ON SIMULTANEOUS LOCAL DIMENSION FUNCTIONS OF SUBSETS OF $\mathbb{R}^d$

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Abstract. For a subset $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the local Hausdorff dimension function of $E$ at $x$ and the local packing dimension function of $E$ at $x$ are defined by

$$\dim_{H, \text{loc}}(x, E) = \lim_{r \to 0} \dim_H(E \cap B(x, r)),$$

$$\dim_{P, \text{loc}}(x, E) = \lim_{r \to 0} \dim_P(E \cap B(x, r)),$$

where $\dim_H$ and $\dim_P$ denote the Hausdorff dimension and the packing dimension, respectively. In this note we give a short and simple proof showing that for any pair of continuous functions $f, g : \mathbb{R}^d \to [0, d]$ with $f \leq g$, it is possible to choose a set $E$ that simultaneously has $f$ as its local Hausdorff dimension function and $g$ as its local packing dimension function.

1. Introduction and statement of results

For a subset $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define the local Hausdorff dimension function of $E$ at $x$ by

$$\dim_{H, \text{loc}}(x, E) = \lim_{r \to 0} \dim_H(E \cap B(x, r)),$$

where $\dim_H$ denotes the Hausdorff dimension. The local packing dimension function of $E$ at $x$ is defined similarly, i.e., by

$$\dim_{P, \text{loc}}(x, E) = \lim_{r \to 0} \dim_P(E \cap B(x, r)),$$

where $\dim_P$ denotes the packing dimension. The reader is referred to [1] for the definitions of the Hausdorff and the packing dimensions. The local Hausdorff dimension function of a set has recently found several applications in fractal geometry and information theory, cf. [2, 4]. In [3] we proved that any continuous function is the local Hausdorff dimension function of some set, i.e., if $f : \mathbb{R}^d \to$
$[0,d]$ is continuous, then there exists a set $E \subseteq \mathbb{R}^d$ such that

$$f(x) = \dim_{H,loc}(x, E)$$

for all $x \in \mathbb{R}^d$. In this note we give a short and simple proof showing that for any pair of continuous functions $f, g : \mathbb{R}^d \to [0,d]$ with $f \leq g$, it is, in fact, possible to choose the set $E$ such that it simultaneously has $f$ as its local Hausdorff dimension function and $g$ as its local packing dimension function, i.e., such that

$$f(x) = \dim_{H,loc}(x, E),$$

$$g(x) = \dim_{P,loc}(x, E),$$

for all $x \in \mathbb{R}^d$. In fact, our result also provides information about the rate at which the dimensions $\dim_{H}(E \cap B(x, r))$ and $\dim_{P}(E \cap B(x, r))$ converge to $f(x)$ and $g(x)$, respectively, as $r \searrow 0$, see (1.1) below. For an arbitrary function $\varphi : \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$, we let

$$\omega_{\varphi}(x, r) = \sup_{x_1, x_2 \in B(x, r)} |\varphi(x_1) - \varphi(x_2)|$$

denote the modulus of continuity of $\varphi$ at $x$, and observe that $\varphi$ is continuous at $x$ if and only if $\omega_{\varphi}(x, r) \to 0$ as $r \searrow 0$.

**Theorem 1.** Let $f, g : \mathbb{R}^d \to [0,d]$ be continuous functions with $f \leq g$. Then there exists an $F_\sigma$ set $E \subseteq \mathbb{R}^d$ such that

$$|f(x) - \dim_{H}(E \cap B(x, r))| \leq \omega_f(x, r),$$

$$|g(x) - \dim_{P}(E \cap B(x, r))| \leq \omega_g(x, r),$$

for all $x \in \mathbb{R}^d$ and all $r > 0$. In particular,

$$f(x) = \dim_{H,loc}(x, E),$$

$$g(x) = \dim_{P,loc}(x, E),$$

for all $x \in \mathbb{R}^d$.

**2. Proof of Theorem 1**

In this section we prove Theorem 1. We need the following well-known result in order to prove Theorem 1.

**Lemma 2.1.** Let $G$ be a non-empty open subset of $\mathbb{R}^d$ and $t, s \in \mathbb{R}$ with $0 \leq t \leq s \leq d$. Then there exists a compact set $E \subseteq G$ such that $\dim_{H}(E) = t$ and $\dim_{P}(E) = s$.

**Proof.** For a proof see, for example, [5]. In fact, the result in [5] is formulated and proved for the case where $d = 1$, but the techniques in [5] can clearly be adapted to prove the same result in the general case. \qed
We can now prove Theorem 1. We first introduce some notation. For a function \( \varphi : \mathbb{R}^d \to \mathbb{R} \) and \( x \in \mathbb{R}^d \) and positive number \( r > 0 \), write
\[
\begin{align*}
m(\varphi; x, r) &= \inf_{y \in B(x, r)} \varphi(y), \\
M(\varphi; x, r) &= \sup_{y \in B(x, r)} \varphi(y).
\end{align*}
\]

**Proof of Theorem 1.** Let \( 0 \leq t < \sup_{x \in \mathbb{R}^d} f(x) \) and \( 0 \leq s < \sup_{x \in \mathbb{R}^d} g(x) \) with \( t \leq s \). Fix \( x \in \{ t < f, s < g \} \) and \( r > 0 \). Since \( f \) and \( g \) are continuous, we conclude that the set \( B(x, r) \cap \{ t < f, s < g \} \) is open, and it therefore follows from Lemma 2.1 that we can find a compact set \( E_{t,s}(x, r) \) satisfying
\[
\begin{align*}
E_{t,s}(x, r) &\subseteq B(x, r) \cap \{ t < f, s < g \}, \\
\text{dim}_H(E_{t,s}(x, r)) &= t, \\
\text{dim}_P(E_{t,s}(x, r)) &= s.
\end{align*}
\]

Next choose a countable dense subset \( U_{t,s} \) of \( \{ t < f, s < g \} \). We now define the set \( E \) as
\[
E = \bigcup_{0 \leq t < \sup_{y \in \mathbb{R}^d} f(y)} \bigcup_{0 \leq s < \sup_{y \in \mathbb{R}^d} g(y)} \bigcup_{t \leq s} \bigcup_{x \in U_{t,s}} E_{t,s}(x, r).
\]

The set \( E \) is clearly \( \mathcal{F}_\sigma \). We will now prove that \( f \) is the local Hausdorff dimension function of \( E \) and that \( g \) is the local packing dimension function of \( E \), i.e., \( f(x) = \text{dim}_H(x, E) \) and \( g(x) = \text{dim}_P(x, E) \) for all \( x \in \mathbb{R}^d \).

**Claim 1.** For all \( x \in \mathbb{R}^d \) and all \( r > 0 \), we have
\[
\begin{align*}
\text{dim}_H(x, E) &\leq M(f; x, r), \\
\text{dim}_P(x, E) &\leq M(g; x, r).
\end{align*}
\]

**Proof of Claim 1.** Fix \( x \in \mathbb{R}^d \) and \( r > 0 \). We now have
\[
E \cap B(x, r) \subseteq \bigcup_{0 \leq t < \sup_{y \in \mathbb{R}^d} f(y)} \bigcup_{0 \leq s < \sup_{y \in \mathbb{R}^d} g(y)} \bigcup_{t \leq s} \bigcup_{x \in U_{t,s}} \left( E_{t,s}(z, r) \cap B(x, r) \right).
\]

Next observe that since \( E_{t,s}(z, r) \subseteq \{ t < f, s < g \} \), we conclude that
\[
E_{t,s}(z, r) \cap B(x, r) \subseteq \{ t < f, s < g \} \cap B(x, r) = \emptyset
\]
for \( M(f; x, r) \leq t \) and \( M(g; x, r) \leq s \). Combining (2.1) and (2.2) yields
\[
E \cap B(x, r) \subseteq \bigcup_{0 \leq t < \sup_{y \in \mathbb{R}^d} f(y)} \bigcup_{0 \leq s < \sup_{y \in \mathbb{R}^d} g(y)} \bigcup_{z \in U_{t,s}} \left( E_{t,s}(z, r) \cap B(x, r) \right).
\]
\[ \subseteq \bigcup_{0 \leq t < M(f; x, r), \rho \in \mathbb{Q}^+} \bigcup_{0 \leq s < M(g; x, r) \in U_{t, s}} E_{t, s}(z, \rho). \]

Since the union in (2.3) is countable, it follows from (2.3) and the fact that the Hausdorff dimension is countable stable that
\[
\dim_H(E \cap B(x, r)) \leq \sup_{0 \leq t < M(f; x, r)} \sup_{\rho \in \mathbb{Q}^+} \dim_H(E_{t, s}(z, \rho))
\]
\[
= \sup_{0 \leq t < M(f; x, r)} \sup_{\rho \in \mathbb{Q}^+} \sup_{0 \leq s < M(g; x, r) \in U_{t, s}} t
\]
\[
= M(f; x, r)
\]
for all \( r > 0 \). Similarly, it follows that
\[
\dim_P(E \cap B(x, r)) \leq M(g; x, r)
\]
for all \( r > 0 \). This completes the proof of Claim 1. \( \square \)

**Claim 2.** For all \( x \in \mathbb{R}^d \) and all \( r > 0 \), we have
\[
m(f; x, r) \leq \dim_{H, \text{loc}}(x, E),
\]
\[
m(g; x, r) \leq \dim_{P, \text{loc}}(x, E).
\]

**Proof of Claim 2.** Fix \( x \in \mathbb{R}^d \) and \( r > 0 \). Next, let \( \varepsilon > 0 \) be such that \( m(f; x, r) - \varepsilon, m(g; x, r) - \varepsilon \in \mathbb{Q}^+ \). Write \( t = m(f; x, r) - \varepsilon \) and \( s = m(g; x, r) - \varepsilon \), and observe that \( t \leq s \). We clearly have \( x \in \{ t < f, s < g \} \), and we can therefore find \( u \in U_{t, s} \) with \( |u - x| \leq \frac{\varepsilon}{2} \). Now, pick any \( \rho \in \mathbb{Q}^+ \) with \( \rho \leq \frac{\varepsilon}{2} \). It now follows that
\[
E_{t, s}(u, \rho) \subseteq E,
\]
and that \( E_{t, s}(u, \rho) \subseteq B(u, \rho) \subseteq B(x, r) \), whence
\[
E \cap B(x, r) \supseteq E_{t, s}(u, \rho) \cap B(x, r) = E_{t, s}(u, \rho).
\]
We therefore conclude that
\[
\dim_H(E \cap B(x, r)) \geq \dim_H(E_{t, s}(u, \rho)) = t \geq m(f; x, r) - \varepsilon.
\]
Similarly, we conclude that
\[
\dim_P(E \cap B(x, r)) \geq \dim_P(E_{t, s}(u, \rho)) = s \geq m(g; x, r) - \varepsilon.
\]
Claim 2 follows from (2.4) and (2.5) by letting \( \varepsilon \searrow 0 \) through values such that \( m(f; x, r) - \varepsilon, m(g; x, r) - \varepsilon \in \mathbb{Q}^+ \). \( \square \)

Theorem 1 follows immediately from Claim 1 and Claim 2. \( \square \)
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References


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