Higher moments for random multiplicative measures

K.J. Falconer

Mathematical Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland

Abstract

We obtain a condition for the L^q-convergence of martingales generated by random multiplicative cascade measures for $q > 1$ without any self-similarity requirements on the cascades.

1 Introduction

Random multiplicative cascades were introduced as a model for turbulence by Mandelbrot [8, 9] in 1974 since when many variants have been studied. These cascades take the form of a sequence of random measures μ_k obtained as the product of random weights indexed by the vertices of an m-ary tree, or equivalently as measures on the hierarchy of m-ary subintervals of the unit interval. The cascades have the property that the sequence $\mu_k(A)$ is a martingale for each set A, so that μ_k converges to a random measure μ_∞. Natural questions about μ_∞ relate to its non-degeneracy, its moments and its Hausdorff dimension. Such questions were addressed for basic self-similar cascades by Mandelbrot [8, 9], Kahane [4] and Kahane and Peyrière [4, 5], and many other aspects and variants of cascades have been developed since, see [1, 6, 7] and the references therein. Much of this work concerns cascades where the weights are self-similar, giving rise to a stochastic functional equation which may be solved to establish properties of the limiting measure. Here we obtain conditions for the qth moments of the measures $\mu_k(A)$ to be bounded, which implies almost sure and L^q-convergence for $q > 1$, without the requirement of self-similarity so that the functional equation methods are not applicable. This was established in the case $1 < q \leq 2$ by Barral and Mandelbrot [1], indeed in a more general setting, but the approach here extends to all $q > 1$. The analysis required seems significantly more awkward when $q > 2$ and in particular when q is not an integer.

To state the main result we need some standard notation. Random multiplicative cascades are indexed by a *symbolic space* of words formed from the symbols $\{1, 2, \ldots, m\}$. For $k = 0, 1, 2, \ldots$ let I_k be the set of all k-term words $I_k = \{(i_1, i_2, \ldots, i_k) : 1 \leq i_j \leq m\}$, taking I_0 to consist of the empty word $\emptyset$. We often abbreviate a word in I_k by $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and write $|\mathbf{i}| = k$ for its length. We let $I = \cup_{k=0}^{\infty} I_k$ denote the set of all finite words with I_∞ for the corresponding set of infinite words, so $I_\infty = \{(i_1, i_2, \ldots) : 1 \leq i_j \leq m\}$. Juxtaposition of $\mathbf{i}$ and $\mathbf{j}$ is written $\mathbf{ij}$. We write $\mathbf{i}|k = (i_1, \ldots, i_k)$ for the *curtailment* after k terms of $\mathbf{i} = (i_1, i_2, \ldots) \in I_\infty$, or of $\mathbf{i} = (i_1, \ldots, i_{k'}) \in I$ if $k \leq k'$, with $\mathbf{i} \preceq \mathbf{j}$ meaning that $\mathbf{i}$ is a curtailment of $\mathbf{j}$. If $\mathbf{i}, \mathbf{j} \in I \cup I_\infty$ then $\mathbf{i} \wedge \mathbf{j}$ is the maximal word such that both $\mathbf{i} \wedge \mathbf{j} \preceq \mathbf{i}$ and $\mathbf{i} \wedge \mathbf{j} \preceq \mathbf{j}$. We may topologise I_∞ in a natural way by the metric $d(\mathbf{i}, \mathbf{j}) = m^{-|\mathbf{i} \wedge \mathbf{j}|}$ for distinct $\mathbf{i}, \mathbf{j} \in I_\infty$ which makes I_∞ into a compact metric space, with the *cylinders* $C_\mathbf{i} = \{\mathbf{j} \in I_\infty : \mathbf{i} \preceq \mathbf{j}\}$ for $\mathbf{i} \in I$ forming a base of open and closed neighbourhoods. Note that these cylinders are often identified with the hierarchy of m-ary subintervals of $[0,1]$ in the natural way, with each $C_\mathbf{i}$ corresponding to a subinterval of length $m^{-|\mathbf{i}|}$; thus the functions and measures introduced below may be thought of as defined on $[0,1]$.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. Let $\{W_\mathbf{i} : \mathbf{i} \in I\}$ be independent strictly positive random variables, not necessarily identically distributed, with $\mathsf{E}(W_\mathbf{i}) = 1$ for each $\mathbf{i} \in I$, and set

$$X_{i_1,\dots,i_k} = W_{i_1} W_{i_1,i_2} \dots W_{i_1,\dots,i_k}.$$

Let $\mathcal{F}_k = \sigma(W_\mathbf{i} : |\mathbf{i}| \leq k)$ be the σ-field underlying the random variables indexed by words of length at most k. Note that $(X_{\mathbf{i}|k}, \mathcal{F}_k)$ is a martingale for each $\mathbf{i} \in I_\infty$; such a family of martingales is termed a *T-martingale*, see [4].

Let μ be a given Borel probability measure on I_∞. We may define a sequence of random measures μ_k on I_∞ by

$$\mu_k(A) = \int_A X_{\mathbf{i}|k} d\mu(\mathbf{i}), \tag{1.1}$$

for μ-measurable A, so in particular for cylinders

$$\mu_k(C_\mathbf{v}) = X_{\mathbf{v}|k} \mu(C_\mathbf{v}) \quad \text{if } k \leq |\mathbf{v}|.$$

It follows that $(\mu_k(A), \mathcal{F}_k)$ is a martingale for each Borel set A. By the martingale convergence theorem $\mu_k(A)$ converges with probability one for each A and μ_k converges weakly to a measure μ_∞ on I_∞.

In particular the sequence of random variables

$$\mu_k(I_\infty) = \int_{I_\infty} X_{\mathbf{i}|k} d\mu(\mathbf{i}) = \sum_{|\mathbf{i}|=k} X_\mathbf{i} \mu(C_\mathbf{i}) \tag{1.2}$$

converges almost surely to $\mu_\infty(I_\infty)$. If $\mu_k(I_\infty)$ is L^q-bounded for some $q > 1$, i.e. if

$$\sup_k \mathsf{E}\Big(\Big(\sum_{|\mathbf{i}|=k} X_\mathbf{i} \mu(C_\mathbf{i})\Big)^q\Big) < \infty, \tag{1.3}$$

then, using Minkowski's inequality,

$$\limsup_{k\to\infty} \Big(\sum_{|\mathbf{i}|=k} \mathsf{E}\big((X_\mathbf{i} \mu(C_\mathbf{i}))^q\big)\Big)^{1/k} \leq 1. \tag{1.4}$$

We seek a converse implication, that is a condition such as (1.4) for L^q convergence of $\mu_k(I_\infty)$. In the self-similar case, that is where $(W_{\mathbf{i}1}, \dots, W_{\mathbf{i}m})$ has the distribution of $(W_1, \dots, W_m)$ for all $\mathbf{i}$, (1.3) holds if and only if $\mathsf{E}(\sum_{i=1}^m W_i^q) < 1$, see, for example, [4, 5, 6]. Here we dispense with any self-similarity assumption.

Theorem 1.1 *Let $X_{\mathbf{i}}$ be a multiplicative random cascade. Let $q > 1$ and suppose that*

$$\limsup_{k\to\infty} \Big(\sum_{|\mathbf{i}|=k} \mathsf{E}\big((X_{\mathbf{i}}\mu(C_{\mathbf{i}}))^q\big)\Big)^{1/k} < 1. \tag{1.5}$$

Then

$$\limsup_{k\to\infty} \mathsf{E}\Big(\Big(\int X_{\mathbf{i}|k} d\mu(\mathbf{i})\Big)^q\Big) = \limsup_{k\to\infty} \mathsf{E}\Big(\Big(\sum_{|\mathbf{i}|=k} X_{\mathbf{i}}\mu(C_{\mathbf{i}})\Big)^q\Big) < \infty. \tag{1.6}$$

Moreover $\int X_{\mathbf{i}|k} d\mu(\mathbf{i})$ converges almost surely and in L^q.

Of course, the final conclusion follows immediately from (1.6) which says that the martingale $\int X_{\mathbf{i}|k} d\mu(\mathbf{i})$ is L^q-bounded.

We will use arguments based on trees and their automorphisms to study the higher moments of multiplicative processes. Tree structures are used in a rather different way to find the L^q-dimensions of self-affine sets [2] and the L^q-dimensions of the images of measures under certain Gaussian processes [3].

2 Trees and automorphisms

This section sets out the notation needed relating to the underlying tree structure of the multiplicative cascades.

The integers $m \geq 2$ and $k \geq 1$ are fixed throughout this and the next section. For $q > 1$ we write $q = n + \epsilon$ where n is an integer and $0 \leq \epsilon < 1$.

We identify the finite words of the symbolic space I with the vertices of the m-ary rooted tree T with root $\emptyset$ in the natural way. We write T_k for the finite rooted tree with vertices $\cup_{l=0}^{k} I_l$. The edges of these trees join each vertex $\mathbf{i} \in \cup_{l=0}^{k-1} I_l$ to its m 'children' $\mathbf{i}1, \ldots, \mathbf{i}m$. Thus the words in I_k are 'bottom' vertices of T_k. We will write T_k to denote these trees regarded both as graphs and as sets of vertices, the context making the usage clear. The estimates in the next section involve automorphisms of the tree T_k, regarded as a graph, which induce permutations of the vertices at each level of the tree.

For each vertex $\mathbf{v} \in T_k$ and $n \geq 1$ we write

$$S_{\mathbf{v}}(n) = \{(\mathbf{i}_1, \ldots, \mathbf{i}_n) \succeq \mathbf{v}\} \subseteq (I_k)^n \tag{2.1}$$

for the set of all ordered n-tuples of I_k (with repetitions allowed) that are descendents of $\mathbf{v}$. Let $\mathrm{Aut}_{\mathbf{v}}$ be the group of automorphisms of the rooted tree T_k that fix $\mathbf{v}$. Define an equivalence relation $\sim$ on $S_{\mathbf{v}}(n)$ by

$$(\mathbf{i}_1, \ldots, \mathbf{i}_n) \sim (\mathbf{i}'_1, \ldots, \mathbf{i}'_n) \text{ if there exists } g \in \mathrm{Aut}_{\mathbf{v}} \text{ such that } g(\mathbf{i}_r) = \mathbf{i}'_r \text{ for all } 1 \leq r \leq n; \tag{2.2}$$

thus the equivalence classes are the orbits of $(I_k)^n$ under $\mathrm{Aut}_{\mathbf{v}}$, and we write $S_{\mathbf{v}}(n)/\sim$ for the set of equivalence classes. We write $[J]_{\mathbf{v}}$ for the equivalence class containing $J = (\mathbf{i}_1, \ldots, \mathbf{i}_n)$.

To work with the case when q is non-integral we need to introduce an extra identified point into these orbits. Let $\mathbf{v} \in T_k$ and let $\mathbf{i}_1, \ldots, \mathbf{i}_n$ be (not necessarily distinct) points of I_k such that $\mathbf{v} \preceq \mathbf{i}_r$ for all $r = 1, \ldots, n$. Write $T_{\mathbf{v}}(\mathbf{i}_1, \ldots, \mathbf{i}_n)$ for the minimal subtree of T_k rooted at $\mathbf{v}$ and containing the points $\{\mathbf{i}_1,\dots,\mathbf{i}_n\}$. For each such $\mathbf{v}\in T_k$ define the ordered set of $(n+1)$-tuples

$$S^+_{\mathbf{v}}(n)=\{(\mathbf{i}_1,\dots,\mathbf{i}_n;\mathbf{p}):(\mathbf{i}_1,\dots,\mathbf{i}_n)\succeq\mathbf{v}\text{ and }\mathbf{p}\in T_{\mathbf{v}}(\mathbf{i}_1,\dots,\mathbf{i}_n)\}\tag{2.3}$$

(note that in (2.3) $\mathbf{p}$ can be a point of $T_{\mathbf{v}}(\mathbf{i}_1,\dots,\mathbf{i}_n)$ at any level). We may define an equivalence relation $\approx$ on each $S^+_{\mathbf{v}}(n)$ by

$$(\mathbf{i}_1,\dots,\mathbf{i}_n;\mathbf{p})\approx(\mathbf{i}'_1,\dots,\mathbf{i}'_n;\mathbf{p}')\text{ if there exists }g\in\mathrm{Aut}_{\mathbf{v}}\\ \text{such that }g(\mathbf{i}_r)=\mathbf{i}'_r\text{ for all }1\le r\le n\text{ and }g(\mathbf{p})=\mathbf{p}'.\tag{2.4}$$

We write $S^+_{\mathbf{v}}(n)/\approx$ for the set of equivalence classes. Observe that, with $\mathbf{p}\in T_{\mathbf{v}}(\mathbf{i}_1,\dots,\mathbf{i}_n)$, the action of $\mathrm{Aut}_{\mathbf{v}}$ on $\mathbf{p}$ is completely determined by the action of $\mathrm{Aut}_{\mathbf{v}}$ on $(\mathbf{i}_1,\dots,\mathbf{i}_n)$.

For notational simplicity, we may omit the subscript when $\mathbf{v}=\emptyset$, so that $S(n)\equiv S_\emptyset(n)$, $S^+(n)\equiv S^+_\emptyset(n)$, $[J]=[J]_\emptyset$ and $T(\mathbf{i}_1,\dots,\mathbf{i}_n)\equiv T_\emptyset(\mathbf{i}_1,\dots,\mathbf{i}_n)$.

We require some terminology relating to the join sets of elements of I_k. Let $J=(\mathbf{i}_1,\dots,\mathbf{i}_n)\in(I_k)^n$. The *join set* of J, denoted by $\bigwedge(J)=\bigwedge(\mathbf{i}_1,\dots,\mathbf{i}_n)$, is the set of vertices $\{\mathbf{v}_1,\dots,\mathbf{v}_{n-1}\}\subseteq T_k$ consisting of the *join points* $\mathbf{i}_i\wedge\mathbf{i}_j$ for all $\mathbf{i}_i,\mathbf{i}_j\in I_k$, with $\mathbf{w}\in\bigwedge(J)$ occurring with *multiplicity* m if there are $(m+1)$ distinct $\mathbf{i}_{i_1},\dots,\mathbf{i}_{i_{m+1}}\in J$ such that $\mathbf{i}_{i_r}\wedge\mathbf{i}_{i_s}=\mathbf{w}$ for all $r\neq s$. Note that if two or more of the $\mathbf{i}_i$ are equal, then this common point is automatically a join point with the appropriate multiplicity.Thus $\bigwedge(\mathbf{i}_1,\dots,\mathbf{i}_n)$ consists of the vertices of the subtree $T_{\mathbf{v}}(\mathbf{i}_1,\dots,\mathbf{i}_n)$ that have at least two offspring together with any repeated $\mathbf{i}_i$. Note that the join set of n points always consists of $n-1$ points counting by multiplicity, and if J consists of a single point its join set is empty. We write $\wedge^T(\mathbf{i}_1,\dots,\mathbf{i}_n)$ for the 'top join point' of $\{\mathbf{i}_1,\dots,\mathbf{i}_n\}$, that is the vertex $\mathbf{v}$ such that $\mathbf{v}\preceq\mathbf{i}_j$ for all $1\le j\le n$ for which $|\mathbf{v}|$ is greatest.

If $\mathbf{j}\succeq\mathbf{v}$ we write $\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1,\dots,\mathbf{i}_n)$ for the vertex $\mathbf{j}|i$ for the largest i such that $\mathbf{j}|i\in T_{\mathbf{v}}(\mathbf{i}_1,\dots,\mathbf{i}_n)$.

The *level* of a vertex $\mathbf{v}\in I$ is just $|\mathbf{v}|$. Thus the *set of join levels* $L(J)$ of $J\in S_{\mathbf{v}}(n)$ is $\{|\mathbf{v}_1|,\dots,|\mathbf{v}_{n-1}|:\mathbf{v}_i\in\bigwedge(J)\}$ with levels repeated according to multiplicity. Notice that if $J\sim J'$ then $L(J)=L(J')\equiv L([J]_{\mathbf{v}})$, i.e. the set of levels is constant across each orbit $[J]_{\mathbf{v}}$ of $(I_k)^n$.

Similarly, given $J=(\mathbf{i}_1,\dots,\mathbf{i}_n;\mathbf{p})\in S^+_{\mathbf{v}}(n)$ we let $L(J)$ be the set of $n-1$ join points of $(\mathbf{i}_1,\dots,\mathbf{i}_n)$ and write $l_0(J)=|\mathbf{p}|$ for the level of the special point $\mathbf{p}$. Again this is independent of the choice of J in any orbit in $S^+_{\mathbf{v}}(n)/\approx$.

3 The main estimates

This section contains the substance of the proof of Theorem 1.1 which involves an inductive argument. The induction is significantly more complicated when $q=n+\epsilon$ is non-integral, that is when $0<\epsilon<1$, when one of the vertices of the underlying tree has to be specifically identified with the 'ϵ' term and we need to work with $S^+(n)$ rather than $S(n)$.

The following identity, which follows from the martingale property and additivity of the measure, will be used repeatedly: for all $\mathbf{w}\in T_k$ and $|\mathbf{w}|\le l\le k$,

$$\sum_{|\mathbf{v}|=l,\mathbf{v}\succeq\mathbf{w}}\mathsf{E}\big(X_{\mathbf{v}}\mu(C_{\mathbf{v}})\big|\mathcal{F}_{\mathbf{w}}\big)=\sum_{|\mathbf{v}|=l,\mathbf{v}\succeq\mathbf{w}}X_{\mathbf{w}}\mu(C_{\mathbf{v}})=X_{\mathbf{w}}\mu(C_{\mathbf{w}}).\tag{3.1}$$

For convenience we define the random variables

$$Y_{\mathbf{v}} \equiv X_{\mathbf{v}}\mu(C_{\mathbf{v}}) \qquad (\mathbf{v} \in I) \tag{3.2}$$

so (3.1) becomes

$$\sum_{|\mathbf{v}|=l,\mathbf{v}\succeq\mathbf{w}} \mathsf{E}\big(Y_{\mathbf{v}}\big|\mathcal{F}_{\mathbf{w}}\big) = Y_{\mathbf{w}} \qquad (l \geq |\mathbf{w}|). \tag{3.3}$$

Thus to prove Theorem 1.1 we must show that for $q \geq 1$,

$$\limsup_{k\to\infty}\Big(\sum_{|\mathbf{j}|=k}\mathsf{E}\big(Y_{\mathbf{j}}^q\big)\Big)^{1/k} < 1 \quad \text{implies} \quad \limsup_{k\to\infty}\mathsf{E}\Big(\big(\sum_{|\mathbf{i}|=k}Y_{\mathbf{i}}\big)^q\Big) < \infty. \tag{3.4}$$

Note that (3.3) implies that for $0 \leq \epsilon \leq 1$

$$\mathsf{E}\Big(\big(\sum_{|\mathbf{v}|=l,\mathbf{v}\succeq\mathbf{w}} Y_{\mathbf{v}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{w}}\Big) \leq \Big(\mathsf{E}\sum_{|\mathbf{v}|=l,\mathbf{v}\succeq\mathbf{w}} Y_{\mathbf{v}}\big|\mathcal{F}_{\mathbf{w}}\Big)^{\epsilon} = Y_{\mathbf{w}}^{\epsilon} \qquad (l \geq |\mathbf{w}|). \tag{3.5}$$

The strategy of the proof, in the simpler case when q is an integer, is first to estimate the sum of the terms $\mathsf{E}(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n})$ over the $(\mathbf{i}_1\dots\mathbf{i}_n)$ within each equivalence class of $S(n)/\sim$ and then sum these estimates over all the equivalence classes. In the case when q is non-integral, that is when $\epsilon > 0$, there is a further initial stage involving summing within the equivalence classes of $S^+(n)/\approx$.

Note that in (3.6) and below, the product is over the set of levels in a join class. The symbol $[n-1]$ above the product sign merely indicates that there are $n-1$ terms in this product; this convention is helpful when keeping track of terms through the proofs. We take the empty product, that is when $n = 1$ in (3.6), to equal 1.

Proposition 3.1 *Let $q = n + \epsilon > 1$ with n an integer and $0 \leq \epsilon < 1$. Let $J \in S^+(n)$. Then*

$$\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n;\mathbf{p})\in[J]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big(\sum_{\substack{|\mathbf{j}|=k\\ \mathbf{j}\wedge T(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}}} Y_{\mathbf{j}}\big)^{\epsilon}\Big) \leq \prod_{l\in L(J)}^{[n-1]}\Big(\sum_{|\mathbf{u}|=l}\mathsf{E}\big(Y_{\mathbf{u}}^q\big)\Big)^{1/(q-1)}\Big(\sum_{|\mathbf{u}|=l(J)}\mathsf{E}\big(Y_{\mathbf{u}}^q\big)\Big)^{\epsilon/(q-1)}. \tag{3.6}$$

(When $\epsilon = 0$ the two terms involving ϵ disappear and $\mathbf{p}$ *becomes redundant.)*

Proposition 3.1 will follow immediately from the following two lemmas which establish inductive hypotheses that specialize to (3.6). The proof of the first lemma, dealing with the case of q an integer (i.e. with $\epsilon = 0$), is simpler, whilst the proof of the second lemma, for non-integral q, both depends on the first result and requires an extension of the approach.

Lemma 3.2 *For all integers $n \geq 1$, for all $q \geq n$ and all $\mathbf{v} \in T_k$, if $J \in S_{\mathbf{v}}(n)$ then*

$$\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big|\mathcal{F}_{\mathbf{v}}\Big) \leq Y_{\mathbf{v}}^{(q-n)/(q-1)}\prod_{l\in L(J)}^{[n-1]}\Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}}\mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)}; \tag{3.7}$$

Proof. We obtain (3.7) by induction on n.

Start of induction If $n = 1$, $\mathbf{v} \in T_k$ and $J = (\mathbf{j}_1) \in S_{\mathbf{v}}(1)$ identity (3.3) gives

$$\sum_{(\mathbf{i}_1)\in[J]_{\mathbf{v}}} \mathsf{E}\big(Y_{\mathbf{i}_1} \big| \mathcal{F}_{\mathbf{v}}\big) = Y_{\mathbf{v}} = Y_{\mathbf{v}}^{(q-1)/(q-1)}$$

which is (3.7) when $n = 1$.

The inductive step Assume that for some integer $n_0 \geq 1$ inequality (3.7) holds for all $1 \leq n \leq n_0$, for all $\mathbf{v} \in T_k$ and all $J \in S_{\mathbf{v}}(n)$. We establish (3.7) when $n = n_0 + 1$. Let $\mathbf{v} \in T_k$ and $J = (\mathbf{j}_1, \dots, \mathbf{j}_n) \in S_{\mathbf{v}}(n)$. We divide the argument into two cases.

Case (a) Assume that $\mathbf{v} = \wedge^T(\mathbf{j}_1, \dots, \mathbf{j}_n)$; thus $\mathbf{v} \preceq \mathbf{j}_i$ for all i and $\mathbf{v}$ is itself the join point of at least two points of $\{\mathbf{j}_1, \dots, \mathbf{j}_n\}$.

If $|\mathbf{v}| = k$, that is $\mathbf{v} = \mathbf{j}_1 = \cdots = \mathbf{j}_n$, then (3.7) is trivially satisfied.

Otherwise J decomposes into $2 \leq r < n$ subsets,

$$J_1 = (\mathbf{j}^1_1, \dots, \mathbf{j}^1_{n_1}) \in S_{\mathbf{v}}(n_1), \dots,\ J_r = (\mathbf{j}^r_1, \dots, \mathbf{j}^r_{n_r}) \in S_{\mathbf{v}}(n_r),$$

say, without loss of generality, where $1 \leq n_i \leq n-1$ for each i, and

$$n_1 + \cdots + n_r = n, \tag{3.8}$$

and such that each tree $T_{\mathbf{v}}(\mathbf{j}^i_1, \dots, \mathbf{j}^i_{n_i})$ has a distinct *single* edge abutting $\mathbf{v}$. Note that the combinatorics of such a decomposition is preserved under every automorphism in $\mathrm{Aut}_{\mathbf{v}}$. We write $L(J_i) \geq 0$ for the set of $(n_i - 1)$ join levels of the trees $T_{\mathbf{v}}(\mathbf{j}^i_1, \dots, \mathbf{j}^i_{n_i})$ (counted by multiplicity) for $i = 1, \dots, r$.

Using independence conditional on $\mathcal{F}_{\mathbf{v}}$ and applying the inductive assumption (3.7) to $J_1, \dots J_r$,

$$\begin{aligned}
&\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1} \dots Y_{\mathbf{i}_n} \big| \mathcal{F}_{\mathbf{v}}\Big) \\
&\leq \mathsf{E}\Big(\sum_{(\mathbf{i}^1_1,\dots,\mathbf{i}^1_{n_1})\in[J_1]_{\mathbf{v}}} Y_{\mathbf{i}^1_1} \cdots Y_{\mathbf{i}^1_{n_1}} \big| \mathcal{F}_{\mathbf{v}}\Big) \times \cdots \times \mathsf{E}\Big(\sum_{(\mathbf{i}^r_1,\dots,\mathbf{i}^r_{n_r})\in[J_r]_{\mathbf{v}}} Y_{\mathbf{i}^r_1} \cdots Y_{\mathbf{i}^r_{n_r}} \big| \mathcal{F}_{\mathbf{v}}\Big) \\
&\leq Y_{\mathbf{v}}^{(q-n_1)/(q-1)} \prod_{l\in L(J_1)}^{[n_1-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y^q_{\mathbf{u}} \big| \mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \Big) \times \cdots \\
&\qquad \times Y_{\mathbf{v}}^{(q-n_r)/(q-1)} \prod_{l\in L(J_r)}^{[n_r-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y^q_{\mathbf{u}} \big| \mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \\
&= Y_{\mathbf{v}}^{(q-n_1-\cdots-n_r)/(q-1)} \big(Y^q_{\mathbf{v}}\big)^{(r-1)/(q-1)} \times \prod_{l\in L(J_1)\cup\cdots\cup L(J_r)}^{[n_1+\cdots+n_r-r]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y^q_{\mathbf{u}} \big| \mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \\
&= Y_{\mathbf{v}}^{(q-n)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y^q_{\mathbf{u}} \big| \mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)},
\end{aligned}$$

where we have used (3.8), and incorporated the terms $Y^q_{\mathbf{v}}$, taken as a trivial sum over the single vertex $\mathbf{v}$, in the main product with multiplicity $(r-1)$, to get (3.7) in this case.

Case (b) Now with $\mathbf{v} \in T_k$ and $J = (\mathbf{j}_1, \ldots, \mathbf{j}_n) \in S_{\mathbf{v}}(n)$, suppose that $\mathbf{v} \preceq \mathbf{w}_0 = \wedge^T(\mathbf{j}_1, \ldots, \mathbf{j}_n)$ and $\mathbf{v} \neq \mathbf{w}_0$. For each $\mathbf{w} \succeq \mathbf{v}$ with $|\mathbf{w}| = l'$ let $g_{\mathbf{w}} \in \mathrm{Aut}_{\mathbf{v}}$ be some automorphism of T_k fixing $\mathbf{v}$ such that $g_{\mathbf{w}}(\mathbf{w}_0) = \mathbf{w}$. Summing (3.7) over each such $\mathbf{w}$, applying Case (a) to each $g_{\mathbf{w}}(J) \in S_{\mathbf{w}}(n)$ noting that $L(g_{\mathbf{w}}(J)) = L(J)$ and using Hölder's inequality for the sums and expectations,

$$\begin{aligned}
&\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&= \mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}} \Big\{\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)\in[g_{\mathbf{w}}(J)]_{\mathbf{w}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big|\mathcal{F}_{\mathbf{w}}\Big)\Big\}\Big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\le \mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}} \Big\{Y_{\mathbf{w}}^{(q-n)/(q-1)} \prod_{l\in L(J)}^{[n-1]}\Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{w}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{w}}\big)\Big)^{1/(q-1)}\Big\}\Big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\le \Big(\mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}} Y_{\mathbf{w}}\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{(q-n)/(q-1)} \prod_{l\in L(J)}^{[n-1]}\Big(\mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}}\ \sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{w}} \mathsf{E}(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{w}})\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{1/(q-1)}\\
&\le Y_{\mathbf{w}}^{(q-n)/(q-1)} \prod_{l\in L(J)}^{[n-1]}\Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}})\Big)^{1/(q-1)}
\end{aligned}$$

using (3.3) and the tower property of conditional expectation, giving (3.7) in this case. □

The next lemma extends Lemma 3.2 to non-integral q. Again conditional independence and Hölder's inequality are used frequently, but the addition of an extra $\mathbf{j} \in I_k$ associated with the 'ϵ' term significantly complicates the argument.

Lemma 3.3 *For all integers $n \ge 1$, for all $0 < \epsilon < 1$, all $q \ge n + \epsilon$ and all $\mathbf{v} \in T_k$, if $J \in S_{\mathbf{v}}^+(n)$ then*

$$\begin{aligned}
&\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n;\mathbf{p})\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1,\ldots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\quad\le Y_{\mathbf{v}}^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]}\Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)}\Big(\sum_{|\mathbf{u}|=l_0(J),\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{\epsilon/(q-1)}.
\end{aligned} \tag{3.9}$$

Proof. We obtain (3.9) by induction on n.

Start of induction Let $\mathbf{v} \in T_k$ and let $J = (\mathbf{j}_1; \mathbf{q}) \in S_{\mathbf{v}}^+(1)$ (so the vertex $\mathbf{q}$ is on the path from $\mathbf{v}$ to $\mathbf{j}_1$). If $\mathbf{j}_1 = \mathbf{q}$ it is simple to check (3.9) as the interior sums are over a single term. Otherwise, using conditional independence, (3.3) and (3.5) and then Hölder's inequality,

$$\begin{aligned}
&\sum_{(\mathbf{i}_1;\mathbf{p})\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\qquad= \sum_{|\mathbf{p}|=l_0(J),\,\mathbf{p}\succeq\mathbf{v}} \mathsf{E}\Big(\sum_{\mathbf{i}_1\succeq\mathbf{p}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{p}}\Big)\Big|\mathcal{F}_{\mathbf{v}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{|\mathbf{p}|=l_0(J),\,\mathbf{p}\succeq\mathbf{v}} \mathsf{E}\Big(\Big(\sum_{\mathbf{i}_1\succeq\mathbf{p}}\mathsf{E}\big(Y_{\mathbf{i}_1}\big|\mathcal{F}_{\mathbf{p}}\big)\Big)\mathsf{E}\Big(\Big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\succeq\mathbf{p}} Y_{\mathbf{j}}\Big)^{\epsilon}\big|\mathcal{F}_{\mathbf{p}}\Big)\Big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\le \sum_{|\mathbf{p}|=l_0(J),\,\mathbf{p}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{p}}\,Y_{\mathbf{p}}^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\big)\\
&= \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J),\mathbf{p}\succeq\mathbf{v}} Y_{\mathbf{p}}^{(q-1-\epsilon)/(q-1)}\big(Y_{\mathbf{p}}^{q}\big)^{\epsilon/(q-1)}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\le \Big(\mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J),\mathbf{p}\succeq\mathbf{v}} Y_{\mathbf{p}}\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{(q-1-\epsilon)/(q-1)}\Big(\mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J),\mathbf{p}\succeq\mathbf{v}} Y_{\mathbf{p}}^{q}\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{\epsilon/(q-1)}\\
&= Y_{\mathbf{v}}^{(q-1-\epsilon)/(q-1)}\Big(\sum_{|\mathbf{p}|=l_0(J),\,\mathbf{p}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{p}}^{q}\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{\epsilon/(q-1)},
\end{aligned}
$$

which is (3.9) when $n=1$.

The inductive step
Assume that for some integer $n_0\ge 1$, inequality (3.9) holds for all $1\le n\le n_0$ and all $J\in S_{\mathbf{v}}^{+}(n)$. We establish (3.9) when $n=n_0+1$. Let $\mathbf{v}\in T_k$ and $J=(\mathbf{j}_1,\ldots,\mathbf{j}_n;\mathbf{q})\in S_{\mathbf{v}}^{+}(n)$. There are three cases.

Case (a) First assume that $\mathbf{v}=\wedge^{T}(\mathbf{j}_1,\ldots,\mathbf{j}_n)$ so $\mathbf{v}$ is the join point of at least two of $\{\mathbf{j}_1,\ldots,\mathbf{j}_n\}$.

Again, if $|\mathbf{v}|=k$, that is $\mathbf{v}=\mathbf{j}_1=\cdots=\mathbf{j}_n=\mathbf{q}$, then (3.7) is straightforward to verify.

Otherwise J decomposes into $2\le r<n$ subsets,

$$
J_1=(\mathbf{j}_1^1,\ldots,\mathbf{j}_{n_1}^1)\in S_{\mathbf{v}}(n_1),\ldots,\ J_{r-1}=(\mathbf{j}_1^{r-1},\ldots,\mathbf{j}_{n_{r-1}}^{r-1})\in S_{\mathbf{v}}(n_{r-1}),\ J_r=(\mathbf{j}_1^{r},\ldots,\mathbf{j}_{n_r}^{r};\mathbf{q})\in S_{\mathbf{v}}^{+}(n_r),
$$

say, without loss of generality, where $1\le n_i\le n-1$ for each r, and

$$
n_1+\cdots+n_r=n, \tag{3.10}
$$

and such that each tree $T_{\mathbf{v}}(\mathbf{j}_1^i,\ldots,\mathbf{j}_{n_i}^i)$ $(i=1,\ldots,r)$ has a distinct *single* edge abutting $\mathbf{v}$. We write $L(J_i)$ for the set of (n_i-1) join levels of the trees $T_{\mathbf{v}}(\mathbf{i}_1^i,\ldots,\mathbf{i}_{n_i}^i)$ (counted by multiplicity) for $i=1,\ldots,r$, and $l_0(J_r)=l_0(J)=|\mathbf{q}|$.

Using conditional independence and applying (3.7) from Lemma 3.2 and the inductive assumption (3.9) to $J_1,\ldots J_r$,

$$
\begin{aligned}
&\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n;\mathbf{p})\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1,\ldots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\le \mathsf{E}\Big(\sum_{(\mathbf{i}_1^1,\ldots,\mathbf{i}_{n_1}^1)\in[J_1]_{\mathbf{v}}} Y_{\mathbf{i}_1^1}\cdots Y_{\mathbf{i}_{n_1}^1}\big|\mathcal{F}_{\mathbf{v}}\Big)\times\cdots\times \mathsf{E}\Big(\sum_{(\mathbf{i}_1^{r-1},\ldots,\mathbf{i}_{n_{r-1}}^{r-1})\in[J_{r-1}]_{\mathbf{v}}} Y_{\mathbf{i}_1^{r-1}}\cdots Y_{\mathbf{i}_{n_{r-1}}^{r-1}}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\qquad\times \mathsf{E}\Big(\sum_{(\mathbf{i}_1^{r},\ldots,\mathbf{i}_{n_r}^{r};\mathbf{p})\in[J_r]_{\mathbf{v}}} Y_{\mathbf{i}_1^{r}}\cdots Y_{\mathbf{i}_{n_r}^{r}}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1^r,\ldots,\mathbf{i}_{n_r}^r)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\Big)\\
&\le Y_{\mathbf{v}}^{(q-n_1)/(q-1)}\prod_{l\in L(J_1)}^{[n_1-1]}\Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^{q}\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)}\Big)\times\cdots\\
&\qquad\times Y_{\mathbf{v}}^{(q-n_{r-1})/(q-1)}\prod_{l\in L(J_{r-1})}^{[n_{r-1}-1]}\Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^{q}\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)}
\end{aligned}
$$

$$\times Y_{\mathbf{v}}^{(q-n_r-\epsilon)/(q-1)} \prod_{l\in L(J_r)}^{[n_r-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0(J_r),\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{\epsilon/(q-1)}$$

$$= Y_{\mathbf{v}}^{(q-n_1-\cdots-n_r-\epsilon)/(q-1)} \big(Y_{\mathbf{v}}^q\big)^{(r-1)/(q-1)}$$

$$\times \prod_{l\in L(J_1)\cup\cdots\cup L(J_r)}^{[n_1+\cdots+n_r-r]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0(J_r),\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{\epsilon/(q-1)}$$

$$= Y_{\mathbf{v}}^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0(J),\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{\epsilon/(q-1)},$$

using (3.10), and incorporating the terms $Y_{\mathbf{v}}^q$ in the main product with multiplicity $(r-1)$ to get (3.9).

Case (*b*) Now with $\mathbf{v} \in T_k$ and $J = (\mathbf{j}_1, \ldots, \mathbf{j}_n; \mathbf{q}) \in S_{\mathbf{v}}^+(n)$, suppose that $\mathbf{v} \preceq \mathbf{w}_0 = \wedge^T(\mathbf{j}_1, \ldots, \mathbf{j}_n)$ and $\mathbf{v} \neq \mathbf{w}_0$. Also suppose that $\mathbf{q} \succeq \mathbf{w}_0$ so that $J \in S_{\mathbf{w}_0}^+(n)$, and let $l' = |\mathbf{w}_0| > |\mathbf{v}|$. For each $\mathbf{w} \succeq \mathbf{v}$ with $|\mathbf{w}| = l'$ let $g_{\mathbf{w}} \in \mathrm{Aut}_{\mathbf{v}}$ be some automorphism of T_k fixing $\mathbf{v}$ such that $g_{\mathbf{w}}(\mathbf{w}_0) = \mathbf{w}$. By Case (a) (3.9) is valid with $\mathbf{v}$ replaced by each such $\mathbf{w}$ in turn, so summing over $\mathbf{w}$ and using Hölder's inequality,

$$\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n;\mathbf{p})\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1,\ldots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\Big)$$

$$= \mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}} \Big\{\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n;\mathbf{p})\in[g_{\mathbf{w}}(J)]_{\mathbf{w}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{w}}(\mathbf{i}_1,\ldots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{w}}\Big)\Big\}\Big|\mathcal{F}_{\mathbf{v}}\Big)$$

$$\leq \mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}} \Big\{Y_{\mathbf{w}}^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{w}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{w}}\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0,\mathbf{u}\succeq\mathbf{w}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{w}}\big)\Big)^{\epsilon/(q-1)}\Big\}\Big|\mathcal{F}_{\mathbf{v}}\Big)$$

$$\leq \Big(\mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}} Y_{\mathbf{w}}\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}}\ \sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{w}} \mathsf{E}(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{w}})\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{1/(q-1)}$$

$$\times \Big(\mathsf{E}\Big(\sum_{|\mathbf{w}|=l',\mathbf{w}\succeq\mathbf{v}}\ \sum_{|\mathbf{u}|=l_0(J),\mathbf{u}\succeq\mathbf{w}} \mathsf{E}(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{w}})\big|\mathcal{F}_{\mathbf{v}}\Big)\Big)^{\epsilon/(q-1)}$$

$$= Y_{\mathbf{v}}^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l,\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0(J),\mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_{\mathbf{u}}^q\big|\mathcal{F}_{\mathbf{v}}\big)\Big)^{\epsilon/(q-1)},$$

using (3.3) to get (3.9) in this case.

Case (*c*) With $\mathbf{v} \in T_k$ and $J = (\mathbf{j}_1, \ldots, \mathbf{j}_n; \mathbf{q}) \in S_{\mathbf{v}}^+(n)$ write $J^- = (\mathbf{j}_1, \ldots, \mathbf{j}_n) \in S_{\mathbf{v}}(n)$. As in Case (b) suppose that $\mathbf{v} \preceq \mathbf{w}_0 = \wedge^T(\mathbf{j}_1, \ldots, \mathbf{j}_n)$ and $\mathbf{v} \neq \mathbf{w}_0$, but now with $\mathbf{v} \preceq \mathbf{q} \preceq \mathbf{w}_0$ ($\mathbf{q} \neq \mathbf{w}_0$) so that $\mathbf{q}$ lies on the path joining $\mathbf{v}$ to $\mathbf{w}_0$. For each $\mathbf{p} \succeq \mathbf{v}$ with $|\mathbf{p}| = |\mathbf{q}| = l_0(J)$ let $g_{\mathbf{p}} \in \mathrm{Aut}_{\mathbf{v}}$ be some tree automorphism fixing $\mathbf{v}$ with $g_{\mathbf{p}}(\mathbf{q}) = \mathbf{p}$. Spliting the sum over $\mathbf{p} \succeq \mathbf{v}$, using conditional independence, (3.7) and (3.5), and again applying Hölder's inequality,

$$\sum_{(\mathbf{i}_1,\ldots,\mathbf{i}_n;\mathbf{p})\in[J]_{\mathbf{v}}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\ldots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T_{\mathbf{v}}(\mathbf{i}_1,\ldots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\big|\mathcal{F}_{\mathbf{v}}\Big)$$

$$= \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} \Big\{ \sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[g_\mathbf{p}(J^-)]_\mathbf{p}} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k, \mathbf{j}\wedge T_\mathbf{v}(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}} Y_\mathbf{j}\big)^\epsilon \Big| \mathcal{F}_\mathbf{p}\Big)\Big\}\Big|\mathcal{F}_\mathbf{v}\Big)$$

$$\leq \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} \Big\{\Big(\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[g_\mathbf{p}(J^-)]_\mathbf{p}} \mathsf{E}\big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big|\mathcal{F}_\mathbf{p}\big)\Big)\Big(\mathsf{E}\big(\sum_{|\mathbf{j}|=k, \mathbf{j}\wedge T_\mathbf{v}(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}} Y_\mathbf{j}\big)^\epsilon \Big| \mathcal{F}_\mathbf{p}\Big)\Big\}\Big|\mathcal{F}_\mathbf{v}\Big)$$

$$\leq \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} \Big\{ Y_\mathbf{p}^{(q-n)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l, \mathbf{u}\succeq\mathbf{p}} \mathsf{E}\big(Y_\mathbf{u}^q\big|\mathcal{F}_\mathbf{p}\big)\Big)^{1/(q-1)} Y_\mathbf{p}^\epsilon \Big\}\Big|\mathcal{F}_\mathbf{v}\Big)$$

$$\leq \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} \Big\{ Y_\mathbf{p}^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l, \mathbf{u}\succeq\mathbf{p}} \mathsf{E}\big(Y_\mathbf{u}^q\big|\mathcal{F}_\mathbf{p}\big)\Big)^{1/(q-1)} (Y_\mathbf{p}^q)^{\epsilon/(q-1)} \Big\}\Big|\mathcal{F}_\mathbf{v}\Big)$$

$$\leq \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} Y_\mathbf{p}\Big|\mathcal{F}_\mathbf{v}\Big)^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} \Big(\sum_{|\mathbf{u}|=l, \mathbf{u}\succeq\mathbf{p}} \mathsf{E}\big(Y_\mathbf{u}^q\big|\mathcal{F}_\mathbf{p}\big)\Big)\Big|\mathcal{F}_\mathbf{v}\Big)^{1/(q-1)}$$

$$\times \mathsf{E}\Big(\sum_{|\mathbf{p}|=l_0(J), \mathbf{p}\succeq\mathbf{v}} Y_\mathbf{p}^q\Big|\mathcal{F}_\mathbf{v}\Big)^{\epsilon/(q-1)}$$

$$= Y_\mathbf{v}^{(q-n-\epsilon)/(q-1)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l, \mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_\mathbf{u}^q\big|\mathcal{F}_\mathbf{v}\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0, \mathbf{u}\succeq\mathbf{v}} \mathsf{E}\big(Y_\mathbf{u}^q\big|\mathcal{F}_\mathbf{v}\big)\Big)^{\epsilon/(q-1)},$$

giving (3.9).

This completes the inductive step and the proof of the lemma. □

Proposition 3.1 follows immediately from Lemmas 3.2 and 3.3 on taking $\mathbf{v} = \emptyset$.

For the case of non-integral q we now sum inequality (3.6) over all $\mathbf{p} \in T(J)$ where now $J \in S(n)$. Recall that $T(J) \equiv T_\emptyset(\mathbf{j}_1, \dots, \mathbf{j}_n)$ for $J = (\mathbf{j}_1, \dots, \mathbf{j}_n) \in S(n)$.

Corollary 3.4 *Let $J \in S(n)$. If $q = n$ is an integer then*

$$\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\Big) \leq \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l} \mathsf{E}\big(Y_\mathbf{u}^q\big)\Big)^{1/(q-1)}. \tag{3.11}$$

If $q = n + \epsilon > 1$ with n an integer and $0 < \epsilon < 1$ then

$$\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k} Y_\mathbf{j}\big)^\epsilon\Big) \leq \sum_{\mathbf{q}\in T(J)} \prod_{l\in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l} \mathsf{E}\big(Y_\mathbf{u}^q\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=|\mathbf{q}|} \mathsf{E}\big(Y_\mathbf{u}^q\big)\Big)^{\epsilon/(q-1)}. \tag{3.12}$$

Proof. Inequality (3.11) is just Proposition 3.1 with $\epsilon = 0$.

For (3.12), if $J \in S(n)$ and $\mathbf{q} \in T(J)$, write $J_\mathbf{q} = (\mathbf{j}_1, \dots, \mathbf{j}_n, q) \in S^+(n)$. With $0 < \epsilon < 1$, breaking up the sum and using that $(\sum a_i)^\epsilon \leq \sum a_i^\epsilon$ for $a_i \geq 0$,

$$\sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k} Y_\mathbf{j}\big)^\epsilon\Big) = \sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\Big(\sum_{\mathbf{p}\in T(\mathbf{i}_1,\dots,\mathbf{i}_n)} \sum_{|\mathbf{j}|=k, \mathbf{j}\wedge T(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}} Y_\mathbf{j}\Big)^\epsilon\Big)$$

$$\leq \sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n} \sum_{\mathbf{p}\in T(\mathbf{i}_1,\dots,\mathbf{i}_n)} \big(\sum_{|\mathbf{j}|=k, \mathbf{j}\wedge T(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}} Y_\mathbf{j}\big)^\epsilon\Big)$$

$$= \sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n)\in[J]} \sum_{\mathbf{p}\in T(\mathbf{i}_1,\dots,\mathbf{i}_n)} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\Big)$$

$$\le \sum_{\mathbf{q}\in T(J)} \sum_{(\mathbf{i}_1,\dots,\mathbf{i}_n,\mathbf{p})\in[J_{\mathbf{q}}]} \mathsf{E}\Big(Y_{\mathbf{i}_1}\dots Y_{\mathbf{i}_n}\big(\sum_{|\mathbf{j}|=k,\,\mathbf{j}\wedge T(\mathbf{i}_1,\dots,\mathbf{i}_n)=\mathbf{p}} Y_{\mathbf{j}}\big)^{\epsilon}\Big)$$

$$\le \sum_{\mathbf{q}\in T(J)} \prod_{l\in L(J_{\mathbf{q}})}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l} \mathsf{E}\big(Y_{\mathbf{u}}^q\big)\Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=l_0(J_{\mathbf{q}})} \mathsf{E}\big(Y_{\mathbf{u}}^q\big)\Big)^{\epsilon/(q-1)},$$

using Proposition 3.1, and this is just (3.12) since $L(J_{\mathbf{q}}) = L(J)$ and $l_0(J_q) = |\mathbf{q}|$. □

4 Completion of the proof

Finally, we have to sum (3.11) and (3.12) over all equivalence classes $[J]$ of $S_{\mathbf{v}}(n)$ under $\sim$, and to do this we need to bound the number of equivalence classes with given sets of levels. Let $0 \le l_1 \le \cdots \le l_n \le k$ and $0 \le l \le k$ be (not necessarily distinct) levels. Write

$$N(l_1,\dots,l_{n-1}) = \#\big\{[J] \in S(n)/\sim \text{ such that } L([J]) = \{l_1,\dots,l_{n-1}\}\big\}$$

and

$$N^+(l_1,\dots,l_{n-1};l) = \#\big\{[J] \in S^+(n)/\approx \text{ such that } L([J]) = \{l_1,\dots,l_{n-1}\}, l_0([J]) = l\big\}.$$

Lemma 4.1 *Let* $0 < \lambda < 1$*. For* $n \in \mathbb{N}$

$$\sum_{0\le l_1\le\cdots\le l_{n-1}\le k} N(l_1,\dots,l_{n-1})\lambda^{l_1+\cdots+l_{n-1}} \le M < \infty, \tag{4.1}$$

and for $n \in \mathbb{N}$ *and* $\epsilon > 0$

$$\sum_{0\le l_1\le\cdots\le l_{n-1}\le k,\,0\le l\le k} N^+(l_1,\dots,l_{n-1};l)\lambda^{l_1+\cdots+l_{n-1}+\epsilon l} \le M^+ < \infty, \tag{4.2}$$

where the bounds M *and* M^+ *are independent of* k*.*

Proof. For $n \ge 1$, every $J \in S(n+1)$ with $L(J) = \{l_1,\dots,l_{n-1},l_n\}$ where $0 \le l_1 \le \cdots \le l_n$ may be obtained by adjoining a vertex $\mathbf{j} \in I_k$ to some $J^- \in S(n)$ where $L(J^-) = \{l_1,\dots,l_{n-1}\}$. For each $J^- \in S(n)$ such a vertex may be adjoined so that the additional join level is l_n in at most n non-equivalent ways under $\mathrm{Aut}_\emptyset$. Thus $N(l_1,\dots,l_n) \le nN(l_1,\dots,l_{n-1})$ and $N(l_1) = 1$, so $N(l_1,\dots,l_n) \le n!$. Thus, for $0 < \lambda < 1$,

$$\sum_{0\le l_1\le\cdots\le l_{n-1}\le k} N(l_1,\dots,l_{n-1})\lambda^{l_1+\cdots+l_{n-1}} \le (n-1)! \sum_{0\le l_1\le\cdots\le l_{n-1}\le k} \lambda^{l_1+\cdots+l_{n-1}}$$

$$\le (n-1)! \sum_{r=0}^{\infty} P(r)\lambda^r \equiv M$$

where $P(r)$ is the number of distinct ways of partitioning the integer r into a sum of n integers $r = l_1 + \cdots + l_n$ where $0 \le l_1 \le \cdots \le l_n$. Since $P(r)$ is polynomially bounded (trivially $P(r) \le (r+1)^{n-1}$) the series is convergent.

Furthermore, for each $J \in S(n)$ there are at most n ways of choosing a vertex $\mathbf{q} \in T(J)$ at level l, so

$$N^+(l_1, \ldots, l_{n-1}; l) \le nN(l_1, \ldots, l_{n-1}) \le n(n-1)! = n!.$$

Thus, for $0 < \lambda < 1$ and $\epsilon > 0$,

$$\begin{aligned} \sum_{0 \le l_1 \le \cdots \le l_{n-1} \le k,\, 0 \le l \le k} N^+(l_1, \ldots, l_{n-1}; l) \lambda^{l_1 + \cdots + l_n + \epsilon l} &\le n! \sum_{0 \le l_1 \le \cdots \le l_{n-1} \le k,\, 0 \le l \le k} \lambda^{l_1 + \cdots + l_{n-1} + \epsilon l} \\ &\le n! \sum_{r=1}^{\infty} \frac{P(r)\lambda^r}{1 - \lambda^\epsilon} \equiv M^+. \end{aligned}$$

□

To get the final conclusion we now sum the inequalities of Corollary 3.4 over all equivalence classes $[J] \in S(n)/\sim$ using Lemma 4.1 to bound the sums.

Proposition 4.2 *Let $q \ge 1$. Suppose*

$$\limsup_{k \to \infty} \Big(\sum_{|\mathbf{i}|=k} \mathsf{E}\big(Y_{\mathbf{i}}^q\big) \Big)^{1/k} < 1. \tag{4.3}$$

Then

$$\limsup_{k \to \infty} \mathsf{E}\Big(\Big(\sum_{|\mathbf{i}|=k} Y_{\mathbf{i}} \Big)^q \Big) < \infty. \tag{4.4}$$

Proof. The case of $q = 1$ is trivial, so assume that $q > 1$. From (4.3) there are numbers $c > 0$ and $0 < \lambda < 1$ such that

$$\sum_{|\mathbf{j}|=k} \mathsf{E}\big(Y_{\mathbf{j}}^q\big) \le c\lambda^{(q-1)k} \quad (k = 1, 2, \ldots), \tag{4.5}$$

choosing λ so that $\lambda^{(q-1)}$ is sufficiently close to 1. In the case of $q = n + \epsilon$, where n is an integer and $0 < \epsilon < 1$, rearranging the summation and using (3.12) gives

$$\begin{aligned} \mathsf{E}\Big(\big(\sum_{|\mathbf{i}|=k} Y_{\mathbf{i}} \big)^q \Big) &= \mathsf{E}\Big(\big(\sum_{|\mathbf{i}_1|=k} Y_{\mathbf{i}_1} \big) \cdots \big(\sum_{|\mathbf{i}_n|=k} Y_{\mathbf{i}_n} \big) \big(\sum_{|\mathbf{j}|=k} Y_{\mathbf{j}} \big)^\epsilon \Big) \\ &= \sum_{(\mathbf{i}_1, \cdots, \mathbf{i}_n) \in S(n)} \mathsf{E}\Big(Y_{\mathbf{i}_1} \cdots Y_{\mathbf{i}_n} \big(\sum_{|\mathbf{j}|=k} Y_{\mathbf{j}} \big)^\epsilon \Big) \\ &= \sum_{[J] \in S(n)/\sim} \; \sum_{(\mathbf{i}_1, \ldots, \mathbf{i}_n) \in [J]} \mathsf{E}\Big(Y_{\mathbf{i}_1} \cdots Y_{\mathbf{i}_n} \big(\sum_{|\mathbf{j}|=k} Y_{\mathbf{j}} \big)^\epsilon \Big) \\ &\le \sum_{[J] \in S(n)/\sim} \; \sum_{\mathbf{q} \in T(J)} \; \prod_{l \in L(J)}^{[n-1]} \Big(\sum_{|\mathbf{u}|=l} \mathsf{E}\big(Y_{\mathbf{u}}^q\big) \Big)^{1/(q-1)} \Big(\sum_{|\mathbf{u}|=|\mathbf{q}|} \mathsf{E}\big(Y_{\mathbf{u}}^q\big) \Big)^{\epsilon/(q-1)} \\ &\le \sum_{[J] \in S(n)/\sim} \; \sum_{\mathbf{q} \in T(J)} \; \prod_{l \in L(J)}^{[n-1]} \big(c\lambda^l\big) \big(c\lambda^{|\mathbf{q}|}\big)^\epsilon \\ &= c^{(n-1+\epsilon)} \sum_{(J, \mathbf{q}) \in S^+(n)/\approx} \; \prod_{l \in L(J)}^{[n-1]} \big(\lambda^l\big) \big(\lambda^{|\mathbf{q}|}\big)^\epsilon \end{aligned}$$

$$
\begin{aligned}
&= c \sum_{0\leq l_1\leq\cdots\leq l_{n-1}\leq k,\, 0\leq l\leq k} N^+(l_1,\ldots,l_{n-1};l)\lambda^{(l_1+\cdots+l_{n-1}+\epsilon l)}\\
&\leq M^+ < \infty,
\end{aligned}
$$

by (4.2).

The case where $q = n$ is an integer is similar but shorter, using (3.11) and (4.1). □.

Proof of Theorem 1.1 Setting

$$Y_{\mathbf{i}} = X_{\mathbf{i}}\mu(C_{\mathbf{v}}) \qquad (\mathbf{i} \in I)$$

in (4.3) and (4.4), Proposition 4.2 gives (1.6) directly. Moreover, (1.6) says that $(\int X_{\mathbf{i}|k} d\mu(\mathbf{i})|\mathcal{F}_k)$ is an L^q-bounded martingale, so the martingale convergence theorem implies almost sure convergence and convergence in L^q for $q > 1$. □

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