences and similarities between Buridan and Kilwardby’s approaches to modality.

As should be clear from what we have written, the accounts of modality offered and defended by Kilwardby and Buridan are very different in nature. Kilwardby’s understanding of the modal syllogism and of modality more generally is fundamentally connected to a statistical understanding of modality. The role of the expository syllogism makes this dependence particularity clear. In order to not run afoul of theological issues, Kilwardby’s modal logic needs to select less general terms for expository proofs in the modal syllogism.

Likewise, Buridan’s modal logic raises a number of interesting philosophical questions

46. Here we will only focus on the philosophically motivated readings of \Box and \Diamond . For example, we will not consider topological or dynamic-programming-based interpretations of these operations.

47. This is because of the constraint that if $A \leq B$ then for all w , $v(A, w) \subseteq v(B, w)$.

6 Comparisons and Philosophical Implications

that we have explored in detail here. Particular attention has been directed to exploring the sort of modal ontology necessary to make sense of Buridan's modal logic. What we have argued for is that Buridan's ontology is contingentist in nature and an interesting form of the position. Likewise we have also seen that it is consistent with Buridan's historical situation that his modal framework is a synchronic notion of possibility. As such, it is possible to view Buridan as working with a proto-possible worlds semantics and to view our formal reconstruction of his logic as showing just how unique and original his contribution to the history of modal logic was.

7 Conclusions & Further Work

7.1 Conclusions

Our main aim in this work was to explore, compare and contrast two different theories of modality. On the one hand, we had the modal theory of Robert Kilwardby. As we have argued, his theory is best situated within the statistical interpretation of modality. Kilwardby's discussion of the modal syllogism represents one of the earliest systematic Latin Medieval engagements with Aristotle's *Prior Analytics*. Kilwardby's theory is systematic in nature and attempts to find a reasonable balance between textual faithfulness to Aristotle and offering a well-motivated interpretation of the *Prior Analytics*.

On the other hand, we had Buridan's account of modality, that, we have argued, is best understood as being in the same vein as the modal theory of Duns Scotus. Buridan's modal logic is a systematic theory developed around his theories of supposition and ampliation. The resulting modal syllogism is one that shows a high degree of innovation and independence. In this work, Buridan shows little regard for Aristotle's modal syllogism and prefers to develop his own theory of the modal syllogism. As we have seen, the resulting system integrates a number of important parts of medieval modal theory in a coherent way. The expository syllogism is used by Buridan to ground the assertoric syllogism, and we see in this theory how it also connects with the modal syllogism.

Formally, we have been able to develop logical systems that capture the ideas in each of Buridan and Kilwardby. In the case of Kilwardby, we develop a formal reconstruction that pairs up the idea of terms being weakly ordered based on meaning with the usual account of necessity to develop a formal model that captures Kilwardby's views about the apodictic fragment of the modal syllogism. The resulting logic draws on a number of different logical features. The key idea is to include relationships that encode the definitional aspects of containment and incompatibility between various terms in a language. We also showed how it is possible to extend this framework to allow for the addition of the usual Boolean operations of \wedge , \neg , \vee , \rightarrow , etc. We then showed how it was possible to define a connexive notion of implication and demonstrated that it seems to capture Kilwardby's notion of natural consequence.

In the case of Buridan's modal logic, we have seen a number of interesting results. First, we have provided a formalisation of Buridan's modal logic using a single domain together with an implicit predicate that tracks the various objects that exist at various worlds. Using a fairly obvious definition of the modal operations which closely follows Buridan's own remarks about modal ampliation, we showed (in the appendix) that it was possible to reconstruct Buridan's modal logic and prove all of the conclusions that he claims are valid in Books Two and Four of the *Treatise On Consequences*. In addition

we provided a syntactic reconstruction of Buridan's modal logic based on his remarks concerning the expository syllogism. While at first glance these systems look different from each other, we were able to show that these two systems are sound and complete relative to each other.

7.2 Some Historical Questions

There do seem to be some natural historical questions that this work has left open. One question concerns the relationship between the notion of meaning containment we attempted to formalise in Kilwardby and its connection to logics that prove various 'connexive theses' such as Aristotle's thesis. As has been well-documented in the literature, e.g. [42][41],[48] there have been a number of different interpretations offered for what is supposed to motivate connexive logic. Some, such as the three referenced above, attempt to change the logical operations in order to obtain a connexive system. What our reading of Kilwardby suggests is a different approach where connexive theorems emerge from the relationships that are formally imposed on terms in the language.

As we have seen, it is the notion of meaning containment that is central to Kilwardby's reason for viewing natural implication as validating the various connexive principles we discussed. This view merits development in at least two directions. The first direction is to go back to two other groups of medieval authors and see if this is a plausible interpretation of how they understood connexive implication. The first group of medieval authors who will need to be looked at are those who subscribe to a connexive logic and we will need to see if this notion of meaning containment plays an important role in their theorising or if Kilwardby's views are idiosyncratic. The most natural starting place for such a project are the writings of Peter Abelard and Peter of Spain. Both authors hold to a connexive theory of implication and write extensively on implication. We have already seen some evidence that for Abelard this is so. Peter of Spain's treatment of implication in his *Synkategoremata* is of particular interest, since, like Kilwardby, he explicitly rejects *ex falso*. Perhaps even more interesting is that he also responds to arguments that are purported to establish the validity of *ex falso* by means of an argument very similar to C.I. Lewis'. The second group are those medieval authors who talk about the validity of logical consequence in terms of meaning containment or who distinguish a number of different senses of logical consequence, one of which is based on meaning containment. The question here is, does this account of logical consequence have connexive elements to it? If it does not, are there features about how meaning containment is understood that render it clearly classical (or some other sort of logic)?

Another set of questions are naturally raised by our exploration of Buridan's modal logic. Two questions and one extension present themselves for further analysis. First, we have set up a framework for exploring how medieval writers could discuss the question of contingentism and necessitism and more general ontological considerations raised by modal logic. It would be interesting to see if there are medieval thinkers whose writings on modal logic either commit them to or seem to suggest a necessitist position. This would be interesting on a few levels. On one level, it would present a prefiguring of the

views defended by Williamson et al. On another level, in so far as there is an objection to necessitism that proceeds by arguing that the view is unintuitive and unnatural, this would, to some degree, tell against it. On a third level, and most importantly, the arguments put forward by medieval authors in defense of this position would prove to be interesting and raise the prospects of exploring the relationship between medieval and modern views on this topic.

A natural second question concerns the relationship between Buridan's modal logic and the modal logic of other nominalists, both those who came before (e.g. Ockham) and those who came after him. We have already briefly touched on the relationship between Buridan and Ockham as it concerns the ampliation of the subject in propositions of possibility and necessity. During the time this thesis was being revised, we have published a paper[23] which begins to address some of these issues. The paper was not included as it was not part of the original text of this thesis.

The obvious natural extension of our analysis of Buridan's logic is to expand the formal treatment of Buridan's modal logic to account for his analysis of composite modal propositions. Ideally, in such a framework we would be able to represent all of the modal inferences that Buridan takes to be valid and be able to verify (or refute) a number of conjectures about Buridan's logic that have been made in the literature. For example, in [25], it is claimed that Buridan's logic requires him to work in S5, while [35] claims that Buridan is committed to the validity of the Barcan and Converse Barcan formulae. While we have argued against the second claim in this thesis informally, it would be helpful to explore Buridan's modal logic in its full generality to be able to attest to the soundness of such claims.

7.3 Some Formal and Technical Questions

Formally, there are a number of interesting issues that should be addressed in future work. The most important one concerns the representation and formalisation of Kilwardby's modal logic. As we have already seen, a possible worlds based reconstruction of Kilwardby's modal logic is somewhat anachronistic and seems to raise a number of difficult problems for how to interpret Kilwardby's logic. As is fairly well-known, there is an alternative framework for formalising syllogistic validity. This is done in terms of preorders and can be found in, e.g. [38]. The basic idea behind this is the following: Suppose that $\mathfrak{T} = \langle T, \leq \rangle$ is a preorder. We build well-formed formulae out of the elements of T , and say that:

- $\mathfrak{T} \models AaB$ if and only if $A \leq B$
- $\mathfrak{T} \models AeB$ if and only if $\neg \exists C \in T (C \leq A \text{ and } C \leq B)$
- $\mathfrak{T} \models AiB$ if and only if $\exists C \in T$ such that $C \leq A$ and $C \leq B$
- $\mathfrak{T} \models AoB$ if and only if $A \not\leq B$

Working in this framework has the benefit of defining the syllogistic propositions by means of the relationship between terms, and makes no reference to the objects that may or may not fall under the terms. However, this also becomes a problem, since for

7 Conclusions & Further Work

medieval authors (such as Kilwardby and Buridan), AaA is not always true. Likewise, we would need to find an interpretation of the modal operations that makes sense of Kilwardby's modal inferences. The natural way of doing this, which we have already seen in [38], is to think of modal operators as designated upsets of a particular family of terms. To deal with the reflexivity issue, we would need to work with a weaker structure than a preorder.

If such a structure could be identified that is suitable for Kilwardby's logic, it would then be interesting to explore when such structures can be expressed using syllogistic propositions. In particular, the following two questions seem like natural generalisations of this approach. First, can any preorder be expressed as a (possibly infinite) set of categorical propositions? If so, can we use this to provide an alternative proof that the usual Boolean operations cannot be defined using categorical formulae? We have already seen that we can give semantics for syllogistic systems in terms of preorders. Does this afford us any new or interesting insights into the nature of syllogistic propositions? Likewise, is there a mathematical structure that can be used to account for the syllogistic systems employed by the medievals that 'abstracts' away the particular objects that fall under a given term?

A second technical question concerns the modal strength that is needed to capture all of Buridan's modal logic. As we stated in Chapter Four, because of the language that we are working in, it is not possible to express inferences stronger than T. In particular, we cannot iterate modal operations. We chose to work in a stronger system so as to simplify a number of the proofs and to leave open the question as to what system Buridan may be working in. In order to provide a proper answer to the question of the strength of Buridan's modal logic, we would need to provide a complete treatment of Buridan's modal logic. This would require a treatment of composite modal propositions, as well as a systematic discussion and exegesis of Buridan's treatment of the interrelationship between divided and modal propositions in Chapter Two of the *Treatise*. These are the places where it is most likely for Buridan to make use of iterative modals, and as such, is a future project for formal analysis.

8 Appendix One: Latin References

Latin references to many of the passages in Buridan we quote can be found below.

8.1 Chapter Four

1. Et ad hoc declarandum, dico quod propositio diuiditur in propositionem categoricam et hypotheticam. Consequentia autem est propositio hypothetica; constituta enim est ex pluribus propositionibus coniunctis per hanc dictionem “si” uel per hanc dictionem “ergo” aut aequiualentem.[4, p.21]
2. Nunc de syllogismis determinandum est ex obliquis terminis. Propter quod primitus supponendum erit quod obliquus terminus quando cum recto construitur a quo regitur est sicut determinatio illius recti, quasi sicut adiectiuum est determinatio substantiui. Sicut enim dicendo ‘Equus albus currit’ haec dictio ‘albus’ determinat hanc dictionem ‘equus’ ad supponendum solum pro illis albis, ita si dico ‘Equus Socratis currit’ haec dictio ‘Socratis’ contrahit hanc dictionem ‘equus’ ad supponendum pro illis qui sunt Socratis solum.[4, p.98]
3. Illa propositio est antecedens ad aliam propositionem quam impossibile est esse ueram illa alia non existente uera illis simul formatis.[4, p.21]
4. Sed haec descriptio deficit uel est incomplete, quia hic est bona consequentia: Omnis homo currit; ergo aliquis homo currit et tamen possibile est primam esse ueram secunda non existente uera, immo secunda non existente.[4, p.21]
5. Deinde, in quinto capitulo, etiam suppono quod propositio affirmatiua designat quod idem sit pro quo termini supponunt, aut fuit aut erit aut potest esse idem, secundum exigentiam propositionum. Si enim dico “A est B”, designo quod idem sit A et B, et si dico “A fuit B”, designo quod A fuit idem quod B, et sic de aliis. Propositio autem negatiua oppositum designat, scilicet quod idem non sit.[4, p.20]
6. Et exponatur totum secundum exigentiam propositionum. Oportet enim, de proprietate sermonis, aliter dicere de uniuersali et aliter de particulari, ut quod nihil est idem pro quo subiectum supponit alicui pro quo praedicatum supponit uel, particulariter, quod aliquid pro quo subiectum supponit non est idem alicui pro quo praedicatum supponit.[4, p.25]
7. Quia si equus Colini est mortuus qui bene ambulauit, haec est uera: ‘Equus Colini bene ambulauit’ et non est ita in re sicut ista propositio significat [4, p.17]

8 Appendix One: Latin References

8. Et intelligo per ‘causas ueritatis’ alicuius propositionis {propositiones} quarum quaelibet sufficeret ad hoc quod propositio esset uera.[4, p.19]
9. Cum enim impossibile sit eandem propositionem esse simul ueram et falsam et quam libet si formetur necesse sit esse ueram uel falsam, necesse est modo contradictorio causam ueritatis et causam falsitatis eiusdem propositionis assignare.[4, p.18]
10. Consequentia “formalis” uocatur quae in omnibus terminis ualet retenta forma consimili. Vel si uis expresse loqui de ui sermonis, consequentia formalis est cui omnis propositio similis in forma quae formaretur esset bona consequenti... [4, pp.22-23]
11. Sed consequentia materialis est cui non omnis propositio consimilis in forma {quae formaretur} esset bona consequentia [4, p.23]
12. Et dico quod in proposito, prout de materia et forma hic loquimur, per ‘materiam’ propositionis aut consequentiae intelligimus terminos pure categorematicos, scilicet subiecta et praedicata, circumscriptis syncategorematicis sibi appositis... sed ad formam pertinere dicimus totum residuum.[4][p.30]
13. Volumus ergo per ‘syllogismum’ in sequentibus intelligere solum consequentiam formalem ad unam conclusionem categoricam per medium ab utraque extremitate dictae conclusionis diuersum.[4, p.82]
14. Quod omnis talis syllogismus exigit in praemissis coniunctionem utriusque extremitatis conclusionis cum medio, propter quam coniunctionem infertur coniunctio extremitatum inter se, uel affirmatiue uel negatiue. Sic igitur manifestum est quod omnis syllogismus, prout hic de syllogismo intendimus, est constitutus ex tribus terminis solum, scilicet ex duabus extremitatibus, quae sunt termini conclusionis, et ex termino medio, cum quo illae extremitates coniunguntur in praemissis... Et ultra sequitur ex his quod huiusmodi syllogismorum sunt solum quattuor figurae. Vocatur enim “figura syllogistica” ordinatio medi ad extremitates in praemissis secundum subiectionem et praedicationem. Hoc autem non potest fieri nisi secundum quattuor combinationes. [4, p.82]
15. Deinde, in quarto capitulo, supponendum est quod syllogismi affirmatiui tenent in uirtute istius principii: ‘Quaecumque uni et eidem sunt eadem inter se sunt eadem’. Unde ex eo quod extremitates designantur in praemissis dici eadem uni medio concluduntur in conclusione dici eadem inter se. Negatiui autem syllogismi tenent per illud aliud principium: ‘Quorumcumque duorum unum est idem alicui cui reliquum non est idem illa non sunt inter se eadem’. Et ob hoc contingit quod affirmatiua conclusio indiget concludi ex ambabus affirmatiuis et negatiua ex una affirmatiua et alia negatiua, quondam propositio affirmatiua designat identitatem et negatiua non identitatem.[4, p.84]

16. Sed notandum est quod propositiones non dicuntur “de necessario” aut “de possibili” ex eo quod sunt possibles aut necessariae, immo ex eo quod in eis ponuntur isti modi “possibile” aut “necessarium”. [4, p.56]
17. Compositae “uocantur in quibus modus subicitur et dictum praedicatur uel econuerso . . . Sed “diuisae” uocantur in quibus pars dicti subicitur et alia pars praedicatur. Modus autem se tenet ex parte copulae, tamquam eius quaedam determinatio. [4, p.57]
18. Supponendum est quod propositio diuisa de possibili habet subiectum ampliatur per modum sequentem ipsum ad supponendum non solum pro his quae sunt sed etiam pro his quae possum esse quamuis non sint. [4, p.58]
19. Deinde, quia possibilitas est ad futura et omnino ad possibilia, ideo similiter hoc uerbum “potest” ampliatur suppositionem subiecti ad omnia quae possunt esse. [4, p.27]
20. Si dico: ‘Omne B potest esse A’ ibi est unicum subiectum et unicum praedicatum et una propositio simpliciter categorica, et subiectum est simul unica distributione distributum. Ideo melius esse uidetur quod exponatur per propositionem unam etiam categoricam, de uno subiecto et uno praedicato, licet propter ampliatur subiecti fiat in subiecto exponentis disiunctio huius uerbi “est” ad hoc uerbum “potest”. [4, p.59]
21. Et capitur hic “possibile” non quia possit esse sed pro propositione possibili, quae ex eo dicitur “possibilis” quia qualitercumque significat ita potest esse. [4, p.69]
22. Ad omnem propositionem de possibili sequi per aequipollentiam aliam de necessario et ad omnem de necessario aliam de possibili, sic se habentes quod si fuerit apposite negatio uel ad modum uel ad dictum uel ad utrumque in una non apponatur ad illud in alia et si non fuerit apposite in una apponatur in alia, aliis manentibus eisdem. [4, p.61]
23. In omni propositione de necessario diuisa subiectum ampliatur ad supponendum pro his quae possunt esse. [4, p.63]
24. Ad nullam propositionem de necessario sequi aliquam de inesse uel econuerso, praeter quod ad uniuersalem negatiuam de necessario sequitur uniuersalis negatiua de inesse. [4, p.64]
25. Ad nullam propositionem de possibili sequi aliquam de inesse uel econtra, praeter quod ad omnem propositionem affirmatiuam de inesse sequitur particularis affirmatiua de possibili. [4, p.65]
26. Ad omnem affirmatiuam de possibili sequi per conuersionem in terminis particularem affirmatiuam de possibili, sed non uniuersalem, et ad nullam negatiuam de possibili sequi per conuersionem in terminis aliam de possibili. [4, p.66]

8 Appendix One: Latin References

27. Ad nullam propositionem de necessario sequi per conuersionem in terminis aliam de necessario, praeter quod ad uniuersalem negatiuam sequitur uniuersalis negatiua.[4, p.67]
28. Omnem propositionem de contingenti ad utrumlibet habentem modum affirmatum conuerti in oppositam qualitatem de modo affirmato, sed nullam sic conuerti si conuertens uel conuersa fuerit de modo negato.[4, p.68]
29. Nullam propositionem de contingenti posse conuerti in terminis in aliam de contingenti, sed omnem habentem modum affirmatum posse conuerti in aliam de possibili. [4][p.68]
30. Tamen licet haec sit possibilis “Omne currens est equus”, non sequitur quod uniuersaliter omne currens possit esse equus.[4, p.76]
31. In omnibus modalibus compositis in quibus dictum subicitur ad particularem sequi uniuersalem caeteris non mutatis.[4, p.70]
32. Multa sunt possibilia quae nunquam sunt, erunt uel fuerunt. [4, pp.27-28]
33. Nulla propositio est negatiua; ergo nullus asinus currit. [4, p.21].

8.2 Chapter Five

1. Deinde, in secundo capitulo, supponam quod omnis talis syllogismus exigit in praemissis coniunctionem utriusque extremitatis conclusionis cum medio, propter quam coniunctionem infertur coniunctio extremitatum inter se, uel affirmatiue uel negatiue. [4, p.82]
2. Sed notandum est quod haec quarta figura non differt a prima nisi secundum transpositionem praemissarum, quae quidem transpositio nihil operatur ad aliam conclusionem inferendam uel ad illationem impediendam, sed solum operatur quod conclusio illata si esset directa in prima figura esset indirecta in quarta et econuerso.[4, p.82]
3. Ex maiore de necessario et minore de inesse in prima figura ualet semper syllogismus ad conclusionem de necessario particularem, sed non ad uniuersalem.[4, p.124]
4. Ex maiore negatiua de necessario et minore de inesse ualet semper <syllogismus> in secunda figura ad conclusionem de necessario particularem, non ad uniuersalem; sed si maior sit affirmatiua de necessario uel de inesse non ualet syllogismus ad conclusionem de necessario, ualet tamen ad conclusionem de inesse..[4, p.125]
5. Quaecumque uni et eidem sunt eadem inter se sunt eadem. [4, p.84]
6. Quaecumque inuicem sunt eadem, a quocumque unum eorum est diuersum ab eodem reliquum est diuersum. [4, p.84]

7. Prima pars patet per syllogismum expositorium. Quia si B potest esse A, signetur illud B et sit hoc C. Tunc sic: hoc C est uel potest esse B et ipsum idem potest esse A; ergo quod potest esse A est uel potest esse B.[4, p.66]
8. Upponendum est quod syllogismi affirmatiui tenent in uirtute istius principii: Quaecumque uni et eidem sunt eadem inter se sunt eadem. Unde ex eo quod extremitates designantur in praemissis dici eadem uni medio concluduntur in conclusione dici eadem inter se. Negatiui autem syllogismi tenent per illud aliud principium: Quorumcumque duorum unum est idem alicui cui reliquum non est idem illa non sunt inter se eadem. [4, p.84]
9. Postea, de quarta reductione, manifestum est quod syllogismi expositorii sunt per se euidentes, maxime in tertia figura; et faciliter omnes sex modi tertiae figurae probantur per reductionem ad syllogismos expositorios. [6, 5.2.4] Second last paragraph.
10. Et dicam primo quod omnis propositio de aliquo termino communi non distributo habet uel habere potest, aut sibi consimilis in forma, plures causas ueritatis quam propositio de eodem termino communi distributo, caeteris similiter manentibus.[4, p.18]
11. In tertia figura ualet semper syllogismus ad conclusionem de necessario ex maiore uniuersali de necessario et minore de inesse, sed ex maiore de inesse non ualet ad conclusionem directam de necessario, nec etiam ex maiore de necessario si sit particularis. Prima pars conclusionis in omnibus modis esset manifesta per syllogismos expositorios, scilicet in Darapti, in Felapton, Datisi et Ferison.[4, p.126]

9 Appendix Two: Kilwardby's Logic

The aim of this Appendix is to show that the properties P1-P7 hold in our semantics as well as some of the other claims that we made in Chapter Two. What we will do here is show that our semantics correctly track the inferences in the assertoric syllogism. This will suffice to establish these properties.

Recall that Kilwardby Model for Immediate Signification is given by the following: Let $\mathfrak{K}_3 = \{D, W, T, R, \leq, \triangleleft, |, c, v\}$ where

D, W , and T are non-empty sets. (Informally, D is our Domain, W is a set of worlds, and T is a set of interpreted terms or predicates)

$$R \subseteq W^2.$$

\sqsubset, \triangleleft , and $|$ are subsets of T^2 and satisfy the conditions given below.

$$c : Terms \rightarrow T.$$

$$v : W \times T \rightarrow \mathcal{P}D.$$

We require that \leq and \triangleleft be preorders on T and that $|$ is irreflexive and symmetric. We further require that our orders have the following properties:

1. $\forall x, y$ if $x \triangleleft y$ then $x \leq y$.
2. $\forall x, y, z$ if $x \leq y$ and $y \triangleleft z$ then $x \triangleleft z$.
3. $\forall x, y$ if $x \leq y$ then not $x|y$
4. $\forall x, y, z$ if $x \leq y$ and $y|z$ then $x|z$.

These will be referred to as order properties 1–4.

As before, we impose the following conditions:

1. For all terms, A,B $c(A) \leq c(B)$ iff for some $w \in W$ $v(w, A) \subseteq v(w, B)$.
2. For all terms, A,B if $c(A) \triangleleft c(B)$ then for all $w \in W$ $v(w, A) \subseteq v(w, B)$.
3. For all terms, A,B if $c(A)|c(B)$ then for all $w \in W$ $v(w, A) \cap v(w, B) = \emptyset$

We will refer to these as valuation conditions 1–3.

9 Appendix Two: Kilwardby's Logic

| | | |
|---|----------------|---|
| $\mathfrak{K}_{\mathcal{J}}, w \models AaB$ | if and only if | $c(A) \leq c(B)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models AeB$ | if and only if | $\neg \exists D \in \mathcal{T} \text{ s.t. } D \leq c(A) \text{ and } D \leq c(B)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models AiB$ | if and only if | $\exists D \in \mathcal{T} \text{ s.t. } D \leq c(A) \text{ and } D \leq c(B)$ |
| $\mathcal{T}, w \models AoB$ | if and only if | $c(A) \not\leq c(B)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.s.}{a} B$ | if and only if | $c(A) \triangleleft c(B)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.s.}{e} B$ | if and only if | $c(A) c(B)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.s.}{i} B$ | if and only if | $\exists D \in \mathcal{T} \text{ s.t. } D \leq c(A) \text{ and } D \triangleleft c(B) \text{ or } D \leq c(B) \text{ and } D \triangleleft c(A)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.s.}{o} B$ | if and only if | $\exists D \in \mathcal{T} \text{ s.t. } D \leq c(A) \text{ and } D c(B)$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.a.}{a} B$ | if and only if | $\forall x \in W \text{ if } wRx \text{ then } v(x, A) \subseteq v(x, B).$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.a.}{e} B$ | if and only if | $\forall v \in W \text{ if } wRv \text{ then } \mathfrak{K}_{\mathcal{J}}, v \models AeB.$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.a.}{i} B$ | if and only if | $\forall v \in W \text{ if } wRv \text{ then } \mathfrak{K}_{\mathcal{J}}, v \models AiB.$ |
| $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.a.}{o} B$ | if and only if | $\forall v \in W \text{ if } wRv \text{ then } \mathfrak{K}_{\mathcal{J}}, v \models AoB.$ |

9.0.1 Single Premise Inferences

We will start by proving some general properties about assertoric propositions in the *Square of Opposition*:

| | |
|--------------------------------------|---------------------------------|
| AaB contradicts AoB | AeB contradicts AiB |
| AaB and AeB are contrary | AiB and AoB are subcontrary |
| AaB implies AiB | AeB implies AoB |
| AeB simply converts to BeA | AiB simply converts to BiA |
| AaB accidentally converts to BiA | |

Recall that two propositions are said to be contradictory if the falsity of the one implies that the other is true, and vice versa. Two propositions are contrary if they cannot both be true, but can both be false. Two propositions are said to be subcontrary if they cannot both be false, but can both be true. In our formal model, we interpret this as follows:

Two formulae ϕ and ψ are said to be contradictory if for all models, \mathfrak{K} and all worlds $w \in W$ $\mathfrak{K}, w \models \phi$ if and only if $\mathfrak{K}, w \not\models \psi$.

Two formulae ϕ and ψ are said to be contrary if there is no model, \mathfrak{K} and no world $w \in W$ such that $\mathfrak{K}, w \models \phi$ and $\mathfrak{K}, w \models \psi$, but there is at least one model and one world where $\mathfrak{K}, w \not\models \phi$ and $\mathfrak{K}, w \not\models \psi$.

Two formulae ϕ and ψ are said to be subcontrary if there is no model, \mathfrak{K} and no world $w \in W$ such that $\mathfrak{K}, w \not\models \phi$ and $\mathfrak{K}, w \not\models \psi$, but there is at least one model and one world where $\mathfrak{K}, w \models \phi$ and $\mathfrak{K}, w \models \psi$.

In the case of simple conversion we will simply prove that the two propositions are equivalent, while in accidental conversion we will treat this as an implication.

That AaB contradicts AoB and that AeB contradicts AiB are immediate from the semantics. To see that AaB and AeB are contraries, first, assume that for some model

\mathfrak{K}_J and some world, $w \in W$ we have $\mathfrak{K}_J, w \models AaB$ and $\mathfrak{K}_J, w \models AeB$. Then we have $c(A) \leq c(B)$ and for no $D \in T$ $D \leq c(A)$ and $D \leq c(B)$. But since \leq is reflexive, we have $c(A) \leq c(A)$ and we already have assumed $c(A) \leq c(B)$. Hence, $\exists D \in T$ such that $D \leq c(A)$ and $D \leq c(B)$ contradicting our assumption.

For the subcontraries, assume that for some model \mathfrak{K}_J and some world, $w \in W$ we have $\mathfrak{K}_J, w \not\models AiB$ and $\mathfrak{K}_J, w \not\models AoB$. Then we have $\mathfrak{K}_J, w \models AeB$ and $\mathfrak{K}_J, w \models AaB$, since these are contradictories. However, we have already shown that $\mathfrak{K}_J, w \models AeB$ and $\mathfrak{K}_J, w \models AaB$ cannot both be true. Hence there is no model where AiB and AoB are both false.

To see that AaB and AeB can both be false, and to see that AiB and AoB can both be true consider the following counter-model:

| | | |
|-------------------------------------|--|------------------|
| $D = \{a, b\}$ | $W = \{w\}$ | $R = \{(w, w)\}$ |
| $T = \{A, B\}$ | $\trianglelefteq = \{(A, A), (B, B)\}$ | $ = \emptyset$ |
| $\leq = \{(A, A), (B, B), (B, A)\}$ | | |
| $c(A) = A$ | $c(B) = B$ | |
| $v(w, A) = \{a, b\}$ | $v(w, B) = \{b\}$ | |

Normally we will simply observe that the model is a Kilwardby Model. However, in a few cases we will verify these conditions. For the order properties we need to show that:

1. $\forall x, y$ if $x \trianglelefteq y$ then $x \leq y$.
2. $\forall x, y, z$ if $x \leq y$ and $y \trianglelefteq z$ then $x \trianglelefteq z$.
3. $\forall x, y$ if $x \leq y$ then not $x|y$
4. $\forall x, y, z$ if $x \leq y$ and $y|z$ then $x|z$.

Condition 1 clearly holds. In the case of 2., observe that there are no distinct terms y, z such that $y \trianglelefteq z$. Hence the only case we need to consider is when $c(B) \leq c(A)$ and $B \trianglelefteq B$. But clearly the consequent is satisfied. Condition 3. clearly holds, since $|$ is empty, and likewise, condition 4 is vacuously satisfied.

Likewise, for the valuation conditions, we need to verify that:

1. For all terms, A, B $c(A) \leq c(B)$ iff for some $w \in W$ $v(w, A) \subseteq v(w, B)$.
2. For all terms, A, B if $c(A) \trianglelefteq c(B)$ then for all $w \in W$ $v(w, A) \subseteq v(w, B)$.
3. For all terms, A, B if $c(A)|c(B)$ then for all $w \in W$ $v(w, A) \cap v(w, B) = \emptyset$

In the case of condition 1, observe that $c(B) \leq c(A)$, $c(A) \leq c(A)$, and $c(B) \leq c(B)$. As we have $v(A) = \{a, b\}$ and $v(B) = \{b\}$. it is easy to verify that $v(w, A) \subseteq v(w, A)$, $v(w, B) \subseteq v(w, B)$, and $v(w, B) \subseteq v(w, A)$.

In the case of condition 2 we only have one world to consider and it clearly holds by inspection of the valuation function, while the case of condition 3 is vacuously true.

In such a model clearly $c(A) \not\leq c(B)$ since $v(w, B)$ is a proper subset of $v(w, A)$. Hence $\mathfrak{M}, w \vDash AaB$. However, since $c(B) \leq c(A)$ and $c(B) \leq c(B)$, it follows that $\exists D D \leq c(A)$ and $D \leq c(B)$ from which it follows that $\mathfrak{M}, w \vDash AeB$. Consequently, it is easy to see that $\mathfrak{M}, w \vDash AiB$ and $\mathfrak{M}, w \vDash AoB$.

To see that AaB implies AiB , assume that for some model \mathfrak{K}_j and some world, $w \in W$ we have $\mathfrak{K}_j, w \vDash AaB$. Then $c(A) \leq c(B)$. Since \leq is reflexive, it follows that $c(A) \leq c(A)$. Hence, there is some D such that $D \leq c(A)$ and $D \leq c(B)$. Therefore $\mathfrak{K}_j, w \vDash AiB$.

To see that AeB implies AoB assume that for some model \mathfrak{K}_j and some world, $w \in W$ we have $\mathfrak{K}_j, w \vDash AeB$. Then it follows that there is no D such that $D \leq c(A)$ and $D \leq c(B)$. But \leq is reflexive, and so $c(A) \leq c(A)$. Hence it would have to be the case that $c(A) \leq c(B)$, but then it would follow that exists D such that $D \leq c(A)$, which would contradict our assumption. Hence $c(A) \not\leq c(B)$ and so $\mathfrak{K}_j, w \vDash AoB$.

We will refer to the previous two inferences as subalternation.

In the cases of simple conversion we need to show:

1. AeB is equivalent to BeA
2. AiB is equivalent to BiA

However, both of these are trivial. Take an arbitrary model \mathfrak{K}_j and arbitrary world w then observe that $\mathfrak{K}_j, w \vDash AeB$ iff it is not the case that there is some D such that $D \leq c(A)$ and $D \leq c(B)$ iff it is not the case that there is some D such that $D \leq c(B)$ and $D \leq c(A)$ iff $\mathfrak{K}_j, w \vDash BeA$. The first and last biconditionals are given by the truth conditions, the middle biconditional follows by basic logic. The proof that AiB is equivalent to BiA is very similar.

Accidental conversion: Take an arbitrary model \mathfrak{K}_j and arbitrary world w and assume that $\mathfrak{K}_j, w \vDash AaB$. Then $c(A) \leq c(B)$. Since \leq is reflexive, it follows that $c(A) \leq c(A)$. But then it follows that there is some D (in this case A) such that $D \leq c(B)$ and $D \leq c(A)$. Hence $\mathfrak{K}_j, w \vDash BiA$.

Square of opposition properties for *Per Se* Necessary Propositions:

$$\begin{array}{ll}
 A \overset{p.s.}{a} B \text{ contradicts } A \overset{p.s.}{o} B & A \overset{p.s.}{e} B \text{ contradicts } A \overset{p.s.}{i} B \\
 A \overset{p.s.}{a} B \text{ and } A \overset{p.s.}{e} B \text{ are contrary} & A \overset{p.s.}{i} B \text{ and } A \overset{p.s.}{o} B \text{ are subcontrary} \\
 A \overset{p.s.}{a} B \text{ implies } A \overset{p.s.}{i} B & A \overset{p.s.}{e} B \text{ implies } A \overset{p.s.}{o} B
 \end{array}$$

To see that $A \overset{p.s.}{a} B$ contradicts $A \overset{p.s.}{o} B$ assume that $A \sqsubseteq B$ and that $\exists D$ such that $D \leq A$ and $D|B$. Assume that $E \leq c(A)$ and $E|B$. By order property 3, it follows that $E \not\leq B$. Then, since $A \sqsubseteq B$ and $E \leq c(A)$ it follows that $E \sqsubseteq B$ by order property 2. Hence $E \leq c(B)$ by order property 1, which is a contradiction.

To see that $A \overset{p.s.}{e} B$ contradicts $A \overset{p.s.}{i} B$, assume that $A|B$ and that $\exists D D \leq c(A)$ and $D \sqsubseteq B$ or $D \leq c(B)$ and $D \sqsubseteq A$. We prove that the right disjunct leads to a contradiction and note that the proof that the left disjunct leads to a contradiction is similar. Assume that $D \leq c(A)$ and $D \sqsubseteq B$. Since $D \leq c(A)$ and $A|B$, it follows by order property 4 that $D|B$, which, by order property 3 entails that $D \not\leq B$. But from $D \sqsubseteq B$ it follows by order property 1 that $D \leq c(B)$, which is a contradiction.

To see that $A \overset{p.s.}{a} B$ and $A \overset{p.s.}{e} B$ are contrary, first assume that for some model \mathcal{M} and some $w \in W$, we have $\mathcal{M}, w \models A \overset{p.s.}{a} B$ and $\mathcal{M}, w \models A \overset{p.s.}{e} B$. Then it follows that $c(A) \sqsubseteq c(B)$ and that $c(A)|c(B)$. By order property 1, $c(A) \sqsubseteq c(B)$ entails $c(A) \leq c(B)$. Likewise, $c(A)|c(B)$ entails $c(A) \not\leq c(B)$, which is a contradiction.

To see that $A \overset{p.s.}{a} B$ and $A \overset{p.s.}{e} B$ can both be false, recall our previous countermodel:

| | | |
|-------------------------------------|------------------------------------|------------------|
| $D = \{a, b\}$ | $W = \{w\}$ | $R = \{(w, w)\}$ |
| $T = \{A, B\}$ | $\sqsubseteq = \{(A, A), (B, B)\}$ | $ = \emptyset$ |
| $\leq = \{(A, A), (B, B), (B, A)\}$ | | |
| $c(A) = A$ | $c(B) = B$ | |
| $v(w, A) = \{a, b\}$ | $v(w, B) = \{b\}$ | |

Clearly $c(A) \not\leq c(B)$. Likewise, it is not the case that $c(A)|c(B)$. Hence, $\mathfrak{M}, w \not\models A \overset{p.s.}{a} B$ and $\mathfrak{M}, w \not\models A \overset{p.s.}{e} B$.

To see that $A \overset{p.s.}{i} B$ and $A \overset{p.s.}{o} B$ are subcontrary, first assume that for some model \mathcal{M} and some $w \in W$, we have $\mathcal{M}, w \not\models A \overset{p.s.}{i} B$ and $\mathcal{M}, w \not\models A \overset{p.s.}{o} B$. As in the previous case, it would then follow that $\mathcal{M}, w \models A \overset{p.s.}{e} B$ and $\mathcal{M}, w \models A \overset{p.s.}{a} B$, which is impossible, since these are contraries.

To see that these can both be true, consider the following model:

| | | |
|---|--|--------------------------|
| $D = \{a, b\}$ | $W = \{w\}$ | $R = \{(w, w)\}$ |
| $T = \{A, B, C\}$ | $\sqsubseteq = \{(A, A), (B, B), (B, A), (C, C)\}$ | $ = \{(C, B), (B, C)\}$ |
| $\leq = \{(A, A), (B, B), (B, A), (C, C), (C, A)\}$ | | |
| $c(A) = A$ | $c(B) = B$ | $c(C) = C$ |
| $v(w, A) = \{a, b\}$ | $v(w, B) = \{b\}$ | $v(w, C) = \{a\}$ |

In this case, since $c(B) \leq c(B)$ and $B \sqsubseteq A$, it follows by basic logic that $\mathfrak{M}, w \models A \overset{p.s.}{i} B$. However, since $c(C) \leq c(A)$ and $C|B$, it follows that $\mathfrak{M}, w \models A \overset{p.s.}{o} B$ as desired.

To see that $A \overset{p.s.}{a} B$ implies $A \overset{p.s.}{i} B$, assume that for some model $\mathfrak{K}_{\mathcal{J}}$ and some world, $w \in W$ we have $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.s.}{a} B$. Then we have $c(A) \sqsubseteq c(B)$. From this it follows that $c(A) \leq c(A)$ (by reflexivity). Hence, there is some D such that $D \sqsubseteq c(B)$ and $D \leq c(A)$. That $\mathfrak{K}_{\mathcal{J}}, w \models A \overset{p.s.}{i} B$ follows by basic logic.

Likewise, to show that AeB implies AoB , assume that for some model $\mathfrak{K}_{\mathcal{J}}$ and some world, $w \in W$ we have $\mathfrak{K}_{\mathcal{J}}, w \models AeB$. From this it follows that $c(A)|c(B)$. Since \leq is reflexive, we have $c(A) \leq c(A)$. Hence, there is some D such that $D \leq c(A)$ and $D|c(B)$, and so $\mathfrak{K}_{\mathcal{J}}, w \models AoB$.

Square of opposition properties of *Per Accidens* Necessary Propositions:

$A \overset{p.a.}{a} B$ is contrary to $A \overset{p.a.}{o} B$ $A \overset{p.a.}{e} B$ is contrary to $A \overset{p.a.}{i} B$
 $A \overset{p.a.}{a} B$ and $A \overset{p.a.}{e} B$ are contrary $A \overset{p.a.}{i} B$ and $A \overset{p.a.}{o} B$ are subcontrary
 $A \overset{p.a.}{a} B$ implies $A \overset{p.a.}{i} B$ $A \overset{p.a.}{e} B$ implies $A \overset{p.a.}{o} B$

The proofs of each of these propositions is a straight-forward. For example, to see that $A \overset{p.a.}{a} B$ is contrary to $A \overset{p.a.}{o} B$, take an arbitrary model \mathfrak{J} , and world $w \in W$. Then $\mathfrak{J}, w \models A \overset{p.a.}{a} B$ if and only if $\forall x \in W$ if wRx then $v(w, A \subseteq v(w, B))$ if and only if $\neg \forall x \in W$ if wRx then $v(w, A \not\subseteq v(w, B))$. However, it follows from $A \overset{p.a.}{o} B$ that $\forall x \in W$ if wRx then $v(w, A \not\subseteq v(w, B))$. Hence these are contrary.

9.1 Inferences Between Per Se, and Assertoric Propositions.

Since Kilwardby restricts his treatment of the modal syllogisms to *per se* necessary proposition, we will only treat inferences between *per se* and assertoric propositions.

$A \overset{p.s.}{a} B$ implies $A a B$

$A \overset{p.s.}{e} B$ implies $A e B$

$A \overset{p.s.}{i} B$ implies $A i B$

$A \overset{p.s.}{o} B$ implies $A o B$

For the proofs of each of these propositions, assume that $\mathfrak{K}_{\mathfrak{J}}$ is an arbitrary model and that $w \in W$.

To see that $A \overset{p.s.}{a} B$ implies $A a B$, assume that $\mathfrak{K}_{\mathfrak{J}}, w \models A \overset{p.s.}{a} B$. Then $c(A) \trianglelefteq c(B)$. By order property 1, it follows that $c(A) \leq c(B)$ and hence $\mathfrak{K}_{\mathfrak{J}}, w \models A a B$.

To see that $A \overset{p.s.}{e} B$ implies $A e B$, assume that $\mathfrak{K}_{\mathfrak{J}}, w \models A \overset{p.s.}{e} B$. Then $c(A) | c(B)$. Now, assume that for some D $D \leq c(A)$ and $D \leq c(B)$. Since $D \leq c(A)$ and $c(A) | c(B)$ it follows by order property 4 that $D | c(B)$ and hence $D \not\leq c(B)$, contradicting our assumption. Hence for no D is $D \leq c(A)$ and $D \leq c(B)$ and so $\mathfrak{K}_{\mathfrak{J}}, w \models A e B$.

To see that $A \overset{p.s.}{i} B$ implies $A i B$, assume that $\mathfrak{K}_{\mathfrak{J}}, w \models A \overset{p.s.}{i} B$. Then there is some D such that $D \leq c(A)$ and $D \trianglelefteq c(B)$ or $D \leq c(B)$ and $D \trianglelefteq c(A)$. That $D \leq c(A)$ and $D \leq c(B)$ follows from both disjuncts by order property 1.

To see that $A \overset{p.s.}{o} B$ implies $A o B$, assume that $\mathfrak{K}_{\mathfrak{J}}, w \models A \overset{p.s.}{o} B$. Hence, there is some D such that $D \leq c(A)$ and $D | c(B)$. Now, assume for a contradiction that $c(A) \leq c(B)$. Then since \leq is transitive, it follows that $D \leq c(B)$. However, from $D | c(B)$ it follows by order property 3 that $D \not\leq c(B)$, which contradicts our assumption. Hence $c(A) \not\leq c(B)$ and so $\mathfrak{K}_{\mathfrak{J}}, w \models A o B$.

9.2 Kilwardby's Validities

In this section we will show that our account of Kilwardby's logic correctly follows the assertoric syllogisms in terms of validity and invalidity for the three figures. With this

in place we will then use this to give proofs for properties P1-P7. In doing this we will not treat the fourth figure, as it can easily be obtained from the first.

We start with the validities: We claim that the following syllogisms are valid:

| | | | | |
|---------------|---------|-----------|---------|---------|
| First Figure | Barbara | Celarent | Ferio | Darii |
| Second Figure | Cesare | Camestres | Festino | Baroco |
| Third Figure | Datisi | Disamis | Ferison | Bocardo |
| | Darapti | Falaption | | |

We will omit proofs of weakened syllogisms (such as Barbari) since they will follow by subalternation.

Proofs: In each case, let \mathfrak{K}_3 be an arbitrary Kilwardby Model and $w \in W$:

First Figure

Barbara

Assume that $\mathfrak{K}_3, w \models BaC$ and $\mathfrak{K}_3, w \models AaB$. Then it follows that $c(B) \leq c(C)$ and that $c(A) \leq c(B)$. It then follows by transitivity that $c(A) \leq c(C)$ and so $\mathfrak{K}_3, w \models AaC$. This proves Barbara. Barbari also follows by subalternation.

Celarent

Assume that $\mathfrak{K}_3, w \models BeC$ and $\mathfrak{K}_3, w \models AaB$. From the first assumption it follows that there is no D such that $D \leq c(B)$ and $D \leq c(C)$. From the second assumption, it follows that $c(A) \leq c(B)$. Assume for reductio that there is some D such that $D \leq c(A)$ and $D \leq c(C)$. Then this together with the second assumption entails $D \leq c(B)$. However, this contradicts the first assumption.

Ferio

Assume that $\mathfrak{K}_3, w \models BeC$ and $\mathfrak{K}_3, w \models AiB$. From the first assumption it follows that there is no D such that $D \leq c(B)$ and $D \leq c(C)$. From the second assumption, it follows that there is some D such that $D \leq c(A)$ and $D \leq c(B)$. Assume for reductio that $c(A) \leq c(C)$. Then by the second assumption it follows that there is some D such that $D \leq c(C)$ and $D \leq c(B)$, clearly contradicting the first assumption.

Darii

Assume that $\mathfrak{K}_3, w \models BaC$ and $\mathfrak{K}_3, w \models AiB$. From the first assumption we have that $c(B) \leq c(C)$. From the second assumption it follows that there is some D such that $D \leq c(A)$ and $D \leq c(B)$. It then clearly follows from the first assumption that there is some D such that $D \leq c(A)$ and $D \leq c(C)$. Hence, $\mathfrak{K}_3, w \models AiC$.

Second Figure

Cesare

Assume that $\mathfrak{K}_3, w \models CeB$ and $\mathfrak{K}_3, w \models AaB$. By accidental conversion, it follows that $\mathfrak{K}_3, w \models BeC$. This is now Celarent, which we already have shown to be valid.

Camestres

Assume that $\mathfrak{K}_3, w \models CaB$ and $\mathfrak{K}_3, w \models AeB$. It then follows that 1) $c(C) \leq c(B)$ and 2) there is no D such that $D \leq c(A)$ and $D \leq c(B)$. Assume for reductio that there is some D such that $D \leq c(A)$ and $D \leq c(C)$. Then by 1, it follows that there is some D such that $D \leq c(B)$ and $D \leq c(A)$, contradicting 2.

Festino

Assume that $\mathfrak{K}_3, w \models CeB$ and $\mathfrak{K}_3, w \models AiB$. It follows by simple conversion that $\mathfrak{K}_3, w \models BeC$. This is now Ferio, which we have shown is valid.

Baroco

Assume that $\mathfrak{K}_3, w \models CaB$ and $\mathfrak{K}_3, w \models AoB$. It then follows that 1) $c(C) \leq c(B)$ and 2) $c(A) \not\leq c(B)$. Assume for reductio that $c(A) \leq c(C)$. From 1) and the assumption it follows that $c(A) \leq c(B)$, contradicting 2).

Third Figure

Datisi

Assume that $\mathfrak{K}_3, w \models BaC$ and $\mathfrak{K}_3, w \models BiA$. It follows by simple conversion that $\mathfrak{K}_3, w \models AiB$. This is Darii, which we have already shown to be valid.

Disamis

Assume that $\mathfrak{K}_3, w \models BiC$ and $\mathfrak{K}_3, w \models BaA$. Then it follows that 1) there is some D such that $D \leq c(B)$ and $D \leq c(C)$ and 2) $c(B) \leq c(A)$. By 1) and 2) (using transitivity) it follows that there is some D such that $D \leq c(A)$ and $D \leq c(C)$. Hence $\mathfrak{K}_3, w \models AiC$.

Ferison

Assume that $\mathfrak{K}_3, w \models BeC$ and $\mathfrak{K}_3, w \models BiA$. Then it follows that 1) there is no D such that $D \leq c(B)$ and $D \leq c(C)$ and 2) there is some D such that $D \leq c(B)$ and $D \leq c(A)$. Assume for reductio for $c(A) \leq c(C)$. Then this together with 2) it follows that there is some D such that $D \leq c(B)$ and $D \leq c(C)$, contradicting 1). Hence $c(A) \not\leq c(C)$ and so $\mathfrak{K}_3, w \models AoC$.

Bocardo

Assume that $\mathfrak{K}_3, w \models BoC$ and $\mathfrak{K}_3, w \models BaA$. Then it follows that 1) $c(B) \not\leq c(C)$ and that 2) $c(B) \leq c(A)$. Assume for reductio that $c(A) \leq c(C)$. Then this together with 2) entails, by the transitivity of \leq that $c(B) \leq c(C)$, contradicting 1).

Darapti Assume that $\mathfrak{K}_3, w \models BaC$ and $\mathfrak{K}_3, w \models BaA$. Then it follows that 1) $c(B) \leq c(C)$ and that 2) $c(B) \leq c(A)$. Then clearly, $\exists D$ such that $D \leq c(A)$ and $D \leq c(C)$. Hence $\mathfrak{K}_3, w \models AiC$.

Felapton Assume that $\mathfrak{K}_3, w \models BeC$ and $\mathfrak{K}_3, w \models BaA$. Then it follows that 1) for no D do both $D \leq c(B)$ and $D \leq c(C)$ hold. Further we have 2) $c(B) \leq c(A)$. Assume that $c(A) \leq c(C)$. Then $c(B) \leq c(C)$ by the transitivity of \leq . Hence $\exists D$ such that $D \leq c(B)$ and $D \leq c(C)$, contradicting 1). So $c(A) \not\leq c(C)$ and so $\mathfrak{K}_3, w \models AoC$.

These are all of the syllogism we needed to show were valid.

The proof of the invalidities is an fairly straightforward exercise in constructing countermodels. We list them in the table that follows in the order Major, Minor, Conclusion. We will only give the counterexamples for the first figure configurations. After this, we will prove a short theorem that tells us how to reduce syllogisms in the other two figures to the first figure. Since AeB implies AoB and AaB implies AiB , if a particular pair of premises does not entail a and i or e and o (as the case may be), we will group countermodels together as is useful.

The following are all invalid on our semantics:

| | | | |
|-----|-----|-----|-----|
| aae | aa0 | | |
| aie | aio | aia | |
| ae0 | aeo | aea | aei |
| aoe | ao0 | aoa | aoi |
| | | ea0 | eai |
| eie | | eia | eii |
| eee | eeo | eea | eei |
| eo0 | eo0 | eo0 | eo1 |
| iae | iao | iaa | iai |
| iie | iio | iia | iii |
| iee | ieo | iea | iei |
| ioe | io0 | ioa | ioi |
| oae | oao | oaa | oai |
| oie | oio | oia | oii |
| oee | o00 | oea | oei |
| ooe | oo0 | ooa | ooi |

Consider the following countermodel:

| | |
|--------------------------|--------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a\}$ |
| $\leq = T^2$ | $\leq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = v(B) = v(C) = D$ | |

Call this model, \mathfrak{J} . Observe that since $v(A) = v(B) = v(C) = D$, we have $\mathfrak{J}, w \models AaB$, $\mathfrak{J}, w \models BaC$, $\mathfrak{J}, w \models AaC$, $\mathfrak{J}, w \models AiB$, $\mathfrak{J}, w \models BiC$, $\mathfrak{J}, w \models AiC$ (as well as some other consequences). It is also easy to see that $\mathfrak{J}, w \not\models AoC$ and $\mathfrak{J}, w \not\models AeC$.

Because of this, the following inferences are clearly invalid:

| | | | |
|------------|------------|------------|------------|
| <i>aae</i> | <i>aa0</i> | <i>aie</i> | <i>aio</i> |
| <i>iae</i> | <i>iao</i> | <i>iie</i> | <i>iio</i> |

Consider the following countermodel:

| | |
|-------------------------------------|--------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a\}$ |
| $\leq = \{(A, A), (B, B), (C, C)\}$ | $\leq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = v(B) = v(C) = \emptyset$ | |

9 Appendix Two: Kilwardby's Logic

Call this model, \mathfrak{J}_1 . Observe that since $v(A) = v(B) = v(C) = \emptyset$, we have $\mathfrak{J}_1, w \models AeB$, $\mathfrak{J}_1, w \models BeC$, $\mathfrak{J}_1, w \models AeC$, $\mathfrak{J}_1, w \models AoB$, $\mathfrak{J}_1, w \models BoC$, $\mathfrak{J}_1, w \models AoC$ (as well as some other consequences). It is also easy to see that $\mathfrak{J}_1, w \not\models AiC$ and $\mathfrak{J}_1, w \not\models AaC$.

Because of this, the following inferences are clearly invalid:

| | | | |
|-------|-------|-------|-------|
| eea | eei | eoA | eoI |
| oea | oei | ooA | ooI |

Consider the following model:

| | |
|---|--------------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a\}$ |
| $\leq = \{(A, A), (B, B), (C, C), (A, C)\}$ | $\triangleq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = v(C) = \{a\}$ | $v(B) = \emptyset$ |

Call this model, \mathfrak{J}_2 . Observe that since $v(A) = v(C) = D$ and $v(B) = \emptyset$ we have $\mathfrak{J}_2, w \models AeB$, $\mathfrak{J}_2, w \models BeC$, $\mathfrak{J}_2, w \models AaC$, $\mathfrak{J}_2, w \models AiB$, $\mathfrak{J}_2, w \models BoC$, $\mathfrak{J}_2, w \models AoC$ (as well as some other consequences). It is also easy to see that $\mathfrak{J}_2, w \not\models AoC$ and $\mathfrak{J}_2, w \not\models AeC$.

Because of this the following are clearly invalid:

| | | | |
|-------|-------|-------|-------|
| eee | eeo | eoe | eoo |
| oee | oeo | ooe | ooo |

Consider the following model:

| | |
|---|--------------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a, b\}$ |
| $\leq = \{(A, A), (B, B), (C, C), (A, B), (C, B)\}$ | $\triangleq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = \{a\}$ | $v(B) = D$ |
| $v(C) = \{b\}$ | |

Call this model, \mathfrak{J}_3 . Observe that since $v(A) = \{a\}$, $v(B) = D$, and $v(C) = \{b\}$ we have $\mathfrak{J}_3, w \models AaB$, $\mathfrak{J}_3, w \models AiB$, $\mathfrak{J}_3, w \models BiC$, $\mathfrak{J}_3, w \models AoC$, $\mathfrak{J}_3, w \models AeB$ (as well as some other consequences). It is also easy to see that $\mathfrak{J}_3, w \not\models AaC$ and $\mathfrak{J}_3, w \not\models AiC$.

Because of this the following are clearly invalid:

| | | | |
|-----|-----|-----|-----|
| aia | iaa | iii | iaa |
| iai | | | |

Consider the following model:

| | |
|-------------------------------------|--------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a, b\}$ |
| $\leq = \{(A, A), (B, B), (C, C)\}$ | $\leq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = \{a\}$ | $v(B) = \{b\}$ |
| $v(C) = D$ | |

Call this model \mathfrak{J}_4 . Observe that since $v(A) = \{a\}$, $v(B) = \{b\}$, and $v(C) = D$ we have $\mathfrak{J}_4, w \models AeB$, $\mathfrak{J}_4, w \models AoB$, $\mathfrak{J}_4, w \models BaC$, $\mathfrak{J}_4, w \models BiC$, $\mathfrak{J}_4, w \models AaC$, and $\mathfrak{J}_4, w \models AiC$. (as well as some other consequences). It is also easy to see that $\mathfrak{J}_4, w \not\models AeC$ and $\mathfrak{J}_4, w \not\models AoC$.

From this it is clear that the following inferences are invalid:

| | | | |
|-----|-----|-----|-----|
| aee | aeo | aoe | aoa |
| iee | ieo | ioe | ioa |

Consider the following model:

| | |
|---|--------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a, b\}$ |
| $\leq = \{(A, A), (B, B), (C, C), (B, C)\}$ | $\leq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = \{a\}$ | $v(B) = \{b\}$ |
| $v(C) = \{b\}$ | |

Call this model \mathfrak{J}_5 . Observe that since $v(A) = \{a\}$, $v(B) = \{b\}$, and $v(C) = \{b\}$ we have $\mathfrak{J}_5, w \models AeB$, $\mathfrak{J}_5, w \models AoB$, $\mathfrak{J}_5, w \models BaC$, $\mathfrak{J}_5, w \models BiC$, $\mathfrak{J}_5, w \models AeC$, and $\mathfrak{J}_5, w \models AoC$. (as well as some other consequences). It is also easy to see that $\mathfrak{J}_5, w \not\models AaC$ and $\mathfrak{J}_5, w \not\models AiC$.

From this it is clear that the following inferences are invalid:

9 Appendix Two: Kilwardby's Logic

| | | | |
|-----|-----|-----|-----|
| aea | aei | aoa | aoi |
| ica | iei | ioa | ioi |

Consider the following model:

| | |
|---|-------------------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a, b\}$ |
| $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ | $\trianglelefteq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = \{b\}$ | $v(B) = \{b\}$ |
| $v(C) = \{a\}$ | |

Call this model \mathfrak{J}_6 . Observe that since $v(A) = \{b\}$, $v(B) = \{b\}$, and $v(C) = \{a\}$ we have $\mathfrak{J}_6, w \models AaB$, $\mathfrak{J}_6, w \models AiB$, $\mathfrak{J}_6, w \models BeC$, $\mathfrak{J}_6, w \models BoC$, $\mathfrak{J}_6, w \models AeC$, and $\mathfrak{J}_6, w \models AoC$. (as well as some other consequences). It is also easy to see that $\mathfrak{J}_6, w \not\models AaC$ and $\mathfrak{J}_6, w \not\models AiC$.

From this it is clear that the following inferences are invalid:

| | | | |
|-----|-----|-----|-----|
| eea | eai | eia | eii |
| oaa | oai | oia | oii |

Consider the following model:

| | |
|---|-------------------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a, b\}$ |
| $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ | $\trianglelefteq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = \{b\}$ | $v(B) = \{a, b\}$ |
| $v(C) = \{b\}$ | |

Call this model \mathfrak{J}_7 . Observe that since $v(A) = \{b\}$, $v(B) = \{b\}$, and $v(C) = \{a\}$ we have $\mathfrak{J}_7, w \models AaB$, $\mathfrak{J}_7, w \models AiB$, $\mathfrak{J}_7, w \models BeC$, $\mathfrak{J}_7, w \models BoC$, $\mathfrak{J}_7, w \models AeC$, and $\mathfrak{J}_7, w \models AoC$. (as well as some other consequences). It is also easy to see that $\mathfrak{J}_7, w \not\models AeC$ and $\mathfrak{J}_7, w \not\models AoC$.

From this it is clear that the following inferences are invalid:

oae oao oie oio

Consider the following model:

| | |
|---|--------------------|
| $W = \{w\}$ | $R = W^2$ |
| $T = \{A, B, C\}$ | $D = \{a, b\}$ |
| $\leq = \{(A, A), (B, B), (C, C), (A, B)\}$ | $\leq = \emptyset$ |
| $ = \emptyset$ | |
| $c(A) = A$ | $c(B) = B$ |
| $c(C) = C$ | |
| $v(A) = \{a, b\}$ | $v(B) = \{a\}$ |
| $v(C) = \{b\}$ | |

Call this model \mathfrak{J}_8 . Observe that since $v(A) = \{b\}$, $v(B) = \{a\}$, and $v(C) = \{b\}$ we have $\mathfrak{J}_8, w \models AiB$, $\mathfrak{J}_8, w \models BeC$, $\mathfrak{J}_8, w \models BoC$, and $\mathfrak{J}_8, w \models AiC$ (as well as some other consequences). It is also easy to see that $\mathfrak{J}_8, w \not\models AeC$

From this it is clear that *eie* is invalid.

This completes the counterexamples for the first figure.

9.2.1 Second and Third Figure Syllogisms

For the second and third figure, we can use the principles of interchange and *proof per impossibile* that we outlined in our treatment of Buridan's modal logic to prove a useful lemma. Recall that:

Interchange: Let $S = \langle M, m, C \rangle$ and $S' = \langle M', m', C' \rangle$ be two syllogisms such that $M = m'$ and $M' = m$, then $\models S$ if and only if $\models S'$.

The proof is trivial.

Proof per impossibile [PPI]:

Formally, let $S = \langle M, m, C \rangle$ and $S' = \langle M', m', C' \rangle$ be two syllogisms such that C' is the contradictory of m , m' is the contradictory of C and $M = M'$, then we claim that $\models S$ if and only if $\models S'$.

The clause for validity is an easy adaption of how we defined validity in Chapter 3. We say that $\models S$ if and only if for all Kilwardby Models, $\mathfrak{K}_{\mathfrak{J}}$ and all worlds $w \in W$, if $\mathfrak{K}_{\mathfrak{J}}, w \models M$ and $\mathfrak{K}_{\mathfrak{J}}, w \models m'$ then $\mathfrak{K}_{\mathfrak{J}}, w \models C$.

The proofs for both of these propositions are completely analogous to what we proved in the case of Buridan models. In what follows, to make the proofs easier to read we write *Contra* for the contradictory of a given proposition.

With this in place we can now prove the following reduction lemma:

Reduction Lemma 1. *Any triple in the second or third figure is equivalent to one triple in the first figure.*

The proof of this lemma is an combinatorial exercise using PPI and interchange. Recall that the second and third figures are of the form:

$$\begin{array}{ll} C \times B & B \times C \\ A \times B & B \times A \\ A \times C & A \times C \end{array}$$

To reduce a second figure syllogism, assume that 'S' is a triple. Then by PPI S is equivalent to a triple S' of the form:

$$\begin{array}{l} C \times B \\ \textit{Contra}(A \times C) \\ \textit{Contra}(A \times B) \end{array}$$

However, this is clearly now in the first figure, where C is now the middle term, A is still the subject and B is now the predicate. Hence S' is a first figure triple that is equivalent to S

To reduce a third figure syllogism, assume that 'S' is a triple. Then by Interchange S is equivalent to a triple of the form:

$$\begin{array}{l} A \times C \\ B \times C \\ B \times A \end{array}$$

By PPI this is equivalent to:

$$\begin{array}{l} A \times C \\ \textit{Contra}(B \times A) \\ \textit{Contra}(B \times C) \end{array}$$

and by Interchange again, this in turn is equivalent to:

$$\begin{array}{l} \textit{Contra}(B \times A) \\ A \times C \\ \textit{Contra}(B \times C) \end{array}$$

which is clearly a first figure syllogism where B is now the subject, C is still the predicate and A is the middle term.

It follows as an easy corollary of this problem that if a syllogism is valid in the first figure, then by using the algorithm outlined above, we can find valid syllogisms in the second and third figure. Similarly, if a syllogism is invalid in the first figure, by applying this algorithm, we can find invalid syllogisms in the second and third figure. As we have considered all possible combinations of terms in the first figure, it follows that we have exhausted all of the terms in the second and third figures also, since the difference between the figures only concerns the order in which the terms occur in the formulae. With this in place, we can now reduce the invalidity of syllogisms in the second and third figure to syllogisms in the first figure.

As such, for the three figures we are considering, the following syllogisms are valid:

| Figure | Valid Syllogism |
|--------|--|
| First | Barbara, Celarent, Darii, Ferio |
| Second | Cesare, Camestres, Festino, Baroco |
| Third | Darapti, Disamis, Datisi, Felapton, Bocardo, Ferison |

With what we have proven above, we are now in a position to see that P1 through P7 all hold for syllogisms. Each case follows by a simple inspection of the table for valid syllogisms.

- P1 In every syllogism, one premise must be universal.
- P2 In every syllogism, one premise must be affirmative.
- P3 In first figure syllogisms, the major must be universal
- P4 In first figure syllogisms, the minor must be affirmative
- P5 In second figure syllogisms, the major must be universal.
- P6 In second figure syllogisms, one of the premises must be negative.
- P7 In third figure syllogisms, the minor must be affirmative.

10 Appendix Three: Buridan's Modal Syllogism

Our aim in this appendix is to verify that all of the syllogisms Buridan claims are valid, are valid on our semantics, and conversely, that if Buridan says a syllogism is not valid then it is not valid in our semantics either. In what follows, we will only treat modal syllogisms, as the assertoric syllogisms are an easy exercise in semantic proofs and counterexample construction and so are left as an exercise to the reader.

In what follows we will work with the semantics for Buridan modal models lacking the function $c : CONS \rightarrow D$ on page 122. This is in part to avoid confusion with the term C . In what follows, we will use the operation *Contra* to denote the contradictory of a proposition. Recall that our semantics are defined as follows:

Buridan Modal Model. A Buridan Modal Model is a tuple:
 $\mathfrak{M} = \langle D, W, R, O, v \rangle$ such that:

D and W are non-empty sets. D is the domain of objects and W is a set of worlds.

R $\subseteq W^2$ which is reflexive.

O : W $\rightarrow \mathcal{P}(D)$ s.t. $O(w) \subseteq D$

v : W $\times PRED \rightarrow \mathcal{P}(D)$

Semantic Abbreviations. Let P be a term, and Q either a term or the negation of a term. Using the semantics we can define the following operations:

$V(w, P) = O(w) \cap v(w, P)$

$V(w, \neg P) = D \setminus (O(w) \cap v(w, P))$

$M(w, Q) = \{d \in D : \text{there is some } z \text{ s.t. } wRz \text{ and } d \in V(z, Q)\}$

$L(w, Q) = \{d \in D : \text{for all } z \text{ if } wRz \text{ then } d \in V(z, Q)\}$

Using these operations we can define the truth for categorical propositions.

Assertoric Categorical Propositions.

$\mathfrak{M}, w \models AaB$ if and only if $V(w, A) \subseteq V(w, B)$ and $V(w, A) \neq \emptyset$

$\mathfrak{M}, w \models AeB$ if and only if $V(w, A) \cap V(w, B) = \emptyset$

$\mathfrak{M}, w \models AiB$ if and only if $V(w, A) \cap V(w, B) \neq \emptyset$

$\mathfrak{M}, w \models AoB$ if and only if $V(w, A) \not\subseteq V(w, B)$ or $V(w, A) = \emptyset$

Modal Categorical Propositions.

| | | |
|--|----------------|---|
| $\mathfrak{M}, w \models A \overset{L}{a} B$ | if and only if | $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{L}{e} B$ | if and only if | $M(w, A) \cap M(w, B) = \emptyset$ |
| $\mathfrak{M}, w \models A \overset{L}{i} B$ | if and only if | $M(w, A) \cap L(w, B) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{L}{o} B$ | if and only if | $M(w, A) \not\subseteq M(w, B)$ or $M(w, A) = \emptyset$ |
| $\mathfrak{M}, w \models A \overset{M}{a} B$ | if and only if | $M(w, A) \subseteq M(w, B)$ and $M(w, A) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{M}{e} B$ | if and only if | $M(w, A) \cap L(w, B) = \emptyset$ |
| $\mathfrak{M}, w \models A \overset{M}{i} B$ | if and only if | $M(w, A) \cap M(w, B) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{M}{o} B$ | if and only if | $M(w, A) \not\subseteq L(w, B)$ or $M(w, A) = \emptyset$ |
| $\mathfrak{M}, w \models A \overset{Q}{a} B$ | if and only if | $M(w, A) \subseteq M(w, B) \cap M(w, \neg B)$ and $M(w, A) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{Q}{e} B$ | if and only if | $\mathfrak{M}, w \models A \overset{Q}{a} B$ |
| $\mathfrak{M}, w \models A \overset{Q}{i} B$ | if and only if | $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{Q}{o} B$ | if and only if | $\mathfrak{M}, w \models A \overset{Q}{i} B$ |
| $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$ | if and only if | $M(w, A) \cap M(w, B) \cap M(w, \neg B) = \emptyset$ |
| $\mathfrak{M}, w \models A \overset{\bar{Q}}{e} B$ | if and only if | $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$ |
| $\mathfrak{M}, w \models A \overset{\bar{Q}}{i} B$ | if and only if | $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$ or $M(w, A) \neq \emptyset$ |
| $\mathfrak{M}, w \models A \overset{\bar{Q}}{o} B$ | if and only if | $\mathfrak{M}, w \models A \overset{\bar{Q}}{i} B$ |

Recall that we defined a syllogistic triple S , to be a triple $\langle M, m, C \rangle$ such that:

1. M , m , and C are all categorical formulae;
2. M , m , and C have exactly three terms;
3. The predicate of C occurs in M ;
4. The subject of C occurs in m ;
5. M and m share a common term that does not occur in C .¹

Syllogistic Validity. *A syllogistic triple S is valid (and called a ‘syllogism’) when the following obtains:*

For all Buridan Modal Models \mathfrak{M} and all worlds $w \in W$ if $\mathfrak{M}, w \models M$ and $\mathfrak{M}, w \models m$ then $\mathfrak{M}, w \models C$.

We will use the term ‘triple’ as a shorthand for ‘syllogistic triple’ to range over S , regardless of whether S is valid or not. We will use the term ‘figure’ here to refer to how the terms are arranged in the various formulae.

1. This definition is standard and can be found in [65].

Before moving onto our discussion of validity and invalidity, the following Reduction Lemma is *extremely* useful for reducing the number of cases that need to be considered for both validity and invalidity.

Lemma 1: Any triple in the second or third figure is equivalent to one triple in the first figure.

The proof of this lemma is an easy combinatorial exercise using *PPI* and *Interchange*. Recall that second and third figure triples are of the form:

| | Second Figure | Third Figure |
|---------------|--------------------------------|--------------------------------|
| Major Premise | $C \overset{\nabla}{\times} B$ | $B \overset{\nabla}{\times} C$ |
| Minor Premise | $A \overset{\nabla}{\times} B$ | $B \overset{\nabla}{\times} A$ |
| Conclusion | $A \overset{\nabla}{\times} C$ | $A \overset{\nabla}{\times} C$ |

To reduce a second figure syllogism or triple, assume that S is a triple. Then by *PPI*, S is equivalent to a triple S' of the form:

$$\frac{C \overset{\nabla}{\times} B}{\text{Contra}(A \overset{\nabla}{\times} C)} \\ \text{Contra}(A \overset{\nabla}{\times} B)$$

However, this is clearly now in the first figure where the term C is now the middle term, A is still the subject and B is now the predicate. Hence S' is a first figure triple that is equivalent to S .

To reduce a third figure triple, assume that S is a triple. Then by *PPI* S is equivalent to a triple of the form:

$$\frac{B \overset{\nabla}{\times} C}{\text{Contra}(A \overset{\nabla}{\times} C)} \\ \text{Contra}(B \overset{\nabla}{\times} A)$$

By *Interchange* this is equivalent to:

$$\frac{\text{Contra}(A \overset{\nabla}{\times} C)}{B \overset{\nabla}{\times} C} \\ \text{Contra}(B \overset{\nabla}{\times} A)$$

and by *PPI* this in turn is equivalent to:

$$\frac{\text{Contra}(A \overset{\nabla}{\times} C) \quad B \overset{\nabla}{\times} A^2}{\text{Contra}(B \overset{\nabla}{\times} C)}$$

which is clearly in the first figure where B is now the subject, C is still the predicate and A is the middle term.

As such, if we want to check that a particular syllogism is valid or invalid in the second or third figure, it suffices to check the corresponding syllogism in the first figure.

Validity

Our aim in this section is to treat the validity of the syllogisms in the three figures.

Recall that Read's tables, which are based on a textual analysis of Buridan's *Treatise on Consequences*, give the following syllogisms as valid[51, p.41,42,44]:

| | L | M | X | Q |
|---|------------------|---------------------|--|---------------------|
| L | L | L, M, Celarent X | Darii L, Fe- rio L, Bar- bara, Celar- ent X | L, M, Celarent X |
| M | M | M | Darii, Ferio, M | M |
| X | M, Celarent X | ∅ | Darii, Ferio M, | ∅ |
| Q | M, Q | M, Q | Darii, Ferio Q, | Q |

Table 10.1: Valid First Figure Syllogisms

| | L | M | X | Q |
|---|---------------------------------------|----------------------------------|--------------------------------------|-----------------------------|
| L | L | L,M, Cesare X, Camestres X | Festino L, Camestres, Baroco X | L,M, Cesare, Camestres X |
| M | L,M,Cesare, <i>Camestres</i> X, | ∅ | ∅ | ∅ |
| X | M, Cesare, Camestres X | ∅ | Festino M | ∅ |
| Q | M, Cesare, Camestres X | ∅ | ∅ | ∅ |

Table 10.2: Valid Second Figure Syllogisms

| | L | M | X | Q |
|---|-------------------------|-------------------------|--|-------------------------|
| L | L,X | L,M | Darapti X, Felapton X, Datisi X, Ferison X Darapti L, Felapton L, Datisi L, Ferison L | L,M |
| M | M | M | Darapti, Felapton, Datisi, Ferison M | M |
| X | Darapti X, Disamis X | Darapti M, Disamis M | Datisi M | Darapti M, Disamis M |
| Q | M, Q | Q | Disamis M, Bocardo M, Datisi Q, Ferison Q | Q |

Table 10.3: Valid Third Figure Syllogisms

We will first show that all of the first figure syllogisms are valid in our semantics. With this and the lemma proved above, we can then check the validities for the tables in the second and third figure. In cases where a particular syllogism or class of syllogisms follows from what we have already shown about single premise inferences, we will group the syllogisms together.

Before we do this, we should note a few obvious semantic consequences:

For all models \mathfrak{M} and all worlds, $w \in W$ we have:

1. $L(w, A) \subseteq V(w, A)$
2. $V(w, A) \subseteq M(w, A)$
3. $L(w, A) \subseteq M(w, A)$
4. $Q(w, A) \subseteq M(w, A)$

Proof of 1 and 2: Take an arbitrary model \mathfrak{M} and world $w \in W$. Assume that $c \in L(w, A)$, then for all v if wRv then $c \in V(v, A)$. Since R is reflexive, it follows that wRw , hence $c \in V(w, A)$ proving 2. Continuing on, since wRw and $c \in V(w, A)$, it follows by the definition of M that $c \in M(w, A)$, proving 1.

Proof 3: Obvious from the second part of the previous proof. Assume that $c \in V(w, A)$. Then since R is reflexive, we have wRw and hence $c \in M(w, A)$.

Proof 4: Obvious from the definition of Q . Take an arbitrary model \mathfrak{M} and world $w \in W$. Assume that $c \in Q(w, A)$. Then $c \in M(w, A) \cap M(w, \neg A)$ and so $c \in M(w, A)$. We will often implicitly appeal to these.

Syllogistic Validity & Triple Invalidity

In this section, we prove the required validities and invalidities for the various figures. The presentation of these results is organised as follows: In each subsection we will treat a particular class of modal propositions. The syllogisms will be grouped so as to cover multiple cases when the inferences are fairly trivial. For example, we group LLL and LLM together because L propositions always entail M propositions. Similarly for the case of Le to Xe and others. Likewise we group propositions such as LML and LQL together, since Q entails M. Because of how these sections are organised, there are some redundancies (for example, how we treat proving both LLL and LML) in our proofs, but there should be no omissions.

First Figure LLL/LLM-Syllogisms

First observe that in each case the LLM syllogism follows from the LLL syllogisms since we have $A \overset{L}{\times} B \vDash A \overset{M}{\times} B$. Since $A \overset{L}{e} B \vDash A e B$, this will handle the case for LLX Celarent.

LLL/LLM Barbara

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \vDash B \overset{L}{a} C$ and $\mathfrak{N}, w \vDash A \overset{L}{a} B$. Then it follows that 1) $M(w, B) \subseteq L(w, C)$, 2) $M(w, B) \neq \emptyset$, 3) $M(w, A) \subseteq L(w, B)$, and 4) $M(w, A) \neq \emptyset$.

We need to show that $M(w, A) \neq \emptyset$ and that $M(w, A) \subseteq L(w, C)$. The first conjunct follows from 4). For the second part of the conjunct observe that from 3) together with semantic consequence 3. (p. 213) it follows that $M(w, A) \subseteq M(w, B)$ and that this together with 1) entails $M(w, A) \subseteq L(w, C)$.

LLL/LLM/LLX Celarent

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \vDash B \overset{L}{e} C$ and $\mathfrak{N}, w \vDash A \overset{L}{a} B$. Then it follows that 1) $M(w, B) \cap M(w, C) = \emptyset$ and 2) $M(w, A) \subseteq L(w, B)$. It suffices to show that $M(w, A) \cap M(w, C) = \emptyset$. Assume that this isn't the case, then $M(w, A) \cap M(w, C) \neq \emptyset$. So take an arbitrary term d such that $d \in M(w, A)$ and $d \in M(w, C)$. By 2) it follows that $d \in M(w, B)$ and so $M(w, B) \cap M(w, C) \neq \emptyset$ contradicting 1).

LLL/LLM Darii

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \vDash B \overset{L}{a} C$ and $\mathfrak{N}, w \vDash A \overset{L}{i} B$. Then we have 1) $M(w, B) \subseteq L(w, C)$, 2) $M(w, B) \neq \emptyset$ 3) $M(w, A) \cap L(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \cap L(w, C) \neq \emptyset$. Take an arbitrary d such that $d \in M(w, A) \cap L(w, B)$. Then $d \in M(w, A)$ and $d \in L(w, B)$, and so $d \in M(w, B)$. By 1), it follows that $d \in L(w, C)$, which with some basic set theory completes the proof.

LLL/LLM Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \stackrel{L}{e} C$ and $\mathfrak{N}, w \models A \stackrel{L}{i} B$. Then we have 1) $M(w, B) \cap M(w, C) = \emptyset$, 2) $M(w, A) \cap L(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \not\subseteq L(w, C)$. Assume for reductio that $M(w, A) \subseteq L(w, C)$. By 2) there is some d such that $d \in M(w, A)$ and $d \in L(w, B)$. Based on our assumption it follows that $d \in L(w, C)$, contradicting 1).

First Figure LML/LMM/LQL/LQM-Syllogisms

There are two things to note here. First, recall that $A \stackrel{Q}{\times} B \models A \stackrel{M}{\times} B$. As such the LQL syllogisms will follow from the validity of the LML syllogisms and the validity of the LQM syllogisms will follow from the validity of the LMM syllogisms. Second, recall that $A \stackrel{L}{\times} B \models A \stackrel{M}{\times} B$. As such the validity of the LMM syllogisms will follow from the validity of the LML syllogisms. As such we will only treat the LML syllogisms.

The cases of LQX Celarent and LMX Celarent will follow from the validity of LML Celarent, since $A \stackrel{L}{e} B \models A e B$.

LML/LLM Barbara

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \stackrel{L}{a} C$ and $\mathfrak{N}, w \models A \stackrel{M}{a} B$. Then it follows that 1) $M(w, B) \subseteq L(w, C)$, 2) $M(w, B) \neq \emptyset$, 3) $M(w, A) \subseteq M(w, B)$, and 4) $M(w, A) \neq \emptyset$.

We need to show that $M(w, A) \neq \emptyset$ and that $M(w, A) \subseteq L(w, C)$. The first conjunct follows from 4). The second conjunct is immediate from 1) and 3).

LML Celarent

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \stackrel{L}{e} C$ and $\mathfrak{N}, w \models A \stackrel{M}{a} B$. Then it follows that 1) $M(w, B) \cap M(w, C) = \emptyset$ and 2) $M(w, A) \subseteq M(w, B)$. It suffices to show that $M(w, A) \cap M(w, C) = \emptyset$. Assume that this isn't the case, then $M(w, A) \cap M(w, C) \neq \emptyset$. So take an arbitrary term d such that $d \in M(w, A)$ and $d \in M(w, C)$. By 2) it follows that $d \in M(w, B)$ and so $M(w, B) \cap M(w, C) \neq \emptyset$ contradicting 1).

LML/LMM Darii

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \stackrel{L}{a} C$ and $\mathfrak{N}, w \models A \stackrel{M}{i} B$. Then we have 1) $M(w, B) \subseteq L(w, C)$, 2) $M(w, B) \neq \emptyset$ 3) $M(w, A) \cap M(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \cap L(w, C) \neq \emptyset$. Take an arbitrary d such that $d \in M(w, A) \cap M(w, B)$. Then $d \in M(w, A)$ and $d \in M(w, B)$. By 1), it follows that $d \in L(w, C)$, which with some basic set theory completes the proof.

LLL/LLM Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \stackrel{L}{e} C$ and $\mathfrak{N}, w \models A \stackrel{M}{i} B$. Then we have 1) $M(w, B) \cap M(w, C) = \emptyset$, 2) $M(w, A) \cap M(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \not\subseteq L(w, C)$. Assume for reductio that $M(w, A) \subseteq L(w, C)$. By 2) there is some d such that $d \in M(w, A)$ and $d \in M(w, B)$. Based on our assumption it

follows that $d \in M(w, C)$, contradicting 1). Hence $M(w, A) \not\subseteq L(w, C)$ and so $\mathfrak{N}, w \models A \overset{L}{o} C$. The proof establishing $M(w, A) \not\subseteq M(w, C)$ is analogous.

MLM

First Figure LXL/LXX-Syllogisms

LXX Barbara

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{L}{a} C$, and 3) $\mathfrak{N}, w \models A a B$. Then 1) $M(w, B) \subseteq L(w, C)$, 2) $V(w, A) \subseteq V(w, B)$ and $V(w, A) \neq \emptyset$. It suffices to show that $V(w, A) \neq \emptyset$ and $V(w, A) \subseteq V(w, C)$. The first conjunct follows from 3). For the second conjunct, take an arbitrary d such that $d \in V(w, A)$ then $d \in V(w, B)$, and so $d \in M(w, B)$. This, with 1) entails $d \in L(w, C)$ and so $d \in V(w, C)$, completing the proof.

LXX Celarent Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{L}{e} C$, and $\mathfrak{N}, w \models A a B$. Then 1) $M(w, B) \cap M(w, C) = \emptyset$, 2) $V(w, A) \subseteq V(w, B)$. We need to show that $V(w, A) \cap V(w, C) = \emptyset$. Assume this is not the case. Then there is some d such that $d \in V(w, A)$ and $d \in V(w, C)$. Then it follows that $d \in V(w, B)$, $d \in M(w, B)$ and $d \in M(w, C)$, contradicting 1).

LXL Darii

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{L}{a} C$, and $\mathfrak{N}, w \models A i B$. Then 1) $M(w, B) \subseteq L(w, C)$, 2) $V(w, A) \cap V(w, B) \neq \emptyset$. We claim that $M(w, A) \cap L(w, C) \neq \emptyset$. Take an arbitrary d such that $d \in V(w, A)$ and $d \in V(w, B)$ (we can do this since their intersection is non-empty). Then $d \in M(w, A)$ and $d \in M(w, B)$. By 1) it follows that $d \in L(w, C)$ and so $M(w, A) \cap L(w, C) \neq \emptyset$.

LXL Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{L}{e} C$, and $\mathfrak{N}, w \models A i B$. Then 1) $M(w, B) \cap M(w, C) = \emptyset$, 2) $V(w, A) \cap V(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \not\subseteq L(w, C)$. Assume that this is not the case. Then $M(w, A) \subseteq L(w, C)$. Using 2) take an arbitrary d such that $d \in V(w, A)$ and $d \in V(w, B)$. Then $d \in M(w, A)$. However, we also have $d \in M(w, B)$, which by our assumption entails that $d \in L(w, C)$ and so $d \in M(w, C)$. But then $M(w, B) \cap M(w, C) \neq \emptyset$, contradicting 1).

While LXX Darii and Ferio are not listed in Read's table, they are valid on our semantics and LXX Darii is needed for our second and third figure reductions as will be seen below. Buridan does not appear to explicitly mention these syllogisms when treating the first figure.

LXX Darii

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{L}{a} C$, and $\mathfrak{N}, w \models A i B$. Then 1) $M(w, B) \subseteq L(w, C)$ and 2) $V(w, A) \cap V(w, B) \neq \emptyset$. So, take an arbitrary d such that $d \in V(w, A)$ and $d \in V(w, B)$. Then $d \in M(w, B)$ and so $d \in L(w, C)$. Hence $d \in V(w, C)$ and so $d \in V(w, A) \cap V(w, C)$ as required.

LXX Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{L}{e} C$, and $\mathfrak{N}, w \models$

$A i B$. Then it follows that $M(w, B) \cap M(w, C) = \emptyset$ and that 1) $V(w, A) \cap V(w, B) \neq \emptyset$. So, take an arbitrary term d such that $d \in V(w, A)$ and $d \in V(w, B)$. Then it follows that $d \in M(w, B)$ and so $d \notin M(w, C)$ which entails $d \notin V(w, C)$. Hence $d \in V(w, A)$ and $d \notin V(w, C)$. From this it follows that $V(w, A) \not\subseteq V(w, C)$. Hence $\mathfrak{J}, w \models AoC$.

First Figure MMM/MLM Syllogisms and MXM Darii and MXM Ferio

Since $A \overset{L}{\times} B \models A \overset{M}{\times} B$, the MLM syllogisms will follow from the MMM syllogisms. Likewise since $AiB \models A \overset{M}{i} B$, this will cover the cases for MXM Darii and MXM Ferio.

MMM/MLM Barbara

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{M}{a} C$, and $\mathfrak{N}, w \models A \overset{M}{a} B$. Then it follows that $M(w, B) \subseteq M(w, C)$, $M(w, A) \subseteq M(w, B)$ and that $M(w, A) \neq \emptyset$. That $\mathfrak{N}, w \models A \overset{M}{a} C$ follows by the transitivity of \subseteq .

MMM/MLM Celarent

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{M}{e} C$, and $\mathfrak{N}, w \models A \overset{M}{a} B$. Then it follows that 1) $M(w, B) \cap L(w, C) = \emptyset$, 2) $M(w, A) \subseteq M(w, B)$, and that 3) $M(w, A) \neq \emptyset$. Now, take an arbitrary term such that $d \in M(w, A)$. Then by 2) $d \in M(w, B)$, which by 3) entails that $d \notin L(w, C)$. Hence $M(w, A) \cap L(w, C) = \emptyset$, as desired.

MMM/MLM Darii

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{M}{a} C$, and $\mathfrak{N}, w \models A \overset{M}{i} B$. Then it follows that 1) $M(w, B) \subseteq M(w, C)$ and 2) $M(w, A) \cap M(w, B) \neq \emptyset$. Using 1), take an arbitrary d such that $d \in M(w, A) \cap M(w, B)$. Then by 2), $d \in M(w, C)$. That $\mathfrak{N}, w \models A \overset{M}{i} C$ follows from the semantic definitions.

MMM/MLM Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{M}{e} C$, and $\mathfrak{N}, w \models A \overset{M}{i} B$. Then it follows that 1) $M(w, B) \cap L(w, C) = \emptyset$ and that 2) $M(w, A) \cap M(w, B) \neq \emptyset$. Assume for reduction that 3) $M(w, A) \subseteq L(w, C)$. Then by 2) it follows that there is some d such that $d \in M(w, A)$ and $d \in M(w, B)$. From the first conjunct and 3) it follows that $d \in L(w, C)$. Hence $M(w, B) \cap L(w, C) \neq \emptyset$, contradicting 1). Hence $M(w, A) \not\subseteq L(w, C)$. It then follows by basic logic that $\mathfrak{N}, w \models A \overset{M}{o} C$ as desired. The case for MLM Ferio is follows by modal subalternation.

First Figure MQM-Syllogisms

These follow from the MMM syllogisms, since Q formulae entail M formulae, and the MMM syllogisms are all valid (as shown above).

First Figure XLM/XLX-Syllogisms

XLM Barbara

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B a C$, and $\mathfrak{N}, w \models A \overset{L}{a} B$. Then it follows that 1) $V(w, B) \subseteq V(w, C)$, 2) $M(w, A) \subseteq L(w, B)$ and 3) $M(w, A) \neq \emptyset$. Together with 3) it suffices to show that $M(w, A) \subseteq M(w, C)$. So, assume that $d \in M(w, A)$. Then by 2) $d \in L(w, B)$. Hence $d \in V(w, B)$ since wRw always holds. Then by 1) it follows that $d \in V(w, C)$ and so $d \in M(w, C)$, which is what we wanted to show.

XLM/XLX Celarent XLM Celarent is very similar.

For XLX Celarent, let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B e C$, and $\mathfrak{N}, w \models A \overset{L}{a} B$. Then it follows that 1) $V(w, B) \cap V(w, C) = \emptyset$ and 2) $M(w, A) \subseteq L(w, B)$.

It suffices to show that $V(w, A) \cap V(w, C) = \emptyset$. Assume that this is false, then there is some d such that $d \in V(w, A)$ and $d \in V(w, C)$. Since wRw it follows that $d \in M(w, A)$ and so by 2) that $d \in L(w, B)$ and so, $d \in V(w, B)$. But then $V(w, B) \cap V(w, C) \neq \emptyset$ contradicting 1).

For XLM Celarent, let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B e C$, and $\mathfrak{N}, w \models A \overset{L}{a} B$. Then it follows that 1) $V(w, B) \cap V(w, C) = \emptyset$ and 2) $M(w, A) \subseteq L(w, B)$.

It suffices to show that $M(w, A) \cap L(w, C) = \emptyset$. Assume that this is false, then there is some d such that $d \in M(w, A)$ and $d \in L(w, C)$. Since wRw it follows that $d \in V(w, C)$. By 2) it also follows that $d \in L(w, B)$ and so, $d \in V(w, B)$. But then $V(w, B) \cap V(w, C) \neq \emptyset$ contradicting 1).

XLM Darii

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B a C$, and $\mathfrak{N}, w \models A \overset{L}{i} B$. Then it follows that 1) $V(w, B) \subseteq V(w, C)$ and 2) $M(w, A) \cap L(w, B) \neq \emptyset$. Using 2) Take an arbitrary d such that $d \in M(w, A)$ and $d \in L(w, B)$. Since wRw always holds, it follows that $d \in V(w, B)$. Hence by 1), it follows that $d \in V(w, C)$ and so $d \in M(w, C)$. Hence $M(w, A) \cap M(w, C) \neq \emptyset$ and so $\mathfrak{N}, w \models A \overset{M}{i} C$.

XLM Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B e C$, and $\mathfrak{N}, w \models A \overset{L}{i} B$. Then it follows that 1) $V(w, B) \cap V(w, C) = \emptyset$ and 2) $M(w, A) \cap L(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \not\subseteq L(w, C)$, so assume that $M(w, A) \subseteq L(w, C)$. Using 2) take an arbitrary d such that 3) $d \in M(w, A)$ and 4) $d \in L(w, B)$. It then follows by 3) and our assumption that $d \in L(w, C)$. Since wRw always holds, it follows that $d \in V(w, C)$ and (from 4)) $d \in V(w, B)$. But then $V(w, B) \cap V(w, C) \neq \emptyset$, contradicting 1).

First Figure XXM Syllogisms

XXM Darii

This follows from XXX Darii since $AiB \models A \overset{M}{i} B$.

XXM Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \text{ e } C$, and $\mathfrak{N}, w \models A \text{ i } B$. Then 1) $V(w, B) \cap V(w, C) = \emptyset$ and 2) $V(w, A) \cap V(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \not\subseteq L(w, C)$. Assume that $M(w, A) \subseteq L(w, C)$. Using 2), take an arbitrary d such that 3) $d \in V(w, A)$ and 4) $d \in V(w, B)$. Then it follows from 3) that $d \in M(w, A)$. From 4) and 1) it follows that $d \notin V(w, C)$ and so $d \notin L(w, C)$. However, by 3) and our assumption, it follows that $d \in L(w, C)$, which is a contradiction. Hence $M(w, A) \not\subseteq L(w, C)$ as desired.

First Figure QMQ Syllogisms

The QQQ, QQM, and QMM syllogisms will all follow from the QMQ syllogisms since Q formulae entail M formulae. The QLQ syllogisms will follow since L formulae entail M formulae. In the cases of QXQ and QXM both Darii and Ferio will follow from QMQ since $AiB \models A \overset{M}{i} B$.

QMQ Barbara and QMQ Celarent

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{Q}{a} C$, and $\mathfrak{N}, w \models A \overset{M}{a} B$. Then $M(w, B) \subseteq (M(w, C) \cap M(w, \neg C))$, $M(w, A) \subseteq M(w, B)$, and $M(w, A) \neq \emptyset$. That $\mathfrak{N}, w \models A \overset{Q}{a} C$, follows from what we just observed by the transitivity of \subseteq .

Celarent follows because $\overset{Q}{a}$ and $\overset{Q}{e}$ are equivalent.

QMQ Darii and QMQ Ferio

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{Q}{a} C$, and $\mathfrak{N}, w \models A \overset{M}{i} B$. Then 1) $M(w, B) \subseteq (M(w, C) \cap M(w, \neg C))$ and 2) $M(w, A) \cap M(w, B) \neq \emptyset$. Using 2) take an arbitrary d such that $d \in M(w, A)$ and $d \in M(w, B)$. From 1) it follows that $d \in (M(w, C) \cap M(w, \neg C))$. Hence $(M(w, A) \cap (M(w, C) \cap M(w, \neg C))) \neq \emptyset$. Hence $\mathfrak{N}, w \models A \overset{Q}{i} C$.

Second and Third Figure Reductions

Given *PPI* and *Interchange*, we need to reduce the second and third figures to the first figure. In the case of the second figure, if we say that this is done using *PPI*, we simply mean that one instance of the rule for *PPI* has been used. In the third figure, if we are reducing the syllogism to the second figure, then when we say that ‘it follows by *PPI* and *Interchange*’ or similar expressions, we mean that *PPI* is applied once and then the major and minor premises are inverted using *Interchange*. If we are reducing the syllogism to the first figure, what we mean by *PPI* and *Interchange* is what was shown in the Reduction Lemma.

Again, we will do this by groups of syllogisms. First, observe that in each combination of modals, Cesare can be reduced to Celarent by converting the major premise (which we have already shown to be valid in Chapter Four).

10 Appendix Three: Buridan's Modal Syllogism

Camestres can be reduced to Darii by *PPI*. For Camestres we need to check the following cases:

LLL, LLM, LML, LMM, LMX, LXX, LQL, LQM, LQX, MLL, MLM, XLM, XLX, QLM, and QLX. We can simplify this to the following cases:

LLM:

LLM Camestres is of the form $C \overset{L}{a} B, A \overset{L}{e} B \vDash A \overset{M}{e} C$. By *PPI* this is valid if and only if $C \overset{L}{a} B, A \overset{L}{i} C \vDash A \overset{M}{i} B$, i.e. if and only if LLM Darii is valid. LLM Darii is valid, etc.

LMM:

LMM Camestres is of the form $C \overset{L}{a} B, A \overset{M}{e} B \vDash A \overset{M}{e} C$. By *PPI* this is valid if and only if $C \overset{L}{a} B, A \overset{L}{i} C \vDash A \overset{L}{i} B$, i.e. if and only if LML Darii is valid. LML Darii is valid, etc.

LMX LMX Camestres is of the form $C \overset{L}{a} B, A \overset{M}{e} B \vDash A e C$. By *PPI* this is valid if and only if $C \overset{L}{a} B, A \overset{X}{i} C \vDash A \overset{L}{i} B$, i.e. if and only if LXL Darii is valid. LXL Darii is valid, etc.

LXX

LXX Camestres is of the form $C \overset{L}{a} B, A e B \vDash A e C$. By *PPI* this is valid if and only if $C \overset{L}{a} B, A \overset{X}{i} C \vDash A i B$, i.e. if and only if LXX Darii is valid. But LXX Darii is valid, etc.

MLL:

MLL Camestres is of the form $C \overset{M}{a} B, A \overset{L}{e} B \vDash A \overset{L}{e} C$. By *PPI* this is valid if and only if $C \overset{M}{a} B, A \overset{M}{i} C \vDash A \overset{M}{i} B$, i.e. if and only if MMM Darii is valid. But MMM Darii is valid, etc.

XLM XLM Camestres is of the form $C a B, A \overset{L}{e} B \vDash A \overset{M}{e} C$. By *PPI* this is valid if and only if $C a B, A \overset{L}{i} C \vDash A \overset{M}{i} B$, i.e. if and only if XLM Darii is valid. But XLM Darii is valid, etc.

XLX XLX Camestres is of the form $C a B, A \overset{L}{e} B \vDash A e C$. By *PPI* this is valid if and only if $C a B, A i C \vDash A \overset{M}{i} B$ i.e. if and only if XXM Darii is valid. But XXM Darii is valid, etc.

MLM MLM Camestres is of the form $C \overset{M}{a} B, A \overset{L}{e} B \vDash A \overset{M}{e} C$. By *PPI* this is valid if and only if $C \overset{M}{a} B, A \overset{L}{i} C \vDash A \overset{M}{i} B$, i.e. if and only if MLM Darii is valid. But MLM Darii is valid, etc.

QLX

QLX Camestres is of the form $C \overset{Q}{a} B, A \overset{L}{e} B \vDash A e C$. By *PPI* this is valid if and only if $C \overset{Q}{a} B, A i C \vDash A \overset{M}{i} B$, i.e. if and only if QXM Darii is valid. But QXM Darii is valid, etc.

This completes the cases for Camestres.

For Festino we need to consider the modal combinations: LLL, LML, LMM, LXL, LQL, LQM, MLL, MLM, XLM, XXM, and QLM. Again, this can be simplified to the

following cases: LML, LXL, LQM, MLL, XLM, XXM, and QLM.

By simple conversion, we can reduce Festino to Ferio. Hence we need to verify that: LML, LXL, LQM, MLL, XLM, XXM, and QLM Ferio are all valid, which can be easily seen by cross-referencing the table and our first figure proofs.

For Baroco we use *PPI* to reduce it to Barbara. We have the following modal combinations to consider:

LLL, LLM, LML, LMM, LXX, LQL, LQM, MLL, MLM, XLM, and QLM

Again we can simplify this to the following inferences:

LML, LXX, MLL, XLM, LQL, and QLM

LML

LML Baroco is of the form $C \overset{L}{a} B, A \overset{M}{o} B \vDash A \overset{L}{o} C$. By *PPI* this is valid if and only if $C \overset{L}{a} B, A \overset{M}{a} C \vDash A \overset{L}{a} B$ i.e. if and only if LML Barbara is valid. But LML Barbara is valid, etc.

For the remaining cases excluding LQL and QLM, it suffices to show that:

LXX, LMM, MMM, and XLM Barbara are valid. To see this, observe that from the validity of LML Barbara, we also have the validity of LMM Barbara, LLM Barbara, MLM Barbara, LLL Barbara, and MLL Barbara since L implies M. By inspection of the Read's table and our proofs, these three clearly hold.

LQL Baroco:

This follows from LML Baroco, observing that $A \overset{Q}{o} B$ entails $A \overset{M}{i} B$.

QLM Baroco:

This follows from MLM Baroco, observing that $C \overset{Q}{a} B$ entails $A \overset{M}{a} B$.

This completes the treatment of Baroco and of the second figure.

For the third figure we need to consider the following syllogisms:

| | L | M | X | Q |
|---|-------------------------|-------------------------|---|-------------------------|
| L | L,X | L,M | X,Darapti L, Felapton L, Datisi L, Ferison L | L,M |
| M | M | M | Darapti, Felapton, Datisi, Ferison M | M |
| X | Darapti X, Disamis X | Darapti M, Disamis M | Datisi M | Darapti M, Disamis M |
| Q | M, Q | Q | Disamis M, Bocardo M, Datisi Q, Ferison Q | Q |

Table 10.4: Valid Third Figure Syllogisms

Before continuing, there is one small caveat that needs to be mentioned here. In the table, the justification for the validity of the LXX figures is based on a reference to Thom's theorem 9.10b, which is derived from 9.10a and was shown to be invalid in Chapter Four for XXM Barbara and XXM Celarent.[51, p.44, fn.e] Using *PPI* and *Interchange* it can be easily checked that XXM Celarent reduces to LXX Ferison and that XXM Barbara reduces to LXX Bocardo and, hence, both of these will be invalid on our semantics. As there is no other reference to Buridan claiming these, we will omit both from our treatment of Buridan.

First, all of the cases for Ferison reduce to Festino by *PPI* and *Interchange*. According to the table, we then need to check the following cases for Ferison: LLL, LLM, LLX, LML, LMM, LXX, LXL, LQL, LQM, MLM, MMM, MXM, MQM, QLM, QLQ, QMQ, QXQ, and QQQ.

We can simplify this to the following modal combinations: LLL, LLX, LML, LXX, MLM, MMM, MXM, QMQ, and QQQ.

We will prove the contingency cases directly; for the rest we reduce them using *PPI* and Inversion.

In the case of Ferison, we need to consider: MLM, XLM, MLL, LMM, LML, and MLX Ferison.

All of these except for MLX Ferison have already been reduced. LMX Ferison can easily be shown to reduce to LXL Darii.

For Datisi we need to check the following cases: LLL, LLM, LLX, LML, LMM, LXL, LQL, LQM, MLM, MMM, MXM, XLX, XMM, XXM, QLM, QLQ, QMQ, QXQ, and QQQ.

We can reduce this to the following modal combinations: LLX, LML, MMM, XLX, XXM, and QMQ

We will prove the contingency modals directly and reduce the rest to the first figure.

LLX Datisi reduces to XLM Ferio, which we have shown to be valid.
 LML Datisi reduces to LML Celarent, which we have shown to be valid.
 MMM Datisi reduces to LMM Darii, which we have shown to be valid.
 XLX Datisi reduces to XXM Celarent, which we have shown to be valid.
 XXM Datisi reduces to LXX Ferio, which we have shown to be valid.

We prove the contingency case directly: **QMQ Datisi**

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{Q}{a} C$ and $\mathfrak{N}, w \models A \overset{M}{i} B$. Then we have 1) $M(w, B) \subseteq Q(w, C)$, 2) $M(w, B) \neq \emptyset$ 3) $M(w, A) \cap M(w, B) \neq \emptyset$. It suffices to show that $M(w, A) \cap Q(w, C) \neq \emptyset$. Take an arbitrary d such that $d \in M(w, A) \cap M(w, B)$. Then $d \in M(w, A)$ and $d \in M(w, B)$. So by 1) it follows that $d \in Q(w, C)$ which suffices to prove the claim.

For Disamis we need to check the following cases: LLL, LLM LLX, LML, LMM, LXX, LQL, LQM, MLM, MMM, MQM, XLX, XMM, XQM, QLM, QLQ, QMQ, QXM, QQQ

Again, we can simplify this to the following modal combinations:
 LML, LLX, LXX, MMM, XLX, QMQ, and QXM.

We will reduce these to the first figure using *PPI* and *Interchange*.

LML Disamis reduces to MLM Celarent, which we have already seen is valid.
 LLX Disamis reduces to XLM Celarent, which we have already seen is valid.
 LXX Disamis reduces to XXM Celarent and is one of the syllogisms we are not treating because the justification for inclusion rests on Thom's mistake.
 MMM Disamis reduces to LML Celarent, which we have shown to be valid.
 XLX Disamis reduces to XLX Celarent, which we have shown to be valid.

We prove the contingency cases directly:

Disamis QMQ

Let \mathfrak{N} be an arbitrary model and let $w \in W$. Assume that $\mathfrak{N}, w \models B \overset{Q}{i} C$ and $\mathfrak{N}, w \models B \overset{M}{a} A$. Then we have 1) $M(w, B) \cap Q(w, C) \neq \emptyset$, 2) $M(w, B) \subseteq M(w, A)$. It suffices to show that $M(w, A) \cap Q(w, C) \neq \emptyset$. Take an arbitrary d such that $d \in M(w, B) \cap Q(w, C)$. Then $d \in M(w, B)$ and $d \in Q(w, C)$. Then by 1) it follows that $d \in M(w, A)$ which is what we needed to prove.

Disamis QXM Read's table states that SD 5.7.4 rule six gives counterexamples to this claim. [51, p.44 fn. z] Our logic agrees with this, rejecting this as invalid. Consider the following countermodel:

| | |
|-----------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = a$ | $v(x, A) = \emptyset$ |
| $v(w, B) = a$ | $v(x, B) = b$ |
| $v(w, C) = \emptyset$ | $v(x, C) = b$ |

Call this model \mathfrak{J} . First, observe that $V(w, A) = V(w, B) = \{a\}$ and so $V(w, B) \subseteq V(w, A)$ and $V(w, B) \neq \emptyset$. Hence $\mathfrak{J}, w \models BaA$. Likewise, observe that since wRx and $V(x, B) = \{b\}$ we have $b \in M(w, B)$ and $b \in M(w, C)$. Likewise, since wRw and $b \notin V(w, C)$ we have $b \in M(w, \neg C)$. Hence $b \in Q(w, C)$ and so $M(w, B) \cap Q(w, C) \neq \emptyset$ and so $\mathfrak{J}, w \models B \overset{Q}{i} C$. However, observe that $M(w, A) \cap M(w, C) = \emptyset$ since $b \notin M(w, A)$ and $a \notin M(w, C)$. Hence $\mathfrak{J}, w \not\models A \overset{M}{i} C$.

For Bocardo we need to consider the following cases:

LLL, LLX, LML, LMM, LXX, LQL, LQM, MLM, MMM, MQM, QLM, QLQ, QMQ, QXM, QQQ

This can be simplified to the following cases: LML, LLX, LXX, MMM, QMQ, QMX, QQQ

As before, we prove the modal cases without contingency by reduction and prove the contingency cases directly.

LML Bocardo reduces to MMM Barbara, which we have already shown is valid.

LLX Bocardo reduces to XLM Barbara, which we have already shown is valid.

LXX Bocardo reduces to XXM Barbara, which is one of the syllogisms we are not treating because the justification for inclusion rests on Thom's mistake.

MMM Bocardo reduces to LML Barbara, which we have shown to be valid.

For the contingency cases observe that since $A \overset{O}{o} B$ is equivalent to $A \overset{Q}{i} B$, and with the exception of QXM Baroco, each of these syllogisms is a form of Disamis, which we have already shown to be valid. QXM Baroco is invalid on our semantics. As in the case of QXM Disamis, our semantics agree with Buridan's observations in SD 5.7.4 that this triple is invalid, as is clear from the following countermodel:

Consider the following countermodel:

| | |
|----------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = a$ | $v(x, A) = \emptyset$ |
| $v(w, B) = b$ | $v(x, B) = b$ |
| $v(w, C) = a$ | $v(x, C) = D$ |

Call this model \mathfrak{J} . First, observe that $V(w, B) \not\subseteq V(w, A)$. As such $\mathfrak{J}, w \models BoA$. Second, observe that $M(w, B) \subseteq M(w, C) \cap M(w, \neg C)$. This follows since $b \in V(x, C)$ and $b \notin V(w, C)$. However, observe that $M(w, A) \subseteq L(w, C)$. To see this, first observe $a \in M(w, A)$ (since $a \in V(w, A)$) and that $a \in L(w, C)$ since $a \in V(x, C)$ and $a \in V(w, C)$. Second, observe that $b \notin M(w, A)$. Hence $\mathfrak{J}, w \not\models A \overset{M}{o} C$

This completes our treatment of the information presented in Read's tables and this

suffices to verify that all of the inferences that Buridan accepts as valid are also valid in our logic.

Invalidity

Our aim in this section is to show that every inference that Buridan claims is invalid is also invalid in our logic. This is a large combinatorial task. In this appendix, we will do two things to shorten our discussion and save unnecessary repetition. First, we will only discuss syllogisms that Buridan discusses. As such, we will not consider modal combinations where the underlying assertoric syllogism is invalid, e.g. the triple $B \overset{L}{i} C, A \overset{L}{i} B \neq A \overset{L}{i} C$. Second, we will produce countermodels for all of the first figure triples that are invalid and exhaust the space of possible combinations. To that end, we need to show that the following are invalid:

| | L | M | X | Q |
|---|-----------------------------------|------------------------|--|---|
| L | Barbara X, Darii X, Q, X, Q | X, Q, Darii X, Ferio X | Barbara X, Darii X, Ferio X | Barbara L, Celarent L, Celarent M, Darii X, Ferio X |
| M | L, Q, X | L, Q, X | L, Q, X, Barbara M, Celarent M | L, Q, X |
| X | L, Q, Barbara X, Darii X, Ferio X | L, M, Q, X | L, Q, Barbara M, Celarent M | L, M, Q, X |
| Q | L, X | L, X | L, Barbara Q, Celarent Q, Barbara M, Celarent M, X | L, X |

I.e. we need to show that the following are all invalid:

LLQ, LMQ, LQQ, MLL, MLQ, MLX, MML, MMQ, MMX, MXL, MXQ, MXX, MQL, MQQ, MQX XLL, XLQ, XML, XMM, XMQ, XMX, XXL, XXQ, XQL, XQM, XQQ, XQX, QLL, QLX, QML, QMX, QXL, QXX, QQL, and QQX
as well as:

LLX Barbara, LLX Darii, LLX Ferio, LMX Barbara, LMX Darii, LMX Ferio, LXL Barbara, LXL Celarent, LXM Barbara, LXM Celarent, MXM Barbara, MXM Celarent, XLX Barbara, XLX Darii, XLX Ferio, XXM Barbara, XXM Celarent, QXQ Barbara, QXQ Celarent, QXM Barbara, and QXM Celarent.

Recall that LXL Barbara and LXL Celarent were treated in Chapter Five, starting on page 114.

At this point we can cull the number of cases we need to consider down to an easier

number by recalling a number of properties of single premise inferences (which we also used in our treatment of validity):

1. $A \overset{L}{\times} B \vDash A \overset{M}{\times} B$
2. $A \overset{Q}{\times} B \vDash A \overset{M}{\times} B$
3. $A \overset{i}{\vDash} B \vDash A \overset{M}{i} B$
4. $A \overset{L}{e} B \vDash A \overset{e}{\vDash} B$
5. $A \overset{L}{a} B \vDash A \overset{L}{i} B$
6. $A \overset{L}{e} B \vDash A \overset{L}{o} B$
7. $A \overset{Q}{a} B \vDash A \overset{Q}{i} B$
8. $A \overset{Q}{e} B \vDash A \overset{Q}{o} B$
9. $A \overset{M}{a} B \vDash A \overset{M}{i} B$
10. $A \overset{M}{e} B \vDash A \overset{M}{o} B$

With that in mind we provide the countermodels as follows:

LLQ, LMQ, MMQ, MLQ, XXQ, MXQ, XMQ Triples

Consider the following countermodel:

| | |
|-----------------------------------|------------|
| $W = \{w\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = D$ | |

Call this model \mathfrak{J} . Then clearly, $M(w, A) = M(w, B) = M(w, C) = D$ and similarly for L and V. Hence, $\mathfrak{J}, w \vDash B \overset{L}{a} C$, $\mathfrak{J}, w \vDash A \overset{L}{a} B$, and $\mathfrak{J}, w \vDash A \overset{L}{a} C$ as well as $\mathfrak{J}, w \vDash B \overset{L}{i} C$, $\mathfrak{J}, w \vDash A \overset{L}{i} B$, and $\mathfrak{J}, w \vDash A \overset{L}{i} C$.

Further we also have: $\mathfrak{J}, w \vDash B \overset{M}{a} C$, $\mathfrak{J}, w \vDash A \overset{M}{a} B$, and $\mathfrak{J}, w \vDash A \overset{M}{a} C$ as well as $\mathfrak{J}, w \vDash B \overset{M}{i} C$, $\mathfrak{J}, w \vDash A \overset{M}{i} B$, and $\mathfrak{J}, w \vDash A \overset{M}{i} C$.

We also have: Further we also have: $\mathfrak{J}, w \vDash B \overset{Q}{a} C$, $\mathfrak{J}, w \vDash A \overset{Q}{a} B$, and $\mathfrak{J}, w \vDash A \overset{Q}{a} C$ as well as $\mathfrak{J}, w \vDash B \overset{Q}{i} C$, $\mathfrak{J}, w \vDash A \overset{Q}{i} B$, and $\mathfrak{J}, w \vDash A \overset{Q}{i} C$.

But $\mathfrak{J}, w \not\vDash A \overset{Q}{i} C$ and $\mathfrak{J}, w \not\vDash A \overset{Q}{a} C$ as $M(w, -C) = \emptyset$. This suffices to give countermodels for LLQ, LMQ, MMQ, MLQ, XXQ Barbara and LLQ, LMQ, MMQ, MLQ, XXQ Darii.

Consider the following countermodel:

| | |
|-------------------------|-----------------------|
| $W = \{w\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = D$ |
| $v(w, A) = v(w, B) = D$ | $v(w, C) = \emptyset$ |

Call this model \mathfrak{J} . Then clearly, $M(w, A) = M(w, B) = D$ and $M(w, C) = \emptyset$ and similarly for L and V. Hence, $\mathfrak{J}, w \models B \overset{L}{e} C$, $\mathfrak{J}, w \models A \overset{L}{a} B$, and $\mathfrak{J}, w \models A \overset{L}{e} C$ as well as $\mathfrak{J}, w \models B \overset{L}{o} C$, $\mathfrak{J}, w \models A \overset{L}{i} B$, and $\mathfrak{J}, w \models A \overset{L}{o} C$.

Further we also have: $\mathfrak{J}, w \models B \overset{M}{e} C$, $\mathfrak{J}, w \models A \overset{M}{a} B$, and $\mathfrak{J}, w \models A \overset{M}{e} C$ as well as $\mathfrak{J}, w \models B \overset{M}{o} C$, $\mathfrak{J}, w \models A \overset{M}{i} B$, and $\mathfrak{J}, w \models A \overset{M}{o} C$.

We also have: $\mathfrak{J}, w \models B e C$, $\mathfrak{J}, w \models A a B$, and $\mathfrak{J}, w \models A e C$ as well as $\mathfrak{J}, w \models B o C$, $\mathfrak{J}, w \models A i B$, and $\mathfrak{J}, w \models A o C$.

But $\mathfrak{J}, w \not\models A \overset{Q}{i} C$ and $\mathfrak{J}, w \not\models A \overset{Q}{a} C$ as $M(w, C) = \emptyset$. This suffices to give countermodels for LLQ, LMQ, MMQ, MLQ, XXQ Celarent and LLQ, LMQ, MMQ, MLQ, MXQ, XXQ Ferio.

QQL, QQX, MQL, MQX, XXL, and XQL Triples

Consider the following countermodel:

| | |
|---|-------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = a, b$ | $O(w) = O(x) = D$ |
| $v(x, A) = v(x, B) = v(x, C) = D$ | |
| $v(w, A) = v(w, B) = v(w, C) = \emptyset$ | |

Call this model \mathfrak{J} . Clearly $M(w, A) = M(w, B) = M(w, C) = D$ and similarly for Q. Also notice that $L(w, A) = L(w, B) = L(w, C) = \emptyset$. Hence $\mathfrak{J}, w \models B \overset{Q}{a} C$ and $\mathfrak{J}, w \models B \overset{Q}{i} C$. Likewise, $\mathfrak{J}, w \models A \overset{Q}{a} B$ and $\mathfrak{J}, w \models A \overset{Q}{i} B$.

In addition we also have $\mathfrak{J}, w \models B \overset{M}{a} C$ and $\mathfrak{J}, w \models B \overset{M}{i} C$. Likewise, $\mathfrak{J}, w \models A \overset{M}{a} B$ and $\mathfrak{J}, w \models A \overset{M}{i} B$.

However, observe that $\mathfrak{J}, w \not\models A \overset{L}{a} C$ and $\mathfrak{J}, w \not\models A \overset{L}{i} C$ since $L(w, C) = \emptyset$ and $M(w, A) \neq \emptyset$. Similarly, because of v, it follows that $\mathfrak{J}, w \not\models A a C$ and $\mathfrak{J}, w \not\models A i C$.

This suffices to give countermodels for QQL, QQX, MQL, MQX, XXL, and XQL Barbara and QQL, QQX, MQL, MQX, XXL, and XQL Darii.

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{b\}$ | $v(w, B) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . Clearly $M(w, A) = M(w, B) = M(w, C) = D$ and similarly for Q. Also notice that $L(w, A) = L(w, B) = L(w, C) = \emptyset$. Hence $\mathfrak{J}, w \models B \overset{Q}{e} C$ and $\mathfrak{J}, w \models B \overset{Q}{o} C$. Likewise, $\mathfrak{J}, w \models A \overset{Q}{e} B$ and $\mathfrak{J}, w \models A \overset{Q}{o} B$.

In addition we also have $\mathfrak{J}, w \models B \overset{M}{a} C$ and $\mathfrak{J}, w \models B \overset{M}{i} C$. Likewise, $\mathfrak{J}, w \models A \overset{M}{a} B$ and $\mathfrak{J}, w \models A \overset{M}{i} B$.

And since $v(w, B) = \emptyset$, we also have $\mathfrak{J}, w \models B e C$ and $\mathfrak{J}, w \models B o C$. Likewise, $\mathfrak{J}, w \models A e B$ and $\mathfrak{J}, w \models A o B$.

However, observe that $\mathfrak{J}, w \models A \overset{M}{i} B$ and that $\mathfrak{J}, w \models A \overset{M}{a} B$. Hence, $\mathfrak{J}, w \not\models A \overset{L}{e} B$ and $\mathfrak{J}, w \not\models A \overset{L}{o} B$. Further, observe that we also have $\mathfrak{J}, w \models A i B$ and $\mathfrak{J}, w \models A a B$. Hence $\mathfrak{J}, w \not\models A e B$ and $\mathfrak{J}, w \not\models A o B$ as desired.

This suffices to give countermodels for QQL, QQX, MQL, MQX, XXL, and XQL Celarent and QQL, QQX, MQL, MQX, XXL, and XQL Ferio.

LQQ and MQQ Triples

For Barbara and Darii:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . First, observe that $\mathfrak{J}, w \models A \overset{Q}{a} B$ since $a \in M(w, A)$ and $a \in M(w, B)$ and $a \notin L(w, B)$. Further observe that $\mathfrak{J}, w \models B \overset{L}{a} C$ since $a \in M(w, B)$ and $a \in L(w, C)$. Based on this it is also easy to see that $\mathfrak{J}, w \models A \overset{Q}{i} B$ $\mathfrak{J}, w \models B \overset{L}{i} C$.

However, observe that $M(w, A) \subseteq L(w, C)$ per construction of the model. Hence $\mathfrak{J}, w \models A \overset{L}{a} C$ and $\mathfrak{J}, w \models A \overset{L}{i} C$. From this it follows that $\mathfrak{J}, w \not\models A \overset{Q}{a} C$ and $\mathfrak{J}, w \not\models A \overset{Q}{i} C$ as required.

For Celarent and Ferio:

| | |
|---|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \{a\}$ | $v(w, C) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = \emptyset$ | |

Call this model \mathfrak{J} . Call this model \mathfrak{J} . First, observe that $\mathfrak{J}, w \models A \overset{Q}{a} B$ since $a \in M(w, A)$ and $a \in M(w, B)$ and $a \notin L(w, B)$. Further observe that $\mathfrak{J}, w \models B \overset{L}{e} C$ since $a \in M(w, B)$ and $a \notin M(w, C)$. Based on this it is also easy to see that $\mathfrak{J}, w \models A \overset{Q}{i} B$ $\mathfrak{J}, w \models B \overset{L}{o} C$.

However, observe that $M(w, A) \not\subseteq M(w, C) \cap M(w, -C)$ since $a \notin M(w, C)$. Hence $\mathfrak{J}, w \not\models A \overset{Q}{e} C$ and $\mathfrak{J}, w \not\models A \overset{Q}{o} C$.

QLL, QLX, QML, QMX, QXL, and QXX Triples

For Barbara, Darii:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \{a\}$ | $v(w, C) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . Now, observe that since $a \in V(w, A)$ and $a \in V(w, B)$ it follows that $a \in M(w, A)$ and $a \in M(w, B)$. Further, $a \in L(w, A)$, $a \in L(w, B)$, $a \in M(w, C)$ and $a \notin L(w, C)$. Hence $\mathfrak{J}, w \models B \overset{Q}{a} B$ and $\mathfrak{J}, w \models A \overset{L}{a} B$. We also have $\mathfrak{J}, w \models A \overset{Q}{i} B$ and $\mathfrak{J}, w \models A \overset{L}{i} B$.

Further, since $\overset{Q}{a}$ and $\overset{Q}{e}$ are equivalent, and because $\overset{Q}{i}$ and $\overset{Q}{o}$ are equivalent, we have $\mathfrak{J}, w \models A \overset{Q}{e} B$ and $\mathfrak{J}, w \models A \overset{Q}{o} B$.

However, $V(w, A) \cap V(w, C) = \emptyset$, $M(w, A) \cap L(w, C) = \emptyset$. Hence $\mathfrak{J}, w \not\models AaC$, $\mathfrak{J}, w \not\models AiC$, $\mathfrak{J}, w \not\models A \overset{L}{a} C$, and $\mathfrak{J}, w \not\models A \overset{L}{i} C$. Hence Barbara and Darii are invalid.

For Celarent and Ferio:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |
| $v(x, A) = v(x, C) = \{a\}$ | $v(x, B) = \emptyset$ |

Call this model \mathfrak{J} . Now observe that $v(w, A) = v(w, B) = v(w, C)$. Hence $\mathfrak{J}, w \models AaB$ and $\mathfrak{J}, w \models AaB$. Further, observe that since $a \in M(w, B)$, $a \in M(w, C)$ and $a \notin L(w, C)$, it follows that $\mathfrak{J}, w \models B \overset{Q}{e} C$ (recall that $\overset{Q}{e}$ is equivalent to $\overset{Q}{a}$). Further, observe that $M(w, A) \subseteq L(w, B)$ and that $a \in M(w, A)$. Hence $\mathfrak{J}, w \models A \overset{L}{a} B$ and $\mathfrak{J}, w \models A \overset{L}{i} B$. The minor formulae of possibility follow from the necessity formulae.

However, observe that $M(w, A) \subseteq L(w, C)$, and that $M(w, A) \cap L(w, C) \neq \emptyset$. Hence $\mathfrak{J}, w \not\models A \overset{L}{e} C$ and $\mathfrak{J}, w \not\models A \overset{L}{o} C$. This gives countermodels for Celarent and Ferio.

XQM, XQX, XMX, XML, and XMM Triples

For Barbara and Darii:

| | |
|-----------------------------|-----------------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = \emptyset$ | $v(w, B) = v(w, C) = \{a\}$ |
| $v(x, A) = v(x, B) = \{b\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . First, observe that $a \in V(w, B)$ and $a \in V(w, C)$. Hence $\mathfrak{J}, w \models BaC$. Next, observe that since $b \in M(w, A)$ and $b \in M(w, B)$ and $b \notin L(w, B)$ (since $b \notin V(w, B)$) so we have $\mathfrak{J}, w \models A \overset{Q}{a} B$ and $\mathfrak{J}, w \models A \overset{M}{a} B$.

However, observe that $M(w, A) \cap M(w, C) = \emptyset$ since $a \notin V(x, A)$ and $a \notin V(w, A)$ and $b \notin V(w, C)$ and $b \notin V(x, C)$. We also have $V(w, A) \cap V(w, C) = \emptyset$. Hence we have $\mathfrak{J}, w \not\models A \overset{M}{a} C$, $\mathfrak{J}, w \not\models A \overset{M}{i} C$, $\mathfrak{J}, w \not\models A a C$, and $\mathfrak{J}, w \not\models A a C$. We also have $\mathfrak{J}, w \not\models A \overset{L}{a} C$ and $\mathfrak{J}, w \not\models A \overset{L}{i} C$, since $M(w, A) \cap L(w, C) = \emptyset$ and $M(w, A) \not\subseteq L(w, C)$, giving the required countermodels.

For Celarent and Ferio:

| | |
|-----------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = \emptyset$ |
| $v(x, B) = v(w, C) = \{a\}$ | $v(x, A) = \emptyset$ |

Call this model \mathfrak{J} . First, observe that $a \notin V(w, B)$ and $a \in V(w, C)$. Hence $\mathfrak{J}, w \models BeC$. Next, observe that since $a \in M(w, A)$ and $a \in M(w, B)$ and $a \notin L(w, B)$. So we have $\mathfrak{J}, w \models A \overset{Q}{a} B$ and $\mathfrak{J}, w \models A \overset{Q}{i} B$. We also have $\mathfrak{J}, w \models A \overset{M}{a} B$ and $\mathfrak{J}, w \models A \overset{M}{i} B$.

However, observe that $V(w, A) \subseteq V(w, C)$ and $V(w, A) \cap V(w, C) \neq \emptyset$. Further, $M(w, A) \subseteq L(w, C)$ and $M(w, A) \cap L(w, C) \neq \emptyset$. Hence $\mathfrak{J}, w \not\models A \overset{M}{e} C$, $\mathfrak{J}, w \not\models A \overset{M}{o} C$, $\mathfrak{J}, w \not\models A e C$, and $\mathfrak{J}, w \not\models A o C$. It also follows that $M(w, A) \subseteq M(w, C)$ and $M(w, A) \cap M(w, C) \neq \emptyset$. Hence $\mathfrak{J}, w \not\models A \overset{L}{e} C$ and $\mathfrak{J}, w \not\models A \overset{L}{o} C$ as well.

MLX, MMX, MXL, and MXX Triples

For Barbara and Darii:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \{a\}$ | $v(w, C) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . Observe $a \in V(w, A)$, $a \in V(w, B)$, $a \in M(w, B)$ $a \in M(w, C)$. Hence $\mathfrak{J}, w \models B \overset{M}{a} C$ and $\mathfrak{J}, w \models A a B$. Further, we have $a \in M(w, A)$ and $a \in L(w, B)$. Hence $M(w, A) \subseteq L(w, B)$ and $M(w, A) \cap L(w, B) \neq \emptyset$. Hence $\mathfrak{J}, w \models A \overset{L}{a} B$ and $\mathfrak{J}, w \models A \overset{L}{i} B$.

However, observe that $V(w, A) \cap V(w, C) = \emptyset$, and that $M(w, A) \cap L(w, C) = \emptyset$. Hence $\mathfrak{J}, w \not\models A \overset{L}{a} C$, $\mathfrak{J}, w \not\models A a C$, $\mathfrak{J}, w \not\models A \overset{L}{i} C$, and $\mathfrak{J}, w \not\models A i C$.

For Celarent and Ferio:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |
| $v(x, A) = v(x, B) = \{a\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . Observe $a \in V(w, A)$, $a \in V(w, B)$, $a \in M(w, B)$ $a \notin L(w, C)$, and $a \in L(w, B)$. Hence $M(w, B) \cap L(w, C) = \emptyset$ and so we have $\mathfrak{J}, w \models B \overset{M}{e} C$. Further, we have $\mathfrak{J}, w \models A \overset{L}{a} B$, $\mathfrak{J}, w \models A \overset{L}{i} B$, $\mathfrak{J}, w \models A \overset{M}{a} B$, $\mathfrak{J}, w \models A \overset{M}{i} B$, $\mathfrak{J}, w \models A a B$, and $\mathfrak{J}, w \models A i B$.

However, observe that $M(w, A) \subseteq M(w, C)$ and that $M(w, A) \cap M(w, C) \neq \emptyset$. Hence $\mathfrak{J}, w \models A \overset{M}{a} C$ and $\mathfrak{J}, w \models A \overset{M}{i} C$. Hence $\mathfrak{J}, w \not\models A \overset{L}{o} C$ and $\mathfrak{J}, w \not\models A \overset{L}{e} C$. Likewise, observe that $V(w, A) \subseteq V(w, C)$ and that $V(w, A) \cap V(w, C) \neq \emptyset$. Hence $\mathfrak{J}, w \not\models A o C$ and $\mathfrak{J}, w \not\models A e C$.

MLL and MML Triples

Observe that the invalidity of the MML triples entails the invalidity of the MLL triples.

For MML Barbara and Darii consider the following model:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |
| $v(x, A) = v(x, B) = \{a\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . Now, observe that we have $a \in M(w, A)$, $a \in L(w, B)$, $a \in M(w, C)$. From this it follows that $\mathfrak{J}, w \models B \overset{M}{a} C$, $\mathfrak{J}, w \models B \overset{M}{i} C$, $\mathfrak{J}, w \models A \overset{L}{a} B$, and $\mathfrak{J}, w \models A \overset{L}{i} B$.

However, we also have that $a \notin L(w, C)$, From this, plus what we observed above, it follows that $\mathfrak{J}, w \not\models A \overset{L}{a} C$, $\mathfrak{J}, w \not\models A \overset{L}{i} C$, which yields the required countermodel.

For MML Celarent and Ferio consider the following model:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |
| $v(x, A) = v(x, B) = \{a\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . Now, observe that we have $a \in M(w, A)$, $a \in L(w, B)$, $a \notin L(w, C)$. From this it follows that $\mathfrak{J}, w \models B \overset{M}{e} C$, $\mathfrak{J}, w \models B \overset{M}{o} C$, $\mathfrak{J}, w \models A \overset{L}{a} B$, and $\mathfrak{J}, w \models A \overset{L}{i} B$.

However, we also have that $a \in M(w, C)$. From this, plus what we observed above, it follows that $\mathfrak{J}, w \not\models A \overset{L}{e} C$, $\mathfrak{J}, w \not\models A \overset{L}{o} C$, which yields the required countermodel.

XQQ Triples

It should be observed that the countermodel given below is also a countermodel for MQQ, although we have already treated that case above.

For Barbara and Darii, consider the following model:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |
| $v(x, A) = v(x, C) = \{a\}$ | $v(x, B) = \emptyset$ |

Call this model \mathfrak{J} . Observe that we have $a \in V(w, A)$, $a \in V(w, B)$, and $a \in V(w, C)$. Hence $\mathfrak{J}, w \models BaC$. We also have $a \in V(w, B)$ and $a \in V(w, C)$ and so $\mathfrak{J}, w \models B \overset{M}{a} C$. Further, observe that $\mathfrak{J}, w \models A \overset{Q}{a} B$ and $\mathfrak{J}, w \models A \overset{Q}{i} B$ since $a \in M(w, A)$, $a \in M(w, B)$, and $a \notin L(w, B)$.

However, we also have $M(w, A) \subseteq L(w, C)$ and $M(w, A) \cap L(w, C)$. From these it follows that $\mathfrak{J}, w \models A \overset{L}{a} C$ and $\mathfrak{J}, w \models A \overset{L}{i} C$ and so $\mathfrak{J}, w \not\models A \overset{Q}{a} C$ and $\mathfrak{J}, w \not\models A \overset{Q}{i} C$.

For Celarent and Ferio, consider the following model:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . Observe that we have $a \in V(w, A)$, $a \notin V(w, B)$, and $a \in V(w, C)$. Hence $\mathfrak{J}, w \models BeC$. We also have $a \notin L(w, B)$ and $a \in M(w, C)$ and so $\mathfrak{J}, w \models B \overset{M}{e} C$. Further, observe that $\mathfrak{J}, w \models A \overset{Q}{a} B$ and $\mathfrak{J}, w \models A \overset{Q}{i} B$ since $a \in M(w, A)$, $a \in M(w, B)$, and $a \notin L(w, B)$.

However, we also have $M(w, A) \subseteq L(w, C)$ and $M(w, A) \cap L(w, C)$. From these it follows that $\mathfrak{J}, w \models A \overset{L}{a} C$ and $\mathfrak{J}, w \models A \overset{L}{i} C$ and so $\mathfrak{J}, w \not\models A \overset{Q}{e} C$ and $\mathfrak{J}, w \not\models A \overset{Q}{o} C$ (recall that $\overset{Q}{a}$ is equivalent to $\overset{Q}{e}$ and similarly for $\overset{Q}{i}$ and $\overset{Q}{o}$ formulae).

XLQ Triples

For Barbara and Darii, consider the following model:

| | |
|---------------------------------------|-------------------|
| $W = \{w\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |

Call this model \mathfrak{J} . Observe that $a \in V(w, A)$, $a \in V(w, B)$, and $a \in V(w, C)$. Hence $\mathfrak{J}, w \models BaC$. Similarly, observe that $a \in M(w, A)$ and $a \in L(w, B)$. Hence $\mathfrak{J}, w \models A \stackrel{L}{a} B$ and $\mathfrak{J}, w \models A \stackrel{L}{i} B$.

However, observe that $M(w, A) \subseteq L(w, C)$ and that $M(w, A) \cap L(w, C) = D$. Hence it follows that $\mathfrak{J}, w \not\models A \stackrel{Q}{a} C$.

For Celarent and Ferio, consider the following model:

| | |
|-----------------------------|-----------------------|
| $W = \{w\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \{a\}$ | $v(w, C) = \emptyset$ |

Call this model \mathfrak{J} . Observe that $a \in V(w, A)$, $a \in V(w, B)$, and $a \notin V(w, C)$. Hence $\mathfrak{J}, w \models BeC$. Similarly, observe that $a \in L(w, A)$, and $a \in L(w, B)$. Hence $\mathfrak{J}, w \models A \stackrel{L}{a} B$ and $\mathfrak{J}, w \models A \stackrel{L}{i} B$.

However, observe that $M(w, A) \subseteq L(w, \neg C)$ and that $M(w, A) \cap L(w, \neg C) = D$. Hence it follows that $\mathfrak{J}, w \not\models A \stackrel{Q}{e} C$ and that $\mathfrak{J}, w \not\models A \stackrel{Q}{o} C$ since $a \notin M(w, C)$.

LLX Barbara, Darii, and Ferio, LMX Barbara, Darii, and Ferio

For Barbara and Darii, consider the following countermodel:

| | |
|---------------------------------------|-----------------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = \emptyset$ | $v(w, B) = v(w, C) = \{a\}$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . Observe that $M(w, B) \subseteq L(w, C)$, since $a \in L(w, B), a \in M(w, B)$ and $a \in L(w, C)$. Likewise, $M(w, A) \subseteq L(w, B)$ since $a \in M(w, A)$ and $a \in L(w, B)$. Hence we have $\mathfrak{J}, w \models B \stackrel{L}{a} C$, $\mathfrak{J}, w \models A \stackrel{L}{a} B$, $\mathfrak{J}, w \models A \stackrel{L}{i} B$. It is also easy to see that $M(w, A) \subseteq M(w, B)$ and $M(w, A) \cap M(w, B) \neq \emptyset$. Hence $\mathfrak{J}, w \models A \stackrel{M}{a} B$ and $\mathfrak{J}, w \models A \stackrel{M}{i} B$.

However, observe that $V(w, A) = \emptyset$ and so $\mathfrak{J}, w \not\models A a C$ and $\mathfrak{J}, w \not\models A i C$.

For Ferio, consider the following countermodel.

| | |
|-----------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = \emptyset$ |
| $v(x, A) = v(x, B) = \{b\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . Observe that $b \in M(w, B)$ and $b \notin M(w, C)$. Likewise, $a \in M(w, C)$ and $a \notin M(w, B)$. Hence $M(w, B) \cap M(w, C) = \emptyset$. Likewise, observe that $b \in M(w, A)$ and $b \in M(w, B)$. Hence $M(w, A) \cap M(w, B) \neq \emptyset$. From this it follows that $\mathfrak{J}, w \models B \overset{L}{e} C$ and $\mathfrak{J}, w \models A \overset{L}{i} B$. However, observe that $V(w, A) \subseteq V(w, C)$ and $V(w, A) \neq \emptyset$ since $a \in V(w, A)$, $a \in V(w, C)$ and $b \notin V(w, A)$. Hence $\mathfrak{J}, w \models AaC$ and so $\mathfrak{J}, w \not\models AoC$.

LXM Barbara, Celarent and MXM Barbara and Celarent, LXQ Darii and Ferio

Observe that if an LXM syllogism is invalid, then the LXL syllogism and the LXQ syllogisms must also be invalid.

For Barbara and LXQ Darii consider the following countermodel:

| | |
|--|-------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | |
| $v(x, A) = \{b\}, v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . First observe, that $a \in M(w, B)$, $a \in L(w, C)$ and that $b \notin M(w, B)$. Hence $M(w, B) \neq \emptyset$ and that $M(w, B) \subseteq L(w, C)$. Hence $\mathfrak{J}, w \models B \overset{L}{a} C$. Similarly, since $b \notin V(w, A)$ and $a \in V(w, A)$ and $a \in V(w, B)$, it follows that $V(w, A) \neq \emptyset$ and that $V(w, B) \subseteq V(w, C)$. Hence $\mathfrak{J}, w \models AaB$.

However, observe that $M(w, A) \not\subseteq M(w, C)$. This follows since $b \in M(w, A)$ and $b \notin M(w, C)$. Hence $\mathfrak{J}, w \models A \overset{L}{o} C$ and so $\mathfrak{J}, w \not\models A \overset{M}{a} C$. For MXM Barbara, simply observe that we also have $\mathfrak{J}, w \models B \overset{M}{a} C$.

For LXQ Darii, observe that $\mathfrak{J}, w \models AiB$. Further, $\mathfrak{J}, w \not\models A \overset{Q}{i} C$ since $b \notin M(w, C)$ and $a \notin M(w, \neg C)$ i.e. $M(w, A) \cap M(w, C) \cap M(w, \neg C) = \emptyset$.

For Celarent consider the following countermodel:

| | |
|-----------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \{a\}$ | $v(w, C) = \{b\}$ |
| $v(x, A) = v(x, C) = \{b\}$ | $v(x, B) = \emptyset$ |

Call this model \mathfrak{J} . First observe that $a \in M(w, B)$, $a \notin M(w, C)$, $b \in L(w, C)$, $b \in M(w, C)$, and $b \notin M(w, B)$. Hence $M(w, B) \cap M(w, C) = \emptyset$. So we have $\mathfrak{J}, w \models B \overset{L}{e} C$ and $\mathfrak{J}, w \models B \overset{M}{e} C$. Observe that we also have $a \in V(w, A)$, $a \in V(w, B)$, and $b \notin V(w, A)$. Clearly then $V(w, A) \subseteq V(w, B)$ and $V(w, A) \neq \emptyset$. Hence $\mathfrak{J}, w \models AaB$. We also have $\mathfrak{J}, w \models AiB$.

However, observe that $M(w, A) \cap L(w, C) = \{b\}$ and so $\mathfrak{J}, w \models A \overset{L}{i} C$ and so $\mathfrak{J}, w \not\models A \overset{M}{e} C$ as desired.

For LXQ Ferio, observe that $M(w, A) \cap (M(w, C) \cap M(w, \neg C)) = \emptyset$ since $b \notin M(w, \neg C)$ and $a \notin M(w, C)$. Hence $\mathfrak{N}, w \vDash A \overset{Q}{o} C$.

XLX Barbara, Darii, and Ferio

For Barbara and Darii, consider the following countermodel:

| | |
|-----------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, B) = v(w, C) = \{a\}$ | $v(w, A) = \emptyset$ |
| $v(x, A) = v(x, B) = v(x, C) = D$ | |

Call this model \mathfrak{J} . First observe that $a \in V(w, B)$, $a \in V(w, C)$, and $b \notin V(w, B)$. Hence $V(w, B) \subseteq V(w, C)$ and $V(w, B) \neq \emptyset$. Hence $\mathfrak{J}, w \vDash BaC$. Also, observe that $b \in M(w, A)$, $b \in L(w, B)$ and $a \in L(w, B)$. Hence we have $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$. Hence $\mathfrak{J}, w \vDash A \overset{L}{a} B$. We also have $\mathfrak{J}, w \vDash A \overset{L}{i} B$.

However, observe that $V(w, A) \cap V(w, C) = \emptyset$ and so $\mathfrak{J}, w \vDash A e C$. Hence $\mathfrak{J}, w \vDash A i C$ and $\mathfrak{J}, w \vDash A a C$.

For Ferio, consider the following countermodel:

| | |
|-----------------------------------|-------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = \{b\}$ |
| $v(x, A) = v(x, B) = v(w, C) = D$ | |

Call this model \mathfrak{J} . First, observe that $b \notin V(w, C)$ and $a \notin V(w, B)$. Hence $V(w, B) \cap V(w, C) = \emptyset$. From this it follows that $\mathfrak{J}, w \vDash BeC$. Likewise, observe that $b \in M(w, A)$ and $b \in L(w, B)$. Hence $M(w, A) \cap L(w, B) \neq \emptyset$. From this it follows that $\mathfrak{J}, w \vDash A \overset{L}{i} B$.

However, observe that $a \in V(w, A)$, $a \in V(w, C)$ and $b \notin V(w, A)$. Hence $V(w, A) \neq \emptyset$ and $V(w, A) \subseteq V(w, C)$. From this it follows that $\mathfrak{J}, w \vDash AaC$ and so $\mathfrak{J}, w \vDash AoC$.

QXQ Barbara, Celarent and QXM Barbara and Celarent

For QXQ and QXM Barbara consider the following countermodel:

| | |
|---------------------------------------|---------------------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | $v(x, B) = v(w, C) = \emptyset$ |
| $v(x, A) = \{b\}$ | |

10 Appendix Three: Buridan's Modal Syllogism

Call this model \mathfrak{J} . First, observe that $a \in M(w, B)$, $a \in M(w, C)$, $a \notin L(w, C)$ and that $b \notin M(w, B)$. Hence $M(w, B) \subseteq (M(w, C) \cap M(w, \neg C))$ and $M(w, B) \neq \emptyset$. Hence $\mathfrak{J}, w \models B \overset{Q}{a} C$. Likewise, observe that since $b \notin V(w, A)$, $a \in V(w, A)$, and $b \in V(w, B)$, it follows that $\mathfrak{J}, w \models AaB$.

However, observe that $M(w, A) \not\subseteq M(w, C) \cap M(w, \neg C)$ since $b \in M(w, A)$ and $b \notin M(w, C)$. Hence $\mathfrak{J}, w \not\models A \overset{Q}{a} C$ and $\mathfrak{J}, w \not\models A \overset{M}{a} C$.

For QXQ and QXM Celarent consider the following countermodel:

| | |
|-----------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = D$ |
| $v(x, A) = v(x, C) = \{b\}$ | $v(x, B) = \emptyset$ |

Call this model \mathfrak{J} . First, observe that $a \in M(w, B)$, $a \in M(w, C)$, $a \notin L(w, C)$ and that $b \notin M(w, B)$. Hence $M(w, B) \subseteq (M(w, C) \cap M(w, \neg C))$ and $M(w, B) \neq \emptyset$. Hence $\mathfrak{J}, w \models B \overset{Q}{e} C$. Likewise, observe that since $b \notin V(w, A)$, $a \in V(w, A)$, and $b \in V(w, B)$, it follows that $\mathfrak{J}, w \models AaB$.

However, observe that $b \in M(w, A)$ and $b \in L(w, C)$. Hence $M(w, A) \not\subseteq (M(w, C) \cap M(w, \neg C))$ and $M(w, A) \cap L(w, C) \neq \emptyset$. Hence it follows that $\mathfrak{J}, w \not\models A \overset{Q}{e} C$ and $\mathfrak{J}, w \not\models A \overset{M}{e} C$.

LQX Barbara, Darii, and Ferio

For Barbara and Darii, consider the following countermodel:

| | |
|---------------------------------------|-------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \emptyset$ | $v(w, C) = \{a\}$ |
| $v(x, A) = v(x, B) = v(x, C) = \{a\}$ | |

Call this model \mathfrak{J} . Clearly $a \in M(w, A)$, $a \in M(w, B)$. To see that $M(w, B) \subseteq L(w, C)$, observe that $a \in V(w, C)$ and $a \in V(x, C)$. Hence $\mathfrak{J}, w \models B \overset{L}{a} C$. Likewise, observe that $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$ since $a \in M(w, A)$ and $a \notin V(w, B)$ and hence in $a \in M(w, \neg B)$. Likewise, since $a \in V(x, B)$ it follows that $a \in M(w, B)$ as claimed.

However, $\mathfrak{J}, w \not\models AaC$ since $V(w, A) = \emptyset$.

For LQX Ferio, consider the following countermodel.

| | |
|-----------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a, b\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, C) = \{a\}$ | $v(w, B) = \emptyset$ |
| $v(x, A) = v(x, B) = \{b\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . Clearly $\mathfrak{J}, w \models B \stackrel{L}{e} C$ since $a \notin M(w, B)$ and $b \notin M(w, C)$. Likewise, $\mathfrak{J}, w \models A \stackrel{Q}{i} B$ since $b \in M(w, A)$, $b \in M(w, B)$ since $b \in V(x, B)$ and $b \in M(w, \neg B)$ since $b \notin V(w, B)$.

However, clearly $V(w, A) \subseteq V(w, C)$ since $b \notin V(w, A)$ and $a \in V(w, A)$ and $a \in V(w, C)$. Hence $\mathfrak{J}, w \not\models A \circ C$.

XLL

For Barbara and Darii consider the following countermodel:

| | |
|---------------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = v(w, C) = \{a\}$ | $v(x, C) = \emptyset$ |
| $v(x, A) = v(x, B) = \{a\}$ | $v(x, C) = \emptyset$ |

Call this model \mathfrak{J} . Observe that clearly $V(w, B) \subseteq V(w, C)$ since $a \in V(w, B)$ and $a \in V(w, C)$. Likewise, observe that $M(w, A) \subseteq L(w, B)$ since $a \in M(w, A)$ and $a \in L(w, B)$. Hence it follows that $\mathfrak{J}, w \models BaC$, $\mathfrak{J}, w \models A \stackrel{L}{a} B$, and $\mathfrak{J}, w \models A \stackrel{L}{i} B$.

However, since $a \notin V(x, C)$ it follows that $a \notin L(w, C)$ and so $M(w, A) \not\subseteq L(w, C)$, i.e. $\mathfrak{J}, w \not\models A \stackrel{L}{a} C$ and $\mathfrak{J}, w \not\models A \stackrel{L}{i} C$.

For XLL Celarent and Ferio consider the following countermodel:

| | |
|---------------------------------|-----------------------|
| $W = \{w, x\}$ | $R = W^2$ |
| $D = \{a\}$ | $O(w) = O(x) = D$ |
| $v(w, A) = v(w, B) = \{a\}$ | $v(w, C) = \emptyset$ |
| $v(x, A) = v(x, B) = \emptyset$ | $v(x, C) = \{a\}$ |

Call this model \mathfrak{J} . First observe that $M(w, A) \subseteq L(w, B)$ since $a \in V(w, B)$ and $a \notin V(x, A)$. That $M(w, B)$ is non-empty follows from $a \in V(w, B)$. Hence $\mathfrak{J}, w \models A \stackrel{L}{a} B$ and $\mathfrak{J}, w \models A \stackrel{L}{i} B$. Next, observe that $V(w, B) \cap V(w, C) = \emptyset$ since $v(w, C) = \emptyset$. Hence $\mathfrak{J}, w \models BeC$.

However, observe that $M(w, A) \subseteq M(w, C)$ since $a \in M(w, C)$, because $a \in V(x, C)$ and $a \in M(w, A)$ as we already observed. Hence $\mathfrak{J}, w \not\models A \stackrel{L}{o} C$ and $\mathfrak{J}, w \not\models A \stackrel{L}{e} C$.

Conclusion

This completes our treatment of invalidities in the first figure. This, together with our treatment of validities in the first figure exhausts all of the possible triple combinations for categorical formulae. By the Reduction Lemma, it follows that we have also exhausted all of the combinations in the other two figures as well. The verification of the invalidities in the second and third figure can be accomplished by reducing these to the first figure. This is a straightforward exercise, however, due to reasons of length we will not include them here.

11 Appendix Four: Soundness

We remarked in Chapter Four that, for the most part, the soundness of our proof rules with respect to our semantics was an easy exercise. For the sake of completeness and for ease of verification, this appendix provides the proofs for the rules not treated in Chapter Four as well as contains the specifications for rule of inference. In this appendix we will proceed as follows: We will start by recalling the semantic and proof-theoretic rules that we are working with, including the full details of how the inferential rules work. We will then prove soundness, starting with the interchange rules, followed by the introduction rules and concluding with the elimination rules.

Semantics

Recall that a Buridan Modal Model is defined as follows:

Buridan Modal Model (Expanded). A Buridan Modal Model is a tuple:
 $\mathfrak{M} = \langle D, W, R, O, c, v \rangle$ such that:

D and W are non-empty sets. D is the domain of objects and W is a set of worlds.

R $\subseteq W^2$ which is universal.

O : W $\rightarrow \mathcal{P}(D)$.

c : CONS $\rightarrow D$ such that c is surjective.

v : W \times PRED $\rightarrow \mathcal{P}(D)$.

As before, we define the following shorthand for use in our semantic definitions model:

Semantic Abbreviations. Let *P* be a term. Using the semantics we can define the following collections:

$$V(w, P) = O(w) \cap v(w, P)$$

$$V(w, \neg P) = D \setminus (O(w) \cap v(w, P))$$

$$M(w, K) = \{d \in D : \text{there is some } z \text{ s.t. } wRz \text{ and } d \in V(z, K)\}$$

$$L(w, K) = \{d \in D : \text{for all } z \text{ if } wRz \text{ then } d \in V(z, K)\}$$

Recall that we are using *K* as a variable to range over terms and their negations.

With these in place, recall that the truth conditions for the various formulae are as follows:

Assertoric Categorical Propositions.

- $\mathfrak{M}, w \models AaB$ if and only if $V(w, A) \subseteq V(w, B)$ and $V(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models AeB$ if and only if $V(w, A) \cap V(w, B) = \emptyset$
 $\mathfrak{M}, w \models AiB$ if and only if $V(w, A) \cap V(w, B) \neq \emptyset$
 $\mathfrak{M}, w \models AoB$ if and only if $V(w, A) \not\subseteq V(w, B)$ or $V(w, A) = \emptyset$

Modal Categorical Propositions.

- $\mathfrak{M}, w \models A \overset{L}{a} B$ if and only if $M(w, A) \subseteq L(w, B)$ and $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{L}{e} B$ if and only if $M(w, A) \cap M(w, B) = \emptyset$
 $\mathfrak{M}, w \models A \overset{L}{i} B$ if and only if $M(w, A) \cap L(w, B) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{L}{o} B$ if and only if $M(w, A) \not\subseteq M(w, B)$ or $M(w, A) = \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{a} B$ if and only if $M(w, A) \subseteq M(w, B)$ and $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{e} B$ if and only if $M(w, A) \cap L(w, B) = \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{i} B$ if and only if $M(w, A) \cap M(w, B) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{M}{o} B$ if and only if $M(w, A) \not\subseteq L(w, B)$ or $M(w, A) = \emptyset$
 $\mathfrak{M}, w \models A \overset{Q}{a} B$ if and only if $M(w, A) \subseteq M(w, B) \cap M(w, \neg B)$ and $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{Q}{e} B$ if and only if $\mathfrak{M}, w \models A \overset{Q}{a} B$
 $\mathfrak{M}, w \models A \overset{Q}{i} B$ if and only if $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{Q}{o} B$ if and only if $\mathfrak{M}, w \models A \overset{Q}{i} B$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$ if and only if $M(w, A) \cap M(w, B) \cap M(w, \neg B) = \emptyset$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{e} B$ if and only if $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{i} B$ if and only if $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$ or $M(w, A) \neq \emptyset$
 $\mathfrak{M}, w \models A \overset{\bar{Q}}{o} B$ if and only if $\mathfrak{M}, w \models A \overset{\bar{Q}}{i} B$

New Propositions.

| | | |
|---|-----------------------|---|
| $\mathfrak{M}, w \models \underline{f}aA$ | <i>if and only if</i> | $c(f) \in V(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}eA$ | <i>if and only if</i> | $c(f) \notin V(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^L a A$ | <i>if and only if</i> | $c(f) \in L(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^M e A$ | <i>if and only if</i> | $c(f) \notin L(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^M a A$ | <i>if and only if</i> | $c(f) \in M(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^L e A$ | <i>if and only if</i> | $c(f) \notin M(w, A)$ |
| $\mathfrak{M}, w \models \underline{f}^Q A$ | <i>if and only if</i> | $c(f) \in M(w, A) \cap M(w, \neg A)$ |
| $\mathfrak{M}, w \models \underline{f}^Q e A$ | <i>if and only if</i> | $\mathfrak{M}, w \models \underline{f}^{\bar{Q}} a A$ |
| $\mathfrak{M}, w \models \underline{f}^{\bar{Q}} A$ | <i>if and only if</i> | $c(f) \in L(w, A) \cup L(w, \neg A)$ |
| $\mathfrak{M}, w \models \underline{f}^{\bar{Q}} e A$ | <i>if and only if</i> | $\mathfrak{M}, w \models \underline{f}^Q a A$ |
| $\mathfrak{M}, w \models \phi \wedge \psi$ | <i>if and only if</i> | $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$ |

We recall here that the operation e places the negation in front of the modal operation when we are dealing with singular terms. So, for example, $\underline{f}^Q e A$ should be read as saying that (the object named by) \underline{f} is not contingently A. With this, the rules and truth conditions that relate Q and \bar{Q} should be clear. In Chapter Four we made use of the operation $C(\phi)$ to pick out the contradictory formula of ϕ , where ϕ is not of the form $\psi \wedge \chi$. Strictly speaking, we should not speak of *the* contradictory formula of ϕ , since in a number of places this is not uniquely defined. However, as we shall prove below, such formulae are logically equivalent we will retain this way of speaking. We here indicate that every formula ϕ appropriately restricted (i.e. restricted to exclude formulae of the form $\phi \wedge \psi$) has a contradictory. Recall that, given two formulae ϕ and ψ , we say that ϕ and ψ are contradictory if and only if for all \mathfrak{M} and $w \in W$ we have $\mathfrak{M}, w \models \phi$ if and only if $\mathfrak{M}, w \not\models \psi$. As an easy corollary of this, observe that if two formulae ϕ and ψ are both contradictory of χ , then ϕ and ψ are equivalent. As \bar{a} and \bar{e} , \bar{i} and \bar{o} , \bar{a} and \bar{e} , and \bar{i} and \bar{o} are all equivalent, $C(\phi)$ is not uniquely defined. However nothing substantial will turn on this, and we stipulate that in these cases $C(\phi)$ returns the affirmative proposition (a or i) if ϕ is affirmative and $C(\phi)$ returns the negative proposition if ϕ is negative (e or o).

Based on these definitions, it is immediate from our semantics that the following are contradictory pairs:

| | | |
|----------------------|-------------|----------------------------|
| $A a B$ | contradicts | $A o B$ |
| $A i B$ | contradicts | $A e B$ |
| $A \overset{L}{a} B$ | contradicts | $A \overset{M}{o} B$ |
| $A \overset{L}{i} B$ | contradicts | $A \overset{M}{e} B$ |
| $A \overset{M}{a} B$ | contradicts | $A \overset{L}{o} B$ |
| $A \overset{M}{i} B$ | contradicts | $A \overset{L}{e} B$ |
| $A \overset{Q}{a} B$ | contradicts | $A \overset{\bar{Q}}{o} B$ |
| $A \overset{Q}{i} B$ | contradicts | $A \overset{\bar{Q}}{e} B$ |

and that in the case of singular propositions, the following obtains::

| | | |
|----------------------------------|-------------|--|
| $\underline{d} a A$ | contradicts | $\underline{d} e A$ |
| $\underline{d} \overset{L}{a} A$ | contradicts | $\underline{d} \overset{L}{e} B$ |
| $\underline{d} \overset{M}{a} A$ | contradicts | $\underline{d} \overset{M}{e} B$ |
| $\underline{d} \overset{Q}{a} A$ | contradicts | $\underline{d} \overset{\bar{Q}}{e} B$ |

Based on the definition of contradictory propositions and $C()$ we have the following easy corollaries:

Let ϕ and ψ be contradictory propositions. Then for no model \mathfrak{M} and $w \in W$ do we have $\mathfrak{M} \models \phi$ and $\mathfrak{M}, w \models \psi$. To see this, assume not. Then for some \mathfrak{M} and $w \in W$, we have $\mathfrak{M} \models \phi$ and $\mathfrak{M}, w \models \psi$. As ϕ and ψ are contradictory, it follows that $\mathfrak{M}, w \not\models \psi$ and $\mathfrak{M}, w \not\models \phi$ which is a contradiction.

That for no model \mathfrak{M} and $w \in W$ do we have $\mathfrak{M} \models \phi$ and $\mathfrak{M}, w \models C(\phi)$, follows by the definition of $C(\phi)$.

As we proved in the previous appendix, we have the following Lemma:

Lemma 1:

For all models \mathfrak{M} and all worlds, $w \in W$ we have:

1. $L(w, A) \subseteq V(w, A)$
2. $V(w, A) \subseteq M(w, A)$
3. $L(w, A) \subseteq M(w, A)$

Proofs of these claims can be found in the appendix on the adequacy of our semantics to capture Buridan's treatment of the modal syllogism.¹ We will often implicitly appeal to this lemma without explicit reference. When we do reference it, the principles will be cited as lemma 1.1, 1.2, and 1.3 respectively.

We will also make use of the following relationships, often without explicitly mentioning it.

For all \mathfrak{M} , $w \in W$, and $d \in D$ we have

1. See page 211.

4. $d \in L(w, A)$ if and only if $d \notin M(w, \neg A)$

5. $d \in M(w, A)$ if and only if $d \notin L(w, \neg A)$

Proof of 4. Take arbitrary \mathfrak{M} , w and d . Then we have $d \in L(w, A)$ if and only if for all $x \in W$ if wRx then $d \in V(x, A)$ if and only if it is not the case that there is some $x \in W$ such that wRx and $d \notin V(x, A)$ if and only if it is not the case that there is some $x \in W$ such that wRx and $d \in V(x, \neg A)$ if and only if $d \notin M(w, \neg A)$.

Proof of 5. Take arbitrary \mathfrak{M} , w and d . Then we have $d \in M(w, A)$ if and only if there is some $x \in W$ such that wRx and $d \in V(x, A)$ if and only if it is not the case that for all $x \in W$ if wRx then $d \notin V(x, A)$ if and only if it is not the case that for all $x \in W$ if wRx then $d \in V(x, \neg A)$ if and only if $d \notin L(w, \neg A)$.

As an easy corollary of 4. and 5. we have the following:
For all \mathfrak{M} , $w \in W$, and $d \in D$ we have

6. $d \in L(w, A) \cup L(w, \neg A)$ if and only if $d \notin M(w, A) \cap M(w, \neg A)$

Take arbitrary \mathfrak{M} , w and d . Then we have $d \in L(w, A) \cup L(w, \neg A)$ if and only if either $d \in L(w, A)$ or $d \in L(w, \neg A)$.

For the left to right direction, observe that if $d \in L(w, A)$ then by 4, $d \notin M(w, \neg A)$ and so $d \notin M(w, A) \cap M(w, \neg A)$. Likewise, if $d \in L(w, \neg A)$ then $d \notin M(w, A)$ by 5. Hence $d \notin M(w, A) \cap M(w, \neg A)$.

For the right to left direction, assume that $d \notin L(w, A) \cup L(w, \neg A)$. Then $d \notin L(w, A)$ and $d \notin L(w, \neg A)$. Hence by 4 and 5 $d \in M(w, A)$ and $d \in M(w, \neg A)$. Hence $d \in M(w, A) \cap M(w, \neg A)$.

11.1 Natural Deduction Rules

It should be recalled that our proof system is a natural deduction framework, composed of three tables of rules. One table is for elimination rules, another for introduction rules, and a third for interaction rules.

| | |
|---|---|
| $\frac{A_i B \quad C}{C} \text{Exposition } +$ $\frac{A_i^L B \quad C}{C} \text{Exposition } L^+$ $\frac{A_i^M B \quad C}{C} \text{Exposition } M^+$ $\frac{A_i^Q B \quad C}{C} \text{Exposition } Q$ | $\frac{A_o B \quad C \quad A_i A}{C} \text{Exposition } -$ $\frac{A_o^L B \quad C \quad A_i^M A}{C} \text{Exposition } L^-$ $\frac{A_o^M B \quad C \quad A_i^M A}{C} \text{Exposition } M^-$ $\frac{A_i^{\bar{Q}} B \quad C \quad A_i^M A}{C} \text{Exposition } \bar{Q}$ |
| $\frac{A a B \quad \underline{d} a A}{\underline{d} a B} DDO$ | $\frac{A e B \quad \underline{d} a A}{\underline{d} e B} DDN$ |
| $\frac{A e B \quad \underline{d} a B}{\underline{d} e A} DDN$ $\frac{A_a^L B \quad \underline{d}_a^M A}{\underline{d}_a^L B} DDO$ | $\frac{A_e^L B \quad \underline{d}_a^M A}{\underline{d}_e^L B} DDN$ |
| $\frac{A_e^L B \quad \underline{d}_a^M B}{\underline{d}_e^L A} DDN$ $\frac{A_a^M B \quad \underline{d}_a^M A}{\underline{d}_a^M B} DDO$ | $\frac{A_e^M B \quad \underline{d}_a^M A}{\underline{d}_e^L B} DDN$ |
| $\frac{A_e^M B \quad \underline{d}_a^L B}{\underline{d}_e^L A} DDN$ $\frac{A_a^Q B \quad \underline{d}_a^M A}{\underline{d}_a^Q B} DDO$ | $\frac{A_a^{\bar{Q}} B \quad \underline{d}_a^M A}{\underline{d}_a^{\bar{Q}} B} DDO$ |
| $\frac{\underline{d}_a^{\bar{Q}} A \quad \underline{d}_a^M A}{\underline{d}_a^L A} \bar{Q}\text{-Out}$ | $\frac{\underline{d}_a^{\bar{Q}} A \quad \underline{d}_e^M A}{\underline{d}_e^L A} \bar{Q}\text{-Out}$ |
| $\frac{\phi \wedge \psi}{\phi} \wedge \text{Elimination}$ | $\frac{\phi \wedge \psi}{\psi} \wedge \text{Elimination}$ |
| $\frac{\phi \quad C(\phi)}{\psi} \text{Ex Falso Quodlibet}$ | |

Table 11.1: Elimination Rules

| | |
|---|--|
| $\frac{\underline{d} a A \wedge \underline{d} a B}{A i B} \text{ Expository Syllogism } ^+$ | $\frac{\underline{d} a A \wedge \underline{d} e B}{A o B} \text{ Expository Syllogism } ^-$ |
| $\frac{\underline{d}^M a A \wedge \underline{d}^L a B}{A^L i B} \text{ Expository Syllogism } L^+$ | $\frac{\underline{d}^M a A \wedge \underline{d}^L e B}{A^L o B} \text{ Expository Syllogism } L^-$ |
| $\frac{\underline{d}^M a A \wedge \underline{d}^M a B}{A^M i B} \text{ Expository Syllogism } M^+$ | $\frac{\underline{d}^M a A \wedge \underline{d}^M e B}{A^M o B} \text{ Expository Syllogism } M^-$ |
| $\frac{\underline{d}^M a A \wedge \underline{d}^Q a B}{A^Q i B} \text{ Expository Syllogism } Q$ | $\frac{\underline{d}^M a A \wedge \underline{d}^{\bar{Q}} a B}{A^{\bar{Q}} i B} \text{ Expository Syllogism } \bar{Q}$ |
| $\frac{A i A \quad \frac{[f a A] \quad f a B}{A a B}}{A a B} a\text{-introduction}$ | $\frac{[f a A] \quad f e B}{A e B} e\text{-introduction}$ |
| $\frac{A^M i A \quad \frac{[f^M a A] \quad f^{\nabla} a B}{A^{\nabla} a B}}{A^{\nabla} a B} a^{\nabla}\text{-introduction}$ | $\frac{[f^M a A] \quad f^{\nabla} e B}{A^{\Delta} e B} e^{\Delta}\text{-introduction}$ |
| $\frac{f e A}{A o B} \text{ Empty Exposition}$ | $\frac{f^L e A}{A^{\nabla} o B} \text{ Empty Exposition}$ |
| $\frac{\underline{d}^L a A}{\underline{d}^{\bar{Q}} a A} \bar{Q}\text{-In}$ | $\frac{\underline{d}^L e A}{\underline{d}^{\bar{Q}} a A} \bar{Q}\text{-In}$ |
| $\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge \text{ Introduction}$ | |
| $\frac{\underline{d} a A}{\underline{d}^L a A} \text{ Necessity Introduction}$ | $\frac{\underline{d} e A}{\underline{d}^L e A} \text{ Necessity Introduction}$ |
| $\frac{\underline{d} a A}{\underline{d}^M a A} \text{ Reflexivity}$ | $\frac{\underline{d}^L e A}{\underline{d} e A} \text{ Reflexivity}$ |
| $\frac{[\phi] \quad \psi \quad C(\psi)}{C(\phi)} \text{ C-Introduction}$ | |

Table 11.2: Introduction Rules

| | |
|---|--|
| $\frac{A \overset{L}{\times} B}{A \overset{M}{\times} B} \text{ Modal Subalternation}^L$ | $\frac{A \overset{Q}{a/e} B}{A \overset{M}{a} B} \text{ Modal Subalternation}^Q$ |
| | $\frac{A \overset{Q}{i/o} B}{A \overset{M}{i} B} \text{ Modal Subalternation}^Q$ |
| $\frac{A a B}{A i B} \text{ Subalternation}^+$ | $\frac{A e B}{A o B} \text{ Subalternation}^-$ |
| $\frac{A \overset{L/M}{a} B}{A \overset{L/M}{i} B} \text{ Subalternation}^+$ | $\frac{A \overset{L/M}{e} A}{A \overset{L/M}{o} B} \text{ Subalternation}^-$ |
| $\frac{A \overset{Q}{a/e} B}{A \overset{Q}{i/o} B} \text{ Subalternation}^+$ | $\frac{A \overset{\bar{Q}}{a/e} A}{A \overset{\bar{Q}}{i/o} B} \text{ Subalternation}^-$ |
| $\frac{\underline{d} \overset{Q}{a} A}{\underline{d} \overset{Q}{e} A} Q \text{ sing Equivalence}$ | $\frac{\underline{d} \overset{\bar{Q}}{a} A}{\underline{d} \overset{\bar{Q}}{e} A} \bar{Q} \text{ sing Equivalence}$ |
| $\frac{A \overset{Q}{a} B}{A \overset{Q}{e} B} Q a - e \text{ Equivalence}$ | $\frac{A \overset{Q}{i} B}{A \overset{Q}{o} B} Q i - o \text{ Equivalence}$ |
| $\frac{A \overset{\bar{Q}}{a} B}{A \overset{\bar{Q}}{e} B} \bar{Q} a - e \text{ Equivalence}$ | $\frac{A \overset{\bar{Q}}{i} B}{A \overset{\bar{Q}}{o} B} \bar{Q} i - o \text{ Equivalence}$ |
| $\frac{\underline{d} \overset{Q}{a} A}{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A} Q - Out$ | $\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A}{\underline{d} \overset{Q}{a} A} Q - In$ |
| $\frac{C(C(\phi))}{\phi} \text{ CC Elimination}$ | |

Table 11.3: Interaction Rules

Here \underline{f} denotes a new constant that does not occur previously in the proof, while \underline{d} has no such restriction.

11.1.1 Inferential Rule Structure

We specify the rules for the introduction and discharging of assumptions as follows:

Elimination rules

The various forms of *Expository Syllogism* are as follows:

For *Exposition*⁺:

$$\langle\langle\Gamma_1, A \text{ i } B\rangle, \langle\Gamma_2, C\rangle, \langle\Lambda, C\rangle\rangle$$

Where $\Lambda = \Gamma_1 \cup (\Gamma_2 - \Gamma_3)$ where Γ_3 consists of some formula $\underline{d} a A \wedge \underline{d} a B \in \Gamma_2$ such that \underline{d} does not occur in Λ or C .

For the modal cases:

$$\langle\langle\Gamma, A \overset{\nabla}{\text{i}} B\rangle, \langle\Gamma_2, C\rangle, \langle\Lambda, C\rangle\rangle$$

Where $\Lambda = \Gamma \cup (\Gamma_2 - \Gamma_3)$ where Γ_3 consists of some formula $\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{\Delta}{a} B \in \Gamma_2$ such that \underline{d} does not occur in Λ or C

For the cases of *Exposition*⁻ we have:

For *Exposition*⁻:

$$\langle\langle\Gamma_1, A \text{ o } B\rangle, \langle\Gamma_2, C\rangle, \langle\Gamma_4, A \text{ i } A\rangle, \langle\Lambda', C\rangle\rangle$$

Where $\Lambda' = \Gamma_1 \cup (\Gamma_2 - \Gamma_3) \cup \Gamma_4$ where Γ_3 consists of some $\underline{d} a A \wedge \underline{d} a B \in \Gamma_2$ such that \underline{d} does not occur in Λ or C .

For the Modal Cases:

$$\langle\langle\Gamma_1, A \overset{\nabla}{\text{o}} B\rangle, \langle\Gamma_2, C\rangle, \langle\Gamma_4, A \overset{M}{\text{i}} A\rangle, \langle\Lambda', C\rangle\rangle$$

Where $\Lambda' = \Gamma_1 \cup (\Gamma_2 - \Gamma_3) \cup \Gamma_4$ where Γ_3 consists of some $\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{\Delta}{e} B \in \Gamma_2$ such that \underline{d} does not occur in Λ or C .

For the various forms of *DDO*:

$$\langle\langle\Gamma_1, A a B\rangle, \langle\Gamma_2, \underline{d} a A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} a B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{L}{a} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{L}{a} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{M}{a} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{M}{a} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{a} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{Q}{a} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{a} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{Q}{a} B\rangle\rangle$$

For the various *DDN* rules:

$$\langle\langle\Gamma_1, A e B\rangle, \langle\Gamma_2, \underline{d} a A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} e B\rangle\rangle$$

$$\langle\langle\Gamma_1, A e B\rangle, \langle\Gamma_2, \underline{d} a B\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} e A\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{L}{e} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{L}{e} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{L}{e} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} B\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{L}{e} A\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{M}{e} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{M}{e} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{M}{e} B\rangle, \langle\Gamma_2, \underline{d} \overset{M}{a} B\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d} \overset{M}{e} A\rangle\rangle$$

For the two \bar{Q} -Out rules we have:

11 Appendix Four: Soundness

$$\begin{aligned} &\langle\langle\Gamma_1, \underline{d}^{\bar{Q}} a A\rangle, \langle\Gamma_2, \underline{d}^M a A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d}^L a A\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^{\bar{Q}} a A\rangle, \langle\Gamma_2, \underline{d}^M e A\rangle, \langle\Gamma_1 \cup \Gamma_2, \underline{d}^L e A\rangle\rangle \end{aligned}$$

For \wedge Elimination we have:

$$\begin{aligned} &\langle\langle\Gamma_1, \phi \wedge \psi\rangle, \langle\Gamma_1, \phi\rangle\rangle \\ &\langle\langle\Gamma_1, \phi \wedge \psi\rangle, \langle\Gamma_1, \psi\rangle\rangle \end{aligned}$$

For *Ex Falso Quodlibet* we have:

$$\langle\langle\Gamma_1, \phi\rangle, \langle\Gamma_2, C(\phi)\rangle, \langle\Gamma_1 \cup \Gamma_2, \psi\rangle\rangle$$

Introduction rules

For our introduction rules we have the following:

For the various forms of *Expository syllogism*⁺ we have:

$$\begin{aligned} &\langle\langle\Gamma_1, \underline{d} a A \wedge \underline{d} a B\rangle, \langle\Gamma_1, A i B\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^M a A \wedge \underline{d}^L a B\rangle, \langle\Gamma_1, A^L i B\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^M a A \wedge \underline{d}^M a B\rangle, \langle\Gamma_1, A^M i B\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^M a A \wedge \underline{d}^Q a B\rangle, \langle\Gamma_1, A^Q i B\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^M a A \wedge \underline{d}^{\bar{Q}} a B\rangle, \langle\Gamma_1, A^{\bar{Q}} i B\rangle\rangle \end{aligned}$$

For the various forms of *Expository Syllogism*⁻ we have:

$$\begin{aligned} &\langle\langle\Gamma_1, \underline{d} a A \wedge \underline{d} e B\rangle, \langle\Gamma_1, A o B\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^M a A \wedge \underline{d}^L e B\rangle, \langle\Gamma_1, A^L o B\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}^M a A \wedge \underline{d}^M e B\rangle, \langle\Gamma_1, A^M o B\rangle\rangle \end{aligned}$$

For the various *a-introduction* rules we have:

$$\langle\langle\Gamma_1, A i A\rangle, \langle\Gamma_2, \underline{f} a B\rangle, \langle\Gamma_1 \cup \Gamma_3, A a A\rangle\rangle$$

Where $\Gamma_3 = \Gamma_2 \setminus \underline{f} a A$ and we require that \underline{f} does not occur in Γ_3 .

$$\langle\langle\Gamma_1, A^M i A\rangle, \langle\Gamma_2, \underline{f}^{\bar{\nabla}} a B\rangle, \langle\Gamma_1 \cup \Gamma_3, A^{\bar{\nabla}} a A\rangle\rangle$$

Where $\Gamma_3 = \Gamma_2 \setminus \{\underline{f}^M a A\}$ and we require that \underline{f} does not occur in Γ_3 .

For the various *e-introduction* rules we have:

$$\langle\langle\Gamma_1, \underline{f} e B\rangle, \langle\Gamma_1 \setminus \{f a A\}, A e B\rangle\rangle$$

Where \underline{f} does not occur in $\Gamma_1 \setminus \{f a A\}$.

$$\langle\langle\Gamma_1, \underline{f}^{\bar{\nabla}} e B\rangle, \langle\Gamma_1 \setminus \{f^M a A\}, A^{\Delta} e B\rangle\rangle$$

Where \underline{f} does not occur in $\Gamma_1 \setminus \{f a A\}$.

For *Empty Exposition* we have in the assertoric case:

$$\langle\langle\Gamma, \underline{f}eA\rangle, \langle\Gamma, AoB\rangle\rangle$$

Where we require that \underline{f} does not occur in Γ .

In the *modal case*:

$$\langle\langle\Gamma, \underline{f} \overset{L}{e} A\rangle, \langle\Gamma, A \overset{\nabla}{o} B\rangle\rangle$$

Where we require that \underline{f} does not occur in Γ .

For \wedge Introduction we have:

$$\langle\langle\Gamma_1, \phi\rangle, \langle\Gamma_2, \psi\rangle, \langle\Gamma_1 \cup \Gamma_2, \phi \wedge \psi\rangle\rangle$$

The rules for *Necessity Introduction* are:

$$\begin{aligned} &\langle\langle\Gamma_1, \underline{d}aA\rangle, \langle\Gamma_1, \underline{d} \overset{L}{a} A\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d}eA\rangle, \langle\Gamma_1, \underline{d} \overset{L}{e} A\rangle\rangle \end{aligned}$$

Where Γ_1 is a set of modalised singular formulae of possibility or necessity. A formula is said to be modalised if and only if it is not of the form $A \overset{\nabla}{\times} B$ or $\underline{d} \overset{\nabla}{\times} B$ where ∇ is any one of L, M, Q , or \bar{Q} and \times is any of a, e, i , or o in the case of categorical propositions and \times is either a or e in the case of singular propositions.

Interaction Rules

The inference rules for the interaction rules are defined as follows:

Modal Subalternation^L

$$\langle\langle\Gamma_1, A \overset{L}{\times} B\rangle, \langle\Gamma_1, A \overset{M}{\times} B\rangle\rangle$$

Modal Subalternation^Q

$$\langle\langle\Gamma_1, A \overset{Q}{\times} B\rangle, \langle\Gamma_1, A \overset{M}{\times} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{e} B\rangle, \langle\Gamma_1, A \overset{M}{a} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{i} B\rangle, \langle\Gamma_1, A \overset{M}{i} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{o} B\rangle, \langle\Gamma_1, A \overset{M}{a} B\rangle\rangle$$

As in the case of ∇ the use of a/e and i/o should be preserved vertically. I.e. From $A \overset{Q}{a} B$ one may conclude $A \overset{M}{a} B$, and similarly from $A \overset{Q}{e} B$ one may conclude $A \overset{M}{e} B$. For clarity we have included all four instances of the rules.

Subalternation⁺ $\langle\langle\Gamma_1, A a B\rangle, \langle\Gamma_1, A i B\rangle\rangle$

$$\langle\langle\Gamma_1, A \overset{L}{a} B\rangle, \langle\Gamma_1, A \overset{L}{i} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{M}{a} B\rangle, \langle\Gamma_1, A \overset{M}{i} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{a} B\rangle, \langle\Gamma_1, A \overset{Q}{i} B\rangle\rangle$$

$$\langle\langle\Gamma_1, A \overset{Q}{e} B\rangle, \langle\Gamma_1, A \overset{Q}{o} B\rangle\rangle$$

As in the case of a/e L/M in the rule should be read vertically. All forms of the rule are written here.

Subalternation⁻

$$\begin{aligned} &\langle\langle\Gamma_1, A e B\rangle, \langle\Gamma_1, A o B\rangle\rangle \\ &\quad \langle\langle\Gamma_1, A \overset{L}{e} B\rangle, \langle\Gamma_1, A \overset{L}{o} B\rangle\rangle \\ &\quad \langle\langle\Gamma_1, A \overset{M}{e} B\rangle, \langle\Gamma_1, A \overset{M}{o} B\rangle\rangle \\ &\quad \langle\langle\Gamma_1, A \overset{\bar{Q}}{a} B\rangle, \langle\Gamma_1, A \overset{\bar{Q}}{i} B\rangle\rangle \\ &\quad \langle\langle\Gamma_1, A \overset{\bar{Q}}{e} B\rangle, \langle\Gamma_1, A \overset{\bar{Q}}{e} B\rangle\rangle \end{aligned}$$

As in the case of a/e , L/M in the rule should be read vertically. All forms of the rule are written here.

Q-Sing Equivalence:

$$\begin{aligned} &\langle\langle\Gamma_1, \underline{d} \overset{Q}{a} A\rangle, \langle\Gamma_1, \underline{d} \overset{Q}{e} A\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d} \overset{Q}{e} A\rangle, \langle\Gamma_1, \underline{d} \overset{Q}{a} A\rangle\rangle \end{aligned}$$

\bar{Q} -Sing Equivalence:

$$\begin{aligned} &\langle\langle\Gamma_1, \underline{d} \overset{\bar{Q}}{a} A\rangle, \langle\Gamma_1, \underline{d} \overset{\bar{Q}}{e} A\rangle\rangle \\ &\langle\langle\Gamma_1, \underline{d} \overset{\bar{Q}}{e} A\rangle, \langle\Gamma_1, \underline{d} \overset{\bar{Q}}{a} A\rangle\rangle \end{aligned}$$

Q a-e Equivalence:

$$\begin{aligned} &\langle\langle\Gamma_1, A \overset{Q}{a} B\rangle, \langle\Gamma_1, A \overset{Q}{e} B\rangle\rangle \\ &\langle\langle\Gamma_1, A \overset{Q}{e} B\rangle, \langle\Gamma_1, A \overset{Q}{a} B\rangle\rangle \end{aligned}$$

Q i-o Equivalence:

$$\begin{aligned} &\langle\langle\Gamma_1, A \overset{Q}{i} B\rangle, \langle\Gamma_1, A \overset{Q}{o} B\rangle\rangle \\ &\langle\langle\Gamma_1, A \overset{Q}{o} B\rangle, \langle\Gamma_1, A \overset{Q}{i} B\rangle\rangle \end{aligned}$$

Q a-e Equivalence:

$$\begin{aligned} &\langle\langle\Gamma_1, A \overset{\bar{Q}}{a} B\rangle, \langle\Gamma_1, A \overset{\bar{Q}}{e} B\rangle\rangle \\ &\langle\langle\Gamma_1, A \overset{\bar{Q}}{e} B\rangle, \langle\Gamma_1, A \overset{\bar{Q}}{a} B\rangle\rangle \end{aligned}$$

\bar{Q} i-o Equivalence:

$$\begin{aligned} &\langle\langle\Gamma_1, A \overset{\bar{Q}}{i} B\rangle, \langle\Gamma_1, A \overset{\bar{Q}}{o} B\rangle\rangle \\ &\langle\langle\Gamma_1, A \overset{\bar{Q}}{o} B\rangle, \langle\Gamma_1, A \overset{\bar{Q}}{i} B\rangle\rangle \end{aligned}$$

Q - Out

$$\langle\langle\Gamma_1, \underline{d} \overset{Q}{a} A\rangle, \langle\Gamma_1, \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A\rangle\rangle$$

Q - In

$$\langle \langle \Gamma_1, \underline{d}^M A \wedge \underline{d}^M A \rangle, \langle \Gamma_1, \underline{d}^Q A \rangle \rangle$$

CC Elimination:

$$\langle \langle \Gamma_1, C(C(\phi)) \rangle, \langle \Gamma, \phi \rangle, \rangle$$

11.2 Soundness

In this section, we show that if $\Gamma \vdash \phi$ then $\Gamma \models \phi$. As is normally the case with soundness proofs, the proof is by induction on the length of the derivation. Before specifying how the soundness proof will proceed, we will often make use of the following (simple) lemma.

Given two assignments $c : SING \rightarrow D$ and $c' : SING \rightarrow D$, we write $c' \stackrel{\sim}{\underline{k}} c$ if and only if c' differs from c in assignment of elements of $SING$ at most with respect to \underline{k} .

Given a (Buridan Modal) Model \mathfrak{M} , define \mathfrak{M}' to be the same model as \mathfrak{M} except that c is replaced with c' .

Observation 1:

For all formulae ϕ , models \mathfrak{M} and $w \in W$ if $c \stackrel{\sim}{\underline{k}} c'$ and \underline{k} does not occur in ϕ then $\mathfrak{M}, w \models \phi$ if and only if $\mathfrak{M}', w \models \phi$.

The proof of this observation is immediate from the definitions of $c \stackrel{\sim}{\underline{k}} c'$ and \mathfrak{M}' .

We now turn to the proof of soundness. We claim:

$$\text{If } \Gamma \vdash \phi \text{ then } \Gamma \models \phi$$

Recall that $\Gamma \models \phi$ if and only if for all \mathfrak{M} and $w \in W$ if $\mathfrak{M}, w \models \Gamma$ then $\mathfrak{M}, w \models \phi$.

To that end, take an arbitrary model \mathfrak{M} and $w \in W$. The proof is by induction on the length of derivation.

Length = 1

If $\Gamma \vdash \phi$ and the derivation is of length 1, then this can only happen if $\phi \in \Gamma$, as our system has no axioms. In that case we need to show that $\Gamma \models \phi$, given $\phi \in \Gamma$. This is immediate from the definition of \models .

Length n

As our induction hypothesis assumes that we have, given $\Gamma \vdash \phi$, that $\Gamma \models \phi$ for all derivations of length $< n$. We claim that this also holds for derivations of length n . To that end, it suffices to show that all of our inference rules preserve validity. We will break this into subsections and start with the interchange rules:

11.2.1 Interchange Rules

Modal Subalternation^L

We claim that the rule:

$$\frac{A \stackrel{L}{\times} B}{A \stackrel{M}{\times} B} \text{Modal Subalternation}^L$$

11 Appendix Four: Soundness

is sound with respect to our semantics.

We have four cases to consider, namely when x is a , x is e , x is o , and x is i .

For x is a :

As the length of the proof that $\Gamma \vdash A \overset{L}{a} B$ is $< n$ it follows by our inductive hypothesis it follows that $\Gamma \vDash A \overset{L}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{L}{a} B$. This holds if and only if $M(w, A) \neq \emptyset$ and $M(w, A) \subseteq L(w, B)$. By Lemma 1.1. we have $L(w, B) \subseteq M(w, B)$ and so $M(w, A) \neq \emptyset$ and $M(w, A) \subseteq M(w, B)$. Hence $\mathfrak{M}, w \vDash A \overset{M}{a} B$ as claimed.

For x is e :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{L}{e} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{L}{e} B$. This holds if and only if $M(w, A) \cap M(w, B) = \emptyset$. By Lemma 1.1. we have $L(w, B) \subseteq M(w, B)$ and so $M(w, A) \cap L(w, B) = \emptyset$. Hence $\mathfrak{M}, w \vDash A \overset{M}{e} B$ as claimed.

For x is i :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{L}{i} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{L}{i} B$. This holds if and only if $M(w, A) \cap L(w, B) \neq \emptyset$. By Lemma 1.1. we have $L(w, B) \subseteq M(w, B)$ and so $M(w, A) \cap M(w, B) \neq \emptyset$. Hence $\mathfrak{M}, w \vDash A \overset{M}{i} B$ as claimed.

For x is o :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{L}{o} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{L}{o} B$. This holds if and only if $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq M(w, B)$. By Lemma 1.1. we have $L(w, B) \subseteq M(w, B)$ and so either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq L(w, B)$. Hence $\mathfrak{M}, w \vDash A \overset{M}{o} B$ as claimed.

Modal Subalternation^Q We claim that the rules:

$$\frac{A \overset{Q}{a/e} B}{A \overset{M}{a} B} \text{Modal Subalternation}^Q$$

and

$$\frac{A \overset{Q}{i/o} B}{A \overset{M}{i} B} \text{Modal Subalternation}^Q$$

are sound with respect to our semantics. We would normally have four cases to consider but since $\overset{Q}{a}$ and $\overset{Q}{e}$ are semantically equivalent and $\overset{Q}{i}$ and $\overset{Q}{o}$ are semantically equivalent, we have only two cases to consider.

For the cases of a and e :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{Q}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{Q}{a} B$. Then $M(w, A) \neq \emptyset$ and $M(w, A) \subseteq$

$(M(w, B) \cap M(w, \neg B))$. It follows by basic set theory that $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$. Hence $\mathfrak{M}, w \models A \overset{M}{a} B$.

For the cases of i and o :

By our inductive hypothesis it follows that $\Gamma \models A \overset{Q}{i} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models A \overset{Q}{i} B$. Then $M(w, A) \cap (M(w, B) \cap M(w, \neg B)) \neq \emptyset$. It follows by basic set theory that $M(w, A) \cap (M(w, B) \cap M(w, \neg B)) \neq \emptyset$. Hence $\mathfrak{M}, w \models A \overset{M}{i} B$.

Subalternation⁺ We claim that the rule:

$$\text{Subalternation}^+ \frac{A a B}{A i B}$$

is sound with respect to our semantics. By our inductive hypothesis it follows that $\Gamma \models A a B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models A a B$. Hence 1) $V(w, A) \subseteq V(w, B)$ and 2) $V(w, A) \neq \emptyset$. Since $V(w, A)$ there is some \underline{d} such that $\underline{d} \in V(w, A)$. by 1) $\underline{d} \in V(w, B)$ also and so $V(w, A) \cap V(w, B) \neq \emptyset$. Hence $\mathfrak{M}, w \models A i B$.

Subalternation⁻ We claim that the rule:

$$\frac{A e B}{A o B}$$

is sound with respect to our semantics. By our inductive hypothesis it follows that $\Gamma \models A e B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models A e B$. Hence $V(w, A) \cap V(w, B) = \emptyset$. It follows by basic set theory that either $V(w, A) = \emptyset$ or $V(w, A) \not\subseteq V(w, B)$. To see this, observe that we have three cases to consider. If $V(w, A) = \emptyset$, then the left disjunct holds. If $V(w, A) \neq \emptyset$ and $V(w, B) = \emptyset$ then the right disjunct holds. Likewise, if $V(w, A) \neq \emptyset$ and $V(w, B) \neq \emptyset$ then by our assumption $V(w, A)$ and $V(w, B)$ are disjoint and so the right disjunct holds.

Subalternation⁺ We claim that the rule:

$$\frac{A \overset{\nabla}{a} B}{A \overset{\nabla}{i} B}$$

is sound with respect to our semantics.

We have four cases to consider, namely the cases where ∇ is L , M , Q , and \bar{Q} respectively.

In the case of L :

By our inductive hypothesis it follows that $\Gamma \models A \overset{L}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models A \overset{L}{a} B$. Hence 1) $M(w, A) \neq \emptyset$ and 2) $M(w, A) \subseteq L(w, B)$. From 1) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$. From this and 2) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$ and $\underline{d} \in L(w, B)$. Hence $M(w, A) \cap L(w, B) \neq \emptyset$ and so $\mathfrak{M}, w \models A \overset{L}{i} B$ as claimed.

11 Appendix Four: Soundness

In the case of M :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{M}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{M}{a} B$. Hence 1) $M(w, A) \neq \emptyset$ and 2) $M(w, A) \subseteq M(w, B)$. From 1) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$. From this and 2) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$ and $\underline{d} \in M(w, B)$. Hence $M(w, A) \cap M(w, B) \neq \emptyset$ and so $\mathfrak{M}, w \vDash A \overset{M}{i} B$ as claimed.

In the case of Q :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{Q}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{Q}{a} B$. Hence 1) $M(w, A) \neq \emptyset$ and 2) $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$. From 1) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$. From this and 2) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$ and $\underline{d} \in M(w, B)$ and $\underline{d} \in M(w, \neg B)$. Hence $M(w, A) \cap M(w, B) \cap M(w, \neg B) \neq \emptyset$ and so $\mathfrak{M}, w \vDash A \overset{Q}{i} B$ as claimed.

In the case of \bar{Q} :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{\bar{Q}}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{\bar{Q}}{a} B$. Hence 1) $M(w, A) \cap M(w, B) \cap M(w, \neg B) = \emptyset$. We claim that either $M(w, A) = \emptyset$ or that $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$. Assume not. Then 2) $M(w, A) \neq \emptyset$ and 3) $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$. From 2) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$. From this and 3) it follows that some \underline{d} such that $\underline{d} \in M(w, A)$, $\underline{d} \in M(w, B)$, and $\underline{d} \in M(w, \neg B)$. But this contradicts 1). Hence either $M(w, A) = \emptyset$ or that $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$ as claimed. It then follows that $\mathfrak{M}, w \vDash A \overset{\bar{Q}}{i} B$.

Subalternation⁻ We claim that the rule:

$$\frac{A \overset{\nabla}{e} A}{A \overset{\nabla}{o} B}$$

is sound with respect to our semantics. We prove the cases for L and M directly. It should be observed that the cases of Q and \bar{Q} are nearly identical to the cases proven in *subalternation⁺*, only requiring the additional observations that $\mathfrak{M}, w \vDash A \overset{Q/\bar{Q}}{a} B$ if and only if $\mathfrak{M}, w \vDash A \overset{Q/\bar{Q}}{e} B$ and that $\mathfrak{M}, w \vDash A \overset{Q/\bar{Q}}{i} B$ if and only if $\mathfrak{M}, w \vDash A \overset{Q/\bar{Q}}{o} B$.

In the case of L :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{L}{e} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{L}{e} B$. Hence $M(w, A) \cap M(w, B) = \emptyset$. We claim that either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq M(w, B)$. Assume not. Then 1) $M(w, A) \neq \emptyset$ and 2) $M(w, A) \subseteq M(w, B)$. From 1) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$. This, together with 2) entails that there is some \underline{d} such that $\underline{d} \in M(w, A)$ and $\underline{d} \in M(w, B)$. But this contradicts $M(w, A) \cap M(w, B) = \emptyset$. Hence either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq M(w, B)$ and so $\mathfrak{M}, w \vDash A \overset{L}{o} B$ as claimed.

In the case of M :

By our inductive hypothesis it follows that $\Gamma \vDash A \overset{M}{e} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash A \overset{M}{e} B$. Hence $M(w, A) \cap L(w, B) = \emptyset$. We claim that either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq L(w, B)$. Assume not. Then 1) $M(w, A) \neq \emptyset$ and 2) $M(w, A) \subseteq L(w, B)$. From 1) it follows that there is some \underline{d} such that $\underline{d} \in M(w, A)$. This, together with 2) entails that there is some \underline{d} such that $\underline{d} \in M(w, A)$ and $\underline{d} \in L(w, B)$. But this contradicts $M(w, A) \cap L(w, B) = \emptyset$. Hence either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq L(w, B)$ and so $\mathfrak{M}, w \vDash A \overset{M}{o} B$ as claimed.

Q sing Equivalence We claim that the rule:

$$\frac{\underline{d} \overset{Q}{a} A}{\underline{d} \overset{Q}{e} A}$$

is sound with respect to our semantics.

In both directions, the soundness follows immediately from the semantic definitions.

I.e. we have $\mathfrak{M}, w \vDash \underline{d} \overset{Q}{a} A$ if and only if $\mathfrak{M}, w \vDash \underline{d} \overset{Q}{e} A$.

\bar{Q} sing Equivalence We claim that the rule:

$$\frac{\underline{d} \overset{\bar{Q}}{a} A}{\underline{d} \overset{Q}{e} A}$$

is sound with respect to our semantics.

In both directions, the soundness follows immediately from the semantic definitions.

I.e. we have $\mathfrak{M}, w \vDash \underline{d} \overset{Q}{e} A$ if and only if $\mathfrak{M}, w \vDash \underline{d} \overset{\bar{Q}}{a} A$.

Q a-e Equivalence We claim that the rule:

$$\frac{A \overset{Q}{a} B}{A \overset{Q}{e} B}$$

is sound with respect to our semantics.

In both directions, the soundness follows immediately from the semantic definitions.

I.e. we have $\mathfrak{M}, w \vDash A \overset{Q}{a} B$ if and only if $\mathfrak{M}, w \vDash A \overset{Q}{e} B$.

Q i-o Equivalence We claim that the rule:

$$\frac{A \overset{Q}{i} B}{A \overset{Q}{o} B}$$

is sound with respect to our semantics.

In both directions, the soundness follows immediately from the semantic definitions.

That is, we have $\mathfrak{M}, w \vDash A \overset{Q}{i} B$ if and only if $\mathfrak{M}, w \vDash A \overset{Q}{o} B$.

Q -Out We claim that the rule:

$$\frac{\underline{d} \overset{Q}{a} A}{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{Q}{a} A$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{Q}{a} A$. This holds if and only if 1) $c(\underline{d}) \in M(w, A)$ and 2) $c(\underline{d}) \in M(w, \neg A)$. 1) holds if and only if $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A$.

2) holds if and only if $c(\underline{d}) \notin L(w, A)$ if and only if $\mathfrak{M}, w \models \underline{d} \overset{M}{e} A$. The first biconditional holds by basic logic, the second because of the semantic definition of $\overset{L}{e}$.

Then clearly, we have $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A$.

Q-In We claim that the rule:

$$\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} A}{\underline{d} \overset{Q}{a} A}$$

is sound with respect to our semantics.

This follows from the previous proof, as each step was a biconditional.

CC Elimination We claim that the rule:

$$\frac{C(C(\phi))}{\phi}$$

is sound with respect to our semantics. Recall that two formulae are said to be contradictory, if for all models, \mathfrak{M} and all $w \in W$ $\mathfrak{M}, w \models \phi$ if and only if $\mathfrak{M}, w \not\models \psi$. We denote ψ by $C(\phi)$.

Observe that, for all formulae ϕ we have $\mathfrak{M}, w \models \phi$ if and only if $\mathfrak{M}, w \not\models C(\phi)$ if and only if $\mathfrak{M}, w \not\models C(C(\phi))$. Both biconditionals follow from the definition of contradictory, and soundness of the rule is an easy corollary of this observation.

11.2.2 Introduction Rules

Expository Syllogism ⁺ We claim that the rule:

$$\frac{\underline{d} a A \wedge \underline{d} a B}{A i B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} a A \wedge \underline{d} a B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} a A \wedge \underline{d} a B$. Hence $c(\underline{d}) \in V(w, A)$ and $c(\underline{d}) \in V(w, B)$. Hence there is some d such that $d \in V(w, A) \cap V(w, B)$. So $\mathfrak{M}, w \models A i B$.

Expository Syllogism ⁻ We claim that the rule:

$$\frac{\underline{d} a A \wedge \underline{d} e B}{A o B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d}aA \wedge \underline{d}eB$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d}aA \wedge \underline{d}eB$. Hence $c(\underline{d}) \in V(w, A)$ and $c(\underline{d}) \notin V(w, B)$. It then follows by basic logic that $V(w, A) \not\subseteq V(w, B)$. Hence either $V(w, A) = \emptyset$ or $V(w, A) \not\subseteq V(w, B)$. So $\mathfrak{M}, w \models AoB$.

Expository Syllogism L^+ We claim that the rule:

$$\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{L}{a} B}{A \overset{L}{i} B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{L}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{L}{a} B$. Hence $c(\underline{d}) \in M(w, A)$ and $c(\underline{d}) \in L(w, B)$. Hence there is some d such that $d \in M(w, A) \cap L(w, B)$. So $\mathfrak{M}, w \models A \overset{L}{i} B$.

Expository Syllogism L^- We claim that the rule:

$$\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{L}{e} B}{A \overset{L}{o} B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{L}{e} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{L}{e} B$. Hence $c(\underline{d}) \in M(w, A)$ and $c(\underline{d}) \notin M(w, B)$. It then follows by basic logic that $M(w, A) \not\subseteq M(w, B)$. Hence either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq M(w, B)$. So $\mathfrak{M}, w \models A \overset{L}{o} B$.

Expository Syllogism M^+ We claim that the rule:

$$\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{a} B}{A \overset{M}{i} B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{a} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{a} B$. Hence $c(\underline{d}) \in M(w, A)$ and $c(\underline{d}) \in M(w, B)$. Hence there is some d such that $d \in M(w, A) \cap M(w, B)$. So $\mathfrak{M}, w \models A \overset{M}{i} B$.

Expository Syllogism M^- We claim that the rule:

$$\frac{\underline{d} \overset{M}{a} A \wedge \underline{d} \overset{M}{e} B}{A \overset{M}{o} B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \vDash \underline{d}^M A \wedge \underline{d}^M B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash \underline{d}^M A \wedge \underline{d}^M B$. Hence $c(\underline{d}) \in M(w, A)$ and $c(\underline{d}) \in L(w, B)$. It then follows by basic logic that $M(w, A) \not\subseteq L(w, B)$. Hence either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq L(w, B)$. So $\mathfrak{M}, w \vDash A \overset{M}{o} B$.

Expository Syllogism Q We claim that the rule:

$$\frac{\underline{d}^M A \wedge \underline{d}^Q B}{A \overset{Q}{i} B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \vDash \underline{d}^M A \wedge \underline{d}^Q B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash \underline{d}^M A \wedge \underline{d}^Q B$. Hence $c(\underline{d}) \in M(w, A)$ and $c(\underline{d}) \in M(w, B) \cap M(w, \neg B)$. Hence there is some d such that $d \in M(w, A) \cap M(w, B) \cap M(w, \neg B)$. So $\mathfrak{M}, w \vDash A \overset{Q}{i} B$.

Expository Syllogism \bar{Q} We claim that the rule:

$$\frac{\underline{d}^M A \wedge \underline{d}^{\bar{Q}} B}{A \overset{\bar{Q}}{i} B}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \vDash \underline{d}^M A \wedge \underline{d}^{\bar{Q}} B$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \vDash \Gamma$. Then $\mathfrak{M}, w \vDash \underline{d}^M A \wedge \underline{d}^{\bar{Q}} B$. Hence $\mathfrak{M}, w \vDash \underline{d}^M A$ and $\mathfrak{M}, w \vDash \underline{d}^{\bar{Q}} B$ and so $c(\underline{d}) \in M(w, A)$ and $c(\underline{d}) \in L(w, B) \cup L(w, \neg B)$.

Assume that $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$. As $c(\underline{d}) \in M(w, A)$, it follows that $c(\underline{d}) \in M(w, B)$ and $c(\underline{d}) \in M(w, \neg B)$.

Hence 1) $\exists x$ such that wRx and $\underline{d} \in V(x, B)$ and 2) $\exists y$ such that wRy and $\underline{d} \in V(y, \neg B)$. However, we already have $c(\underline{d}) \in L(w, B) \cup L(w, \neg B)$ hence either $c(\underline{d}) \in L(w, B)$ or $c(\underline{d}) \in L(w, \neg B)$. If the first disjunct holds, then we have:

$\forall x$ if wRx then $c(\underline{d}) \in V(w, B)$

which contradicts 2). However, if the second disjunct holds, a similar contradiction follows from 1). Hence $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$. Hence either $M(w, A) = \emptyset$

or $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$. So $\mathfrak{M}, w \vDash A \overset{\bar{Q}}{i} B$.

a-introduction We claim that the rule:

$$\frac{A \overset{f}{i} A \quad \frac{[f a A]}{f a B}}{A a B}$$

(where f does not occur in any open assumption other than the discharged assumption $f a A$) is sound with respect to our semantics.

In order to show this rule is sound, note that by the inductive hypothesis (since their proofs are shorter):

- 1) $\Gamma \models A \text{ i } A$
- 2) $\Delta \models \underline{f}aB$

We need to show that $\Gamma \cup (\Delta \setminus \{\underline{f} a A\}) \models A a B$. To that end, take an arbitrary model \mathfrak{M} and world $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma \cup (\Delta \setminus \{\underline{f} a A\})$. Then $\mathfrak{M}, w \models A \text{ i } A$ and so $V(w, A) \neq \emptyset$.

We need to show that $V(w, A) \subseteq V(w, B)$. Take an arbitrary $s \in D$ and assume that $s \in V(w, A)$.

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \underset{\underline{f}}{\sim} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f}aA\}$ and $\mathfrak{M}', w \models \Gamma$ since \underline{f} does not occur in $\Delta \setminus \{\underline{f}aA\}$ nor in Γ .

Moreover, $\mathfrak{M}', w \models \underline{f}aA$ since $c'(\underline{f}) \in V(w, A)$. Hence $\mathfrak{M}', w \models \Delta$ and so by 2) it follows that $\mathfrak{M}', w \models \underline{f} a B$, whence $s = c'(\underline{f}) \in V(w, B)$. As s was arbitrary, it follows that $V(w, A) \subseteq V(w, B)$. Hence $\mathfrak{M}', w \models A a B$. But \underline{f} does not occur in $A a B$ and $c' \underset{\underline{f}}{\sim} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models A a B$.

e-introduction We claim that the rule:

$$\frac{\frac{[\underline{f}aA]}{\underline{f}eB}}{AeB}$$

is sound with respect to our semantics.

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f}aA$).

In order to show this rule is sound, note that by the inductive hypothesis (since the proof is shorter):

- 1) $\Gamma \models \underline{f}eB$

We need to show that $\Gamma \setminus \{\underline{f} a A\} \models AeB$. To that end, take an arbitrary model \mathfrak{M} , $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma \setminus \{\underline{f} a A\}$. We need to show that $\mathfrak{M}, w \models AeB$, i.e. $V(w, A) \cap V(w, B) = \emptyset$.

To that end, assume that $s \in V(w, A)$. Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \underset{\underline{f}}{\sim} c$. Since \underline{f} does not occur in $\Gamma \setminus \{\underline{f} a A\}$, we have $\mathfrak{M}', w \models \Gamma \setminus \{\underline{f} a A\}$.

Moreover, $\mathfrak{M}', w \models \underline{f}aA$ since $c'(\underline{f}) \in V(w, A)$. Hence $\mathfrak{M}', w \models \Gamma$. Hence $\mathfrak{M}', w \models \underline{f}eB$ that is to say, $s = \underline{f} \in \bar{V}(w, B)$. As s was arbitrary, it follows that $V(w, A) \cap V(w, B) = \emptyset$. Hence $\mathfrak{M}', w \models Ae\bar{B}$. But \underline{f} does not occur in $Ae\bar{B}$ and $c' \underset{\underline{f}}{\sim} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models AeB$.

a[∇]-introduction We claim that the rule:

$$\frac{A \overset{M}{i} A \quad \underline{f} \overset{\nabla}{a} B}{A \overset{\nabla}{a} B} \quad [\underline{f} \overset{M}{a} A]$$

is sound with respect to our semantics. The proof in each case is analogous to the assertoric case. We prove the case for L .

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f} \overset{M}{a} A$).

In order to show this rule is sound, note that by the inductive hypothesis (since their proofs are shorter):

- 1) $\Gamma \models A \overset{M}{i} A$
- 2) $\Delta \models \underline{f} \overset{L}{a} B$

We need to show that $\Gamma \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A\}) \models A \overset{L}{a} B$. To that end, take an arbitrary model \mathfrak{M} and world $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A\})$. Then $\mathfrak{M}, w \models A \overset{M}{i} A$ and so $M(w, A) \neq \emptyset$.

We need to show that $M(w, A) \subseteq L(w, B)$. Take an arbitrary $s \in D$ and assume that $s \in M(w, A)$.

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f} \overset{M}{a} A\}$ and $\mathfrak{M}', w \models \Gamma$ since \underline{f} does not occur in $\Delta \setminus \{\underline{f} \overset{M}{a} A\}$ nor in Γ .

Moreover, $\mathfrak{M}', w \models \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$. Hence $\mathfrak{M}', w \models \Delta$ and so by 2) it follows that $\mathfrak{M}', w \models \underline{f} \overset{L}{a} B$, whence $s = c'(\underline{f}) \in L(w, B)$. As s was arbitrary, it follows that $M(w, A) \subseteq L(w, B)$. But \underline{f} does not occur in $A \overset{L}{a} B$ and $c' \sim_{\underline{f}} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models A \overset{L}{a} B$.

e^Δ -introduction We claim that the rule:

$$\frac{\underline{f} \overset{\nabla}{e} B}{A \overset{\Delta}{e} B} \quad [\underline{f} \overset{M}{a} A]$$

is sound with respect to our semantics. The proof in each case is analogous to the assertoric case. We prove the case for $\overset{L}{e}$. i.e we show

$$\frac{[\underline{f} \overset{M}{a} A]}{\frac{\underline{f} \overset{L}{e} B}{A \overset{L}{e} B}}$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f}aA$).

In order to show this rule is sound, note that by the inductive hypothesis (since the proof is shorter):

$$1) \Gamma \models \underline{f} \overset{L}{e} B$$

We need to show that $\Gamma \setminus \{\underline{f} \overset{M}{a} A\} \models A \overset{L}{e} B$. To that end, take an arbitrary model \mathfrak{M} , $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma \setminus \{\underline{f} \overset{M}{a} A\}$. We need to show that $\mathfrak{M}, w \models A \overset{L}{e} B$, i.e. $M(w, A) \cap M(w, B) = \emptyset$.

To that end, assume that $s \in M(w, A)$. Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \underset{\underline{f}}{\sim} c$. Since \underline{f} does not occur in $\Gamma \setminus \{\underline{f} \overset{M}{a} A\}$, we have $\mathfrak{M}', w \models \Gamma \setminus \{\underline{d} \overset{M}{a} A\}$.

Moreover, $\mathfrak{M}', w \models \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$. Hence $\mathfrak{M}', w \models \Gamma$. Hence $\mathfrak{M}', w \models \underline{f} \overset{L}{e} B$ that is to say, $s = \underline{f} \notin M(w, B)$. As s was arbitrary, it follows that $M(w, A) \cap M(w, B) = \emptyset$. Hence $\mathfrak{M}', w \models A \overset{L}{e} B$. But \underline{f} does not occur in $A \overset{L}{e} B$ and $c' \underset{\underline{f}}{\sim} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models A \overset{L}{e} B$.

Empty Exposition We claim that the rule:

$$\frac{\underline{f}eA}{A \circ B}$$

(where \underline{f} does not occur in any open assumption) is sound with respect to our semantics.

In order to show this rule is sound, note that by the inductive hypothesis (since the proof is shorter):

$$1) \Gamma \models \underline{f}eA$$

We want to show that from this we can infer that $V(w, A) = \emptyset$ and then use that to conclude $A \circ B$. To that end, take an arbitrary model \mathfrak{M} , $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma$. We need to show that $\Gamma \models A \circ B$.

Assume that $V(w, A) \neq \emptyset$. Then there is some $t \in D$ such that $t \in V(w, A)$. Call this element s . Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. It then follows that $c'(\underline{f}) \in V(w, A)$. However, by construction we have $c' \underset{\underline{f}}{\sim} c$. Since \underline{f} does not occur in Γ we have $\mathfrak{M}', w \models \Gamma$ and so $\mathfrak{M}', w \models \underline{f}eA$. Hence $s = \underline{f} \notin V(w, A)$. This contradicts our assumption that $s \in V(w, A)$. Hence $V(w, A) = \emptyset$. It then follows by

basic logic that $V(w, A) = \emptyset$ or $V(w, A) \not\subseteq V(w, B)$. Hence $\mathfrak{M}', w \models AoB$. Since \underline{f} does not occur in AoB and $c' \sim_{\underline{f}} c$, it follows by Observation 1 that $\mathfrak{M}, w \models AoB$.

Empty Exposition We claim that the rule:

$$\frac{\underline{f} \overset{L}{e} A}{A \overset{\nabla}{o} B}$$

is sound with respect to our semantics. The proof in each case is analogous to the assertoric case. We prove the case for L . The general proof strategy is the same as before.

To that end, take an arbitrary model \mathfrak{M} , $w \in W$ and assume that $\mathfrak{M}, w \models \Gamma$. We need to show that $\Gamma \models A \overset{L}{o} B$.

Then assume that $M(w, A) \neq \emptyset$ and see that there is some $t \in D$ such that $t \in M(w, A)$. Call this element s . Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. It then follows that $c'(\underline{f}) \in M(w, A)$. However, by construction we have $c' \sim_{\underline{f}} c$. Since \underline{f} does not occur in Γ we have $\mathfrak{M}', w \models \Gamma$ and so $\mathfrak{M}, w \models \underline{f} \overset{L}{e} A$. Hence $s = \underline{f} \notin M(w, A)$. This contradicts our assumption that $s \in M(w, A)$. Hence $M(w, A) = \emptyset$. It then follows by basic logic that $M(w, A) = \emptyset$ or $M(w, A) \subseteq M(w, B)$. Hence $\mathfrak{M}', w \models A \overset{L}{o} B$. Since \underline{f} does not occur in $A \overset{L}{o} B$ and $c' \sim_{\underline{f}} c$, it follows by Observation 1 that $\mathfrak{M}, w \models A \overset{L}{o} B$.

\bar{Q} -In We claim that the rule:

$$\frac{\underline{d} \overset{L}{a} A}{\underline{d} \overset{\bar{Q}}{a} A}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{L}{a} A$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{L}{a} A$. Hence $c(\underline{d}) \in L(w, A)$. By basic set theory it follows that $c(\underline{d}) \in L(w, A) \cup L(w, \neg A)$. Hence $\mathfrak{M}, w \models \underline{d} \overset{\bar{Q}}{a} A$.

\bar{Q} -In We claim that the rule:

$$\frac{\underline{d} \overset{L}{e} A}{\underline{d} \overset{\bar{Q}}{e} A}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{L}{e} A$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{L}{e} A$. So, $\mathfrak{M}, w \models \underline{d} \overset{L}{e} A$. Hence $c(\underline{d}) \notin M(w, A)$. So $c(\underline{d}) \in L(w, \neg A)$. By basic set theory it follows that $c(\underline{d}) \in L(w, A) \cup L(w, \neg A)$. Hence $\mathfrak{M}, w \models \underline{d} \overset{\bar{Q}}{e} A$.

\wedge **Introduction** We claim that the rule:

$$\frac{\phi \quad \psi}{\phi \wedge \psi}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \phi$ and $\Delta \models \mathfrak{M}, w \models \psi$. So, take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$. By the semantic definition of \wedge it follows that $\mathfrak{M}, w \models \phi \wedge \psi$. Hence $\Gamma \cup \Delta \models \phi \wedge \psi$.

Necessity Introduction We claim that the rule:

$$\frac{\underline{f}aA}{\underline{f} \overset{L}{a} B}$$

is sound with respect to our semantics.

The proof of this claim can be found in Chapter Five 135.

Necessity Introduction We claim that the rule:

$$\frac{\underline{f}eA}{\underline{f} \overset{L}{e} A}$$

is sound with respect to our semantics.

The proof of this claim can be found in Chapter Five 135.

Reflexivity We claim that the rule:

$$\frac{\underline{d}aA}{\underline{d} \overset{M}{a} A}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d}aA$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d}aA$. Hence $c(\underline{d}) \in V(w, A)$. Since R is reflexive, it follows that wRw . Hence $c(\underline{d}) \in M(w, A)$ and so $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A$.

Reflexivity We claim that the rule:

$$\frac{\underline{d} \overset{L}{e} A}{\underline{d}eA}$$

is sound with respect to our semantics.

By our inductive hypothesis it follows that $\Gamma \models \underline{d} \overset{L}{e} A$. Take an arbitrary model \mathfrak{M} and world w such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \underline{d} \overset{L}{e} A$. Hence $c(\underline{d}) \notin M(w, A)$. Since R is reflexive, it follows that wRw . Hence $c(\underline{d}) \notin V(w, A)$ and so $\mathfrak{M}, w \models \underline{d}eA$.

C-Introduction We claim that the rule:

$$\frac{[\phi] \quad \psi \wedge C(\psi)}{C(\phi)}$$

11 Appendix Four: Soundness

is sound with respect to our semantics.

First, observe that for all \mathfrak{M} and for all $w \in W$, we have $\mathfrak{M}, w \not\models \phi \wedge C(\phi)$, as we have shown on page 240 above.

From our inductive hypothesis it follows that $\Gamma \models \psi \wedge C(\psi)$. We will show that $\Gamma \setminus \{\phi\} \models (C\phi)$.

Take an arbitrary model, \mathfrak{M} and $w \in W$ and assume $\mathfrak{M}, w \models \Gamma \setminus \{\phi\}$. Assume for a contradiction that, $\mathfrak{M}, w \models \phi$. Then $\mathfrak{M}, w \models \Gamma$ and so $\mathfrak{M}, w \models \psi \wedge C(\psi)$. But this is impossible, as was shown previously. Hence $\mathfrak{M}, w \not\models \phi$. It then follows by the definition of C that $\mathfrak{M}, w \models C(\phi)$

11.2.3 Elimination Rules

Exposition ⁺ We claim that the rule:

$$\frac{AiB \quad \frac{[faA \wedge faB]}{C}}{C}$$

(where f does not occur in any open assumption other than the discharged assumption $\underline{faA} \wedge \underline{faB}$) is sound with respect to our semantics.

To that end, take an arbitrary model \mathfrak{M} and $w \in W$. In order to show this rule is sound, note that by the inductive hypothesis (since the proofs are shorter):

- 1) $\Gamma \models AiB$,
- 2) $\Delta \models C$

We need to show that $\Gamma \cup (\Delta \setminus \{\underline{faA} \wedge \underline{faB}\}) \models C$.

To that end, assume that $\mathfrak{M}, w \models \Gamma$, and $\mathfrak{M}, w \models \Delta \setminus \{\underline{faA} \wedge \underline{faB}\}$.

By 1) we know that $\exists t \in D$ such that $t \in V(w, A)$ and $t \in V(w, B)$. Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{faA} \wedge \underline{faB}\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{faA} \wedge \underline{faB}\}$.

Moreover, $\mathfrak{M}', w \models \underline{faA}$ since $c'(\underline{f}) \in V_{\mathfrak{M}'}(w, A)$ and $\mathfrak{M}', w \models \underline{faB}$ since $c'(\underline{f}) \in V_{\mathfrak{M}'}(w, B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 2).

But \underline{f} does not occur in C and $c' \sim_{\underline{f}} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup (\Delta \setminus \{\underline{faA} \wedge \underline{faB}\}) \models C$. This proves the soundness of the rule.

Exposition ⁻ We claim that the rule:

$$\frac{AoB \quad \frac{[faA \wedge feB]}{C}}{C} \quad AiA$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{faA} \wedge \underline{feB}$) is sound with respect to our semantics.

In order to show this rule is sound, note that by the inductive hypothesis (since the proofs are shorter):

- 1) $\Xi \models AiA$,
- 2) $\Gamma \models AoB$, and
- 3) $\Delta \models C$

From 1) $\Xi \models AiA$ and 2) $\Gamma \models AoB$, it follows that (for all \mathfrak{M} and $w \in W$) if $\mathfrak{M}, w \models \Gamma \cup \Xi$ then $V(w, A) \neq \emptyset$ and either $V(w, A) = \emptyset$ or $V(w, A) \not\subseteq V(w, B)$. This then entails that $V(w, A) \not\subseteq V(w, B)$ and so

- 4) $\exists t \in D$ such that $t \in V(w, A)$ and $t \notin V(w, B)$.

We need to show that $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f}aA \wedge \underline{f}eB\}) \models C$.

To that end, take an arbitrary model \mathfrak{M} and world w , and assume that $\mathfrak{M}, w \models \Gamma$, $\mathfrak{M}, w \models \Xi$, and $\mathfrak{M}, w \models \Delta \setminus \{\underline{f}aA \wedge \underline{f}eB\}$.

By 4) we know that $\exists t \in D$ such that $t \in V(w, A)$ and $t \notin V(w, B)$. Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f}aA \wedge \underline{f}eB\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f}aA \wedge \underline{f}eB\}$.

Moreover, $\mathfrak{M}', w \models \underline{f}aA$ since $c'(\underline{f}) \in V_{\mathfrak{M}'}(w, A)$ and $\mathfrak{M}', w \models \underline{f}eB$ since $c'(\underline{f}) \notin V_{\mathfrak{M}'}(w, B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 3).

But \underline{f} does not occur in C and $c' \sim_{\underline{f}} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f}aA \wedge \underline{f}eB\}) \models C$. This proves the soundness of the rule.

Exposition L^+ We claim that the rule:

$$\frac{\frac{A \quad \underline{f} \quad B}{A \quad \underline{f} \quad B} \quad \frac{[\underline{f} \quad \underline{a} \quad A \wedge \underline{f} \quad \underline{a} \quad B]}{C}}{C}$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f} \quad \underline{a} \quad A \wedge \underline{f} \quad \underline{a} \quad B$) is sound with respect to our semantics.

To that end, take an arbitrary model, \mathfrak{M} and $w \in W$. Then note that by the inductive hypothesis (since the proofs are shorter):

- 1) $\Gamma \models A \quad \underline{f} \quad B$,
- 2) $\Delta \models C$

We need to show that $\Gamma \cup (\Delta \setminus \{\underline{f} \quad \underline{a} \quad A \wedge \underline{f} \quad \underline{a} \quad B\}) \models C$.

To that end, assume that $\mathfrak{M}, w \models \Gamma$, and $\mathfrak{M}, w \models \Delta \setminus \{\underline{f} \quad \underline{a} \quad A \wedge \underline{f} \quad \underline{a} \quad B\}$.

By 1) we know that $\exists t \in D$ such that $t \in M(w, A)$ and $t \in L(w, B)$. Call one such object s .

11 Appendix Four: Soundness

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{a} B\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{a} B\}$.

Moreover, $\mathfrak{M}', w \models \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$ and $\mathfrak{M}', w \models \underline{f} \overset{L}{a} B$ since $c'(\underline{f}) \in L(w, B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 2).

But \underline{f} does not occur in C and $c' \sim_{\underline{f}} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{a} B\}) \models C$. This proves the soundness of the rule.

Exposition L^- We claim that the rule:

$$\frac{\begin{array}{ccc} & [\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{e} B] & \\ A \overset{L}{o} B & C & A \overset{M}{i} A \end{array}}{C}$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{e} B$) is sound with respect to our semantics.

In order to show this rule is sound, note that by the inductive hypothesis (since the proofs are shorter):

$$1) \Xi \models A \overset{M}{i} A,$$

$$2) \Gamma \models A \overset{L}{o} B, \text{ and}$$

$$3) \Delta \models C$$

From 1) $\Xi \models A \overset{M}{i} A$ and 2) $\Gamma \models A \overset{L}{o} B$, it follows that (for all \mathfrak{M} and $w \in W$) if $\mathfrak{M}, w \models \Gamma \cup \Xi$ then $M(w, A) \neq \emptyset$ and either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq M(w, B)$. This then entails that $M(w, A) \not\subseteq M(w, B)$ and so

$$4) \exists t \in D \text{ such that } t \in M(w, A) \text{ and } t \notin M(w, B).$$

We need to show that $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{e} B\}) \models C$.

To that end, take an arbitrary model \mathfrak{M} and world w , and assume that $\mathfrak{M}, w \models \Gamma$, $\mathfrak{M}, w \models \Xi$, and $\mathfrak{M}, w \models \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{e} B\}$.

By 4) we know that $\exists t \in D$ such that $t \in M(w, A)$ and $t \notin M(w, B)$. Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{e} B\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{L}{e} B\}$.

Moreover, $\mathfrak{M}', w \models \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$ and $\mathfrak{M}', w \models \underline{f} \overset{L}{e} B$ since $c'(\underline{f}) \notin M(w, B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 3).

But \underline{f} does not occur in C and $c \sim_{\underline{f}} c'$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f}^M A \wedge \underline{f}^L B\}) \models C$. This proves the soundness of the rule.

Exposition M^+ We claim that the rule:

$$\frac{\frac{A \overset{M}{i} B \quad [\underline{f}^M A \wedge \underline{f}^M B]}{C}}{C}$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f}^M A \wedge \underline{f}^M B$) is sound with respect to our semantics. To that end, take an arbitrary model \mathfrak{M} and world $w \in W$ and note that by the inductive hypothesis (since the proofs are shorter):

- 1) $\Gamma \models A \overset{M}{i} B$,
- 2) $\Delta \models C$

We need to show that $\Gamma \cup (\Delta \setminus \{\underline{f}^M A \wedge \underline{f}^M B\}) \models C$.

To that end, assume that $\mathfrak{M}, w \models \Gamma$, and $\mathfrak{M}, w \models \Delta \setminus \{\underline{f}^M A \wedge \underline{f}^M B\}$.

By 1) we know that $\exists t \in D$ such that $t \in M(w, A)$ and $t \in M(w, B)$. Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f}^M A \wedge \underline{f}^M B\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f}^M A \wedge \underline{f}^M B\}$.

Moreover, $\mathfrak{M}', w \models \underline{f}^M A$ since $c'(\underline{f}) \in M(w, A)$ and $\mathfrak{M}', w \models \underline{f}^M B$ since $c'(\underline{f}) \in M(w, B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 2).

But \underline{f} does not occur in C and $c \sim_{\underline{f}} c'$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup (\Delta \setminus \{\underline{f}^M A \wedge \underline{f}^M B\}) \models C$. This proves the soundness of the rule.

Exposition M^- We claim that the rule:

$$\frac{\frac{A \overset{M}{o} B \quad [\underline{f}^M A \wedge \underline{f}^M B]}{C} \quad A \overset{M}{i} A}{C}$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f}^M A \wedge \underline{f}^M B$) is sound with respect to our semantics.

In order to show this rule is sound, note that by the inductive hypothesis (since the proofs are shorter):

- 1) $\Xi \models A \overset{M}{i} A$,

11 Appendix Four: Soundness

2) $\Gamma \vDash A \overset{M}{o} B$, and

3) $\Delta \vDash C$

From 1) $\Xi \vDash A \overset{M}{i} A$ and 2) $\Gamma \vDash A \overset{L}{o} B$, it follows that (for all \mathfrak{M} and $w \in W$) if $\mathfrak{M}, w \vDash \Gamma \cup \Xi$ then $M(w, A) \neq \emptyset$ and either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq L(w, B)$. This then entails that $M(w, A) \not\subseteq L(w, B)$ and so

4) $\exists t \in D$ such that $t \in M(w, A)$ and $t \notin L(w, B)$.

We need to show that $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{M}{e} B\}) \vDash C$.

To that end, take an arbitrary model \mathfrak{M} and world w , and assume that $\mathfrak{M}, w \vDash \Gamma$, $\mathfrak{M}, w \vDash \Xi$, and $\mathfrak{M}, w \vDash \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{M}{e} B\}$.

By 4) we know that $\exists t \in D$ such that $t \in M(w, A)$ and $t \notin L(w, B)$. Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \vDash \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{M}{e} B\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{M}{e} B\}$.

Moreover, $\mathfrak{M}', w \vDash \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$ and $\mathfrak{M}', w \vDash \underline{f} \overset{M}{e} B$ since $c'(\underline{f}) \notin L(w, B)$.

So $\mathfrak{M}', w \vDash \Delta$, and so $\mathfrak{M}', w \vDash C$ by 3).

But \underline{f} does not occur in C and $c' \sim_{\underline{f}} c$, so it follows by Observation 1 that $\mathfrak{M}, w \vDash C$.

Hence $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{M}{e} B\}) \vDash C$. This proves the soundness of the rule.

Exposition Q We claim that the rule:

$$\frac{A \overset{Q}{i} B \quad [\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B]}{C}$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B$) is sound with respect to our semantics.

To that end, take an arbitrary model, \mathfrak{M} and $w \in W$ and note that by the inductive hypothesis (since the proofs are shorter):

1) $\Gamma \vDash A \overset{Q}{i} B$,

2) $\Delta \vDash C$

We need to show that $\Gamma \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B\}) \vDash C$.

To that end, assume that $\mathfrak{M}, w \vDash \Gamma$, and $\mathfrak{M}, w \vDash \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B\}$.

By 1) we know that $\exists t \in D$ such that $t \in M(w, A)$, $t \in M(w, B)$, and $t \in M(w, \neg B)$ (i.e. $t \in Q(w, B)$). Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B\}$.

Moreover, $\mathfrak{M}', w \models \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$ and $\mathfrak{M}', w \models \underline{f} \overset{Q}{a} B$ since $c'(\underline{f}) \in M(w, B) \cap M(w, \neg B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 2).

But \underline{f} does not occur in C and $c' \sim_{\underline{f}} c$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B\}) \models C$. This proves the soundness of the rule.

Exposition \bar{Q} We claim that the rule:

$$\frac{A \overset{\bar{Q}}{i} B \quad [\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B] \quad A \overset{M}{i} A}{C} C$$

(where \underline{f} does not occur in any open assumption other than the discharged assumption $\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{Q}{a} B$) is sound with respect to our semantics.

In order to show this rule is sound, note that by the inductive hypothesis (since the proofs are shorter):

- 1) $\Xi \models A \overset{M}{i} A$,
- 2) $\Gamma \models A \overset{\bar{Q}}{i} B$, and
- 3) $\Delta \models C$

From 1) $\Xi \models A \overset{M}{i} A$ and 2) $\Gamma \models A \overset{\bar{Q}}{i} B$, it follows that (for all \mathfrak{M} and $w \in W$) if $\mathfrak{M}, w \models \Gamma \cup \Xi$ then $M(w, A) \neq \emptyset$ and either $M(w, A) = \emptyset$ or $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$. This then entails that $M(w, A) \not\subseteq (M(w, B) \cap M(w, \neg B))$ and so

- 4) $\exists t \in D$ such that $t \in M(w, A)$ and $t \in L(w, B) \cup L(w, \neg B)$ i.e. $t \notin Q(w, B)$.

We need to show that $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{\bar{Q}}{a} B\}) \models C$.

To that end, take an arbitrary model \mathfrak{M} and world w , and assume that $\mathfrak{M}, w \models \Gamma$, $\mathfrak{M}, w \models \Xi$, and $\mathfrak{M}, w \models \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{\bar{Q}}{a} B\}$.

By 4) we know that $\exists t \in D$ such that $t \in M(w, A)$ and $t \notin Q(w, B)$. Call one such object s .

Let c' be an assignment that differs from c at most in that $c'(\underline{f}) = s$. By construction we have $c' \sim_{\underline{f}} c$. Then $\mathfrak{M}', w \models \Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{\bar{Q}}{a} B\}$, since \underline{f} does not occur in $\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{\bar{Q}}{a} B\}$.

11 Appendix Four: Soundness

Moreover, $\mathfrak{M}', w \models \underline{f} \overset{M}{a} A$ since $c'(\underline{f}) \in M(w, A)$ and $\mathfrak{M}', w \models \underline{f} \overset{\bar{Q}}{a} B$ since $c'(\underline{f}) \in L(w, B) \cup L(w, \neg B)$.

So $\mathfrak{M}', w \models \Delta$, and so $\mathfrak{M}', w \models C$ by 3).

But \underline{f} does not occur in C and $c_{\underline{f}}^{\sim} c'$, so it follows by Observation 1 that $\mathfrak{M}, w \models C$.

Hence $\Gamma \cup \Xi \cup (\Delta \setminus \{\underline{f} \overset{M}{a} A \wedge \underline{f} \overset{\bar{Q}}{a} B\}) \models C$. This proves the soundness of the rule.

DDO:

We claim that the rule:

$$\frac{A a B \quad \underline{d} a A}{\underline{d} a B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A a B$ and $\Delta \models \underline{d} a A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A a B$ and $\mathfrak{M}, w \models \underline{d} a A$. It then follows that $V(w, A) \subseteq V(w, B)$ and $c(\underline{d}) \in V(w, A)$. Hence $c(\underline{d}) \in V(w, B)$ and so $\mathfrak{M}, w \models \underline{d} a B$. Hence $\Gamma \cup \Delta \models \underline{d} a B$.

DDN We claim that the rule:

$$\frac{A e B \quad \underline{d} a A}{\underline{d} e B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A e B$ and $\Delta \models \underline{d} a A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A e B$ and $\mathfrak{M}, w \models \underline{d} a A$. It then follows that $V(w, A) \cap V(w, B) = \emptyset$ and $c(\underline{d}) \in V(w, A)$. Hence $c(\underline{d}) \notin V(w, B)$ and so $\mathfrak{M}, w \models \underline{d} e B$. Hence $\Gamma \cup \Delta \models \underline{d} e B$.

DDN We claim that the rule:

$$\frac{A e B \quad \underline{d} a B}{\underline{d} e A}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A e B$ and $\Delta \models \underline{d} a B$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A e B$ and $\mathfrak{M}, w \models \underline{d} a B$. It then follows that $V(w, A) \cap V(w, B) = \emptyset$ and $c(\underline{d}) \in V(w, B)$. Hence $c(\underline{d}) \notin V(w, A)$ and so $\mathfrak{M}, w \models \underline{d} e A$. Hence $\Gamma \cup \Delta \models \underline{d} e A$.

DDO We claim that the rule:

$$\frac{A \overset{L}{a} B \quad \underline{d} \overset{M}{a} A}{\underline{d} \overset{L}{a} B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{L}{a} B$ and $\Delta \models \underline{d} \overset{M}{a} A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{L}{a} B$ and

$\mathfrak{M}, w \models \underline{d} \overset{M}{a} A$. It then follows that $M(w, A) \subseteq L(w, B)$ and $c(\underline{d}) \in M(w, A)$. Hence $c(\underline{d}) \in L(w, B)$ and so $\mathfrak{M}, w \models \underline{d} \overset{L}{e} B$. Hence $\Gamma \cup \Delta \models \underline{d} \overset{L}{e} B$.

DDN We claim that the rule:

$$\frac{A \overset{L}{e} B \quad \underline{d} \overset{M}{a} A}{\underline{d} \overset{L}{e} B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{L}{e} B$ and $\Delta \models \underline{d} \overset{M}{a} A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{L}{e} B$ and $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A$. It then follows that $M(w, A) \cap M(w, B) = \emptyset$ and $c(\underline{d}) \in M(w, A)$. Hence $c(\underline{d}) \notin M(w, B)$ and so $\mathfrak{M}, w \models \underline{d} \overset{L}{e} B$. Hence $\Gamma \cup \Delta \models \underline{d} \overset{L}{e} B$.

DDN We claim that the rule:

$$\frac{A \overset{L}{e} B \quad \underline{d} \overset{M}{a} B}{\underline{d} \overset{L}{e} A}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{L}{e} B$ and $\Delta \models \underline{d} \overset{M}{a} B$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{L}{e} B$ and $\mathfrak{M}, w \models \underline{d} \overset{M}{a} B$. It then follows that $M(w, A) \cap M(w, B) = \emptyset$ and $c(\underline{d}) \in M(w, B)$. Hence $c(\underline{d}) \notin M(w, A)$ and so $\mathfrak{M}, w \models \underline{d} \overset{L}{e} A$. Hence $\Gamma \cup \Delta \models \underline{d} \overset{L}{e} A$.

DDO We claim that the rule:

$$\frac{A \overset{M}{a} B \quad \underline{d} \overset{M}{a} A}{\underline{d} \overset{M}{a} B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{M}{a} B$ and $\Delta \models \underline{d} \overset{M}{a} A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{M}{a} B$ and $\mathfrak{M}, w \models \underline{d} \overset{M}{a} A$. It then follows that $M(w, A) \subseteq M(w, B)$ and $c(\underline{d}) \in M(w, A)$. Hence $c(\underline{d}) \in M(w, B)$ and so $\mathfrak{M}, w \models \underline{d} \overset{M}{a} B$. Hence $\Gamma \cup \Delta \models \underline{d} \overset{M}{a} B$.

DDN We claim that the rule:

$$\frac{A \overset{M}{e} B \quad \underline{d} \overset{M}{a} A}{\underline{d} \overset{M}{e} B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{M}{e} B$ and $\Delta \models \underline{d} \overset{M}{a} A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{M}{e} B$ and

11 Appendix Four: Soundness

$\mathfrak{M}, w \models \underline{d}^M a A$. It then follows that $M(w, A) \cap L(w, B) = \emptyset$ and $c(\underline{d}) \in M(w, A)$. Hence $c(\underline{d}) \notin L(w, B)$ and so $\mathfrak{M}, w \models \underline{d}^M e B$. Hence $\Gamma \cup \Delta \models \underline{d}^M e B$.

DDN We claim that the rule:

$$\frac{A \overset{M}{e} B \quad \underline{d}^L a B}{\underline{d}^L e A}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{M}{e} B$ and $\Delta \models \underline{d}^L a B$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{M}{e} B$ and $\mathfrak{M}, w \models \underline{d}^L a B$. It then follows that $M(w, A) \cap L(w, B) = \emptyset$ and $c(\underline{d}) \in L(w, B)$. Hence $c(\underline{d}) \notin M(w, A)$ and so $\mathfrak{M}, w \models \underline{d}^L e A$. Hence $\Gamma \cup \Delta \models \underline{d}^L e A$.

DDO We claim that the rule:

$$\frac{A \overset{Q}{a} B \quad \underline{d}^M a A}{\underline{d}^Q a B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{Q}{a} B$ and $\Delta \models \underline{d}^M a A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{Q}{a} B$ and $\mathfrak{M}, w \models \underline{d}^M a A$. It then follows that $M(w, A) \subseteq (M(w, B) \cap M(w, \neg B))$ and $c(\underline{d}) \in M(w, A)$. Hence $c(\underline{d}) \in (M(w, B) \cap M(w, \neg B))$ and so $\mathfrak{M}, w \models \underline{d}^Q a B$. Hence $\Gamma \cup \Delta \models \underline{d}^Q a B$.

DDO We claim that the rule:

$$\frac{A \overset{\bar{Q}}{a} B \quad \underline{d}^M a A}{\underline{d}^{\bar{Q}} a B}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models A \overset{\bar{Q}}{a} B$ and $\Delta \models \underline{d}^M a A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models A \overset{\bar{Q}}{a} B$ and $\mathfrak{M}, w \models \underline{d}^M a A$. It then follows that $M(w, A) \cap (M(w, B) \cap M(w, \neg B)) = \emptyset$ and $c(\underline{d}) \in M(w, A)$. Hence $c(\underline{d}) \notin (M(w, B) \cap M(w, \neg B))$ and so $c(\underline{d}) \in (L(w, B) \cup L(w, \neg B))$. $\mathfrak{M}, w \models \underline{d}^{\bar{Q}} a B$. Hence $\Gamma \cup \Delta \models \underline{d}^{\bar{Q}} a B$.

\bar{Q} -Out We claim that the rule:

$$\frac{\underline{d}^{\bar{Q}} a A \quad \underline{d}^M a A}{\underline{d}^L a A}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models \underline{d}^{\bar{Q}} a A$ and $\Delta \models \underline{d}^M a A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models \underline{d}^{\bar{Q}} a A$ and $\mathfrak{M}, w \models \underline{d}^M a A$. Then it follows that $c(\underline{d}) \in L(w, A) \cup L(w, \neg A)$ and $c(\underline{d}) \in M(w, A)$. From $c(\underline{d}) \in M(w, A)$ it follows that $c(\underline{d}) \notin L(w, \neg A)$. Hence $c(\underline{d}) \in L(w, A)$ and so $\mathfrak{M}, w \models \underline{d}^L a A$. Hence $\Gamma \cup \Delta \models \underline{d}^L a A$.

\bar{Q} -Out We claim that the rule:

$$\frac{\underline{d}^{\bar{Q}} a A \quad \underline{d}^M e A}{\underline{d}^L e A}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models \underline{d}^{\bar{Q}} a A$ and $\Delta \models \underline{d}^M e A$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models \underline{d}^{\bar{Q}} a A$ and $\mathfrak{M}, w \models \underline{d}^M e A$. Then it follows that $c(\underline{d}) \in L(w, A) \cup L(w, \neg A)$ and $c(\underline{d}) \notin L(w, A)$. Hence $c(\underline{d}) \in L(w, \neg A)$ and so $c(\underline{d}) \notin M(w, A)$. So $\mathfrak{M}, w \models \underline{d}^L e A$. Hence $\Gamma \cup \Delta \models \underline{d}^L e A$.

\wedge **Elimination** We claim that the rule:

$$\frac{\phi \wedge \psi}{\phi}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models \phi \wedge \psi$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \phi \wedge \psi$. Hence $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$. So $\mathfrak{M}, w \models \phi$ and $\Gamma \models \phi$.

\wedge **Elimination** We claim that the rule:

$$\frac{\phi \wedge \psi}{\psi}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models \phi \wedge \psi$. Take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$. Then $\mathfrak{M}, w \models \phi \wedge \psi$. Hence $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$. So $\mathfrak{M}, w \models \psi$ and $\Gamma \models \psi$.

Ex Falso Quodlibet We claim that the rule:

$$\frac{\phi \quad C(\phi)}{\psi}$$

is sound with respect to our semantics.

By our inductive hypothesis we have $\Gamma \models \phi$ and $\Delta \models C(\phi)$. We claim that $\Gamma \cup \Delta \models \psi$ for any formula ψ . To see this, take an arbitrary model \mathfrak{M} , and $w \in W$ such that $\mathfrak{M}, w \models \Gamma$ and $\mathfrak{M}, w \models \Delta$. Then $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models C(\phi)$. But this is impossible, as ϕ and $C(\phi)$ are contradictory. So the antecedent is false, and it follows that $\Gamma \cup \Delta \models \psi$ regardless of choice of ψ .

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*I waited patiently for the LORD;
He inclined to me and heard my cry.*

Psalm 40, English Standard Version