ON GENERATORS, RELATIONS AND D-SIMPLICITY OF DIRECT PRODUCTS, BYLEEN EXTENSIONS, AND OTHER SEMIGROUP CONSTRUCTIONS

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On generators, relations and $\mathcal{D}$-simplicity of direct products, Byleen extensions, and other semigroup constructions

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I, Samuel Baynes, hereby certify that this thesis, which is approximately 45 000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for a higher degree.

I was admitted as a research student in September 2010 and as a candidate for the degree of PhD in September 2010; the higher study for which this is a record was carried out in the University of St Andrews between 2010 and 2015.

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Abstract

In this thesis we study two different topics, both in the context of semigroup constructions. The first is the investigation of an embedding problem, specifically the problem of whether it is possible to embed any given finitely presentable semigroup into a $\mathcal{D}$-simple finitely presentable semigroup. We consider some well-known semigroup constructions in Chapter 2, investigating their properties to determine whether they might prove useful for finding a solution to our problem. In Chapter 3 we carry out a more detailed study into a more complicated semigroup construction, the Byleen extension, which has been used to solve several other embedding problems. We prove several results regarding the structure of this extension, finding necessary and sufficient conditions for an extension to be $\mathcal{D}$-simple and a very strong necessary condition for an extension to be finitely presentable. Though we ultimately do not find a solution to our motivating embedding problem.

The second topic covered in this thesis is relative rank, specifically the sequence obtained by taking the rank of incremental direct powers of a given semigroup modulo the diagonal subsemigroup. This can be considered a measure of growth of the direct powers with respect to the embedded copy of the base semigroup. In Chapter 4 we investigate the relative rank sequences of infinite Cartesian products of groups and of semigroups. We characterise all semigroups for which the relative rank sequence of an infinite Cartesian product is finite, and show that if the sequence is finite then it is bounded above by a logarithmic function. We will find sufficient conditions for the relative rank sequence of an infinite Cartesian product to be logarithmic, and sufficient conditions for it to be constant. Chapter 4 ends with the introduction of a new topic, relative presentability, which follows naturally from the topic of relative rank.

Chapter 1

Introduction

1.1 Summary

In this thesis we will investigate semigroup constructions from two different perspectives, first we will consider a variety of semigroup constructions to determine whether they might help to solve a specific embedding problem, then we will concentrate on a specific sequence of semigroup extensions and determine for which semigroups it will satisfy certain finiteness conditions.

A typical embedding problem has the form: Given a semigroup S with property A, is it possible to find a semigroup T with property B such that S embeds in T?

For S to embed in T means that T has a subsemigroup which is isomorphic to S, often denoted $S \hookrightarrow T$.

Some embedding problems are true almost from the definitions, for example; for any property A any semigroup with property A can be embedded in a semigroup with property A (by the fact that any semigroup embeds in itself), any semigroup without identity can be embedded in a monoid (by simply adjoining an identity). There are many non-trivial results, here are just a few of the easier to state ones:

- Any group can be embedded in a group of symmetries (Cayley's Theorem).

- In 1949 G. Higman, B. H. Neumann and H. Neumann proved that any countable semigroup can be embedded in a semigroup generated by two elements [10], though this was essentially proven by W. Sierpiński in 1935 [21].

- In 1958 R. H. Bruck proved that any semigroup can be embedded in a simple semigroup with identity [2].

- In 1966 J. M. Howie proved that any semigroup can be embedded in an idempotent generated semigroup [11].

The embedding problem which motivates our work is this:

Question. Is it possible to embed any given finitely presented semigroup into a finitely presented $\mathcal{D}$-simple semigroup?

So given a finite presentation for a semigroup we are interested in finding a new finite presentation which contains a subsemigroup isomorphic to the given semigroup and satisfies the strong structural condition of $\mathcal{D}$-simplicity.

Here, $\mathcal{D}$-*simple* means that the semigroup comprises exactly one $\mathcal{D}$-class.

The following examples demonstrate that some related embedding problems have been solved:

- In 1959 G. B. Preston proved that any semigroup can be embedded in a $\mathcal{D}$-simple semigroup [17].

- In 1977 F. Pastijn proved that any semigroup can be embedded in an idempotent generated $\mathcal{D}$-simple semigroup.

When Preston proved that any semigroup can be embedded in a $\mathcal{D}$-simple semigroup, he did so by taking an arbitrary semigroup, constructing a new semigroup which contains the original wholly inside a $\mathcal{D}$-class, then repeating this process countably many times and taking the union over all the iterations. The end result is a $\mathcal{D}$-simple semigroup in which the original semigroup embeds. Unfortunately this end result is not finitely presentable for any non-trivial starting point, and is not very easy to work with.

Many of the other results which embed a given semigroup in a $\mathcal{D}$-simple semigroup with some other property(ies) rely first on Preston's result to embed into a $\mathcal{D}$-simple semigroup, then embed this semigroup into a semigroup with the desired properties and use the transitivity of embeddings, never regaining finite presentability in the process.

In Chapter 2, we will investigate a selection of semigroup constructions, paying particular interest to when they might be $\mathcal{D}$-simple, with a view to assessing whether they might lead to a solution to our embedding problem.

We will see that direct products will not afford us a way to solve our embedding problem. If S is a non-$\mathcal{D}$-simple semigroup, then any direct product $S \times T$ is not $\mathcal{D}$-simple (Theorem 2.1.6).

We will investigate semidirect products, noting that a semidirect product $S \rtimes_\varphi T$ does not necessarily contain an embedded copy of either S or T, but if T is not $\mathcal{D}$-simple then nor is $S \rtimes_\varphi T$. We will see that there are examples of semidirect products $S \rtimes_\varphi T$ which are $\mathcal{D}$-simple for non-$\mathcal{D}$-simple semigroups S, but embedding S in $S \rtimes_\varphi T$ in such examples is far from easy.

We will consider some conditions which will ensure that S embeds in $S \rtimes_\varphi T$, and then show that in this case if S has more than one, but not infinitely many $\mathcal{J}$-classes then $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple (Corollary 2.2.5).

We will see that Bruck-Reilly extensions cannot be directly used to solve our embedding problem, but we will see that they have been modified to solve a very specific case of our problem, by A. Clement and F. Pastijn [6].

Free products and monoid free products will be briefly investigated and we will see quite easily that neither will solve our embedding problem.

The chapter will end with a more substantial study of Rees matrix semigroups. We will see in Theorem 2.5.3 that if $\mathcal{M}(S; I, \Lambda; P)$ is a $\mathcal{D}$-simple Rees matrix semigroup, then S is $\mathcal{D}$-simple, and so Rees matrix semigroups do not immediately afford us a solution to our embedding problem. However, we will go on to investigate the subsemigroups of Rees matrix semigroups in order to support some results which will come in Chapter 3.

One of the results regarding subsemigroups of Rees matrix semigroups will lead to Corollary 2.5.10, which states that if a semigroup T embeds in a finite and $\mathcal{D}$-simple semigroup S, then T is $\mathcal{D}$-simple. Equivalently, if T is a finite non-$\mathcal{D}$-simple semigroup, and S is a $\mathcal{D}$-simple semigroup such that T embeds in S, then S is not finite. This means that for our embedding problem if there is a solution, then it will never be finite.

In Chapter 3, we will investigate the semigroup construction used by Byleen to solve a number of embedding problems, in the hope that it might provide a solution for ours. The construction is defined in terms of a semigroup, S, a countable matrix indexed by disjoint sets, and semigroup actions of S on the index sets of the matrix, satisfying a simple condition on the matrix entries in terms of the actions. The semigroup used in the construction always embeds in the construction, so it is an extension.

Byleen used this construction, which we will call the *Byleen extension*, and a monoid version which we will call the *Byleen monoid extension*, to prove the following results:

- Any countable semigroup can be embedded in a two-generated $\mathcal{D}$-simple monoid [3].
- Any countable semigroup can be embedded in $\mathcal{D}$-simple semigroup which can be generated by 3 idempotents [3].
- Any countable semigroup without idempotents can be embedded in a two-generated simple semigroup without idempotents [4].
- Any countable semigroup can be embedded in a two-generated semigroup which is congruence-free [5].

We will see that no Byleen extension is $\mathcal{D}$-simple (Corollary 3.3.3), and so these clearly cannot be used directly to solve our embedding problem. However, this is not the case for the monoid extension as these can be $\mathcal{D}$-simple in certain circumstances. In Theorem 3.3.5, we will see that if a Byleen monoid extension of a monoid S is $\mathcal{D}$-simple then S must be $\mathcal{D}$-simple, and so the Byleen monoid extension will not provide a direct embedding of a non-$\mathcal{D}$-simple semigroup in a $\mathcal{D}$-simple one.

Similarly we will see that for a Byleen monoid extension of a monoid S to be finitely presentable then S must be finitely presentable (Theorem 3.3.9), and we will see that the matrix used in the extension must satisfy quite a restrictive finiteness condition (Theorem 3.3.10).

However, it remains conceivable that for a given non-$\mathcal{D}$-simple semigroup S we might be able to find a monoid T and a Byleen monoid extension thereof which is $\mathcal{D}$-simple such that S embeds in the extension. With this in mind we will carry out an investigation into the possible subsemigroups of a Byleen monoid extension, concentrating on periodic subsemigroups and ultimately describing their structure in terms of Rees matrix semigroups over the monoid used in the extension (Theorem 3.4.10).

Chapter 4 will take us away from the embedding problem and onto the topic of relative rank sequences, or relative d-sequences. In order to understand what these are, we first need to be reminded of the definition of the rank of a semigroup, and introduce a generalisation thereof.

The *rank* of a semigroup S, denoted $d(S)$, is the minimum number of elements required to generate S, that is $d(S) = \min\{|X| : X \subset S, S = \langle X \rangle\}$.

This can be generalised to the relative rank in the following way: The *relative rank* of a semigroup S with respect to subset A, denoted $d(S{:}A)$ is the minimum number of elements required to add to the set A and make a generating set for S, that is $d(S{:}A) = \min\{|X| : X \subseteq S, S = \langle X, A \rangle\}$.

Note that $d(S{:}A) \leq d(S{:}\emptyset) = d(S)$, for all $A \subseteq S$.

From the rank we can measure some notion of growth of a semigroup, by defining the d-sequence. The *d-sequence* of a semigroup S is the sequence obtained by taking the rank of incremental direct products of S with itself, that is $\mathbf{d}(S) = (d(S), d(S^2), d(S^3), \ldots)$.

J. Wiegold investigated the d-sequences of groups thoroughly in a series of papers from the 1970s and 1980s, on the topic of finite groups [25], [26], [27], [28], [14] (co-authored with D. Meier), and on the topic of finitely generated groups [30] (co-authored with J. S. Wilson), and [22] (co-authored with A. G. R. Stewart).

Wiegold also started the investigation of d-sequences of semigroups by determining the nature of the d-sequence of any finite semigroup in [29]. The study of d-sequences of infinite finitely generated semigroups was taken up by J. T. Hyde, N. J. Loughlin, M. Quick, N. Ruškuc and A. R. Wallis in [13].

Of course, if S is not finitely generated, then the d-sequence of S is not finite and so does not allow us to gauge the growth of S. This task is taken up by the relative rank sequence, but first we must define the diagonal subsemigroup of a direct product of a semigroup.

The *diagonal* of S^n, denoted Δ_{S^n}, is the subsemigroup of S^n which comprises all the elements which have the same element of S in each of their components, that is $\Delta_{S^n} = \{(s, s, \ldots, s) \in S^n : s \in S\}$. This diagonal is isomorphic to S for any n, and so may be referred to as the *diagonal copy* of S in S^n.

The relative rank of S^n with respect to the diagonal copy of S is related to the rank of S^n by the following inequalities, as will be proven in Proposition 4.1.5:

$$d(S^n{:}\Delta_{S^n}) \leq d(S^n) \leq d(S^n{:}\Delta_{S^n}) + d(S).$$

So it is apparent that the behaviour of the sequence $(d(S^n{:}\Delta_{S^n}))_{n\in\mathbb{N}}$ is the same as the behaviour of the d-sequence of S.

The *relative rank sequence*, or *relative d-sequence*, of S, denoted $\mathbf{d}_\Delta(S)$, is the sequence obtained by taking the relative rank of incremental direct products of S with itself, each with respect to the diagonal copy of S, that is $\mathbf{d}_\Delta(S) = (d_\Delta(S), d_\Delta(S^2), d_\Delta(S^3), \ldots)$, where $d_\Delta(S^i) = d(S^i{:}\Delta_{S^i})$.

The relative rank sequence clearly has the same behaviour as the d-sequence for finitely generated semigroups, by the inequalities above. We can consider $d_\Delta(S^n)$ to be a measure of how much must be added to S (or the isomorphic Δ_{S^n}) in order to generate the extension S^n. In Chapter 4 we will see that there are non-finitely generated semigroups which have finite relative rank sequence, and so the relative rank sequence offers a similar concept of growth for semigroups as the d-sequence offers for finitely generated semigroups.

We will investigate the relative rank sequences of infinite Cartesian products of groups, and then of semigroups, as this will afford us a rich field of non-finitely generated (semi)groups to study and because they have risen to prominence in the potentially related area of semigroups with the Bergman property [8]. Given an infinite cardinal I and a semigroup S, the infinite Cartesian product is denoted S^I and comprises the set of all I-tuples over S with the operation defined in the familiar component wise manner.

A useful tool for working with relative rank sequences of infinite Cartesian products of groups will be a property $P(m)$, which is defined in the following way: For a group G and a natural number m, we say that G has property $P(m)$ if there exist $g_1, g_2, \ldots, g_m \in G$ and $n_1, n_2, \ldots, n_m \in \mathbb{N}_0$ such that $G = (g_1^G)^{n_1}(g_2^G)^{n_2} \ldots (g_m^G)^{n_m}$.

While this may not appear to have any immediate consequence for infinite Cartesian products, Theorem 4.3.13 will show that a group G has property $P(m)$ if and only if $d_\Delta((G^I)^2) \leq m$, and Lemma 4.3.16 will show that if this is the case then the whole sequence $\mathbf{d}_\Delta(G^I)$ is finite (and in particular, at most logarithmic).

To summarise the results for infinite Cartesian products of groups:

- If G is not a perfect group, then $\mathbf{d}_\Delta(G^I)$ is infinite (Theorem 4.3.5).
- If G is finite and perfect, then $\mathbf{d}_\Delta(G^I)$ is logarithmic (Theorem 4.3.10).
- If G does not have property $P(m)$ for any $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(G^I)$ is infinite (Theorem 4.3.13).
- If G is infinite, has property $P(m)$ for some $m \in \mathbb{N}$, and has a non-trivial finite homomorphic image, then $\mathbf{d}_\Delta(G^I)$ is logarithmic (Lemmas 4.3.4, 4.3.3, 4.3.16, and Theorem 4.3.10).
- If G is infinite, has property $P(m)$ and has no non-trivial finite homomorphic image, then $\mathbf{d}_\Delta(G^I)$ is bounded below by a constant function and above by a logarithmic one (Lemma 4.3.16).

Further to this we will widen the investigation to the relative rank sequences of infinite Cartesian products of semigroups. In order to do so we will generalise the definition of the property $P(m)$ of groups to apply to semigroups, using this we will prove the following:

- If S does not have property $P(m)$ for any $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(S^I)$ is infinite (Theorem 4.4.11).
- If S has property $P(m)$, for some $m \in \mathbb{N}$, and has a non-trivial finite perfect group as a homomorphic image then $\mathbf{d}_\Delta(S^I)$ is logarithmic (Theorems 4.4.13 and 4.3.10, Lemmas 4.3.3 and 4.3.4).
- If S has property $P(m)$, for some $m \in \mathbb{N}$ and has no non-trivial finite perfect groups as homomorphic images, then $\mathbf{d}_\Delta(S^I)$ is at least constant and at most logarithmic (Theorem 4.4.13).

In this final case we will find examples which have constant relative rank sequence, but none with non-constant relative rank sequence, though there will be no proof that such semigroups always have constant relative rank sequence.

We will see that if a semigroup S satisfies either of the following criteria, then S does not have property $P(m)$ for any $m \in \mathbb{N}$, and so $\mathbf{d}_\Delta(S^I)$ is infinite (though not meeting either of the criteria does not necessarily imply $P(m)$ for some m):

- S has a homomorphic image which is finite and not a perfect group (Theorem 4.4.4, Lemmas 4.3.3 and 4.3.4).
- S has a non-trivial and commutative homomorphic image (Theorem 4.4.6, Lemmas 4.3.3 and 4.3.4).

In Lemma 4.4.15 we will see that if a monoid has a cyclic diagonal bi-act then it has property $P(1)$ (and so S^I has finite relative rank sequence). However, the subsequent examples will demonstrate that the converse is not true, diagonal bi-acts are insufficient when trying to characterise all infinite Cartesian products of semigroups with finite relative rank sequences.

In Theorem 4.4.18 we will see that infinite Cartesian products of Byleen monoid extensions can have constant relative rank if the matrix used the construction meets certain criteria, and then in Corollary 4.4.19 we will see an example of how relative rank results regarding finitely generated semigroups can be used to prove results about the non-relative rank sequences.

This work on relative rank sequences will finish with a pair of lemmas which suppose that for a given semigroup S the relative rank of $(S^I)^2$ is finite and then prove that S must be generated by one of its $\mathcal{J}$-classes, J, and that $\mathbf{d}_\Delta(S^I) = \mathbf{d}_\Delta((S/(S \setminus J))^I)$, that is we can take the Rees quotient of S by the ideal $S \setminus J$ without changing the relative rank sequence. These could be useful in further study of this topic.

The chapter will end with an introduction to a natural follow-up concept to relative rank, *relative presentability*, serving to indicate a potential avenue of further study.

While it may seem that the study of relative rank sequences and relative presentability in Chapter 4 is disjoint from the embedding problem which motivates Chapters 2 and 3, they can be thought of as complementary topics. In the study of relative rank sequences and relative presentability we concern ourselves with whether or not we can describe a direct power of a semigroup with just finitely more generators or relations, this can be thought of as whether the constructed semigroup is only finitely more complex than the base semigroup. Our embedding problem seeks to embed any finitely presentable semigroup in a $\mathcal{D}$-simple finitely presented semigroup, that is we are trying to extend the semigroup just enough to attain $\mathcal{D}$-simplicity, but without going so far as to lose finite presentability, we are trying to gain a new property whilst only adding a finite amount of complexity.

1.2 Preliminary Semigroup Theory

To start at the very beginning: A *semigroup* $(S, .)$ is a non-empty set S with associative binary operation $. : S \times S \to S$. In almost all contexts we will denote the semigroup $(S, .)$ simply by S and the operation simply by juxtaposition, $s.t = st$.

We will say that elements s and t of a semigroup *commute* if $st = ts$, and we will say that a semigroup is *commutative* if all pairs of elements of S commute.

If $st = s$ then we say that t is a *right-identity for* s, similarly if $ts = s$ then we say that t is a *left-identity for* s. If an element is a right identity for every element of the semigroup, then we call it simply a *right-identity*, and of course *left-identity* is defined analogously. If a semigroup S contains an element which is both a *left-* and a *right-identity*, then we call that element an *identity*, and usually denote it 1 (or 1_S if there is any ambiguity). A semigroup with identity is called a *monoid*.

In many situations we will call on the semigroup S^1 which is the smallest monoid which contains S, that is $S^1 = S$ if S is a monoid and $S^1 = S \cup \{1\}$ if S is not a monoid.

If s is an element of a semigroup S such that $st = s$ for all $t \in S$, then we say that s is a *left zero*, with *right zero* defined in the analogous manner. In the event that an element is both a left and a right zero then we say that it is a *zero*, and typically denote it by 0.

For example, the set of all mappings from the set $\{1, 2, 3\}$ to the set $\{1, 2\}$ is a semigroup with composition of mappings from the left as the operation, and the element α defined by $\alpha{:}1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 2$, is a right identity for the semigroup, but not a left identity as the element $\beta{:}1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 1$ demonstrates, $(3)\alpha\beta = (2)\beta = 2 \neq 1 = (3)\beta$ hence $\alpha\beta \neq \beta$.

For an element e of a semigroup, if $ee = e$ we will say that e is *idempotent*, or that e is an *idempotent*. While identity and zero elements are examples of idempotents, it is not true that all idempotents necessarily fall into one of these categories. In fact, it is easy to see that no semigroup can have more than one identity or more than one zero, but there are semigroups with more than two idempotents. The semigroup in the example above has four idempotents, but no identity and no zero.

A *semilattice* is a commutative semigroup in which every element is idempotent.

There exists a partial order on the idempotents of any semigroup which we will call upon, particularly when investigating Rees matrix semigroups. If e and f are idempotents the *partial order of idempotents*, denoted $\leq$, is defined such that $e \leq f$ if and only if $ef = fe = e$. Note that for any two idempotents e and f it is not necessarily the case that $ef = fe$ and so they are not necessarily comparable with respect to $\leq$.

An idempotent e is said to be *primitive* if in the partial order of idempotents there are no non-zero idempotents below e. That is, if $f \in S \setminus \{0\}$ is an idempotent such that $ef = fe = f$, then $e = f$. A semigroup S is said to be *completely simple* if it is *simple* and contains a primitive idempotent.

A semigroup element s is said to be *periodic* if it has finite order, that is if there exist $0 < i < j$ such that $s^i = s^j$. If every element of S is periodic, then S may be described as a *periodic semigroup*.

Clearly all idempotents are periodic, and any finite semigroup is periodic. There are, however, plenty of semigroups with elements which are not periodic, for example *free semigroups* which can be defined as the set of all words over a given alphabet with concatenation as the operation have no periodic elements.

In a semigroup S, a subset $L \subseteq S$ is called a *left ideal* if $SL \subseteq L$, or equivalently if $sl \in L$ for all $s \in S$ and all $l \in L$. Similarly, a *right ideal* is a subset $R \subseteq S$ which has the property that $RS \subseteq R$. If a subset satisfies both of these conditions then it is called a *two-sided ideal*, or more often just an *ideal*.

When we have an ideal $K \subseteq S$ we can take the *Rees quotient* of S by K, denoted S/K. This is defined as the quotient of S by the congruence $\{(s, s) : s \in S\} \cup (K \times K)$. In fact, as K is an ideal the resulting quotient is isomorphic to $(S \setminus K) \cup \{0\}$ with the operation defined as in S, but with every element of K replaced by 0.

For a semigroup S and an element $a \in S$, the *principal left ideal generated by a* is defined to be the smallest left ideal which contains a. It is easy to see that the principal left ideal generated by a is $S^1a = \{sa : s \in S^1\}$. Similarly the *principal right ideal generated by* a is aS^1. The *principal (two-sided) ideal generated by* a, S^1aS^1, is the smallest two-sided ideal which contains a.

A key topic for this thesis will be $\mathcal{D}$-simplicity, and in order to understand what this is we must first introduce *Green's relations*. *Green's relations* are a set of equivalence relations on a semigroup, denoted $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{J}$, and defined as follows: Let $a, b \in S$;

- $a\mathcal{L}b$ if and only if a and b generate the same principal left-ideal,
- $a\mathcal{R}b$ if and only if a and b generate the same principal right-ideal,
- $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, that is, $a\mathcal{H}b$ if and only if $a\mathcal{L}b\mathcal{R}a$,
- $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, that is, $a\mathcal{D}b$ if and only if there exists $c \in S$ such that $a\mathcal{L}c\mathcal{R}b$,
- $a\mathcal{J}b$ if and only if a and b generate the same principal two-sided ideal.

An alternative, and often more convenient, way to consider the $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ relations is the following:

- $a\mathcal{L}b$ if and only if there exists $u, v \in S^1$ such that $a = ub$ and $b = va$,
- $a\mathcal{R}b$ if and only if there exists $u, v \in S^1$ such that $a = bu$ and $b = av$,
- $a\mathcal{J}b$ if and only if there exists $u, v, w, x \in S^1$ such that $a = ubv$ and $b = wax$.

We will say that any semigroup for which the $\mathcal{R}$ relation is trivial is an $\mathcal{R}$-trivial semigroup, that is a semigroup is $\mathcal{R}$-trivial if its $\mathcal{R}$-classes are all singletons. At the other extreme, we will say that any semigroup for which the $\mathcal{R}$ relation is full is an $\mathcal{R}$-simple semigroup, that is a semigroup is $\mathcal{R}$-simple if every pair of elements are $\mathcal{R}$ related. These conventions will be applied to other equivalence relations, for example $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$. However, the property of a semigroup to be $\mathcal{J}$-simple is equivalent to it being *simple* and so in this case we will defer to the established nomenclature. These two conditions on equivalence relations will be used frequently throughout the rest of this thesis, most often applied to Green's relations.

In some publications you may find the term *bisimple* referring to a semigroup (in fact in many of the papers we will reference), this is defined to mean that the semigroup comprises exactly one $\mathcal{D}$-class, which is to say the semigroup is $\mathcal{D}$-simple. We will use $\mathcal{D}$-simple to describe this property, though in every case it could be supplanted by bisimple without changing the meaning.

Green's $\mathcal{L}, \mathcal{R}$ and $\mathcal{J}$ relations have associated preorders, denoted $\leq_L, \leq_R$ and $\leq_J$ respectively. For elements s and t of a semigroup S,

- $s \leq_L t$ if and only if there exists $u \in S^1$ such that $s = ut$,
- $s \leq_R t$ if and only if there exists $u \in S^1$ such that $s = tu$,
- $s \leq_J t$ if and only if there exist $u, v \in S^1$ such that $s = utv$.

Clearly, if $s \leq_L t$ and $t \leq_L s$ then $s\mathcal{L}t$, and if $s\mathcal{L}t$ then $s \leq_L t \leq_L s$, and so the $\mathcal{L}$ relation is the equivalence relation associated with $\leq_L$, and so $\leq_L$ can be considered a partial order on the $\mathcal{L}$-classes. Similarly, $\leq_R$ and $\leq_J$ correspond to partial orders on the $\mathcal{R}$-classes and $\mathcal{J}$-classes, respectively.

The term *maximal $\mathcal{L}$-class* will refer to any $\mathcal{L}$-class of the semigroup which is maximal with respect to $\leq_L$. We similarly define *maximal $\mathcal{R}$-classes* and *maximal $\mathcal{J}$-classes*.

Example 1.2.1. The *bicyclic monoid* is the set $B = \{c^i b^j : i, j \in \mathbb{N}_0\}$ along with the operation $(c^i b^j)(c^k b^l) = c^{i-j+k} b^l$ if $j \leq k$ and $(c^i b^j)(c^k b^l) = c^i b^{j-k+l}$ if $j > k$. This description of the operation appears cumbersome, but it boils down to $bc = 1$ and its consequences, as formalised in Section 1.4.

The idempotents of the bicyclic monoid are the elements of the form $c^i b^i$, and so there are countably many of them. Consider the product of any two, $c^i b^i c^j b^j = c^{i+j-i} b^j = c^j b^j$ if $i \leq j$ and $c^i b^i c^j b^j = c^i b^{i-j+j} = c^i b^i$ if $j \leq i$, and so all the idempotents commute and $c^i b^i \leq c^j b^j$ if and only if $j \leq i$.

If $c^i b^j \leq_L c^k b^l$ then there exists $c^m b^n \in B$ such that $c^i b^j = (c^m b^n)(c^k b^l)$, this implies that $j \geq l$. If $j \geq l$ then $c^i b^j = (c^i b^{k+j-l})(c^k b^l)$ and this implies $c^i b^j \leq_L c^k b^l$. Hence, $c^i b^j \leq_L c^k b^l$ if and only if $j \geq l$, and in turn $c^i b^j \mathcal{L} c^k b^l$ if and only if $j = l$.

Similarly, $c^ib^j \leq_R c^kb^l$ if and only if $i \geq k$ and in turn $c^ib^j\mathcal{R}c^kb^l$ if and only if $i = k$.

Two elements are $\mathcal{H}$ related if and only if they are both $\mathcal{L}$ and $\mathcal{R}$ related, and so $c^ib^j\mathcal{H}c^kb^l$ if and only if $i = k$ and $j = l$, hence B is $\mathcal{H}$-trivial.

For any $c^ib^j, c^kb^l \in B$, we have $c^ib^j\mathcal{L}c^kb^j\mathcal{R}c^kb^l$ which implies $c^ib^j\mathcal{D}c^kb^l$, and so B is $\mathcal{D}$-simple.

The $\mathcal{L}$-class which contains the identity, $\{c^i : i \in \mathbb{N}_0\}$, is the unique maximal $\mathcal{L}$-class, and the $\mathcal{R}$-class which contains the identity, $\{b^i : i \in \mathbb{N}_0\}$, is the unique maximal $\mathcal{R}$-class.

A mapping from one semigroup to another, $\varphi : S \to T$, is called a *homomorphism* if it respects the operation, that is if $(st)\varphi = (s)\varphi(t)\varphi$ for all $s, t \in S$. A homomorphism from a semigroup to itself is called an *endomorphism*, and the set of all endomorphisms of a semigroup S will be denoted $\mathrm{End}(S)$. If an endomorphism is bijective then we will call it an *automorphism* and it will belong to the set $\mathrm{Aut}(S)$.

A mapping $\varphi : S \times X \to X$ is a *left semigroup action* of the semigroup S on the set X if $(s, (t, x)\varphi) = (st, x)\varphi$ for all $s, t \in S$ and all $x \in X$. In this situation we say that S *acts on* X (from the left). *Right actions* are defined analogously. A *monoid action* is a semigroup action of a monoid on a set such that the identity fixes every element of the set, that is $(1, x)\varphi = x$ for all $x \in X$.

1.3 Preliminary Group Theory

From the perspective of semigroup theory, a *group*, G, is just an $\mathcal{H}$-simple semigroup, or equivalently a monoid such that for every $g \in G$ there exists a unique $g^{-1} \in G$ such that $gg^{-1} = g^{-1}g = 1$.

If a group is commutative, then we will say that it is *abelian* in order to comply with popular convention.

A large part of our deductions with regards to groups will involve *conjugates* and *conjugacy classes*. For group elements g and h the *conjugate of g* *by* h, denoted g^h, is the product $h^{-1}gh$. The set of all conjugates of g will be called the *conjugacy class of* g and denoted g^G.

We will be paying special attention to *simple groups*, the definition of which relies on *normal subgroups.* A subgroup N of a group G is *normal* if it is closed under conjugation, that is if for all $n \in N$ we have $n^G \subseteq N$. The subgroups $\{1\}$ and G, for example, are always normal subgroups of G. A non-trivial group is *simple* if it has no non-trivial proper normal subgroups, that is if the trivial and full subgroups are the only normal subgroups.

Special care should be taken with *simplicity* as it has two different definitions when applied to groups or to semigroups, but of course groups *are* semigroups. However, groups are by definition simple semigroups as groups are $\mathcal{H}$-simple, $\mathcal{H} \subseteq \mathcal{J}$, and $\mathcal{J}$-simple is exactly the definition of simple for semigroups. So if a group is described as simple this will always mean that it has no non-trivial proper normal subgroups.

Cyclic groups are groups which can be generated by just one element, which is to say every element can be found as a power of one specific element. They are necessarily abelian, and so every subgroup is normal but if they have prime order (which is to say they have p elements for some prime number p), then they have no non-trivial proper subgroups and so are simple. In fact these are the only abelian simple groups.

The alternating group on n points is the set of all even permutations of n points with composition of mappings as the operation, denoted A_n. It is well-known that A_n is simple for all $n \geq 5$. There are infinite simple groups too, for example the finitary alternating group on the natural numbers, $A_{\mathbb{N}}^{\text{fin}}$, is the group of all even permutations of the natural numbers and is simple.

The *normal closure* of an element g in a group G is the smallest normal subgroup of G which contains g, that is $\langle g^G \rangle$.

Most of the group theory used in the text will revolve around *perfect groups*, and in order to understand what these are we must first define what a *commutator* is. If g and h are elements of a group, the *commutator of* g *and* h is denoted $[g, h]$ and defined to be the product $g^{-1}h^{-1}gh$. Note that two elements commute if and only if their commutator is the identity.

The *commutator subgroup* or *derived subgroup* of a group G, denoted G' is the subgroup generated by all commutators of G. That is $G' = \langle [G, G] \rangle$. The derived subgroup is always a normal subgroup as any conjugate of a commutator will always be a commutator. A group is said to be *perfect* if it is equal to its derived subgroup, that is if every element can be expressed as a finite product of commutators. For a group G, the derived subgroup G' is the smallest normal subgroup such that the quotient group G/G' is abelian. This means that if G is perfect and N is a proper normal subgroup of G, then G/N is not abelian, and so a group is perfect if and only if it has no non-trivial abelian homomorphic images.

As simple groups do not have any non-trivial proper normal subgroups, it is clear that the derived subgroup of a simple group G must either be $\{1\}$ or G. If $G' = \{1\}$, then G is abelian, and if $G' = G$ then G is perfect. Hence, every non-abelian simple group is perfect.

There are non-simple perfect groups, for example if G and H are perfect (possibly simple) groups, then their direct product $G \times H$ is perfect but not simple. See Section 1.5 for the definition of a direct product.

The *commutator width* of a group element g is the minimum length of an expression for g in terms of commutators. A perfect group is said to have *bounded commutator width* if there exists $n \in \mathbb{N}$ such every element of the group has commutator width less than or equal to n.

1.4 Generation and Presentations

Let X be a subset of the semigroup S. The subsemigroup generated by X, denoted $\langle X \rangle$, is the set of all elements of S which can be expressed as a finite product of elements of X.

If the subsemigroup generated by X is the whole semigroup S, that is if $S = \langle X \rangle$, then the subset X is said to be a *generating set* for S. In this case the elements of X are referred to as *generators*. A semigroup S is said to be *finitely generated* if there exists a finite generating set for S.

It is easy to see that any finite semigroup is finitely generated as the whole semigroup is always a generating set and if the semigroup is finite then this is a finite generating set. There are also finitely generated infinite semigroups, for example $(\mathbb{N}, +)$ the natural numbers under addition is an infinite semigroup which is generated by 1 (in this case 1 denotes the number one, not the identity element). We can see that it is generated by 1 as every natural number can be expressed as a finite sum of ones.

The natural numbers under multiplication also form an infinite semigroup, though this one is not finitely generated. In order to see this we need only recall that there are infinitely many prime numbers, and so there are infinitely many elements of our semigroup which cannot be expressed as a product of other elements, clearly all of these must occur in any generating set and so no finite generating set can exist.

If a semigroup is uncountable then it cannot be finitely generated, this can be seen by considering the maximum size of $\langle X \rangle$ for a finite set X. The subgroup generated by X was defined to be the set of all finite products of elements of X, and so is at most countably infinite. For example, the group of all permutations of the natural numbers, $S_{\mathbb{N}}$, is an uncountable (semi)group and so is not finitely generated.

The following is a well-known natural result regarding finite generation of semigroups.

Proposition 1.4.1. *Let S be a finitely generated semigroup, and let $X \subseteq S$ such that $S = \langle X \rangle$.*

Then there exists $Y \subseteq X$ such that Y is a finite generating set for X.

Proof. As S is finitely generated, there exists a finite generating set $A \subseteq S$.

As X generates S, for each element of A there exists a finite expression in terms of elements from X. Fix such an expression for each element of A and let Y comprise the elements of X which occur in any of those expressions.

Clearly, $Y \subseteq X$, and $A \subseteq \langle Y \rangle$. Hence $S = \langle A \rangle \subseteq \langle Y \rangle \subseteq S$. □

A property relating to generating sets is the *Bergman property*, a semigroup S has the *Bergman property* if for any generating set X of S, there exists $n \in \mathbb{N}$ such that $S = X \cup X^2 \cup \ldots \cup X^n$. That is every element of

S can be expressed as a word over X of length at most n. The property was introduced by G. Bergman when he proved that for any infinite set Ω, the symmetric group S_Ω has the property when generated as a group or as a monoid [1]. The property clearly holds for all finite semigroups.

The bicyclic monoid, introduced in Section 1.2, is the set $B = \{c^i b^j : i, j \in \mathbb{N}_0\}$, and so it is clear that every element of B can be expressed as a product of bs and cs, hence $B = \langle b, c\rangle$. Suppose that B has the Bergman property, this implies that there exists $n \in \mathbb{N}$ such that $B = \{b, c\} \cup \{b, c\}^2 \cup \ldots \cup \{b, c\}^n$, or equivalently every element of B can be expressed as a product of length at most n over $\{b, c\}$. This is clearly not the case as $b^{n+1} \in B$, and so the bicyclic monoid does not have the Bergman property.

In fact, the Bergman property does not hold for any finitely generated infinite semigroups, in order to see this we simply consider the size of the set $X \cup X^2 \cup \ldots \cup X^n$ for a finite generating set X, it is bounded by $n|X|^n$ and so can never be infinite.

There are non-finitely generated semigroups for which the Bergman property holds, for example $S_{\mathbb{N}}$ has the Bergman property [1] and, as we saw earlier, is not finitely generated.

A *semigroup presentation* is an ordered pair $\langle A \mid R\rangle$, where A is an alphabet and R is a set of relations, $R \subseteq A^+ \times A^+$. The semigroup this defines is the universal semigroup generated by A which satisfies all of the relations R, that is $\langle A \mid R\rangle$ is the unique semigroup generated by A which satisfies R and any other semigroup which has a generating set satisfying all the relations of R is necessarily a homomorphic image of $\langle A \mid R\rangle$. Equivalently, if ρ is the congruence generated by R then $\langle A \mid R\rangle = A^+/\rho$.

Given any semigroup S we can find a presentation for that semigroup easily, let $R = \{(u, v) \in S^+ \times S^+ : u = v \text{ in } S\}$, then $S \cong \langle S \mid R\rangle$, though the value of considering presentations in place of the semigroups they represent is when we can take simpler presentations, where simpler may mean finite, or that the relations all fit some kind of pattern.

We will say that a semigroup S is finitely presentable if there exist a finite alphabet A and a finite set of relations $R \subseteq A^+ \times A^+$ such that $\langle A \mid R\rangle$ is a presentation for S. Of course any finite semigroup is necessarily finitely presentable, one could simply construct a set of relations by taking each entry from its Cayley table and using the whole semigroup as a generating set.

A key tool for working with presentations will be the following well-known proposition, a proof of which can be found in [20] (Proposition 2.2).

Proposition 1.4.2. *Let $\langle A \mid R \rangle$ be a presentation, let S be the semigroup defined by it, and let $w_1, w_2 \in A^+$. Then $w_1 = w_2$ in S if and only if w_2 can be deduced from w_1.*

For one word to be deduced from another means that it can be obtained from the other after finitely many applications of relations from R.

This is not the only fundamental result regarding semigroup presentations which we will make use of. We will also use the fact that finite presentability is independent of the finite generating set. This result exists in the folk-lore of semigroup presentations, and again a proof can be found in [20] (Proposition 1.3.1).

Proposition 1.4.3. *Let S be a semigroup, and let A and B be two finite generating sets for S. If S can be defined by a finite presentation in terms of generators A, then S can be defined by a finite presentation in terms of generators B as well.*

The last part of semigroup presentation folk-lore we need is the following proposition. We include a proof to give an indication of the kinds of deductions which will be made when proving finite presentability.

Proposition 1.4.4. *Let S be a semigroup with presentation $\langle A \mid R \rangle$ such that A is finite.*

If S is finitely presentable then there exists a finite subset $T \subseteq R$ such that $S = \langle A \mid T \rangle$.

Proof. Suppose S is finitely presentable. By Proposition 1.4.3 there exists a finite set of relations $U = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\} \subseteq A^+ \times A^+$ such that $S = \langle A \mid U \rangle$.

As $S = \langle A \mid R \rangle$, for each $1 \leq i \leq n$ we have $u_i = v_i$ as a consequence of R. This means that there are finitely many relations of R necessary to imply that $u_i = v_i$, let $R_i \subseteq R$ be a finite subset which contains sufficient relations to imply $u_i = v_i$.

Let $T = R_1 \cup R_2 \cup \ldots \cup R_n$. Of course, $T \subseteq R$ and T is finite, and as every relation in U comes as a consequence of the relations in T it holds that $S = \langle A \mid T \rangle$. □

We saw in Section 1.2 that the bicyclic monoid is the set $B = \{c^i b^j : i, j \in \mathbb{N}_0\}$ with the operation defined by $(c^i b^j)(c^k b^l) = c^{i+k-j} b^l$ if $j \leq k$ and $(c^i b^j)(c^k b^l) = c^i b^{j-k+l}$ otherwise, and in Section 1.4 we observed that $\{b, c\}$ is a generating set for this semigroup. In fact the bicyclic monoid is finitely presentable and $\langle b, c \mid bc = 1 \rangle$ is a valid presentation for it. To see that this is the case we must prove that every element of $\langle b, c \rangle$ can be expressed as an element of B as a consequence of $bc = 1$, and observe that the operation defined above agrees with $bc = 1$. Consider an expression from the set $\langle b, c \rangle$, if there exists a b immediately before a c then apply the relation to remove them both, repeating this until there are no such occurrences must result in an element of the form $c^i b^j$. Hence, $B = \langle b, c \mid bc = 1 \rangle$.

Examples such as this illustrate the elegance of finite presentability; while the semigroup may be relatively easy to describe and understand, a finite presentation can distil the semigroup into a pure form without losing any detail.

There are of course semigroups which cannot be finite presented. We have seen non-finitely generated semigroups, and as finite presentability implies finite generation it is clear that these cannot be finitely presentable.

There are finitely generated non-finitely presentable semigroups too. For example, the semigroup $S = \langle a, b \mid ab^i a = aba, i \in \mathbb{N} \rangle$ is (clearly) finitely generated, but cannot be finitely presented. In order to see this we suppose that it is finitely presentable with a view to finding a contradiction. By Proposition 1.4.4, if S is finitely presentable then there exists a finite $R \subset \{ab^i a = aba : i \in \mathbb{N}\}$ such that $S = \langle a, b \mid R \rangle$, or equivalently there exists a finite $A \subset \mathbb{N}$ such that $S = \langle a, b \mid ab^i a = aba, i \in A \rangle$. Let $j \in \mathbb{N}$ such that $j > i$ for all $i \in A$. Then $ab^j a = aba$ as a consequence of the relations

$\{ab^ia = aba : i \in A\}$, but none of these can be applied to ab^ja, and so no deductions can be made from ab^ja using the finite set of relations and we have a contradiction.

1.5 Constructions

In semigroup theory there are many ways to find new semigroups from known semigroups, these will be referred to collectively as *constructions*. Constructions may take one semigroup and possibly some other structure to create a new semigroup, or they may take multiple semigroups and combine them in a specific way. In the event that a construction takes only one semigroup as input and that semigroup embeds in the construction, we will refer to this construction as an *extension* of the semigroup.

One of the simplest ways to construct a semigroup from others is the *free product*. For semigroups S and T, the *free product* of S and T, denoted $S*T$, is the set of all words over $S \cup T$, with the binary operation simply the union of the operations of S and T, along with concatenation for any product not defined therein. The presentation of a free product is easy to find from the presentations of its constituent semigroups, if $S = \langle X \mid R\rangle$ and $T = \langle Y \mid U\rangle$ then $S * T = \langle X, Y \mid R, U\rangle$.

In the event that S and T are monoids we can construct a *monoid free product*. The *monoid free product* of S and T, denoted $S *_1 T$, is the free product of S and T modulo the congruence $1_S = 1_T$. That is if $S = \langle X \mid R\rangle$ and $T = \langle Y \mid U\rangle$ then $S *_1 T = \langle X, Y \mid R, U, 1_S = 1_T\rangle$.

Another relatively simple way to construct a semigroup from others is the *direct product*. Denoted $S \times T$, the *direct product of S and T* is the Cartesian product of the two semigroups as sets, along with the binary operation defined to be the component-wise application of the semigroups' respective operations. That is, $S \times T = \{(s, t) : s \in S, t \in T\}$, and for all $(s_1, t_1), (s_2, t_2) \in S \times T$ their product is $(s_1, t_1)(s_2, t_2) = (s_1s_2, t_1t_2)$.

The *semidirect product* is (unsurprisingly) a generalisation of the *direct* *product*. In order to construct a semidirect product of two semigroups S and T, we also require a homomorphism from T to the endomorphisms of S, $\varphi : T \to \text{End}(S)$. To simplify notation, if $t \in T$ and $s \in S$ we will use ${}^{t}s$ to denote the image of s under $(t)\varphi$.

The underlying set is the Cartesian product $S \times T = \{(s,t) : s \in S, t \in T\}$ as before, and for two elements $(s_1,t_1),(s_2,t_2)$ the operation incorporates φ in the following way, $(s_1,t_1)(s_2,t_2) = (s_1({}^{t_1}s_2), t_1t_2)$. We denote the semidirect product of S and T with respect to φ by $S \rtimes_\varphi T$.

Given a semigroup S, non-empty sets Λ and I and a $\Lambda \times I$ matrix $P = (p_{\lambda i})_{(\lambda,i)\in\Lambda\times I}$ with entries from S, we can define the *Rees matrix semigroup* $\mathcal{M}(S; I, \Lambda; P)$ to be the set of triples $I \times S \times \Lambda$ with the operation defined such that $(i,x,\lambda)(j,y,\mu) = (i, xp_{\lambda j}y, \mu)$.

If the semigroup used in the construction of a Rees matrix semigroup is a group then the Rees matrix semigroup is a completely simple semigroup, and in fact every completely simple semigroup is isomorphic to a Rees matrix semigroup over a group. This result is usually attributed to Rees [19], although essentially determined by Suschkewitsch [23]. As we will see in Section 2.5, this result is very useful when working with Rees matrix semigroups as it allows us to consider Rees matrix semigroups over groups from a different perspective.

For a monoid M, and an endomorphism $\theta : M \to M$, the *Bruck-Reilly extension of M with respect to θ*, denoted $BR(M,\theta)$, is the set $\mathbb{N}_0 \times M \times \mathbb{N}_0$ with the operation defined such that if $t = \max(n,p)$ then $(m,a,n)(p,b,q) = (m-n+t, (a\theta^{t-n})(b\theta^{t-p}), q-p+t)$. It is easy to see that $\{0\} \times M \times \{0\}$ is always a subsemigroup of $BR(M,\theta)$ which is isomorphic to M. If the image of the endomorphism θ is contained in the group of units of the semigroup ($\mathcal{H}_1$), then the Bruck-Reilly extension is a simple monoid and so Bruck-Reilly extensions can be used to embed any semigroup S in a simple monoid (by taking a Bruck-Reilly extension of S^1 by the endomorphism which maps every element to the identity) (Proposition 5.6.6, [12]).

Much like for Rees matrix semigroups, there is a characterisation of

Bruck-Reilly extensions of groups; every Bruck-Reilly extension of a group is a $\mathcal{D}$-simple inverse ω-semigroup, and every $\mathcal{D}$-simple inverse ω-semigroup is isomorphic to a Bruck-Reilly extension of a group (Theorem 5.6.7, [12]). For a semigroup to be *inverse* means for every element s, there exists a unique inverse s' such that $s = ss's$ and $s' = s'ss'$. An ω-semigroup is a semigroup whose idempotents form an infinite descending chain with respect to the partial order of idempotents, $\{e_0 > e_1 > e_2 > \ldots\}$.

There are more complicated constructions, for example the *Byleen extension* which we will be investigating in Chapter 3. In order to define a *Byleen extension*, we first need the following:

- a semigroup S,
- two disjoint non-empty sets A and B,
- actions of S on A and B from the right and left, respectively:
  $\rho : A \times S \to A, (a, s) \mapsto a^s$; $\sigma : S \times B \to B, (s, b) \mapsto {}^s b$,
- and $M = (m_{ij})_{A\times B}$ a matrix with entries in $A \cup B \cup S$ which respects the actions, which is to say $m_{a^s,b} = m_{a,{}^s b}$ for all $a \in A, b \in B, s \in S$.

These combine to form the *Byleen extension of S by the matrix M and actions σ and ρ*, denoted $\mathcal{C}(S; \sigma, \rho; M)$. If R is the set of all relations of S, then the extension is defined by the following presentation:

$$\mathcal{C}(S; \sigma, \rho; M) = \langle S, A, B | R, as = a^s, sb = {}^s b, ab = m_{a,b}, (a \in A, b \in B, s \in S)\rangle.$$

These semigroups can be thought of as a generalisation of Rees matrix semigroups, in fact if $\mathcal{M}(S; I, \Lambda; P)$ is a Rees matrix semigroup then $\mathcal{M}(S; I, \Lambda; P) \hookrightarrow \mathcal{C}(S; \sigma, \rho; P)$ for σ any right action of S on Λ and ρ any left action of S on I.

Chapter 2

$\mathcal{D}$-simplicity

In this chapter we will investigate whether some well-known semigroup constructions conserve $\mathcal{D}$-simplicity, or can produce $\mathcal{D}$-simple semigroups from non-$\mathcal{D}$-simple constituents.

The constructions investigated here by no means represent all known constructions, but do represent some of the more widely used ones.

2.1 Direct Product

The *direct product* is perhaps the simplest way to construct a semigroup from others, the definition of which was covered earlier (Section 1.5).

In this section we will investigate the structure of direct products and see that, at least for finitely generated $\mathcal{D}$-simple semigroups, the Green's relations structure of a direct product will be the intuitive extension of the structure of the constituent semigroups (Theorem 2.1.2, Corollaries 2.1.3 and 2.1.4).

In Example 2.1.5 we see that for more general semigroups, $\mathcal{D}$-simplicity of constituents does not necessarily imply $\mathcal{D}$-simplicity of a direct product. Ultimately, we will see that a direct product is $\mathcal{D}$-simple only if its constituent semigroups are $\mathcal{D}$-simple (Theorem 2.1.6).

The following lemma will prove useful when studying the Green's relations of direct products of finitely generated $\mathcal{D}$-simple semigroups.

Lemma 2.1.1. *Let S be a finitely generated $\mathcal{D}$-simple semigroup.*
Then for any $x \in S$ there exist $u, v \in S$ such that $xu = x$ and $vx = x$.

Proof. Suppose first that S has two $\mathcal{R}$-related elements. As a consequence of Green's Lemma and the fact that S is $\mathcal{D}$-simple, every $\mathcal{R}$-class has the same size (which is at least 2), and so for any $x \in S$, there exists $y \in S \setminus \{x\}$ such that $x\mathcal{R}y$. Then there exists $a, b \in S$ such that $xa = y$, and $yb = x$, from which it immediately follows that $xab = x$.

So if the size of the $\mathcal{R}$-classes in S is non-trivial then for any $x \in S$ we can immediately find $u \in S$ such that $xu = x$.

On the other hand, suppose all $\mathcal{R}$-classes are trivial and that there exists $x \in S$ such that $xy \neq x$ for all $y \in S$. Clearly S must be $\mathcal{R}$-trivial, and so $\mathcal{L}$-simple (any two elements in S are $\mathcal{L}$-related). If there exist $s, t \in S$ such that $st = s$, then since $s\mathcal{L}x$ for all $x \in S$, we have $xt = x$. So it must be that there is no right identity for any element.

Let $X \subseteq S$, be a finite generating set for S. If $|X| = 1$, then S is commutative, so $\mathcal{L} = \mathcal{R}$, and since S is $\mathcal{R}$-trivial it must also be $\mathcal{L}$-trivial, and so S must be trivial, a contradiction. Hence, $|X| \geq 2$.

Let $x_1 \in X, y \in S \setminus \{x_1\}$.
Since $x_1\mathcal{L}y$, there exists $a \in S$ such that $x_1 = ay$. Expressing a in terms of the generating set X yields $a = x_2w$ where $x_2 \in X$, $w \in X^*$.

Let $z_1 = wy$ and we have an expression for x_1 which begins with x_2, that is $x_1 = x_2z_1$.

We repeat this process, choosing $y \in S \setminus \{x_2\}$, to get an expression for x_2 which begins with $x_3 \in X$, that is there exists $z_2 \in S$ such that $x_2 = x_3z_2$, and in turn $x_1 = x_3z_2z_1$.

We can repeat this indefinitely, but X is finite so there must come a point when the new first generator in the expression has been seen before, $x_i = x_j$ for some $i < j$, that is we can express x_i as x_is for some $s \in S$. Here s is a right identity for x_i, a contradiction.

Therefore for any finitely generated $\mathcal{D}$-simple semigroup, S, has the property that for each $x \in S$ there exists $y \in S$ such that $xy = x$.
The symmetrical argument can be made to complete the proof of the lemma.

□

In the following theorem we will see that Green's $\mathcal{L}$- and $\mathcal{R}$-relations of a direct product of finitely generated $\mathcal{D}$-simple semigroups can be easily expressed in terms of the $\mathcal{L}$- and $\mathcal{R}$-relations of the constituent semigroups.

Theorem 2.1.2. *Let S and T be finitely generated $\mathcal{D}$-simple semigroups, and let $s_1, s_2 \in S$, $t_1, t_2 \in T$.*

Then $(s_1, t_1)\mathcal{R}(s_2, t_2)$ in $S \times T$ if and only if $s_1\mathcal{R}s_2$ in S and $t_1\mathcal{R}t_2$ in T, that is $R_{(s,t)} = R_s \times R_t$.
The analogous assertion holds for the $\mathcal{L}$-classes of $S \times T$.

Proof. The forward implication is easily demonstrated:
Let $(s_1, t_1)\mathcal{R}(s_2, t_2)$. Then there exist $(u_1, v_1), (u_2, v_2) \in (S \times T)^1$ such that

$$(s_2, t_2) = (s_1, t_1)(u_1, v_1),\ (s_1, t_1) = (s_2, t_2)(u_2, v_2).$$

If $(u_1, v_1) = 1$, then $s_1 = s_2$ and $t_1 = t_2$, and certainly $s_1\mathcal{R}s_2$ in S, $t_1\mathcal{R}t_2$ in T. Otherwise, $u_1, u_2 \in S$, $v_1, v_2 \in T$ and

$$s_2 = s_1u_1,\ s_1 = s_2u_2,\ t_2 = t_1v_1,\ t_1 = t_2v_2,$$

and so $s_1\mathcal{R}s_2$ in S and $t_1\mathcal{R}t_2$ in T.

The reverse implication comes with just a little more work:
Let $s_1\mathcal{R}s_2$ in S and $t_1\mathcal{R}t_2$ in T.
There exist $u_1, u_2 \in S^1$ and $v_1, v_2 \in T^1$, such that

$$s_2 = s_1u_1,\ s_1 = s_2u_2,\ t_2 = t_1v_1,\ t_1 = t_2v_2.$$

By Lemma 2.1.1, there exist $x_1, x_2 \in S$ and $y_1, y_2 \in T$ such that $s_1x_1 = s_1$,

$s_2x_2 = s_2$, $t_1y_1 = t_1$ and $t_2y_2 = t_2$. Using these right identities,

$$s_2 = s_1x_1u_1,\ s_1 = s_2x_2u_2,\ t_2 = t_1y_1v_1,\ t_1 = t_2y_2v_2.$$

Note that $x_1u_1, x_2u_2 \in S$ and $y_1v_1, y_2v_2 \in T$, and we have (x_1u_1, y_1v_1), $(x_2u_2, y_2v_2) \in S \times T$ such that

$$(s_2, t_2) = (s_1, t_1)(x_1u_1, y_1t_1),\ (s_1, t_1) = (s_2, t_2)(x_2u_2, y_2v_2).$$

Hence, $(s_1, t_1)\mathcal{R}(s_2, t_2)$. □

The finitely generated condition was necessary to ensure that we could use Lemma 2.1.1 for the proof of the reverse implication. For example, if S contained an element x such that $sx \neq x$ for all $s \in S$, and T contained distinct t_1, t_2 such that $t_1\mathcal{L}t_2$ (such as the semigroups in Example 2.1.5), then (x, t_1) and (x, t_2) would not be $\mathcal{L}$-related in $S \times T$, despite their components being related in the respective semigroups.

This theorem makes it easy to determine the rest of Green's relations on the direct product of two finitely generated $\mathcal{D}$-simple semigroups.

Corollary 2.1.3. *Let S and T be finitely generated $\mathcal{D}$-simple semigroups, and let $(s_1, t_1), (s_2, t_2) \in S \times T$.*

Then $(s_1, t_1)\mathcal{H}(s_2, t_2)$ if and only if $s_1\mathcal{H}s_2$ in S and $t_1\mathcal{H}t_2$ in T.

Corollary 2.1.4. *Let S and T be finitely generated $\mathcal{D}$-simple semigroups.*

Then $S \times T$ is $\mathcal{D}$-simple.

It is not true, however, that the direct product of any two $\mathcal{D}$-simple semigroups is $\mathcal{D}$-simple, as the following example will demonstrate:

Example 2.1.5. There exist $\mathcal{D}$-simple semigroups S and T, such that their direct product $S \times T$ is not $\mathcal{D}$-simple.

Let A be a countably infinite set, and let S be the set of all injective mappings $\phi : A \to A$ such that $|A \setminus A\phi| = |A|$. Along with the operation composition of mappings, S is a semigroup. In fact, S is the Baer-Levi semigroup with $p = q = \aleph_0$, see Clifford and Preston, The algebraic theory of semigroups vol.2, [7], Chapter 8, for more details.

S is $\mathcal{R}$-simple and $\mathcal{L}$-trivial. That is, all elements in S are $\mathcal{R}$-related, but no distinct elements are $\mathcal{L}$-related. Also, S has the useful property that for all $\alpha, \beta \in S, \alpha\beta \neq \beta$.

Let T be the bicyclic monoid, $T = \langle b, c \mid bc = 1 \rangle$ and suppose $S \times T$ is $\mathcal{D}$-simple.

Let $\alpha \in S$. Then $(\alpha, 1)\mathcal{D}(\alpha, c)$. This means we can find $\beta \in S$ and $c^i b^j \in T$ such that

$$(\alpha, 1)\mathcal{L}(\beta, c^i b^j)\mathcal{R}(\alpha, c).$$

The $\mathcal{L}$ relation implies that $\alpha\mathcal{L}\beta$, but S is $\mathcal{L}$-trivial, so $\alpha = \beta$, and

$$(\alpha, 1)\mathcal{L}(\alpha, c^i b^j)\mathcal{R}(\alpha, c).$$

The first half of the above asserts the existence of $(\mu, u), (\pi, v) \in (S \times T)^1$ such that

$$(\mu, u)(\alpha, 1) = (\alpha, c^i b^j),\ (\pi, v)(\alpha, c^i b^j) = (\alpha, 1).$$

Observing the first component in each of these tells us that $\mu = \pi = 1$, and so $u = v = 1$, and $c^i b^j = 1$, and in turn

$$(\alpha, 1)\mathcal{L}(\alpha, 1)\mathcal{R}(\alpha, c).$$

An immediate consequence of this is that $1\mathcal{R}c$ in T, which is a contradiction. Hence, $S \times T$ is not $\mathcal{D}$-simple, despite the fact that both S and T are.

Of course this counterexample could have been constructed using any $\mathcal{D}$-simple semigroup with more than one $\mathcal{L}$-class in place of the bicyclic monoid.

The following result demonstrates that the direct product will not provide an easy way to embed a non-$\mathcal{D}$-simple semigroup in a $\mathcal{D}$-simple one.

Theorem 2.1.6. *Let S and T be semigroups.*

If $S \times T$ is $\mathcal{D}$-simple, then both S and T are $\mathcal{D}$-simple.

Proof. Let S and T be semigroups such that $S \times T$ is $\mathcal{D}$-simple, and let $r, s \in S$, and $t, u \in T$.

As $S \times T$ is $\mathcal{D}$-simple, $(r,t)\mathcal{D}(s,u)$. That is, there exists $(p,v) \in S \times T$ such that $(r,t)\mathcal{L}(p,v)\mathcal{R}(s,u)$.

This implies that $r\mathcal{L}p\mathcal{R}s$ in S and $t\mathcal{L}v\mathcal{R}u$ in T. Hence S and T are $\mathcal{D}$-simple. □

We have seen that for a direct product of two semigroups to be $\mathcal{D}$-simple, both constituent semigroups must be $\mathcal{D}$-simple, this means that direct products do not afford us an easy way to embed non-$\mathcal{D}$-simple semigroups in those with the property.

2.2 Semidirect Product

Recall from Section 1.5 that the semidirect product of a semigroup S and a semigroup T with respect to a homomorphism $\varphi : T \to \mathrm{End}(S)$, denoted $S \rtimes_\varphi T$ is the set $S \times T$ such that the product is defined by $(s_1, t_1)(s_2, t_2) = (s_1({}^{t_1}s_2), t_1t_2)$, where ${}^{t_1}s_2$ denotes the image of s_2 under $(t_1)\varphi$.

In this section we will prove some results regarding the $\mathcal{D}$-class structure of semidirect products, given certain restrictions on the associated homomorphism. We will also investigate how S might embed in $S \rtimes_\varphi T$, determining that if it is as a projection onto the first component then there must be a homomorphic image of S which is a subsemigroup of T. With this kind of embedding in mind we will demonstrate that if this corresponding homomorphic image is trivial and S is not simple and satisfies a condition on its $\mathcal{J}$-order under automorphisms, then $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple (Theorem 2.2.4).

A corollary to that theorem, Corollary 2.2.6, will show that the theorem's complicated set of criteria can be simplified for finite semigroups to show that if S embeds in $S \rtimes_\varphi T$ in this nice way, and S is not $\mathcal{D}$-simple, then $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple.

The following lemma will be useful in showing that the semidirect product of two $\mathcal{D}$-simple semigroups is not necessarily $\mathcal{D}$-simple:

Lemma 2.2.1. *Let S be a semigroup with idempotent $e \in S$, let T be a finitely generated $\mathcal{D}$-simple semigroup, and let $\varphi : T \to \mathrm{End}(S)$ be such that ${}^{t}s = e$ for all $t \in T$ and $s \in S$.*

Then $S \rtimes_{\varphi} T$ is $\mathcal{D}$-simple if and only if e is a right identity for S.

Proof. Beginning with the forward implication, suppose $S \rtimes_{\varphi} T$ is $\mathcal{D}$-simple.

Let $s \in S$, $t \in T$. As $S \times_{\varphi} T$ is $\mathcal{D}$-simple, $(s,t)\mathcal{D}(e,t)$, and so there exists $(s_1, t_1) \in S \rtimes_{\varphi} T$ such that $(s,t)\mathcal{L}(s_1,t_1)\mathcal{R}(e,t)$.

The $\mathcal{R}$-relation implies that either $(s_1,t_1) = (e,t)$ or there exists $(u,v) \in S \rtimes_{\varphi} T$ such that

$$(s_1,t_1) = (e,t)(u,v) = (e\,{}^{t}u, tv) = (ee, tv) = (e, tv).$$

In either case, $s_1 = e$.

The $\mathcal{L}$-relation implies that either $(s,t) = (s_1,t_1)$ or there exists $(x,y) \in S \rtimes_{\varphi} T$ such that

$$(s,t) = (x,y)(s_1,t_1) = (xe, yt_1).$$

That is, either $s = s_1 = e$ or $s = xe$ for some $x \in S$. In either case, $se = s$, and so e is a right identity for s, and in turn for all of S.

For the reverse implication, suppose that $se = s$ for all $s \in S$.

Let $s \in S$, $t_1, t_2 \in T$. As T is $\mathcal{D}$-simple, $t_1 \mathcal{D} t_2$ in T, that is there exist $t_3 \in T$ and $v_1, v_2, v_3, v_4 \in T^1$ such that

$$t_1 = t_3 v_1,\ t_3 = t_1 v_2,\ t_2 = v_3 t_3,\ t_3 = v_4 t_2.$$

If $v_1 = 1$ or $v_2 = 1$ then $t_1 = t_3$, and $(s,t_1) = (s,t_3)$, otherwise $(s,t_1)\mathcal{R}(s,t_3)$ as a consequence of the following:

$$(s,t_1) = (s,t_3)(e,v_1),\ (s,t_3) = (s,t_1)(e,v_2).$$

If $v_3 = 1$ or $v_4 = 1$ then $t_3 = t_2$, and by Lemma 2.1.1 there exists $v \in T$

such that $vt_2 = t_2$, and

$$(s, t_2) = (s, v)(e, t_2),\ (e, t_2) = (e, v)(s, t_2).$$

Otherwise,

$$(s, t_3) = (s, v_4)(e, t_2),\ (e, t_2) = (e, v_3)(s, t_3).$$

In any case, $(s, t_3)\mathcal{L}(e, t_2)$, and in turn $(s, t_1)\mathcal{D}(e, t_2)$. Hence, $S \rtimes_\varphi T$ is $\mathcal{D}$-simple. □

This lemma serves to provide the following example:

Example 2.2.2. There exist $\mathcal{D}$-simple semigroups S and T, and a homomorphism $\varphi : T \to \mathrm{End}(S)$ such that $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple.

Let $S = \langle b, c \mid bc = 1\rangle$, the bicyclic monoid, let T be any finitely generated $\mathcal{D}$-simple semigroup, and let $\varphi : T \to \mathrm{End}(S)$ be such that ${}^t s = cb$ for all $t \in T$, $s \in S$. Note that $1 \notin Scb$, and so $Scb \neq S$.

By Lemma 2.2.1, $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple.

The more interesting question is the converse: Is it possible to find a semidirect product which is $\mathcal{D}$-simple when one of the components is not?

Of course T must be $\mathcal{D}$-simple in order for $S \rtimes_\varphi T$ to be $\mathcal{D}$-simple, as the following demonstrates:

Suppose that $S \rtimes_\varphi T$ is $\mathcal{D}$-simple for some semigroups S and T.

Let $s \in S$ and $t_1, t_2 \in T$. As $S \rtimes_\varphi T$ is $\mathcal{D}$-simple, $(s, t_1)\mathcal{D}(s, t_2)$. Following the obvious deductive path would prove the existence of $t_3 \in T$ such that $t_1\mathcal{R}t_3\mathcal{L}t_2$ in T. So if T is not $\mathcal{D}$-simple then $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple.

As for the possibility of $S \rtimes_\varphi T$ being $\mathcal{D}$-simple for a non-$\mathcal{D}$-simple S, Lemma 2.2.1 is again useful for finding an example:

Example 2.2.3. There exist semigroups S and T, and homomorphism $\varphi : T \to \mathrm{End}(S)$, such that S is not $\mathcal{D}$-simple but $S \rtimes_\varphi T$ is.

Let S be a monoid which is not $\mathcal{D}$-simple, for example any semigroup with an identity appended, let T be any finitely generated $\mathcal{D}$-simple semigroup, and let $\varphi : T \to \mathrm{End}(S)$ such that ${}^{t}s = 1$ for all $t \in T$ and $s \in S$.

Since $S1 = S$, applying Lemma 2.2.1 we see that $S \rtimes_\varphi T$ is $\mathcal{D}$-simple.

This example is not as exciting as it may seem at first as it does not offer an embedding of S in $S \rtimes_\varphi T$. In fact, $S \rtimes_\varphi T$ is isomorphic to $L \times T$ where L is a left-zero semigroup such that $|L| = |S|$.

In order to ensure an embedding of S in $S \rtimes_\varphi T$ there must be restrictions put on φ. The obvious way we would hope to find an embedding would be where the S component of the embedded copy of S in $S \rtimes_\varphi T$ corresponds to the element of S it represents. That is, for all $s \in S$ there exists $s' \in T$ such that

$$\theta : S \hookrightarrow S \rtimes_\varphi T,\ s \mapsto (s, s')$$

is an injective homomorphism.

Suppose this is the case and consider the set $S' = \{s' \in T : s \in S\} \subseteq T$. This is the image of $S\theta$ under the projection which maps $S \rtimes_\varphi T$ onto T according to $(s, t) \mapsto t$, hence S' is a homomorphic image of S inside T. If this image is faithful then $S \hookrightarrow T$ and we already have an embedding of S in a $\mathcal{D}$-simple semigroup.

Consider the multiplication in this embedded copy of S in $S \rtimes_\varphi T$:

$$(rs, (rs)') = (r, r')(s, s') = (r\, {}^{r'}s, (rs)').$$

That is, for all $r, s \in S$, we have $rs = r\, {}^{r'}s$. This puts a heavy restriction on which endomorphisms φ can map elements of the form s' to.

Since we want to consider what happens in general, and we don't want T to depend on S, consider the trivialising homomorphism which maps all of S to a single element $e \in T$. Of course e must be an idempotent so we have at least a small restriction on which semigroups can be used for T.

The embedding now looks like this:

$$\theta : S \hookrightarrow S \rtimes_{\varphi} T, \ s \mapsto (s, e).$$

In order for the multiplication to respect the embedding we need $rs = r\,{}^{e}s$ for all $r, s \in S$, the obvious choice for $(e)\varphi$ here is to have it act as the identity.

The following theorem will demonstrate that for S from a large class of semigroups, semidirect products $S \rtimes_{\varphi} T$ with S embedded in this way can not be $\mathcal{D}$-simple.

Theorem 2.2.4. *Let S be a semigroup with more than one $\mathcal{J}$-class, T a $\mathcal{D}$-simple semigroup with idempotent $e \in T$, and let $\varphi : T \to \mathrm{End}(S)$ be a homomorphism such that $e\varphi$ acts as the identity on S.*

If S has the property that $s \not<_J (s)\pi$ for all $s \in S$ and $\pi \in \mathrm{Aut}(S)$, then $S \rtimes_{\varphi} T$ is not $\mathcal{D}$-simple.

Proof. Let S have the property that $s \not<_J (s)\pi$ for all $s \in S$ and $\pi \in \mathrm{Aut}(S)$, and suppose that $S \rtimes_{\varphi} T$ is $\mathcal{D}$-simple with a view to finding a contradiction. Since S has more than one $\mathcal{J}$-class, there exists $s_1, s_2 \in S$ such that $s_2 <_J s_1$ and by the $\mathcal{D}$-simplicity of $S \rtimes_{\varphi} T$, we have $(s_1, e)\mathcal{D}(s_2, e)$.

There exist $s_3 \in S$ and $t \in T$ such that

$$(s_1, e)\mathcal{L}(s_3, t)\mathcal{R}(s_2, e).$$

From this we can conclude that $s_3\mathcal{R}s_2$ in S, and $e\mathcal{L}t\mathcal{R}e$ in T, that is $e\mathcal{H}t$. This second deduction along with the fact that $e\varphi \in \mathrm{Aut}(S)$ implies that $t\varphi \in \mathrm{Aut}(S)$.

From the $\mathcal{L}$-relation, there exists $(u, v) \in S \rtimes_{\varphi} T$ such that

$$(s_1, e) = (u, v)(s_3, t) = (u\,{}^{v}s_3, vt).$$

The first component gives us that $s_1 \leq_L {}^{v}s_3$.

Applying φ to the second component we see that $e\varphi = v\varphi t\varphi$. Rearranging, we see that $v\varphi = e\varphi(t\varphi)^{-1}$, that is $v\varphi$ can be expressed as a product of automorphisms, hence $v\varphi \in \mathrm{Aut}(S)$.

Now we have that

$$s_3 \mathcal{R} s_2 <_J s_1 \leq_L {}^v s_3,$$

which implies that

$$s_3 =_J s_2 <_J s_1 \leq_J {}^v s_3,$$

and so

$$s_3 <_J {}^v s_3,$$

a contradiction to the property imposed on the automorphism and the $\mathcal{J}$-order on S. $\square$

This property that $s \not<_J (s)\pi$ for all $s \in S$ and $\pi \in \mathrm{Aut}(S)$ is not one which is particularly easy to check, and so this theorem is not so easy to directly apply. However, if S has finitely many $\mathcal{J}$-classes then it necessarily has this property as we will see in the next corollary.

Corollary 2.2.5. *Let S be a semigroup with more than one but finitely many $\mathcal{J}$-classes, T a $\mathcal{D}$-simple semigroup with idempotent $e \in T$, and let $\varphi : T \to \mathrm{End}(S)$ be a homomorphism such that $e\varphi$ acts as the identity on S.*

Then $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple.

Proof. Suppose there exist $s \in S$ and $\pi \in \mathrm{Aut}(S)$ such that $s <_J s\pi$. Then there exist $u, v \in S^1$ (not both 1) such that $s = u(s\pi)v$.

Applying π to this, if $u, v \neq 1$ then $s\pi = (u\pi)(s\pi^2)(v\pi)$, if $u = 1$ then $s\pi = (s\pi^2)(v\pi)$, or if $v = 1$ then $s\pi = (u\pi)(s\pi^2)$. In any case, $s\pi \leq_J s\pi^2$.

Applying π repeatedly will yield $s\pi^i \leq_J s\pi^{i+1}$ for all $i \in \mathbb{N}$.

If $s\pi^i \mathcal{J} s\pi^{i+1}$ for some $i \in \mathbb{N}$, then applying the automorphism π^{-1} must map both of these elements to the same $\mathcal{J}$-class, that is

$$s\pi^{i-1} = (s\pi^i)\pi^{-1} \mathcal{J} (s\pi^{i+1})\pi^{-1} = s\pi^i.$$

This can be repeated to get $s\pi^j \mathcal{J} s\pi^{j+1}$ for all $0 \leq j \leq i$, which contradicts the condition that $s <_J s\pi$. Hence, $s\pi^i <_J s\pi^{i+1}$ for all $i \in \mathbb{N}_0$.

However, S has finitely many $\mathcal{J}$-classes, and so we reach a contradiction. Hence, there do not exist $s \in S$ and $\pi \in \mathrm{Aut}(S)$ such that $s <_J s\pi$, and applying the theorem we see that $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple. $\square$

In particular, we can restrict the case to S being finite and get a far simpler statement about the $\mathcal{D}$-simplicity of $S \rtimes_\varphi T$ when the homomorphism meets the condition.

Corollary 2.2.6. *Let S be a finite semigroup, T a semigroup with idempotent $e \in T$, and let $\varphi : T \to \mathrm{End}(S)$ be a homomorphism such that $e\varphi$ acts as the identity on S.*

If S is not $\mathcal{D}$-simple then $S \rtimes_\varphi T$ is not $\mathcal{D}$-simple.

Proof. As S is not $\mathcal{D}$-simple, S has more than one $\mathcal{J}$-class (for finite semigroups $\mathcal{D} = \mathcal{J}$), and of course S has finitely many $\mathcal{J}$-classes. □

These results all rely on φ mapping a subsemigroup, T', of T which is a homomorphic image of S, to the endomorphism of S which maps all elements to the right-identity $e \in S$. This is the strictest possible case for the embedding. As we saw earlier, if the embedding works in such a way that S can be found as a projection onto the first component we have $s\, {}^{s'}t = st$ for all $s, t \in S$.

Question. Does there exist a non-$\mathcal{D}$-simple semigroup S, a semigroup T, and a homomorphism $\varphi : T \to \mathrm{End}(S)$, such that $S \rtimes_\varphi T$ is $\mathcal{D}$-simple and $S \hookrightarrow S \rtimes_\varphi T$?

2.3 Bruck-Reilly Extensions

Refer to Section 1.5 for the definition of a *Bruck-Reilly extension*.

In 1958 Bruck [2] proved that Bruck-Reilly extensions can be used to embed any semigroup in a simple monoid, by appending an identity if the semigroup does not already contain one and then taking the extension with respect to the endomorphism which maps every element to the identity.

Unfortunately, it was later proven by Munn [15] that for two elements in a Bruck-Reilly extension $BR(M, \theta)$ to be $\mathcal{D}$-related, their M-components must be $\mathcal{D}$-related in M. This comes from the fact that two elements $(m, a, n), (p, b, q) \in BR(M, \theta)$ are $\mathcal{R}$-related if and only if $m = p$ and $a\mathcal{R}b$ in M, and they are $\mathcal{L}$-related if and only if $n = q$ and $a\mathcal{L}b$ in M. Of course, this means that Bruck-Reilly extensions cannot be directly used to embed non-$\mathcal{D}$-simple semigroups in $\mathcal{D}$-simple ones.

In 1984, Clement and Pastijn [6] presented a generalisation of the Bruck-Reilly extension which allowed them to embed 0-bisimple monoids into bisimple semigroups. Recall that *bisimple* means exactly the same as $\mathcal{D}$-simple. A semigroup with zero is said to be 0-*bisimple* if every element is $\mathcal{D}$-related except for the zero element, which must of course always form a $\mathcal{D}$-class of its own. The equivalent to 0-bisimple in our terminology would be the rather cumbersome 0-$\mathcal{D}$-simple, but fortunately this concept will not be revisited hereafter.

The thing to note here is that they found a way to embed a non-$\mathcal{D}$-simple semigroup, in particular a non-($\mathcal{J}$-)simple semigroup, into a $\mathcal{D}$-simple semigroup.

While limited, this approach seems promising and leads to some questions:

Question. When is this generalised Bruck-Reilly extension finitely generated, and when is it finitely presented?

Question. Is it possible to generalise Clement and Pastijn's generalisation of the Bruck-Reilly extension? In particular, can it be used to embed semigroups with a longer chain of trivial $\mathcal{J}$-classes, or to embed semigroups with exactly two $\mathcal{J}$-classes in which the lower is not just a zero, or both?

2.4 Free Products

It doesn't take a great deal of work to determine when a semigroup free product is $\mathcal{D}$-simple.

First we should recall that the semigroup free product of S and T, $S * T$, is the set of all words over $S \cup T$ such that any adjacent letters from the same semigroup are reduced according the multiplication of that semigroup. So the elements of $S * T$ are all the words over $S \cup T$ such that no adjacent letters in the word are both from S or both from T. Multiplication is defined in the obvious way, by concatenating the words and reducing in the event that the first word ended with an element from the same semigroup as the first element of the second word.

Theorem 2.4.1. *Let S and T be semigroups.*

Then $S * T$ is not $\mathcal{D}$-simple.

Proof. Let $s \in S$, $t \in T$.
Suppose $s\mathcal{D}t$. Then there exists $z \in S * T$ such that $s\mathcal{L}z\mathcal{R}t$.
There exist $u, v \in (S * T)^1$ such that $s = uz$ and $t = zv$. The first of these implies $z \in S$ and the second implies $z \in T$, a contradiction to $s\mathcal{D}t$. □

Often we consider a slightly different free product when dealing with monoids, the *monoid free product.* The elements of the monoid free product, $S *_1 T$, are the words over $S \cup T$, reduced whenever two adjacent elements are from the same semigroup and with the extra rule that the identities from each of S and T are combined to become the identity for $S *_1 T$.

The key difference here is that the length of a product can be less than either of the constituents, and can even be trivial. Of course, in the original free product the length of a product $l(uv)$ was either $l(u) + l(v)$ if the final element in the word u was from S and the first in the word v was from T (or vice versa) or $l(u) + l(v) - 1$ in the event that the final element of u and the first of v come from the same semigroup. In particular, a product in the original free product was always at least as long as both the elements, so we never stood a chance of $\mathcal{D}$-relating an element of length two to an element of length one.

Unfortunately, this ability to reduce the length is not sufficient to show that $\mathcal{D}$-simple monoids are closed under monoid free products.

Theorem 2.4.2. *Let S and T be $\mathcal{D}$-simple monoids such that S has at least two $\mathcal{L}$-classes and at least two $\mathcal{R}$-classes, and T is non-trivial.*

Then $S *_1 T$ is not $\mathcal{D}$-simple.

Proof. Since S has multiple $\mathcal{L}$- and $\mathcal{R}$-classes, there exists $s \in S$ such that s is neither $\mathcal{L}$- nor $\mathcal{R}$-related to the identity 1_S.

Let $t \in T$ such that $t \neq 1_T$.

Now suppose, seeking contradiction, that $S *_1 T$ is $\mathcal{D}$-simple. In particular, $sts\mathcal{D}1$. That is there exists $z \in S *_1 T$ such that $sts\mathcal{R}z\mathcal{L}1$.

Since s is not $\mathcal{R}$-related to 1_S in S, we see that for all $x \in S *_1 T$, we have $l(sx) \geq l(s) = 1$, and in turn $l(stsx) \geq l(sts) = 3$. Hence $l(z) \geq 3$, and the first two elements in the word z are st, let $y \in S *_1 T$ such that $z = sty$.

Arguing symmetrically, as s is not $\mathcal{L}$-related to 1_S in S, for all $x \in S *_1 T$ we have $l(xs) \geq l(s)$, and in turn $l(xsty) \geq l(sty) = l(z) \geq 3$. Hence any element which is $\mathcal{L}$-related to z must have length at least 3, the identity has length 0 and so we have a contradiction to $z\mathcal{L}1$. □

What if neither monoid has multiple $\mathcal{L}$- and multiple $\mathcal{R}$-classes?
It is quite easy to see that if both are $\mathcal{L}$-trivial (or both are $\mathcal{R}$-trivial) then $S *_1 T$ is $\mathcal{D}$-simple.

The following theorem shows that these trivial cases are the only monoid free products which are $\mathcal{D}$-simple.

Theorem 2.4.3. *Let S be a monoid with at least two $\mathcal{R}$-classes, and let T be a monoid with at least two $\mathcal{L}$-classes.*

Then $S *_1 T$ is not $\mathcal{D}$-simple.

Proof. Choose $s \in S, t \in T$ such that s is not $\mathcal{R}$-related to 1_S in S and t is not $\mathcal{L}$-related to 1_T in T.

Suppose that $S *_1 T$ is $\mathcal{D}$-simple. Then in particular, $ts\mathcal{D}1$.
That is, there exists $z \in S *_1 T$ such that $ts\mathcal{R}z\mathcal{L}1$.

Arguing along the same lines as the previous theorem, $l(z) \geq l(ts) = 2$ as $ts\mathcal{R}z$ and s is not $\mathcal{R}$-related to 1_S in S. Similarly, as $z = tsy$ for some $y \in S *_1 T$ and t is not $\mathcal{L}$-related to 1_T in T, any element of $S *_1 T$ which is $\mathcal{L}$-related to z must have length at least that of z. This implies that

$$0 = l(1) \geq l(z) \geq l(ts) = 2,$$

which is of course a contradiction to the $\mathcal{D}$-simplicity of $S *_1 T$. □

2.5 Rees Matrix Semigroups

Refer to Section 1.5 for the definition of a *Rees matrix semigroup.*

A matrix P with entries from a semigroup S is said to be *regular* if S is a monoid with group of units $G(S)$ and every row and every column of P contains at least one element from $G(S)$.

Theorem 2.5.1. *Let S be a $\mathcal{D}$-simple monoid, let P be a regular $\Lambda \times I$ matrix over S, and let $M = \mathcal{M}(S; I, \Lambda; P)$.*

Then M is $\mathcal{D}$-simple.

Proof. Let $(i, x, \lambda), (j, y, \mu) \in M$.
Since S is $\mathcal{D}$-simple we can find $z \in S$ such that $x\mathcal{R}z\mathcal{L}y$, and the associated $t, u, v, w \in S$ such that $x = zt,\ z = xu,\ z = vy,\ y = wz$.

Choose $k, l \in I, \rho, \pi \in \Lambda$ such that $p_{\lambda k}, p_{\mu l}, p_{\rho i}, p_{\pi j} \in G(S)$. This is possible since P is regular.

Now we can see that $(i, x, \lambda)\mathcal{R}(i, z, \mu)$:

$$(i, x, \lambda)(k, p_{\lambda k}^{-1}u, \mu) = (i, xp_{\lambda k}p_{\lambda k}^{-1}u, \mu) = (i, xu, \mu) = (i, z, \mu),$$

$$(i, z, \mu)(l, p_{\mu l}^{-1}t, \lambda) = (i, zp_{\mu l}p_{\mu l}^{-1}t, \lambda) = (i, zt, \lambda) = (i, x, \lambda).$$

Similarly, $(i, z, \mu)\mathcal{L}(j, y, \mu)$, and so $(i, x, \lambda)\mathcal{D}(j, y, \mu)$. □

It is easy to see that if P is not regular, then the resulting Rees matrix semigroup is not necessarily $\mathcal{D}$-simple, but it is possible to loosen the regularity condition, as is demonstrated in the following theorem.

Theorem 2.5.2. *Let S be a $\mathcal{D}$-simple monoid and let P be a $\Lambda \times I$-matrix over S such that every row contains an element $\mathcal{R}$-related to the identity and every column contains an element $\mathcal{L}$-related to the identity.*

The Rees matrix semigroup $M = \mathcal{M}(S; I, \Lambda; P)$ is $\mathcal{D}$-simple.

Proof. Let $(i, x, \lambda), (j, y, \mu) \in M$.

Since S is $\mathcal{D}$-simple we can find $z \in S$ such that $x\mathcal{R}z\mathcal{L}y$, and the associated $t, u, v, w \in S$ such that $x = zt,\ z = xu,\ z = vy,\ y = wz$.

Choose $k, l \in I, \rho, \pi \in \Lambda$ such that $p_{\lambda k}, p_{\mu l} \in R_1$ and $p_{\rho i}, p_{\pi j} \in L_1$. There exist $p'_{\lambda k}, p'_{\mu l} \in S$ such that $1 = p_{\lambda k} p'_{\lambda k} = p_{\mu l} p'_{\mu l}$.

Now we can see that $(i, x, \lambda)\mathcal{R}(i, z, \mu)$:

$$(i, x, \lambda)(k, p'_{\lambda k} u, \mu) = (i, x p_{\lambda k} p'_{\lambda k} u, \mu) = (i, xu, \mu) = (i, z, \mu),$$

$$(i, z, \mu)(l, p'_{\mu l} t, \lambda) = (i, z p_{\mu l} p'_{\mu l} t, \lambda) = (i, zt, \lambda) = (i, x, \lambda).$$

Similarly, $(i, z, \mu)\mathcal{L}(j, y, \mu)$, and so $(i, x, \lambda)\mathcal{D}(j, y, \mu)$. □

Unfortunately, it is not possible to use Rees matrix semigroups to directly embed general semigroups in $\mathcal{D}$-simple semigroups.

Theorem 2.5.3. *Let $M = \mathcal{M}(S; I, \Lambda; P)$ be a Rees matrix semigroup.*

If M is $\mathcal{D}$-simple, then S is $\mathcal{D}$-simple too.

Proof. Let $M = \mathcal{M}(S; I, \Lambda; P)$ be a $\mathcal{D}$-simple Rees matrix semigroup, and let $s, t \in S$, $i, j \in I$, $\lambda, \mu \in \Lambda$.

By the $\mathcal{D}$-simplicity, $(i, s, \lambda)\mathcal{D}(j, t, \mu)$. That is, there exists $(k, r, \nu) \in M$ such that

$$(i, s, \lambda)\mathcal{L}(k, r, \nu)\mathcal{R}(j, t, \mu),$$

and in turn, there exist $(l_1, u_1, \pi_1), (l_2, u_2, \pi_2), (l_3, u_3, \pi_3), (l_4, u_4, \pi_4) \in M^1$ such that

$$(i, s, \lambda) = (l_1, u_1, \pi_1)(k, r, \nu) = (l_1, u_1 p_{\pi_1 k} r, \nu),$$

$$(k, r, \nu) = (l_2, u_2, \pi_2)(i, s, \lambda) = (l_2, u_2 p_{\pi_2 i} s, \lambda),$$

$$(k, r, \nu) = (j, t, \mu)(l_3, u_3, \pi_3) = (j, t p_{\mu l_3} u_3, \pi_3),$$

$$(j, t, \mu) = (k, r, \nu)(l_4, u_4, \pi_4) = (k, r p_{\nu l_4} u_4, \pi_4).$$

Considering only the middle component of each of these we see that $s\mathcal{L}r\mathcal{R}t$ in S, and in turn $s\mathcal{D}t$ in S.

Hence, S is $\mathcal{D}$-simple. □

Of course, the Rees matrix semigroup may contain other subsemigroups apart from the base semigroup, but categorising them all is not an easy task. As we are interested primarily in embeddings in $\mathcal{D}$-simple semigroups, we can restrict the investigation to Rees matrix semigroups over $\mathcal{D}$-simple semigroups.

The restriction of taking a Rees matrix semigroup over a group is an appropriate starting point, as it is necessarily $\mathcal{D}$-simple by Theorem 2.5.1 and the fact that the matrix must clearly be regular.

This brings us to the study of *completely simple semigroups* as these coincide with Rees matrix semigroups over groups, usually attributed to Rees [19], although essentially determined by Suschkewitsch [23].

Theorem 2.5.4 (Suschkewitsch 1928, Rees 1940)**.** *Let G be a group, let I and Λ be non-empty sets, and let $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries in G.*

The Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ is completely simple.
Conversely, every completely simple semigroup is isomorphic to a Rees matrix semigroup over a group.

This theorem will be used to deduce information about possible subsemigroups of completely simple semigroups.

Lemma 2.5.5. *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup over the group G.*

The idempotents of S form an antichain with respect to the partial order of idempotents.

Proof. Let $a, b \in S$ be idempotents such that $a \leq b$.
Let $a = (i, g, \lambda)$, $b = (j, h, \mu)$. As $a \leq b$, it holds that $a = ab, a = ba$,

$$(i, g, \lambda) = (i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu),$$

$$(i, g, \lambda) = (j, h, \mu)(i, g, \lambda) = (j, hp_{\mu i}g, \lambda).$$

The third component of the first equality implies that $\mu = \lambda$ and the first component of the second equality implies that $i = j$. The central component of either yields $p_{\lambda i} = h^{-1}$.

$$(i, g, \lambda) = a = a^2 = (i, g, \lambda)(i, g, \lambda) = (i, gp_{\lambda i}g, \lambda) = (i, gh^{-1}g, \lambda)$$

From the central component, $g = gh^{-1}g$, rearranging we see that $g = h$.

Hence, $b = (j, h, \mu) = (i, g, \lambda) = a$, and in turn no distinct idempotents are comparable, that is they form an antichain. □

Corollary 2.5.6. *Let S be a $\mathcal{D}$-simple semigroup.*

If S has distinct idempotents $a, b \in S$ such that $a \leq b$, then S does not contain any primitive idempotents. That is, for every idempotent there exists an infinite descending chain of idempotents below it.

Proof. If S contains a primitive idempotent, then S is completely simple, and by Theorem 2.5.4 is isomorphic to a Rees matrix semigroup over a group. Applying Lemma 2.5.5 a contradiction arises to the comparability of a and b. □

An immediate consequence of this is the following corollary about finite $\mathcal{D}$-simple semigroups.

Corollary 2.5.7. *Let S be a $\mathcal{D}$-simple semigroup.*

Then either every pair of idempotents is incomparable under $\leq$ or there is an infinite descending chain of idempotents. In particular, the existence of comparable idempotents implies that S is not finite.

Proof. By Corollary 2.5.6, if S is $\mathcal{D}$-simple and has a pair of comparable idempotents then it contains an infinite descending chain of idempotents. □

Note that this corollary makes no mention of primitive idempotents or Rees matrix semigroups.

Now we turn our attention to the subsemigroups of completely simple semigroups.

Theorem 2.5.8. *Let S be a completely simple semigroup.*

If T is a periodic subsemigroup of S, then T is completely simple.

Proof. As S is completely simple, it is isomorphic to some Rees matrix semigroup over a group $S \cong \mathcal{M}[G; I, \Lambda; P]$.

Let K be the image of T under the isomorphism, let $I' = \{i \in I : (\exists \lambda \in \Lambda, g \in G : (i, g, \lambda) \in K)\}$, and define Λ' similarly.

Note that for all $i \in I', \lambda \in \Lambda'$, there exists $(i, g, \lambda) \in K$.

Let $t = (i, g, \lambda) \in K$. Since K has the property that all elements have finite order, there exists $0 < j < k$ such that $t^j = t^k$, that is

$$(i, g(p_{\lambda i}g)^{j-1}, \lambda) = (i, g, \lambda)^j = t^j = t^k = (i, g, \lambda)^k = (i, g(p_{\lambda i}g)^{k-1}, \lambda).$$

The group components must be equal, and in groups we have cancellativity, hence $1 = (p_{\lambda i}g)^{k-j}$.

Clearly, $(i, g, \lambda)^{k-j}$ is a right-identity for the set of elements with Λ-component λ and a left-identity for the set of elements with I-component i, and of course, a two-sided identity for the set of elements with Λ-component λ and I-component i.

Hence, K contains at least one idempotent, and since S has no comparable idempotents (Lemma 2.5.5), this idempotent must be primitive.

Let $(i, g, \lambda), (j, h, \mu) \in K$.
Above we saw that for any element of K we can find a power of that element which is a left-identity for any element with the same I-component and a right-identity for any element with the same Λ-component, hence there exist $m \in \mathbb{N}$ such that

$$((i, g, \lambda)(j, h, \mu))^m(i, g, \lambda) = (i, g, \lambda).$$

Hence, $(i, g, \lambda) \leq_J (j, h, \mu)$, and by symmetry, K is simple.

Since $T \cong K$ is simple and has a primitive idempotent, it is completely simple. □

Corollary 2.5.9. *Let S be a completely simple semigroup with a maximal subgroup which is periodic.*

Every subsemigroup of S is completely simple.

Proof. By Theorem 2.5.4, S is isomorphic to a Rees matrix semigroup over a group, G, and since a maximal subgroup of S is periodic, G must be periodic. In turn, any element of S is necessarily periodic, and so any subsemigroup is periodic and we can apply Theorem 2.5.8. □

Corollary 2.5.10. *Let S be a finite $\mathcal{D}$-simple semigroup.*

If $T \hookrightarrow S$ then T is $\mathcal{D}$-simple.

Proof. As S is finite and $\mathcal{D}$-simple, S has a primitive idempotent and is simple, hence completely simple, and since it is finite, any subsemigroups must also be finite and in turn periodic.

If $T \hookrightarrow S$ then T is isomorphic to a subsemigroup of S, so by Theorem 2.5.8, T is completely simple.

By Theorem 2.5.4, T is isomorphic to a Rees matrix semigroup over a group, which is $\mathcal{D}$-simple by Theorem 2.5.1. □

This last corollary confirms that the solution to the problem of trying to find a $\mathcal{D}$-simple semigroup which contains a given non-$\mathcal{D}$-simple semigroup is never finite, a result which is intuitive once you have tried the simplest examples.

Of course questions remain for embedding in $\mathcal{D}$-simple Rees matrix semigroups which are not completely simple, for example:

Question. Does there exist a finite semigroup which does not embed in any $\mathcal{D}$-simple Rees matrix semigroup?

Or if not:

Question. Does there exist a finitely generated semigroup which does not embed in any $\mathcal{D}$-simple Rees matrix semigroup?

Chapter 3

Byleen Semigroups

The *Byleen semigroup construction*, developed by Karl Byleen in the 1980s to solve some embedding problems, provides the topic for this chapter. In short, he proved the following using this construction (recall that *bisimple* and $\mathcal{D}$-*simple* are synonymous):

- Using an early version of the construction, he proved that any countable semigroup can be embedded in a 2-generated bisimple monoid, [3].
- Introducing the full construction in [4], Byleen used it to demonstrate that any countable semigroup without idempotents can be embedded in a 2-generated simple semigroup without idempotents.
- In [5], Byleen introduced a slight modification to produce a monoid and demonstrated that this can be used to embed any countable semigroup in a 2-generated congruence-free bisimple monoid.

In 2010 Martyn Quick and Nik Ruškuc [18] modified the construction slightly and introduced some conditions which resulted in an infinite simple module with constant d-sequence, amongst other properties.

In Section 3.1 we prove some structural results regarding the construction used by Quick and Ruškuc, showing that it is congruence-free (Propostion 3.1.1), determining exactly when two such constructions are isomorphic

(Theorem 3.1.2), and determining Green's relations (Theorem 3.1.3, Corollary 3.1.4, Corollary 3.1.5, Corollary 3.1.6).

In Section 3.2 we will then modify the construction slightly to obtain a monoid which is $\mathcal{D}$-simple (Theorem 3.2.2).

In Theorem 3.2.4, we see that for either of these constructions to be finitely presentable the matrix which underpins them must contain only finitely many repeated entries, which will greatly impact on the other properties which they can have.

In Section 3.3, we introduce the full extension introduced by Byleen, and observe that the two used so far are special cases of this, using the trivial semigroup as a zero or an identity.

We will see that no Byleen extension is $\mathcal{D}$-simple (Corollary 3.3.3), but that it is possible for a Byleen monoid extension to be $\mathcal{D}$-simple (Theorem 3.2.2).

We will see that if a Byleen extension of a semigroup is $\mathcal{D}$-simple, then the semigroup must have been $\mathcal{D}$-simple too (Theorem 3.3.5). Also if a Byleen extension of a semigroup is finitely presentable then the semigroup must have been finitely presentable (Theorem 3.3.9), and the matrix must satisfy very restrictive property on its repeated entries (Theorem 3.3.10).

In Section 3.4, we will investigate the structure of possible subsemigroups of Byleen extensions, in particular subsemigroups in which all elements have finite order, with a view to characterising which semigroups can be embedded in Byleen extensions (Theorem 3.4.10).

3.1 Byleen 0-semigroup

We begin with the modified version of Byleen's construction as used by Quick and Ruškuc in [18]. This serves as a relatively gentle introduction to the construction allowing us to get to grips with the fundamental aspects without worrying about some of the more complicated components.

The construction goes as follows: Let A and B be two disjoint sets, and let $M = (m_{ij})_{A\times B}$ be a matrix with entries from the set $A \cup B \cup \{0\}$. The semigroup $\mathcal{B}_M$ is generated by the sets A and B, has zero element 0, and has the operation which identifies any pair ab with the matrix entry $m_{a,b}$ for all $a \in A$, $b \in B$. The semigroup satisfies the following presentation:

$$\mathcal{B}_M = \langle A, B, 0 \mid ab = m_{a,b}, 0a = a0 = 0, 0b = b0 = 0, (a \in A, b \in B)\rangle.$$

It is easy to determine a set of unique normal forms for $\mathcal{B}_M$: Let $W = \{\beta\alpha : \alpha \in A^*, \beta \in B^*, \beta\alpha \neq \epsilon\} \cup \{0\}$.

Let $s \in \mathcal{B}_M$. As $\mathcal{B}_M$ is generated by $A \cup B \cup \{0\}$, this means s can be expressed as a finite product of elements of $A \cup B \cup \{0\}$, that is $s = x_1 x_2 \dots x_k$ for some $k \in \mathbb{N}$ and $x_1, \dots, x_k \in A \cup B \cup \{0\}$. Of course if any x_i in the expression is 0 then $s = 0 \in W$.

Suppose that 0 does not occur in the expression of s.
If $x_1 x_2 \dots x_k \notin W$, then there exists $1 \leq i < k$ such that $x_i \in A$ and $x_{i+1} \in B$. Applying the relation $(x_i x_{i+1}, m_{x_i,x_{i+1}})$ affords us a new expression for s in terms of the generating set which is necessarily shorter than the previous expression, $s = x_1 \dots x_{i-1} m_{x_i,x_{i+1}} x_{i+2} \dots x_k$. If this expression is not in W then we can repeat the process to find another expression for s of shorter length. This process must terminate as the length of the first expression for s was finite, hence W is a set of normal forms for $\mathcal{B}_M$.

Uniqueness of this set of normal forms comes as a consequence of the fact that the non-zero relations have the form (ab, c) where $a \in A$, $b \in B$, and $c \in A \cup B$, the relations do not overlap and so if there is ever a choice of the order in which to apply the relations to get to the normal form, the choice does not change the outcome. This means that given an expression for an element in terms of the generating set we can never reduce it into two different elements of W.

In [18], Quick and Ruškuc introduced the following conditions on the matrix M in order to infer properties of $\mathcal{B}_M$:

(P1) For every $n \geq 1$, every collection $a_1, \dots, a_n \in A$ of distinct indices, and every collection $c_1, \dots, c_n \in A \cup B \cup \{0\}$ there exist infinitely many distinct $b \in B$ such that $m_{a_i,b} = c_i$ for all $i = 1, \ldots, n$.

(P2) For every $n \geq 1$, every collection $b_1, \ldots, b_n \in B$ of distinct indices, and every collection $c_1, \ldots, c_n \in A \cup B \cup \{0\}$ there exist infinitely many distinct $a \in A$ such that $m_{a,b_i} = c_i$ for all $i = 1, \ldots, n$.

(P3) There exist enumerations $a'_1, a'_2, \ldots$ and $b'_1, b'_2, \ldots$ of A and B respectively, such that $m_{a'_i,b'_i} = b'_{i+1}$ and $m_{a'_i,b'_{i+1}} = a'_{i+1}$, for all $i = 1, 2, \ldots$.

Quick and Ruškuc demonstrated that if M satisfies $(P1), (P2), (P3)$, then $\mathcal{B}_M$ is finitely generated, congruence free and has a zero but no identity (Lemma 6.7, [18]), however the proof that it is congruence-free came as a consequence of Lemma 6.6 in the same paper which is unfortunately erroneous. It is an easy mistake to make; in the statement of the lemma there is a choice of any $t_1, \ldots, t_n \in S$, where S is the semigroup $\mathcal{B}_M$, and in the proof the penultimate step finds these t_i as entries of the matrix M, but the matrix has entries from the set $A \cup B \cup \{0\}$ and so there are elements of S which do not occur as entries in the matrix.

Fortunately, it is true that the semigroup is congruence-free, as the following will demonstrate.

Proposition 3.1.1. *Let $M = (m_{ij})_{A \times B}$ be a matrix with entries from $A \cup B \cup \{0\}$ satisfying $(P1), (P2), (P3)$.*

The semigroup $\mathcal{B}_M$ is congruence-free.

Proof. Let $\sim$ be a non-trivial congruence of $\mathcal{B}_M$.
In order to prove that $\mathcal{B}_M$ is congruence-free it will suffice to show that $\sim$ is the universal relation.

Since $\sim$ is non-trivial, there exists $u, v \in \mathcal{B}_M$ such that $u \neq v$ and $u \sim v$. At least one of u, v is non-zero, so without loss of generality let $u \neq 0$.

Suppose $v \neq 0$. Expressing u and v in their normal form, $u = b_u a_u, v = b_v a_v$, where $b_u \in B^*$, $a_u \in A^*$, not both empty, and $b_v \in B^*$, $a_v \in A^*$, not both empty.

If $a_v \neq \epsilon$ and $a_u \neq \epsilon$, then by $(P1)$ there exists $b \in B$ such that $a_v b = 0$, $a_u b = a_u$, and in turn $0 \neq ub \sim vb = 0$.

If $a_v \neq \epsilon$ and $a_u = \epsilon$, then by $(P1)$ there exists $b \in B$ such that $a_v b = 0$, and in turn $0 \neq b_u b = ub \sim vb = 0$.

If $a_v = \epsilon$ then $b_v \neq \epsilon$, using $(P2)$ in similar arguments as above, there exists $a \in A$ such that $0 \neq au \sim av = 0$.

So the existence of a non-trivial congruence $\sim$ on $\mathcal{B}_M$ implies the existence of $w \in \mathcal{B}_M \setminus \{0\}$ such that $w \sim 0$.
Expressing w in its normal form, there exists $a_w \in A^*$, $b_w \in B^*$, not both empty, such that $w = b_w a_w$.

If $b_w = \epsilon$, let $b \in B$ and note that $0 \neq bw \sim b0 = 0$. So without loss of generality, $b_w \in B^+$.

If $a_w \neq \epsilon$, using a lemma by Quick and Ruškuc ([18], Lemma 6.4), there exists $b \in B$ such that $a_w b \in B$, and in particular $wb \in B^+$ and $wb \sim 0b = 0$. So without loss of generality, $w \in B^+$.

Let $w = b_1 b_2 \ldots b_k$, where $k \in \mathbb{N}$ and $b_1, \ldots, b_k \in B$.
Again using the lemma by Quick and Ruškuc ([18], Lemma 6.4), there exists $a \in A$ such that $ab_1 b_2 \ldots b_k = b_k \in B$, and so,

$$b_k = ab_1 b_2 \ldots b_k = aw \sim a0 = 0.$$

Let $c \in A \cup B$. By $(P2)$, there exists $a \in A$ such that $ab_k = c$,

$$c = ab_k \sim a0 = 0.$$

Hence, $(A \cup B) \times \{0\} \subseteq \sim$. Of course, $A \cup B$ is a generating set for $\mathcal{B}_M$, and so $\sim$ is the universal relation. $\square$

For the coming results we will concentrate on semigroups $\mathcal{B}_M$ where M satisfies the properties $(P1)$, $(P2)$, and $(P3)$, determining first when a change in the matrix might result in an isomorphic semigroup, and then determining Green's relations to better understand the structure.

Theorem 3.1.2. *If N and M are $A \times B$ matrices satisfying $(P1), (P2), (P3)$, then $\mathcal{B}_N \cong \mathcal{B}_M$ if and only if there exist re-orderings of A and B which take N to M.*

Proof. The reverse implication of this theorem is easily observed to be true, if π and τ are the permutations of A and B respectively then we can combine them to make φ, a permutation of $A \cup B$, and extend this to a homomorphism $\mathcal{B}_N \to \mathcal{B}_M$, which would be bijective and so an isomorphism.

To show the forward implication, suppose that $\mathcal{B}_N$ is isomorphic to $\mathcal{B}_M$, with the isomorphism $\varphi : \mathcal{B}_N \to \mathcal{B}_M$. Obviously $(0)\varphi = 0$ and $(0)\varphi^{-1} = 0$.

Let $a \in A$, by the normal form described above $(a)\varphi = d_1 \dots d_k c_1 \dots c_l$ where $c_1, \dots, c_l \in A$, $d_1, \dots, d_k \in B$ and at least one of k or l is greater than zero.

Of course, a is the image of the isomorphism φ^{-1} when applied to $(a)\varphi$, and so

$$a = (d_1 \dots d_k c_1 \dots c_l)\varphi^{-1} = (d_1)\varphi^{-1} \dots (d_k)\varphi^{-1}(c_1)\varphi^{-1} \dots (c_l)\varphi^{-1}.$$

Suppose $k \neq 0$. We can see that the normal form of $(d_1)\varphi^{-1}$ cannot start with an element of B (and of course it is not 0), so $(d_1)\varphi^{-1} \in A^+$.
Let $(d_1)\varphi^{-1} = a_1 \dots a_m$. By $(P2)$, there exists $\alpha \in A$ such that $\alpha d_1 = d_1$.
Apply φ^{-1};

$$(\alpha)\varphi^{-1} a_1 \dots a_m = (\alpha)\varphi^{-1}(d_1)\varphi^{-1} = (\alpha d_1)\varphi^{-1} = (d_1)\varphi^{-1} = a_1 \dots a_m.$$

This cannot happen, as we can only reduce the length of a string in $\mathcal{B}_N$ when we see elements of A before elements of B, so the left hand side of the above equation is a string of length at least $m+1$ and the right hand side has length m. So this contradicts with the existence of α and in turn d_1, hence $k = 0$ and $(a)\varphi \in A^+$. That is,

$$(a)\varphi = c_1 \dots c_l, \quad a = (c_1)\varphi^{-1} \dots (c_l)\varphi^{-1}.$$

We have seen that the image of an element of A under an isomorphism must be in A^+, but of course φ^{-1} is an isomorphism too and applying this deduction in the opposite direction we see that $(a)\varphi = c_1$, and the image of any element of A under any isomorphism must be an element of A.

A similar argument can be made for B, hence, φ is a bijection from A to itself, and a bijection from B to itself.

Now if we enumerate A and B as described in $(P3)$ for N, so that $a_i b_i = b_{i+1}$, $a_i b_{i+1} = a_{i+1}$, we can enumerate A and B in M as the image of this ordering,

$$a'_i = (a_i)\varphi, \; b'_i = (b_i)\varphi.$$

Suppose that $m_{a'_i, b'_j} \neq (n_{a_i, b_j})\varphi$ for some i, j, then $(a_i b_j)\varphi \neq a'_i b'_j = (a_i)\varphi(b_j)\varphi$. So these two matrices, N and M, must be identical after the re-ordering φ effects on A and B. □

One way to further understand these semigroups is to investigate the structure with respect to Green's relations, and in the next theorem we will see that Green's $\mathcal{L}$ and $\mathcal{R}$ relations are easily described.

Theorem 3.1.3. *Let $M = (m_{ij})_{A\times B}$ be a matrix with entries from $A \cup B \cup \{0\}$ satisfying $(P1), (P2), (P3)$.*

The $\mathcal{L}$-classes of $\mathcal{B}_M$ are:
$\{0\}$*;* $\{\alpha\}$*, for* $\alpha \in A^+$*;* $\{\beta\alpha : \beta \in B^+\}$*, for* $\alpha \in A^*$*.*

Similarly, the $\mathcal{R}$-classes of $\mathcal{B}_M$ are:
$\{0\}$*;* $\{\beta\}$*, for* $\beta \in B^+$*;* $\{\beta\alpha : \alpha \in A^+\}$*, for* $\beta \in B^*$*.*

Proof. We will only prove the statement regarding $\mathcal{L}$-classes, as the $\mathcal{R}$-classes will be apparent by symmetry.

Obviously, $\{0\}$ forms an $\mathcal{L}$-class as 0 is a zero.

Let $\alpha \in A^+$ and suppose $\alpha \mathcal{L} x$ for some $x \in \mathcal{B}_M$.
Then, there exist $u, v \in \mathcal{B}^1_M$ such that $u\alpha = x$, $vx = \alpha$, that is

$$b_u a_u \alpha = u\alpha = x = b_x a_x,$$

$$b_v a_v b_x a_x = vx = \alpha,$$

where $b_u a_u, b_v a_v$ and $b_x a_x$ are the normal forms of u, v and x, respectively.

Given that the normal forms are unique, we can immediately infer that $a_u \alpha = a_x$ from the first equation. From the second we see that $b_v = \epsilon$ and

$a_v b_x \in A^*$, let $c = a_v b_x \in A^*$, and we have

$$\alpha = v b_x a_x = c a_x = c a_u \alpha.$$

Hence, $c a_u = \epsilon$, and so $x = a_x = \alpha$, that is Green's $\mathcal{L}$-relation splits A^+ into singletons.

Let $\alpha \in A^*$, $\beta \in B^+$ and suppose $\beta\alpha\mathcal{L}x$ for some $x \in \mathcal{B}_M$.
Then there exist $u, v \in \mathcal{B}_M^1$ such that $u\beta\alpha = x$ and $vx = \beta\alpha$, that is

$$b_u a_u \beta\alpha = u\beta\alpha = x = b_x a_x,$$

$$b_v a_v b_x a_x = vx = \beta\alpha,$$

following the convention that for $t \in \mathcal{B}_M^1$, we denote the normal form by $b_t a_t$.

As we know that $a_u\beta$ is either in A^+ or in B^+, and similarly $a_v b_x \in A^+ \cup B^+$, from the uniqueness of the normal forms the first equation implies that α is a suffix of a_x and the second implies that a_x is a suffix of α. Hence, $a_x = \alpha$ and

$$u\beta = b_x, \; vb_x = \beta.$$

So two elements are $\mathcal{L}$-related if their A-components are the same, and their B-components are $\mathcal{L}$-related.

Let $b_1, \ldots, b_n, \beta_1, \ldots, \beta_m \in B$ such that $b_x = b_1 \ldots b_n$ and $\beta = \beta_1 \ldots \beta_m$. By repeated application of (P2), there exist $a_1, \ldots, a_m \in A$ such that $a_j\beta_j = a_{j+1}$ for $1 \leq j < m$ and $a_m\beta_m = b_n$.
Then $u = b_1 \ldots b_{n-1} a_1$ will satisfy $u\beta = b_x$, and the same can be done to find $v \in \mathcal{B}_M$ such that $vb_x = \beta$.

Hence, any two elements in B^+ are $\mathcal{L}$-related, and in turn $x\mathcal{L}\beta\alpha$ if and only if $x_a = \alpha$. □

Of course the $\mathcal{L}$- and $\mathcal{R}$-class structure can be easily used to determine the $\mathcal{H}$- and $\mathcal{D}$-classes.

Corollary 3.1.4. *Let $M = (m_{ij})_{A \times B}$ be a matrix with entries from $A \cup B \cup \{0\}$ satisfying $(P1), (P2), (P3)$.*

Then $\mathcal{B}_M$ is $\mathcal{H}$-trivial.

Corollary 3.1.5. *Let $M = (m_{ij})_{A \times B}$ be a matrix with entries from $A \cup B \cup \{0\}$ satisfying $(P1), (P2), (P3)$.*

Then $\mathcal{B}_M$ comprises the following four $\mathcal{D}$-classes:
$D_0 = \{0\}$, $D_a = A^+$, $D_b = B^+$, $D_{ba} = \{\beta\alpha : \alpha \in A^+, \beta \in B^+\}$.

Proof. As $\{0\} = L_0 = R_0$, it is immediate that $\{0\}$ is a $\mathcal{D}$-class.

Let $a \in A$, and let $x \in \mathcal{B}_M$ such that $a\mathcal{D}x$. Then there exists $y \in \mathcal{B}_M$ such that $a\mathcal{L}y\mathcal{R}x$. The theorem tells us that the $\mathcal{L}$-class of a is just $\{a\}$, so $y = a$. The $\mathcal{D}$-class of a is exactly the $\mathcal{R}$-class of a, $D_a = R_a = A^+$.

Let $b \in B$, and let $x \in \mathcal{B}_M$ such that $b\mathcal{D}x$. Then there exists $y \in \mathcal{B}_M$ such that $b\mathcal{R}y\mathcal{L}x$. Again, by the Theorem, $y \in R_b = \{b\}$, and so $y = b$. Hence, the $\mathcal{D}$-class of b is exactly its $\mathcal{L}$-class, $D_b = L_b = B^+$.

Now we will see that all the elements not in $D_0 \cup D_a \cup D_b$ are $\mathcal{D}$-related. Let $\alpha_1, \alpha_2 \in A^+$, and $\beta_1, \beta_2 \in B^+$. Observe that by the theorem, $\beta_1\alpha_1\mathcal{L}\beta_2\alpha_1$ and $\beta_2\alpha_1\mathcal{R}\beta_2\alpha_2$. Hence, $\beta_1\alpha_1\mathcal{D}\beta_2\alpha_2$. □

The remaining Green's relation is $\mathcal{J}$, and the following will show that this relation is as big as possible, which is to say that the semigroup $\mathcal{B}_M$ is 0-simple.

Corollary 3.1.6. *Let $M = (m_{ij})_{A \times B}$ be a matrix with entries from $A \cup B \cup \{0\}$ satisfying $(P1), (P2), (P3)$.*

The $\mathcal{J}$-classes of $\mathcal{B}_M$ are $\{0\}$ and $\mathcal{B}_M \setminus \{0\}$.

Proof. Of course, 0 is a zero and so $\{0\}$ must be a $\mathcal{J}$-class.

Let $a \in A$ and $b \in B$.
By $(P1)$, there exists $b' \in B$ such that $ab' = b$, and by $(P2)$, there exists $a' \in A$ such that $a'b = a$. That is $b \leq_R a \leq_L b$, and in turn $a\mathcal{J}b$.

By $(P1)$, there exists $b'' \in B$ such that $ab'' = b$, and by $(P2)$, there exists $a'' \in A$ such that $a''b = a'$.

Now we see that $a''bab'' = a'b = a$, and in turn $a \leq_J ab$. Clearly, $ba \leq_J a$. Hence $a\mathcal{J}ba$.

As we know that $\mathcal{D} \subseteq \mathcal{J}$, this implies that $D_a \cup D_b \cup D_{ba} \subseteq J_a$, and by Corollary 3.1.5, $D_a \cup D_b \cup D_{ba} = \mathcal{B}_M \setminus \{0\}$. □

Since $\mathcal{D}$-simplicity is our underlying interest, the question arises as to whether we can modify this construction to create a $\mathcal{D}$-simple semigroup.

Of course, for the semigroup to be $\mathcal{D}$-simple would preclude the existence of a zero, and so the matrix would have to have entries only from $A \cup B$. If the matrix satisfied the properties $(P1), (P2)$ and $(P3)$, each with the zero omitted, then the Green's relations would remain as described above (with $\{0\}$ removed from each set of classes).

We could consider removing other unwanted $\mathcal{D}$-classes until we had just one and checking whether it is closed. It is easy to see that D_a and D_b are both free semigroups, $\langle A \rangle$ and $\langle B \rangle$ respectively, and so are not $\mathcal{D}$-simple on their own.

The interesting $\mathcal{D}$-class is D_{ba}. We can see that D_{ba} is closed by letting $\beta_1\alpha_1, \beta_2\alpha_2 \in D_{ba}$ and considering their product, $\beta_1\alpha_1\beta_2\alpha_2 = \beta_1(\alpha_1\beta_2)\alpha_2$. The subproduct $\alpha_1\beta_2$ must be in A^+ or B^+ when reduced to normal form, and so the normal form of the bigger product still has non-trivial A and B components, and so sits in D_{ba}.

We will see that this subsemigroup is not $\mathcal{D}$-simple by considering the length of products:
Let $\beta_1\alpha_1, \beta_2\alpha_2 \in D_{ba}$. Consider their product $(\beta_1\alpha_1)(\beta_2\alpha_2) = \beta_1(\alpha_1\beta_2)\alpha_2$, and note that $\alpha_1\beta_2 \in A^+ \cup B^+$. This implies that the length of the product is at least 3, as $l((\beta_1\alpha_1)(\beta_2\alpha_2)) \geq l(\beta_1) + 1 + l(\alpha_2) \geq 3$. Let $a_1 \in A$ and $b_1 \in B$ and we can immediately see that the element $b_1a_1 \in D_{ba}$ cannot be expressed as a product of any two elements of D_{ba}, and so b_1a_1 cannot be $\mathcal{L}$- or $\mathcal{R}$-related to any other elements in the subsemigroup.

Another way to get a $\mathcal{D}$-simple semigroup from $\mathcal{B}_M$ could be to introduce a new relation which would have to be of the form $\beta\alpha = c$ for some $\alpha \in A^*, \beta \in B^*$, not both trivial, $c \in \mathcal{B}_M \setminus \{\beta\alpha\}$. This would give rise to a congruence on $\mathcal{B}_M$, which is congruence-free as we saw in Proposition 3.1.1, and so this would trivialise the entire semigroup.

3.2 Byleen Monoid

All is not lost, there is a way to change the matrix construction from above to give rise to a $\mathcal{D}$-simple monoid with a very similar structure. To do this we construct a matrix, $M = (m_{ij})_{A\times B}$, indexed by countable sets A and B as before, with entries this time from $A\cup B\cup\{1\}$ where 1 will be an identity. The new presentation is the same as before,

$$\mathcal{B}_M = \langle A, B \mid ab = m_{a,b}, (\forall a \in A, b \in B)\rangle,$$

with the obvious adjustment that 1 is an identity where 0 was a zero before.

Of course this will be a monoid, and when Byleen's full construction is introduced in Section 3.3 we will see that this is in fact a *Byleen monoid extension of the trivial monoid.*

The following theorem will demonstrate that these monoids have a simple unique normal form, similar to that of the semigroups we worked with in the previous section.

Theorem 3.2.1. *Let $M = (m_{i,j})_{A\times B}$ be a matrix over $A \cup B \cup \{1\}$.*

Then $\{\beta\alpha : \alpha \in A^*, \beta \in B^*\}$ is a set of unique normal forms for $\mathcal{B}_M$.

Proof. Let $W = \{\beta\alpha : \alpha \in A^*, \beta \in B^*\}$.

From the presentation of $\mathcal{B}_M$ we see that any element can be expressed as a finite product of elements of $A \cup B$ (with the identity being the empty product), so let $s \in \mathcal{B}_M$ and we know there exist $k \in \mathbb{N}_0$ and $s_1, \ldots, s_k \in A\cup B$ such that $s = s_1 \ldots s_k$.

Clearly, if $k = 0$ then $s = \epsilon = 1 \in W$.

Suppose $k \neq 0$ and $s \notin W$. Then there must be an element of A preceding an element of B in the expression of s, that is there exists $1 \leq i < k$ such that $s_i \in A$ and $s_{i+1} \in B$. Applying the relation $(s_i s_{i+1}, m_{s_i,s_{i+1}})$ we get another expression for s which has length strictly less than k.

It is clear that this process must terminate with an expression for s which is an element of W, hence W is a set of normal forms for $\mathcal{B}_M$.

Uniqueness of the normal forms can be seen in the following way:
For each $c \in \{1\} \cup A$, define $\tau_c : W \to W$ by $(w)\tau_c = w$ if $c = 1$, and $(w)\tau_c = wc$ if $c \in A$.
For $c \in B$ we define $w_c : W \to W$ iteratively. If $w \in B^*$, then $(w)\tau_c = wc$. If $w \notin B^*$ then w has the form $\beta\alpha$ for some $\alpha \in A^+$ and $\beta \in B^*$. Let $a_1, \ldots, a_k \in A$ be such that $\alpha = a_1 \ldots a_k$. Then let $(w)\tau_c = (\beta a_1 \ldots a_{k-1})\tau_{m_{a_k,c}}$.

This iterative definition must terminate for any $w \in W$ and $c \in B$ as with each step the length of the input is reduced by 1, and the functions are explicitly defined on the empty input.

For $u = u_1 \ldots u_n \in (A \cup B)^*$, define $\tau_u = \tau_{u_1} \circ \ldots \circ \tau_{u_n}$, and let $\pi = \{\tau_u : u \in W\}$.

Let $a \in A$ and $b \in B$.
Consider the composition of τ_a and τ_b:

$$(w)\tau_a \circ \tau_b = ((w)\tau_a)\tau_b = (wa)\tau_b = (w)\tau_{ab}.$$

Hence, $\tau_a \circ \tau_b = \tau_{ab}$ and so we have a homomorphism $\varphi : \mathcal{B}_M \to \pi$ such that $(s)\varphi = \tau_s$.

Let $w_1, w_2 \in W$ such that $w_1 \not\equiv w_2$ (in the sense that w_1 and w_2 are not the same word).

If these distinct elements of the set of normal forms represent the same element of $\mathcal{B}_M$ then the functions τ_{w_1} and τ_{w_2} must be equal. Evaluating them at the identity we see that they are not, $(1)\tau_{w_1} = w_1 \neq w_2 = (1)\tau_{w_2}$.

Hence, W is a set of unique normal forms for $\mathcal{B}_M$. □

We modified the construction by removing the zero and including an identity in a bid to find a $\mathcal{D}$-simple semigroup with an otherwise similar structure to those in the previous section, and so it makes sense to modify the properties $(P1), (P2), (P3)$ to accommodate the removal of 0 and inclusion of 1.

To that end, we define the following properties:

(P1′) For every $n \geq 1$, every collection $a_1, \dots, a_n \in A$ of distinct indices, and every collection $c_1, \dots, c_n \in A \cup B \cup \{1\}$ there exist infinitely many distinct $b \in B$ such that $m_{a_i,b} = c_i$ for all $i = 1, \dots, n$.

(P2′) For every $n \geq 1$, every collection $b_1, \dots, b_n \in B$ of distinct indices, and every collection $c_1, \dots, c_n \in A \cup B \cup \{1\}$ there exist infinitely many distinct $a \in A$ such that $m_{a,b_i} = c_i$ for all $i = 1, \dots, n$.

(P3) There exist enumerations $a'_1, a'_2, \dots$ and $b'_1, b'_2, \dots$ of A and B respectively, such that $m_{a'_i,b'_i} = b'_{i+1}$ and $m_{a'_i,b'_{i+1}} = a'_{i+1}$, for all $i = 1, 2, \dots$.

Note that $(P3)$ is exactly the same statement as in the previous section as the zero element played no part in it, and in fact any matrix which satisfies $(P1), (P2), (P3)$ can be used to find a matrix which satisfies $(P1'), (P2'), (P3)$ simply by replacing every 0 in the matrix with a 1, or vice versa.

Theorem 3.2.2. *Let $M = (m_{ij})_{A\times B}$ be a matrix over $A \cup B \cup \{1\}$, satisfying $(P1'), (P2'), (P3)$, let $\alpha \in A^*$, and let $\beta \in B^*$.*

Then in $\mathcal{B}_M$,

$$L_\alpha = \{\beta'\alpha : \beta' \in B^*\},$$

and similarly

$$R_\beta = \{\beta\alpha' : \alpha' \in A^*.$$

Proof. We will prove that the $\mathcal{L}$-class containing α is as described, then the $\mathcal{R}$-class of β can be found by symmetry.

Let $\beta_1\alpha_1 \in \mathcal{B}_M$ such that $\alpha \mathcal{L} \beta_1\alpha_1$.

Then there exist $u, v \in \mathcal{B}_M$ such that $\alpha = u\beta_1\alpha_1$ and $\beta_1\alpha_1 = v\alpha$. The normal form for $u\beta_1\alpha_1$ is α and so α must have α_1 as a suffix, and similarly the second equality implies that α is a suffix of α_1. Hence, $\alpha_1 = \alpha$ and $\beta_1\alpha_1 \in \{\beta_1\alpha : \beta_1 \in B^*\}$, that is $L_\alpha \subseteq \{\beta'\alpha : \beta' \in B^*\}$.

Let $\beta_1 \in B^+$, and let $b_1, \ldots, b_k \in B$ such that $\beta_1 = b_1 \ldots b_k$.
Using $(P2')$ k times, there exist $a_1, \ldots, a_k \in A$ such that $a_k b_k = 1$ and for $1 \leq i < k$ we have $a_i b_i = a_{i+1}$. Now,

$$a_1\beta_1\alpha = a_1 b_1 \ldots b_k\alpha = a_2 b_2 \ldots b_k\alpha = \ldots = a_k b_k\alpha = \alpha.$$

Hence $\beta_1\alpha\mathcal{L}\alpha$, and so $\{\beta_1\alpha : \beta_1 \in B^+\} \subset L_\alpha$, and in turn, $L_\alpha = \{\beta'\alpha : \beta' \in B^*\}$. □

This result was already known as a consequence of Byleen's work (Theorem 2.4, [3]), but proving it here was much simpler than defining the terms used by Byleen and proving that this semigroup met the criteria of his theorem.

Of course from the $\mathcal{L}$ and $\mathcal{R}$ relations we can immediately derive the $\mathcal{H}$ and $\mathcal{D}$ relations.

Corollary 3.2.3. *Let $M = (m_{ij})_{A\times B}$ be a matrix over $A \cup B \cup \{1\}$, satisfying $(P1'), (P2'), (P3)$.*

Then $\mathcal{B}_M$ is $\mathcal{H}$-trivial and $\mathcal{D}$-simple, and in turn simple ($\mathcal{J}$-simple).

Clearly the way we have been presenting these semigroups is an infinite presentation, and an obvious question is: What conditions must the matrix satisfy in order for the semigroup to be finitely presented?

The following theorem demonstrates that if M has any of the properties $(P1), (P2), (P3), (P1')$ or $(P2')$ then $\mathcal{B}_M$ is not finitely presentable.

Theorem 3.2.4. *Let $M = (m_{ij})_{A\times B}$ be a matrix with entries from $A \cup B \cup \{0\}$ or $A \cup B \cup \{1\}$, such that $\mathcal{B}_M$ is finitely generated.*

Then $\mathcal{B}_M$ is finitely presentable if and only if there are only finitely many repeated entries in M.

Proof. The backward implication comes as a consequence of the following reasoning:

Suppose that M has finite submatrix T in which all of the repetition of entries of M occur.
Pick a finite generating set $X \subset A \cup B$, let $X_A = X \cap A,\ X_B = X \cap B$. Let T_A be the subset of A which indexes T and T_B be the subset of B which indexes T.

Let $U = \{a \in A : aB \cap X \neq \emptyset\}$, $V = \{b \in B : Ab \cap X \neq \emptyset\}$, that is U indexes the set of rows of M which contain an element of X, and V indexes the set of columns of M which contain an element of X.

Let $R' = \{(ab, m_{a,b}) : a \in X_A \cup T_A \cup U, b \in X_B \cup T_B \cup V\}$. As X is a generating set for $\mathcal{B}_M$, we can express any element of $\mathcal{B}_M$ in terms of elements of X. Let $R = \{(u, v) : (ab, m_{a,b}) \in R', u, v \in X^+, u = ab, v = m_{a,b}\}$, and note that R is finite.

Now, we will see that $\mathcal{B}_M = \langle X \mid R \rangle$ is a finite presentation of $\mathcal{B}_M$. Obviously, X is finite and will generate all of $\mathcal{B}_M$.

Suppose there is a distinct pair of words $u, v \in X^+$ such that $u = v$ in $\mathcal{B}_M$ but not as a consequence of R. Then there exists $a_1, a_2 \in A,\ b_1, b_2 \in B$ such that $a_1 b_1 = a_2 b_2$ in $\mathcal{B}_M$, but not as a consequence of R. However, $a_1 b_1 = a_2 b_2$ implies $m_{a_1,b_1} = m_{a_2,b_2}$ and T was chosen to contain all repeated entries of M, so $a_1, a_2 \in T_A,\ b_1, b_2 \in T_B$ and $(a_1 b_1, m_{a_1,b_1}), (a_2 b_2, m_{a_2,b_2}) \in R$, a contradiction.

For the forward implication: Let $\mathcal{B}_M$ be finitely presentable. Of course $\mathcal{B}_M$ is finitely generated, and by Proposition 1.4.1, there exists a finite $X \subset A \cup B$ such that X is a generating set for $\mathcal{B}_M$.

We have a valid presentation $\mathcal{B}_M = \langle X \mid ab = m_{a,b}, (\forall a \in A, b \in B) \rangle$, where every element of $A \cup B$ in the relations has been rewritten in terms of the generating set X.

Since $\mathcal{B}_M$ is finitely presentable, by Proposition 1.4.4 there exists a finite subset R of the relations such that $\mathcal{B}_M = \langle X \mid R \rangle$.

Let $A' = \{a \in A : (\exists b \in B : (ab, m_{a,b}) \in R)\}$, $B' = \{b \in B : (\exists a \in A : (ab, m_{a,b}) \in R)\}$.

Let $A'' = A' \cup (X \cap A)$, $B'' = B' \cup (X \cap B)$ and let $T = (m_{ij})_{A'' \times B''}$ be the submatrix of M indexed by A'' and B''.
Since $X \subset A'' \cup B''$ and $R \subseteq \{(ab, m_{a,b}) : a \in A'', b \in B''\}$, we have the following finite presentation for $\mathcal{B}_M$:

$$\mathcal{B}_M = \langle A'', B'' \mid ab = m_{a,b}, (\forall a \in A'', b \in B'') \rangle.$$

Note that all the relations here have the form $ab = c$ where $a \in A''$, $b \in B''$ and either $c \in A'' \cup B''$ or $c \equiv ab$. Clearly if $c \equiv ab$ then it is a redundant relation.

Note too that any expression for an element of $(A \cup B) \setminus (A'' \cup B'')$ as a product of elements from the generating set $A'' \cup B''$ must start with an element of A'' and end with an element of B''.

Suppose there is a repeated entry in the matrix M, that is $m_{a_1,b_1} = m_{a_2,b_2}$ for some $(a_1, b_1) \neq (a_2, b_2)$. Then, when expressed in terms of the generating set $A'' \cup B''$, we must get $a_1b_1 = a_2b_2$ as a consequence of the relations from R. Which is to say that for either $i = 1$ or $i = 2$, there is a suffix, α, of some expression for a_i and prefix, β, of some expression for b_i such that $(\alpha\beta, c) \in R$ for some c. Without loss of generality let it be for $i = 1$.

Any expression for an element of A either ends with an element of B'' or is in A'', similarly, any expression for an element of B either begins with an element of A'' or is in B'', hence $a_1 \in A''$, $b_1 \in B''$.

Since $a_2b_2 = a_1b_1$, $a_1b_1 \not\equiv a_2b_2$, and a_1b_1 has length at most 2 in terms of the generating set, there must be an expression for a_2 which ends with an element from A'' and an expression for b_2 which begins with an element from B'', arguing as above, $a_2 \in A''$, $b_2 \in B''$.

Hence, the repeated entry of M occurred in the finite submatrix T. $\square$

A consequence of this is that if $\mathcal{B}_M$ is finitely presentable, then there exists a finite matrix N such that $\mathcal{B}_M$ is isomorphic to $\mathcal{B}_N$.

3.3 Byleen Extensions

Now we are ready to investigate the full construction which Byleen created, first introduced in [4].

The full Byleen construction takes the following components:

- a semigroup S,
- two disjoint non-empty sets A and B,
- actions of S on A and B from the right and left, respectively: $\rho : A \times S \to A, (a, s) \mapsto a^s$; $\sigma : S \times B \to B, (s, b) \mapsto {}^s b$,
- and $M = (m_{ij})_{A \times B}$ a matrix with entries in $A \cup B \cup S$ which respects the actions, which is to say $m_{a^s,b} = m_{a,{}^s b}$ for all $a \in A, b \in B, s \in S$.

These combine to form the *Byleen extension of S by the matrix M and actions σ and ρ*, denoted $\mathcal{C}(S; \sigma, \rho; M)$. The semigroup is defined by the following presentation:

$$\mathcal{C}(S; \sigma, \rho; M) = \langle S, A, B | R, as = a^s, sb = {}^b s, ab = m_{a,b}, (\forall a \in A, b \in B, s \in S) \rangle,$$

where R denotes the relations of the semigroup S, that is $R = \{(st, u) : (s, t, u \in S : u = st \text{ in } S)\}$.

Byleen used α and β to represent the actions, but here we are using ρ and σ in order to maintain the convention that α denotes a word over A and β a word over B.

The construction introduced in Section 3.1 on first glance might look like it fits this construction with $S = \{0\}$, however this 0 acted as a zero for the whole construction. What was used was $\mathcal{B}_M = \mathcal{C}(\{0\}; \sigma, \rho; M')/\sim$ where M' is the matrix M with an extra row and column indexed by new elements a_0 and b_0, respectively, with 0 in all entries, σ and ρ are both constant mappings taking all of their domain to b_0 and a_0, and $\sim$ is the congruence generated by $\{(0a, 0), (b0, 0) : a \in A \cup \{a_0\}, b \in B \cup \{b_0\}\}$.

The version introduced in Section 3.2, which permitted 1 to occur in the matrix is an example of a more refined construction, the *Byleen monoid extension* which is possible when the semigroup used is a monoid and the actions ρ and σ are monoid actions, which is to say that the identity corresponds to the actions which fixed all elements.

Let $\sim$ be the congruence on $\mathcal{C}(S;\sigma,\rho;M)$ generated by $\{(1a,a),(b1,b) : a \in A, b \in B\}$, note that it is already true that $a1 = a$ and $1b = b$ for all $a \in A, b \in B$ if ρ and σ are monoid actions. Now we can define the *Byleen monoid extension*:

$$\mathcal{C}^1(S;\sigma,\rho;M) = \mathcal{C}(S;\sigma,\rho;M)/\sim .$$

This is not as complicated as it may seem, the congruence only serves to allow the identity of S to act as an identity for the extension.

It is now clear to see that in Section 3.2 when the matrix M had entries from $A \cup B \cup \{1\}$, the semigroup $\mathcal{B}_M$ was an example of a Byleen monoid extension of the trivial monoid by the matrix M.

The following unique normal forms were found by Byleen [4], but are easy to confirm following a similar proof to Theorem 3.2.1.

Theorem 3.3.1 (Byleen)**.** *Any Byleen extension $\mathcal{C}(S;\sigma,\rho;M)$ admits the unique normal form $B^+A^* \cup B^*A^+ \cup B^*SA^*$.*

Any Byleen monoid extension $\mathcal{C}^1(S;\sigma,\rho;M)$ admits the unique normal form B^*SA^*.

As a consequence of this, from now on if we have $x \in \mathcal{C}(S;\sigma,\rho;M)$ we will use the convention $x = b_x s_x a_x$, with the implied condition that $b_x \in B^*$, $s_x \in S \cup \{\epsilon\}$, $a_x \in A^*$, and they are not all trivial.

Similarly the convention for $x \in \mathcal{C}^1(S;\sigma,\rho;M)$ will be $x = b_x s_x a_x$ with the implied condition that $b_x \in B^*$, $s_x \in S$, and $a_x \in A^*$.

It is clear to see from the unique normal forms that $S \hookrightarrow \mathcal{C}(S;\sigma,\rho;M)$, and $S \hookrightarrow \mathcal{C}^1(S;\sigma,\rho;M)$ (if it is defined), for any σ,ρ and M, and so the

Byleen extension and Byleen monoid extension really are extensions of the base semigroup S.

The motivation behind the investigation into this construction was to see whether this construction can be used to embed finitely presented semigroups in finitely presented $\mathcal{D}$-simple semigroups, so we will begin by investigating the Green's relations of these semigroups.

The following theorems will be useful to better understand the $\mathcal{L}$- and $\mathcal{R}$-class structure of Byleen extensions.

Theorem 3.3.2. *Let $C = \mathcal{C}(S; \sigma, \rho; M)$, and let $\alpha \in A^+$, $s \in S$ and $\beta \in B^*$. Then:*

(i) $R_{\beta s \alpha} \subseteq \{\beta s_1 \alpha_1 : s_1 \mathcal{R} s \text{ in } S,\ \alpha_1 \in A^*\}$,

(ii) $R_{\beta\alpha} \subseteq \{\beta\alpha_1 : \alpha_1 \in A^+\}$,

(iii) *If* $\beta \neq \epsilon$, *then* $R_\beta = \{\beta\}$.

The conditions imposed on the $\mathcal{L}$-classes can be found symmetrically.

Proof. Let $x \in C$.

Suppose $x \mathcal{R} \beta s \alpha$.
That is, there exist $u, v \in C^1$ such that

$$b_x s_x a_x = x = \beta s \alpha u = \beta s \alpha b_u s_u a_u, \ \beta s \alpha = xv = b_x s_x a_x b_v s_v a_v.$$

From the first, considering the unique normal form, we see that β is a prefix of b_x and from the second, b_x is a prefix of β, hence $b_x = \beta$.

As β is a common prefix to both equalities, and we have left cancellativity for elements of B, we have

$$s_x a_x = s \alpha b_u s_u a_u, \ s\alpha = s_x a_x b_v s_v a_v$$

Clearly, $\alpha b_u, a_x b_v \notin B^+$, and so $\alpha b_u s_u, a_x b_v s_v \in A^+ \cup S$. If either is in A^+, then $s_x = s$, and if both are in S then $s_x \mathcal{R} s$ in S.

Hence, $R_{\beta s\alpha} \subseteq \{\beta s_1 \alpha_1 : s_1 \mathcal{R} s \text{ in } S,\ \alpha_1 \in A^*\}$.

Suppose $x\mathcal{R}\beta\alpha$. Then for some $u, v \in C$,

$$b_x s_x a_x = x = \beta\alpha u = \beta\alpha b_u s_u a_u,\ \beta\alpha = b_x s_x a_x v = b_x s_x a_x b_v s_v a_v.$$

Again we see that β is a prefix of b_x which is a prefix of β, and so $b_x = \beta$. In turn $s_x a_x b_v s_v a_v = \alpha$, which implies that $s_x = \epsilon$, and $a_x = \alpha b_u s_u a_u$. As α is non-trivial, we see that a_x is non-trivial too.

Hence, $R_{\beta\alpha} \subseteq \{\beta\alpha_1 : \alpha_1 \in A^+\}$.

Suppose that $\beta \in B^+$ and $x\mathcal{R}\beta$. Then for some $u, v \in C$

$$b_x s_x a_x = \beta u = \beta b_u s_u a_u,\ \beta = b_x s_x a_x v = b_x s_x a_x b_v s_v a_v.$$

As before, it is immediate that $b_x = \beta$. Then the second equality implies that $\epsilon = s_x a_x b_v s_v a_v$, which in turn implies that $s_x, a_x = \epsilon$.

Hence, $R_\beta = \{\beta\}$. □

The following corollary demonstrates that even in the best case scenario, where the $\mathcal{L}$- and $\mathcal{R}$-classes are as big as possible, $\mathcal{C}(S; \sigma, \rho; M)$ is never $\mathcal{D}$-simple.

Corollary 3.3.3. *Let $C = \mathcal{C}(S; \sigma, \rho, M)$ be a Byleen extension.*

Then C is not $\mathcal{D}$-simple.

Proof. Let $a \in A$, $b \in B$. Suppose that C is $\mathcal{D}$-simple. Then there exists $\beta r\alpha \in C$ such that $a\mathcal{R}\beta r\alpha\mathcal{L}b$. Which implies that $\beta r\alpha \in R_a \cap L_b$.

By Theorem 3.3.2, $R_a \subseteq A^+$, and $L_b \subseteq B^+$, and so

$$\beta r\alpha \in R_a \cap L_b \subseteq A^+ \cap B^+ = \emptyset.$$

A clear contradiction to the existence of $\beta r\alpha$, and in turn to the $\mathcal{D}$-simplicity of C. □

As the normal forms simplify from $\mathcal{C}(S; \sigma, \rho; M)$ to $\mathcal{C}^1(S; \sigma, \rho; M)$, the upper bounds on the $\mathcal{L}$- and $\mathcal{R}$-classes simplify too.

Theorem 3.3.4. *Let $C = \mathcal{C}^1(S; \sigma, \rho; M)$, and let $\beta \in B^*$, $s \in S$, $\alpha \in A^*$. Then:*

$$R_{\beta s \alpha} \subseteq \{\beta s_1 \alpha_1 : s_1 \mathcal{R} s \text{ in } S,\ \alpha_1 \in A^*\},$$

$$L_{\beta s \alpha} \subseteq \{\beta_1 s_1 \alpha : s_1 \mathcal{L} s \text{ in } S,\ \beta_1 \in B^*\}.$$

Proof. Suppose $\beta_1 s_1 \alpha_1 \mathcal{R} \beta s \alpha$ for some $\beta_1 \in B^*$, $s \in S$, $\alpha_1 \in A^*$. Then there exists $u, v \in C$ such that

$$\beta_1 s_1 \alpha_1 = \beta s \alpha u = \beta s \alpha b_u s_u a_u,\ \beta s \alpha = \beta_1 s_1 \alpha_1 v = \beta_1 s_1 \alpha_1 b_v s_v a_v.$$

Reducing these into normal form we see that β is a prefix of β_1 which is a prefix of β, and so $\beta_1 = \beta$.

Either $\alpha b_u \in A^+$, in which case $s_1 = s$, or $\alpha b_u \in S$ and $s_1 = s(\alpha b_u)s_u$, $s_1 \leq_R s$ in S. If the former is the case, then $s_1 \mathcal{R} s$, or if the latter is the case, then either $s_1 = s$ or $\alpha_1 b_v \in S$ and $s \leq_R s_1$ in S, which combined with $s_1 \leq_R s$ implies $s_1 \mathcal{R} s$ in S.

In any case, $s_1 \mathcal{R} s$. Hence, $R_{\beta s \alpha} \subseteq \{\beta s_1 \alpha_1 : s_1 \mathcal{R} s \text{ in } S,\ \alpha_1 \in A^*\}$.

The argument works symmetrically to show that the condition on the $\mathcal{L}$-classes holds. □

In Theorem 3.2.2 we saw that one can impose conditions on the matrix that will make $\mathcal{C}^1(\{1\}; \sigma, \rho; M)$ $\mathcal{D}$-simple, the following theorem will demonstrate that the choice of the semigroup is important too.

Theorem 3.3.5. *Let $C = \mathcal{C}^1(S; \sigma, \rho; M)$ be a Byleen monoid extension of S. If C is $\mathcal{D}$-simple, then so too is S.*

Proof. Suppose that C is $\mathcal{D}$-simple and let $s, t \in S$.

Then in C, we have $s \mathcal{D} t$. That is, there exists $\beta \in B^*$, $\alpha \in A^*$, $r \in S$ such that

$$s \mathcal{R} \beta r \alpha \mathcal{L} t;$$

in other words

$$\beta r \alpha \in R_s \cap L_t.$$

By Theorem 3.3.4, $R_s \subseteq \{s_1\alpha_1 : s_1\mathcal{R}s \text{ in } S, \alpha_1 \in A^*\}$, and $L_t \subseteq \{\beta_1 s_1 : s_1\mathcal{L}t \text{ in } S, \beta_1 \in B^*\}$. Hence,

$$\beta r\alpha \in \{s_1\alpha_1 : s_1\mathcal{R}s \text{ in } S, \alpha_1 \in A^*\} \cap \{\beta_1 s_1 : s_1\mathcal{L}t \text{ in } S, \beta_1 \in B^*\}.$$

Clearly, $\beta = \alpha = \epsilon$, and $s\mathcal{R}r\mathcal{L}t$ in S, that is $s\mathcal{D}t$ in S, and so S is $\mathcal{D}$-simple. □

This alone does not prove that this construction cannot be used to embed an arbitrary finitely presented semigroup into a finitely presented $\mathcal{D}$-simple Byleen monoid, perhaps it may be possible to embed our finitely presented semigroup S into a (non-finitely presentable) $\mathcal{D}$-simple monoid T, and then find σ, ρ, M such that $\mathcal{C}^1(T; \sigma, \rho; M)$ is finitely presentable and $\mathcal{D}$-simple.

So this leads to the question: Given a non-finitely presentable $\mathcal{D}$-simple semigroup, T, can we construct $\mathcal{C}^1(T; \sigma, \rho; M)$ which is finitely presentable?

Any finitely presentable semigroup must be finitely generated, and so we begin by investigating when a Byleen extension might be finitely generated.

For the purposes of finding a finite subset which will generate all of A and B with S we will derive partially ordered sets (posets) from the actions of S on A and B and consider maximal elements of this poset. This will help to find conditions which will ensure $\mathcal{C}(S; \sigma, \rho; M)$ is finitely generated, regardless of the content of the matrix M.

Definition 3.3.6. We will say a partially ordered set $(P, \geq)$ is *bounded above* if for any $p \in P$ there exists a maximal element $m \in P$ such that $m \geq p$. Also, we will say that the poset is *finitely bounded above* if it is bounded above, and there are only finitely many maximal elements of P.

Define a relation $\rightarrow \subseteq (A \cup B) \times (A \cup B)$ on elements of A by $a \rightarrow c$ if and only if there exists $s \in S$ such that $a^s = c$, or $a = c$, and on elements of B by $b \rightarrow c$ if and only if there exists $s \in S$ such that ${}^s b = c$, or $b = c$.

This is reflexive and transitive, thus a pre-order. We can construct a partially ordered set from this: Let $\sim$ be the equivalence relation on $A \cup B$

given by $c \sim d$ if and only if $c \to d$ and $d \to c$. Define $\geq$ on $(A \cup B)/\!\sim$ by $c/\!\sim\, \geq d/\!\sim$ if and only if $c \to d$. This is a partial order on the set $(A \cup B)/\!\sim$.

Lemma 3.3.7. *If the poset $((A \cup B)/\!\sim, \geq)$ is finitely bounded above and S is finitely generated then $\mathcal{C}(S; \sigma, \rho; M)$ is finitely generated.*

Proof. Let $\mathcal{C}(S; \sigma, \rho; M)$ be a Byleen semigroup as before, such that S is finitely generated and $((A \cup B)/\!\sim, \geq)$ is finitely bounded above.

That is, for all $a \in A, b \in B, c \in A \cup B$:

- $a \sim c$ if and only if $c \in A$ and there exists $s_1, s_2 \in S^1$ such that $a = c^{s_1}$, $c = a^{s_2}$,
- $b \sim c$ if and only if $c \in B$ and there exists $s_1, s_2 \in S^1$ such that $b = {}^{s_1}c$, $c = {}^{s_2}b$,
- $(a/\!\sim) \geq (c/\!\sim)$ if and only if there exists $s \in S^1$ such that $c = a^s$,
- $(b/\!\sim) \geq (c/\!\sim)$ if and only if there exists $s \in S^1$ such that $c = {}^s b$,
- there exists maximal element $(d/\!\sim)$ such that $(d/\!\sim) \geq (c/\!\sim)$,
- and there are only finitely many maximal elements.

Let X be a finite generating set for S. Let N be a set of representatives of the maximal elements of $((A \cup B)/\!\sim, \geq)$. Then $S \subseteq \langle X \rangle$, and $A \cup B \subset \langle S, N \rangle$, and so

$$\mathcal{C}(S; \sigma, \rho; M) = \langle S, A, B \rangle \subseteq \langle X, N \rangle \subseteq \mathcal{C}(S; \sigma, \rho; M).$$

As X and N are both finite sets, $\mathcal{C}(S; \sigma, \rho; M)$ is finitely generated. $\square$

Obviously, applying this lemma in the event that S is a monoid and σ, ρ are monoid actions, we see that if the poset $((A \cup B)/\!\sim, \geq)$ is finitely bounded above and S is finitely generated then $\mathcal{C}^1(S; \sigma, \rho; M)$ is finitely generated as a consequence of the fact that $\mathcal{C}^1(S; \sigma, \rho; M)$ is a homomorphic image of $\mathcal{C}(S; \sigma, \rho; M)$.

This is the only way we can guarantee finite generation in either case without knowing about the structure of the matrix M. Similarly we can put conditions on the structure of M which will ensure finite generation of $\mathcal{C}(S;\sigma,\rho;M)$ for any semigroup S and actions σ,ρ.

For example, if the matrix has property $(P3)$ as defined in Section 3.1 and every element of S occurs at least once in the matrix, then all of $A \cup B$ will be generated by the elements a_1', b_1' as defined in $(P3)$, and $S \subset AB$, and so $\mathcal{C}(S;\sigma,\rho;M) = \langle S, A, B\rangle \subseteq \langle A, B\rangle \subseteq \langle a_1', b_1'\rangle$.

None of the conditions so far are absolutely necessary to ensure finite generation of $\mathcal{C}(S;\sigma,\rho;M)$ as we will see in the following example.

Example 3.3.8. There exists a Byleen extension $\mathcal{C}(S;\sigma,\rho;M)$ such that:

(i) $\mathcal{C}(S;\sigma,\rho;M)$ is finitely generated,

(ii) $((A \cup B)/{\sim}, \geq)$ is not finitely bounded above,

(iii) S is not finitely generated,

(iv) M does not satisfy $(P3)$, and

(v) there exist elements of S which do not occur in M.

In order to find such a Byleen extension, let T and V be disjoint semigroups such that T is finite and V is not finitely generated, and let S be the semigroup with elements $T \cup V \cup \{0\}$ with the operation defined such that $t_1.t_2 = t_1t_2$ for all $t_1, t_2 \in T$, and $v_1.v_2 = v_1v_2$ for all $v_1v_2 \in V$, and $t.v = v.t = 0$ for all $t \in T, v \in V$. Note that (iii) is satisfied.

Let A_1, A_2, B_1, B_2 be disjoint countable sets, and let $A = \{a_0\} \cup A_1 \cup A_2$, $B = \{b_0\} \cup B_1 \cup B_2$ where a_0 and b_0 are not in any of the other sets. Let $\varphi_A : A_1 \to A_2$ and $\varphi_B : B_1 \to B_2$ be bijections.

Let $\rho : A \times S \to A$ be defined by

- $(a, v)\rho = a_0$, for all $a \in A$ and $v \in V \cup \{0\}$,
- $(a, t)\rho = (a)\varphi_A$ for all $a \in A_1$ and $t \in T$,
- $(a, t)\rho = a$ for all $a \in A_2$ and $t \in T$, and

- $(a_0, s) = a_0$ for all $s \in S$.

Let $\sigma : S \times B \to B$ be defined by

- $(v, b)\sigma = b_0$, for all $v \in V \cup \{0\}$ and $b \in B$,
- $(t, b)\sigma = (b)\varphi_B$ for all $t \in T$ and $b \in B_1$,
- $(t, b)\sigma = b$ for all $t \in T$ and $b \in B_2$, and
- $(s, b_0) = b_0$ for all $s \in S$.

It is easy to verify that ρ and σ are right and left semigroup actions, respectively.

Note that (ii) is satisfied due to the fact that $a/\!\sim\, = \{a\}$ is a maximal element of $((A \cup B)/\!\sim, \geq)$ for each $a \in A_1$.

Let $N = (n_{i,j})_{A_1 \times B_1}$ be a matrix with entries from $A_1 \cup B_1 \cup V$ such that there exist enumerations $A_1 = \{a_1, a_2, \ldots\}$ and $B_1 = \{b_1, b_2, \ldots\}$ such that $n_{a_i,b_i} = b_{i+1}$ and $n_{a_i,b_{i+1}} = a_{i+1}$ for $i = 1, 2, \ldots$, and such that every element of V occurs in the matrix at least once.

Let M be the unique $A \times B$ matrix determined by the conditions;

- $m_{a,b} = n_{a,b}$ for all $a \in A_1, b \in B_1$,
- $m_{a_0,b_0} = 0$, and
- $m_{a^s,b} = m_{a,{}^s b}$ for all $a \in A$, $s \in S$ and $b \in B$.

The entries of the matrix M are $A_1 \cup B_1 \cup V \cup \{0\}$, and so clearly we have (v), and M does not have the property $(P3)$ (even though the submatrix N did), that is we have satisfied (iv).

It remains to find a finite generating set for $\mathcal{C}(S; \sigma, \rho; M)$:
By the conditions on the matrix N above, and recalling that N is exactly the submatrix of M indexed by A_1 and B_1, there exist $a_1 \in A_1$ and $b_1 \in B_1$ such that $A_1 \cup B_1 \subseteq \langle a_1, b_1 \rangle$.

As every element of V occurs in the $A_1 \times B_1$ submatrix of M, we see that $V \subseteq \langle A_1, B_1 \rangle$, and so $V \subseteq \langle a_1, b_1 \rangle$.

Consider the subsemigroup generated by $T \cup \{a_1, b_1\}$. As we have seen, it contains V, and as $S = \langle T, V \rangle$, it contains S. Recall that every element of A_2 occurs as the image of $(a,t)\rho$ for some $a \in A_1$ and $t \in T$, and so we also have all of A_2. Symmetrically, σ implies that B_1 and T generates all of B_2. Similarly, a_0 and b_0 can be found using the actions.

Hence,

$$\mathcal{C}(S; \sigma, \rho; M) = \langle A_1, A_2, a_0, B_1, B_2, b_0, S \rangle \subseteq \langle A_1, B_1, S \rangle \subseteq \langle a_1, b_1, T \rangle.$$

As T is finite, this demonstrates that $\{a_1, b_1\} \cup T$ is a finite generating set, and so we have satisfied (i).

This example illustrates the fact that necessary and sufficient conditions for finite generation of Byleen extensions cannot refer exclusively to the actions, semigroup or matrix.

Moving on to finite presentability we see that any Byleen extension of a non-finitely presentable monoid is not finitely presentable.

Theorem 3.3.9. *Let S be a finitely generated, but not finitely presentable monoid.*

Any Byleen monoid extension of S is not finitely presentable.

Proof. Let X be a finite generating set for S, and let R be a (necessarily infinite) set of relations such that $S = \langle X \mid R \rangle$. Let $C = \mathcal{C}^1(S; \sigma, \rho; M)$ be a Byleen monoid extension of S.

We have the following presentation

$$C = \langle X, A, B \mid R, ax = a^x, xb = {}^x b, ab = m_{a,b}, (\forall a \in A, b \in B, x \in X) \rangle.$$

Suppose C is finitely presentable. Then by Proposition 1.4.4 there exists a finite subset of the relations which is sufficient to define the semigroup, and so we must be able to find a finite subset $R_0 \subset R$ such that

$$C = \langle X, A, B \mid R_0, ax = a^x, xb = {}^x b, ab = m_{a,b}, (\forall a \in A, b \in B, x \in X) \rangle.$$

Of course every relation of S must still hold in C, and so we see that every element of $R \setminus R_0$ comes as a consequence of $R_0 \cup \{ax = a^x, xb = {}^x b, ab = m_{a,b} : a \in A, b \in B, x \in X\}$.

Since S is not finitely presentable, there exists $(u, v) \in R$ such that (u, v) does not come as a consequence of R_0 alone.

Using relations from $R_0 \cup \{ax = a^x, xb = {}^x b, ab = m_{a,b} : a \in A, b \in B, x \in X\}$ we get a finite sequence

$$u = u_1 = u_2 = \ldots = u_n = v.$$

Observe that u and v are strings over X, but not all of u_i are (if they were then $u = v$ as a result of R_0 alone).

Consider the first step at which u_i contains letters not from X; the previous string must have contained a subword which occurred in $\{ax = a^x, xb = {}^x b, ab = m_{a,b} : a \in A, b \in B, x \in X\}$, but the only possibility for this is one of the relations of the form $ab = m_{a,b}$ because $m_{a,b}$ can be in S. That is, $u_{i-1} = w_1 w_2 w_3$ for some $w_1, w_3 \in X^*$ and $w_2 \in X^+$, and $v_i = w_1 a_1 b_1 w_3$ where $a_1 \in A$ and $b_1 \in B$ such that $m_{a_1,b_1} = w_2$.

From here we can use the relations of the form $ax = a^x$ and $xb = {}^x b$ to get

$$w_1 w_2 w_3 = w_1 a_2 x_1 \ldots x_k b_2 w_3,$$

where $a_2 \in A$, $B_2 \in B$, and $x_1, \ldots, x_k \in X$. The other relations which can change this subword are those of the form $ab = m_{a,b}$ which could replace any subword with ab for some $a \in A$, $b \in B$. Ultimately, $w_1 w_2 w_3 = w_1 a_3 y_1 \ldots y_n b_3 w_3$ for $a_3 \in A$, $b_3 \in B$, and $y_1, \ldots, y_n \in A \cup B \cup X$.

From here no relations can straddle $w_1 a_3$ or $b_3 w_3$ as there are no relations which end with an element of A (and every prefix of $a_3 y_1 \ldots y_n$ is in A), there are no relations which begin with an element of B (and every suffix of $y_1 \ldots y_n b_3$ is an element of B). That is, the left most element of A and the right most element of B act as buffers.

Of course, the sequence must return to an element of X^+ at some point, as $v \in X^+$, but whichever relations we use, the only way to remove elements of A and B is to have them juxtaposed and apply a rule of the form $ab = m_{a,b}$.

So we replaced a subword w_2 with ab for some $a \in A, b \in B$, then expanded this and contracted to get $a'b'$ for some $a' \in A, b' \in B$ not necessarily the same as a, b. The construction puts one big restriction on the structure of the matrix M, that is, $m_{a^s,b} = m_{a,{}^sb}$ and so $m_{a,b} \equiv m_{a',b'}$ and we have not changed the string over X.

This implies that the $u = u_1 = u_2 = \ldots = u_n = v$ contained a subsequence which held true using only relations from R_0 and we have a contradiction. □

Earlier we saw that if S is not $\mathcal{D}$-simple then any Byleen extension of S could not be $\mathcal{D}$-simple, and now we see that the same can be said for finite presentability.

There are also conditions which can be put on the underlying matrix which will result in a Byleen extension not being finitely presentable.

Let S be a semigroup and M be an $A \times B$ matrix with entries from $A \cup B \cup S$ which respects that actions ρ and σ of the semigroup on the sets A and B, that is $m_{a^s,b} = m_{a,{}^sb}$ for all $a \in A$, $b \in B$, $s \in S$.

Let $\pi \subseteq (A \times B) \times (A \times B)$ be the relation defined by $(a_1, b_1)\pi(a_2, b_2)$ if and only if there exists $s \in S^1$ such that $a_1^s = a_2$ and $b_1 = {}^sb_2$.

Let $\tau \subseteq (A \times B) \times (A \times B)$ be the symmetric and transitive closure of π and note that τ is the equivalence relation which describes exactly which positions in the matrix M must be equal as a consequence of the actions ρ and σ.

Theorem 3.3.10. *Let $C = \mathcal{C}(S; \sigma, \rho; M)$, a Byleen extension.*

If C is finitely presentable, then only finitely many repetitions can occur in M/τ.

Proof. Suppose C is finitely presentable.

By Theorem 3.3.9, S is finitely presented, let $S = \langle X \mid R \rangle$ be a finite presentation.

As C is finitely generated, there exist finite $A' \subset A$ and $B' \subset B$ such that $C = \langle X, A', B' \rangle$.

Let ρ denote the set of relations (as, a^s) for all $a \in A$ and all $s \in S$, similarly let σ denote the set of all relations $(sb, {}^sb)$, and let M denote the set of all the relations $(ab, m_{a,b})$, all expressed in terms of the generating set $X \cup A' \cup B'$. Then,

$$C = \langle X, A', B' \mid R, \rho, \sigma, M\rangle.$$

As C is finitely presentable, there exists a finite subset of these relations which will afford a finite presentation, that is $C = \langle X, A', B' \mid R, \rho', \sigma', M'\rangle$ for some finite subsets $\rho' \subset \rho$, $\sigma' \subset \sigma$, and $M' \subset M$.

In order to simplify some coming deductions we need a slightly larger (but still finite) presentation, which will still be valid as a consequence of containing this one:

Let X' be the union of X with the set of all elements of S which occur in any relation from $\rho' \cup \sigma' \cup M'$.

Similarly, let A'' be the union of A' with the set of all elements of A which occur in any relation from $\rho' \cup M'$, and let B'' be the union of B' with the set of all elements of B which occur in any relation from $\sigma' \cup M'$.

In order to expand the generating set and keep a valid presentation, it is necessary to extend the set of relations to identify all the new generators: For each $w \in (X' \cup A'' \cup B'') \setminus (X \cup A' \cup B')$ let v_w be an expression for w in terms of $X \cup A' \cup B'$, and let $T = \{(w, v_w) : w \in (X' \cup A'' \cup B'') \setminus (X \cup A' \cup B')\}$.

This new set of generators and relations gives rise to the following presentation,

$$C = \langle X', A'', B'' \mid R, \rho', \sigma', M', T\rangle.$$

Let $M'' = \{(axb, m_{a^x,b}) : a \in A'', b \in B'', x \in X'\}$, where each $m_{a^x,b}$ is expressed in terms of $X' \cup A'' \cup B''$, and note that $M' \subseteq M''$. Now the following is a finite presentation,

$$C = \langle X', A'', B'' \mid R, \rho', \sigma', M'', T\rangle.$$

Suppose two entries of the matrix M are the same, but not as a consequence of the actions, that is there exist $a_1, a_2 \in A$, $b_1, b_2 \in B$ such that $m_{a_1,b_1} = m_{a_2,b_2}$ and $((a_1, b_1), (a_2, b_2)) \notin \tau$.

Then $a_1b_1 = a_2b_2$ can be shown using finitely many applications of relations from $R \cup \rho' \cup \sigma' \cup M'' \cup T$, and not exclusively $R \cup \rho' \cup \sigma' \cup T$.

For $i \in \{0, 1\}$, there must exists an expression for a_i in terms of the generating set $X' \cup A'' \cup B''$ which has an element a of A, as a suffix, and an expression for b_i in terms of the generating set $X' \cup A'' \cup B''$ which has an element b of B, as a prefix, such that $(ab, m_{a,b}) \in M''$. Without loss of generality, let $i = 1$.

The fact that the product can be evaluated by M'' implies that $a \in A''S$ and $b \in SB''$, and that the product of the two S-components is in X' or is the identity. Let $a = a_3s_1$ and $b = s_2b_3$, such that $a_3 \in A''$, $b_3 \in B''$, $s_1, s_2 \in S$ and $s_1s_2 = x \in X' \cup \{1\}$. Clearly, $(a, b)\tau(a_3x, b_3)$.

For an expression of an element a_1 of A, to have a suffix a in A, means that $a_1 = a$, similarly $b_1 = b$.

Hence, $(a_1, b_1)\tau(a_3x, b_3)$, and in turn $a_1b_1 = a_3xb_3 = m_{a_3^x,b_3} \in M''$. This is an expression for a_2b_2 of length less than or equal to 3.

If $l(a_2) + l(b_2) \leq 3$, then one of the following cases must hold true:

(i) $a_2 \in A''$ and $b_2 \in B'' \cup X'B''$,

(ii) $a_2 \in A''X'$ and $b_2 \in B''$,

(iii) $a_2 \in A''$ and $b_2 \in A''B''$,

(iv) $a_2 \in A''B''$ and $b_2 \in B''$.

The first two cases imply that $a_1b_1 = a_2b_2$ as a direct consequence of a repeated entry in the finite submatrix M''.

Case (iii), let $b_2 = ab$ where $a \in A''$, $b \in B''$. Any attempt to equate a_1b_1 with a_2ab using only $R \cup \rho' \cup \sigma' \cup M''$ will fail unless $m_{a,b} \in B'' \cup X'B''$, which would imply case (i).

Case (iv) resolves to case (ii), in the same way that case (iii) resolves to case (i).

If $l(a_2) + l(b_2) > 3$, then there must be an expression for a_2 which has an element of $A''S$ as a suffix and an expression for b_2 which has an element of SB'' as a prefix, otherwise their product could not be reduced in length to 3 or less. This implies that $a_2 \in A''S$ and $b_2 \in SB''$, and that the product of the S components is in X' or is the identity. That is, there exist $a_4 \in A'' \cup A''X'$ and $b_4 \in B''$ such that $(a_2, b_2)\tau(a_4, b_4)$.

Hence, if (a_1, b_1) and (a_2, b_2) represent two repeated entries in the matrix M, then there exist positions (a_3, b_3) and (a_4, b_4) in the submatrix M'' such that $(a_1, b_1)\tau(a_3, b_3)$, $(a_2, b_2)\tau(a_4, b_4)$, and $m_{a_3,b_3} = m_{a_4,b_4}$. That is, the repeated entry comes as a direct consequence of the equivalence τ and a finite set of repetitions. $\square$

We have seen that for a Byleen extension, $\mathcal{C}(S; \sigma, \rho; M)$, to be finitely presentable S must be finitely presentable and M must be, in some sense, finite (specifically, M/τ must have finitely many repeated entries), and for it to be $\mathcal{D}$-simple then S must be $\mathcal{D}$-simple too.

We now know that embedding a given finitely presentable semigroup, T, into another semigroup, S, and then taking a Byleen extension, $\mathcal{C}(S; \sigma, \rho; M)$ in a bid to find a finitely presentable $\mathcal{D}$-simple semigroup does not solve our embedding problem. That is, if

$$\mathcal{T} \hookrightarrow S \hookrightarrow \mathcal{C}(S; \sigma, \rho; M),$$

and $\mathcal{C}(S; \sigma, \rho; M)$ is finitely presentable and $\mathcal{D}$-simple, then S is finitely presentable and $\mathcal{D}$-simple, and we must have solved the problem already in order to use this method to solve the problem, a wasted effort.

However, this does not mean that Byleen extensions are necessarily useless for this endeavour: It may be that $T \hookrightarrow \mathcal{C}(S; \sigma, \rho; M)$ for some finitely presentable $\mathcal{D}$-simple semigroup S such that $T \not\hookrightarrow S$.

With that in mind, we come to the final section of this chapter.

3.4 Periodic Subsemigroups

It has become apparent that understanding the possible subsemigroups of a Byleen extension could shed light on our embedding problem, and in this section we will determine the structure of periodic subsemigroups of Byleen extensions and show that they are constructed in a predictable way from Rees matrix semigroups.

The following lemma will be useful in determining what subsemigroups can be found of any given Byleen extension.

Lemma 3.4.1. *Let $C = \mathcal{C}(S; \sigma, \rho; M)$ be a Byleen extension of the semigroup S, and let $c \in C$ be an element of finite order.*

If $c = \beta s \alpha$ in its normal form, then $\alpha\beta \in S$ or $\alpha = \beta = \epsilon$.

Proof. Let $0 < i < j$ such that $c^i = c^j$. Expressing c in its normal form we have $c = \beta s \alpha$ where $\beta \in B^*, s \in S \cup \{\epsilon\}, \alpha \in A^*$, and $\beta s \alpha \neq \epsilon$.

By Proposition 1.3 [4], we see $\alpha\beta \in S \cup A^+ \cup B^+ \cup \{\epsilon\}$. Obviously, $\alpha\beta = \epsilon$ if and only if $\alpha = \beta = \epsilon$.

If $\alpha\beta = \alpha_1 \in A^+$, then we see that

$$\beta s (\alpha_1^s)^{i-1} \alpha = \beta s (\alpha\beta s)^{i-1} \alpha = (\beta s \alpha)^i = (\beta s \alpha)^j = \beta s (\alpha_1^s)^{j-1} \alpha.$$

As both sides of this are in the unique normal form, we can infer that $(\alpha_1^s)^{i-1}\alpha = (\alpha_1^s)^{j-1}\alpha$. We can cancel identical strings over A from the right as a consequence of the unique normal form to get $\epsilon = (\alpha_1^s)^{j-i}$. This is clearly a contradiction as $j - i \neq 0$.

A similar argument will provide a contradiction to the possibility $\alpha\beta \in B^+$.

Hence, if $\alpha\beta \neq \epsilon$ then $\alpha\beta \in S$. □

Of course we can apply this lemma when considering a Byleen monoid extension, as this is a homomorphic image of a (non-monoid) Byleen extension to see that elements of finite order have the property that the product of their A component with their B component must be in S.

Lemma 3.4.2. *Let $C = \mathcal{C}^1(S;\sigma,\rho;M)$ be a Byleen monoid extension of S, and let $c = \beta s\alpha \in C$.*

If c has finite order, then $\alpha\beta \in S$.

Proof. Of course, if $c \in S$ then $\alpha = \beta = \epsilon$ and so their product is $\epsilon = 1 \in S$.

Recall that $\mathcal{C}^1(S;\sigma,\rho;M)$ is the image of $\mathcal{C}(S;\sigma,\rho;M)$ after applying the congruence which makes the identity of S act as an identity on A and B.

The only case for c in which its preimage is not unique is if $s = 1$, the preimages of this are $\beta 1\alpha$ and $\beta\alpha$ in $\mathcal{C}(S;\sigma,\rho;M)$. Applying Lemma 3.4.1 to either of these possibilities will imply that $\alpha\beta \in S$, or $\alpha\beta = 1 \in S$.

For all the cases where c has unique preimage in $\mathcal{C}(S;\sigma,\rho;M)$, Lemma 3.4.1 can be applied immediately, implying that $\alpha\beta \in S$ in $\mathcal{C}(S;\sigma,\rho;M)$, and then of course it must hold true in C. □

For a Byleen monoid extension $C = \mathcal{C}^1(S;\sigma,\rho;M)$, let $\sim \subseteq C \times C$ be the relation defined by $\beta_1 s_1\alpha_1 \sim \beta_2 s_2\alpha_2$ if and only if $\alpha_1\beta_2 \in S$.

Note that $\sim$ is not necessarily reflexive on C, however the following lemma will demonstrate that it is useful when considering certain subsemigroups.

Recall that for a semigroup to be *periodic* means that every element therein has finite order, which is to say that for any element s, there exist $1 \le i < j$ such that $s^i = s^j$.

Lemma 3.4.3. *Let T be a periodic subsemigroup of $C = \mathcal{C}^1(S;\sigma,\rho;M)$.*

Then $\sim$ is a congruence when restricted to T.

Proof. Let $\beta_1 s_1\alpha_1, \beta_2 s_2\alpha_2, \beta_3 s_3\alpha_3, \beta_4 s_4\alpha_4 \in T$.

As T is periodic, $\beta_1 s_1\alpha_1$ must have finite order. Applying Lemma 3.4.2, $\alpha_1\beta_1 \in S$, hence $\beta_1 s_1\alpha_1 \sim \beta_1 s_1\alpha_1$, and so $\sim$ is reflexive when restricted to T.

If $\beta_1 s_1\alpha_1 \sim \beta_2 s_2\alpha_2$, then $s_1\alpha_1\beta_2 s_2 \in S$, and

$$\beta_1 s_1\alpha_1\beta_2 s_2\alpha_2 = \beta_1(s_1\alpha_1\beta_2 s_2)\alpha_2 \in T.$$

This implies that $\alpha_2\beta_1 \in S$, and in turn $b_2 s_2 a_2 \sim b_1 s_1 a_1$. Hence $\sim$ is symmetric on T.

If $\beta_1 s_1 \alpha_1 \sim \beta_2 s_2 \alpha_2$ and $\beta_2 s_2 \alpha_2 \sim \beta_3 s_3 \alpha_3$, then

$$\beta_1 s_1 \alpha_1 \beta_2 s_2 \alpha_2 \beta_3 s_3 \alpha_3 = \beta_1 (s_1 \alpha_1 \beta_2 s_2 \alpha_2 \beta_3 s_3) \alpha_3 \in T.$$

As this is the normal form for the product, and it sits in T, by Lemma 3.4.2, $\alpha_3 \beta_1 \in S$. Hence $\beta_3 s_3 \alpha_3 \sim \beta_1 s_1 \alpha_1$, and so $\sim$ is transitive on T.

Hence, $\sim$ is an equivalence relation on T.

Suppose $\beta_1 s_1 \alpha_1 \sim \beta_2 s_2 \alpha_2$ and $\beta_3 s_3 \alpha_3 \sim \beta_4 s_4 \alpha_4$. There are three cases which hinge on $\alpha_1 \beta_3$ as it is a member of $A^+ \cup B^+ \cup S$.

If $\alpha_1 \beta_3 \in A^+$, then expressing $\beta_1 s_1 \alpha_1 \beta_3 s_3 \alpha_3$ in its unique normal form,

$$\beta_1 s_1 \alpha_1 \beta_3 s_3 \alpha_3 = \beta_1 s_1 ((\alpha_1 \beta_3)^{s_3} \alpha_3)$$

Of course this product is in T, and so has finite order. Applying Lemma 3.4.2 again, we see that $(\alpha_1 \beta_3)^{s_3} \alpha_3 \beta_1 \in S$. From this it is clear that $\alpha_3 \beta_1 \in B^+$.

Expressing $\beta_4 s_4 \alpha_4 \beta_3 s_3 \alpha_3 \beta_1 s_1 \alpha_1 \beta_2 s_2 \alpha_2$ in its unique normal form we see

$$\beta_4 s_4 \alpha_4 \beta_3 s_3 \alpha_3 \beta_1 s_1 \alpha_1 \beta_2 s_2 \alpha_2 = (\beta_4 {}^{s_4 \alpha_4 \beta_3 s_3}(\alpha_3 \beta_1))(s_1 (\alpha_1 \beta_2) s_2) \alpha_2.$$

Using Lemma 3.4.2 once more, $\alpha_2 (\beta_4 {}^{s_4 \alpha_4 \beta_3 s_3}(\alpha_3 \beta_1)) \in S$, and so $\alpha_2 \beta_4 \in A^+$. This allows us to express $\beta_2 s_2 \alpha_2 \beta_4 s_4 \alpha_4$ in its normal form,

$$\beta_2 s_2 \alpha_2 \beta_4 s_4 \alpha_4 = \beta_2 s_2 ((\alpha_2 \beta_4)^{s_4} \alpha_4).$$

Tying this all together we see that

$$\beta_1 s_1 ((\alpha_1 \beta_3)^{s_3} \alpha_3) \sim \beta_1 s_1 \alpha_1 \sim \beta_2 s_2 \alpha_2 \sim \beta_2 s_2 ((\alpha_2 \beta_4)^{s_4} \alpha_4).$$

Hence, $(\beta_1 s_1 \alpha_1)(\beta_3 s_3 \alpha_3) \sim (\beta_2 s_2 \alpha_2)(\beta_4 s_4 \alpha_4)$.

The same holds by symmetry if $\alpha_1 \beta_3 \in B^+$.

If $\alpha_1 \beta_3 \in S$, then $\beta_2 s_2 \alpha_2 \sim \beta_1 s_1 \alpha_1 \sim \beta_3 s_3 \alpha_3 \sim \beta_4 s_4 \alpha_4$ and it is easy to see that $\sim$-classes are closed under products.

Hence, $\sim$ is a congruence on T. □

Let T be a periodic subsemigroup of $C = \mathcal{C}^1(S; \sigma, \rho; M)$, then the previous lemma demonstrates that $\sim$ is a congruence when restricted to T. Let $\leq$ denote the relation on $T/\sim$ such that $u_1 \leq u_2$ if and only if there exists $\beta_1 s_1 \alpha_1 \in u_1$ and $\beta_2 s_2 \alpha_2 \in u_2$ such that $\alpha_1 \beta_2 \notin B^+$.

Lemma 3.4.4. *Let T be a periodic subsemigroup of $C = \mathcal{C}^1(S; \sigma, \rho; M)$. Then $\leq$ is a total order on $T/\sim$.*

Proof. First it will be necessary to note that if $u_1 \leq u_2$ then not only is there a pair of elements from them which satisfy the condition, but the condition is met for any pair:

Let $u_1, u_2 \in T/\sim$ such that $u_1 \leq u_2$. Then there exists $\beta_1 s_1 \alpha_1 \in u_1$, $\beta_2 s_2 \alpha_2 \in u_2$ such that $\alpha_1 \beta_2 \notin B^+$.

Let $\beta_3 s_3 \alpha_3 \in u_1$, $\beta_4 s_4 \alpha_4 \in u_2$, and suppose that $\alpha_3 \beta_4 \in B^+$. Then

$$(\beta_3 s_3 \alpha_3)(\beta_4 s_4 \alpha_4) = (\beta_3 \, {}^{s_3}(\alpha_3 \beta_4)) s_4 \alpha_4 \sim \beta_2 s_2 \alpha_2.$$

As $\sim$ is a congruence, this implies that $\beta_1 s_1 \alpha_1 \beta_2 s_2 \alpha_2 \sim \beta_2 s_2 \alpha_2$, which in turn implies that $\alpha_1 \beta_2 \notin A^+$. Hence, $\alpha_1 \beta_2 \in S$, and $\alpha_3 \beta_4 \in S$, a contradiction.

If $u \in T/\sim$ and $\beta s \alpha \in u \subset T$, then $\alpha\beta \in S$, and so $\alpha\beta \notin B^+$, hence $u \leq u$ and $\leq$ is reflexive.

If $u \leq v$ and $v \leq u$, then for any $\beta_1 s_1 \alpha_1 \in u$, $\beta_2 s_2 \alpha_2 \in v$, it holds that $\alpha_1 \beta_2 \notin B^+$ and $\alpha_2 \beta_1 \notin B^+$. The former implies $\alpha_2 \beta_1 \notin A^+$, so combined with the latter, $\alpha_2 \beta_1 \in S$. Hence, $\beta_2 s_2 \alpha_2 \sim \beta_1 s_1 \alpha_1$, this implies that $u = v$, and so $\leq$ is antisymmetric.

If $u \leq v \leq w$, then there exist $\beta_1 s_1 \alpha_1 \in u$, $\beta_2 s_2 \alpha_2 \in v$, $\beta_3 s_3 \alpha_3 \in w$ such that $\alpha_1 \beta_2, \alpha_2 \beta_3 \notin B^+$. Consider the product

$$(\beta_1 s_1 \alpha_1)(\beta_2 s_2 \alpha_2)(\beta_3 s_3 \alpha_3) = \beta_1 s_1 (\alpha_1 \beta_2) s_2 (\alpha_2 \beta_3) s_3 \alpha_3.$$

The normal form for this has B component β_1 and α_3 is a suffix of the A component. Since this product sits in T it must have finite order, and by Lemma 3.4.2 the product of the A component with the B component must be in S, which means $\alpha_3\beta_1 \notin A^+$. Considering the normal form of the product $\beta_3 s_3 \alpha_3 \beta_1 s_1 \alpha_1$ will then result in the conclusion that $\alpha_1\beta_3 \notin B^+$. Hence, $\beta_1 s_1 \alpha_1 \leq \beta_3 s_3 \alpha_3$ and $\leq$ is transitive.

As $\leq$ is a reflexive, antisymmetric and transitive relation on $T/\sim$, it is a partial order on $T/\sim$.

Let $u, v \in T/\sim$ such that $u \not\leq v$, and let $\beta_1 s_1 \alpha_1 \in u$, $\beta_2 s_2 \alpha_2 \in v$. Then it holds that $\alpha_1\beta_2 \in B^+$, and

$$(\beta_1 s_1 \alpha_1)(\beta_2 s_2 \alpha_2) = (\beta_1 \, {}^{s_1}(\alpha_1\beta_2)) s_2 \alpha_2 \in T.$$

By Lemma 3.4.2, again, $\alpha_2(\beta_1 \, {}^{s_1}(\alpha_1\beta_2)) = (\alpha_2\beta_1) \, {}^{s_1}(\alpha_1\beta_2) \in S$, hence $\alpha_2\beta_1 \in A^+$. Hence, $v \leq u$.

Any two elements of $T/\sim$ are comparable with $\leq$, which is to say, $\leq$ is a total order on $T/\sim$. □

The following lemma will show that $T/\sim$ is a semilattice, which is to say it is a commutative semigroup in which every element is an idempotent.

Lemma 3.4.5. *Let $C = \mathcal{C}^1(S; \sigma, \rho; M)$, let T be periodic a subsemigroup of C, and let $t_1, t_2 \in T$.*

If $t_1/\sim \leq t_2/\sim$, then $t_1 t_2 \sim t_2 t_1 \sim t_1$.

Proof. Let $t_1 = \beta_1 s_1 \alpha_1, t_2 = \beta_2 s_2 \alpha_2 \in T$ such that $t_1/\sim \leq t_2/\sim$.
Then $\alpha_1\beta_2 \notin B^+$. That is, $\alpha_1\beta_2 \in S \cup A^+$.

If $\alpha_1\beta_2 \in S$, then $t_1 t_2 = \beta_1(s_1\alpha_1\beta_2 s_2)\alpha_2$ and $t_2 t_1 = \beta_2(s_2\alpha_2\beta_1 s_1)\alpha_1$, and since $\alpha_2\beta_2 \in S$, it is clear that $t_1 t_2 \sim t_2 t_1 \sim t_1$.

If $\alpha_1\beta_2 \in A^+$, then $t_1 t_2 = \beta_1 s_1(\alpha_1\beta_1 s_2 \alpha_2)$ and $t_2 t_1 = (\beta_2 s_2 \alpha_2 \beta_1) s_1 \alpha_1$, and since $\alpha_1\beta_1 \in S$, we see $t_2 t_1 \sim t_1 \sim t_1 t_2$. □

By Lemma 3.4.4, we know that $\leq$ is a total order on $T/\sim$, and by Lemma 3.4.5 we see that $T/\sim$ is a semilattice, hence $T/\sim$ is a chain in which any product is equal to the minimum of the elements it comprises.

The following lemma demonstrates that $\leq$ is related to the $\mathcal{J}$-order, $\leq_J$, on T.

Lemma 3.4.6. *Let $C = \mathcal{C}^1(S;\sigma,\rho;M)$, and let T be a periodic subsemigroup of C.*

If $t_1, t_2 \in T$ such that $t_1 \leq_J t_2$, then $t_1/{\sim} \leq t_2/{\sim}$.

Proof. Let $t_1, t_2 \in T$ such that $t_1 \leq_J t_2$ in T.
This implies that there exist $u, v \in T^1$ such that $t_1 = ut_2v$.

By Lemma 3.4.5, $(ut_2v)/{\sim} \leq t_2/{\sim}$, hence $t_1/{\sim} \leq t_2/{\sim}$. □

Now we are equipped to show that $\mathcal{J} \subseteq \sim$.

Lemma 3.4.7. *Let $C = \mathcal{C}^1(S;\sigma,\rho;M)$, and let T be a periodic subsemigroup of C.*

If $t_1, t_2 \in T$ such that $t_1 \mathcal{J} t_2$ in T, then $t_1 \sim t_2$.

Proof. Let $t_1, t_2 \in T$.

If $t_1 \mathcal{J} t_2$ in T, then $t_1 \leq_J t_2 \leq_J t_1$ and by Lemma 3.4.6, $t_1/{\sim} \leq t_2/{\sim} \leq t_1/{\sim}$. Clearly, $t_1/{\sim} = t_2/{\sim}$ and so $t_1 \sim t_2$. □

This tells us that the $\mathcal{J}$-class structure of any periodic subsemigroup of a Byleen monoid extension is contained in its $\sim$-class structure, and by Lemmas 3.4.4 and 3.4.5 we know that the $\sim$-classes form a very simple structure, a chain.

It remains to consider how the $\mathcal{J}$-classes inside a $\sim$-class behave, and the following lemma demonstrates that if we are working in a Byleen monoid extension of a group they are as well behaved as possible.

Lemma 3.4.8. *Let $C = \mathcal{C}^1(S;\sigma,\rho;M)$, where S is a group, and let T be a periodic subsemigroup of C.*

If $t_1, t_2 \in T$ then $t_1 \mathcal{J} t_2$ if and only if $t_1 \sim t_2$.

Proof. The 'only if' component comes as a consequence of Lemma 3.4.7.

Let $t_1 = \beta_1 s_1 \alpha_1, t_2 = \beta_2 s_2 \alpha_2 \in T$ such that $t_1 \sim t_2$.
This implies that $\alpha_1\beta_2, \alpha_2\beta_1 \in S$. Let $s_3 = s_1(\alpha_1\beta_2)s_2$, and consider the product

$$(t_1t_2)^k = (\beta_1 s_3 \alpha_2)^k = \beta_1(s_3((\alpha_2\beta_1)s_3)^{k-1})\alpha_2.$$

As $t_1t_2 \in T$, it must have finite order. That is, there exists $0 < i < j$ such that $(t_1t_2)^i = (t_1t_2)^j$, and so

$$\beta_1(s_3((\alpha_2\beta_1)s_3)^{i-1})\alpha_2 = \beta_1(s_3((\alpha_2\beta_1)s_3)^{j-1})\alpha_2.$$

As these are both in normal form, we can equate the S components:

$$s_3(\alpha_2\beta_1 s_3)^{i-1} = s_3(\alpha_2\beta_1 s_3)^{j-1}.$$

Since S is a group we can use cancellativity, and so

$$1 = (\alpha_2\beta_1 s_3)^{j-i}.$$

Let $n = j - i$. In a group, conjugate elements have the same order, hence

$$1 = (s_3(\alpha_2\beta_1 s_3)s_3^{-1})^n = (s_3\alpha_2\beta_1)^n,$$

and so,

$$(t_1t_2)^n t_1 = \beta_1(s_3(\alpha_2\beta_1 s_3)^{n-1}\alpha_2\beta_1 s_1)\alpha_1) = \beta_1((s_3\alpha_2\beta_1)^n s_1)\alpha_1 = \beta_1 s_1 \alpha_1 = t_1.$$

Hence, $t_1 \leq_J t_2$, and by symmetry, $t_2 \leq_J t_1$, and in turn $t_1 \mathcal{J} t_2$. □

In the more general case we can see that each $\mathcal{J}$-class of a periodic subsemigroup of a Byleen extension of a semigroup, S, is a subsemigroup of a Rees matrix semigroup of S.

Lemma 3.4.9. *Let $C = \mathcal{C}^1(S; \sigma, \rho, M)$.*

If T is a periodic subsemigroup of C, then each $\sim$-class of T is a subsemigroup of a Rees matrix semigroup over S, and in turn, each $\mathcal{J}$-class is *a subsemigroup of a Rees matrix semigroup over S.*

Proof. Let u be a $\sim$-class of T. Lemma 3.4.5 implies that u is closed, and so is a subsemigroup of T.

Let $A' = \{\alpha \in A^* : (\exists \beta \in B^*, s \in S : \beta s \alpha \in u)\}$, and let $B' = \{\beta \in B^* : (\exists \alpha \in A^*, s \in S : \beta s \alpha \in u)\}$.

If $\epsilon \in A'$, then there exists $c = \beta s \alpha \in u$ such that $\alpha = \epsilon$, then by Lemma 3.4.2, $\beta = \epsilon$ and $c = s \in S$.

Let $d = \beta_1 s_1 \alpha_1 \in u$, then as $d \sim c$ we see $\beta_1 s_1 \alpha_1 \sim s$, which means $\alpha_1 \in S$. Clearly $\alpha_1 = \beta_1 = \epsilon$, and so $A' = B' = \{\epsilon\}$. Hence, $u \subseteq S$, which is (trivially) a subsemigroup of a Rees matrix semigroup over S.

If $\epsilon \notin A'$, then $\epsilon \notin B'$ and we can construct the following matrix: Let $Q = (q_{i,j})_{A' \times B'}$ such that $q_{a,b} = m_{a,b}$ and note that all entries in the matrix Q are elements of S.

Let $R = \mathcal{M}[S; B', A'; Q]$, let $\varphi : u \to M$ be defined by $(\beta s \alpha)\varphi = (\beta, s, \alpha)$, and let $\beta_1 s_1 \alpha_1, \beta_2 s_2 \alpha_2 \in u$.

We can see that φ is a homomorphism by the following:

$$((\beta_1 s_1 \alpha_1)(\beta_2 s_2 \alpha_2))\varphi = (\beta_1 s_1 (\alpha_1 \beta_1) s_2 \alpha_2)\varphi = (\beta_1, s_1(\alpha_1\beta_2)s_2, \alpha_2),$$

$$(\beta_1 s_1 \alpha_1)\varphi(\beta_2 s_2 \alpha_2)\varphi = (\beta_1, s_1, \alpha_1)(\beta_2, s_2, \alpha_2) = (\beta_1, s_1(\alpha_1\beta_2)s_2, \alpha_2).$$

If $(\beta_1 s_1 \alpha_1)\varphi = (\beta_2 s_2 \alpha_2)\varphi$, then $(\beta_1, s_1, \alpha_1) = (\beta_2, s_2, \alpha_2)$ and clearly $\beta_1 s_1 \alpha_1 = \beta_2 s_2 \alpha_2$, and so φ is injective.

Hence, u is isomorphic to a subsemigroup of R, a Rees matrix semigroup over S. □

We can now tie all we know about the structure of periodic subsemigroups of Byleen monoid extensions into the following theorem.

Theorem 3.4.10. *Let $C = \mathcal{C}^1(S; \sigma, \rho; M)$.*

If T is a periodic subsemigroup of C, then T is a chain of $\sim$-classes, each of which is a subsemigroup of Rees matrix semigroups over S.

Proof. By Lemmas 3.4.4, T is a chain of $\sim$-classes. By Lemma 3.4.5, any product $st \in T$ is in the least $\sim$-class of $s/\sim$ and $t/\sim$. By Lemma 3.4.9, each $\sim$-class is a subsemigroup of a Rees matrix semigroup over S. □

As we are interested in embedding in $\mathcal{D}$-simple Byleen monoid extensions, by Theorem 3.3.5, we see that if it is possible to embed any given countable semigroup T in a Byleen monoid extension which is $\mathcal{D}$-simple, then there exists a $\mathcal{D}$-simple monoid S such that every periodic subsemigroup of T is a chain of subsemigroups of Rees matrix semigroups over S.

It remains to consider exactly what semigroups occur as subsemigroups of Rees matrix semigroups over $\mathcal{D}$-simple monoids.

Corollary 3.4.11. *Let G be a group, and let $C = \mathcal{C}^1(G; \sigma, \rho; M)$, a Byleen monoid extension of G.*

If T is a finite subsemigroup of C, then the $\mathcal{J}$-classes of T form a chain, in which a product always evaluates to the lower element, and each $\mathcal{J}$-class is a completely simple semigroup.

Proof. Of course, if T is a finite semigroup then it is periodic.
By Lemma 3.4.8, the $\mathcal{J}$-classes of T are exactly the $\sim$-classes.
Applying the theorem, the $\mathcal{J}$-classes form a chain, Lemma 3.4.5 implies that products evaluate in the lower $\mathcal{J}$-class of their constituents.
Applying the second implication of the theorem, each $\mathcal{J}$-class is a subsemigroup of a Rees matrix semigroup over G, and by Theorems 2.5.4 and 2.5.8 each $\mathcal{J}$-class is a completely simple semigroup. □

Of course, completely simple semigroups are simple, this means that the $\mathcal{J}$-class structure of a periodic subsemigroup of a Byleen extension of a group is a chain, and in particular there are no incomparable elements.

Corollary 3.4.12. *Let T be a finite semigroup with at least two incomparable $\mathcal{J}$-classes, that is there exists $x, y \in T$ such that $x \not\leq_J y \not\leq_J x$.*

Then T does not embed in a Byleen monoid extension of a group.

We can apply the theorem to an even more specialised case to describe the periodic subsemigroups of the construction used in Section 3.2.

Corollary 3.4.13. *Let $C = \mathcal{C}^1(\{1\}; \sigma, \rho; M)$.*

If T is a subsemigroup of C comprising elements of finite order, then T is a chain of rectangular bands.

The question of whether any finitely presentable $\mathcal{D}$-simple Byleen extension could contain an embedded copy of any given periodic semigroup is now reduced to the question of whether any given periodic semigroup can be shown to be a chain of subsemigroups of some Rees matrix semigroup over a finitely presentable $\mathcal{D}$-simple monoid. This has not clarified the situation as much as was hoped.

Further investigation into the possible structures of periodic subsemigroups of Rees matrix semigroups over finitely presentable $\mathcal{D}$-simple monoids is warranted.

Chapter 4

Relative Rank Sequences

4.1 Introduction

In a slight change of topic we have come to the final chapter of this thesis. Where before we investigated embeddings, studying when semigroup constructions were extensions and finitely presentable or $\mathcal{D}$-simple or both, now we turn our attention to a sequence of semigroup extensions, specifically the sequence of incremental direct powers $(S^n)_{n\in\mathbb{N}}$, and measure some notion of growth on the sequence.

Recall the following definition:

Definition 4.1.1. For a semigroup S the *rank*, $d(S)$, is defined to be the minimum number of elements required to generate S, that is

$$d(S) = \min\{|X| : X \subseteq S, S = \langle X \rangle\}.$$

A lot of work has been done to study the behaviour of the rank as we take direct products of the initial semigroup.

Definition 4.1.2. The *d-sequence* of a semigroup S, denoted $\mathbf{d}(S)$, is the sequence obtained by taking the ranks of incremental direct powers of S,

$$\mathbf{d}(S) = (d(S), d(S^2), d(S^3), d(S^4), \ldots).$$

Extensive study of d-sequences has been carried out for groups by Wiegold and a number of co-authors, see [25], [26], [27], [28], [14], [30], and [22]. They determined the behaviour precisely for all finitely generated groups except for infinite perfect groups without non-trivial finite homomorphic images, in which case they bounded the behaviour:

- If G is trivial, then $\mathbf{d}(G)$ is (obviously) constant.
- If G is non-perfect, then $\mathbf{d}(G)$ grows linearly.
- If G is finite and perfect, then $\mathbf{d}(G)$ grows logarithmically.
- If G is infinite and simple, then $\mathbf{d}(G)$ is eventually constant.
- If G is infinite, perfect and has non-trivial finite images, then $\mathbf{d}(G)$ grows logarithmically.
- If G is infinite, perfect and has no non-trivial finite images then $\mathbf{d}(G)$ is at least logarithmic and at most linear.

Further to his work on the d-sequences of groups, Wiegold also turned his attention to finite semigroups, [29], and demonstrated that the d-sequence of a finite non-group semigroup is linear if the semigroup has an identity and exponential otherwise. This suggests that the study of d-sequences of infinite semigroups might be similarly simple, but this is not the case.

The study of d-sequences of infinite semigroups was taken up by Hyde, Loughlin, Quick, Ruškuc and Wallis [13]. They demonstrated that the behaviour of d-sequences of infinite semigroups is more varied than their finite counterparts. Examples were found exhibiting the following behaviours; constant, logarithmic, linear, exponential, and eventually infinite. Also, they proved that no semigroups have d-sequence strictly super-linear and strictly sub-exponential.

Quick and Ruškuc, [18], also investigated the d-sequences of other classical algebraic structures (rings, modules, algebras, Lie algebras), finding results much more in line with groups.

Of course, investigating the d-sequence of a semigroup S gives us an insight into the rate at which some notion of size of the semigroup increases as we take more copies of it, however it only makes sense if S is finitely generated. In order to investigate the behaviour for non-finitely generated semigroups we can introduce *relative rank*.

Definition 4.1.3. The *relative rank* of a semigroup S with respect to a subset $A \subseteq S$ is the least number of elements required to form a generating set when added to A, that is,

$$d(S{:}A) = \min\{|X| : X \subseteq S, S = \langle A, X \rangle\}.$$

As with *rank*, we are interested in the sequence achieved by taking increasing direct powers of a given semigroup, but in order to do so we must decide upon a sensible subset to take the rank *relative* to.

The following theorem by Wiegold and Wilson (Theorem 4.2, [30]) provides some motivation for our choice of which subset of S^n to find the rank relative to:

Theorem 4.1.4 (Wiegold, Wilson 1978)**.** *If G is a perfect group having a sequence (g_n) of elements such that G is the normal closure of the element $g_i^{-1}g_j$ whenever $i \neq j$, then every finite direct power of G is a homomorphic image of the free product of G and an infinite cyclic group. In particular, if G is also finitely generated then $d(G^n) \leq d(G) + 1$ for all $n \geq 1$.*

What they actually proved was $d(G^n{:}\Delta_{G^n}) = 1$ for all $n \geq 2$ and their statement is a consequence, where Δ_{G^n} is the diagonal copy of G in G^n, that is $\Delta_{G^n} = \{(g, g, \ldots, g) \in G^n : g \in G\}$.

This is not the only link between $d(S^n{:}\Delta_{S^n})$ and $d(S^n)$, as we can see in the following proposition.

Proposition 4.1.5. *Let S be a semigroup, and let $n \in \mathbb{N}$.*

Then

$$d(S^n{:}\Delta_{S^n}) \leq d(S^n) \leq d(S^n{:}\Delta_{S^n}) + d(S).$$

Proof. Let $X \subseteq S^n$ be a generating set of S^n such that $|X| = d(S^n)$.

Then, $S^n = \langle X \rangle$, and of course adding elements to a generating set will always result in a generating set, and so $S^n = \langle X, \Delta_{S^n} \rangle$. That is, X is a relative generating set for S^n with respect to Δ_{S^n}.

Hence, $d(S^n{:}\Delta_{S^n}) \leq |X| = d(S^n)$.

Let $Y \subseteq S^n$ be a relative generating set for S^n with respect to Δ_{S^n} such that $|Y| = d(S^n{:}\Delta_{S^n})$, and let $Z \subseteq S$ be a generating set of S such that $|Z| = d(S)$. Let $U = \{(z, z, \ldots, z) \in \Delta_{S^n} : z \in Z\}$, and note that $\Delta_{S^n} = \langle U \rangle$.

Then, $S^n = \langle Y, \Delta_{S^n} \rangle \subseteq \langle Y, U \rangle \subseteq S^n$, and so $Y \cup U$ is a generating set for S^n of size $|Y \cup U| \leq |Y| + |U| = |Y| + |Z| = d(S^n{:}\Delta_{S^n}) + d(S)$.

Hence, $d(S^n) \leq d(S^n{:}\Delta_{S^n}) + d(S)$. $\square$

An immediate consequence of this is that if S is finitely generated, then the behaviour of the relative rank sequence of S with respect to its diagonal is the same as the behaviour of the d-sequence of S.

Of course, if the semigroup S is not finitely generated, then its d-sequence is infinite, but the inequalities still hold and $d(S^n{:}\Delta_{S^n})$ describes the growth of the d-sequence of S.

From now on we will denote $d(S^n{:}\Delta_{S^n})$ by $d_\Delta(S^n)$ for brevity. Note that the symbols representing the semigroup are now important, for example if we let $T = S^7$ then $0 = d_\Delta(T^1) = d_\Delta(T) \neq d_\Delta(S^7) > 0$, and so special care will be taken to avoid such ambiguity.

Definition 4.1.6. The *relative rank sequence*, or *d_Δ-sequence*, of a semigroup S is defined to be the sequence of relative ranks of incremental direct powers of S, each with respect to the diagonal, that is,

$$\mathbf{d}_\Delta(S) = (d_\Delta(S), d_\Delta(S^2), d_\Delta(S^3), \ldots).$$

Note that the first entry in $\mathbf{d}_\Delta(S)$ is 0 for any S, and so when we describe $\mathbf{d}_\Delta(S)$ in any way we will have the implied caveat that the first entry is 0, for example if every entry of $\mathbf{d}_\Delta(S)$ is infinite (except for the first entry) we will simply say $\mathbf{d}_\Delta(S)$ is infinite.

The *relative rank sequence* will be the topic for this chapter.

Of course, as a consequence of Proposition 4.1.5, any behaviour which is possible for a d-sequence is possible for a relative rank sequence of a finitely generated semigroup. In Section 4.2 we will see some examples of non-finitely generated semigroups which demonstrate different behaviours, in particular constant, logarithmic and infinite.

Section 4.3 is an investigation into the relative rank sequences of infinite Cartesian products of groups. That is, for an infinite cardinal I, and a group G, we will see results regarding $\mathbf{d}_\Delta(G^I)$. To summarise these results, using a property, $P(m)$, to be defined later (Definition 4.3.12):

- If G is not a perfect group, then $\mathbf{d}_\Delta(G^I)$ is infinite (Theorem 4.3.5).
- If G is finite and perfect, then $\mathbf{d}_\Delta(G^I)$ is logarithmic (Theorem 4.3.10).
- If G is infinite and does not have property $P(m)$ for any $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(G^I)$ is infinite (Theorem 4.3.13).
- If G is infinite, has property $P(m)$ for some $m \in \mathbb{N}$, and has a non-trivial finite homomorphic image, then $\mathbf{d}_\Delta(G^I)$ is logarithmic (Lemmas 4.3.3, 4.3.4, 4.3.16, Theorem 4.3.10).
- If G is infinite, has property $P(m)$ and has no non-trivial finite homomorphic image, then $\mathbf{d}_\Delta(G^I)$ is bounded below by a constant function and above by a logarithmic one (Lemma 4.3.16).

This is nearly a complete characterisation of the behaviour of the relative rank sequences of infinite Cartesian products of groups but for the bounded rather than specific nature of the final case. In this case we will see examples of constant sequences, but none which are super-constant so far.

Note the similarities between this almost complete characterisation of relative rank sequences of infinite Cartesian products of groups, and the almost complete characterisation of rank sequences of finitely generated groups due to Wiegold et al.

Section 4.4 is a semigroup parallel of Section 4.3, which is to say it is an investigation into the relative rank sequences of infinite Cartesian products of semigroups.

The first noteworthy result is that for a finite non-group semigroup, S, any infinite Cartesian product of S has infinite relative rank sequence (Theorem 4.4.4). This of course settles the case for any infinite semigroups which happen to have a non-trivial finite (non-group) semigroup as a homomorphic image.

The next result states that for any non-trivial commutative semigroup, the relative rank sequence is infinite (Theorem 4.4.6). This leaves only the infinite semigroups without finite, commutative, or non-perfect group homomorphic images, and in order to deal with these we will introduce the property $P'(m)$ (Definition 4.4.7), and then see that this is a generalisation of the group property $P(m)$ (Definition 4.3.12, Theorem 4.4.8).

We will see that this is a useful generalisation of the group specific property as Theorem 4.4.11 provides the semigroup analogue of Theorem 4.3.13, which is to say that a semigroup S has property $P'(m)$ for some $m \in \mathbb{N}$ if and only if $d_{\Delta}((S^I)^2) \leq m$. We will also see, in Theorem 4.4.13, that if a semigroup S has property $P'(m)$ for some $m \in \mathbb{N}$, then $\mathbf{d}_{\Delta}(S^I)$ is at most logarithmic.

In Lemma 4.4.15, and the subsequent examples, we will see that if a semigroup has a cyclic diagonal bi-act then it has property $P(1)$, but the converse is not true, demonstrating that diagonal bi-acts are insufficient when trying to characterise all infinite Cartesian products of semigroups with finite relative rank sequences.

In Theorem 4.4.18 we will see that infinite Cartesian products of Byleen monoid extensions can have constant relative rank if the matrix used in the construction has certain properties, and then in Corollary 4.4.19 we will see an example of how relative rank results of finitely generated semigroups can be used to prove results about the non-relative rank sequences.

This section will finish with a pair of lemmas which suppose that for a given semigroup S the relative rank of $(S^I)^2$ is finite and then prove that S must be generated by one of its $\mathcal{J}$-classes, J, and that $\mathbf{d}_\Delta(S^I) = \mathbf{d}_\Delta((S/(S \setminus J))^I)$, that is we can take the Rees quotient of S by the ideal $S \setminus J$ without changing the relative rank sequence.

The final section, Section 4.5, will serve to introduce a new concept which is a natural follow-up to relative rank, *relative presentability.* We will prove some simple lemmas on the topic, and then prove that no direct power of the full transformation monoid on the natural number, $T_{\mathbb{N}}^n$, has a finite relative presentation (Proposition 4.5.3).

4.2 Examples

The symmetric group $S_{\mathbb{N}}$, consisting of all permutations on the set $\mathbb{N}$ of natural numbers, is uncountably infinite and so is not finitely generated, and in turn has infinite d-sequence. Our first example will show that it has the lowest possible relative d-sequence:

Example 4.2.1. In order to see that $\mathbf{d}_\Delta(S_{\mathbb{N}}) = (0, 1, 1, 1, \ldots)$ we will begin by proving that $d_\Delta(S_{\mathbb{N}}^2) = 1$:

Clearly, $d_\Delta(S_{\mathbb{N}}^2) \geq 1$ as $S_{\mathbb{N}}$ is non-trivial, so it remains to prove that there exist $(\alpha, \beta) \in S_{\mathbb{N}}^2$ such that $S_{\mathbb{N}}^2 = \langle (\alpha, \beta), \Delta_{S_{\mathbb{N}}^2} \rangle$.

Let $\alpha \in S_{\mathbb{N}}$ with infinite support, which is to say there are infinitely many $n \in \mathbb{N}$ such that $n\alpha \neq n$.

Fix $X \subset \operatorname{supp}(\alpha)$ such that X is a moiety in $\operatorname{supp}(\alpha)$ and $X\alpha \cap X = \emptyset$. Such an X can be found in the following way: if α contains an infinite cycle then take every second entry therein to form X, if α has no infinite cycles then it has infinitely many finite cycles, take one element from each to form X.

Let $\gamma \in S_X$, where $S_X = \{\sigma \in S_{\mathbb{N}} : \operatorname{supp}(\sigma) \subseteq X\}$. A theorem by Ore (Theorem 6, [16]) asserts that γ can be expressed as a commutator of two other elements of S_X, that is, there exist $\pi, \tau \in S_X$ such that $[\pi, \tau] = \gamma$.

Consider $[(\pi,\pi)^{(\alpha,1)},(\tau,\tau)] \in \langle(\alpha,1),\Delta_{S^2_{\mathbb{N}}}\rangle$,

$$[(\pi,\pi)^{(\alpha,1)},(\tau,\tau)] = ([\pi^\alpha,\tau],[\pi^1,\tau]) = ([\pi^\alpha,\tau],\gamma).$$

Since $\mathrm{supp}(\pi) \subset X$ and $X\alpha \cap X$ is empty, $\mathrm{supp}(\pi^\alpha) \cap X$ is also empty and in turn $\mathrm{supp}(\pi^\alpha) \cap \mathrm{supp}(\tau)$ is empty. As a consequence, π^α and τ commute, $[\pi^\alpha,\tau] = 1$ and $(1,\gamma) \in \langle(\alpha,1),\Delta_{S^2_{\mathbb{N}}}\rangle$.

As $\gamma \in S_X$ was an arbitrary choice, $\{1\} \times S_X \subset \langle(\alpha,1),\Delta_{S^2_{\mathbb{N}}}\rangle$.

Let Y be a moiety of X, and let $Z = Y \cup \{\mathbb{N} \setminus X\}$.
Now $Z \cap X = Y$ is a moiety in each X and Z, and $X \cup Z = \mathbb{N}$.

Let $\sigma \in S_{\mathbb{N}}$ such that σ fixes every point in Y and maps $X \setminus Y$ to $Z \setminus Y$ bijectively. Then

$$\{1\} \times S_Z = (\{1\} \times S_X)^{(\sigma,\sigma)} \subset \langle(\alpha,1),\Delta_{S^2_{\mathbb{N}}}\rangle.$$

Now since X and Z intersect in a moiety and their union covers $\mathbb{N}$ we can use a lemma by Galvin (Lemma 2.1, [9]) which states $S_{\mathbb{N}} = S_X S_Z S_X \cup S_Z S_X S_Z$ to derive the following:

$$S^2_{\mathbb{N}} \subseteq \langle\{1\} \times S_{\mathbb{N}},\Delta_{S^2_{\mathbb{N}}}\rangle = \langle\{1\} \times S_X,\{1\} \times S_Z,\Delta_{S^2_{\mathbb{N}}}\rangle \subseteq \langle(\alpha,1),\Delta_{S^2_{\mathbb{N}}}\rangle \subseteq S^2_{\mathbb{N}}.$$

Hence $d_\Delta(S^2_{\mathbb{N}}) = 1$.

This method generalises: Let $\alpha_1,\ldots,\alpha_{k-1} \in S_{\mathbb{N}}$ such that each has infinite support and $\mathrm{supp}(\alpha_i) \cap \mathrm{supp}(\alpha_j) = \emptyset$, for each $i \neq j$. Let X_i be a moiety of $\mathrm{supp}(\alpha_i)$ such that $X_i\alpha_i \cap X_i$ is empty, as before, for each i.
Let $1 \leq j \leq k$ and let $\gamma \in S_{X_j}$, there exist $\pi,\tau \in S_{X_j}$ such that $\gamma = [\pi,\tau]$.

As $\mathrm{supp}(\tau) \cap \mathrm{supp}(\alpha_i) = \emptyset$ for all $i \neq j$, and $\mathrm{supp}(\tau^{\alpha_j}) \cap \mathrm{supp}(\pi) \subseteq X_j\alpha_j \cap X_j = \emptyset$, we see that

$$[(\tau,\ldots,\tau)^{(\alpha_1,\ldots,\alpha_{k-1},1)},(\pi,\ldots,\pi)] = ([\tau,\pi],\ldots,[\tau,\pi],1,[\tau,\pi],\ldots,[\tau,\pi]),$$

where the identity is in the j^{th} component.

Multiplying by $([\pi,\tau],\ldots,[\pi,\tau]) \in \Delta_{S_{\mathbb{N}}^k}$ we get

$$(1,\ldots,1,\gamma,1,\ldots,1) \in \langle(\alpha_1,\ldots,\alpha_{k-1},1),\Delta_{S_{\mathbb{N}}^k}\rangle.$$

This allows us to generate S_{X_j} in the j^{th} component and in turn $S_{\mathbb{N}}$ in the j^{th} component.

Hence,

$$\{1\}^{j-1} \times S_{\mathbb{N}} \times \{1\}^{k-j} \subset \langle(\alpha_1,\ldots,\alpha_{k-1},1),\Delta_{S_{\mathbb{N}}^k}\rangle,$$

for each $1 \le j \le k$, and so $S_{\mathbb{N}}^k = \langle(\alpha_1,\ldots,\alpha_{k-1},1),\Delta_{S_{\mathbb{N}}^k}\rangle$.

Examples of semigroups which are not finitely generated but have finite relative rank sequence are not restricted to groups, the following demonstrates that $T_{\mathbb{N}}$, the full transformation monoid on the natural numbers, affords us another nice example.

Example 4.2.2. Let $n \in \mathbb{N}$. For $1 \le i \le n$, let $\alpha_i \in T_{\mathbb{N}}$ be defined by $(m)\alpha_i = nm + i$, and let $\alpha = (\alpha_1,\ldots,\alpha_n) \in T_{\mathbb{N}}^n$. Note that $\text{im}(\alpha_i) \cap \text{im}(\alpha_j)$ is empty for all $i \neq j$.

Let $(\gamma_1,\ldots,\gamma_n) \in T_{\mathbb{N}}^n$, and let $\rho \in T_{\mathbb{N}}$ such that $\rho : nm + i \mapsto m\gamma_i$.

Now

$$(\alpha_1,\alpha_2,\ldots,\alpha_n)(\rho,\rho,\ldots,\rho) = (\gamma_1,\gamma_2,\ldots,\gamma_n).$$

Hence, $T_{\mathbb{N}}^n = \langle\alpha,\Delta_{T_{\mathbb{N}}^n}\rangle$, and in turn $\mathbf{d}_\Delta(T_{\mathbb{N}}) = (0,1,1,\ldots)$.

Using a similar proof we can show that $\mathbf{d}_\Delta(\mathcal{B}_{\mathbb{N}}) = \mathbf{d}_\Delta(\mathcal{P}_{\mathbb{N}}) = \mathbf{d}_\Delta(\mathcal{F}_{\mathbb{N}}) = (0,1,1,\ldots)$, where $\mathcal{B}_{\mathbb{N}}$ is the semigroup of binary relations on the natural numbers, $\mathcal{P}_{\mathbb{N}}$ is the semigroup of partial transformations on the natural numbers, and $\mathcal{F}_{\mathbb{N}}$ is the semigroup of full finite-to-one transformations on the natural numbers.

So far we have seen only constant relative rank sequences, but this is not the only possible behaviour. In the following example we see that relative rank sequences can be logarithmic.

Example 4.2.3. Let I be an infinite cardinal, and let $T = A_5^I$, the Cartesian product of I copies of the alternating group on 5 points.

Then $d_\Delta(T^n) = \lceil \log_{60} n \rceil$ as an immediate consequence of Theorem 4.3.9.

It is possible for a semigroup to have infinite relative rank sequence.

Example 4.2.4. Let $S = \langle x \mid \rangle$, the free monogenic semigroup. We will see that, $\mathbf{d}_\Delta(S)$ is infinite.

It will suffice to show that $d_\Delta(S^2)$ is not finite as relative rank sequences are non-decreasing.

Let $U \subseteq S^2$ such that $S^2 = \langle U, \Delta_{S^2} \rangle$. Let $n \in \mathbb{N} \setminus 1$. The element $(x, x^n) \in S^2$ and so can be expressed in terms of the generators. Of course x cannot be expressed as a product of more than one element, and so the expression for (x, x^n) in terms of the generators cannot contain more than one term, that is $(x, x^n) \in U \cup \Delta_{S^2}$.

The choice of $n \neq 1$ means $(x, x^n) \notin \Delta_{S^2}$, and so $(x, x^n) \in U$. Hence, $|U| \geq |\mathbb{N} \setminus \{1\}| = |\mathbb{N}|$ and $d_\Delta(S^2)$ can not be finite.

In fact, many results in Section 4.4 will provide sufficient conditions on S for $\mathbf{d}_\Delta(S)$ to be infinite and we will see that it is much harder to find semigroups which have finite relative rank sequence.

4.3 Infinite Cartesian Products of Groups

As we are interested in using relative rank as some measure of growth on things which do not exhibit growth behaviour according to non-relative rank, we will be considering non-finitely generated groups (and later semigroups). One way to find lots of non-finitely generated groups is to take an infinite Cartesian product of non-trivial groups, since these must have uncountably many elements they certainly cannot be finitely generated.

Also, infinite Cartesian products have risen to prominence in work on the Bergman property [8], and we will see that if an infinite simple group G has the Bergman property, then $\mathbf{d}_\Delta(G^I) = (0, 1, 1, \ldots)$, where I is an infinite cardinal (Corollary 4.3.20).

Before we can prove our first result regarding infinite Cartesian products of groups, we have the following theorem. Note that when comparing sequences we will use $(a_i) \leq (b_i)$ to denote $a_n \leq b_n$ for all n, and similarly $(a_i) \geq (b_i)$ if and only if $a_n \geq b_n$ for all n.

Theorem 4.3.1. *Let T be a non-finitely generated commutative group, with rank μ.*

Then

$$\mathbf{d}_\Delta(T) \geq (0, \mu, \mu, \ldots).$$

Proof. Let $U \subset T \times T$ such that $T \times T = \langle U, \Delta_{T \times T} \rangle$.

We can find another relative generating set $U' \subseteq \{1\} \times T$ by taking each $(u, v) \in U$ and multiplying by $(u^{-1}, u^{-1}) \in \Delta_{T^2}$, that is $U' = \{(1, vu^{-1}) : (u, v) \in U\}$. This is clearly a relative generating set as $U \subset U'\Delta_{T^2}$.

Since T is commutative, we have the following normal form for $T \times T$: let $(r, s) \in T \times T$, there exist $(1, u) \in \langle U' \rangle$ and $t \in T$ such that $(r, s) = (1, u)(t, t)$ (in particular, $t = r$ and $u = sr^{-1}$).

Let $r \in T$ and consider the normal form for the element $(r^{-1}, 1)$, there exist $(1, u) \in \langle U' \rangle$ and $t \in T$ such that

$$(r^{-1}, 1) = (1, u)(t, t) = (t, ut).$$

The first component implies that $t = r^{-1}$, and then the second reads $1 = ur^{-1}$, which implies that $u = r$.

Hence, $\{1\} \times T \subseteq \langle U' \rangle$ and since we already had that $U' \subseteq \{1\} \times T$ we see that $\langle U' \rangle = \{1\} \times T \cong T$.

So $|U| = |U'| \geq d(\{1\} \times T) = d(T) = \mu$. □

For the remainder of this chapter, let I be an infinite cardinal.

We can use this theorem to find our first result about infinite Cartesian products of groups:

Corollary 4.3.2. *Let G be a non-trivial commutative group.*

Then $\mathbf{d}_\Delta(G^I) \geq (0, 2^I, 2^I, \ldots)$.

Proof. Let $T = G^I$. The theorem shows that the relative rank of T^n is at least the rank of T.

Let $X \subseteq T$ be a generating set for T, that is $T = \langle X \rangle$, and let Z be the set of all finite subsets of $X \times \mathbb{N}$.

Let $\varphi : Z \to \langle X \rangle$ be defined by $z \mapsto \prod_{(x,i) \in z} x^i$. As T is commutative the order of products of generators does not matter, just the total number of occurrences of each generator, and so φ is a surjection.

Hence, $|Z| \geq |\langle X \rangle| = |T| \geq 2^I$.

Note that $|Z| = |X \times \mathbb{N}| = \max\{|X|, |\mathbb{N}|\}$, and since $2^I > |\mathbb{N}|$, it is clear that $|X| = |Z| \geq 2^I$. □

As we are working with infinite Cartesian products of semigroups, the following lemma will be useful by proving that for any pair of semigroups, S, T, if T is a homomorphic image of S then T^I is a homomorphic image of S^I.

Lemma 4.3.3. *Let S and T be semigroups such that T is a homomorphic image of S, and let $n \in \mathbb{N}$.*

Then, T^n is a homomorphic image of S^n and T^I is a homomorphic image of S^I.

Proof. Let $\varphi : S \to T$ be a surjective homomorphism.

Let $\Phi : S^n \to T^n$ be defined by $\Phi : (s_1, s_2, \ldots, s_n) \mapsto (s_1\varphi, s_2\varphi, \ldots, s_n\varphi)$.

Let $\theta : S^I \to T^I$ be defined by $\theta : (s_i)_{i \in I} \mapsto (s_i\varphi)_{i \in I}$.

It is clear to see that both Φ and θ are surjective homomorphisms, hence T^n is a homomorphic image of S^n and T^I is a homomorphic image of S^I. □

The next lemma be used frequently with Lemma 4.3.3 throughout the rest of this chapter in order to find lower bounds on relative rank sequences of infinite Cartesian products.

Lemma 4.3.4. *Let S and T be semigroups such that T is a homomorphic image of S.*

Then, $\mathbf{d}_\Delta(T) \leq \mathbf{d}_\Delta(S)$.

Proof. Let $n \in \mathbb{N}$, and let $U \subset S^n$ such that $|U| = d_\Delta(S^n)$ and $S^n = \langle U, \Delta_{S^n} \rangle$. By Lemma 4.3.3 there exists a surjective homomorphism $\Phi : S^n \to T^n$.

As Φ is surjective, we see that $S^n\Phi = T^n$ and $\Delta_{S^n}\Phi = \Delta_{T^n}$, and so $T^n = S^n\Phi = \langle U, \Delta_{S^n} \rangle \Phi = \langle U\Phi, \Delta_{S^n}\Phi \rangle = \langle U\Phi, \Delta_{T^n} \rangle$.

Hence, $U\Phi$ is a relative generating set for T^n of size $|U\Phi| \leq |U| = d_\Delta(S^n)$, and so $d_\Delta(T^n) \leq d_\Delta(S^n)$. □

Lemmas 4.3.3 and 4.3.4 will most often be used in conjunction to demonstrate that if T is a homomorphic image of S then $\mathbf{d}_\Delta(T^I) \leq \mathbf{d}_\Delta(S^I)$.

We are now equipped to prove that if $d_\Delta((G^I)^2)$ is finite for a group G, then G is perfect, which narrows the field of investigation significantly.

Theorem 4.3.5. *Let G be a non-perfect group.*

Then $\mathbf{d}_\Delta(G^I) \geq (0, 2^I, 2^I, \ldots)$.

Proof. As G is not perfect, G has a non-trivial commutative group as a homomorphic image, let H be that image. We can apply Lemma 4.3.3 to see that H^I is a homomorphic image of G^I, then Lemma 4.3.4 to see that $\mathbf{d}_\Delta(H^I) \leq \mathbf{d}_\Delta(G^I)$, and then Lemma 4.3.2 to see that $\mathbf{d}_\Delta(H^I) \geq (0, 2^I, 2^I, \ldots)$.

Hence, $\mathbf{d}_\Delta(G^I) \geq (0, 2^I, 2^I, \ldots)$. □

So we have seen that if we take an infinite Cartesian product of a non-perfect group the relative rank sequence is not finite, the following theorem will demonstrate that if we take an infinite Cartesian product of a finite group the relative rank sequence is at least logarithmic. This result is basically an application of the *pigeonhole principle.*

In the proof of the following theorem (and throughout the chapter) we will need notation to describe a specific component of an element of an infinite Cartesian product in an element of a direct product. We will achieve this through a double index on the element, the first denoting which component of the direct product we are referring to and the second denoting which component of the infinite Cartesian product. For example if $u \in (S^J)^n$ for some semigroup S, infinite cardinal J and natural number n, then $u_1, u_2, \ldots, u_n$

will refer to the elements of S^J such that $u = (u_1, u_2, \ldots, u_n)$, and $u_{i,j}$ will denote the j component of u_i ($u_{i,j} = (u_i)_j$).

This notation style works if both the products are finite or infinite, or if the products are nested deeper than just two levels, though we will not have need of this here.

Theorem 4.3.6. *Let G be a non-trivial finite group, let $m = |G|$, and let $T = G^I$.*

Then $d_\Delta(T^n) \geq \lceil \log_m n \rceil$.

Proof. In order to prove this we will demonstrate that $d_\Delta(T^{m^n+1}) \geq n+1$ and use the fact that relative d-sequences are non-decreasing.

Clearly the inequality holds for $n = 0$, so let $n = 1$ and let $U \subset T^{m^n+1} = T^{m+1}$, such that $|U| = 1$.

Let $i \in I$, and let $\varphi : T^{m+1} \to G^{m+1}$ be defined such that $\varphi : u \mapsto (u_{1,i}, u_{2,i}, \ldots, u_{m+1,i})$.

Of course φ is a surjective homomorphism, and so the image of U must be a relative generating set for G^{m+1}. Let $V = U\varphi$.

Let $v \in V$. Note that v has $m+1$ components, each of which is from G. As $|G| = m$, by the pigeonhole principle, there are two components with the same value, that is there exist $1 \leq j < k \leq m+1$ such that $v_j = v_k$. Clearly, for all $\delta \in \Delta_{G^{m+1}}$, we have $\delta_j = \delta_k$.

So, for all $t \in \langle V, \Delta_{G^{m+1}} \rangle$, we have $t_j = t_k$, which means that $G^{m+1} \neq \langle V, \Delta_{G^{m+1}} \rangle$, so V is not a relative generating set for G^{m+1} and in turn U is not a relative generating set for T^{m+1}. Hence, $d_\Delta(T^{m+1}) \geq 2$.

Assume that $d_\Delta(T^{m^k+1}) \geq k+1$, for some $k \geq 1$.
Suppose that $d_\Delta(T^{m^{k+1}+1}) \leq k+1$, which is to say there exists $U \subset T^{m^{k+1}+1}$, such that $|U| = k+1$ and $T^{m^{k+1}+1} = \langle U, \Delta_{T^{m^{k+1}+1}} \rangle$.

Let $u \in U$. For each $i \in I$ by the pigeonhole principle there exists $A_i \subseteq \{1, 2, \ldots, m^{k+1}+1\}$ such that $|A_i| = m^k + 1$ and $u_{j,i} = u_{k,i}$ for all $j, k \in A_i$.

There are only finitely many subsets of $\{1, 2, \ldots, m^{k+1}+1\}$, and so there are only finitely many possibilities for distinct A_i. This implies that there exists $B \subseteq I$ such that $|B| = |I|$ and $A_b = A_c$ for all $b, c \in B$, let $A = A_b$ for some $b \in B$.

Let $\Phi : T^{m^{k+1}+1} \to T^{m^k+1}$ be the projection on both index sets such that $(t_{a,b})_{a\in\{1,\ldots,m^{k+1}+1\}, b\in I} \mapsto (t_{a,b})_{a\in A, b\in B}$.

Note that $U\Phi$ is a relative generating set for T^{m^k+1}, and that $u\Phi \in \Delta_{T^{m^k+1}}$. Hence, $(U \setminus \{u\})\Phi$ is a relative generating set for T^{m^k+1} of size $|U| - 1 = k$, a contradiction to the assumption that $d_\Delta(T^{m^k+1}) \geq k + 1$.

So by induction on n, we see that $d_\Delta(T^{m^n+1}) \geq n+1$, which is equivalent to $d_\Delta(T^n) \geq \lceil \log_m n \rceil$ as relative rank sequences are non-decreasing. □

While not about infinite Cartesian products, the following lemma will be useful for finding relative generating sets for copies of G^n inside $(G^I)^n$, where G is a non-commutative simple group.

Lemma 4.3.7. *Let G be a non-abelian simple group, let $n \in \mathbb{N}$ and let $u = (a_1, a_2, \ldots, a_n) \in G^n$.*

Then $G^n = \langle u, \Delta_{G^n} \rangle$ if and only if $a_i \neq a_j$ for all $i \neq j$.

Proof. This was proven by Wiegold and Wilson in [30], in particular it was proven in the course of proving Theorem 4.2 of that paper. □

For $g \in G$, we will use $\mathbf{g}$ to denote the element of G^I with g in every component, that is $\mathbf{g} = (g)_{i\in I}$, this will hopefully make it much easier to understand a lot of the results and proofs.

The next lemma will show that for any finite perfect group G, a specific subset, V, of $(G^I)^n$ will be a relative generating set. This will be useful when checking whether a finite subset is a relative generating set as it will suffice to show that we can generate all of V.

Lemma 4.3.8. *Let G be a finite perfect group, $T = G^I$, $n \geq 2$, and let $V = \{(\mathbf{g}, \mathbf{1}, \ldots, \mathbf{1}), (\mathbf{1}, \mathbf{g}, \mathbf{1}, \ldots, \mathbf{1}), \ldots, (\mathbf{1}, \ldots, \mathbf{1}, \mathbf{g}) : g \in G\} \subset T^n$.*

Then V is a relative generating set for T^n.

Proof. For any subset A of the index set I, let g_A denote the element of T with g in every component with index in A and the identity in every other component.

Consider the commutator of $(\mathbf{f}, \mathbf{1}, \ldots, \mathbf{1}) \in V$ with $(g_A, \ldots, g_A) \in \Delta_{T^n}$,

$$[(\mathbf{f}, \mathbf{1}, \ldots, \mathbf{1}), (g_A, \ldots, g_A)] = ([f, g]_A, \mathbf{1}, \ldots, \mathbf{1}).$$

This commutator demonstrates that we can find such elements in $\langle V, \Delta_{T^n} \rangle$ for any $f, g \in G$ and for any $A \subseteq I$. Since G is perfect it is generated by its commutators and we have

$$\{h_A : h \in G\} \times \{\mathbf{1}\} \times \ldots \times \{\mathbf{1}\} \subset \langle V, \Delta_{T^n} \rangle$$

for all $A \subseteq I$.

Of course G is finite, and so every element of $T = G^I$ can be expressed as a product of finitely many elements of $\{h_A : h \in G, A \subseteq I\}$. Hence,

$$T \times \{\mathbf{1}\} \times \ldots \times \{\mathbf{1}\} \subset \langle V, \Delta_{T^n} \rangle.$$

We can do the same for each of the n components, and so

$$\{\{\mathbf{1}\}^i \times T \times \{\mathbf{1}\}^{n-i-1} : 0 \leq i < n\} \subset \langle V, \Delta_{T^n} \rangle.$$

Obviously, these generate all of T^n, and so $T^n = \langle V, \Delta_{T^n} \rangle$. □

This lemma provides the support in the following theorem (Theorem 4.3.9), which determines the relative rank sequence of any infinite Cartesian product of any non-abelian finite simple group.

In the proof of this theorem we will use elements of T^m which follow specific patterns, and in order to succinctly describe them we must first introduce some new notation.

In a tuple, $(a)^{(j)}$ will denote the element a occurring in j consecutive components, for example in A^6, if $a, b \in A$, then $((a)^{(3)}, (b)^{(2)}, a) = (a, a, a, b, b, a)$.

As well as using this notation to represent repeating a single entry a certain amount of times, we will apply it to tuples to represent repeating that block of entries a certain number of times, for example in A^7, if $a, b \in A$, then $((a, b)^{(3)}, b) = (a, b, a, b, a, b, b)$ and $(((a)^{(2)}, b)^{(2)}, a) = (a, a, b, a, a, b, a)$.

Theorem 4.3.9. *Let G be a non-abelian finite simple group of order m.*
If $T = G^I$, then $d_\Delta(T^n) = \lceil \log_m n \rceil$.

Proof. We have the lower bound as a consequence of Theorem 4.3.6 so it remains to find a generating set of that size for a given n.

Let $a_1, \dots, a_m \in G$ such that $a_i \neq a_j$ for all $i \neq j$.

Starting with $n = m$:
Let $u = (\mathbf{a_1}, \mathbf{a_2}, \dots, \mathbf{a_m}) \in T^m$, let $D = \{(\mathbf{g}, \dots, \mathbf{g}) : g \in G\}$, and let $S = \langle u, D\rangle \subset \langle u, \Delta_{T^n}\rangle$.

By Lemma 4.3.7 we see that $S = \{(\mathbf{g_1}, \mathbf{g_2}, \dots, \mathbf{g_m}) : g_i \in G\}$, and in particular S contains the set V from Lemma 4.3.8, hence

$$T^m = \langle V, \Delta_{T^m}\rangle \subseteq \langle S, \Delta_{T^m}\rangle \subseteq \langle u, \Delta_{T^m}\rangle,$$

and in turn, $d_\Delta(T^m) = 1$.

Now let $n = m^k$ for some $k \geq 2$:
For i from 1 to k let $u_i = (((\mathbf{a_1})^{(m^{i-1})}, (\mathbf{a_2})^{(m^{i-1})}, \dots, (\mathbf{a_m})^{(m^{i-1})})^{(m^{k-i})})$, and let $U = \{u_1, \dots, u_k\}$.

We know that for each u_i using diagonal elements we can generate elements with $\mathbf{g}$ in place of any a_j and identity everywhere else as a consequence of the first part of this proof. That is, for any $g \in G$, $1 \leq i \leq k$, $0 \leq j < m$,

$$(((\mathbf{1})^{(jm^{i-1})}, (\mathbf{g})^{(m^{i-1})}, (\mathbf{1})^{((m-1-j)m^{i-1})})^{(m^{k-i})}) \in \langle U, \Delta_{T^n}\rangle.$$

Let $x, y \in T^n$ such that x is an n-tuple over $\{\mathbf{1}, \mathbf{g}\}$ and y is an n-tuple over $\{\mathbf{1}, \mathbf{h}\}$, for some $g, h \in G$. Note that our more structured elements above fit this form. Consider the commutator $[x, y]$, it is element with the commutator $[\mathbf{g}, \mathbf{h}]$ in the components corresponding to the intersection of the supports of x and y and the identity everywhere else. So taking commutators of elements of the form $(((\mathbf{1})^{(jm^{i-1})}, (\mathbf{g})^{(m^{i-1})}, (\mathbf{1})^{((m-1-j)m^{i-1})})^{(m^{k-i})})$ for different values i, j, g yields tuples with support equal to the intersection of the two and with the commutator of the group elements in the non-identity components. As G is perfect, we can use such tuples to generate elements with support equal to the intersection of the support of any elements we have so far, and $\mathbf{g}$ in the remaining components, for any $g \in G$.

For each of the m^k components and each of the k possibilities for i, there exists a unique j such that the component in question is in the support of the element $(((\mathbf{1})^{(jm^{i-1})}, (\mathbf{g})^{(m^{i-1})}, (\mathbf{1})^{((m-1-j)m^{i-1})})^{(m^{k-i})})$, and in particular the component is the only one which is in the intersection of these k supports.

Hence, for all $0 \leq i < m^k$, and all $g \in G$,

$$((\mathbf{1})^{(i)}, \mathbf{g}, (\mathbf{1})^{(m^k-i-1)}) \in \langle U, \Delta_{T^{m^k}} \rangle,$$

and by Lemma 4.3.8, $T^{m^k} = \langle U, \Delta_{T^{m^k}} \rangle$. Hence, $d_\Delta(T^{m^k}) \leq k$.

This is equivalent to $d_\Delta(T^n) \leq \lceil \log_m n \rceil$. □

The only remaining case for finite groups is non-simple finite perfect groups, and this case is settled in the following theorem. We will see that infinite Cartesian products of non-trivial finite perfect groups have logarithmic relative rank sequence, shown by bounding the sequence between two logarithmic functions. This is not as clean cut as the result for simple groups which found the explicit function for the sequence, but we are more concerned with behaviour than specifics.

Theorem 4.3.10. *Let G be a non-trivial finite perfect group of order m, let $c = d(G)$, and let $T = G^I$.*

Then $\mathbf{d}_\Delta(T)$ is logarithmic, and in particular, for $n \geq 2$,

$$\lceil \log_m n \rceil \leq d_\Delta(T^n) \leq c\lceil \log_2 n \rceil.$$

Proof. Theorem 4.3.6 affords us the lower bound $d_\Delta(T^n) \geq \lceil \log_m n \rceil$, so it remains to establish the upper bound.

Let X be a generating set for G of size c, let $U = \{(\mathbf{x}, \mathbf{1}) : x \in X\} \subset T^2$, and let $V = \{(\mathbf{g}, \mathbf{h}) : g, h \in G\}$. Note that $|U| = d(G) = c$, and that $V \subset \langle U, \Delta_{T^2} \rangle$.

Clearly V contains the relative generating set from Lemma 4.3.8, hence $T^2 = \langle U, \Delta_{T^2} \rangle$, and $d_\Delta(T^2) \leq c$.

Let $k \geq 2$. We will see that $d_\Delta(T^{2^k}) \leq ck$.

Let $W = \{(((\mathbf{x})^{(2^{i-1})}, (\mathbf{1})^{(2^{i-1})})^{(2^{k-i})}) : x \in X, 1 \leq i \leq k\} \subset T^{2^k}$.

Since $\langle U, \Delta_{T^2} \rangle = T^2$, we see that W will allow us to generate all of the elements of the form $(((\mathbf{f})^{(2^{i-1})}, (\mathbf{g})^{(2^{i-1})})^{(2^{k-i})})$ for any $f, g \in G, 1 \leq i \leq k$. In particular,

$$Y = \{(((\mathbf{g})^{(2^{i-1})}, (\mathbf{1})^{(2^{i-1})})^{(2^{k-i})}) : g \in G, 1 \leq i \leq k\} \subseteq \langle V, \Delta_{T^{2^k}} \rangle.$$

A lemma courtesy of Wiegold (Lemma 3.1, [30]) asserts that Y will generate $2^k - 1$ direct copies of G and have the identity in the final component, that is $\{(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_{2^k-1}}, \mathbf{1}) : g_1, g_2, \ldots, g_{2^k-1} \in G\} \subseteq \langle V \rangle$, and of course the diagonal elements can be used to populate the final component, and so

$$\{(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_{2^k}}) : g_1, g_2, \ldots, g_{2^k} \in G\} \subseteq \langle V, \Delta_{T^{2^k}} \rangle.$$

By Lemma 4.3.8, again, we have $T^{2^k} = \langle W, \Delta_{T^{2^k}} \rangle$.

Hence, $d_\Delta(T^n) \leq |W| = c\lceil \log_2 n \rceil$. □

Of course, since we know that relative rank sequence does not increase when we take a homomorphic image, by Lemma 4.3.4, this affords us a nice corollary.

Corollary 4.3.11. *Let G be a group.*

If G has any non-trivial finite homomorphic images, then $\mathbf{d}_\Delta(G^I)$ is at least logarithmic.

Proof. If G is not perfect, then $\mathbf{d}_\Delta(G^I)$ is infinite by Theorem 4.3.5.

Of course if G is finite, perfect and has non-trivial homomorphic image, then G is not trivial, and $\mathbf{d}_\Delta(G^I)$ is logarithmic by Theorem 4.3.10.

Suppose G is infinite, perfect and has non-trivial finite homomorphic image H, then H is perfect and by Lemmas 4.3.3 and 4.3.4, $\mathbf{d}_\Delta(G^I) \geq \mathbf{d}_\Delta(H^I)$. By theorem, $\mathbf{d}_\Delta(H^I)$ is logarithmic, and so $\mathbf{d}_\Delta(G^I)$ is at least logarithmic. $\square$

The following definition will be pivotal to the rest of the results in this section, it came about after an investigation into the necessary and sufficient conditions on a group G such that $d_\Delta((G^I)^n)$ is finite.

Definition 4.3.12. We say that a group G has property $P(m)$ if there exist $g_1, g_2, \ldots, g_m \in G$ and $n_1, n_2, \ldots, n_m \in \mathbb{N}_0$ such that

$$G = (g_1^G)^{n_1}(g_2^G)^{n_2} \ldots (g_m^G)^{n_m}.$$

Note that $P(m)$ implies $P(n)$ for all $n \geq m$.

While this definition may not look like it has any place in the study of infinite Cartesian products of groups, the next theorem demonstrates that it does exactly what it was intended to.

Theorem 4.3.13. *Let G be a group, $T = G^I$ and $m \in \mathbb{N}$.*
Then G has property $P(m)$ if and only if $d_\Delta(T^2) \leq m$.

Proof. Beginning with the forward implication: Suppose G has property $P(m)$.

By the definition, there exist $g_1, g_2, \ldots, g_m \in G,\ n_1, n_2, \ldots, n_m \in \mathbb{N}_0$ such that

$$G = (g_1^G)^{n_1}(g_2^G)^{n_2} \ldots (g_m^G)^{n_m}.$$

Let $U = \{(\mathbf{g_1}, \mathbf{1}), (\mathbf{g_2}, \mathbf{1}), \ldots, (\mathbf{g_m}, \mathbf{1})\} \subset T^2$.
As $G = (g_1^G)^{n_1}(g_2^G)^{n_2} \ldots (g_m^G)^{n_m}$, it follows that

$$T = (\mathbf{g_1}^T)^{n_1}(\mathbf{g_2}^T)^{n_2} \ldots (\mathbf{g_m}^T)^{n_m},$$

and of course

$$\{\mathbf{1}\} = (\mathbf{1}^T)^{n_1}(\mathbf{1}^T)^{n_2}\dots(\mathbf{1}^T)^{n_m}.$$

Hence,

$$T \times \{\mathbf{1}\} = ((\mathbf{g_1}, \mathbf{1})^{\Delta_{T^2}})^{n_1}((\mathbf{g_2}, \mathbf{1})^{\Delta_{T^2}})^{n_2}\dots((\mathbf{g_m}, \mathbf{1})^{\Delta_{T^2}})^{n_m}.$$

That is,

$$T \times \{\mathbf{1}\} \subset \langle U, \Delta_{T^2}\rangle.$$

And in turn,

$$T^2 = (T \times \{\mathbf{1}\})\Delta_{T^2} \subseteq \langle U, \Delta_{T^2}\rangle.$$

Hence, $T^2 = \langle U, \Delta_{T^2}\rangle$, and so $d_\Delta(T^2) \leq m$.

The reverse implication: Suppose that $d_\Delta(T^2) = m$ and that G does not have property $P(m)$.

There exists $U \subset T^2$ such that $|U| = m$ and $T^2 = \langle U, \Delta_{T^2}\rangle$.
Let $V = \{xy^{-1} : (x, y) \in U\}$, and note that $U \subset (V \times \{\mathbf{1}\})\Delta_{T^2}$. Hence,

$$T^2 = \langle V \times \{\mathbf{1}\}, \Delta_{T^2}\rangle.$$

Let $t \in T$. As $(t, \mathbf{1}) \in T^2 = \langle (V \times \{\mathbf{1}\}), \Delta_{T^2}\rangle$, there exists a finite expression for $(t, \mathbf{1})$ in terms of the generators, and as $\langle \Delta_{T^2}\rangle = \Delta_{T^2}$ it can be expressed as

$$(t, \mathbf{1}) = (d_1, d_1)(v_1, \mathbf{1})(d_2, d_2)(v_2, \mathbf{1})\dots(d_k, d_k)(v_k, \mathbf{1})(d_{k+1}, d_{k+1}),$$

where $d_1, \dots, d_{k+1} \in T$ and $v_1, \dots, v_k \in V$.

Let $c_{k+1} = d_{k+1}$, and for $i = k, k-1, \dots, 1$ let $c_i = d_{i+1}c_{i+1}$.
Now for $1 < i \leq k+1$, we have $d_i = c_{i-1}c_i^{-1}$, and we can substitute these in the expression for $(t, \mathbf{1})$,

$$(t, \mathbf{1}) = (d_1, d_1)(v_1, \mathbf{1})(c_1c_2^{-1}, c_1c_2^{-1})\dots(c_kc_{k+1}^{-1}, c_kc_{k+1}^{-1})(v_k, \mathbf{1})(c_{k+1}, c_{k+1}).$$

The second component, implies that $\mathbf{1} = d_1c_1c_2^{-1}c_2c_3^{-1}\dots c_kc_{k+1}^{-1}c_{k+1} =$

$d_1c_1(c_2^{-1}c_2)(c_3^{-1}c_3)\ldots(c_{k+1}^{-1}c_{k+1}) = d_1c_1$, and so $d_1 = c_1^{-1}$, and

$$(t, \mathbf{1}) = (v_1, \mathbf{1})^{(c_1,c_1)}(v_2, \mathbf{1})^{(c_2,c_2)} \ldots (v_k, \mathbf{1})^{(c_{k+1},c_{k+1})}.$$

Hence, $T \times \{\mathbf{1}\} = \langle (V \times \{\mathbf{1}\})^{\Delta_{T^2}} \rangle$, and considering only the first component, $T = \langle V^T \rangle$.

Note that $|V| = |U| = m$ and let $V = \{v_1, v_2, \ldots, v_m\}$.

Let $\varphi : I \to \mathbb{N}_0^m$ be a surjection.

For each $i \in I$, let $t_i \notin (v_{1,i}^G)^{(i\varphi)_1}(v_{2,i}^G)^{(i\varphi)_2} \ldots (v_{m,i}^G)^{(i\varphi)_m}$, a choice made possible by the fact that G does not have property $P(m)$, and let $t = (t_i)_{i \in I} \in T$.

As $T = \langle V^T \rangle$, there exist $k_1, k_2, \ldots, k_m \in \mathbb{N}_0$ such that

$$t \in (v_1^T)^{k_1}(v_2^T)^{k_2} \ldots (v_m^T)^{k_m}.$$

Since φ is a surjection, there exists $\iota \in I$ such that $\iota\varphi = (k_1, k_2, \ldots, k_m)$ and so

$$t_\iota \in (v_{1,\iota}^G)^{(\iota\varphi)_1}(v_{2,\iota}^G)^{(\iota\varphi)_2} \ldots (v_{m,\iota}^G)^{(\iota\varphi)_m},$$

a contradiction. □

This theorem allows us to draw an immediate conclusion about the groups with property $P(m)$ as we already had a result about which set of groups can possibly have finite relative rank sequence.

Corollary 4.3.14. *If G is a group with property $P(m)$ for some $m \in \mathbb{N}$, then G is perfect.*

Proof. Let G be a group with property $P(m)$ for some $m \in \mathbb{N}$, and let $T = G^I$.

By the theorem, $d_\Delta(T^2)$ is finite. If G is not perfect then we attain a contradiction by Theorem 4.3.5. □

So if a group has property $P(m)$ for some $m \in \mathbb{N}$ it is necessarily perfect. In fact for finite groups we can see that a group has property $P(m)$ for some $m \in \mathbb{N}$ if and only if it is perfect, as by Theorems 4.3.5 and 4.3.10 we see that a finite group G is perfect if and only if $d_\Delta((G^I)^2)$ is finite.

The next lemma demonstrates that if it is in fact simple then it has property $P(1)$, and hence $P(m)$ for all $m \in \mathbb{N}$.

Lemma 4.3.15. *Let G be a simple group.*

If G has property $P(m)$ for some $m \in \mathbb{N}$, then G has property $P(1)$, moreover, for each $g \in G \setminus \{1\}$, there exists $n \in \mathbb{N}$ such that $G = (g^G)^n$.

Proof. Let G be a simple group with property $P(m)$, for some $m \in \mathbb{N}$. That is, there exist $g_1, g_2, \ldots, g_m \in G$, $n_1, n_2, \ldots, n_m \in \mathbb{N}_0$ such that

$$G = (g_1^G)^{n_1}(g_2^G)^{n_2} \ldots (g_m^G)^{n_m}.$$

Let $g \in G \setminus \{1\}$. Since G is simple, each g_i is in the normal closure of g. That is, there exist $k_1, k_2, \ldots, k_m \in \mathbb{N}$ such that $g_i \in (g^G)^{k_i}$ for each i. In turn, $(g_i^G)^{n_i} \subseteq ((g^G)^{k_i})^{n_i} = (g^G)^{k_i n_i}$.

Let $n = k_1 n_1 + k_2 n_2 + \ldots + k_m n_m$. Then

$$G = (g_1^G)^{n_1}(g_2^G)^{n_2} \ldots (g_m^G)^{n_m} \subseteq (g^G)^n.$$

Hence, $G = (g^G)^n$. □

Note that if for each $g \in G \setminus \{1\}$, there exists $n \in \mathbb{N}$ such that $G = (g^G)^n$, then G is the normal closure of each of its non-identity elements and so must be simple.

Theorem 4.3.13 states that a group, G, has property $P(m)$ if and only if $d_\Delta((G^I)^2) \leq m$, but it doesn't say anything about the rest of the sequence. The following lemma does exactly that. We will see that if a group has property $P(m)$ for some $m \in \mathbb{N}$, then the relative rank sequence of any infinite Cartesian product of it is finite, and bounded by a logarithmic function.

Lemma 4.3.16. *If G is a group with property $P(m)$ for some $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(G^I)$ is at most logarithmic.*

Proof. Let G be a group with property $P(m)$ for some $m \in \mathbb{N}$, and let $T = G^I$. Then G is perfect by Corollary 4.3.14.

There exist $g_1, g_2, \ldots, g_m \in G,\ n_1, n_2, \ldots, n_m \in \mathbb{N}_0$ such that

$$G = (g_1^G)^{n_1}(g_2^G)^{n_2}\ldots(g_m^G)^{n_m}.$$

Let $V = \{\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_m}\} \subset T$. Clearly, $T^2 = \langle V \times \{\mathbf{1}\}, \Delta_{T^2}\rangle$.

Let $k \in \mathbb{N}$, let $U = \{(((v)^{(2^i)}, (\mathbf{1})^{(2^i)})^{(2^{k-i-1})}) : 0 \leq i < k, v \in V\}$, and let $S = \langle U, \Delta_{T^{2^k}}\rangle$. Note that $|U| = k|V| = km$.

The earlier part of the proof generalises to

$$\{(((\mathbf{f})^{(2^i)}, (\mathbf{h})^{(2^i)})^{(2^{k-i-1})}) : f, h \in G, 0 \leq i < k\} \subseteq \langle U, \Delta_{T^{2^k}}\rangle = S.$$

In particular,

$$\{(((\mathbf{f})^{(2^i)}, (\mathbf{1})^{(2^i)})^{(2^{k-i-1})}) : f \in G, 0 \leq i < k\} \subseteq \langle U, \Delta_{T^{2^k}}\rangle = S.$$

By (Lemma 3.1, [30]), this set generates constant copies of G in all but the last component, and of course the final component can be isolated easily from this set and the diagonal. Hence,

$$\{(\mathbf{h_1}, \mathbf{h_2}, \ldots, \mathbf{h_{2^k}}) : h_1, h_2, \ldots, h_{2^k} \in G\} \subseteq \langle U, \Delta_{T^{2^k}}\rangle \subseteq S.$$

Unfortunately, since G is not necessarily finite we cannot apply Lemma 4.3.8 and be finished, however, the final step is fairly straightforward:

$$\{((\mathbf{1})^{(i)}, \mathbf{g_j}, (\mathbf{1})^{(2^k-i-1)}) : 0 \leq i < 2^k, 1 \leq j \leq m\} \subseteq S.$$

By conjugating by elements of the diagonal and taking products, we see that

$$\{((\mathbf{1})^{(i)}, t, (\mathbf{1})^{(2^k-i-1)}) : t \in T, 0 \leq i < 2^k\} \subseteq S.$$

Of course, any element of T^{2^k} can be expressed as a product of 2^k elements from the above set. Hence, $d_\Delta(T^{2^k}) \leq |U| = km$.

This is equivalent to $d_\Delta(T^n) \leq m\lceil\log_2 n\rceil$ as the relative rank sequence is non-decreasing. □

One immediate consequence of this lemma is that for a group G, if

$d_\Delta((G^I)^n)$ is infinite for some n, which is to say that the sequence is eventually infinite, then $d_\Delta((G^I)^2)$ is infinite, and the whole sequence is infinite (apart from the first entry which is always 0).

The next theorem is a surprising result which simplifies the study of relative rank sequences of infinite Cartesian products of groups, demonstrating that if $\mathbf{d}_\Delta(G^I)$ is finite then it is equal to $\mathbf{d}_\Delta(G)$.

Theorem 4.3.17. *If G is a group with property $P(m)$ for some $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(G^I) = \mathbf{d}_\Delta(G)$.*

Proof. There is an abundance of homomorphisms from G^I onto G, so by Lemma 4.3.4, $\mathbf{d}_\Delta(G^I) \geq \mathbf{d}_\Delta(G)$.

The upper bound takes only a little more work:
Let G be a group with property $P(m)$ for some $m \in \mathbb{N}$ and let $T = G^I$.

By Lemma 4.3.16 we know that $\mathbf{d}_\Delta(T)$ is at most logarithmic, but in particular it is a finite sequence. Given the established lower bound we see that $\mathbf{d}_\Delta(G)$ is also a finite sequence.

Let $n \geq 2$, and let $k = d_\Delta(G^n) \in \mathbb{N}$. There exist $g_1, g_2, \ldots, g_k \in G^n$ such that $G^n = \langle g_1, g_2, \ldots, g_k, \Delta_{G^n}\rangle$.

Let $h_i = (\mathbf{g_{i,1}}, \mathbf{g_{i,2}}, \ldots, \mathbf{g_{i,n}})$ for $1 \leq i \leq k$, where $(g_{i,1}, g_{i,2}, \ldots, g_{i,n}) = g_i$ for each i. Clearly,

$$\{(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) : x_i \in G\} \subseteq \langle h_1, h_2, \ldots, h_k, \Delta_{T^n}\rangle.$$

As G has property $P(m)$, from the definition there exist $f_1, f_2, \ldots, f_m \in G$ and $n_1, n_2, \ldots, n_m \in \mathbb{N}_0$ such that

$$G = (f_1^G)^{n_1}(f_2^G)^{n_2}\ldots(f_m^G)^{n_m}.$$

For each $1 \leq i \leq m$, $(\mathbf{f_i}, \mathbf{1}, \ldots, \mathbf{1}) \in \langle h_1, h_2, \ldots, h_k, \Delta_{T^n}\rangle$, and taking a product of n_i conjugates of each we see that

$$T \times \{\mathbf{1}\} \times \ldots \times \{\mathbf{1}\} \subset \langle h_1, h_2, \ldots, h_k, \Delta_{T^n}\rangle.$$

Repeating this for each of the n components affords us

$$T^n \subseteq \langle h_1, h_2, \ldots, h_k, \Delta_{T^n} \rangle \subseteq T^n.$$

Hence, $d_\Delta(T^n) \leq k = d_\Delta(G^n)$, and so $\mathbf{d}_\Delta(G^I) \leq \mathbf{d}_\Delta(G)$. □

This theorem helps in determining the relative rank sequence of infinite Cartesian products of infinite simple groups, as we see in the next corollary.

Corollary 4.3.18. *Let G be an infinite group with property $P(m)$ for some $m \in \mathbb{N}$.*

If G is simple, then $\mathbf{d}_\Delta(G^I) = (0, 1, 1, \ldots)$.

Proof. Let G be an infinite simple group with property $P(m)$ for some $m \in \mathbb{N}$. By Theorem 4.3.17 and Lemma 4.3.7, $\mathbf{d}_\Delta(G^I) = \mathbf{d}_\Delta(G) = (0, 1, 1, \ldots)$. □

Recall that for G to have the *Bergman property* means that for every generating set X of G, there exists $n \in \mathbb{N}$ such that $G = X \cup X^2 \cup \ldots \cup X^n$. This property looks quite similar to property $P(m)$, and the following theorem shows that, at least for non-abelian simple groups, the Bergman property implies property $P(1)$.

Theorem 4.3.19. *Let G be a non-abelian simple group with the Bergman property.*

Then G has property $P(1)$.

Proof. Let $g \in G \setminus \{1\}$.

Since g is non-identity and G is simple, the normal closure of g is G, that is $G = \langle g^G \rangle$, and so g^G is a generating set for G. As G has the Bergman property, there exists $n \in \mathbb{N}$ such that

$$G = g^G \cup (g^G)^2 \cup \ldots \cup (g^G)^n.$$

As $1 \in G$ and $1 \notin g^G$, there exists $1 < k \leq n$ such that $1 \in (g^G)^k$ and $1 \notin (g^G)^j$ for all $1 \leq j < k$.

As G is non-abelian and simple, and $g \neq 1$, g is not in the centre of G. That is there exists $f \in G$ such that g and f do not commute, and in particular $g^f \neq g$. Now $g^f g^{k-1}$ and g^k are both elements of $(g^G)^k$, and they are necessarily distinct due to cancellativity of the group G. Hence, there exists $h \in (g^G)^k \setminus \{1\}$.

Again using that the normal closure of any non-identity element is the whole group G, there exists $m \in \mathbb{N}$ such that $g^{-1} \in (h^G)^m$, and of course

$$(h^G)^m \subseteq (((g^G)^k)^G)^m = (g^G)^{km}.$$

So $g^{-1} \in (g^G)^{km}$, and in turn,

$$1 = gg^{-1} \in g^G(g^G)^{km} = (g^G)^{km+1}.$$

Since $1 \in (g^G)^k$ and $1 \in (g^G)^{km+1}$, it holds that for all $x, y, z \in \mathbb{N}$ we have $(g^G)^x \subseteq (g^G)^{x+yk}, (g^G)^{x+z(km+1)}$.

Let $1 \leq j \leq n$. Now

$$(g^G)^j \subseteq (g^G)^{j+jm(k)+(n-j)(km+1)} = (g^G)^{n(km+1)}.$$

Hence,

$$G = g^G \cup (g^G)^2 \cup \ldots \cup (g^G)^n \subseteq (g^G)^{n(km+1)},$$

and so G has property $P(1)$. □

Corollary 4.3.20. *If G is an infinite simple group with the Bergman property, then $\mathbf{d}_\Delta(G^I) = (0, 1, 1, \ldots)$.*

Proof. This comes as an immediate consequence of the theorem and Corollary 4.3.18. □

So far we have that for a group to have property $P(m)$ for some $m \in \mathbb{N}$ implies that the relative rank sequence is at most logarithmic, and if it is in fact simple then the sequence is constant at 1.

The following examples demonstrate that in the set of infinite non-simple groups with property $P(m)$ for some $m \in \mathbb{N}$ we can find both kinds of behaviour, in particular this dichotomy occurs in the subset of groups with $P(1)$.

Example 4.3.21. $S_{\mathbb{N}}$ is an infinite perfect, but not simple, group and we will see that it has property $P(1)$, and $\mathbf{d}_{\Delta}(S_{\mathbb{N}}^{I}) = (0, 1, 1, \ldots)$.

In order to see that $S_{\mathbb{N}}$ has property $P(1)$, let $\rho \in S_{\mathbb{N}}$ such that the cycle structure of ρ comprises countably many cycles of each possible length (including trivial and infinite cycles). The set of all conjugates of ρ is the set of all elements of $S_{\mathbb{N}}$ with the same cycle structure as ρ.

Let $\sigma \in S_{\mathbb{N}}$ be an element with infinitely many fixed points, let x_i be the number of cycles of length i in σ for each $i \in \mathbb{N}$ and let x_0 be the number of infinite cycles.

Let $\tau \in \rho^{S_{\mathbb{N}}}$ such that on the support of ρ we have $\tau = \rho^{-1}$, and on the complement of the support τ has exactly x_i many i-cycles for each $i \in \mathbb{N} \cup \{0\}$.

It is clear that the product $\tau\rho \in (\rho^{S_{\mathbb{N}}})^2$ will have cycle structure the same as σ and so is conjugate to it, hence $\sigma \in ((\rho^{S_{\mathbb{N}}})^2)^{S_{\mathbb{N}}} = (\rho^{S_{\mathbb{N}}})^2$. Hence, $(\rho^{S_{\mathbb{N}}})^2$ contains all the elements of $S_{\mathbb{N}}$ with infinitely many fixed points.

Any element with infinitely many cycles can be easily expressed as a product of two elements with infinitely many fixed points, and so $(\rho^{S_{\mathbb{N}}})^4$ contains all elements with infinitely many cycles.

The only elements of $S_{\mathbb{N}}$ not yet accounted for are those with finitely many fixed points and finitely many cycles, these are exactly the elements with finitely many fixed points, finitely many finite cycles and finitely many, but not zero, infinite cycles. If such an element contained more than one infinite cycle we could find it as a product of two elements with infinitely many fixed points.

It remains to consider the elements with exactly one infinite cycle which accounts for all but finitely many of the points. Let $\pi \in S_{\mathbb{N}}$ be such an element, and let $\nu \in S_{\mathbb{N}}$ be the element which agrees with π on the finite set of points not in the infinite cycle and fixes all of the points in the cycle. Clearly ν has infinitely many fixed points, and so is in $(\rho^{S_{\mathbb{N}}})^2$. Now $\pi\nu^{-1}$ is a single infinite cycle. Such a cycle can be expressed as a product of two elements, each comprising infinitely many 2-cycles. Hence, $\pi\nu^{-1} \in (\rho^{S_{\mathbb{N}}})^8$, and in turn $\pi \in (\rho^{S_{\mathbb{N}}})^{10}$.

Of course, as ρ and the identity each have infinitely many fixed points, $\rho^{S_{\mathbb{N}}} \subset (\rho^{S_{\mathbb{N}}})^2$, and $(\rho^{S_{\mathbb{N}}})^2 \subseteq (\rho^{S_{\mathbb{N}}})^4 \subseteq (\rho^{S_{\mathbb{N}}})^{10}$. Hence, $S_{\mathbb{N}} = (\rho^{S_{\mathbb{N}}})^{10}$.

Hence, $S_{\mathbb{N}}$ has property $P(1)$.

By Theorem 4.3.17, $\mathbf{d}_\Delta(S_{\mathbb{N}}^I) = \mathbf{d}_\Delta(S_{\mathbb{N}})$, and as we saw in Example 4.2.1, $\mathbf{d}_\Delta(S_{\mathbb{N}}) = (0, 1, 1, \ldots)$.

Note that the exponent 10 found in this example is not necessarily least possible, but when considering property $P(m)$ we are not concerned with the size of the exponents, simply that they are finite.

Example 4.3.22. Let G and H be simple groups with property $P(1)$, such that G is infinite and H is finite. Then $G \times H$ is an infinite perfect group and we will see that it has property $P(1)$, and that $\mathbf{d}_\Delta((G \times H)^I)$ is logarithmic.

Let $g \in G \setminus \{1_G\}$, $h \in H \setminus \{1_H\}$. Since G and H both have property $P(1)$, there exist $m, n \in \mathbb{N}$ such that $G = (g^G)^m$ and $H = (h^H)^n$. Note that $G = G^n = (g^G)^{mn}$ and $H = H^m = (h^H)^{mn}$.

Let $(x, y) \in G \times H$. Then there exist $c_1, c_2, \ldots, c_{mn} \in G$, $d_1, d_2, \ldots, d_{mn} \in H$ such that

$$(x, y) = ((g, h)^{G \times H})^{(c_1, d_1)}((g, h)^{G \times H})^{(c_2, d_2)} \ldots ((g, h)^{G \times H})^{(c_{mn}, d_{mn})},$$

that is, $(x, y) \in ((g, h)^{G \times H})^{mn}$. Hence, $G \times H$ has property $P(1)$.

Let $T = (G \times H)^I$. Projecting each $G \times H$ in T onto $\{\mathbf{1_G}\} \times H$ yields a homomorphism from T onto H^I, and by Lemma 4.3.4, $\mathbf{d}_\Delta(T) \geq \mathbf{d}_\Delta(H^I)$.

As a consequence of Theorem 4.3.10, $\mathbf{d}_\Delta(H^I)$ is logarithmic. Hence $\mathbf{d}_\Delta(T)$ is at least logarithmic.

The logarithmic upper bound for $\mathbf{d}_\Delta(T)$ comes as a consequence of the fact that $G \times H$ has property $P(1)$ and an application of Lemma 4.3.16.

So further study into the properties of non-simple groups with property $P(m)$ is warranted. Intuition and the few examples we have worked through lead us to the following conjecture:

Conjecture 4.3.23. *Let G be an infinite group with property $P(m)$ for some $m \in \mathbb{N}$.*

If G has no non-trivial finite homomorphic image, then $\mathbf{d}_\Delta(G^I)$ is constant.

The following lemma may be of some use to anybody who investigates this topic further.

Recall that the *commutator width* of an element of G is length of the shortest expression for the element in terms of commutators. If G is perfect then every element can be expressed as a product of commutators and so every element has finite commutator width. If there exists an upper bound on the commutator width for all elements of a group, then that group is said to have *bounded commutator width.*

Lemma 4.3.24. *If G is a group with property $P(m)$ for some $m \in \mathbb{N}$ then G has bounded commutator width.*

Proof. Let G be a group with property $P(m)$ for some m. Then there exist $g_1, g_2, \ldots, g_m \in G,\ n_1, n_2, \ldots, n_m \in \mathbb{N}$ such that $G = (g_1^G)^{n_1}(g_2^G)^{n_2}\ldots(g_m^G)^{n_m}$.

By Corollary 4.3.14, G is perfect, and so each g_i can be expressed as a finite product of commutators, let k_i denote the commutator width of g_i for each $1 \leq i \leq m$.

Commutator width remains constant under conjugation:

$$[a,b]^c = c^{-1}a^{-1}b^{-1}abc = (a^{-1})^c(b^{-1})^c a^c b^c = [a^c, b^c],$$

$$([a,b][c,d])^e = e^{-1}[a,b][c,d]e = e^{-1}[a,b]ee^{-1}[c,d]e = [a^e,b^e][c^e,d^e].$$

Hence, the commutator width of G is bounded by $k_1n_1+k_2n_2+\ldots+k_mn_m$. $\square$

We end this section with a question, which if answered in the affirmative would disprove the above conjecture:

Question. Does there exist a group G, such that $\mathbf{d}_\Delta(G^I)$ is not eventually constant and is strictly sub-logarithmic?

4.4 Infinite Cartesian Products of Semigroups

Now we can generalise to infinite Cartesian products of semigroups. As in the previous section, let I be an infinite cardinal throughout.

Let X be the two element zero semigroup and let Y be the two element semilattice. So $X = \{a, 0\}$ such that every product is equal to 0, and $Y = \{a, 0\}$ such that a is an identity and 0 is a zero.

The first lemma will show that an infinite Cartesian product of X or Y will have infinite relative rank sequence, and will frequently be used with Lemmas 4.3.3 and 4.3.4 to find an infinite lower bound on the relative rank sequence of certain semigroups.

Lemma 4.4.1. $\mathbf{d}_\Delta(X^I) = \mathbf{d}_\Delta(Y^I) = (0, 2^I, 2^I, \ldots)$.

Proof. For all $n \in \mathbb{N}$, $(X^I)^n$ is a relative generating set for $(X^I)^n$, and $(Y^I)^n$ is a relative generating set for $(Y^I)^n$, and so we can see that $d_\Delta((X^I)^n) \leq |(X^I)^n| = 2^I$ and $d_\Delta((Y^I)^n) \leq |(Y^I)^n| = 2^I$. Hence, $\mathbf{d}_\Delta(X^I), \mathbf{d}_\Delta(Y^I) \leq (0, 2^I, 2^I, \ldots)$.

Let $T = X^I$ and let $U \subset T \times T$ such that $T \times T = \langle U, \Delta_{T\times T}\rangle$.
Since X is a zero semigroup, so too is T, and in turn $T \times T$. So any product is the zero element. Hence, $T \times T = \langle U, \Delta_{T\times T}\rangle = U \cup \Delta_{T\times T}$, and in turn, $T \times T \setminus \Delta_{T\times T} \subseteq U$. Considering the cardinality of the sets on either side we see that $|U| = 2^I$.

Let $S = Y^I$ and let $V \subset S \times S$ such that $S \times S = \langle V, \Delta_{S\times S}\rangle$. Note that a is an identity for Y.
Of course $\{\mathbf{a}\} \times S \subset S \times S = \langle V, \Delta_{S\times S}\rangle$, but any element of $\Delta_{S^2} \setminus \{(\mathbf{a}, \mathbf{a})\}$ in a product will result in neither component being $\mathbf{a}$ as it will necessarily contain a 0 in one of the components, hence

$$\{\mathbf{a}\} \times S \subset \langle V\rangle.$$

The same reasoning again implies that there exists a subset $W \subseteq V$ such that $W \subseteq \{\mathbf{a}\} \times S$ and $\{\mathbf{a}\} \times S = \langle W\rangle$. Clearly, W cannot be finite as

$\{\mathbf{a}\} \times S \cong S$ and S is not finitely generated.

Let Z be the set of all finite, non-empty subsets of W. Every element of S, and in turn of W, is idempotent, and S is commutative. This means that we can find a surjection from Z to $\langle W \rangle$, specifically the mapping which takes the product of all the elements in the subset. Hence, $|Z| \geq |\langle W \rangle| = |S|$. The number of finite subsets of an infinite set is just the cardinality of that set.

Hence, $2^I = |S| \leq |\langle W \rangle| = |W| \leq |V|$, and $d_\Delta(S^2) \geq 2^I$. □

In order to put this lemma to use we need the following piece of semigroup theory folk-lore, we include a proof for completeness.

Lemma 4.4.2. *Let S be a finite monoid.*

Then the $\mathcal{J}$-class which contains the identity, is a group, that is $J_1 = H_1$.

Proof. Green's Theorem tells us that H_1 is a group. Of course, as S is finite $\mathcal{D} = \mathcal{J}$, and so J_1 contains exactly one $\mathcal{D}$-class, it remains to show that it contains only one $\mathcal{L}$-class and one $\mathcal{R}$-class.

Suppose J_1 has more than one $\mathcal{L}$-class.

Let $y \in S$ such that $y \in R_1$ and $y \notin L_1$. Since S is finite, there exists $k \in \mathbb{N}$ such that y^k is an idempotent, let $z = y^k$.

As $y\mathcal{R}1$, there exists $y' \in S$ such that $yy' = 1$, and

$$y^k(y')^k = y^{k-1}1(y')^{k-1} = y^{k-1}(y')^{k-1} = \ldots = yy' = 1.$$

Hence $z = y^k\mathcal{R}1$.

So there exists $z' \in S$ such that $zz' = 1$, but

$$z = z1 = z(zz') = z^2z' = zz' = 1.$$

In turn, $y\mathcal{L}z$:

$$y^{k-1}y = y^k = z = 1,$$

and so $y\mathcal{L}1$, a contradiction to J_1 having multiple $\mathcal{L}$-classes.

Hence, $J_1 = L_1$.

By symmetry, J_1 is also $\mathcal{R}$-simple and so $J_1 = L_1 = R_1 = H_1$, hence J_1 is a group. □

Let S be a semigroup and let R be an $\mathcal{R}$-class of S. We say that R is a maximal $\mathcal{R}$-class if for all $r \in R$ and all $s \in S$ we have, $r \leq_R s$ if and only if $s \in R$. That is, R is maximal with respect to the $\mathcal{R}$ partial order, $\leq_R$.

Of course, maximal $\mathcal{L}$-classes are defined in the same way, with respect to $\leq_L$.

Lemma 4.4.3. *Let S be a semigroup with more than one maximal $\mathcal{R}$-class, or more than one maximal $\mathcal{L}$-class.*

Then $\mathbf{d}_\Delta(S^I) \geq (0, 2^I, 2^I, \ldots)$.

Proof. Let S be a semigroup with distinct maximal $\mathcal{R}$-classes, R_1 and R_2, let $T = S^I$, and let $U \subseteq T^2$ such that $T^2 = \langle U, \Delta_{T^2} \rangle$.
Let $a \in R_1, b \in R_2$, let A be a non-empty proper subset of I, and let $t_A = (t_i)_{i \in I} \in T$ such that $t_i = a$ for all $i \in A$ and $t_i = b$ for all $i \notin A$.

Since $(\mathbf{a}, t_A) \in T^2$, there exist $n \in \mathbb{N}$, $\delta_1, \ldots, \delta_{n+1} \in \Delta_{T^2} \cup \{(\mathbf{1}, \mathbf{1})\}$, $u_1, \ldots, u_n \in U$ such that

$$(\mathbf{a}, t_A) = \delta_1 u_1 \delta_2 u_2 \ldots \delta_n u_n \delta_{n+1}.$$

For each $1 \leq i \leq n+1$ let $d_i \in T^1$ such that $\delta_i = (d_i, d_i)$ and for each $1 \leq i \leq n$ let $v_i, w_i \in T$ such that $u_i = (v_i, w_i)$. Considering the two components of the above equation we see that for all $j \in I$,

$$a = d_{1,j} v_{1,j} d_{2,j} v_{2,j} \ldots v_{n,j} d_{n+1,j}, \; t_j = d_{1,j} w_{1,j} d_{2,j} w_{2,j} \ldots w_{n,j} d_{n+1,j}.$$

Since $a \in R_1$ and R_1 is maximal with respect to $\leq_R$ it must hold that every prefix of every expression for a is in R_1, and in particular if $d_1 \neq (\mathbf{1}, \mathbf{1})$ then $d_1 \in R_1^I$. However, if $d_1 \in R_1^I$ then $t_k \in R_1 S$ for all k and in particular $b \in R_1 S$ a contradiction to the choice of $b \in R_2$. Hence, $\delta_1 = (\mathbf{1}, \mathbf{1})$, and

$$a = v_{1,j} d_{2,j} v_{2,j} \ldots v_{n,j} d_{n+1,j}, \; t_j = w_{1,j} d_{2,j} w_{2,j} \ldots w_{n,j} d_{n+1,j}.$$

Every prefix of every expression of a is in R_1, so in particular $v_1 \in R_1^I$ and $w_{1,j} \in R_1$ for all $j \in A$ and similarly $w_{1,j} \in R_2$ for all $j \notin A$.

Hence, for each of the 2^I choices for A there exists $(v, w) \in U$ such that $v_j \in R_1$ for all $j \in A$ and $v_j \in R_2$ for all $j \notin A$, and so $|U| \geq 2^I$ and in turn $d_\Delta(T^2) \geq 2^I$.

Of course the symmetrical argument holds if S has more than one maximal $\mathcal{L}$-class. □

The combination of these three lemmas allows us to determine the relative rank sequence of any infinite Cartesian product of a finite semigroup which is not a group. Recall that the finite group case was determined in Section 4.3.

Theorem 4.4.4. *Let S be a finite non-group semigroup.*
Then $\mathbf{d}_\Delta(S^I) = (0, 2^I, 2^I, \ldots)$.

Proof. First suppose S is a monoid. Lemma 4.4.2 implies that the $\mathcal{J}$-class containing the identity is a group, and since S is not a group, $S \setminus J_1 \neq \emptyset$. Let $K = S \setminus J_1$ and note that K is an ideal.

Let $\varphi : S \to \{a, 0\}$ be the homomorphism defined by $(s)\varphi = a$ for all $s \in J_1$ and $(s)\varphi = 0$ for all $s \in K$. As J_1 is closed and K is an ideal we see that the image of this homomorphism is the two element semilattice, Y.

Applying Lemmas 4.3.3, 4.3.4 and 4.4.1 we see that

$$\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta(Y^I) = (0, 2^I, 2^I, \ldots).$$

If S has more than one maximal $\mathcal{R}$-class or more than one $\mathcal{L}$-class then Lemma 4.4.3 would yield $\mathbf{d}_\Delta(S^I) \geq (0, 2^I, 2^I, \ldots)$.

The remaining case is that S is a finite non-monoid semigroup with unique maximal $\mathcal{R}$-class, R, and unique maximal $\mathcal{L}$-class, L.

Let $l \in L, r \in R$.
As $l \leq_R r \leq_L l$, it is apparent that $l\mathcal{J}r$. Since S is finite, $\mathcal{J} = \mathcal{D}$, hence $l\mathcal{D}r$.

Let $a \in L \cap R$ and suppose that $|R| > 1$. Then there exist $u, v \in S$ such that $a = auv$. Let $uv = b$, and we see that

$$a = ab = ab^2 = ab^3 = \dots.$$

This implies that $a \leq_L b^m$ for all $m \in \mathbb{N}$, but $a \in L$, a maximal $\mathcal{L}$-class, so $\{b^m : m \in \mathbb{N}\} \subseteq L$.

As S is finite, there exists $n \in \mathbb{N}$ such that b^n is idempotent and so D_a is regular. Each $\mathcal{L}$- and $\mathcal{R}$-class contains at least one idempotent, let $i \in L, j \in R$ be idempotents. Now $j \leq_L i,\ i \leq_R j$, which implies that

$$j = ji = i.$$

Thus $i = j \in L \cap R$.

Now for any $s \in S$, we have $s \leq_L i$ and $s \leq_R i$, which imply that

$$si = s = is,$$

which is to say i is an identity for S. This is a contradiction as S is not a monoid. Hence $|R| = 1$. Using a symmetrical argument, $|L| = 1$, and in turn $D_a = \{a\}$.

Since S is not a monoid, $a^2 \neq a$. Taking the Rees quotient of S by the ideal $S \setminus \{a\}$ we get the two element zero semigroup, X, from Lemma 4.4.1. Applying this lemma and Lemmas 4.3.3 and 4.3.4, we see that $\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta(X^I) \geq (0, 2^I, 2^I, \dots)$.

The upper bound is much easier to attain:

$$\mathbf{d}_\Delta(S^I) \leq (0, |(S^I)^2|, |(S^I)^3|, \dots) = (0, 2^I, 2^I, \dots).$$

Hence $\mathbf{d}_\Delta(S^I) = (0, 2^I, 2^I, \dots)$. □

Recall that Wiegold proved that the d-sequence of any finite monoid is linear and of any finite semigroup without identity is exponential, [29], comparing these results we see that, as with groups, the finite semigroups which have super-logarithmic d-sequences correspond to infinite Cartesian products with infinite relative rank sequences.

Of course, by the fact that any Cartesian product of a semigroup S^I has the semigroup S as a homomorphic image, using Lemma 4.3.4 we know that $\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta(S)$, and by Proposition 4.1.5 the behaviour of $\mathbf{d}_\Delta(S)$ is the same as that of $\mathbf{d}(S)$ if S is finitely generated, we see that the behaviour of the relative rank sequence of an infinite Cartesian product must be at least that of the (non-relative) d-sequence of its base semigroup, if that semigroup is finitely generated.

As we now know the behaviour of the relative rank sequences of all infinite Cartesian products of finite semigroups, and we know that relative rank sequences are non-increasing under homomorphic images, we can state the following corollary.

Corollary 4.4.5. *Let S be a semigroup with finite homomorphic image R.*

If R is not a perfect group, then $\mathbf{d}_\Delta(S^I) \geq (0, 2^I, 2^I, \ldots)$.

Proof. If R is a non-perfect group then by applying Theorem 4.3.5 we see that $\mathbf{d}_\Delta(R^I) \geq (0, 2^I, 2^I, \ldots)$.

If R is a not a group then it is a finite non-group semigroup and by Theorem 4.4.4 we see that $\mathbf{d}_\Delta(R^I) = (0, 2^I, 2^I, \ldots)$.

Lemmas 4.3.3 and 4.3.4 imply that $\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta(R^I)$, and so

$$\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta(R^I) \geq (0, 2^I, 2^I, \ldots).$$

□

This corollary narrows our pool of infinite Cartesian products of semigroups with potentially finite relative rank sequence, but there are of course many semigroups without any finite homomorphic images, or whose only finite images are perfect groups, and so we continue.

The following theorem will determine that the relative rank sequence of an infinite Cartesian products of any non-trivial commutative semigroup is always infinite, narrowing our interest a little further.

Theorem 4.4.6. *Let S be a non-trivial commutative semigroup.*

Then $\mathbf{d}_\Delta(S^I)$ is infinite.

Proof. Note that the commutativity implies that $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$, consider the preorder $(S, \leq)$, where $a \leq b$ if and only if $a \leq_J b$.

If $(S, \leq)$ is bounded above, then $(S, \leq)$ has at least one maximal element. That is S has a $\mathcal{J}$-class, J, such that $ab \in J$ implies $a, b \in J$. Either $S = J$ or $S \setminus J$ is a non-empty ideal. In the first case S is a non-trivial commutative group and Corollary 4.3.2 asserts that $\mathbf{d}_\Delta(S^I) \geq (0, 2^I, 2^I, \ldots)$. In the second case, let $K = S \setminus J$ and let $\varphi : S \to \{a, 0\}$ be the homomorphism defined by $s \mapsto a$ if $s \in J$ and $s \mapsto 0$ if $s \in K$. Note that φ is a homomorphism from S to X or Y from Lemma 4.4.1 since either $J^2 = J$ or $J^2 \subseteq K$.

Applying Lemmas 4.3.3 and 4.3.4 to the homomorphic image X or Y of S, we see that $\mathbf{d}_\Delta(S^I) \geq (0, 2^I, 2^I, \ldots)$.

Suppose that $(S, \leq)$ is not bounded above and S has an idempotent i such that there exists $s \in S$, $s < i$.

Let $K = \{s \in S : i \not\leq s\}$. Note that K is an ideal. Let $s, t \in S \setminus K$, then there exist $x, y \in S^1$ such that $i = sx = ty$, and so

$$i = i^2 = sxty = stxy,$$

hence $st \notin K$. That is $S \setminus K$ is closed and we can find a homomorphism from S to the two element semilattice, Y from Lemma 4.4.1 by mapping $S \setminus K$ to a and K to 0. Thus, by Lemmas 4.4.1, 4.3.3 and 4.3.4, $\mathbf{d}_\Delta(T) \geq (0, 2^I, 2^I, \ldots)$.

Suppose that $(S, \leq)$ is not bounded above, there are no idempotents outwith the minimum $\mathcal{J}$-class if $(S, \leq)$ is bounded below and no idempotents at all otherwise, and there exist $s, t \in S$ such that s is not minimal and $st \mathcal{J} s$.

Let $K = \{r \in S : t^i \not\leq r, \forall i \in \mathbb{N}\}$. Note that if $(t^i)^2 \mathcal{J} t^i$ then there would be an idempotent in J_{t^i}, so $(t^i)^2 < t^i$ for all $i \in \mathbb{N}$ and so $s \mathcal{J} st^{2i} \leq t^{2i} < t^i$ for all $i \in \mathbb{N}$, hence $s \in K$.

Let $a, b \in S$. If $ab \in S \setminus K$ then there exists $i \in \mathbb{N}$ such that $t^i \leq ab \leq a, b$, hence K is an ideal.

Let $a, b \in S \setminus K$. Then there exist $i, j \in \mathbb{N}$ such that $t^i \leq a$ and $t^j \leq b$.

There exist $x, y \in S^1$ such that $t^i = ax$ and $t^j = by$. Considering their product,

$$t^{i+j} = t^i t^j = axby = abxy,$$

we see that $t^{i+j} \leq ab$ and in turn $ab \in S \setminus K$.

Hence, $S \setminus K$ is closed. Now S clearly has the two element semilattice as a homomorphic image and so applying Lemmas 4.4.1, 4.3.3 and then Lemma 4.3.4, we see that $\mathbf{d}_\Delta(S^I) \geq (0, 2^I, 2^I, \ldots)$.

The remaining case is that $(S, \leq)$ is not bounded above and for all elements s, t outwith the minimum $\mathcal{J}$-class (if there is one) $st < s$.

Let $T = S^I$, and suppose that there exists finite $U \subset T^2$ such that $T^2 = \langle U, \Delta_{T^2} \rangle$.

Since U is finite and $(S, \leq)$ is not bounded above, there exist $s \neq t \in T$, such that $(v, w) < (s, t)$ for all $(v, w) \in U$.

There exists an expression for (s, t) in terms of $U \cup \Delta_{T^2}$, and the above inequality demonstrates that U can play no part in it, hence $(s, t) \in \langle \Delta_{T^2} \rangle$ which implies $s = t$, a contradiction, hence U cannot be finite. □

Note that in the above theorem, all but the final case had a stronger lower bound on the relative rank sequence than simply infinite.

We can extend the definition of property $P(m)$ from Definition 4.3.12, in order to apply to semigroups, which will be useful for some coming results.

Definition 4.4.7. Let S be a semigroup and let $m \in \mathbb{N}$.

We say S has property $P'(m)$ if there exists $A \subseteq S^2$ such that $|A| = m$, $k \in \mathbb{N}$, $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in A$, and $\epsilon_1, \ldots, \epsilon_{k+1} \in \{0, 1\}$, such that for any $s, t \in S$, there exist $a_1, a_2, \ldots, a_{k+1} \in S$ such that

$$s = a_1^{\epsilon_1} x_1 a_2^{\epsilon_2} x_2 \ldots x_k a_{k+1}^{\epsilon_{k+1}} \ \& \ t = a_1^{\epsilon_1} y_1 a_2^{\epsilon_2} y_2 \ldots y_k a_{k+1}^{\epsilon_{k+1}}.$$

Note that if S is a monoid all the ϵ terms can be set to 1 as $a_i^0 = 1 \in S$. While this property may look different to property $P(m)$ defined in Definition 4.3.12, it is in fact the same when applied to a group:

Theorem 4.4.8. *Let G be a group, and let $m \in \mathbb{N}$.*

Then G has property $P'(m)$ if and only if G has property $P(m)$.

Proof. Beginning with the forward implication: Let G be a group with property $P'(m)$ for some $m \in \mathbb{N}$.

Pulling directly from the definition of $P'(m)$, there exists $A \subseteq G^2$ such that $|A| = m$, $k \in \mathbb{N}$, $(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k) \in A$ such that for any $s, t \in G$, there exist $a_1, a_2, \dots, a_{k+1} \in G$ such that

$$s = a_1 x_1 a_2 x_2 \dots x_k a_{k+1} \ \& \ t = a_1 y_1 a_2 y_2 \dots y_k a_{k+1}.$$

Note that the ϵ terms were superfluous as G has an identity.

Let $g \in G$. There exist $a_1, a_2, \dots, a_{k+1} \in G$ such that

$$g = a_1 x_1 a_2 \dots x_k a_{k+1} \ \& \ 1 = a_1 y_1 a_2 \dots y_k a_{k+1}.$$

Let $B = \{(xy^{-1}, 1) : (x, y) \in A\}$. The following demonstrates that we can use B in place of A:

$$g = a_1(x_1 y_1^{-1}) y_1 a_2 (x_2 y_2^{-1}) y_2 a_3 \dots (x_k y_k^{-1}) y_k a_{k+1},$$

$$1 = a_1 1 (y_1 a_2) 1 (y_2 a_3) \dots 1 (y_k a_{k+1}).$$

The definition now states that there exist $z_1, z_2, \dots, z_k \in \{b : (b, 1) \in B\}$ such that for any $g \in G$, there exist $a_1, a_2, \dots, a_{k+1} \in G$ such that

$$g = a_1 z_1 a_2 z_2 \dots z_k a_{k+1} \ \& \ 1 = a_1 a_2 \dots a_{k+1}.$$

For $1 \leq i \leq k$ let $c_i = a_{i+1} a_{i+2} \dots a_{k+1}$. Note that the second condition above means that $c_i^{-1} = a_1 a_2 \dots a_i$, and in turn $c_i c_{i+1}^{-1} = a_{i+1}$. Combining all this:

$$g = c_1^{-1} z_1 c_1 c_2^{-1} z_2 c_2 \dots c_k^{-1} z_k c_k = z_1^{c_1} z_2^{c_2} \dots z_k^{c_k}.$$

Which implies, $G = z_1^G z_2^G \dots z_k^G$.

Let $g_1, g_2 \in G$, and let $h \in g_1^G g_2^G$. That is, there exist $b_1, b_2 \in G$ such that $h = (g_1^{b_1})(g_2^{b_2})$. We can conjugate this by $b_3 = (g_2^{b_2})^{-1}$ to see that

$h^{b_3} = (g_2^{b_2})(g_1^{b_1}) \in g_2^G g_1^G$. We can conjugate this by b_3^{-1} to see that

$$h = (g_2^{b_2})(g_1^{b_1})^{b_3^{-1}} = b_3(g_2^{b_2})(g_1^{b_1})b_3^{-1} = b_3(g_2^{b_2})b_3^{-1}b_3(g_1^{b_1})b_3^{-1} = (g_2^{b_2 b_3^{-1}})(g_1^{b_1 b_3^{-1}}),$$

and so $h \in g_2^G g_1^G$, and products of conjugacy classes commute.

As products of conjugacy classes commute, we can gather like terms in the expression $G = z_1^G z_2^G \dots z_k^G$ and we have exactly the statement of $P(l)$ for G, for some $l \leq m$.

Recall that $P(x)$ implies $P(y)$ whenever $x \leq y$. Hence, G has property $P(m)$.

The reverse implication is much simpler: Let G be a group with property $P(m)$ for some $m \in \mathbb{N}$. From the definition, there exist $g_1, g_2, \dots, g_m \in G$ and $n_1, n_2, \dots, n_m \in \mathbb{N}_0$ such that

$$G = (g_1^G)^{n_1}(g_2^G)^{n_2} \dots (g_m^G)^{n_m}.$$

Let $A = \{(g_i, 1) : 1 \leq i \leq m\}$, and let $g \in G = (g_1^G)^{n_1}(g_2^G)^{n_2} \dots (g_m^G)^{n_m}$. Hence, for each $1 \leq i \leq m$ there exist $a_{i,1}, a_{i,2}, \dots, a_{i,n_i} \in G$ such that

$$g = (a_{1,1}^{-1} g_1 a_{1,1} \dots a_{1,n_1}^{-1} g_1 a_{1,n_1}) \dots (a_{m,1}^{-1} g_m a_{m,1} \dots a_{m,n_m}^{-1} g_m a_{m,n_m}),$$

and clearly the product of all the coefficients here equals 1.

Clearly, A provides the necessary set to demonstrate that G has property $P'(m)$. □

This theorem demonstrates that property $P'(m)$ is a semigroup generalisation of the group specific property $P(m)$, so from now on we won't distinguish between the two.

The following example demonstrates that this generalised property may help to find non-group semigroups with finite relative rank sequences. When referring to partial injective mappings, we will use the term n-cycle where $n \in \mathbb{N}$, to describe a cycle of length n, and the term n-path to describe a sequence of length n such that the mapping maps each entry in the sequence to the next, doesn't map anything to the first and doesn't map the last to anything.

Example 4.4.9. The semigroup $\mathcal{I}_{\mathbb{N}}$ of all partial injective mappings from the natural numbers, $\mathbb{N}$, to itself has property $P(1)$, and $\mathbf{d}_{\Delta}(\mathcal{I}_{\mathbb{N}}^{I}) = \mathbf{d}_{\Delta}(\mathcal{I}_{\mathbb{N}}) = (0, 1, 1, \ldots)$.

To see that this is the case, let $\rho \in \mathcal{I}_{\mathbb{N}}$ such that ρ comprises countably many n-cycles, countably many n-paths for each $n \in \mathbb{N}$, countably many infinite cycles and infinite paths, and also doesn't map countably many elements of the domain anywhere.

Let $\tau \in \mathcal{I}_{\mathbb{N}}$. Consider the structure of τ in terms of cycles, paths and undefined elements. There exists $M \subseteq \mathbb{N}$ such that when restricted to M, ρ has the same structure as τ does on all of $\mathbb{N}$.

Let $\sigma \in \mathcal{I}_{\mathbb{N}}$ such that σ maps M to $\mathbb{N}$ bijectively. Now $\sigma^{-1}\rho\sigma$ has the same cycle/path/undefined structure as τ, which means there exists $\alpha \in S_{\mathbb{N}} \subset \mathcal{I}_{\mathbb{N}}$ such that

$$\tau = \alpha^{-1}\sigma^{-1}\rho\sigma\alpha = \rho^{\sigma\alpha}.$$

Of course, $\alpha^{-1}\sigma^{-1}\sigma\alpha = 1$, the identity map.

Let $N \subset \mathbb{N}$ be a moiety of the moiety on which ρ fixes every point, let $\sigma_1 : \mathbb{N} \to \mathbb{N}$ be a bijection which maps N to $(\mathbb{N} \setminus N)$ and $(\mathbb{N} \setminus N)$ to N and let $\gamma = \sigma_1^{-1}\rho\sigma_1$.

Let $A = \{(\rho, \gamma)\}$, this is a sufficient set to satisfy the definition of $P(1)$.

Let $n \geq 2$, and consider $\mathcal{I}_{\mathbb{N}}^{n}$. Let $M_1, M_2, \ldots, M_n$ be disjoint moieties of $\mathbb{N}$, and let $\rho_1, \rho_2, \ldots, \rho_n \in \mathcal{I}_{\mathbb{N}}$ such that, for each i, ρ_i has the same structure as ρ on the domain M_i and is the identity map everywhere else.

Let $\tau \in \mathcal{I}_{\mathbb{N}}$, and fix i. As before, there exists $A \subseteq M_i$ such that ρ_i has the same structure on A as τ has on all of $\mathbb{N}$. We can find $\sigma \in \mathcal{I}_{\mathbb{N}}$ such that σ is a bijection from A to $\mathbb{N}$.

Now ρ_i^{σ} has the same structure as τ, and ρ_j^{σ} is the identity mapping for each $j \neq i$.

There exists $\alpha \in S_{\mathbb{N}} \subset \mathcal{I}_{\mathbb{N}}$ such that $\tau = \rho_i^{\sigma\alpha}$, and conjugating the identity by α will have no effect. Hence,

$$(\rho_1, \ldots, \rho_{i-1}, \rho_i, \rho_{i+1}, \ldots, \rho_n)^{(\sigma\alpha, \ldots, \sigma\alpha)} = (1, \ldots, 1, \tau, 1, \ldots, 1).$$

Of course the same process can be followed for any $\tau \in \mathcal{I}_{\mathbb{N}}$ and for each i, and so

$$\mathcal{I}_{\mathbb{N}}^n = \langle (\rho_1, \ldots, \rho_n), \Delta_{\mathcal{I}_{\mathbb{N}}^n} \rangle,$$

and in turn

$$\mathbf{d}_\Delta(\mathcal{I}_{\mathbb{N}}) = (0, 1, 1, \ldots).$$

Let $T = \mathcal{I}_{\mathbb{N}}^I$, with n and $\rho_1, \ldots, \rho_n \in \mathcal{I}_{\mathbb{N}}$ still defined as above.
Let $C = \{(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) : x_i \in \mathcal{I}_{\mathbb{N}}\}$, $\pi_i = ((\rho_i)^I) \in T$ for each i, and by the above result,

$$C \subset \langle (\pi_1, \ldots, \pi_n), \Delta_{T^n} \rangle.$$

In particular,

$$(\pi_1, \mathbf{1}, \ldots, \mathbf{1}) \in \langle (\pi_1, \ldots, \pi_n), \Delta_{T^n} \rangle.$$

Since $\mathcal{I}_{\mathbb{N}}$ has property $P(1)$,

$$T \times \mathbf{1} \times \ldots \times \mathbf{1} \subset \langle (\pi_1, \mathbf{1}, \ldots, \mathbf{1}), \Delta_{T^n} \rangle \subseteq \langle (\pi_1, \ldots, \pi_n), \Delta_{T^n} \rangle$$

The same process can be undertaken for each of the n components,

$$T^n = \langle (\pi_1, \ldots, \pi_n), \Delta_{T^n} \rangle.$$

Hence, $\mathbf{d}_\Delta(\mathcal{I}_{\mathbb{N}}^I) = \mathbf{d}_\Delta(T) = (0, 1, 1, \ldots)$.

The following lemma begins to convince us that property $P(m)$ is a worthwhile property to investigate as it demonstrates that if S has property $P(m)$ then $d_\Delta((S^I)^2) \leq m$, and it gives us a particularly nice relative generating set for $(S^I)^2$ of size m.

Lemma 4.4.10. *Let S be a semigroup with property $P(m)$ for some $m \in \mathbb{N}$.*
If $T = S^I$, then $d_\Delta(T^2) \leq m$.
Moreover, there exists a relative generating set, U, for T^2, of size m, such that $U \subseteq \{(\mathbf{u}, \mathbf{v}) : u, v \in S\}$.

Proof. As S has property $P(m)$, applying the definition, there exist $A \subseteq S^2$, $k \in \mathbb{N}$, $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in A$, and $\epsilon_1, \ldots, \epsilon_{k+1} \in \{0, 1\}$, such that $|A| = m$ and for any $s, t \in S$, there exist $a_1, a_2, \ldots, a_{k+1} \in S$ such that

$$s = a_1^{\epsilon_1} x_1 a_2^{\epsilon_2} x_2 \ldots x_k a_{k+1}^{\epsilon_{k+1}} \ \& \ t = a_1^{\epsilon_1} y_1 a_2^{\epsilon_2} y_2 \ldots y_k a_{k+1}^{\epsilon_{k+1}}.$$

Let $T = S^I$, and let $U = \{(\mathbf{x}, \mathbf{y}) : (x, y) \in A\} \subset T^2$. Note that in particular, we have $(\mathbf{x_i}, \mathbf{y_i}) \in U$ for $1 \leq i \leq k$.

Let $s, t \in T$. For each $i \in I$, there exist $a_{1,i}, a_{2,i}, \ldots, a_{k+1,i} \in S$ such that

$$s_i = a_{1,i}^{\epsilon_1} x_1 a_{2,i}^{\epsilon_2} x_2 \ldots a_{k,i}^{\epsilon_k} x_k a_{k+1,i}^{\epsilon_{k+1}} \ \& \ t_i = a_{1,i}^{\epsilon_1} y_1 a_{2,i}^{\epsilon_2} y_2 \ldots y_k a_{k+1,i}^{\epsilon_{k+1}}.$$

For each j such that $1 \leq j \leq k+1$, let $a_j = (a_{j,i})_{i \in I} \in T$, and we have

$$(s, t) = (a_1, a_1)^{\epsilon_1} (\mathbf{x_1}, \mathbf{y_1}) (a_2, a_2)^{\epsilon_2} \ldots (\mathbf{x_k}, \mathbf{y_k}) (a_{k+1}, a_{k+1})^{\epsilon_{k+1}} \in \langle U, \Delta_{T^2} \rangle.$$

Hence, $T^2 = \langle U, \Delta_{T^2} \rangle$, and in turn,

$$d_\Delta(T^2) \leq |U| = m.$$

Of course, $U \subseteq \{(\mathbf{u}, \mathbf{v}) : u, v \in S\}$. □

Theorem 4.3.13 states that for a group, G, property $P(m)$ is equivalent to $d_\Delta((G^I)^2) \leq m$, the following theorem will demonstrate that the same is true for the more general property applied to semigroups.

Theorem 4.4.11. *Let S be a semigroup, $T = S^I$ and $m \in \mathbb{N}$.*

Then S has property $P(m)$ if and only if $d_\Delta(T^2) \leq m$.

Proof. The forward implication is exactly the statement of Lemma 4.4.10.

The reverse implication runs on similar lines to the analogous group specific proof: Let S be a semigroup such that if $T = S^I$ then $d_\Delta(T^2) \leq m$. That is, there exists $U \subset T^2$ such that $|U| = m$ and $T^2 = \langle U, \Delta_{T^2} \rangle$.

Suppose that S does not have property $P(m)$.

Let Σ denote the set of all finite sequences comprising elements of U,

$$\Sigma = \{(u_1, u_2, \ldots, u_k) : k \in \mathbb{N}, u_i \in U\}.$$

Note that Σ is countable, so has cardinality less than or equal to I.

Let $\varphi : I \to \Sigma$ be a surjection.
For each $i \in I$, let k_i denote the length of $i\varphi$ and let

$$X_i = \{(a_1, a_1)i\varphi_{1,i}(a_2, a_2)i\varphi_{2,i} \ldots (a_{k_i}, a_{k_i})i\varphi_{k_i,i}(a_{k_i+1}, a_{k_i+1}) : a_j \in S^1\}.$$

Since S does not have property $P(m)$, it holds that $X_i \neq S^2$ for any $i \in I$. For each $i \in I$, let $(s_i, t_i) \in S^2 \setminus X_i$, and let $s = (s_i)_{i\in I}, t = (t_i)_{i\in I}$.

Since $T^2 = \langle U, \Delta_{T^2}\rangle$, there exist $k \in \mathbb{N}$, $u_1, \ldots, u_k \in U$, $d_1, \ldots, d_{k+1} \in T^1$ such that

$$(s, t) = (d_1, d_1)u_1(d_2, d_2)u_2 \ldots u_k(d_{k+1}, d_{k+1}).$$

As φ is a surjection, there exists $\iota \in I$ such that $\iota\varphi = (u_1, u_2, \ldots, u_k)$, however this means that $(s_\iota, t_\iota) \in X_\iota$, a contradiction.

Hence, S has property $P(m)$. □

Theorem 4.3.17 states that if a group G has property $P(m)$ for some $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(G^I) = \mathbf{d}_\Delta(G)$, and we will now see that the same is true of semigroups.

Theorem 4.4.12. *Let $m, n \in \mathbb{N}$, let S be a semigroup with property $P(m)$ and let U be a relative generating set for S^n.*

Then $\{(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}) : (u_1, u_2, \ldots, u_n) \in U\}$ is a relative generating set for $(S^I)^n$, and so $\mathbf{d}_\Delta(S^I) = \mathbf{d}_\Delta(S)$.

Proof. Let $T = S^I$, let $V = \{(\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_n}) : (u_1, u_2, \ldots, u_n \in U\}$, let $D = \{(\mathbf{s}, \mathbf{s}, \ldots, \mathbf{s}) \in T^n : s \in S\}$, and let $R = \{(\mathbf{s_1}, \mathbf{s_2}, \ldots, \mathbf{s_n}) \in T^n : s_i \in S\}$. As $S^n = \langle U, \Delta_{S^n}\rangle$ it is clear that $R = \langle V, D\rangle \subset \langle V, \Delta_{T^n}\rangle$.

We will see that R is a relative generating set for T^n.

For $i \in \mathbb{N}$ let $W_i = \{(t_1, t_2, \ldots, t_n) \in T^n : |\{t_1, t_2, \ldots, t_n\}| \leq i\}$, the set of all elements of T^n which comprise at most i elements of T, note that $W_i = T^n$ for all $i \geq n$. Clearly $W_1 = \Delta_{T^n} \subseteq \langle R, \Delta_{T^n}\rangle$.

As S has property $P(m)$ we can use the set

$$\{(\mathbf{s_1}, \dots, \mathbf{s_n}) : |\{s_1, \dots, s_n\}| \leq 2, s_i \in S\} \subseteq R$$

along with the diagonal to generate W_2.

In order to see that $W_i \subseteq \langle R, \Delta_{T^n} \rangle$ for all $i \in \mathbb{N}$ we will carry out induction on i. Assume $W_j \subseteq \langle R, \Delta_{T^n} \rangle$ for some $j \geq 2$, and let $t_1, \dots, t_{j+1} \in T$.

For $i = 1, 2, \dots, j$ applying $P(m)$ to the pair (t_1, t_i), we see that there exist $k \in \mathbb{N}$, $x_1, y_1, \dots, x_k, y_k \in S$, $\epsilon_1, \dots, \epsilon_{k+1} \in \{0, 1\}$ and $a_{i,1}, \dots, a_{i,k+1} \in \Delta_{T^n}$ such that,

$$t_1 = a_{1,1}^{\epsilon_1} x_1 a_{1,2}^{\epsilon_2} x_2 \dots x_k a_{1,k+1}^{\epsilon_{k+1}},$$

$$t_{i+1} = a_{i,1}^{\epsilon_1} y_1 a_{i,2}^{\epsilon_2} y_2 \dots y_k a_{i,k+1}^{\epsilon_{k+1}}.$$

Consider how many elements appear in any given component of these expressions for $t_1, \dots, t_{j+1}$. If the component corresponds to an x_i or y_i component then it contains either x_i or y_i, a maximum of two distinct entries. If the component corresponds to an $a_{*,i}$ component, then it must come from set $\{a_{1,i}, a_{2,i}, \dots, a_{j,i}\}$, a choice of at most j distinct entries.

This implies that any element of $\{t_1, \dots, t_{j+1}\}^n$ can be expressed as a product of elements from W_j. Hence, $W_{j+1} \subseteq \langle W_j \rangle$, and by induction $W_i \subseteq \langle R, \Delta_{T^n} \rangle$ for all $i \in \mathbb{N}$.

Hence, $T^n = \langle R, \Delta_{T^n} \rangle = \langle V, \Delta_{T^n} \rangle$, and so $d_\Delta(T^n) \leq |V| = |U|$ and in turn, $\mathbf{d}_\Delta(S^I) \leq \mathbf{d}_\Delta(S)$.

By the fact that S is a homomorphic image of S^I, and Lemma 4.3.4, we see that

$$\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta(S).$$

□

In fact, we will see that if S has property $P(m)$ for some $m \in \mathbb{N}$, then $\mathbf{d}_\Delta(S^I)$ is finite, and in particular at most logarithmic.

Theorem 4.4.13. *Let $m \in \mathbb{N}$, let S be a semigroup with property $P(m)$.*

Then $d_\Delta((S^I)^n) \leq m\lceil \log_2 n\rceil$ *for all* $n \in \mathbb{N}$*, and in particular* $\mathbf{d}_\Delta(S^I)$ *is at most logarithmic.*

Proof. By Lemma 4.4.10, $d_\Delta((S^I)^2) \leq m$, and by Theorem 4.4.12 $\mathbf{d}_\Delta(S^I) = \mathbf{d}_\Delta(S)$.

Let U be a relative generating set for S^2 of size m, and let $k \in \mathbb{N}$.

Let $V = \{(((u_1)^{(2^i)}, (u_2)^{(2^i)})^{(2^{k-i-1})}) : (u_1, u_2) \in U, 0 \leq i < k\} \subset S^{2^k}$, and note that $|V| = mk$. By the fact S has property $P(m)$, we can see that $V \cup \Delta_{S^{2^k}}$ generates the set $\{(((s_1)^{(2^i)}, (s_2)^{(2^i)})^{(2^{k-i-1})}) : s_1, s_2 \in S, 0 \leq i < k\}$.

Let $s_1, s_2, s_3, s_4 \in S$. By the fact that S has property $P(m)$, we can find the element $((s_1)^{(2^{k-2})}, (s_2)^{(2^{k-2})}, (s_3)^{(2^{k-2})}, (s_4)^{(2^{k-2})})$ as a product of elements from the set $\{(((t_1)^{(2^{k-2})}, (t_2)^{(2^{k-2})})^{(2)}) : t_1, t_2 \in S\}$ and from the set $\{(((t_1)^{(2^{k-1})}, (t_2)^{(2^{k-1})})) : t_1, t_2 \in S\}$. We can use elements of this form and the set $\{(((t_1)^{(2^{k-3})}, (t_2)^{(2^{k-3})})^{(2^2)}) : t_1, t_2 \in S\}$ to generate elements of the form $(((s_1)^{(2^{k-3})}, \ldots, (s_8)^{(2^{k-3})})^{(2^2)})$ for all $s_1, \ldots, s_8 \in S$.

We can repeat this process until we see that $(s_1, s_2, \ldots, s_{2^k}) \in \langle V, \Delta_{S^{2^k}}\rangle$, and so $S^{2^k} = \langle V, \Delta_{S^{2^k}}\rangle$.

Hence, $d_\Delta(S^{2^k}) \leq mk$. By Theorem 4.4.12 and as relative rank sequences are non-decreasing, $d_\Delta((S^I)^n) = d_\Delta(S^n) \leq m\lceil \log_2 n\rceil$ for all $n \in \mathbb{N}$. □

This means that if for an infinite Cartesian product the relative rank sequence starts finite then it is at most logarithmic. We can use this, and Lemma 4.3.4, to see that if a semigroup S has strictly super-logarithmic relative rank, then the relative rank sequence of S^I is infinite.

Recall that Proposition 4.1.5 showed that for finitely generated semigroups, the behaviour of the d-sequence was the same as that of the relative rank sequence, and so we can deduce that for any finitely generated semigroup S with d-sequence which is strictly super-logarithmic, the relative rank of S^I is infinite.

The property $P(1)$ resembles the property of having a cyclic diagonal bi-act, which is defined as follows:

Definition 4.4.14. The *diagonal bi-act* of a semigroup S is the set $S \times S$

with the actions of S on the left and right acting as follows: $s(x,y) = (sx, sy)$, $(x,y)s = (xs, ys)$.

A diagonal bi-act is said to be *finitely generated* if there exists a finite subset $A \subseteq S \times S$ such that $S \times S = S^1 A S^1$.

A diagonal bi-act is said to be *cyclic* if there exists $a \in S \times S$ such that $S \times S = S^1 a S^1$.

The following lemma demonstrates that the two properties are related.

Lemma 4.4.15. *If S is a monoid with cyclic diagonal bi-act, then S has property $P(1)$.*

Proof. Let S be a monoid with cyclic diagonal bi-act. That is, there exists $(u,v) \in S \times S$ such that $S \times S = S^1(u,v)S^1 = S(u,v)S$.

For any $(s,t) \in S \times S$ there exist $x, y \in S$ such that

$$(s,t) = x(u,v)y = (xuy, xvy) = (x,x)(u,v)(y,y).$$

Hence, S has property $P(1)$. □

The converse, however, is not true as the following example demonstrates.

Example 4.4.16. $S_{\mathbb{N}}$ has property $P(1)$ but does not have a finitely generated diagonal bi-act, let alone a cyclic diagonal bi-act.

Recall from Example 4.3.21 that the symmetric group on the natural numbers, $S_{\mathbb{N}}$, has property $P(1)$.

In his Ph.D. thesis, Thomson proved that a group has finitely generated diagonal bi-act if and only if the group has finitely many conjugacy classes (Proposition 6.7, [24]), and of course $S_{\mathbb{N}}$ has infinitely many conjugacy classes. Hence, $S_{\mathbb{N}}$ does not have a finitely generated cyclic diagonal bi-act.

Another example of a semigroup with property $P(1)$ but no finitely generated diagonal bi-acts comes as a consequence of the following results, Proposition 4.4.17 and Theorem 4.4.18, (and Theorem 4.4.11).

Recall the definition of the Byleen monoid extension, from Section 3.3.

Proposition 4.4.17. *Let $C = \mathcal{C}^1(\{1\}; \sigma, \rho; M)$ a Byleen monoid extension of the trivial monoid $\{1\}$ by the matrix M.*

Then C has no finitely generated diagonal bi-act.

Proof. Suppose that C has a finitely generated diagonal bi-act, which is to say there exists a finite subset $X \subset C \times C$ such that $C^2 = \Delta_{C^2} X \Delta_{C^2}$.

Recall that C admits a unique normal form B^*A^*, where 1 is identified with the empty word.

Let $a_1 \neq a_2 \in A, b_1 \neq b_2 \in B$, such that b_1 does not appear in the normal form for any element in any component of any element of X, this choice is possible as X is a finite set of pairs of elements, and each element can be expressed uniquely as a finite product of elements of $A \cup B \cup \{1\}$.

As $(b_1b_1a_1, b_2a_2) \in S^2$, there exist $(u, v) \in X, w, x \in C$ such that

$$(b_1b_1a_1, b_2a_2) = (w, w)(u, v)(x, x).$$

This implies that

$$b_1b_1a_1 = wux \ \& \ b_2a_2 = wvx.$$

Expressing u, v, w, x in the normal form following the convention, $w = b_wa_w$, where $b_w \in B^*, a_w \in A^*$, we see that

$$b_1b_1a_1 = b_wa_wb_ua_ub_xa_x \ \& \ b_2a_2 = b_wa_wb_va_vb_xa_x.$$

This implies that b_w is a prefix of both b_1b_1 and b_2, but they are distinct in the first entry, hence $b_w = \epsilon$. Similarly, $a_x = \epsilon$.

Now,

$$b_1b_1a_1 = a_wb_ua_ub_x \ \& \ b_2a_2 = a_wb_va_vb_x.$$

Consider the possibilities for a_wb_u: when reduced to normal form it must be an element of A^+ agreeing with a prefix of a_w on all but it's final component, or an element of B^+ agreeing with a suffix of b_u on all but it's first component, or it is the identity. That is, either $a_wb_u = a'_wa$ where $a'_w \in A^+$ is a prefix of a_w and $a \in A$, or $a_wb_u = bb'_u$ where $b \in B$ and $b'_u \in B^+$ is a suffix of b_u, or $a_wb_u = 1$.

Similarly, either $a_u b_x = a'_u a$ where $a'_u \in A^+$ is a prefix of a_u and $a \in A$, or $a_u b_x = bb'_x$ where $b \in B$ and $b'_x \in B^+$ is a suffix of b_x, or $a_u b_x = 1$.

Since $b_1 b_1 a_1 = (a_w b_u)(a_u b_x)$, it is clear that neither can be the identity, and that they can't both be in A^+ or both be in B^+.

If $a_w b_u \in A^+$ and $a_u b_x \in B^+$ then their product is in $A^+ \cup B^+ \cup 1$, a contradiction.

The only remaining possibility is that $a_w b_u \in B^+$ and $a_u b_x \in A^+$, that is $a_w b_u = bb'_u$, $a_u b_x = a'_u a$, and so,

$$b_1 b_1 a_1 = bb'_u a'_u a.$$

This is in the unique normal form so we can equate components,

$$b_1 = b,\ b_1 = b'_u,\ a'_u = \epsilon,\ a_1 = a.$$

However, $b_1 = b'_u$ is a suffix of b_u and so appears in $(u, v) \in X$, a contradiction to the choice of b_1 and in turn to the supposition that C has finitely generated diagonal bi-act. □

Recall the following properties of an $A \times B$ matrix, M, (originally defined for the semigroup $\{0\}$ but here generalised for any semigroup S):

(P1) For every $n \geq 1$, every collection $a_1, \ldots, a_n \in A$ of distinct indices, and every collection $c_1, \ldots, c_n \in A \cup B \cup S$ there exist infinitely many distinct $b \in B$ such that $m_{a_i,b} = c_i$ for all $i = 1, \ldots, n$.

(P2) For every $n \geq 1$, every collection $b_1, \ldots, b_n \in B$ of distinct indices, and every collection $c_1, \ldots, c_n \in A \cup B \cup S$ there exist infinitely many distinct $a \in A$ such that $m_{a,b_i} = c_i$ for all $i = 1, \ldots, n$.

(P3) There exist enumerations $a'_1, a'_2, \ldots$ and $b'_1, b'_2, \ldots$ of A and B respectively, such that $m_{a'_i,b'_i} = b'_{i+1}$ and $m_{a'_i,b'_{i+1}} = a'_{i+1}$, for all $i = 1, 2, \ldots$.

With these in mind we can demonstrate that an infinite Cartesian product of any Byleen monoid extension by a matrix satisfying these properties will have constant relative rank sequence, in fact the least possible sequence $(0, 1, 1, \ldots)$.

Theorem 4.4.18. *Let $C = \mathcal{C}^1(S;\sigma,\rho;M)$ a Byleen monoid extension of the monoid S by the matrix M.*

If M has properties $(P1),(P2),(P3)$, then $\mathbf{d}_\Delta(C^I) = (0,1,1,\ldots)$.

Proof. Let $n \geq 2$, let $x_1, x_2, \ldots, x_n \in A$, all distinct, let $\alpha \in A^*$, and let $a \in A$.

As a consequence of $(P1)$, there exist $b_1, b_2, \ldots, b_n \in B$ such that $\alpha a b_1 = \alpha$ and $\alpha a b_i = 1$ for $i = 2, \ldots, n$. Similarly, there exists $b \in B$ such that $x_i b = b_i$ for $i = 1, \ldots, n$, and so,

$$(\alpha a, \alpha a, \ldots, \alpha a)(x_1, x_2, \ldots, x_n)(b, b, \ldots, b) = (\alpha, 1, \ldots, 1).$$

This can be repeated to gain any element of A^* in any of the components, hence

$$(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\Delta_{C^n}(x_1, x_2, \ldots, x_n)\Delta_{C^n})^n$$

for all $\alpha_i \in A^*$.

Let $\beta \in B^*$, $b \in B$. By $(P2)$, there exist $a_1, a_2, \ldots, a_n \in A$ such that $a_1 b\beta = \beta$ and $a_i b\beta = 1$ for $i = 2, \ldots, n$.

By $(P1)$, there exists $b' \in B$ such that $x_i b' = a_i$ for $i = 1, \ldots, n$, and so

$$(x_1, x_2, \ldots, x_n)(b'b\beta, b'b\beta, \ldots, b'b\beta) = (\beta, 1, \ldots, 1).$$

Repeating for the other components we see,

$$(\beta_1, \beta_2, \ldots, \beta_n) \in ((x_1, x_2, \ldots, x_n)\Delta_{C^n})^n,$$

for all $\beta_i \in B^*$.

Let $s_1, s_2, \ldots s_n \in S$. By $(P2)$, there exists $b \in B$ such that $x_i b = s_i$ for each i, and so

$$(s_1, s_2, \ldots, s_n) = (x_1, x_2, \ldots, x_n)(b, b, \ldots, b) \in (x_1, x_2, \ldots, x_2)\Delta_{C^n}.$$

Hence,

$$C^n = ((x_1, x_2, \ldots, x_n)\Delta_{C^n})^{n+1}(\Delta_{C^n}(x_1, x_2, \ldots, x_n)\Delta_{C^n})^n.$$

It is easy to see that if we let $T = C^I$ and let $u = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}) \in T^n$, it follows that $T^n = \langle u, \Delta_{T^n} \rangle$, and so $\mathbf{d}_\Delta(T) = (0, 1, 1, \ldots)$. □

From this we can draw a nice result about the non-relative rank sequence of Byleen monoid extensions using matrices which satisfy $(P1), (P2)$ and $(P3)$.

Corollary 4.4.19. *Let $C = \mathcal{C}^1(S; \sigma, \rho; M)$ be a Byleen extension of the monoid S by the matrix M.*

If M satisfies $(P1), (P2)$ and $(P3)$, then the (non-relative) d-sequence of C is eventually constant. In fact, $\mathbf{d}(C) = (2, 2, 2, \ldots)$.

Proof. In the proposition we saw that $\mathbf{d}_\Delta(C^I) = (0, 1, 1, \ldots)$. There exist homomorphisms from C^I to C, so by Lemma 4.3.4, $\mathbf{d}_\Delta(C) \leq (0, 1, 1, \ldots)$. Obviously this is the lowest possible relative rank sequence as C is not trivial, hence $\mathbf{d}_\Delta(C) = (0, 1, 1, \ldots)$.

Let $n \geq 2$. There exists $u \in C^n$ such that $C^n = \langle u, \Delta_{C^n} \rangle$.

$(P3)$ ensured that there exist $a \in A, b \in B$ such that $C = \langle a, b \rangle$, and so $\Delta_{C^n} = \langle (a, a, \ldots, a), (b, b, \ldots, b) \rangle$, and in turn,

$$C^n = \langle u, (a, a, \ldots, a), (b, b, \ldots, b) \rangle.$$

In fact, we can go one better than this using the freedom afforded to C by condition $(P1)$: Let $a_1, \ldots, a_n \in A$ be distinct elements, also distinct from a. There exists $b_1 \in C$ such that $a_i b_1 = a$ for $i = 1, \ldots, n$ and $ab_1 = b$. Now,

$$(a, \ldots, a) = (a_1, \ldots, a_n)(b_1, \ldots, b_1),$$

$$(b, \ldots, b) = (a, \ldots, a)(b_1, \ldots, b_1).$$

Hence,

$$C^n = \langle (a_1, \ldots, a_n), (a, \ldots, a), (b, \ldots, b) \rangle \subseteq \langle (a_1, \ldots, a_n), (b_1, \ldots, b_1) \rangle \subseteq C^n,$$

and so, $\mathbf{d}(C) = (2, 2, 2, \ldots)$. $\square$

This corollary serves as evidence to the fact that the study of relative rank sequences can yield interesting results regarding (non-relative) d-sequences.

We can narrow the field of study a little by excluding semigroups which are not generated by any of their $\mathcal{J}$-classes, as the next lemma will demonstrate that if a semigroup S is not generated by any of its $\mathcal{J}$-classes then $\mathbf{d}_\Delta(S^I)$ is infinite.

Lemma 4.4.20. *Let S be a semigroup, and let $T = S^I$.*
If $d_\Delta(T^2)$ is finite, then $S = \langle J \rangle$ for some $\mathcal{J}$-class J.

Proof. Suppose that $d_\Delta(T^2)$ is finite.

By Theorem 4.4.11, S has property $P(m)$ for some $m \in \mathbb{N}$. That is there exist $A \subset S^2$, $k \in \mathbb{N}$, and $(x_1, y_1), \ldots, (x_k, y_k) \in A$, such that $|A| = m$ and for any $s, t \in T$ there exist $a_1, a_2, \ldots, a_{k+1} \in T^1$ such that

$$(s, t) = (a_1, a_1)(\mathbf{x_1}, \mathbf{y_1})(a_2, a_2) \ldots (\mathbf{x_k}, \mathbf{y_k})(a_{k+1}, a_{k+1}).$$

We can find such expressions of $(\mathbf{x_i}, \mathbf{y_i})$ and $(\mathbf{y_i}, \mathbf{x_i})$ for all $i = 1, \ldots k$ demonstrating that $x_i \mathcal{J} x_l \mathcal{J} y_i \mathcal{J} y_l$ for all $1 \leq i, l \leq k$, call this $\mathcal{J}$-class J.

Let $j \in J$ and let $z \in S$.
Since $(\mathbf{j}, \mathbf{z}), (\mathbf{j}, \mathbf{j}) \in T^2$, there exist $a_1, \ldots, a_{k+1}, b_1, \ldots, b_{k+1} \in T^1$ such that,

$$(\mathbf{j}, \mathbf{z}) = (a_1, a_1)(\mathbf{x_1}, \mathbf{y_1})(a_2, a_2) \ldots (\mathbf{x_k}, \mathbf{y_k})(a_{k+1}, a_{k+1}),$$

$$(\mathbf{j}, \mathbf{j}) = (b_1, b_1)(\mathbf{x_1}, \mathbf{y_1})(b_2, b_2) \ldots (\mathbf{x_k}, \mathbf{y_k})(b_{k+1}, b_{k+1}).$$

As $S \setminus J$ is an ideal and each $a_{i,l}$ and $b_{i,l}$ appear in an expression for $j \in J$, no component of a_i or b_i can be from $S \setminus J$, that is $a_{i,l}, b_{i,l} \in J$ for all $1 \le i \le k$ and all $l \in I$. We have already seen that $x_i, y_i \in J$ for all $1 \le i \le k$.

So we have expressions for z in terms of elements of J, that is $z \in \langle J \rangle$. Hence, $S = \langle J \rangle$. □

When investigating semigroups whose infinite Cartesian products might have finite relative rank sequence we need only investigate a semigroup S if all of the following hold:

- S has no finite homomorphic images which are not perfect groups, (Corollary 4.4.5),
- S does not have more than one maximal $\mathcal{L}$- or $\mathcal{R}$-classes, (Lemma 4.4.3),
- S does not have any non-trivial commutative homomorphic images, (Lemmas 4.4.6, 4.3.3 and 4.3.4),
- S has property $P(m)$ for some $m \in \mathbb{N}$, (Theorem 4.4.11),
- S is generated by one of its $\mathcal{J}$-classes, (Lemma 4.4.20).

The next result will not reduce the field of study further, but it will make it so that we only need to concern ourselves with the top $\mathcal{J}$-class of any semigroup we investigate.

Recall that the *Rees quotient* of a semigroup S by an ideal K is the image of a homomorphism which maps elements of $S \setminus K$ to themselves and maps elements of K to 0, we denote the image of this homomorphism S/K.

Lemma 4.4.21. *Let S be a non-simple semigroup with $\mathcal{J}$-class J such that $S = \langle J \rangle$, and let $K = S \setminus J$.*

If $\mathbf{d}_\Delta(S^I)$ is finite, then $\mathbf{d}_\Delta(S^I) = \mathbf{d}_\Delta((S/K)^I)$.

Proof. Suppose $\mathbf{d}_\Delta(S)$ is finite, then by Theorem 4.4.11, S has property $P(m)$ for some $m \in \mathbb{N}$.

Observing that S/K is a homomorphic image of S and applying Lemma 4.3.3 and then Lemma 4.3.4 we see that

$$\mathbf{d}_\Delta(S^I) \geq \mathbf{d}_\Delta((S/K)^I).$$

To demonstrate that this inequality is in all cases an equality we will take a relative generating set for $((S/K)^I)^n$ of least possible size and find a relative generating set for $(S^I)^n$ of the same size:

Let $n \in \mathbb{N}$, let $U \subset ((S/K)^I)^n$ be a relative generating set for $((S/K)^I)^n$ of least possible size and let $j \in K$.

Let $\varphi : S/K \to S$ such that φ fixes any elements in $J = S \backslash K$ and $0\varphi = j$, and let $V = \{((u_{1,i}\varphi)_{i \in I}, (u_{2,i}\varphi)_{i \in I}, \ldots, (u_{n,i}\varphi)_{i \in I}) : (u_1, u_2, \ldots, u_n) \in U\}$.

As S has property $P(m)$ there exists $A \subseteq S^2$ such that $|A| = m$, $k \in \mathbb{N}$, $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k) \in A$, and $\epsilon_1, \ldots, \epsilon_{k+1} \in \{0, 1\}$, such that for any $s, t \in S$, there exist $a_1, a_2, \ldots, a_{k+1} \in S$ such that

$$s = a_1^{\epsilon_1} x_1 a_2^{\epsilon_2} x_2 \ldots x_k a_{k+1}^{\epsilon_{k+1}} \ \& \ t = a_1^{\epsilon_1} y_1 a_2^{\epsilon_2} y_2 \ldots y_k a_{k+1}^{\epsilon_{k+1}}.$$

As K is an ideal, it is clear that $x_i, y_i \in S \setminus K = J$ for each i.

Let $s \in J$ and $t \in K$, and apply the above condition. It is clear that both s and t are expressed as a product of a specific number of elements of J, that number being $l = 2k + 1 - |\{i : \epsilon_i = 0\}|$. This means that any element of $(S^I)^n$ can be expressed as a product of l elements of $(J^I)^n$, and since

$$(J^I)^n \subset \langle U, \Delta_{((S/K)^I)^n} \rangle$$

it holds that

$$(J^I)^n \subset \langle V, \Delta_{(S^I)^n} \rangle,$$

and in turn,

$$(S^I)^n = \langle V, \Delta_{(S^I)^n} \rangle.$$

Hence, $\mathbf{d}_\Delta(S^I) \leq \mathbf{d}_\Delta((S/K)^I)$. □

So when we are interested in the relative rank sequence of an infinite Cartesian product of a semigroup S, we need only concern ourselves with the infinite Cartesian product of the Rees quotient of S by the set of elements not in the top $\mathcal{J}$-class, and so we are reduced to considering only 0-simple semigroups.

We have seen that for semigroups, as for groups, if $\mathbf{d}_\Delta(S^I)$ is finite then $\mathbf{d}_\Delta(S^I) = \mathbf{d}_\Delta(S)$ and is at most logarithmic, and of course the relative rank sequence is non-decreasing and so at least constant. We have found some sufficient conditions on the base semigroup which ensure logarithmic behaviour, and seen some examples of infinite Cartesian products which demonstrate constant relative rank sequence.

The only remaining case for which the behaviour of the relative rank sequence of an infinite Cartesian product has not been determined is if the base semigroup has property $P(m)$ and no non-trivial finite homomorphic images. For this case we offer the following conjecture:

Conjecture 4.4.22. *If S is a semigroup with property $P(m)$ for some $m \in \mathbb{N}$ and no finite homomorphic images, then $\mathbf{d}_\Delta(S^I)$ is constant.*

4.5 Relative Presentations

The relative rank sequence allowed us to assess, in some sense, the rate at which a semigroup grew as we took direct powers, by determining how many extra generators were needed. We can turn this idea to how many extra *relations* might be needed to describe the direct power, in order to assess, in some sense, the rate at which the direct powers become more complex.

As the relative generating sets were taken with respect to the diagonal, we will do the same to establish the notion of a *relative presentation.*

If X is a relative generating set for S^n, and R_Δ denotes the set of all relations which hold on the set Δ_{S^n}, then a *relative presentation of S^n* would have the form

$$S^n = \langle X, \Delta_{S^n} \mid T, R_\Delta \rangle.$$

The key piece of information about a relative presentation would be the minimum size necessary for T in order to have a valid presentation for S^n, if there existed a finite relative generating set X and a finite set of relations T such that $S^n = \langle X, \Delta_{S^n} \mid T, R_\Delta \rangle$, then we would say that S^n is *finitely relatively presentable.*

Of course, if S^n is finitely presentable then S^n is finitely relatively presentable, but it remains to be seen whether there might be a semigroup S and an integer $n \geq 2$ such that S^n is not finitely presentable but it is finitely relatively presentable.

This section serves simply to introduce this new topic as a potential avenue for further study.

We begin by showing that *finite relative presentability* is invariant under the change of finite relative generating set, paralleling the well-known result that if a semigroup is finitely presentable then there exists a finite presentation for any finite generating set.

Lemma 4.5.1. *Let $n \in \mathbb{N}$, and let S be a semigroup such that S^n is finitely relatively presentable.*

Then S^n is finitely relatively presentable with respect to any finite relative generating set.

Proof. Let $X \subseteq S^n$ such that S^n is finitely relatively presentable with respect to X, and let T be a finite set of relations such that $S^n = \langle X, \Delta_{S^n} \mid T, R_\Delta \rangle$.

Let Y be another finite relative generating set for S^n. Every element of X can be expressed as a product of elements from $Y \cup \Delta_{S^n}$, so for each $x \in X$ let p_x be one such expression.

Let $\varphi : X \to (Y \cup \Delta_{S^n})^+$ be defined by $(x)\varphi = p_x$. This can be extended to a mapping Φ from $(X \cup \Delta_{S^n})^+ \times (X \cup \Delta_{S^n})^+$ to $(Y \cup \Delta_{S^n})^+ \times (Y \cup \Delta_{S^n})^+$ by fixing the elements of the diagonal. Let $P = (T)\Phi$, and note that P is finite.

Every relation of S^n which holds true due to T must also hold true due to P, hence $\langle Y, \Delta_{S^n} \mid P, R_\Delta \rangle$ is a valid presentation for S^n, and so S^n is finitely relatively presentable with respect to Y. $\square$

So if a direct power S^n is finitely relatively presentable, then for any finite relative generating set X of S^n, there exists a finite set of relations T such that $S^n = \langle X, \Delta_{S^n} \mid T, R_\Delta \rangle$.

The following lemma will show that the property of finite relative presentability descends the chain of direct powers.

Lemma 4.5.2. *Let S be a semigroup and let $n \geq 2$.*

If S^n is finitely relatively presentable, then S^m is finitely relatively presentable for all $2 \leq m \leq n$.

Proof. Suppose that S^n is finitely relatively presentable. Then there exist finite sets X and T such that $S^n = \langle X, \Delta_{S^n} \mid T, R_\Delta \rangle$.

Let $2 \leq m \leq n$, let $\varphi : S^n \to S^m$ be the homomorphism defined by $(s_1, s_2, \ldots, s_n)\varphi = (s_1, s_2, \ldots, s_m)$, and let $Y = X\varphi$ and $P = \{(u\varphi, v\varphi) : (u, v) \in T\}$. Note that $\Delta_{S^m} = \Delta_{S^n}\varphi$.

If Y is not a relative generating set for S^m, then there exists an element $(s_1, s_2, \ldots, s_m) \in S^m$ which cannot be expressed as a product of elements from Y and Δ_{S^m}, but then for any $s_{m+1}, \ldots, s_n \in S$ the element $(s_1, s_2, \ldots, s_n) \in S^n$ cannot be expressed as a product of elements from X and Δ_{S^n}, a contradiction. Hence, Y is a relative generating set for S^m.

It is clear that every relation in P must be true in S^m, otherwise the preimage in T would not hold true in S^n. It remains to see that there are no relations which hold true in S^m which do not come as a consequence of P, but this must be the case as any such relation could be extended to a relation in S^n which would hold true as a consequence of T and would have its image hold true as a consequence of the image of T.

Hence, $S^m = \langle Y, \Delta_{S^m} \mid P, R_\Delta \rangle$, and so S^m is finitely relatively presentable. □

In Example 4.2.2 we saw that $T_{\mathbb{N}}^n$ has a finite relative generating set for all $n \in \mathbb{N}$, so we can ask the question as to whether there exists a relative finite presentation for $T_{\mathbb{N}}^n$.

Proposition 4.5.3. *$T_{\mathbb{N}}^n$ is not finitely relatively presentable for any $n \geq 2$.*

Proof. Let $\alpha, \beta \in T_{\mathbb{N}}^2$ such that $\alpha : n \mapsto 2n$ and $\beta : n \mapsto 2n - 1$.

For any $(\pi, \tau) \in T_{\mathbb{N}}^2$, there exists a unique $\delta \in T_{\mathbb{N}}$ such that $(\pi, \tau) = (\alpha, \beta)(\delta, \delta)$. In particular, δ defined by $\delta : 2n \mapsto (n)\pi$ and $\delta : 2n - 1 \mapsto (n)\tau$. Hence, $(\alpha, \beta)\Delta_{T_{\mathbb{N}}^2}$ is a set of unique normal forms for $T_{\mathbb{N}}^2$, and $T_{\mathbb{N}}^2 = \langle (\alpha, \beta), \Delta_{T_{\mathbb{N}}^2} \rangle$.

Suppose there is a finite P such that we get a valid presentation

$$T_{\mathbb{N}}^2 = \langle (\alpha, \beta), \Delta_{T_{\mathbb{N}}^2} \mid P, R_\Delta \rangle.$$

This means that we can take any finite string over $\{(\alpha, \beta)\} \cup \Delta_{T_{\mathbb{N}}^2}$ and apply a finite number of relations from $P \cup R_\Delta$ to reach something in the normal form. In particular, for each $\rho \in T_{\mathbb{N}}$ we can use finitely many relations from $P \cup R_\Delta$ to rewrite $(\rho, \rho)(\alpha, \beta)$ to normal form.

Let $A \subset T_{\mathbb{N}}$ be the set of elements which occur in any of the diagonal elements in relations from P. Clearly A is finite since P is a finite collection of finite statements. Let $\rho \in T_{\mathbb{N}} \setminus \langle A \rangle$. Since $T_{\mathbb{N}}$ is not finitely generated we know that such a ρ can be found, and since $\rho \notin \langle A \rangle$ we know that $(\rho, \rho)(\alpha, \beta)$ cannot be rewritten to normal form by $P \cup R_\Delta$, a contradiction to the existence of a finite set P.

Applying the contrapositive of Lemma 4.5.1, we see that $T_{\mathbb{N}}^2$ is not finitely relatively presentable.

Lemma 4.5.2 shows that if $T_{\mathbb{N}}^n$ is finitely relatively presentable for some $n > 2$ then $T_{\mathbb{N}}^2$ is finitely relatively presentable, and as this is not the case we see that $T_{\mathbb{N}}^n$ is not finitely relatively presentable for any $n \geq 2$. $\square$

Just because direct powers of the full transformation monoid do not have finite relative presentations does not necessarily imply that there are no semigroups of which direct powers have finite relative presentations, this leads us to a question which ends this chapter:

Question. Does there exist a semigroup S which is not finitely presentable, such that S^2 is finitely relatively presentable?

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