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On the microeconomic foundations of linear demand for differentiated products *

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Abstract

This paper provides a thorough exploration of the microeconomic foundations for the multi-variate linear demand function for differentiated products that is widely used in industrial organization. A key finding is that strict concavity of the quadratic utility function is critical for the demand system to be well defined. Otherwise, the true demand function may be quite complex: Multi-valued, non-linear and income-dependent. The solution of the first order conditions for the consumer problem, which we call a local demand function, may have quite pathological properties. We uncover failures of duality relationships between substitute products and complementary products, as well as the incompatibility between high levels of complementarity and concavity. The two-good case emerges as a special case with strong but non-robust properties. A key implication is that all conclusions derived via the use of linear demand that does not satisfy the law of Demand ought to be regarded with some suspicion.

JEL codes: D43, L13, C72.

Key words and phrases: linear demand, substitutes, complements, representative consumer, Law of Demand.

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1 Introduction

The emergence of the modern theory of industrial organization owes much to the development of game theory. Due to its privileged position as the area where novel game theoretic advances found their initial application in an applied setting, industrial organization then served as a further launching ground for these advances to spread to other areas of economics. Yet to explain the success of industrial organization in reaching public policy makers, antitrust practitioners, and undergraduate students, one must mention the role played by the fact that virtually all of the major advances in the theory have relied on an accessible illustration of the underlying analysis using the convenient framework of linear demand.

While this framework goes back all the way to Bowley (1924), it received its first well-known treatment in two visionary books that preceded the revival of modern industrial organization, and yet were quite precise in predicting the intimate link to modern game theory: Shubik (1959) and Shubik and Levitan (1980). Then early on in the revival period, Dixit (1979), Deneckere (1983) and Singh and Vives (1984) were among the first users of the linear demand setting. Subsequently, this framework has become so widely invoked that virtually no author nowadays cites any of these early works when adopting this convenient setting.

Yet, despite this ubiquitous and long-standing reliance on linear demand, the present paper will argue that some important foundational and robustness aspects of this special demand function remain less than fully understood.¹ Often limiting consideration to the two-good case, the early literature on linear demand offered a number of clear-cut conclusions both on the structure of linear demand systems as well as on its potential to deliver unambiguous conclusions for some fundamental questions in oligopoly theory. Among the former, one can mention the elegant duality features uncovered in the influential paper by Singh and Vives (1984), namely (i) the dual linear structure of inverse and direct demands (along with the clever use of roman and greek parameters), (ii) the duality between substitute and complementary products and the invariance of the associated cross-slope parameter range of length one for each, and (iii) the resulting dual structure of Cournot

¹One is tempted to attribute this oversight to the fact that industrial economists' strong interest in linear demand is not shared by general microeconomists (engaged either in theoretical or in empirical work), as evidenced by the fact that quadratic utility hardly ever shows up in basic consumer theory or in general equilibrium theory.

and Bertrand competition. In the way of important conclusions, Singh and Vives (1984) showed that, under linear demand and symmetric firms, competition is always tougher under Bertrand than under Cournot. In addition, were the mode of competition to be endogenized in a natural way, both firms would always prefer to compete in a Cournot rather than in a Bertrand setting. (Singh and Vives, 1984 inspired a rich literature still active today). Subsequently, Hackner (2000) showed that with three or more firms and unequal demand intercepts, the latter conclusion is not universally valid in that there are parameter ranges for which competition is tougher under a Cournot setting, and that consequently some firms might well prefer a Bertrand world. Hsu and Wang (2005) show that consumer surplus and social welfare are nevertheless higher under Bertrand competition for any number of firms.

With this as its starting point, the present paper provides a thorough investigation of the micro-economic foundations of linear demand. Following the aforementioned studies, linear demand is derived in the most common manner as the solution to a representative consumer maximizing a utility function that is quadratic in the n consumption goods and quasi-linear in the numeraire. When this utility function is strictly concave in the quantities consumed, the first order conditions for the consumer problem do give rise to linear demand, as is well known. Our first main result is to establish that this is the only way to obtain such a micro-founded linear demand. In other words, we address the novel question of integrability of linear demand, subject to the quasi-linearity restriction on candidate utility functions and find that linear demand can be micro-founded in the sense of a representative consumer if and only if satisfies the strict Law of Demand in the sense of decreasing operators (see Hildenbrand, 1994), i.e., if and only if the associated substitution/complementarity matrix is positive definite. As a necessary first step, we derive some general conclusions about the consumer problem with quasi-linear preferences that do not necessarily satisfy the convexity axiom. In so doing, we explicitly invoke some powerful results from the theory of monotone operators and convex analysis (see e.g., Vainberg, 1973 and Hildenbrand, 1994), as well as a mix of basic and specialized results from linear algebra.

We also observe that strict concavity of the utility function imposes significant restrictions on the range of complementarity of the n products. For the symmetric substitution matrix of Hackner (2000), we show that the valid parameter range for the complementarity cross-slope is $(-\frac{1}{n-1}, 0)$,

which coincides with the standard range of $(-1, 0)$ if and only if there are exactly two goods ($n = 2$). In contrast, the valid range for the cross parameter capturing substitute products is indeed $(0, 1)$, independently of the number of products, which is in line with previous belief. We also explore the relationship between the standard notions of gross substitutes/complements and the alternative definition of these relationships via the utility function. Here again, the conclusions diverge for substitutes and for complements as soon as one has three or more products. All together then, the neat duality between substitute and complementary products breaks down in multiple ways for the case of three or more products.

Another point of interest is that, in the case of complements, as one approaches from above the critical value of $-\frac{1}{n-1}$, the usual necessary assumption of enough consumer wealth for an interior solution becomes strained as the amount of wealth needed is shown to converge to infinity! This further reinforces, in a sense that is hard to foresee, the finding that linear demand is not robust to the presence of high levels of inter-product complementarity!

Since many studies have used linear demand in applied work without concern for microeconomic foundations,² it is natural to explore the nature of linear demand when the strict concavity conditions on utility do not hold. In other words, we investigate the properties of the solution to the first order conditions of the consumer problem. For simplicity we do so for the n -good fully symmetric case (i.e., all off-diagonal terms of the substitution matrix are equal). Due to the lack of strict concavity, this will be only a local extremum (with no global optimality properties), which we term a local demand function. We find that, depending on which violation of strict concavity one allows, several rather unexpected exotic phenomena might arise. Demand functions might then fail the Law of Demand, even though each individual demand might remain downward-sloping in own price. For another parameter violation, individual demands might even be upward-sloping in own price (i.e., Giffen goods with a linear demand). In particular, we explicitly solve for the global solution of the utility maximization problem with a symmetric quadratic utility function that barely fails strict concavity, and show that the resulting demand can be multi-valued, highly non-linear and overall quite complex even for the two-good case. (Though similar effects will apply, higher

²A classical example appears in Okuguchi's (1987) early work on the comparison between Cournot and Bertrand equilibria, which is discussed in some detail in the present paper.

values of n appear to be intractable as far as a closed-form solutions are concerned).

As a final point, we investigate one special case of linear demand with a local interaction structure. This is characterized by the fact that the representative consumer is postulated as viewing products as imperfect substitutes if they are direct neighbors in a horizontal attribute space and as unrelated otherwise. Though intended as a model of vertical differentiation, the well-known model of the car industry due to Bresnahan (1987) has the same local interaction structure.

This paper is organized as follows. Section 2 gathers all the microeconomic preliminaries for general quasi-linear preferences. Section 3 specializes to quadratic utility and investigates the integrability properties of linear demand. Section 4 narrows consideration further to symmetric quadratic utility for tractability reasons. Section 5 explores the relationship between the notions of gross substitutes/complements and the alternative definition of these relationships via the utility function. Finally Section 6 offers a brief conclusion.

2 Some basic microeconomic preliminaries

In this section, we work with the two standard models from the textbook treatment of consumer theory, but allowing for general preferences that are quasi-linear in the numeraire good, but do not necessarily satisfy the convexity axiom (i.e, the utility function is not necessarily strictly quasi-concave). The main goal is to prove that Marshallian demands are decreasing in the sense of monotone operators (Hildenbrand, 1994), and thus also decreasing in own price.

2.1 On consumer theory with quasi-linear utility

Let $x \in R_+^n$ denote the consumption levels of the n goods and $y \in R_+$ be the numeraire good. The agent is endowed with a utility function $U : R_+^n \rightarrow R$ over the n goods and the numeraire y appears in an additively separable manner in the overall utility. The agent has income $m \geq 0$ to spend on purchasing the $(n + 1)$ goods.

The utility maximization problem is, given a price vector $p \in R_+^n$ and the numeraire price normalized to 1,

$$\max U(x) + y \tag{1}$$

subject to (throughout, " \cdot " denotes the usual dot product between vectors)

$$p \cdot x + y \leq m. \tag{2}$$

We shall refer to the solution vector (i.e., the argmax) as the Marshallian demands, denoted $(x^*(p, m), y^*(p, m))$ or simply (x^*, y^*) . We shall also use the notation $D(p) = (D_1(p), D_2(p), \dots, D_n(p))$ for this direct demand function since the argument m will be immaterial in what follows.

The (dual problem of) expenditure minimization is (with u being a fixed utility level)

$$\min p \cdot x + y \tag{3}$$

subject to

$$U(x) + y \geq u.$$

We shall refer to the solution vector as the Hicksian demands $(x^h(p, u), y^h(p, u))$ or simply (x^h, y^h) . We shall also use the notation $D^h(p)$ for this direct demand function since the argument u will not matter below. Recall that the (minimal) value function is the so-called expenditure function, denoted $e(p, u)$.

The following assumption is maintained throughout the paper.³

(A1). *The utility function U is twice continuously differentiable and satisfies $U_i \triangleq \frac{\partial U_i}{\partial x_i} > 0$, for $i = 1, 2, \dots, n$.*

Since U is not necessarily strictly quasi-concave, the solutions to the two problems above, the Marshallian demands (x^*, y^*) and the Hicksian demands (x^h, y^h) , may be correspondences in general.⁴ By Weirstrass's Theorem, both correspondences are non-empty valued for each (p, m) .

2.2 On the Law of Demand

In standard microeconomic demand theory, though not always explicitly recognized, the downward monotonicity of multi-variate demand is usually meant in the sense of monotone operators

³Smoothness is assumed only for convenience here, and is not critical to any of the conclusions of the paper.

⁴It is important in this paper to allow for utility functions that do not satisfy the ubiquitous quasi-concavity assumptions since we shall be concerned in some parts of this paper with maximizing quadratic, but non-concave, utility functions.

(for a thorough introduction, see Vainberg, 1973). This is a central concept in the theory of demand aggregation in economics (Hildenbrand, 1994) as well as in several contexts in applied mathematics (Vainberg, 1973). We begin with its definition and a brief summary of some simple implications.

Let S be an open convex subset of R^n and F be a function from S into R^n . We shall say that F is (strictly) aggregate-monotonic if F satisfies (here " \cdot " denotes dot product) ⁵

$$[F(s) - F(s')] \cdot (s - s') \leq (<)0 \text{ for every } s, s' \in S. \quad (4)$$

This notion of downward monotonicity is quite distinct from the more prevalent notion of monotonicity in the coordinate-wise (or product) Euclidean order that arises naturally in the theory of super-modular optimization and games (Topkis, 1998, Vives, 1999). Nonetheless, for the special case of a scalar function, both notions boil down to the usual notion of monotonicity, and thus constitute alternative but distinct natural generalizations.

The following characterization of aggregate monotonicity in this context is well known. Let $\partial F(s)$ denote the Jacobian matrix of $F(s)$, i.e., the ij th entry of the matrix $\partial D(s)$ is $\partial_{ij}F(s) = \frac{\partial F_i(s)}{\partial s_j}$, which captures of the effect of a change in the price of the j th good on the demand for the i th good. This is a well-known result; for a proof, see e.g., Vainberg (1973) or [Hildenbrand (1994), Appendix].

Lemma 1 *Let S be an open convex subset of R^n and $F : S \rightarrow R^n$ be a continuously differentiable map. Then the following two properties hold.*

- (i) *F is aggregate-monotonic if and only if the Jacobian matrix $\partial F(s)$ is negative semi-definite.*
- (ii) *If the Jacobian matrix $\partial F(s)$ is negative definite, then F is strictly aggregate-monotonic.*

In Part (ii), the equivalence between the two strict notions need not hold. There are examples of strictly aggregate-monotonic maps with a Jacobian matrix whose determinant is not everywhere non-zero.

⁵In the mathematics literature, functions with this property are simply referred to as monotone functions (or operators). The choice of the terminology "aggregate-monotonic" is ours, and is motivated by two considerations. One is that this is the standard notion of monotonic demand in aggregation theory in economics. The other is a desire to distinguish this monotonicity notion from the more prevalent one of coordinatewise monotonicity.

An important direct implication of Lemma 1 is that the diagonal terms of $\partial F(s)$ must be negative. However, this monotonicity concept does not impose restrictions on the signs of the off-diagonal elements of $\partial F(s)$. In contrast, monotonicity in the coordinate-wise order requires that every element of the Jacobian $\partial F(s)$ be (weakly) negative.

Definition 2 *The Marshallian demand $D(p)$ satisfies the (strict) Law of Demand if $D(p)$ is (strictly) aggregate-monotonic, i.e., for any two price vectors p and p' , D satisfies*

$$[D(p) - D(p')] \cdot (p - p')(<) \leq 0 \tag{5}$$

In classical consumer theory, this property is well-known not to hold under very general conditions on the utility function, but sufficient conditions that validate it are available, see Hildenbrand (1994) for details and discussion.

Consistent with Lemma 1, a demand function that satisfies the Law of Demand necessarily has the property that each demand component is downward-sloping in own price (i.e., the diagonal elements of the Jacobian matrix are all ≤ 0). In other words, no good can be a Giffen good. In addition, as Lemma 1 makes clear, the Law of Demand entails significantly more restrictions on the demand function.

The following general result reflects a key property of demand that constitutes the primary motivation for postulating a quasi-linear utility function in industrial organization. This result will prove very useful below.

Proposition 3 *Under Assumption A1, the Marshallian demand $D(p)$ satisfies the Law of Demand.*

Proof. We first prove that the Hicksian demand satisfies the Law of Demand. In the expenditure minimization problem, the expenditure function $e(p, u)$, as defined in (3), is the pointwise infimum of a collection of affine functions in p . Hence, by a standard result in convex analysis (see e.g., Rockafellar, 1970, Theorem 5.5 p. 35), for an arbitrary such collection, $e(p, u)$ is a concave function of the price vector p . Since the Hicksian demand $D^h(p)$ is the gradient of $e(p, u)$, i.e., $\frac{\partial e(p, u)}{\partial p_i} = D_i^h(p) = x_i^h$ (in other words, this is just the standard Hotelling's Lemma), it follows from a well-known result in convex analysis, which characterizes the subgradients of concave functions (Rockafellar, 1966), that $D^h(p)$ satisfies (5).

Since the overall utility is quasi-linear in the numeraire, it is well known that the Marshallian demand inherits the properties of the Hicksian demand. Hence $D(p)$ too satisfies the Law of Demand (5). ■

Recall that in the standard textbook treatment of these monotonicity issues, the utility function is assumed to be strictly quasi-concave. The main advantage of using the given general results from convex analysis is to bring to light the fact that quasi-concavity of the utility function is not needed for this basic result.

3 The case of quadratic utility

In this section, we investigate the implications of the general results from the previous section that hold when we specialize the utility function U to be a quadratic function in problem (1). Along the way, we also review and build on the basic existing results for the case of a concave utility.

Using the same notation as above, the representative consumer's utility function is now given by (here $'$ denotes the transpose operation)

$$U(x) = a'x - \frac{1}{2}x'Bx \tag{6}$$

where a is a positive n -vector and B is an $n \times n$ matrix. Without loss of generality, assume B is symmetric and has all its diagonal entries equal to 1.

3.1 A strictly concave quadratic utility

For this subsection, we shall assume that the matrix B is positive definite, which implies that the utility function is strictly concave. This constitutes the standard case in the broad literature in industrial organization that relies on quadratic utility.

It is well known that such a utility function gives rise to a generalized Bowley-type demand function. We allow a priori for the off-diagonal entries of the matrix to have any sign, although different restrictions will be introduced for some more definite results. Thus, this formulation nests different inter-product relationships, including substitute goods, complementary goods, and hybrid cases.

The consumer's problem is to choose x to solve

$$\max\{a'x - \frac{1}{2}x'Bx + y\} \quad \text{subject to } p'x + y = m \quad (7)$$

As a word of caution, we shall follow the standard abuse of terminology in referring to the demand function at hand as linear demand, although a more precise description would clearly refer to it as being an affine function whenever positive and zero otherwise.

The following result is well known (see e.g. Amir and Jin, 2001), but included for the sake of stressing the need to make explicit the following basic assumption.

(A2). *The primitive data in (7) satisfy $B^{-1}(a - p) > 0$ and $pB^{-1}(a - p) \leq m$.*

As will become clear below, this Assumption is needed not only to obtain an interior solution to the consumer problem (in each product), but also to preserve the linear nature of the resulting demand function.

Lemma 4 *Assume that (A2) holds and that the matrix B is positive definite. Then the inverse demand is given by*

$$P(x) = a - Bx \quad (8)$$

and the direct demand is

$$D(p) = B^{-1}(a - p) \quad (9)$$

Proof. Since the utility function is quasi-linear in y , the consumer's problem (1) can be rewritten as $\max\{a'x - \frac{1}{2}x'Bx + m - p \cdot x\}$. Since B is positive definite, this maximand is strictly concave in x . Therefore, whenever the solution is interior, the usual first-order condition with respect to x , i.e., $a - Bx - p = 0$, is sufficient for global optimality. Solving the latter matrix equation directly yields the inverse demand function (8). It is easy to check that this solution is interior under Assumption (A2), as the part $B^{-1}(a - p) > 0$ says that each quantity demanded is strictly positive, and the part $pB^{-1}(a - p) \leq m$ simply says that $p \cdot D(p) \leq m$, i.e., that the optimal expenditure is feasible.

Since B is positive definite, the inverse matrix B^{-1} exists and is also positive definite (see e.g., McKenzie, 1960). Inverting in (8) then yields (9). ■

At this point, it is worthwhile to remind the reader about three hidden points that will play a clarifying role in what follows. The first two points elaborate on the tacit role of Assumption (A2).

Remark 5 *In the common treatment of the derivation of linear demand in industrial organization, one tacitly assumes that the representative consumer is endowed with a sufficiently high income. The main purpose of Assumption (A2) is simply to provide an explicit lower bound on how much income is needed for an interior solution. We shall see later on that when Assumption (A2) is violated, the resulting demand is not only non-linear, it is also income-dependent. Thus income effects are then necessarily present, a key departure from the canonical case in industrial organization.*

The second point explains the absence of income effects, and thus captures the essence of a quasi-linear utility.

Remark 6 *Suppose we have a solution (8) and (9) for some m such that $pB^{-1}(a-p) \leq m$. Then it can be easily shown that, for every $m' > m$, the solution of the consumer problem is still given by (8)-(9).*

The third point explains the need for the *strict* concavity of U .

Remark 7 *The reason one cannot simply work with a quadratic utility function that is just concave (but not strictly so) is that, then, a matrix B that is just positive semi-definite (and not positive definite) may fail to be invertible. One immediate implication then is that the direct demand need not be well defined (unless one uses some suitable notion of generalized inverse).*

It is well-known that when B is positive definite, direct and inverse demands are both decreasing in own price (see e.g., Amir and Jin, 2001). In fact, we now observe that a stronger property holds.

Corollary 8 *If the matrix B is positive definite, both the inverse demand and the direct demand satisfy the strict Law of Demand, i.e., (4).*

Proof. This follows directly from Lemma 1, since the Jacobian matrices of the inverse demand and the direct demand are clearly B and B^{-1} respectively, both of which are positive definite. ■

The Law of Demand includes joint restrictions on the dependence of one good's price on own quantity as well as on all cross quantities. It captures in particular the well known property that own effect dominates cross effects.

3.2 Integrability of linear demand

In this subsection, we consider the reverse question from the one treated in the previous subsection. Namely, suppose one is given a linear inverse demand function of the form $D(p) = d - Mp$, where d is an $n \times 1$ vector and M is an $n \times n$ matrix, along with the corresponding inverse demand. The issue at hand is to identify minimal sufficient conditions on d and M that will guarantee the existence of a utility function of the form 1, a priori satisfying only continuity and quasi-linearity in the numeraire good, such that $D(p)$ can be obtained as a solution of maximizing that utility function subject to the budget constraint (2)?

The framing of the issue under consideration here is directly reminiscent of the standard textbook treatment of integrability of demand, but there are two important distinctions. In the present treatment, on the one hand, we limit consideration to quasi-linear utility, but on the other hand, we do not a priori require the underlying utility function to reflect convex preferences. The latter point is quite important in what follows, in view of the fact that one of the purposes of the present paper is to shed light on the role that the concavity of the quadratic utility function (or lack thereof) plays in determining some relevant properties of the resulting linear demand function. The second distinction from the textbook treatment is that the starting primitives here include both the direct and the inverse demand functions. It turns out that this is convenient for a full characterization.

Proposition 9 (i) *Let there be given a linear demand function $D(p) = d - Mp$ with $d_i \geq 0$ for each i , along with the corresponding inverse demand $P(\cdot)$. Then there exists a continuous utility function $U : R_+^n \rightarrow R$ such that $D(p)$ can be obtained by solving*

$$\max\{U(x) + y\} \quad \text{subject to} \quad px + y \leq m$$

if and only if M is positive definite and Assumption A2 holds.

Then the desired U is given by the strictly concave quadratic function (6) with $B = M^{-1}$, and both the demand and the inverse demand function satisfy the strict Law of Demand.

Proof. Part (i) The "if" part was already proved in Lemma 4, with being the quadratic utility given in (6).

For the "only if" part, recall that by Proposition 3, every direct demand function that is the solution to the consumer problem when U is continuous and quasi-linear in the numeraire good (but

not necessarily quadratic) satisfies the Law of Demand. Therefore, via Lemma 1, the Jacobian of $(d - Mp)$, which is equal to M , must be positive semi-definite.

Now, since both direct and inverse demands are given, the matrix M must be invertible, and hence has no zero eigenvalue. Therefore, since M is Hermitian, M must in fact be positive definite. This implies in turn that the system of linear equations $Ma = d$ possesses a unique solution a with $a_i > 0$, i.e., such that $a = M^{-1}d$. Finally, identifying M with B^{-1} yields the fact that the demand function can be expressed in the desired form, i.e., $D(p) = d - Mp = M(a - d) = B^{-1}(a - d)$, as given in (9).

Inverting the direct demand $D(p)$ yields the inverse demand (8). Integrating the latter yields the utility function (6), which is then strictly concave since the matrix B is positive definite.

Finally, the fact that both $D(p)$ and $P(x)$ satisfy the strict Law of Demand, then follows directly from Corollary 8. ■

The main message of this Proposition is that any linear demand that is micro-founded in the sense of maximizing the utility of a representative consumer necessarily possesses strong regularity properties. Provided the utility function is quasi-linear (but not even quasi-concave a priori), the linear demand must necessarily satisfy the Law of Demand, and originate from a strictly concave quadratic utility function.

This clear-cut conclusion carries some strong implications, some of which are well understood, including in particular that (i) the demand for each product must be downward-sloping in own price (i.e., no Giffen goods are possible), and (ii) demand cross effects must be dominated by own effects.

On the other hand, the following implication is remarkable, and arguably quite surprising.

Corollary 10 *If a quadratic utility function of the form given in (6) is not concave, i.e., if the matrix B is not positive semi-definite, then this utility could not possibly give rise to a linear demand function.*

We emphasize that this conclusion holds despite the fact that the utility function is concave in each good separately (indeed, recall that the matrix B is assumed to have all 1's on the diagonal). The key point here is that joint concavity fails.

This immediately raises a natural question: What solution is implied by the first order conditions for utility maximization in case the matrix B is not positive semi-definite, and how does this solution

fit in with the Corollary? This question is addressed in the next section, in the context of a fully symmetric utility function, postulated as a simplifying assumption, as in Singh and Vives (1984), Hackner (2000) and others .

4 Symmetric non-concave quadratic utility

In this section, we investigate a common specification of linear demand in industrial organization along the lines suggested by the results of the previous section. We also elaborate on the question raised there about the meaning of first order conditions when the substitution matrix fails to be positive semi-definite.

4.1 A common special case

A widely used utility specification for a representative consumer foundation is characterized by a fully symmetric substitution/complementarity matrix, i.e. one in which all cross terms are identical for all pairs of goods and represented by a parameter $\gamma \in [-1, 1]$ (e.g., Singh and Vives, 1984 and Hackner, 2000). The substitution matrix is thus

$$B = \begin{bmatrix} 1 & \gamma & \dots & \gamma \\ \gamma & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma \\ \gamma & \dots & \gamma & 1 \end{bmatrix}, \quad (10)$$

which can be reformulated as $B \equiv (1 - \gamma)I_n + \gamma J_n$, where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix of all ones.

It is common in the literature to postulate that the meaningful range for the possible values of γ is a priori $[-1, 1]$, with $\gamma \in [-1, 0)$ corresponding to (all goods being) complements, $\gamma \in (0, 1)$ to substitutes, and $\gamma = 0$ to independent goods. While we begin with $[-1, 1]$ being the a priori possible range, we shall see below that for the case of complements, further restrictions will be needed.

As previously stated, concavity of U is sufficient for the first-order condition to provide a solution to the consumer's problem. It turns out that for the special substitution matrix at hand, concavity of U can easily be fully characterized.

Lemma 11 *The quadratic utility function in (6) with B as in (10) is strictly concave if only if $\gamma \in (-\frac{1}{n-1}, 1)$.*

Proof. For U to be concave, it is necessary and sufficient that B be positive semi-definite. To prove the latter is equivalent to showing that all the eigenvalues of B are positive. To this end, consider

$$\begin{aligned} B - \lambda I_n &= \begin{bmatrix} 1 - \lambda & \gamma & \dots & \gamma \\ \gamma & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \gamma \\ \gamma & \dots & \gamma & 1 - \lambda \end{bmatrix} \\ &= \gamma J_n + (1 - \lambda - \gamma)I_n \end{aligned}$$

By the well known matrix determinant lemma, we have

$$\begin{aligned} \det[B - \lambda I_n] &= \det[\gamma J_n + (1 - \lambda - \gamma)I_n] \\ &= (1 - \lambda - \gamma)^{n-1}[1 - \lambda + (n - 1)\gamma] \end{aligned} \tag{11}$$

The solutions of $\det[B - \lambda I_n] = 0$ are then $\lambda = 1 - \gamma$ and $\lambda = 1 + (n - 1)\gamma$. Since a priori $\gamma \in [-1, 1]$, by simple inspection, these solutions are > 0 if and only if $-1/(n - 1) < \gamma < 1$. ■

The following observation follows directly from the Proposition and the results of the previous section.

Corollary 12 *Given a linear demand $D(p) = B^{-1}a - B^{-1}p$ with B as given in (10), $D(p)$ can be derived from a quadratic utility function of the form (6) if only if $\gamma \in (-\frac{1}{n-1}, 1)$, in which case both $D(p)$ and the corresponding inverse demand satisfy the Law of Demand.*

It follows that the range of values of the parameter that validate a linear demand function is not $(0, 1]$, but rather $(-\frac{1}{n-1}, 1)$. One important direct implication is that there is a fundamental asymmetry between the cases of substitutes and complements. For substitutes, the valid range is indeed $(0, 1)$, as is widely believed, and this range is independent of the number of goods n . However, for complements, the valid range is $(-\frac{1}{n-1}, 0)$, which monotonically shrinks with the number of goods n , and converges to the empty set as the number of goods $n \rightarrow +\infty$.

This Corollary uncovers an exceptional feature of the ubiquitous two-good case, as reported next.

Remark 13 *The special case of two goods ($n = 2$) is the only case for which the valid range of the parameter γ for a concave utility, and thus for a well-founded demand, i.e. $(-\frac{1}{n-1}, 1)$, is equivalent to the interval $(-1, 1)$, as commonly (and correctly) believed (e.g., Singh and Vives, 1984).⁶*

Before moving on to explore the properties of the solutions of the first order conditions when the latter are not sufficient for global optimality, we report a remarkable result about the hidden regularizing effects of strict concavity.

Proposition 14 *Consider the quadratic utility function in (6) with B as in (10). As $\gamma \downarrow -\frac{1}{n-1}$, the level of income required to obtain an interior linear demand function converges to ∞ .*

Proof. We first derive a simplified version of Assumption (A2) for the case where the matrix B as in (10). As $a_i = a > p_i = p$, one clearly has $x > 0$.

To check that $px \leq m$, first note that $b_{ii} = 1$ and $b_{ij} = \gamma$ (for $i \neq j$). In addition, for the matrix B^{-1} , each diagonal element is equal to $1 + \frac{(n-1)\gamma^2}{(1-\gamma)[1+(n-1)\gamma]}$, and each off-diagonal term is $\frac{-\gamma}{(1-\gamma)[1+(n-1)\gamma]}$. Therefore, upon a short computation, $p \cdot x = \frac{np(a-p)}{1+(n-1)\gamma}$. The latter fraction converges to $+\infty$ as $\gamma \downarrow -\frac{1}{n-1}$ (since its numerator is > 0).

Since Assumption (A2) requires that $p \cdot x = \frac{np(a-p)}{1+(n-1)\gamma} \leq m$, the conclusion follows. ■

This Proposition is a powerful criticism of the assumption that the representative consumer is endowed with a sufficient income level to allow for an interior solution to the utility maximization problem, in cases where the products under consideration are strong complements (i.e., for γ close to the maximum allowed value of $-\frac{1}{n-1}$). If one needs to require infinite wealth to rationalize a linear demand system for complements, then perhaps it is time to start questioning the well-foundedness of such demand functions. Put differently, perhaps industrial economists have been overly valuing the analytical tractability of linear demand.

We next move on to other pathological features that might emerge in the absence of a strictly concave utility function.

⁶ Actually, in industrial organization, it is not uncommon to find studies that postulate the valid range as being the closed interval $[-1, 1]$, instead of the open $(-1, 1)$.

4.2 The solution to the first order conditions

Continuing with our investigation of the robustness of the linear demand specification, we now address the following key issue: What properties are satisfied by the solution implied by the first order conditions for utility maximization in case the utility is given by (6), but with a matrix B in (10) that is not positive semi-definite? The answer, as we shall justify in some detail below, is that although the said solution looks exactly like the familiar linear demand function, it is actually not the true global solution to the utility maximization problem, due to the lack of concavity of the utility function.

We know from Lemma 11 that for U not to be concave requires exactly the following assumption, which we make throughout this subsection.

(A3) *The parameter γ satisfies $-1 < \gamma < -1/(n-1)$.*

In such a case, it is obvious that the first order conditions for utility maximization continue to give rise to a *candidate solution*, which looks just like the standard linear inverse demand (8). However, no longer being a priori the actual global argmax of the consumer problem (due to the absence of concavity of the consumer's objective function), we shall refer to (8) as a local inverse demand function in this context. We stress that this just an extremum of the consumer problem, and not the actual demand function. The true demand function in such cases will actually be non-linear and quite complex, as illustrated in an example below.

In light of Assumption (A3) and Lemma 11, this candidate solution need not, a priori, be invertible so as to yield a corresponding direct demand function. Thus the first issue we tackle is the invertibility of the matrix B for this local demand, as given in (10) but without positive definiteness. Due to the special structure at hand, we obtain a full characterization via a closed-form inverse.

Lemma 15 *As long as $\gamma \neq 1$ and $\gamma \neq -\frac{1}{n-1}$, the matrix $B = (1-\gamma)I_n + \gamma J_n$ is non-singular and has the inverse*

$$B^{-1} = \frac{1}{1-\gamma} \left[I_n - \frac{\gamma}{(n-1)\gamma + 1} J_n \right]. \quad (12)$$

Proof. The proof follows from the identity that the inverse of a matrix of the form $aI_n + bJ_n$ takes the form $\tilde{a}I_n + \tilde{b}J_n$. The unknown coefficients can be identified directly by setting the product

of B and B^{-1} equal to I_n as follows (removing n subscripts for notational convenience):

$$\begin{aligned}
& [(1 - \gamma)I + \gamma J][aI + bJ] = I \\
\Rightarrow & (1 - \gamma)IaI + (1 - \gamma)IbJ + \gamma JaI + \gamma JbJ = I \\
\Rightarrow & (1 - \gamma)aI + (1 - \gamma)bJ + \gamma aJ + \gamma bnJ = I \\
\Rightarrow & (1 - \gamma)aI + [(1 - \gamma)b + \gamma a + \gamma bn]J = I,
\end{aligned}$$

the third step following from the fact that $J_n J_n = nJ_n$. Letting $a = \frac{1}{1-\gamma}$, all that remains to be done is solve for b with

$$(1 - \gamma)b + \frac{\gamma}{1 - \gamma} + \gamma bn = 0,$$

which has the solution

$$b = -\frac{\gamma}{(1 - \gamma)[(n - 1)\gamma + 1]},$$

as long as $\gamma \neq 1$ and $\gamma \neq -\frac{1}{n-1}$. This concludes the proof. ■

To recapitulate, for all values of $\gamma \in [-1, 1]$ other than $\gamma = 1$ and $\gamma = -\frac{1}{n-1}$, the solution of the first order conditions for utility maximization yields a well-defined local inverse demand and direct demand of the forms (8) and (9). However, the main results of the present paper indicate that this pair cannot be the actual solution to the utility maximization problem unless $\gamma \in (-\frac{1}{n-1}, 1)$! In other words, whenever $\gamma \in [-1, -\frac{1}{n-1}]$, the utility maximization problem here is a well-defined non-concave quadratic optimization problem for which the first order conditions do not yield an actual solution, due to the absence of concavity.⁷

Nonetheless, it is worth investigating the properties of the local inverse and direct demand functions, despite the fact that they do not arise from the actual solution to the consumer problem. In particular, any study that postulates a demand function with a B matrix that is not positive definite might be viewed as a local demand function in the present sense. One such example appears in Okuguchi (1987), and is reviewed below.

The following results will underscore the importance of strict concavity of the utility function for consumer theory even in the quadratic case.

⁷In the general theory of quadratic programming, this feature is well known to arise when suitable second order conditions do not hold (see e.g., Burer and Letchford, 2009).

The first set of properties of the local direct demand function already contain some major departures from familiar characteristics in industrial organization.

Proposition 16 *Under Assumption (A2), the following hold.*

(i) *Though not aggregate-monotonic, the local inverse demand function is such that the price of each good is downward-sloping in own quantity.*

(ii) *If $\gamma \in [-1, -\frac{1}{n-2})$, the local direct demand of every good is decreasing in own price, even though D does not satisfy the Law of Demand overall.*

(iii) *If $\gamma \in (-\frac{1}{n-2}, -\frac{1}{n-1}]$, the local direct demand of every good is increasing in own price, i.e., all goods are Giffen goods.*

Proof. (i) It is obvious that the local inverse demand function $P(x) = a - Bx$ is such that each price is downward-sloping in own quantity since all the diagonal entries of B are 1's.

(ii)-(iii) Since the slope of the demand for each good is determined by the diagonal elements of B^{-1} , using (12), the following condition is equivalent to a downward-sloping (upward-sloping) demand curve:

$$1 - \frac{\gamma}{(n-1)\gamma + 1} > (<)0. \quad (13)$$

Upon a simple computation, using Assumption (A3), the two desired conclusions follow (the details are left out). ■

4.3 Two examples

Here, we shall solve explicitly for the demand function corresponding to a quadratic utility function in the two-good case when the underlying utility function is not strictly concave. This is meant as an example to illustrate the fact that without strict concavity, the resulting true demand function can be quite complex and non-linear. The restriction here to two goods is due to tractability. This example will further highlight the importance of strict concavity when working with a linear demand. Of particular interest is that violations of concavity for a quadratic utility easily lead to drastic departures from the usual properties one spontaneously associates with linear demand.

Example 17 *Here we consider a substitution parameter $\gamma = -1$. We find that even though this is*

a boundary case, the optimal solution departs in substantial ways from the familiar linear demand function.

For inverse demand function $p_i = 1 - x_i - rx_j$, with $\gamma = -1$, the underlying utility function is

$$U = x_0 + x_1 + x_2 - 0.5x_1^2 - 0.5x_2^2 + x_1x_2.$$

Without loss of generality, assume that $p_1 < p_2$. If $p_1 \geq 1$, substituting from the budget constraint $x_0 = m - p_1x_1 - p_2x_2$ into U yields

$$U = m + (1 - p_1)x_1 + (1 - p_2)x_2 - 0.5(x_1 - x_2)^2.$$

Hence the demand is $x_1 = x_2 = 0, x_0 = m$.

Now we let $p_1 < 1$, and consider the Lagrangian:

$$L = x_0 + x_1 + x_2 - 0.5(x_1 - x_2)^2 - \lambda(x_0 + p_1x_1 + p_2x_2 - m).$$

We need to compare three cases to determine the optimal demand.

(i) Choose $x_0 = m$, so $x_1 = x_2 = 0$, with $U = m$.

(ii) Choose $x_0 = 0$, so $p_1x_1 + p_2x_2 = m$. Then substitute $x_2 = (m - p_1x_1)/p_2$ into the utility function to obtain

$$U = m + (1 - p_1)x_1 + (1 - p_2)(m - p_1x_1)/p_2 - 0.5[x_1 - (m - p_1x_1)/p_2]^2.$$

a) The optimal demand can be obtained by the first-order condition as the second-order condition is easily seen to hold. If $m \geq (p_2 - p_1)p_1/(p_1 + p_2)$, we have

$$x_1 = \frac{m}{p_1 + p_2} + \frac{(p_2 - p_1)p_2}{(p_1 + p_2)^2} \text{ and } x_2 = \frac{m}{p_1 + p_2} - \frac{(p_2 - p_1)p_1}{(p_1 + p_2)^2}.$$

This leads to the corresponding $U = \frac{2m}{p_1 + p_2} + \frac{(p_2 - p_1)^2}{2(p_1 + p_2)^2}$.

b) If $m < (p_2 - p_1)p_1/(p_1 + p_2)$, the demands are $x_1 = m/p_1$ and $x_2 = 0$, and then $U = m/p_1 - 0.5(m/p_1)^2$.

(iii) Choose $x_0 \in (0, m)$, and x_1 and x_2 must satisfy the first-order conditions: $1 - x_1 + x_2 = p_1$, and $1 - x_2 + x_1 = p_2$. This is possible only if $p_1 + p_2 = 2$. The budget constraint implies $x_1 = 0.5[m - x_0 + p_2(1 - p_1)]$, $x_2 = 0.5[m - x_0 + p_1(1 - p_2)]$ where $p_1 + p_2 = 2$. This yields $U = m + (p_2 - p_1)^2/8$.

Finally the true demand function must be chosen from the solutions (i) – (iii), which yield the highest utility, depending on the parameters m , p_1 and p_2 .

It can be shown that if $p_1 + p_2 = 2$, (iii) is indeed the optimal demand (equally good as (ii.a)). However, if $p_1 + p_2 < 2$, it is dominated by (ii.a) and cannot be the true demand.

This demand function may be discontinuous. For instance, let $p_1 = \frac{3}{4}$, $p_2 = 9/4$, $m = 3/8$. Then case (iii) is not relevant. As $m = (p_2 - p_1)p_1/(p_1 + p_2)$, both (ii.a) and (ii.b) can apply as well as (i). All three cases yield the same utility of $3/8$, so all these three demand functions are valid.

If p_1 falls marginally, (ii.a) is valid, but not (ii.b). The utility given by (ii.a) rises while that in (i) remains constant. So we should have $x_1 = m/p_1 = 0.5$ and $x_2 = 0$.

However, if p_1 rises marginally, (ii.b) applies, but not (ii.a). But the utility given by (ii.b) falls, while that in (i) remains constant. Then the true demand becomes $x_1 = x_2 = 0$. In this case the demand for good 2 remains continuous, but not for good one.

To obtain a linear demand function, we assume prices are sufficiently low such that $p_1 + p_2 \leq 2$.

This implies some of x_1 or x_2 will be demanded, so case (i) is ruled out. Furthermore we assume there is sufficient income ($m \geq (p_2 - p_1)p_1/(p_1 + p_2)$), so that (ii.b) is excluded. Then we only have two cases of (ii.a) and (iii) left.

If $p_1 + p_2 < 2$, the solution (ii.a) dominates (iii). The demand functions are non-linear, and the income effect exists. Different from normal cases of concave utility functions, the income effect never disappears regardless of how high the income is. This is because two goods are perfect complements, the marginal utility of income can be kept above 1, so x_0 is never consumed.

If $p_1 + p_2 = 2$, (ii.a) and (iii) become identical when $x_0 = 0$. The demand for x_1 and x_2 will be lower in (iii) when $x_0 > 0$, but the utility is same (for both x_1 and x_2 to be positive, x_0 cannot be equal to m unless both p_1 and $p_2 = 1$). Even in this case, (iii) cannot be a truly linear demand. First, $p_1 + p_2 = 2$ implies prices cannot change independently. Secondly, x_1 and x_2 cannot be determined as both depend on x_0 . So a linear demand is not feasible.

Given these conditions, the demand is a continuous function as long as $p_1 + p_2 < 2$, but becomes an upper-semi continuous correspondence along the boundary $p_1 + p_2 = 2$.

The second example appears in a classic study in the literature on the comparison between Cournot and Bertrand equilibria.

Example 18 *Okuguchi (1987) uses the following demand specification to show that equilibrium prices may be lower under Cournot than under Bertrand.*

Consider the symmetric inverse/direct demand pair (for $i \neq j$):

$$p_i = \frac{1}{8}(2 + x_i - 3x_j) \text{ and } x_i = 1 - p_i - 3p_j \quad (14)$$

Two violations of standard properties stand out: (i) The inverse demand is upward-sloping, (ii) the two products appear to be complements in the inverse demand function, but substitutes in the inverse demand function.

The candidate utility function to conjecture as the origin of this demand pair is clearly

$$U = \frac{1}{8}(2x_1 + 2x_2 - 3x_1x_2 + 0.5x_1^2 + 0.5x_2^2) + x_0.$$

This is not a concave function. In fact, in contrast to our treatment so far, this utility function is actually strictly convex (and not concave) in each good separately, though not jointly strictly convex.

It can easily be shown by solving the usual consumer problem with this utility function that the resulting demand solution is not the one given in (14). The true solution includes some of the same complex features encountered in the previous example (the solution is not derived here for brevity).

This confirms what the results of the present paper directly imply for this demand pair, namely that it cannot be micro-founded in the sense of maximizing the utility of a representative consumer.

Therefore, this demand pair is essentially invalid, and thus the fact that it leads to Bertrand prices that are higher than their Cournot counterparts does not a priori constitute a valid counter-argument to the well known positive result under symmetry (Vives, 1985; Amir and Jin, 2001).

4.4 Cournot and Bertrand Oligopoly

Here we consider the standard models of Cournot and Bertrand oligopolies with linear (fully symmetric) demand, and linear costs normalized to zero. The issues already discussed concerning the relationship between concavity of the underlying utility function and the extent of product differentiation arise for oligopoly as well. To understand the role of the concavity of the utility function, we also consider local demand functions that are not necessarily global solutions to the consumer problem. After characterizing the concavity of each firm's profit function in own action, we will show how strategic substitutes or complements arise for different values of γ .

Under Assumption A2, the profit functions for firm i under Cournot and Bertrand competition are respectively

$$\Pi_i^C(q) = q_i(a - b_i \cdot q) \quad (15)$$

$$\Pi_i^B(p) = p_i b_i^{-1}(a - p) \quad (16)$$

where b_i and b_i^{-1} are the i th row of B and B^{-1} respectively. We allow a priori $\gamma \in (-1, 1)$ since we wish to investigate the behavior of firms' reaction curves even when the demand and inverse demand are only valid in a local sense.

Lemma 19 *For the profit functions in (15) and (16)*

- $\Pi_i^C(q)$ is strictly concave in own output q_i if and only if $\gamma \in \left(-\frac{1}{n-1}, 1\right)$.
- $\Pi_i^B(p)$ is strictly concave in own price p_i if and only if $\gamma \in \left(-1, -\frac{1}{n-2}\right) \cup \left(-\frac{1}{n-1}, 1\right)$.

Proof. The proofs of concavity come directly from the second-order conditions of Equation (15) and Equation (16), which are straight forward to derive for firm i as $-2b_{ii}$ and $-2b_{ii}^{-1}$, respectively. That these terms are negative is easy to see from the fact that the diagonal elements of B are 1 and B^{-1} are given by Equation (12). The details are omitted. ■

Hackner (2000) imposes similar restrictions on the range of γ , as second order conditions when investigating the properties of firms' reaction curves.

As pointed out in Singh and Vives (1984), Bertrand reaction curves should be upward-sloping under substitutes and downward-sloping under complements and the converse should hold for Cournot. While this is clearly the case under Cournot competition since the off-diagonal elements of B are simply γ , it is not necessarily true for Bertrand.

Proposition 20 *Under Bertrand competition, reaction curves are*

- (i) always upward-sloping for substitutes ($\gamma > 0$), and
- (ii) downward sloping for complements ($\gamma < 0$) if and only if $\gamma > -1/(n-2)$.

Proof. The reaction function for firm i under Bertrand competition is given by

$$p_i = \frac{B_i^{-1}a}{2B_{ii}} - \frac{B_{-i}^{-1}p_{-i}}{2B_{ii}}$$

with $-i$ indicating that the i th element has been removed. We want to show that

$$\frac{\partial p_i}{\partial p_j} = -\frac{B_{ij}^{-1}}{2B_{ii}} = \frac{\gamma}{2[(n-1)\gamma + 1 - \gamma]} < 0$$

For $\gamma > 0$, the result holds trivially since $\gamma < 1$. For $\gamma < 0$, all that is required is that $(n-1)\gamma + 1 > 0$, which implies the result. ■

Proposition 20 highlights the fact that excessive complementarity, i.e., $-1 \leq \gamma < -1/(n-2)$, is also not compatible with a standard property of the behavior of firms in price competition with complements and more than three firms, namely strategic substitutes.

5 Gross substitutes/complements vs substitutes/complements in utility

The purpose of this subsection is to explore the relationship between the standard notions of gross substitutes/complements and the alternative definition of substitute/complement relationships via the utility function. The main finding argues that two products may well appear as substitutes in a quadratic utility function, even though they constitute gross complements in demand. On the other hand, we also establish that when all goods are complements in a quadratic utility function, then any two goods necessarily appear as gross complements in demand as well.

We begin with the formal definitions of the underlying notions, along with some general remarks.

Definition 21 (a) *Two goods, i and j , are said to be gross substitutes (gross complements) if $\partial x_i^*/\partial p_j = \partial x_j^*/\partial p_i > (<)0$*

(b) *Two goods, i and j , are said to be substitutes (complements) in utility if the utility function U has increasing differences in (x_i, x_j) , or for smooth utility, if $\partial^2 U(x)/\partial x_i \partial x_j \leq (\geq 0)$ for all x .*

Here, Part (a) is a standard notion in microeconomics. On the other hand, though a useful and well defined notion, part (b) is not as widely used in demand theory. The following remark will prove useful below.

Remark 22 *One can also define substitutes (complements) with respect to the inverse demand function, in the obvious way: i and j are substitutes (complements) if $\partial P_i/\partial x_j = \partial P_j/\partial x_i < (>)0$.*

However, since the inverse demand is simply the gradient of the utility function here, this new definition would simply coincide with part (b).⁸

It is generally known that for two-good linear demand, the two definitions are equivalent, namely two goods that are gross substitutes (complements) are always substitutes (complements) in utility as well, and vice versa (see e.g., Singh and Vives, 1984). On the other hand, this is not necessarily the case for three or more goods, as we now demonstrate.

Example 23 Consider a quadratic utility function $U(x) = a'x - \frac{1}{2}x'Bx$ with any strictly positive vector a and

$$B = \begin{bmatrix} 1 & 3/4 & 0.5 \\ 3/4 & 1 & 3/4 \\ 0.5 & 3/4 & 1 \end{bmatrix}.$$

It is easy to verify that this matrix is positive definite, so that U is strictly concave. Hence the first order conditions define a valid inverse demand function, and we are thus in the standard situation.

It is also easy to check that the inverse of B is

$$B^{-1} = \begin{bmatrix} 7/3 & -2 & 1/3 \\ -2 & 4.0 & -2 \\ 1/3 & -2 & 7/3 \end{bmatrix}$$

Recall that the slopes of both inverse and direct demand functions do not depend on the vector a (though both intercepts do depend on a). Hence, invoking the above Remark, one sees by inspection that all goods are substitutes in utility (or according to inverse demand), including in particular goods 1 and 3. On the other hand, the latter two goods are clearly gross complements (according to B^{-1}).

This possibility is actually quite an intuitive feature, as we shall argue below by providing a basic intuition for it. Nonetheless, this might well appear paradoxical at first sight because we tend

⁸In other words, one always has directly from the first order conditions

$$\frac{\partial^2 U(x)}{\partial x_i \partial x_j} = \frac{\partial P_i}{\partial x_j} = \frac{\partial P_j}{\partial x_i}.$$

to be over-conditioned by observations that hold clearly for the standard two-good case, but are actually not fully robust when moving to a multi-good setting (similar counter-examples are easy to construct whenever $n \geq 3$).

The intuition behind this switch is quite easy to grasp. Assume for concreteness that we consider an exogenous increase in p_3 . This leads to a lower demand for good 3, but a higher demand for goods 1 and 2 through substitution. The latter effect impacts good 2 relatively more than for good 1 (due to a constant of .75 for 3-2 versus .5 for 3-1). A second effect is that the large increase in the consumption of good 2 ends up driving down that of good 1 (as the two are substitutes in utility). The overall effect of the increase in p_3 is a decrease in the consumption of both goods 3 and 1, which thus emerge as gross complements.

Consider next a three good utility function with all goods being complements in utility instead. Adapting the foregoing intuition to this case will make it clear that any two goods will then emerge as gross complements too. In fact, we now prove that this constitutes a general conclusion for the n -good case.

Proposition 24 *Consider an n -good concave quadratic utility U that is supermodular in x (i.e., $\frac{\partial^2 U(x)}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$). Then all the goods are gross complements.*

Proof. Since $\frac{\partial^2 U(x)}{\partial x_i \partial x_j} \geq 0$ for all $i \neq j$, all the off-diagonal elements of B are negative. Since B is positive semi-definite, it follows from a well known result (see e.g., Mc Kenzie, 1960) that all the off-diagonal elements of B^{-1} are positive. This in turn implies directly, via (9), that all goods are gross complements. ■

In this case, all the reactions to a given price change move in the same direction, in a mutually reinforcing manner, so complementarity in utility across all goods always carries over to gross complementarity between any two goods.

6 Linear demand with local interaction

In this section, we introduce one more alternative form of the substitution matrix B that may be of interest in particular economics applications. Specifically, we suggest a particular substitution

matrix based on product similarities, place it in context of the broader study of linear demand, and highlight interesting properties of the resulting inverse and direct demands.

Consider a consumer with a preference ordering over goods based on their similarities. The main idea consists in capturing the intuitive notion that the closer products are in their characteristics, the closer substitutes they ought to be. Specifically, the consumer has preferences over n goods horizontally differentiated along one dimension, with the goods uniformly dispersed over a compact interval in that dimension. Without loss of generality, let $i = 1, \dots, n$ represent the order of the products over the interval. Consider a quadratic utility function (6) where B is now a Kac-Murdock-Szegö matrix, that is, a symmetric n -Toeplitz matrix whose ij -th term is

$$b_{ij} = \gamma^{|i-j|}, \quad i, j = 1, \dots, n. \quad (17)$$

As such matrices were first defined in Kac, Murdock, and Szegö (1953), we will refer to this as the KMS model. We focus on the case $\gamma \in (0, 1)$, so that all products are substitutes. In this specification, the price of any good j responds to changes in the quantity of every other good i with a magnitude that decreases with the distance between i and j in characteristic space.

It is well known⁹ that this matrix is positive-definite for $\gamma \in (0, 1)$ and has the inverse¹⁰

$$B^{-1} = \frac{1}{1 - \gamma^2} \begin{bmatrix} 1 & -\gamma & 0 & \dots & 0 \\ -\gamma & 1 + \gamma^2 & -\gamma & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & -\gamma & 1 + \gamma^2 & -\gamma \\ 0 & \dots & 0 & -\gamma & 1 \end{bmatrix}. \quad (18)$$

While indirect demand facing a given firm is a function of all other goods, direct demand is only a function of the two adjacent substitutes for interior firms, and one adjacent substitute for the two firms at the edges. As an example of a demand system with such structure in empirical industrial organization, consider the vertically differentiated model for the automobile industry due to Bresnahan (1987), with equal quality increments. The key idea in this model is to capture the intuitive fact that a given car is in direct competition only with cars of similar qualities.

⁹See, for example, Horn and Johnson (1991, Section 7.2, Problems 12-13)

¹⁰Due to the location of two products at the end points of the segment (that can thus have only one neighbor instead of two), direct demand is no longer fully symmetric.

From our previous general results, we easily deduce this demand system is well-defined for all $\gamma \in (0, 1)$.

Corollary 25 *Both inverse and direct demand in the KMS model satisfy the strict Law of Demand, i.e. Equation (4).*

Proof. The proof follows directly from Corollary 8, since the matrix B is positive definite. ■

This result has different implications for oligopolistic competition between firms (when each firm sells one of the varieties), depending on the mode of competition. Under Cournot competition, each firm competes with all other firms, but reacts more intensely to those whose products are more similar to its own. In contrast, under Bertrand competition, each firm directly competes only with its one or two adjacent rivals, i.e., those with very similar products (with respect to horizontal differentiation). With respect to those similar firms, previous results still hold. Specifically, as in Singh and Vives (1984), the Bertrand reaction curve for a firm with respect to its direct neighbors is upward sloping.

Proposition 26 *In the KMS model, each firm i price competes only with its closest substitutes, $i + 1$ and $i - 1$. With respect to these two rivals, firm i 's reaction curve is upward sloping.*

Proof. Reaction curves can be derived as in Proposition 20, yielding the derivative (here, b_{ij}^{-1} is the ij -th term of the matrix $(1 - \gamma^2)B^{-1}$)

$$\frac{\partial p_i}{\partial p_j} = -\frac{b_{ij}^{-1}}{2b_{ii}} = \begin{cases} \frac{\gamma}{K} & \text{for } |i - j| = 1 \\ 0 & \text{for } |i - j| \neq 1 \end{cases}$$

with $K = 2 > 0$ for boundary firms and $K = 2(1 + \gamma^2) > 0$ for interior firms. The conclusion follows from the fact that $\frac{\gamma}{K} > 0$. ■

The KMS model thus highlights another interesting lack of duality between oligopolistic price and quantity competition, which is a result of a lack of duality between indirect and direct demands. When firms compete over price, a type of local strategic interaction takes place in that each firm *directly* takes into account the behavior of only their direct neighbors (though in equilibrium, every firm's action will still end up indirectly being a function of all the rivals' actions). However, when

firms compete over quantity, they directly take into consideration the behavior of all the other firms (as in the standard case).

7 Conclusion

This paper provides a thorough exploration of the theoretical foundations of the multi-variate linear demand function for differentiated products that is widely used in industrial organization. For the question of integrability of linear demand, a key finding is that strict concavity of the quadratic utility function of the representative consumer is critical for the resulting demand system to be well defined. Without strict concavity, the true demand function may be quite complex, non-linear and income-dependent. In addition, the solution of the first order conditions for the consumer problem, which we call a local demand function, may have quite pathological properties.

The paper uncovers a number of failures of duality relationships between substitute products and complementary products, as well as the incompatibility of high levels of complementarity and concavity. The two-good case often investigated since the pioneering work of Singh and Vives (1984) emerges as a special case with strong but non-robust properties.

A key implication of our results is that all conclusions and policy prescriptions derived via the use of a linear multi-variate demand function that does not satisfy the law of Demand ought to be regarded a priori with some suspicion.

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