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ON SIMULTANEOUS LOCAL DIMENSION FUNCTIONS OF SUBSETS OF $\mathbb{R}^d$

Lars Olsen

Abstract. For a subset $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the local Hausdorff dimension function of E at x and the local packing dimension function of E at x are defined by

$$\dim_{\mathsf{H},\mathsf{loc}}(x,E) = \lim_{r \searrow 0} \dim_{\mathsf{H}}(E \cap B(x,r)),$$
$$\dim_{\mathsf{P},\mathsf{loc}}(x,E) = \lim_{r \searrow 0} \dim_{\mathsf{P}}(E \cap B(x,r)),$$

where $\dim_{\mathsf{H}}$ and $\dim_{\mathsf{P}}$ denote the Hausdorff dimension and the packing dimension, respectively. In this note we give a short and simple proof showing that for any pair of continuous functions $f, g : \mathbb{R}^d \to [0,d]$ with $f \leq g$, it is possible to choose a set E that simultaneously has f as its local Hausdorff dimension function and g as its local packing dimension function.

1. Introduction and statement of results

For a subset $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define the local Hausdorff dimension function of E at x by

$$\dim_{\mathsf{H},\mathsf{loc}}(x,E) = \lim_{r \searrow 0} \dim_{\mathsf{H}}(E \cap B(x,r)),$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. The local packing dimension function of E at x is defined similarly, i.e., by

$$\dim_{\mathsf{P},\mathsf{loc}}(x,E) = \lim_{r \searrow 0} \dim_{\mathsf{P}}(E \cap B(x,r)),$$

where $\dim_{\mathsf{P}}$ denotes the packing dimension. The reader is referred to [1] for the definitions of the Hausdorff and the packing dimensions. The local Hausdorff dimension function of a set has recently found several applications in fractal geometry and information theory, cf. [2, 4]. In [3] we proved that any continuous function is the local Hausdorff dimension function of some set, i.e., if $f : \mathbb{R}^d \to$

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$[0,d]$ is continuous, then there exists a set $E \subseteq \mathbb{R}^d$ such that

$$f(x) = \dim_{\mathsf{H,loc}}(x, E)$$

for all $x \in \mathbb{R}^d$. In this note we give a short and simple proof showing that for any pair of continuous functions $f, g : \mathbb{R}^d \to [0,d]$ with $f \leq g$, it is, in fact, possible to choose the set E such that it simultaneously has f as its local Hausdorff dimension function and g as its local packing dimension function, i.e., such that

$$f(x) = \dim_{\mathsf{H,loc}}(x, E)\,,$$
$$g(x) = \dim_{\mathsf{P,loc}}(x, E)\,,$$

for all $x \in \mathbb{R}^d$. In fact, our result also provides information about the rate at which the dimensions $\dim_{\mathsf{H}}(E \cap B(x,r))$ and $\dim_{\mathsf{P}}(E \cap B(x,r))$ converge to $f(x)$ and $g(x)$, respectively, as $r \searrow 0$, see (1.1) below. For an arbitrary function $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$, we let

$$\omega_\varphi(x,r) = \sup_{x_1, x_2 \in B(x,r)} |\varphi(x_1) - \varphi(x_2)|$$

denote the modulus of continuity of φ at x, and observe that φ is continuous at x if and only if $\omega_\varphi(x,r) \to 0$ as $r \searrow 0$.

Theorem 1. *Let $f, g : \mathbb{R}^d \to [0,d]$ be continuous functions with $f \leq g$. Then there exists an $\mathcal{F}_\sigma$ set $E \subseteq \mathbb{R}^d$ such that*

$$\begin{aligned} |f(x) - \dim_{\mathsf{H}}(E \cap B(x,r))| &\leq \omega_f(x,r)\,, \\ |g(x) - \dim_{\mathsf{P}}(E \cap B(x,r))| &\leq \omega_g(x,r)\,, \end{aligned} \tag{1.1}$$

for all $x \in \mathbb{R}^d$ and all $r > 0$. In particular,

$$f(x) = \dim_{\mathsf{H,loc}}(x, E)\,,$$
$$g(x) = \dim_{\mathsf{P,loc}}(x, E)\,,$$

for all $x \in \mathbb{R}^d$.

2. Proof of Theorem 1

In this section we prove Theorem 1. We need the following well-known result in order to prove Theorem 1.

Lemma 2.1. *Let G be a non-empty open subset of $\mathbb{R}^d$ and $t, s \in \mathbb{R}$ with $0 \leq t \leq s \leq d$. Then there exists a compact set $E \subseteq G$ such that $\dim_{\mathsf{H}}(E) = t$ and $\dim_{\mathsf{P}}(E) = s$.*

Proof. For a proof see, for example, [5]. In fact, the result in [5] is formulated and proved for the case where $d = 1$, but the techniques in [5] can clearly be adapted to prove the same result in the general case. $\square$

We can now prove Theorem 1. We first introduce some notation. For a function $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$ and positive number $r > 0$, write

$$\begin{aligned} m(\varphi; x, r) &= \inf_{y \in B(x,r)} \varphi(y)\,, \\ M(\varphi; x, r) &= \sup_{y \in B(x,r)} \varphi(y)\,. \end{aligned}$$

Proof of Theorem 1. Let $0 \le t < \sup_{x \in \mathbb{R}^d} f(x)$ and $0 \le s < \sup_{x \in \mathbb{R}^d} g(x)$ with $t \le s$ Fix $x \in \{t < f\,,\, s < g\}$ and $r > 0$. Since f and g are continuous, we conclude that the set $B(x, r) \cap \{t < f\,,\, s < g\}$ is open, and it therefore follows from Lemma 2.1 that we can find a compact set $E_{t,s}(x, r)$ satisfying

$$\begin{aligned} &E_{t,s}(x, r) \subseteq B(x, r) \cap \{t < f\,,\, s < g\}\,, \\ &\dim_{\mathsf{H}}(E_{t,s}(x, r)) = t\,, \\ &\dim_{\mathsf{P}}(E_{t,s}(x, r)) = s\,. \end{aligned}$$

Next choose a countable dense subset $U_{t,s}$ of $\{t < f\,,\, s < g\}$. We now define the set E as

$$E \;=\; \bigcup_{\substack{0 \le t < \sup_{y \in \mathbb{R}^d} f(y) \\ 0 \le s < \sup_{y \in \mathbb{R}^d} g(y) \\ t,s \in \mathbb{Q}_+ \\ t \le s}} \;\bigcup_{\substack{r \in \mathbb{Q}_+ \\ x \in U_{t,s}}} E_{t,s}(x, r)\,.$$

The set E is clearly $\mathcal{F}_\sigma$. We will now prove that f is the local Hausdorff dimension function of E and that g is the local packing dimension function of E, i.e., $f(x) = \dim_{\mathsf{H,loc}}(x, E)$ and $g(x) = \dim_{\mathsf{P,loc}}(x, E)$ for all $x \in \mathbb{R}^d$.

Claim 1. *For all $x \in \mathbb{R}^d$ and all $r > 0$, we have*

$$\begin{aligned} \dim_{\mathsf{H,loc}}(x, E) &\le M(f; x, r)\,, \\ \dim_{\mathsf{P,loc}}(x, E) &\le M(g; x, r)\,. \end{aligned}$$

Proof of Claim 1. Fix $x \in \mathbb{R}^d$ and $r > 0$. We now have

$$E \cap B(x, r) \subseteq \bigcup_{\substack{0 \le t < \sup_{y \in \mathbb{R}^d} f(y) \\ 0 \le s < \sup_{y \in \mathbb{R}^d} g(y) \\ t,s \in \mathbb{Q}_+ \\ t \le s}} \;\bigcup_{\substack{\rho \in \mathbb{Q}_+ \\ z \in U_{t,s}}} \Big(E_{t,s}(z, \rho) \cap B(x, r)\Big)\,. \tag{2.1}$$

Next observe that since $E_{t,s}(z, \rho) \subseteq \{t < f\,,\, s < g\}$, we conclude that

$$E_{t,s}(z, \rho) \cap B(x, r) \subseteq \{t < f\,,\, s < g\} \cap B(x, r) = \emptyset \tag{2.2}$$

for $M(f; x, r) \le t$ and $M(g; x, r) \le s$. Combining (2.1) and (2.2) yields

$$E \cap B(x, r) \subseteq \bigcup_{\substack{0 \le t < M(f;x,r) \\ 0 \le s < M(g;x,r) \\ t,s \in \mathbb{Q}_+ \\ t \le s}} \;\bigcup_{\substack{\rho \in \mathbb{Q}_+ \\ z \in U_{t,s}}} \Big(E_{t,s}(z, \rho) \cap B(x, r)\Big) \tag{2.3}$$

$$\subseteq \bigcup_{\substack{0\le t<M(f;x,r)\\ 0\le s<M(g;x,r)\\ t,s\in\mathbb{Q}_+\\ t\le s}} \bigcup_{\substack{\rho\in\mathbb{Q}_+\\ z\in U_{t,s}}} E_{t,s}(z,\rho)\,.$$

Since the union in (2.3) is countable, it follows from (2.3) and the fact that the Hausdorff dimension is countable stable that

$$\begin{aligned}\dim_{\mathsf{H}}(E\cap B(x,r)) &\le \sup_{\substack{0\le t<M(f;x,r)\\ 0\le s<M(g;x,r)\\ t,s\in\mathbb{Q}_+\\ t\le s}} \sup_{\substack{\rho\in\mathbb{Q}_+\\ z\in U_{t,s}}} \dim_{\mathsf{H}}(E_{t,s}(z,\rho))\\ &= \sup_{\substack{0\le t<M(f;x,r)\\ 0\le s<M(g;x,r)\\ t,s\in\mathbb{Q}_+\\ t\le s}} \sup_{\substack{\rho\in\mathbb{Q}_+\\ z\in U_{t,s}}} t\\ &= M(f;x,r)\end{aligned}$$

for all $r>0$. Similarly, it follows that

$$\dim_{\mathsf{P}}(E\cap B(x,r)) \le M(g;x,r)$$

for all $r>0$ This completes the proof of Claim 1. $\square$

Claim 2. *For all $x\in\mathbb{R}^d$ and all $r>0$, we have*

$$\begin{aligned} m(f;x,r) &\le \dim_{\mathsf{H},\mathsf{loc}}(x,E)\,,\\ m(g;x,r) &\le \dim_{\mathsf{P},\mathsf{loc}}(x,E)\,. \end{aligned}$$

Proof of Claim 2. Fix $x\in\mathbb{R}^d$ and $r>0$. Next, let $\varepsilon>0$ be such that $m(f;x,r)-\varepsilon, m(g;x,r)-\varepsilon\in\mathbb{Q}_+$. Write $t=m(f;x,r)-\varepsilon$ and $s=m(g;x,r)-\varepsilon$, and observe that $t\le s$. We clearly have $x\in\{t<f\,,\,s<g\}$, and we can therefore find $u\in U_{t,s}$ with $|u-x|\le\frac{r}{2}$. Now, pick any $\rho\in\mathbb{Q}_+$ with $\rho\le\frac{r}{2}$. It now follows that

$$E_{t,s}(u,\rho)\subseteq E\,,$$

and that $E_{t,s}(u,\rho)\subseteq B(u,\rho)\subseteq B(x,r)$, whence

$$E\cap B(x,r)\supseteq E_{t,s}(u,\rho)\cap B(x,r)=E_{t,s}(u,\rho)\,.$$

We therefore conclude that

$$\dim_{\mathsf{H}}(E\cap B(x,r))\ge \dim_{\mathsf{H}}(E_{t,s}(u,\rho))=t\ge m(f;x,r)-\varepsilon\,. \tag{2.4}$$

Similarly, we conclude that

$$\dim_{\mathsf{P}}(E\cap B(x,r))\ge \dim_{\mathsf{P}}(E_{t,s}(u,\rho))=s\ge m(g;x,r)-\varepsilon\,. \tag{2.5}$$

Claim 2 follows from (2.4) and (2.5) by letting $\varepsilon\searrow 0$ through values such that $m(f;x,r)-\varepsilon, m(g;x,r)-\varepsilon\in\mathbb{Q}_+$. $\square$

Theorem 1 follows immediately from Claim 1 and Claim 2. $\square$

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Department of Mathematics
University of St. Andrews
St. Andrews, Fife KY16 9SS, Scotland
E-mail address: `lo@st-and.ac.uk`