RESONANCES FOR GRAPH DIRECTED MARKOV SYSTEMS, AND GEOMETRY OF INFINITELY GENERATED DYNAMICAL SYSTEMS

Martial R. Hille

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I, Martial R. Hille, hereby certify that this thesis, which is approximately 26.000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

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Contents

Abstract

In the first part of this thesis we transfer a result of Guillopé *et al.* concerning the number of zeros of the Selberg zeta function for convex cocompact Schottky groups to the setting of certain types of graph directed Markov systems (GDMS). For these systems the zeta function will be a type of Ruelle zeta function. We show that for a finitely generated primitive conformal GDMS S, which satisfies the strong separation condition (SSC) and the nestedness condition (NC), we have for each $c > 0$ that the following holds, for each $w \in \mathbb{C}$ with $\mathrm{Re}(w) > -c$, $|\mathrm{Im}(w)| > 1$ and for all $k \in \mathbb{N}$ sufficiently large:

$$\log|\zeta(w)| \ll \mathrm{e}^{\delta(S)\cdot\log(\mathrm{Im}\,|w|)} \text{ and } \operatorname{card}\{w \in Q(k) \mid \zeta(w) = 0\} \ll k^{\delta(S)}.$$

Here, $Q(k) \subset \mathbb{C}$ denotes a certain box of height k, and $\delta(S)$ refers to the Hausdorff dimension of the limit set of S.

In the second part of this thesis we show that in any dimension $m \in \mathbb{N}$ there are GDMSs for which the Hausdorff dimension of the uniformly radial limit set is equal to a given arbitrary number $d \in (0, m)$ and the Hausdorff dimension of the Jørgensen limit set is equal to a given arbitrary number $j \in [0, m)$.

Furthermore, we derive various relations between the exponents of convergence and the Hausdorff dimensions of certain different types of limit sets for iterated function systems (IFS), GDMSs, pseudo GDMSs and normal subsystems of finitely generated GDMSs.

Finally, we apply our results to Kleinian groups and generalise a result of Patterson by showing that in any dimension $m \in \mathbb{N}$ there are Kleinian groups for which the Hausdorff dimension of their uniformly radial limit set is less than a given arbitrary number $d \in (0, m)$ and the Hausdorff dimension of their Jørgensen limit set is equal to a given arbitrary number $j \in [0, m)$.

1 Introduction

Graph directed Markov systems (GDMS) were introduced by Mauldin and Urbanski (see e.g. [63]). These systems form a significant generalisation of the concept of an iterated function system (IFS) in fractal geometry. A large class of fractals can be described as limit sets obtained by iterating the maps of such systems. Examples range from the well known middle third Cantor set to limit sets of certain types of Kleinian groups. In this thesis we consider various aspects of GDMSs. In particular, we relate these aspects to certain problems and facts in the theory of Kleinian groups and in the analysis on their associated hyperbolic manifolds.

In the first part of this thesis we consider a certain kind of zeta function associated to a GDMS. This type of zeta function will be a kind of Ruelle zeta function, and hence, can be considered as a dynamical zeta function. The zeros of this function will be called resonances. We generalise the recent result in [42] on zeros of the Selberg zeta function to this type of Ruelle zeta function for certain GDMSs.

In the second part of this thesis we study infinitely generated GDMSs and their generalisations, the so-called pseudo GDMSs.

1.1 Statement of results

This thesis consists of two main parts. These are given in the sections *Resonances for GDMSs* and *Geometry of infinitely generated function schemes*. In the first part we give an upper bound for the growth of the number of zeros of a particular Ruelle zeta function associated to a conformal GDMS. The second part will be concerned with investigations of various aspects of certain types of infinitely generated function schemes. The following summarises the main results of this thesis.

In the first part we show how to transfer a result of Guillopé *et al.* on the zeros of the Selberg zeta function of a convex cocompact Schottky group to a zeta function associated to a certain type of GDMSs. As already mentioned before, this zeta function will be a type of Ruelle zeta function, and will be defined via the determinant of the identity operator minus the complexified FPR-operator. The latter operator will act on a Hilbert space of complex valued functions defined in a complex neighbourhood of the limit set of the GDMS. The main results of this part of the thesis are summarised in the following theorem. Throughout, we write $a(w) \ll b(w)$ if there is a universal constant $c > 0$ with $a(w) \leq c \cdot b(w)$. We also write $a(w) \asymp b(w)$ if $a(w) \ll b(w)$ and $b(w) \ll a(w)$.

Main Theorem 1. *Let S be a finitely generated primitive conformal GDMS acting on $\mathbb{R}^m$, satisfying the strong separation condition (SSC) and the nestedness condition (NC). For each $c>0$ and $w \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) > -c, |\operatorname{Im}(z)| > 1\}$, we then have*

$$\log|\zeta(w)| \ll \mathrm{e}^{\delta(S)\cdot\log(|\operatorname{Im}(w)|)}.$$

Here, $\delta(S)$ denotes the Hausdorff dimension of the limit set of S. Moreover, for all $k>0$ sufficiently large, we then have the following upper bound for the growth of the number of resonances

$$\operatorname{card}\left\{w \in Q_{k,k+1}^{-c,\infty} \mid \zeta(w) = 0\right\} \ll k^{\delta(S)}.$$

Here, $Q_{k,k+1}^{-c,\infty} := \{z \in \mathbb{C} \mid -c \le \operatorname{Re}(z) < \infty,\ k \le \operatorname{Im}(z) \le k+1\}$.

In the second part of this thesis we shall investigate various aspects of limit sets of infinitely generated function schemes. This will include considerations of GDMSs, pseudo GDMSs and IFSs. In particular, we adapt the notion of the Jørgensen limit set $L_J(S)$ of [67] and the notion of the uniformly radial limit set $L_{ur}(S)$ of [86] to a wider class of function schemes (for the definitions of $L_J(S)$ and $L_{ur}(S)$ see Definition 3.1.1). Also, we shall consider the dynamical limit set $L_{dyn}(S) := L(S) \setminus L_J(S)$.
The first main result of this part of the thesis will be the following theorem. Here, $\dim_{\mathrm{H}}$ refers to the Hausdorff dimension.

Main Theorem 2. *For every $m \in \mathbb{N}$ and every $d, j \in (0,m)$, there exists an GDMS S acting on $\mathbb{R}^m$ such that*

$$\dim_{\mathrm{H}} L_{ur}(S) = d \quad \textit{and} \quad \dim_{\mathrm{H}} L_J(S) = j.$$

In particular, S can be chosen to be an IFS.

Note that the statement in this theorem can clearly be extended such that the case $j=0$ is included. Indeed, if $j=0$, then every finitely generated GDMS which satisfies SSC and for which $\delta(S) = d$ serves as an example. In order to state the next main theorem, we introduce some notation. Let $\Delta(S)$ and $\Lambda(S)$ refer to the exponents of convergence of the two series $\sum_{\underline{e}\in E^*(S)}(\operatorname{diam}(\phi_{\underline{e}}(X_{i(\underline{e})})))^s$ and $\sum_{\underline{e}\in E^*(S)}(\|\phi'_{\underline{e}}\|)^s$ respectively. Here, $\|\phi'_{\underline{e}}\|$ denotes the norm of the derivative of $\phi_{\underline{e}}$. These exponents will be crucial in our investigations of $L_{ur}(S)$, $L_{dyn}(S)$, and the radial limit set $L_r(S)$ (see Definition 3.1.1). We refer to Definition 3.2.15 for the slightly technical concept of a normal subsystem, which was motivated by the notion of a normal covering of a hyperbolic manifold. The main results here are summarised in the following theorem.

Main Theorem 3. *Assuming that S and N are finitely primitive and satisfy the bounded distortion condition (BDC) and the strong separation condition (SSC), the following hold.*

- *If S is a finitely or infinitely generated IFS, then*

$$\dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S) = \dim_{\mathrm{H}} L_{dyn}(S) = \Delta(S) = \Lambda(S).$$

- *If S is a GDMS, then*

$$\dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S) = \dim_{\mathrm{H}} L_{dyn}(S) \leq \Delta(S) = \Lambda(S).$$

- *If N is a normal subsystem of a finitely generated GDMS, then*

$$\dim_{\mathrm{H}} L_{ur}(N) \leq \dim_{\mathrm{H}} L_{dyn}(N) = \Delta(N) \leq \Lambda(N).$$

- *If S is a pseudo GDMS, then*

$$\dim_{\mathrm{H}} L_{ur}(S) \leq \dim_{\mathrm{H}} L_{dyn}(S) \leq \Delta(S) \leq \Lambda(S).$$

It is hoped that in general one has $\dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S)$, as well as $\dim_{\mathrm{H}} L_{dyn}(S) = \Delta(S)$. Additionally, our investigations suggest that $\Delta(S)$ and $\Lambda(S)$ do not coincide in general (see Lemma 3.2.4 and the remark thereafter).

Finally, we apply the results of the second part of this thesis to Kleinian groups Γ of Schottky type, and in this way we derive the following theorem. Here, $L_{ur}(\Gamma)$ denotes the uniformly radial limit set of Γ and $L_J(\Gamma)$ denotes the Jørgensen limit set of Γ, which are defined similar as for GDMSs (see Definition 3.4.3).

Main Theorem 4. *For every $m \in \mathbb{N}$ and for every $d, j \in (0, m)$, there exists a Kleinian group Γ acting on $(m+1)$-dimensional hyperbolic space such that*

$$\dim_{\mathrm{H}} L_{ur}(\Gamma) \leq d \quad \textit{and} \quad \dim_{\mathrm{H}} L_J(\Gamma) = j.$$

In particular, Γ can be chosen to be of Schottky type .

Note again that the statement clearly holds for $j = 0$ as well. Let us remark that the latter theorem represents a generalisation of a result of Patterson in [70], where this result was obtained in the situation in which the Hausdorff dimension of the limit set of Γ is equal to m.

1.2 Resonances - A brief motivation

In [53] Kaç stated his famous question "Can one hear the shape of a drum?" Though derived independently, this question might be considered to be a more popular version of a conjecture by Gelfand and Piatetski-Shapiro [35]. In [65] Milnor was the first to give a negative answer to this question (by giving a counterexample in dimension 16). Nevertheless, this question inspired many mathematicians around the world. Meanwhile, the most satisfying answers are probably given in the context of hyperbolic geometry. But even in this context the answers are in general negative, as observed for instance by Vignéras in [94], [95] and Buser in [18]. Continuing this tradition, we consider Kaç's original question from a slightly different perspective.

Basic hyperbolic geometry

The upper half-space model of the hyperbolic space $\mathbb{H}^{m+1} := \{(x_0, x_1, \ldots, x_m) \in \mathbb{R}^{m+1} \mid x_0 > 0\}$ is determined by the metric which is given by $ds^2 := |dz|^2/\operatorname{Im}(z)^2$, where $\operatorname{Im}(x_0, \ldots, x_m) := x_0$. We call $\partial\mathbb{H}^{m+1} := \mathbb{R}^m \cup \{\infty\}$ its boundary at infinity. In the 2-dimensional case the group $\operatorname{Iso}(\mathbb{H}^2)$ of isometries on $\mathbb{H}^2$ consists of Möbius transformations, that is, maps $\varphi : \mathbb{H}^2 \to \mathbb{H}^2$ with $z \mapsto \frac{az+b}{cz+d}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{R})$.

For a properly discontinuous subgroup $\Gamma \subset \operatorname{Iso}(\mathbb{H}^{m+1})$, that is, for a group Γ with the property that the orbit $\Gamma(z)$ of some arbitrary $z \in \mathbb{H}^{m+1}$ has no accumulation points in $\mathbb{H}^{m+1}$ (this property guarantees that $M = \mathbb{H}^{m+1}/\Gamma$ is Hausdorff), the quotient space $M := \mathbb{H}^{m+1}/\Gamma$ is a hyperbolic manifold. A fundamental domain D of Γ is an open connected convex subset of $\mathbb{H}^{m+1}$ such that

- $D \cap \gamma(D) = \emptyset$, for all $\gamma \in \Gamma \setminus \{\mathbf{id}\}$;
- $\bigcup_{\gamma \in \Gamma} \gamma(\overline{D}) = \mathbb{H}^{m+1}$.

A hyperbolic manifold is said to be geometrically finite if it has a fundamental domain with finitely many sides. In dimension two this is equivalent to the fundamental group of the hyperbolic manifold being finitely generated. This is no longer true if $m = 2$ (see for example [1], [50]).

Recall that geodesics in $\mathbb{H}^{m+1}$ are either Euclidean half-circles with centres satisfying $\operatorname{Im}(x) = 0$, or straight lines parallel to the x_0-axis. Since $\mathbb{H}^{m+1}$ is the universal cover of M, geodesics in M are given by the projection of the geodesics in $\mathbb{H}^{m+1}$ to M. We call a geodesic *prime* if it is a primitive closed geodesic, that is, a closed geodesic that traces out its image exactly once. The length spectrum $\mathit{length}(M)$ is defined to be the set of lengths of all prime geodesics, ordered according to their lengths.

Basic spectral theory

Let (M,g) denote a Riemannian manifold. Let g_{ij} be the entries of the matrix corresponding to the Riemannian metric g, and let ∂_i be the i-th basis vector of the tangent space of M. Using the Einstein summation convention, the Laplace-Beltrami operator on M is then given by $\Delta : f \mapsto -\mathrm{div}\ \mathrm{grad}(f) = \frac{1}{\sqrt{|g|}}\partial_j \sqrt{|g|} g_{ij} \partial^i f$ and is formally defined on the space $C_c^2(M)$ of twice differentiable functions on M with compact support. Since $C_c^2(M)$ is dense in the space $L^2(M)$ of square integrable functions on M, one considers Δ as a differential operator on $L^2(M)$. Then Δ is formally self-adjoint, that is, $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$ for all $f, g \in \mathrm{Dom}(\Delta) = C_c^\infty(M)$, the domain of Δ. Here, $\langle , \rangle$ denotes the usual inner product on $L^2(M)$. It turns out that the Laplace-Beltrami operator is an elliptic, positive, unbounded, essentially self-adjoint differential operator acting on $L^2(M)$ (see for example [75]). If f satisfies the equation $\Delta f = \lambda f$, then f is called eigenfunction of Δ with corresponding eigenvalue $\lambda \in \mathbb{C}$. The set of all eigenvalues will be denoted by $\sigma_{pp}(M)$. (For a more detailed introduction to the Laplace-Beltrami operator in the case of general Riemannian manifolds we refer to [52], and for the particular case of hyperbolic manifolds we refer to the books [20] and [21] of Chavel.)

Remark: Let us recall the following basic property of the Laplace-Beltrami operator. On $\mathbb{H}^{m+1}$ this operator is given by

$$\Delta = -x_0^2 \left(\frac{\partial^2}{\partial x_0^2} + \ldots + \frac{\partial^2}{\partial x_m^2} \right) + (m-1) x_0 \frac{\partial}{\partial x_0},$$

and one easily verifies that Δ has eigenfunctions given by x_0^s with eigenvalues $s(m-s)$ (see for example [20, Section XI.2]).

Spectral theory for compact hyperbolic manifolds

If M is a compact manifold, an easy argument using Sobolev spaces and the Sobolev embedding theorem shows that the eigenvalues of Δ satisfy $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots < \infty$. Furthermore, each eigenvalue has finite multiplicity and infinity is the only accumulation point. Moreover, the eigenfunctions form an orthogonal basis of $L^2(M)$ (see for example [51, p. 355] for a proof of this fact for compact subsets of $\mathbb{R}^m$, which easily generalises to compact manifolds). Let us also mention the following useful analogy.

> Any sound from a drum is built up in the following way. Every eigenfunction is an eigenmode (or base state) of the drum, and every eigenvalue (or frequency) of this eigenmode is a base sound. In this respect, every sound one hears can be thought of as corresponding to a combination

of the eigenvalues, and it therefore corresponds to a combination of the base states.

1.2.1 Spectral theory for non-compact hyperbolic manifolds

In the case of non-compact hyperbolic manifolds, the spectrum no longer consists solely of eigenvalues. This changes the spectral geometry dramatically. In order to inspect this situation more closely, let us review the definition of the spectrum. The eigenvalue equation $\Delta f = \lambda f$ can be rewritten as $\Delta f - \lambda f = 0$. (Here, we consider the Laplacian with Dirichlet boundary conditions.) This way one sees the connection with the resolvent set $\mathcal{R} = \{\lambda \in \mathbb{C} \mid (\Delta - \lambda \cdot \mathbf{id})^{-1} \text{ exists}\}$. If $\lambda \in \mathcal{R}$, then the operator $R(\lambda) := (\Delta - \lambda \cdot \mathbf{id})^{-1}$ is called the resolvent of λ. Here, the inverse $(\cdot)^{-1}$ is meant in the operator sense. The complement $\sigma := \mathbb{C} \setminus \mathcal{R}$ of the resolvent set is called the spectrum. Note that the eigenvalues are clearly elements of the spectrum. In the case of compact manifolds, the eigenvalues form the whole spectrum, that is $\sigma(M) = \sigma_{pp}(M)$. This is no longer true in the non-compact case.

If M is a hyperbolic manifold with finite volume, then M consists of a compact part and possibly finitely many cusps. In this case, Eisenstein series can be introduced, and these allow a good control over the cusps (for example one obtains a basis for $L^2(M)$). However, we do not go into the details here (see for example [92]).

If M is a geometrically finite hyperbolic manifold with infinite volume, then it has at least one funnel, that is, a hyperbolic cylindrical end. In this case the spectrum decomposes into $\sigma(M) = \sigma_{pp}(M) \sqcup [m^2/4; \infty)$. Here, $\sqcup$ denotes the disjoint union. It is well known that there is no eigenvalue in $(m^2/4; \infty)$, that the bottom of the continuous spectrum is an eigenvalue of infinite multiplicity, and that $\mathrm{card}(\sigma_{pp}(M)) < \infty$ (see e.g. [73][Theorem 3.1]). (Note that these results are variously due to several authors including Elstrodt, Fay, Lax-Phillips and Patterson and we refer to [56] and [73] and references given therein). Hence, in this case, it is clear that the finite number of eigenvalues cannot completely determine the geometry of M. To rectify this, one introduces the concept of resonances as a generalisation of eigenvalues. We again work with the resolvent map R, which is well defined on the resolvent set $\mathcal{R}$. Let us now make the substitution $\lambda = s(m - s)$. In this way the continuous spectrum $[m^2/4; \infty)$ is mapped to the line $\{z \in \mathbb{C} \mid \mathrm{Re}(z) = m/2\}$, and hence the corresponding resolvent map R_s is meromorphic on the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > m/2\}$ with poles coming from the eigenvalues. Since there were only finitely many eigenvalues, each with finite multiplicity, there are only finitely many poles each having finite rank. Guillopé and Zworski showed in [44] that in dimension 2 the map R_s admits a meromorphic extension to $\mathbb{C}$, and that this extension has poles of finite rank only. The poles of this meromorphic extension are the resonances, where the multiplicity of a resonance is defined to be equal to the order of the corresponding pole.

REMARK: Note that all of the statements above are restricted to geometrically finite groups. In order to emphasise this note that already in dimension 2 it is a direct consequence of the author's diploma thesis [47] (see also [34]) that for hyperbolic surfaces with infinitely generated fundamental groups there can be infinitely many eigenvalues smaller than $1/4$. In particular, this shows that from a spectral theoretical point of view there is a significant difference between the infinitely generated situation and the geometrically finite situation.

1.2.2 The Selberg zeta function

For a compact manifold M, the Selberg trace formula gives a precise quantitative and qualitative relation between the set of eigenvalues and the length spectrum. Inspired by this formula, Selberg considered a particular function, which nowadays is referred to as Selberg's zeta function. If M is a convex cocompact manifold, then the fundamental group Γ of $M = \mathbb{H}^{m+1}/\Gamma$ consists of loxodromic elements. Each loxodromic element γ can be written in the form $(x_0, x_1, \ldots, x_m) \mapsto \mathrm{e}^{l(\gamma)}(x_0, A_\gamma(x_1, \ldots, x_m))$, where $A_\gamma \in O(m)$ is a rotation matrix and $l(\gamma) > 0$ is the length of the closed geodesic on M corresponding to the conjugacy class of Γ represented by γ. Let $\lambda_1(\gamma), \ldots, \lambda_m(\gamma)$ denote the eigenvalues of A_γ. With this notation, the Selberg zeta function is given by

$$\zeta_M(s) := \prod_{\gamma} \prod_{k_1,\ldots,k_m=0}^{\infty} \Big(1 - (\lambda_1(\gamma))^{k_1} \cdot \ldots \cdot (\lambda_m(\gamma))^{k_m} \cdot \mathrm{e}^{-l(\gamma)(s+k_1+\ldots+k_m)}\Big). \quad (1)$$

Here, the outer product is taken over representatives γ of all conjugacy classes of Γ, and hence, over elements $\gamma \in \Gamma$ corresponding to the closed geodesics on M. One can check that the zeros of this zeta function are in one-to-one correspondence with the eigenvalues of Δ, in the sense that s is a zero if and only if $s(m-s)$ is an eigenvalue of the Laplace-Beltrami operator on M.
Also, one can easily verify that the product in (1) is a well defined holomorphic function for $\mathrm{Re}(s) > m$.
For compact manifolds M with Euler characteristic $\chi(M)$ it follows from the Selberg trace formula that ζ_M has an analytic extension ζ to the whole complex plane, and moreover, we have that the following properties hold.

- If $\mathrm{Re}(s) \geq m/2$, then ζ has a spectral zero at s if and only if $s(m-s)$ is an eigenvalue of Δ.
- The function ζ has a zero at $s = 0$ (of order $1 - \chi(M)$).

- The function ζ has topological zeros at $s = k$ (of order $(2k-1)\chi(M)$), for each $-k \in \mathbb{N}$.

The following non-compact analogue for surfaces, which is quoted from [15], follows from results of several authors.

Theorem 1.2.1. *For a geometrically finite hyperbolic surface of infinite volume, the function ζ admits a meromorphic extension to $\mathbb{C}$ and its zeros are characterized as follows.*

- *If s is a resonance of a certain order, then s is a spectral zero of ζ of the same order.*

- *The function ζ has a topological zero at each non-positive integer k (of order $(2k-1)\chi(M)$). Additionally, for each $-k \in \mathbb{N} \cup \{0\}$ there are topological zeros at $1/2 + k$ (of order equal to the number of cusps of M).*

Namely, the first results seem to have appered in [69], where Patterson showed how a Selberg zeta function could be defined and investigated for a convex cocompact Fuchsian group. Later results were due to Colin de Verdiére [22] and Guillopé [39]. According to [72] the meromorphic continuation of ζ is due to Guillopé [40], while, according to [73], Borthwick, Judge and Perry [15] gave the characterisation of the zeros of the zeta function.
The following theorem of Borthwick, Judge and Perry [15] explains the significance of resonances.

Theorem 1.2.2 (Borthwick, Judge and Perry, [15])**.** *For a hyperbolic surface, the set of resonances determines the length spectrum, the Euler characteristic and the number of cusps. Moreover, the converse is also true, that is, the length spectrum, the Euler characteristic and the number of cusps together determine the set of resonances.*

Combining this theorem with the fact that the length spectrum almost determines the surface M, it follows that the set of resonances almost determines M. Here, "almost" refers to the fact that there are only finitely many possible choices for M (each having the same spectrum of resonances).
In higher dimensions the history of the analogue of Theorem 1.2.1 is rather contorted as the spectral theory took some time to develop (see the discussion in [72]). Namely, there has been work of Mandouvalos ([58], [59]), on which Patterson based his [71], where he extended the results of [69] to a certain class of convex cocompact groups. (Also see the discussion in [69] for the relation to work of Mazzeo/Melrose, Perry, Elstrodt/Grunewald/Mennicke, and, in particular, to work of Fried and Ruelle and the connection to the theory of

dynamical systems.) Finally, the higher dimensional anaogue is due to Patterson and Perry [72] in even dimensions. In odd dimensions it has been obtained in this form by Bunke and Olbrich in [17], where the results of Patterson and Perry in even dimensions play a key role. Parallelly, Patterson and Perry [72] also managed to resolve the even dimensional case, but obtained a slightly weaker result in that they could not identify a series of singularities at negative integer point as multiples of the Euler charactersitic (see the discussion in [72] for further details). In higher dimensions, the zeros of the zeta function are related to the poles of the scattering operator instead of to the poles of the resolvent. The statement of this relationship is slightly involved, and since we are not going to work with it here, we do not go into further details. For an introduction into scattering theory we refer to [56]. There is also the work of Faddeev and Pavlov [27], which seems to be availible in Russian only.

1.2.3 Asymptotics for the counting function for resonances

For compact hyperbolic surfaces we have the following Weyl law for the eigenvalues of the Laplacian (see for example [92, 3.7 Theorem 5]).

$$\mathrm{card}\{|\lambda| < r \mid \lambda \text{ is an eigenvalue of } \Delta\} \sim const. \cdot r.$$

Similarly, the growth of the number of resonances are of interest. However, as we shall see, for resonances we have different rates of growth, depending on the type of region we consider.

Namely, we shall compare some upper bounds for the counting function of resonances and we shall explain why they appear to be so different. In what follows, let $\mathcal{R}$ denote the set of resonances. Let us begin with $\mathrm{card}\{\lambda \in \mathbb{C} \cap \mathcal{R} \mid |\lambda| < r\}$, the number of resonances in a ball of radius r around the origin. In the case of compact hyperbolic manifolds it is well known that $\mathrm{card}\{\lambda \in \mathbb{C} \cap \mathcal{R} \mid |\lambda| < r\} \ll r^{m+1}$ and that this bound is optimal (see [73] and references given there). In fact, the same is true for convex cocompact hyperbolic manifolds of Schottky type (see [42], see also [43], [96]). At first glance, this seems not to match with the results for $\mathrm{card}\{\lambda \in \mathbb{C} \cap \mathcal{R} \mid \mathrm{Im}(\lambda) < r; \mathrm{Re}(\lambda) < c\}$ the number of resonances in the half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < c\}$, with imaginary part less than r. That is, for compact hyperbolic manifolds it is well known that $\mathrm{card}\{\lambda \in \mathbb{C} \cap \mathcal{R} \mid \mathrm{Im}(\lambda) < r; \mathrm{Re}(\lambda) < c\} \ll r^{m+1}$. For convex cocompact hyperbolic manifolds the result in [42] gives that $\mathrm{card}\{\lambda \in \mathbb{C} \cap \mathcal{R} \mid \mathrm{Im}(\lambda) < r; \mathrm{Re}(\lambda) < c\} \ll r^{\delta+1}$. However, this difference is easy to explain. First, note that in the compact case it is well known (see [8],[9],[10]) that $L(\Gamma) = S^m = L_{ur}(\Gamma)$, and hence (by the result in [14]), we have that $\delta(\Gamma) = \dim_{\mathrm{H}} L_{ur}(\Gamma) = m$. Thus, the upper bound for the growth within strips for compact hyperbolic manifolds and for convex cocompact hyperbolic manifolds can be expressed by the same formula.

However, in order to understand the difference between the results for counting in a strip and for counting in a ball, we have to recall some deep results from the last decade. Namely, Patterson/Perry [72] and Bunke/Olbrich [17] obtained the following result.

Theorem 1.2.3. *Let Γ be an orientation-preserving, torsion-free, convex co-compact discrete group and let $X = H^{m+1}/\Gamma$. We then have that the Selberg zeta function Z_Γ has a zero (or pole) of order $h_m(k)\chi(X)$ at $-k$, where $\chi(X)$ denotes the Euler characteristic of X and $h_m(k) := (2k+m)(k+m-1)!/k!m!$*

We call the zeros (or poles) in the above theorem the *topological zeros*. Moreover, note that this theorem shows that if one counts the zeros of Z_Γ in a ball of radius r centered at the origon, one at least has to take these topological zeros into account. That is, we have that

$$\begin{aligned} \mathrm{card}\{\lambda \in \mathbb{C} \cap \mathcal{R} \mid |\lambda| < r\} &\geq \sum_{k=1}^{r} h_m(k) = \sum_{k=1}^{r} (2k+m)(k+m-1)!/k!m! \\ &\geq \sum_{k=1}^{r} (2k+m) \frac{(k+1)(k+2)\dots(k+m-1)}{m!} \\ &\geq \sum_{k=1}^{r} (2k+m) \frac{1}{m!}(k+1)^{m-1} \\ &\geq \frac{1}{m!} \sum_{k=1}^{r} (2k+m) k^{m-1} \gg \frac{1}{m!} r^{m+1}. \end{aligned}$$

1.3 Introduction to infinitely generated function schemes

1.3.1 Infinitely generated Kleinian groups

In hyperbolic geometry, infinitely generated groups have produced very interesting and, in their time, surprising examples of intricate fractal sets. Currently, we are far away from a meaningful complete classification of infinitely generated Kleinian groups (see e.g. [1], [60], [61]). We now give a very short overview of the history and motivations behind the subject.

Recall the definitions of the hyperbolic space $\mathbb{H}^{m+1}$ and fundamental group Γ given above. Suppose the group $\Gamma \subset \mathrm{Iso}(\mathbb{H}^{m+1})$ acts discontinuously on $\mathbb{H}^{m+1}$. Then there can be accumulation points of $\Gamma(z)$ in $\partial\mathbb{H}^{m+1}$, and the set $L(\Gamma)$ of these accumulation points is called the limit set of Γ. Note that $L(\Gamma)$ is either empty (then Γ is said to be trivial), consists of either one or two points (then Γ is said to be elementary), or else consists of uncountably many points (then Γ is said to be non-elementary). A Kleinian group Γ is said to be of the first kind if $L(\Gamma) = \partial\mathbb{H}^{m+1}$, and of the second kind otherwise. Also, Γ is called cocompact if the manifold $\mathbb{H}^{m+1}/\Gamma$ is compact. It is well known

that a cocompact group is of the first kind, and in this case the Hausdorff dimension of $L(\Gamma)$ is equal to m.
Another important and well studied quantity attached to a Kleinian group Γ is the exponent of convergence $\delta(\Gamma)$ of its Poincaré series $\sum_{\gamma\in\Gamma} \mathrm{e}^{-d(z,\gamma(w))}$. (It is easy to see that the Poincaré series does not depend on the choice of $z, w \in \mathbb{H}^{m+1}$.) Here, d denotes the hyperbolic distance in $\mathbb{H}^{m+1}$. The exponent of convergence $\delta(\Gamma)$ is sometimes also referred to as the Poincaré exponent of Γ. It is a classical result that if Γ is cocompact, then $\delta(\Gamma) = m$. This represents a first (trivial) example of a class of Kleinian groups for which we have $\dim_{\mathrm{H}}(L(\Gamma)) = \delta(\Gamma)$.
Let us recall the following result of Beardon (see [8],[9] and [10]). If Γ is a non-elementary, geometrically finite Fuchsian group of the second kind, then $0 < \dim_{\mathrm{H}} L(\Gamma) \le \delta(\Gamma) < 1$. The equality $\dim_{\mathrm{H}} L(\Gamma) = \delta(\Gamma)$ was eventually proved in 1976 by Patterson in [68] for geometrically finite Fuchsian groups of the second kind without parabolic elements. There, Patterson further proved this equality for Γ with parabolic elements, on the assumption that $\delta(\Gamma) \ge 2/3$. In fact, the lower bound $2/3$ resulted from a slight imprecision in the calculations, and a careful reconsideration of the arguments in [68] shows that the proof does indeed work for all non-elementary geometrically finite Fuchsian groups of the second kind. Since in dimension 2 we have that geometrically finite is equivalent to finitely generated, it was then 'essentially clear' that $\dim_{\mathrm{H}} L(\Gamma) = \delta(\Gamma)$ holds for all non-elementary, finitely generated Fuchsian groups. Recall that for $m = 2$ we do not have equivalence between a Kleinian group being geometrically finite and being finitely generated (see for example [1], [50]). Also note that for infinitely generated Kleinian groups a counterexample for $\delta(\Gamma) = \dim_{\mathrm{H}} L(\Gamma)$ was obtained in [70]. In that paper, Patterson constructed groups of the first kind for which $\delta(\Gamma)$ is arbitrarily small. (Note that the construction of infinitely generated GDMSs which we give in Section 3.2.3 is motivated by the construction in [70].) In [14] Bishop and Jones improved the relationship of $\dim_{\mathrm{H}} L(\Gamma)$ and $\delta(\Gamma)$ even further. Namely, they proved that $\delta(\Gamma) = \dim_{\mathrm{H}} L_r(\Gamma)$ for *all* non-elementary Kleinian groups Γ. Here, $L_r(\Gamma)$ denotes the radial limit set (see Definition 3.4.3). Note that if Γ is geometrically finite, then every limit point is either a radial point or a parabolic fixed point.
On the basis of these results, Falk and Stratmann introduced in [30] the concept of a discrepancy group (abbreviated d-group) for Kleinian groups which satisfy $\dim_{\mathrm{H}} L(\Gamma) > \delta(\Gamma)$. In that paper, they also introduced the Jørgensen limit set $L_J(\Gamma)$ and the transient limit set $L_t(\Gamma)$. Note that the name for the Jørgensen limit set was inspired by Sullivan's notion "Jørgensen end", which was introduced in [90, Figure 1] (see also [67, p.172]). Let us give a brief outlook on the definitions of these subsets of $L(\Gamma)$. For these we will use the

Poincaré model $\mathbb{D}^{m+1} := \{x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \mid \|x\| < 1\}$, (equipped with the metric given by $ds^2 = dx^2/(1 - \|x\|^2)$) of the $(m+1)$-dimensional hyperbolic space. Its boundary will be denoted by $\mathbb{S}^m$. Moreover, for $x \in \mathbb{S}^m$ let s_x refer to the geodesic rayfrom the origin to x and let $b(g(0), c)$ refer to the hyperbolic ball centred at $\gamma(0)$ of hyperbolic radius c. With this notation, we then have the following descriptions of the various types of limit sets of Γ.

- An element $x \in L(\Gamma)$ is called uniformly radial limit point if for some positive $c = c(x)$ we have that $s_x \subset \bigcup_{\gamma \in \Gamma} b(\gamma(0), c)$. The set $L_{ur}(\Gamma)$ of uniformly radial limit points is called the uniformly radial limit set of Γ.

- An element $x \in L(\Gamma)$ is called radial limit point if for some positive $c = c(x)$ we have that $s_x \cap b(\gamma(0), c) \neq \emptyset$ for infinitely many different orbit points $\gamma(0) \in \Gamma(0)$. The set $L_{ur}(\Gamma)$ of radial limit points is called the radial limit set of Γ.

- The transient limit set is defined by $L_t(\Gamma) := L(\Gamma) \setminus L_r(\Gamma)$.

- An element $x \in L(\Gamma)$ is called Jørgensen limit point if and only if, for some Dirichlet domain D_z of Γ based at some point $z \in \mathbb{D}^{m+1}$, there exists $\gamma \in \Gamma$ such that $\gamma(D_z)$ contains the hyperbolic geodesic ray from $\gamma(z)$ to x. The set $L_J(\Gamma)$ of Jørgensen limit points is called the Jørgensen limit set of Γ.

In [30] Falk and Stratmann decomposed $L_t(\Gamma)$ into the Jørgensen limit set $L_J(\Gamma)$ and the dissipative limit set $L_d(\Gamma) := L_t(\Gamma) \setminus L_J(\Gamma)$. However, in this thesis we use a slightly different decomposition, which will play a crucial role in our investigations. Namely, instead of $L_t(\Gamma)$ we introduce the following limit sets.

- The dynamical limit set is defined by

$$L_{dyn}(\Gamma) := L_d(\Gamma) \cup L_r(\Gamma).$$

- The Jørgensen limit set is defined by

$$L_J(\Gamma) = L(\Gamma) \setminus L_{dyn}(\Gamma).$$

We employ a construction similar to the one used by Patterson in [70] to construct Kleinian groups $\Gamma \subset \mathrm{Iso}(\mathbb{H}^{m+1})$ with $d \leq \dim_{\mathrm{H}} L_{ur}(\Gamma)$ and $j = \dim_{\mathrm{H}} L_J(\Gamma)$, for arbitrary elements $d \in (0, m)$ and $j \in [0, m]$. In fact, these will be derived as applications of the more general formalism for GDMSs. Also, in the second part we show how to transfer the various notions of limit sets to pseudo GDMSs.

1.3.2 The normal covering of a convex cocompact Kleinian group

In addition to the construction in [70], there is another important class of infinitely generated Kleinian groups, namely, normal subgroups of finitely generated Kleinian groups. Here, we are mainly interested in normal coverings of convex cocompact hyperbolic manifolds, which provide a fruitful source of examples. Firstly, the limit set of each nontrivial normal subgroup N of a Kleinian group Γ coincides with the limit set of Γ. We always assume that N/Γ is infinite. Secondly, there is an exact description of the Jørgensen limit set of a normal subgroup of a convex cocompact Kleinian group. In order to present this description let $\Gamma = \Gamma_1 \star \Gamma_2$ be the free product of two non-elemantary freely generated convex cocompact Kleinian groups Γ_1 and Γ_2 acting on $\mathbb{H}^{m+1}$ with (open) fundamental domains F_1 and F_2 respectively, such that $F_1^c \cap F_2^c = \emptyset$. With N referring to the normal subgroup of Γ generated by Γ_1 we have that N/Γ is isomorphic to Γ_2. Then the Jørgensen limit set of N is equal to $N(L(\Gamma_2))$, the N-orbit of the limit set $L(\Gamma_2)$. This description leads to the very useful formula $\dim_{\mathrm{H}} L_J(N) = \dim_{\mathrm{H}} L(\Gamma_2)$ for the Hausdorff dimension of the Jørgensen limit set. In particular, this implies that $\dim_{\mathrm{H}} L_J(N) < \dim_{\mathrm{H}} L(N)$. Thirdly, Brooks showed in [16], under the assumption that the exponent of convergence $\delta(\Gamma)$ exceeds $m/2$, that the Hausdorff dimension of the radial limit set of a normal subgroup of a convex cocompact Kleinian group is strictly smaller than the Hausdorff dimension of the limit set of the convex cocompact group if and only if the quotient group is non-amenable. In fact, in [16] it was actually shown that the bottom of the spectrum of the two manifolds do not agree if N/Γ is non-amenable. To translate this result of Brooks into our language here, recall that it is known that $\delta(\Gamma)(m - \delta(\Gamma))$ is equal to the first eigenvalue of the Laplace-Beltrami operator (see [24],[25],[26]; see also [68]). Let us remark that there is unpublished work by Stadlbauer, in which a measure theoretical proof of the result of Brooks has been obtained. Stadlbauer's approach uses skew products and a result by Kesten [54].

We shall generalise the concept of a normal covering to what we call a normal subsystem of a GDMS in Section 3.2.2. Moreover, in the spirit of Falk and Stratmann [30], we shall say that a (pseudo) GDMS is of discrepancy type if the Hausdorff dimension of the uniformly radial limit set of Γ is strictly less than the Hausdorff dimension of the dynamical limit set of Γ. The main reason for considering normal coverings is that they represent a class of pseudo GDMS for which both the Jørgensen and the radial limit set can be of smaller Hausdorff dimension than the limit set itself. Hence, this class provides very interesting examples of pseudo GDMS of discrepancy type.

1.3.3 A journey into pseudo GDMSs

The results for infinitely generated Kleinian groups stated above can be seen as a motivation for investigating similar phenomena for general pseudo GDMSs. This should also provide a deeper understanding of the geometric mechanisms which produce the phenomena responsible for a system to be of discrepancy type. In particular, the main question was whether or not there exists an affine pseudo GDMS S satisfying $\dim_{\mathrm{H}} L_{dyn}(S) > \dim_{\mathrm{H}} L_{ur}(S)$, as it is the case for certain normal coverings of convex cocompact Kleinian groups (as mentioned above).

First step of the journey: IFSs

In fractal geometry a standard example of an IFS is the middle third Cantor set. To define it, start with the closed unit interval $[0,1]$ and remove the middle third, leaving $[0,1/3]\cup[2/3,1]$. Now remove the middle third of each of the two intervals, leaving $[0,1/9]\cup[2/9,1/3]\cup[2/3,7/9]\cup[8/9,1]$. Iterating this infinitely often gives the so-called middle third Cantor set. Nowadays, this method of construction is normally described by the family of maps $\{f_1(x) := x/3, f_2(x) := x/3+2/3\}$ acting on the initial set $X := [0,1]$. This collection is usually called an IFS. The notion of an IFS was introduced by Hutchinson in [49], whereas the terminalogy is due to Barnsley (see e.g. [7]). Nowadays it is widely used not only by mathematicians but also by researchers working in applied sciences (see for example the book of Barnsley [7]). With the notation above, the middle third Cantor set is the unique compact non-empty set $C \subset [0,1]$ such that $C = f_1(C) \cup f_2(C)$. (For a rigorous introduction to IFSs, we refer to the books of Falconer [28], [29], Pesin [74] or Mattila [62].) Note that in the literature an IFS is usually defined by a finite number of generating functions. In Section 3 we extend this notion to infinitely generated IFSs following the approach of [63] (Infinitely generated IFSs are for instance also studied in [45] and [33], amongst many others.) An IFS is called *affine* if all of its generating maps are similarities. We prove that infinitely generated affine IFSs are not of discrepancy type, that is, for such a system S we prove that $\dim_{\mathrm{H}} L_{ur}(S)$ and $\dim_{\mathrm{H}} L_{dyn}(S)$ always coincide. The main ingredient in this proof is the side series, which gives rise to a generalized analogue of the Hutchinson formula.

Second step of the journey: GDMSs

Since affine IFSs turn out not to be of discrepancy type, we continue by investigating the more general concept GDMS. This was introduced by Mauldin and Urbański in [63] and it is well studied from a measure-theoretical point of view. These studies rely heavily on the pressure function and the FPR-operator. The thermodynamical formalism was also introduced in the setting of GDMSs. However, our investigations of the limit set and its subsets differs from those of Mauldin and Urbański.

Roughly speaking, a GDMS consists of a finite collection of non-empty compact connected metric spaces $\{X_v\}_{v\in V}$, a countable collection of maps $\{\phi_e\}_{e\in E}$ between the spaces in $\{X_v\}_{v\in V}$ and an information which compositions are admissible. This information is stored in an incidence matrix $A=\left(A_{e_i,e_j}\right)$, which is a square matrix of order $\mathrm{card}(E)\times\mathrm{card}(E)$ with entries in $\{0,1\}$. In fact, the combinatorics of the system is described by a directed multi-graph with vertices $v\in V$ and directed edges $e\in E$ such that e goes from its initial vertex $v_1\in V$ to its terminal vertex $v_2\in V$ if and only if ϕ_e maps X_{v_1} to X_{v_2}. Also, there are two maps $i,t:E\to V$, called the initial map and the terminal map respectively, which are given by $i:e\mapsto v_1$ and $t:e\mapsto v_2$. Finally, for the incidence matrix we have that if $A_{e_i,e_j}=1$ then $t(e_i)=i(e_j)$, for $e_i,e_j\in E$.

More precisely, a GDMS is defined as follows. A GDMS is an octuple $(V,E,i,t,A,\{X_v\}_{v\in V},s,\{\phi_e\}_{e\in E})$ of a finite set V of vertices, a countable set E of directed edges, two maps $i,t:E\to V$ and a $(\mathrm{card}\,E)\times(\mathrm{card}\,E)$ matrix A with entries in $\{0,1\}$, a collection of non-empty compact connected metric spaces $\{X_v\}_{v\in V}$ (which we shall allways assume to be subsets of $\mathbb{R}^m$ and which are closures of open sets), a number $s\in(0,1)$, and injective contractions $\phi_e:X_{i(e)}\to X_{t(e)}$ with Lipschitz constants less than s.

One of the advantages of GDMSs is that they can be used to describe the action of a convex cocompact (and hence finitely generated) Schottky group, as was shown in [63, Example 5.1.5]. Furthermore, an infinitely generated GDMS (that is, a system with an infinite number of edges) can be used to describe the function scheme of certain finitely generated Kleinian groups containing parabolic elements. However, in Section 3.4 of this thesis, we consider infinitely generated Kleinian groups and show (see Lemma 3.4.5) that these can not be described by GDMSs. In fact, for these we require the concept of pseudo GDMSs.

Organisation of Part II

Section 3 is organised as follows. Inspired by the theory of Kleinian groups as well as by the fractal geometry of GDMSs, we introduce rather carefully the relevant concepts of the fractal geometry of GDMSs. In particular, we introduce two series, namely the *side series* and the *distortion series*. The exponent of convergence $\Delta(S)$ of the side series will be called the side exponent, and the exponent of convergence $\Lambda(S)$ of the distortion series will be referred to as the distortion exponent. Also, we introduce the Poincaré exponent $\delta(S)$. Note that one can immediately verify that in the finitely generated case these two series are comparable, and that these three exponents coincide. Our analysis will show that this is also the case for arbitrary IFSs (see Corollary 3.2.5 and Corollary 3.3.4), as well as for arbitrary GDMSs (see Corollary 3.2.5 and Corollary 3.3.6). Nevertheless, for pseudo GDMSs these three exponents can all be different. As far as we know, the side series has not been studied in the literature before. The reason for this might be that the side series and the distortion series are comparable for GDMSs, as mentioned above.

The main aim of Section 3.2.3 will be to show that for each $m \in \mathbb{N}$ and for arbitrary $d, j \in (0, m)$, one can find a GDMS S on $\mathbb{R}^m$ such that the Hausdorff dimension of the uniformly radial limit set of S is equal to d, whereas the Hausdorff dimension of the Jørgensen limit set of S is equal to j. Note that S does not need to be an IFS. However, as we shall see, we can always find an IFS with the same properties. In particular, for these we have that d is equal to the Hausdorff dimension of the dynamical limit set of the IFS. This main theorem shows that the Jørgensen limit set is independent of the dynamical limit set in terms of Hausdorff dimension. Therefore, we adapt an idea of Falk and Stratmann in [30] and say that a function scheme is of *discrepancy type* if $\dim_{\mathrm{H}} L_{ur}(S) \neq \dim_{\mathrm{H}} L_{dyn}(S)$. As Theorem 3.3.5 will show, GDMSs cannot be of discrepancy type. However, the existence of affine pseudo GDMSs of discrepancy type still seems to be an open problem.

In Section 3.2.2, we introduce the notion of a normal subsystem of a GDMS, which generalises the idea of a normal covering of a convex cocompact hyperbolic manifold. Using this notion, we then establish further interesting results, which in particular will also allow applications to Kleinian groups of Schottky type. In particular, we show that the side exponent of N is equal to $\dim_{\mathrm{H}} L_{dyn}(N)$ for every normal subsystem of a finitely generated GDMS. We conjecture that this equality holds in general.

In Section 3.4 we consider non-elementary Kleinian groups Γ of Schottky type and study the side series in this context. We want to mention a result of [30], where it was shown that $\dim_{\mathrm{H}} L_r(N) \geq \dim_{\mathrm{H}} L_r(\Gamma)/2$, for any non-trivial normal subgroup N of a non-elementary Kleinian group Γ. We suspect a similar relation holds for arbitrary Kleinian groups of Schottky type.

2 Resonances for GDMSs

Throughout this section we introduce and study a finitely generated primitive conformal GDMS S acting on $\mathbb{R}^m$. The aim is to give a proof of Main Theorem 1.

2.1 Basic notions and statement of results

2.1.1 Basic notions of finitely generated GDMSs

In this section we collect some of the important basic geometric concepts necessary for the proof of Main Theorem 1. We begin by giving a detailed definition of a GDMS. Note that each of these systems is based on a directed multigraph and *not* a graph. The multigraph consists of a finite set V of vertices and a countable set of directed edges E.

Definition 2.1.1. *A* graph directed Markov system (GDMS) S *is defined by an octuple* $(V, E, i, t, A, \{X_v\}_{v\in V}, \ell, \{\phi_e\}_{e\in E})$ *given by the following list.*

- *A non-empty finite set* V *of vertices.*
- *A countable set* E *of directed edges.*
- *Two maps* $i, t : E \to V$, *which assign to each edge* $e \in E$ *its initial vertex* $i(e)$ *and terminal vertex* $t(e)$.
- *A* $(\operatorname{card} E) \times (\operatorname{card} E)$*-matrix* A *with entries in* $\{0, 1\}$, *which is also called transition matrix or edge incident matrix, since it determines which paths are to be admissible, that is, which edges may follow a given edge, and which satisfies that whenever* $A_{e,f} = 1$ *then* $t(e) = i(f)$.
- *A collection* $\{X_v\}_{v\in V}$ *of non-empty compact connected metric spaces which we assume to be pairwise disjoint sets* $X_v \subset \mathbb{R}^m$, *which are closures of open sets, that is* $X_v = \overline{\operatorname{Int}(X_v)}$.
- *Some constant* $\ell \in (0, 1)$.
- *Injective contractions* $\phi_e : X_{i(e)} \to X_{t(e)}$ *with Lipschitz constants less than* $\ell \in (0, 1)$.

Moreover, if E *is finite, then* S *is called finitely generated.*

Remark: From now on we always assume that our GDMSs are primitive. Namely, we assume that there exists a $p \geq 1$ such that all entries of A^p are positive. Note that in order for a GDMS to be primitive it is necessary that the multigraph (V, E) is connected.

We now recall some basic facts about GDMSs. For finitely generated GDMSs these are well known, and we refer to the textbook [63] for the proofs and details.
For a GDMS S, define the set of admissible words of length $n \in \mathbb{N}$ by

$$E^n := \left\{(e_1, \ldots, e_n) \mid e_i \in E \text{ such that } A_{e_i, e_{i+1}} = 1 \text{ for all } i \geq 1\right\}.$$

Also, let E^∞ denote the set of infinite (admissible) words, and define the set of finite (admissible) words by $E^* := \bigcup_{n \in \mathbb{N}} E^n$.
Remark: Note that for $\underline{e} = (e_1, \ldots, e_n)$ we have

$$\phi_{\underline{e}} = \phi_{e_n} \circ \ldots \circ \phi_{e_1} : X_{i(e_1)} \to X_{t(e_n)}.$$

This notation differs from that of [63].

Definition 2.1.2. *We define the* limit set $L(S)$ *of a GDMS* S *by*

$$L(S) := \bigcap_{n \in \mathbb{N}} \bigcup_{(e_1, \ldots, e_n) \in E^n} \phi_{e_n} \circ \ldots \circ \phi_{e_1} \left(X_{i(e_1)}\right).$$

Clearly, the set $L(S)$ can be identified with the set of infinite words E^∞ (see Lemma 3.1.5 for details). An important parameter associated to a GDMS is given by the Hausdorff dimension of its limit set.

Definition 2.1.3. *The* Hausdorff dimension $\dim_H$ *of a subset* Y *in* $\mathbb{R}^m$ *is defined by*

$$\dim_H(Y) := \sup\{s \in \mathbb{R} | \lim_{\epsilon \to 0} \inf_{\{U_i\}} \sum_{i \in \mathbb{N}} \operatorname{diam}(U_i)^s = \infty\},$$

where the infimum is taken over all countable coverings $\{U_i\}$ *of* Y *with diameter at most* ϵ.

We require the following three properties.

Definition 2.1.4. *A GDMS* S *satisfies the* strong separation condition (SSC) *if for all* $e, f \in E$ *with* $e \neq f$,

$$\phi_e\left(X_{i(e)}\right) \cap \phi_f\left(X_{i(f)}\right) = \emptyset.$$

Definition 2.1.5. *A GDMS* S *satisfies the* bounded distortion property (BDP) *if there exists a constant* $c \geq 1$ *such that*

$$\frac{1}{c} \cdot \|\phi'_{\underline{e}}(y)\| \leq \|\phi'_{\underline{e}}(x)\| \leq c \cdot \|\phi'_{\underline{e}}(y)\|,$$

for all $\underline{e} \in E^*$ *and* $x, y \in X_{i(\underline{e})}$. *Here,* $\|\phi'_{\underline{e}}(x)\|$ *is any norm on the linear mappings on* $\mathbb{R}^m$ *(all such norms are equivalent).*

Definition 2.1.6. *A GDMS S satisfies the* nestedness condition (NC) *if for each $e \in E$ there is an open set $U \subset X_{t(e)}$ such that $\phi_e \left(X_{i(e)} \right) \subset U$.*

Finally, we adapt the definition of a conformal GDMS of [63] to our setting as follows.

Definition 2.1.7. *A finitely generated GDMS S is said to be* conformal *if the following conditions are satisfied.*

- *For every vertex $v \in V$ there exists an open set W_v such that $X_v \subset W_v$. Moreover, for every $e \in E$ the map ϕ_e extends to a C^1-conformal diffeomorphism from $W_{i(e)}$ to $W_{t(e)}$.*
- *There exists a constant $c > 1$ such that for every $e \in E$ and all $x, y \in X_{i(e)}$ the following holds:*

$$| \, \|\phi_e'(x)\| - \|\phi_e'(y)\| \, | \leq c|x - y|.$$

- *The strong separation condition (SSC) is satisfied.*

Remark: Definition 2.1.7 is a restricted version of the definition of a finitely generated conformal GDMS in [63]. The only difference is that we require SSC, while in [63] only the weaker open set condition was assumed. Hence, the conformal GDMSs as defined above are conformal GDMSs in the sense of [63]. It is well known that every conformal GDMS satisfies BDP (see [63][(4f)]). Let us recall that a C^1 diffeomorphism $\phi : U \to \mathbb{R}^m$, where $m \geq 1$, from an open connected set $U \subset \mathbb{R}^m$ to $\mathbb{R}^m$ is conformal if its derivative at every point of U is a similarity map (cf. [63][p.62]). Note that for $m = 1$, C^1-conformality means that the maps ϕ_e are monotone C^1 diffeomorphisms, for each $e \in E$. For $m = 2$, C^1-conformal maps are holomorphic or antiholomorphic. For $m \geq 3$, conformal maps between domains in $\mathbb{R}^m$ are of the form $x \mapsto \lambda A i(x) + b$, where $\lambda > 0$, $b \in \mathbb{R}^m$, $A \in O(m)$ and i is either the identity or an inversion. Here, $O(m)$ denotes the orthogonal group. (A proof of this can be found for example in [11] where it is referred to as Liouville's Theorem (Theorem A.3.7).) Recall that the inversion at the unit circle around zero is given by $x \mapsto \frac{x}{\|x\|^2}$, and this is C^∞ in $\mathbb{R}^m \setminus \{0\}$ (see [11][Proposition A.3.1]). Since $\|x\|^2 = x_1^2 + \ldots + x_m^2$, it immediately follows that this inversion is real analytic on $\mathbb{R}^m \setminus \{0\}$. Also, since it maps zero to ∞ and since in a GDMS the maps ϕ_e map compact sets to compact sets, it follows that the centre of the circle associated to the inversion is not included in the corresponding compact domain. This implies that for conformal GDMSs the maps ϕ_e are real analytic on the open sets $W_{i(e)}$, and that the $\|\phi_e'\|$ are non-zero.

In what follows we always assume that our GDMSs satisfy the nestedness condition (NC).

2.1.2 Basic notions of functional analysis

In this section we recall the main definitions and facts from functional analysis which will be required later. (For a comprehensive introduction to functional analysis we refer to [75], [76] and [36].) In what follows, let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ denote Hilbert spaces. The inner product in $\mathcal{H}$ will be denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and we drop the subscript if it is clear from the context which Hilbert space is meant. The operator norm of $A : \mathcal{H}_1 \to \mathcal{H}_2$ is defined by $\|A\| := \sup \{\|Af\|_{\mathcal{H}_2} \mid f \in \mathcal{H}_1,\ \|f\|_{\mathcal{H}_1} = 1\}$. A linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ is called compact if for any bounded subset $X \subset \mathcal{H}_1$ the image $A(X)$ is relatively compact in $\mathcal{H}_2$, that is, the closure $\overline{A(X)}$ is compact. Such an operator is necessarily a bounded operator, and it is therefore continuous. To each such A corresponds a unique $A^* : \mathcal{H}_2 \to \mathcal{H}_1$, called the adjoint of A, which is compact and satisfies $\langle Ax, y\rangle_{\mathcal{H}_2} = \langle x, A^* y\rangle_{\mathcal{H}_1}$, for all $y \in \mathcal{H}_2$ and all $x \in \mathcal{H}_1$. Furthermore, $\|A^*\| = \|A\|$ (see [77][4.10]). Also, for a compact operator $A : \mathcal{H} \to \mathcal{H}$ there is an expansion

$$A = \sum_{n=0}^{N} \chi_n (A) \langle x_n, \cdot \rangle y_n,$$

where $N \in \mathbb{N} \cup \{-1, 0, \infty\}$, $\chi_n (A) \in \mathbb{R}$ and $\chi_n (A) \geq \chi_{n+1} (A) > 0$, for all $n \in \mathbb{N} \cup \{0\}$ (see [83][Theorem 1.4]). Moreover, $\{x_n\}_{n \in \mathbb{N} \cup \{0\}}$ and $\{y_n\}_{n \in \mathbb{N} \cup \{0\}}$ are orthonormal sets in $\mathcal{H}$. Furthermore, the $\chi_n (A)$ are uniquely determined, and they are referred to as *singular values* (see [83][Theorem 1.4]). Here, we have used the convention $\sum_{i=0}^{-1} =: 0$. For ease of exposition, let us only consider the case $N = \infty$. The resolvent set of A is defined by $\rho (A) := \{\mu \in \mathbb{C} \mid (\mu - A)^{-1} \text{ exists}\}$. The spectrum of A is defined by $\sigma (A) := \mathbb{C} \setminus \rho (A)$. If $A(f) = \lambda \cdot f$, then we call $\lambda = \lambda (A)$ an eigenvalue of A and $f \in \mathcal{H}$ its associated eigenvector. The dimension of $\{f \in \mathcal{H} \mid A(f) = \lambda f\}$ is called the geometric multiplicity of λ. Note that if λ is an eigenvalue of A, we necessarily have that $\lambda \subset \sigma (A)$. By the well known spectral theorem for compact operators (see [83][Theorem 1.1]), we have that each non-zero $\lambda \in \sigma (A)$ is an eigenvalue of A of finite multiplicity, that $\sigma (A)$ is countable, that 0 is the only accumulation point of the non-zero eigenvalues, and hence, the function $z \mapsto (z - A)^{-1}$ has a pole at λ. The order of the pole is called the algebraic multiplicity. In what follows, we always refer to the algebraic multiplicity only, unless stated otherwise. Furthermore, $\{\lambda_n (A)\}_{n \in \mathbb{N} \cup \{0\}}$ refers to the collection of all non-zero eigenvalues of A repeated according their algebraic multiplicity.

Let us make a few more comments about compact operators between two Hilbert spaces. For this let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator. Then $A^* A : \mathcal{H}_1 \to \mathcal{H}_1$ is self-adjoint, and hence has only real, non-negative eigenvalues which are equal to the eigenvalues of AA^*. Therefore, the eigenvectors

of A^*A form an orthonormal basis (see [83][Theorem 1.1]). Let $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ be an orthonormal basis of eigenvectors of $(A^*A)^{1/2}$ and $\{y_n\}_{n\in\mathbb{N}\cup\{0\}}$ be an orthonormal basis of eigenvectors of $(AA^*)^{1/2}$. Then we have an expansion $A = \sum_{n=0}^{\infty} \sqrt{\lambda\left(A^*A\right)}\langle x_n, \cdot\rangle_{\mathcal{H}_2} y_n$ (cf. [83][Proof of Theorem 1.4]). From this one also sees that the singular values $\{\chi_n(A)\}_{n\in\mathbb{N}\cup\{0\}}$ are the non-zero eigenvalues of $(A^*A)^{1/2}$, counted according to their multiplicity.
We now recall the min-max principle for singular values (see [83][Theorem 1.5]), which follows from the fact that the singular values of A are exactly the non-zero eigenvalues of $(A^*A)^{1/2}$ and from the min-max-Theorem for eigenvalues (see [76][Theorem XIII.1]).

Lemma 2.1.8. *Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a compact operator. Then the singular values of A form a decreasing sequence with 0 being the only accumulation point. Also, the n-th singular value $\chi_n(A)$ of A is given by*

$$\chi_n(A) = \min_{\dim(\mathcal{H}_{1,n})=n} \max_{f\in\mathcal{H}_{1,n}^{\perp}} \frac{\|A(f)\|_{\mathcal{H}_2}}{\|f\|_{\mathcal{H}_1}}.$$

Here, the minimum is taken over all n-dimensional subspaces $\mathcal{H}_{1,n}$ of $\mathcal{H}_1$, while the maximum is taken over all elements in the orthogonal complement of $\mathcal{H}_{1,n}$.

From this it immediately follows that for any orthonormal basis $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of $\mathcal{H}_1$ we have

$$\chi_n(A) \leq \sum_{j=n}^{\infty} \|Ax_j\|. \tag{2}$$

Remark: Note that in the literature one often finds $\dim(\mathcal{H}_{1,n}) = n+1$, rather than $\dim(\mathcal{H}_{1,n}) = n$. Consequently, one then has that χ_1 is the first singular value, while in our definition the first singular value is χ_0.

Recall that a compact operator $A : \mathcal{H} \to \mathcal{H}$ is said to be of trace-class if

$$\|A\|_1 := \sum_{n=0}^{\infty} \chi_n(A) < \infty.$$

Let $A = \sum_{n=0}^{\infty} \chi_n(A)\langle x_n, \cdot\rangle y_n$ be of trace-class. Then, for any orthonormal basis $\{\eta_n\}_{n\in\mathbb{N}\cup\{0\}}$, the sum $\sum_{n=0}^{\infty} |\langle \eta_n, A\eta_n\rangle|$ converges. Moreover,

$$\mathrm{Tr}(A) := \sum_{n=0}^{\infty} \langle \eta_n, A\eta_n\rangle = \sum_{n=0}^{\infty} \chi_n(A)\langle x_n, y_n\rangle$$

is independent of the basis (cf. [83][Theorem 3.1]). Furthermore, one can show that if A is of trace-class, then the series $\sum_{n\in\mathbb{N}\cup\{0\}} \lambda_n(A) \leq \sum_{n\in\mathbb{N}\cup\{0\}} \chi_n(A)$ is

absolutely convergent, and the trace of A satisfies $\operatorname{Tr}(A) = \sum_{n\in\mathbb{N}\cup\{0\}} \lambda_n(A)$. This is often referred to as Lidskii's equality (see [83][Theorem 3.7]).
The following definition is adopted from [37].

Definition 2.1.9. *For an operator A of trace-class we define the* determinant $\det(1+A)$ *by*

$$\det(1+A) := \prod_{n=0}^{\infty} (1+\lambda_n(A)).$$

REMARK: There are several ways to define $\det(1+A)$ for a trace-class operator A. For example, in [23] one finds $\det(1+zA) := \exp(\operatorname{Tr}(\ln(1+zA)))$, for $z \in \mathbb{C}$ with $|z|$ small, and one then considers an analytic continuation of this locally holomorphic function.

We finish this section by recalling a well known fact which we need later. The following is an immediate implication of [83][(5.12)].

Lemma 2.1.10. *Let A be of trace-class and $\|A\|_1 < 1$, then we have that*

$$\det(1-A) = \exp\left(-\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{Tr}\left(A^k\right)\right).$$

2.1.3 Definition of the zeta function

In this section we introduce the zeta function for a conformal GDMS used in Main Theorem 1. This zeta function is a type of Artin-Mazur zeta-function (cf. [2]), which was generalized by Ruelle in [82], and hence is a type of dynamical zeta function. For a more comprehensive introduction to dynamical zeta-functions we refer to [82] and [6]. Roughly speaking, the zeta function is defined as the Fredholm determinant of the difference of the identity and the complexified FPR-operator. In the definition of this function, the particular choice of the underlying function space will be essential. Here, the functions under consideration are holomorphic, square-integrable functions defined on a complex neighbourhood of the limit set, rather than on the limit set only. One of the key facts in our investigation will be that the space of holomorphic L^2-functions is a Hilbert space. Furthermore, it turns out that this function space is somehow more natural, since the so obtained zeta-function coincides with the Selberg zeta function in the case in which S represents the action of a convex cocompact Schottky group (see [42]).

Definition of the real valued FPR-operator

We start by recalling the definition of the usual (that is, not complexified) version of the FPR-operator. At this point we would like to remark that one can find many different names attached to this operator in the literature. Often it is referred to as the Ruelle transfer operator or just the Ruelle operator, since it was formally introduced by Ruelle in [78]. We refer to it as the Frobenius-Perron-Ruelle-operator (FPR-operator) in order to stress that one can prove a kind of Frobenius-Perron theorem for it.

Definition 2.1.11 (Frobenius-Perron-Ruelle-operator)**.** *The Frobenius-Perron-Ruelle operator (FPR-operator) $\mathcal{L}_s$ for a GDMS S is defined for $s \in \mathbb{R}$, $x \in L(S)$ and $u : L(S) \to \mathbb{R}$ by*

$$\mathcal{L}_s(u)(x) := \sum_{e\in E} \|\phi_e'(x)\|^s u(\phi_e(x)).$$

Here, we use the convention that if $x \notin X_{i(e)}$, then $u(\phi_e(x)) := 0$.

Definition of the complexified FPR-operator

Recall that the compact sets X_v of a conformal GDMS S are subsets of $\mathbb{R}^m$. We now want to embed $\mathbb{R}^m$ into $\mathbb{C}^m$. For this, let $e \in E$ be fixed and recall that the maps ϕ_e and $\|\phi_e'\|$ are real analytic on $W_{i(e)}$ (see the discussion following Definition 2.1.1). Hence, we can complexify the real power series of ϕ_e and in this way we obtain a complex power series which converges in a complex neighbourhood of its real domain (see [55][Proposition 2.3.15], see also the discussion at the beginning of [55][Section 2.3.1]). Choose a domain of convergence for the complexified power series of $\|\phi_e'\|$ and intersect it with a domain for the complexified power series of ϕ_e. This gives a complex domain, say $(X_e)_{\mathbb{C}}$, on which both, ϕ_e and $\|\phi_e'\|$, have holomorphic extensions. These holomorphic extensions will be denoted by $(\phi_e)_{\mathbb{C}}$, and $(\|\phi_e'\|)_{\mathbb{C}}$ respectively.
Since ϕ_e is a contraction on $X_{i(e)}$ with $\|\phi_e'\| < \ell < 1$, it follows that $(\phi_e)_{\mathbb{C}}$ is contracting on some sufficiently small complex domain containing $(X_e)_{\mathbb{C}}$ (with Lipschitz constant less than $\frac{\ell+1}{2}$). Therefore, we have that $|(\|\phi_e'\|)_{\mathbb{C}}(z)| < 1$ on some sufficiently small complex domain containing $(X_e)_{\mathbb{C}}$. Without loss of generality, we can assume that this domain is $(X_e)_{\mathbb{C}}$, since otherwise we can choose $(X_e)_{\mathbb{C}}$ to be the intersection of both domains. For $v \in V$ let $(X_v)_{\mathbb{C}} := \bigcap_{e\in E, i(e)=v} (X_e)_{\mathbb{C}}$. Note that this intersection is an open domain, since E is finite.
For $\epsilon > 0$, let $B_\epsilon\left(X_{i(e)}\right) := \{z \in \mathbb{C}^m \mid \operatorname{dist}\left(z, X_{i(e)}\right) \leq \epsilon\}$. Combining the observation that $(\phi_e)_{\mathbb{C}}\left(B_\epsilon\left(X_{i(e)}\right) \cap \left(X_{i(e)}\right)_{\mathbb{C}}\right) \subset B_{\ell\cdot\epsilon}\left(X_{t(e)}\right)$ with the nestedness condition (NC) and the fact that E is finite, it follows that one can choose the elements of the sequence $\{(X_v)_{\mathbb{C}}\}_{v\in V}$ sufficiently small such that the sequence $\left\{(\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right\}_{e\in E}$ is nested in $\left(X_{t(e)}\right)_{\mathbb{C}}$. That is, we have that $(\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right) \subset \tilde{U} \subset \left(X_{t(e)}\right)_{\mathbb{C}}$ for some open set $\tilde{U} \subset \left(X_{t(e)}\right)_{\mathbb{C}}$. Note

that by choosing the sets $(X_v)_{\mathbb{C}}$ sufficiently small, if necessary, we can further assume that the sets $(X_v)_{\mathbb{C}}$ are pairwise disjoint. This can be done, since the finitely many sets X_v are pairwise disjoint and hence have a positive distance to each other. Finally, let $X_{\mathbb{C}} := \bigcup_{v \in V} (X_v)_{\mathbb{C}} \subset \mathbb{C}$ be the union of the pairwise disjoint complex sets $(X_v)_{\mathbb{C}}$.

Definition 2.1.12 (Complexified FPR-operator)**.** *Let $\mathcal{H}\left((X)_{\mathbb{C}}\right)$ denote the Hilbert space of holomorphic L^2-functions on $X_{\mathbb{C}}$. For $w \in \mathbb{C}$, $z \in X_{\mathbb{C}}$ and $u \in \mathcal{H}(X_{\mathbb{C}})$, we define the* complexified FPR-operator $\mathcal{L}_w : \mathcal{H}(X_{\mathbb{C}}) \to \mathcal{H}(X_{\mathbb{C}})$ *by*

$$\mathcal{L}_w(u)(z) := \sum_{e \in E} (D_e(z))^w (\Phi_e(u))(z).$$

Here, Φ_e denotes the composition operator given by $(\Phi_e(u))(z) := u((\phi_e)_{\mathbb{C}}(z))$, for each $u \in \mathcal{H}(X_{\mathbb{C}})$, and $D_e : \mathbb{C} \to \mathbb{C}$ is given by $D_e(z) := (\|\phi_e'\|)_{\mathbb{C}}(z)$. Similarly to Definition 2.1.11, we have used the convention that if $z \notin \left(X_{i(e)}\right)_{\mathbb{C}}$, then $(\Phi_e(u))(z) := 0$.

Note that $\mathcal{L}_w$ is a compact operator (see eg. [79]). In Lemma 2.2.7 we shall show that $\mathcal{L}_w$ is of trace-class. With this in mind, we can now define the zeta function as follows.

Definition 2.1.13. *The* zeta function $\zeta : \mathbb{C} \to \mathbb{C}$ *is given for $w \in \mathbb{C}$ by*

$$\zeta(w) := \det(1 - \mathcal{L}_w).$$

The zeros of ζ will be referred to as resonances*.*

2.2 Preparations for the proofs of the main results

2.2.1 Geometric preliminaries

An important tool for studying fractal sets is provided by fractal measures. Well studied examples of these measures are the Frostman measure for IFSs and the Patterson measure for Kleinian groups. As a consequence of the mass distribution principle, there is a direct connection between these measures and the Hausdorff dimension of the limit set. We use this connection when we apply the following theorem.

Theorem 2.2.1. *Let S be a finitely generated primitive conformal GDMS. Then there exists an Ahlfors-regular Borel probability measure μ supported on $L(S)$. Here, Ahlfors-regular means that the measure μ satisfies the following condition:*

$$\mu\left(B\left(x,r\right)\right) \asymp r^{\dim_{\mathrm{H}} L(S)},$$

for all $x \in L(S)$ and $0 < r < \frac{1}{2}\min\{\operatorname{diam} X_v \mid v \in V\}$.

For the proof we refer to [63] (proof of Theorem 4.2.11, page 79). A well written introduction can be found in [74](Section 7).

Using this theorem, we can prove the following lemma.

Lemma 2.2.2. *For a finitely generated primitive conformal GDMS S acting on $\mathbb{R}^m$, let $\phi_{\min} := \min_{e\in E} \|\phi'_e\|_\infty$, and for $r > 0$ define*

$$E\left(r\right) := \{\underline{e} \in E^* \mid r \geq \operatorname{diam}\left(\phi_{\underline{e}}\left(X_{i(\underline{e})}\right)\right) \geq r \cdot \phi_{\min}\}.$$

We then have that

$$\operatorname{card} E\left(r\right) \asymp r^{-\delta(S)}.$$

Here, $\delta\left(S\right) := \dim_{\mathrm{H}} L(S)$ refers to the Hausdorff dimension of the limit set $L(S)$ of S.

Proof. Clearly, $\bigcup_{e\in E(r)} \phi_e\left(X_{i(e)}\right)$ is a cover of $L_{dyn}(S)$ and hence of $L(S)$, since the $\phi_e(X_{i(e)})$ are compact and E is finite. For $1 > \phi_{\max} := \max_{e\in E} \|\phi'_e\|_\infty$ note that $\phi_{\max} \cdot \operatorname{diam}(A) \leq \operatorname{diam} \phi_f(A) \leq \phi_{\min} \cdot \operatorname{diam}(A)$, for all sets $A \subset X_{i(f)}$ and all $f \in E$. Hence, a straight forward calculation shows that the multiplicity of this cover is at most $\frac{\log(|\phi_{\max}|)}{\log(|\phi_{\min}|)}$. By Theorem 2.2.1, we have that there exists an Ahlfors-regular Borel probability measure μ on $L(S)$. Hence, we have

$$1 = \mu\left(L(S)\right) = \mu\left(\bigcup_{e\in E(r)} \phi_e\left(X_{i(e)}\right)\right) \asymp \sum_{e\in E(r)} \mu\left(\phi_e\left(X_{i(e)}\right)\right) \asymp \operatorname{card} E\left(r\right) \cdot r^{\delta(S)}.$$

$\square$

2.2.2 Functional analytic preliminaries

In this section we present some important facts from functional analysis which will be required later. The main aim is to show the following inequality, which will be crucial in the proof of Main Theorem 1. Namely, for non-empty finite index sets I and J, and for a family of trace-class operators $\{A_{i,j}\}_{(i,j)\in I\times J}$, we have that

$$\left|\det\left(1-\sum_{i\in I}\bigoplus_{j\in J}A_{i,j}\right)\right|\leq\prod_{l=0}^{\infty}\left(1+\sharp\left(I\right)\max_{i\in I,j\in J}\chi_{\left[\frac{l}{\sharp(I)\cdot\sharp(J)}\right]}\left(A_{i,j}\right)\right). \qquad (3)$$

Here, $[x]$ denotes the Gauss bracket (or floor function), and $\sharp I$ refers to the cardinality of I.
In what follows we require the following lemma.

Lemma 2.2.3. *Let $A:\mathcal{H}_A\to\mathcal{H}_A$ and $B:\mathcal{H}_B\to\mathcal{H}_B$ be compact operators. For all $l\in\mathbb{N}\cup\{0\}$, we have for the l-th singular value $\chi_l\left(A\oplus B\right)$ of the direct sum of A and B that*

$$\chi_l\left(A\oplus B\right)=\min\left\{\max\left\{\chi_j\left(A\right);\chi_n\left(B\right)\right\}\mid j+n=l\right\}. \qquad (4)$$

Proof. By applying Lemma 2.1.8, we have

$$\begin{aligned}\chi_l\left(A\oplus B\right) &= \inf_{\substack{\dim(\mathcal{H}_l)=l\\ \mathcal{H}_l\subset\mathcal{H}_A\oplus\mathcal{H}_B}}\ \sup_{\substack{z\in\mathcal{H}_l^{\perp}\\ \|z\|=1}}\sqrt{\left\langle\left(A\oplus B\right)^{*}\left(A\oplus B\right)z,z\right\rangle}\\ &= \inf_{\substack{\dim(\mathcal{H}_l)=l\\ \mathcal{H}_l\subset\mathcal{H}_A\oplus\mathcal{H}_B}}\ \sup_{\substack{z\in\mathcal{H}_l^{\perp}\\ \|z\|=1}}\frac{\|Az_{|\mathcal{H}_A}\|+\|Bz_{|\mathcal{H}_B}\|}{\|z\|}\\ &= \inf_{\substack{\dim(\mathcal{H}_l)=l\\ \mathcal{H}_l\subset\mathcal{H}_A\oplus\mathcal{H}_B}}\ \sup_{\substack{z\in\mathcal{H}_l^{\perp}\\ \|z\|=1}}\max\left\{\frac{\|Az_{|\mathcal{H}_A}\|}{\|z_{|\mathcal{H}_A}\|};\frac{\|Bz_{|\mathcal{H}_B}\|}{\|z_{|\mathcal{H}_B}\|}\right\}\end{aligned}$$

.

This shows that the set of singular values of $A\oplus B$ is equal to the union of the singular values of A and B. The assertion in (4) now follows by a straight forward combinatorical argument. □

Proposition 2.2.4. *For a compact operator $A:\mathcal{H}\to\mathcal{H}$, let $\{\chi_n\left(A\right)\}_{n\in\mathbb{N}\cup\{0\}}$ denote the decreasing set of singular values. Moreover, let $\{A_j\}_{j\in\{1,\ldots,k\}}$ refer to some finite family of compact operators. Then the following inequalities hold for all $l\in\mathbb{N}\cup\{0\}$.*

1. $|\det\left(1+A\right)|\leq\prod_{n=0}^{\infty}\left(1+\chi_n\left(A\right)\right)$

2. $\chi_l\left(\sum_{j=1}^{k}(A_j)\right) \leq k \cdot \max\left\{\chi_{\left[\frac{l}{k}\right]}(A_j) \mid j \in \{1,\ldots,k\}\right\}$

3. $\chi_l\left(\bigoplus_{j=1}^{k}(A_j)\right) \leq \max\left\{\chi_{\left[\frac{l}{k}\right]}(A_j) \mid j \in \{1,\ldots,k\}\right\}$

Proof. The assertion in *1* is well known and can be found in the literature, for example in [83] (see there the equation following (3.8), where one has to set $z = 1$). In contrast, the assertions of *2* and *3* are less well known, and we therefore include their proofs.
To prove the assertion in *2*, we use Fan's inequality [83][Theorem 1.7] (see also [31] and [32]), which states that for compact operators A and B we have for all $l, j \in \mathbb{N} \cup \{0\}$ that

$$\chi_{l+j}(A+B) \leq \chi_l(A) + \chi_j(B). \tag{5}$$

Now, let $\{A_j\}_{j\in\{1,\ldots,k\}}$ be some family of compact, normal operators of trace-class. For all $l \in \mathbb{N} \cup \{0\}$, we then have that

$$\begin{aligned}\chi_l\left(\sum_{j=1}^{k}(A_j)\right) &\leq \min_{j_1+\ldots+j_k=l}\{\chi_{j_1}(A_1)+\ldots+\chi_{j_k}(A_k)\}\\ &\leq k \cdot \max_{j=1,\ldots,k}\chi_{\left[\frac{l}{k}\right]}(A_j).\end{aligned}$$

This completes the proof of the assertion in point 2.
In order to prove the assertion in point *3*, observe that it is implied by (4), since for all $l \in \mathbb{N} \cup \{0\}$ we have

$$\begin{aligned}\chi_l\left(\bigoplus_{j=1}^{k}A_j\right) &= \min\left\{\max\{\chi_{j_1}(A_1),\ldots,\chi_{j_k}(A_k)\} \mid j_1+\ldots+j_k=l\right\}\\ &\leq \max_{j=1,\ldots,k}\chi_{\left[\frac{l}{k}\right]}(A_j).\end{aligned}$$

This finishes the proof of the proposition. □

Note that we can now use Proposition 2.2.4 to obtain the statement in (3) as follows. By applying first part *1*, then part *2*, and finally part *3* of Proposition 2.2.4, we derive for the family $\{A_{i,j}\}$ of bounded normal operators

$$\begin{aligned}\left|\det\left(1-\sum_{i\in I}\bigoplus_{j\in J}A_{i,j}\right)\right| &\leq \prod_{l=0}^{\infty}\left(1+\chi_l\left(\sum_{i\in I}\bigoplus_{j\in J}A_{i,j}\right)\right)\\ &\leq \prod_{l=0}^{\infty}\left(1+\sharp(I)\max\left\{\chi_{\left[\frac{l}{\sharp(I)}\right]}\left(\bigoplus_{j\in J}A_{i,j}\right) \mid i \in I\right\}\right)\\ &\leq \prod_{l=0}^{\infty}\left(1+\sharp(I)\max\left\{\chi_{\left[\frac{l}{\sharp(I)\cdot\sharp(J)}\right]}(A_{i,j}) \mid i\in I, j\in J\right\}\right).\end{aligned}$$

In the proof of Main Theorem 1 we also need the following lemma.

Lemma 2.2.5. *Let $\{A_j\}_{j\in J}$ be a finite family of trace-class operators. Then, for each $c_0 > 0$, there exists a constant $c_1 > 0$ such that for all $j \in J$ and $k \in \mathbb{N}$, we have that*

$$\sum_{l=0}^{\infty} \log\left(1 + c_0 \cdot \max_{j\in J} \chi_{\left[\frac{l}{k}\right]}(A_j)\right) \leq c_1 \cdot k \cdot \max_{j\in J} \sum_{l=0}^{\infty} \chi_l(A_j) \ll k.$$

Proof. Recall that the singular values are positive and bounded from above. This implies that for each $j \in J$ we have

$$c_0 \cdot \chi_{\left[\frac{l}{k}\right]}(A_j) \asymp \log\left(1 + c_0 \cdot \chi_{\left[\frac{l}{k}\right]}(A_j)\right).$$

Since J is finite, we obtain that

$$\begin{aligned}\sum_{l=0}^{\infty} \log\left(1 + c \cdot \max_{j\in J} \chi_{\left[\frac{l}{k}\right]}(A_j)\right) &\asymp \sum_{l=0}^{\infty} c_0 \cdot \max_{j\in J} \chi_{\left[\frac{l}{k}\right]}(A_j) \\ &= c_0 \sum_{l=0}^{\infty} k \cdot \max_{j\in J} \chi_l(A_j) = c_0 k \sum_{l=0}^{\infty} \max_{j\in J} \chi_l(A_j).\end{aligned}$$

Now we have $\sum_{l=0}^{\infty} \max_{j\in J} \chi_l(A_j) \leq \sum_{l=0}^{\infty} \sum_{j\in J} \chi_l(A_j)$, which is finite, since J is finite and the operators A_j are of trace-class. This completes the proof. $\square$

We end this section by giving an estimate for the determinant of a particular type of finite dimensional matrix, which we shall use in Section 2.2.3.

Lemma 2.2.6. *Let $U \subset \mathbb{R}^m$ be open, and let $g : U \to U$ be differentiable and Lipschitz with Lipschitz constant less than $0 < \ell < 1$. Let g' denote the Jacobian of g. We then have for all $x \in U$ that*

$$|\det(1 - g'(x))| \geq (1 - \ell)^m. \tag{6}$$

Proof. Since $\ell > 0$ is the Lipschitz-constant of g, we have that each eigenvalue $\lambda(g')$ of g' satisfies the inequality $|\lambda(g')| < \ell$. Therefore, $|1 - \lambda(g')| \geq 1 - \ell$. Since the Jacobian g' is an $m \times m$-matrix, it has exactly m (complex) eigenvalues. Hence, $\det(1 - g') = \prod_{j=1}^{m} (1 - \lambda_j(g'))$. Combining these observations, we obtain $|\det(1 - g')| \geq (1 - \ell)^m$. $\square$

2.2.3 Combining geometric and analytic facts

In this section we show how the nestedness condition (NC) of the conformal GDMS S comes into play. Namely, we show that $\mathcal{L}_w$ is of trace-class, and the proof mainly relies on the nestedness condition (NC). Furthermore, we show that there is some half-space $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > c\}$ on which ζ has no zeros. The proof of this fact will mainly rely on the nestedness condition (NC) and on an Atiyah-Bott-type fixed point theorem of Ruelle from [79].

Lemma 2.2.7. *Let S be a finitely generated conformal GDMS satisfying the nestedness condition (NC). Then $\mathcal{L}_w$ is of trace-class for all $w \in \mathbb{C}$.*

Proof. The proof will be given in several steps. First fix $w \in \mathbb{C}$ and note that it is enough to show that the sum $\sum_{l=1}^{\infty} \chi_l(\mathcal{L}_w)$ is finite. Hence, it is enough to find appropriate bounds for $\chi_l(\mathcal{L}_w)$. Recall that $\mathcal{L}_w = \sum_{e\in E} (D_e)^w \Phi_e$. Combining this with (4) (Fan's inequality), we have that

$$\chi_l(\mathcal{L}_w) \leq \operatorname{card}(E) \cdot \max\{\chi_l((D_e)^w \Phi_e) \mid e \in E\}.$$

Clearly, we have that $\chi_l((D_e)^w \Phi_e) \leq \| (D_e)^w \|_\infty \cdot \chi_l(\Phi_e)$ (see [83][Theorem 1.6]). Also, one immediately verifies, that for every $w \in \mathbb{C}$, we have that $\| (D_e)^w \|_\infty = \sup_{z\in X_{\mathbb{C}}} |(D_e(z))^w|$ and that the latter supremum is bounded from above by some finite constant, since $(D_e)^w$ is a continuous map defined on a compact set. Therefore, we only have to find bounds for $\chi_i(\Phi_e)$.
For this, we study $\Phi_e : \mathcal{H}\left(\left(X_{t(e)}\right)_{\mathbb{C}}\right) \to \mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$. Without loss of generality, we can assume that for each $e \in E$ there exists a family of open sets $\{B_k(e)\}_{k=1}^N$,for some $N \in \mathbb{N}$, which does not depend on $e \in E$, such that the following holds. Let $\overline{B}_k(e)$ denote the closure of $B_k(e)$, then $\left(X_{t(e)}\right)_{\mathbb{C}} = \bigcup_{k=1}^N \overline{B}_k(e)$ and $\left(X_{t(e)}\right)_{\mathbb{C}} \setminus \partial\left(X_{t(e)}\right)_{\mathbb{C}} = \bigcup_{k=1}^N B_k(e)$, and each $\overline{B}_k(e)$ is biholomorphic to $\overline{B}_1(0)$, the closed unit ball in $\mathbb{C}^m$. Let $b_{e,k}$ denote this biholomorphic map, so that $b_{e,k} : \overline{B}_k(e) \to \overline{B}_1(0)$. Recall from the discussion preceding Definition 2.1.12 that $(\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$ is nested in $\left(X_{t(e)}\right)_{\mathbb{C}}$ and note that this implies that $(\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right) \subset \bigcup_{k=1}^N {b_{e,k}}^{-1}\left(\overline{B}_\rho(0)\right)$ for some $\rho \in (0,1)$. We can take ρ to be independent of $e \in E$ (by taking the maximum of the ρ's), since E is finite.
We can express Φ_e as a composition of maps in the following way.

$$\begin{tikzcd}
\mathcal{H}\left(\bigcup_{k=1}^N \overline{B}_k(e)\right) \arrow[r] & \bigoplus_{k=1}^N \mathcal{H}\left(\overline{B}_k(e)\right) \arrow[r] & \bigoplus_{k=1}^N \mathcal{H}\left(\overline{B}_1(0)\right) \arrow[d, "\bigoplus_{k=1}^N R_\rho"] \\
\mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \arrow[d, "\widetilde{\Phi_e}"] & \mathcal{H}\left(\bigcup_{k=1}^N b_{e,k}\left(\overline{B}_\rho(0)\right)\right) \arrow[l] & \bigoplus_{k=1}^N \mathcal{H}\left(\overline{B}_\rho(0)\right) \arrow[l] \\
\mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right) & &
\end{tikzcd}$$

Here, the restriction operator $R_\rho : \mathcal{H}(B_1(0)) \to \mathcal{H}(B_\rho(0))$ is given by $R_\rho(f) := f_{|_{B_\rho(0)}}$.

Note that the norms of the biholomorphic maps $b_{e,k}$ are uniformly bounded, as are the norms of the natural restrictions (that is, all maps corresponding to horizontal arrows in the above diagram are uniformly bounded). In order to see that $\widetilde{\Phi_e}$ is bounded, note that by substitution one has

$$\begin{aligned}
\sup_{\|u\|=1} \|u \circ (\phi_e)_{\mathbb{C}}\| &= \sup_{\|u\|=1} \int_{X_{\mathbb{C}}} u^2((\phi_e)_{\mathbb{C}}(z))dz \\
&= \sup_{\|u\|=1} \int_{(\phi_e)_{\mathbb{C}}(X_{\mathbb{C}})} \frac{1}{\|(\phi_e)'_{\mathbb{C}}(y)\|} u^2(y)dy \\
&\leq \sup_{\|u\|=1} \frac{1}{\|(\phi_e)'_{\mathbb{C}}\|_\infty} \|u\| = \frac{1}{\|(\phi_e)'_{\mathbb{C}}\|_\infty}.
\end{aligned}$$

Note that the latter expression is uniformly bounded for each $e \in E$. Hence, since E is finite, there exists a universal upper bound. Therefore, it is enough to find bounds for the singular values of the operator R_ρ. We now show that there exists $\gamma \in (0,1)$ such that $\chi_l(R_\rho) \ll \gamma^{l^{\frac{1}{m}}}$, for all $l \in \mathbb{N} \cup \{0\}$. This is sufficient, since $\sum_{l=0}^{\infty} \gamma^{l^{\frac{1}{m}}}$ is dominated by a geometric series, which then implies that R_ρ is of trace-class, and hence that Φ_e is of trace-class as well. Since the polynomials form a basis of $\mathcal{H}$, it follows from (2) that it is sufficient to consider, for each multi index $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{N} \cup \{0\})^m$, the normalised polynomials u_α, given by $u_\alpha(z) := c_\alpha \prod_{i=1}^m z_i^{\alpha_i}$ for $c_\alpha \in \mathbb{C}$, $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$. With $|\alpha| := \sum_{i=1}^m \alpha_i$, we have that

$$\begin{aligned}
\|R_\rho(u_\alpha)\|_{L^2}^2 &= \int_{B(0,\rho)} \left| c_\alpha \prod_{i=1}^m z_i^{\alpha_i} \right|^2 dz \\
&\leq \int_{B(0,\rho)} |c_\alpha|^2 \left| \prod_{i=1}^m |z|^{\alpha_i} \right|^2 dz \\
&= \int_{B(0,\rho)} |c_\alpha|^2 |z|^{2\sum_{i=1}^m \alpha_i} dz \\
&= |c_\alpha|^2 \int_{\mathbb{S}^{2m-1}} \int_0^\rho r^{2|\alpha|} \cdot r^{2m-1}\, dr\, d\omega \\
&= |c_\alpha|^2 \cdot vol\left(\mathbb{S}^{2m-1}\right) \cdot \frac{1}{2(|\alpha|+m)} \rho^{2(|\alpha|+m)}
\end{aligned}$$

.

Clearly, we can assume that the basis of polynomials is ordered such that $\{\hat{u}_j\}_{j\in\mathbb{N}} := \{u_\alpha\}_\alpha$ and $\deg(\hat{u}_j) \leq \deg(\hat{u}_{j+1})$, where deg refers to the degree of a polynomial. Combining this with the estimate above and with (2), we have for all $l \in \mathbb{N} \cup \{0\}$ that

$$\chi_l(R_\rho) \leq \sum_{j=l}^{\infty} \|R_\rho \hat{u}_j\|_{L^2} \ll \sum_{|\alpha| \geq l^{1/m}} \rho^{|\alpha|+m}.$$

Also, observe that $\operatorname{card}\{|\alpha| = k\} \ll k^{m-1}$. Hence, we have

$$\sum_{|\alpha| \geq l^{1/m}} \rho^{|\alpha|+m} \ll \sum_{k \geq l^{1/m}} k^{m-1} \rho^{k+m}.$$

We require the following estimate:

$$\int_{l^{1/m}}^{\infty} x^{m-1} \rho^x dx \ll l \cdot \rho^{\left(l^{1/m}\right)}, \text{ for all } l \in \mathbb{N} \cup \{0\}. \tag{7}$$

This estimate is well known to experts in the area, since the left hand side of the latter inequality is equal to the well known upper incomplete gamma function. (For an introduction to the incomplete gamma function we refer to [91][Chapter 11.2].) However, for sake of completeness, we include an elementary proof of this inequality. Indeed, this estimate can be obtained by integration by parts, as follows.

$$\int_{l^{1/m}}^{\infty} x^{m-1} \rho^x dx = \left[\sum_{i=1}^{m} x^{m-i} \ln(\rho)^{-i} \rho^x \frac{(m-1)!}{(m-i)!} (-1)^{i-1} \right]_{l^{1/m}}^{\infty} =: [I(x)]_{l^{1/m}}^{\infty}$$

Since $\lim_{x \to \infty} x^{m-i} \rho^x = 0$ for all $i \in \{1, \ldots, m\}$, one can immediately verify that $\lim_{x \to \infty} I(x) = 0$. Hence, the integral in the equation above is equal to $-I\left(l^{1/m}\right)$. Finally, observe that

$$\begin{aligned} -I\left(l^{1/m}\right) &= -\sum_{i=1}^{m} \left(l^{1/m}\right)^{m-i} (\ln(\rho))^{-i} \rho^{\left(l^{1/m}\right)} \frac{(m-1)!}{(m-i)!} (-1)^{i-1} \\ &\leq l \cdot \rho^{\left(l^{1/m}\right)} \cdot (m-1)! \cdot \sum_{i=1}^{m} |\ln(\rho)|^{-i} \\ &\ll l \cdot \rho^{\left(l^{1/m}\right)}. \end{aligned}$$

This verifies (7).

To finish the proof of the lemma, note that there exists some $\tilde{\rho} \in (\rho, 1)$ such that

$$l \cdot \rho^{\left(l^{1/m}\right)} \leq \tilde{c} \tilde{\rho}^{(l^{1/m})}.$$

Hence, for all $l \in \mathbb{N} \cup \{0\}$ we have that

$$\chi_l(R_\rho) \ll \sum_{k \geq l^{1/m}} k^{m-1} \rho^{k+m} \ll \int_{l^{1/m}}^{\infty} x^{m-1} \rho^x dx \ll l \cdot \rho^{\left(l^{1/m}\right)} \ll \tilde{\rho}^{(l^{1/m})}.$$

Since $\sum_l \tilde{\rho}^{(l^{1/m})}$ is dominated by a geometric series, this implies that R_ρ is of trace-class. Recall that we expressed Φ_e as a composition of several operators, and these were all bounded. Hence, we can use the fact that $\chi_l(AB) \leq \|A\| \chi_l(B)$ to bound the singular values of Φ_e by $\chi_l(R_\rho)$. Hence, Φ_e is of trace-class, since R_ρ is of trace-class. From the discussion at the beginning of this proof it now follows that $\sum_{l=0}^{\infty} \chi_l(\mathcal{L}_w) < \infty$, and hence, that $\mathcal{L}_w$ is of trace-class. $\square$

For the proof of the next Proposition we need the following theorem, which is due to Ruelle ([79]). Its proof is based on a fixed point theorem by Atiyah and Bott [3](see also [4],[5] and [46][Theorem 4.1]).

Theorem 2.2.8. *Let $U \subset \mathbb{C}^m$ be a non-empty open bounded complex domain. Let $\psi : U \to \mathbb{C}$ and $\phi : U \to U$ be holomorphic functions with continuous extensions to $\overline{U}$, and assume that $\phi\left(\overline{U}\right) \subset U$. Then ϕ has a unique fixed point $z^* \in U$, and the weighted composition operator $T : \mathcal{H}(U) \to \mathcal{H}(U)$, given by $(Tu)(z) := \psi(z)(u \circ \phi)(z)$, is of trace-class with trace given by the Atiyah-Bott type fixed point formula*

$$\operatorname{Tr}(T) = \frac{\psi(z^*)}{\det(1 - \phi'(z^*))}.$$

We use this theorem to prove the following proposition.

Proposition 2.2.9. *With the notion as above we have that there exists a constant $c \in \mathbb{R}$ such that for all w with $\operatorname{Re}(w) > c$, we have that $|\zeta(w)| > \widetilde{c}$, for some $\widetilde{c} > 0$. In particular, there are no zeros of ζ in the half-space $\{w \in \mathbb{C} \mid \operatorname{Re}(w) > c\}$.*

Proof. Combining Lemma 2.1.10 and the fact that $\det(1-\mathcal{L}_w)$ is analytic (cf. [38], see also [79]), one easily verifies that

$$\det(1 - \mathcal{L}_w) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}(\mathcal{L}_w^n)\right).$$

In order to evaluate the traces, we use the notation from Definition 2.1.12 and write

$$\mathcal{L}_w(u)(z) = \sum_{e \in E} (D_e(z))^w \cdot \Phi_e(u)(z),$$

where $\Phi_e : \mathcal{H}\left(\left(X_{t(e)}\right)_{\mathbb{C}}\right) \to \mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$. We use an idea similar to the one of Ruelle in [79]. For ease of notation, we define the operator L_e by $L_e(u)(z) := (D_e(z))^w \cdot \Phi_e(u(z))$. With this notation we then have that (see [79][p. 235])

$$\operatorname{Tr}(\mathcal{L}_w^n) = \sum_{\substack{(e_1,\dots,e_n) \in E^n \\ i(e_1) = t(e_n)}} \operatorname{Tr}(L_{e_1} \dots L_{e_n}).$$

For each $\underline{e} := (e_1, \dots, e_n) \in E^n$ we have that

$$L_{e_1} \dots L_{e_n}(u)(z) = ((D_{e_n} \circ \dots \circ D_{e_1})(z))^w \cdot (u \circ (\phi_{e_n})_{\mathbb{C}} \circ \dots \circ (\phi_{e_1})_{\mathbb{C}})(z).$$

Note that $L_{\underline{e}} := L_{e_1} \dots L_{e_n}$ has the form of a weighted composition operator T given by $T(u)(z) = h(z) \cdot (u \circ g_{\mathbb{C}})(z)$, for functions $h : U \to \mathbb{C}$ and

$g_{\mathbb{C}} : U \to U$. Further, note that $g_{\mathbb{C}}(\overline{U}) \subset U$, since S satisfies the nestedness condition (NC). Hence, by Theorem 2.2.8, we have that

$$\mathrm{Tr}\left(L_{e_1} \dots L_{e_n}\right) = \frac{\left(\left(D_{e_n} \circ \dots \circ D_{e_1}\right)\left(z^*_{\underline{e}}\right)\right)^w}{\det\left(1 - \left(g_{\mathbb{C}}\right)'\left(z^*_{\underline{e}}\right)\right)},$$

where $z^*_{\underline{e}}$ is the unique fixed point of $g_{\mathbb{C}} = (\phi_{\underline{e}})_{\mathbb{C}} = (\phi_{e_n})_{\mathbb{C}} \circ \dots \circ (\phi_{e_1})_{\mathbb{C}}$ and $(g_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)$ denotes the Jacobian of $g_{\mathbb{C}}$ at $z^*_{\underline{e}}$. Recall that the maps ϕ_e are real analytic and that $(\phi_e)_{\mathbb{C}}$ are holomorphic maps defined via exactly the same power series. Therefore, the fixed point $z^*_{\underline{e}}$ of $g_{\mathbb{C}}$ is equal to the fixed point of $g := \phi_{e_n} \circ \dots \circ \phi_{e_1}$, and thus $z^*_{\underline{e}}$ belongs to $\mathbb{R}^m$ (in particular, $z^*_{\underline{e}} \in L(S)$). Let us now evaluate the determinant $\det\left(1 - (g_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)\right)$. Since $(\phi_e)_{\mathbb{C}}$ is defined via the power series of ϕ_e , it follows that the entries of their Jacobian coincide (in the sense that each entry of the Jacobian is an analytic function and so it is a power series; for g' and $g'_{\mathbb{C}}$ the coefficients of these power series coincide). Hence, the Jacobian of $g_{\mathbb{C}}$ evaluated at $z^*_{\underline{e}} \in \mathbb{R}^m$ equals the Jacobian of g at $z^*_{\underline{e}}$. It is therefore suffices to evaluate $\det\left(1 - g'\left(z^*_{\underline{e}}\right)\right)$. By Lemma 2.2.6, we now have

$$\left|\det\left(1 - ((\phi_{\underline{e}})_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)\right)\right| = \left|\det\left(1 - (g_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)\right)\right| \geq (1-\ell)^m . \tag{8}$$

This follows, since all maps of the GDMS S are contracting at least by some factor $\ell < 1$.
Recall that $D_e(z) = (\|(\phi'_e)\|)_{\mathbb{C}}(z)$. Since $(\|(\phi'_e)\|)_{\mathbb{C}}$ is a holomorphic extension of $\|(\phi'_e)\|$, we have that $(\|(\phi'_e)\|)_{\mathbb{C}}$ evaluated at a real $z^*_{\underline{e}}$ is the same as $\|(\phi'_e)\|$ evaluated at $z^*_{\underline{e}}$, that is $\|(\phi'_e)\|\left(z^*_{\underline{e}}\right) = (\|(\phi'_e)\|)_{\mathbb{C}}\left(z^*_{\underline{e}}\right)$. Therefore, we have that $D_e\left(z^*_{\underline{e}}\right) = \left\|(\phi'_e)\left(z^*_{\underline{e}}\right)\right\| \in \mathbb{R}$. Hence, $|D_e\left(z^*_{\underline{e}}\right)^w| = \left\|(\phi'_e)\left(z^*_{\underline{e}}\right)\right\|^{\mathrm{Re}(w)} \leq \ell^{\mathrm{Re}(w)}$.
We are now ready to complete the proof. First recall that

$$\zeta(w) = \det(1 - \mathcal{L}_w) \;=\; \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left(\mathcal{L}^n_w\right)\right).$$

Now we can bound the exponent of the right hand side of the above equation in the following way:

$$\begin{aligned}\left|\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}\left(\mathcal{L}^n_w\right)\right| &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{(e_1,\dots,e_n)} \left|\frac{\left(\left(D_{e_n} \circ \dots \circ D_{e_1}\right)\left(z^*_{\underline{e}}\right)\right)^w}{\det\left(1 - (g_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)\right)}\right| \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n} \sum_{(e_1,\dots,e_n)} \left|\frac{\left(\left\|\left(\phi'_{e_n}\right)\left(z^*_{\underline{e}}\right)\right\| \cdot \dots \cdot \left\|\left(\phi'_{e_1}\right)\left(z^*_{\underline{e}}\right)\right\|\right)^w}{(1-\ell)^m}\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{(1-\ell)^m} \sum_{(e_1,\ldots,e_n)} \left|(\ell^n)^w\right| \\
&= \frac{1}{(1-\ell)^m} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\underline{e}\in E^n} \ell^{n\cdot \mathrm{Re}(w)} \\
&= \frac{1}{(1-\ell)^m} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{card}\left(E^n\right) \cdot \ell^{n\cdot \mathrm{Re}(w)} \\
&\leq \frac{1}{(1-\ell)^m} \sum_{n=1}^{\infty} \operatorname{card}\left(E\right)^n \cdot \ell^{n\cdot \mathrm{Re}(w)} \\
&= \frac{1}{(1-\ell)^m} \sum_{n=1}^{\infty} \left(\operatorname{card}\left(E\right) \cdot \ell^{\mathrm{Re}(w)}\right)^n .
\end{aligned}
$$

The series in the latter expression is a geometric series, so it converges for $\mathrm{Re}\,(w)$ large enough. This shows that there are no zeros of ζ for $\mathrm{Re}\,(w)$ large enough. Furthermore, there exists a positive constant $\tilde{c}$ such that for $\mathrm{Re}\,(w)$ sufficiently large, we have $|\zeta\,(w)\,| > \tilde{c} > 0$. □

2.3 Proofs of main results

2.3.1 Refinement of the FPR-operator

For the following lemma, recall that in Definition 2.1.12 we defined $\mathcal{H}\,(Y_{\mathbb{C}})$ to be the Hilbert space of holomorphic L^2-functions on a complex neighbourhood $Y_{\mathbb{C}} \subset \mathbb{C}^m$.

Lemma 2.3.1. *Let S be a finitely generated primitive conformal GDMS. For each $e \in E$ and $r > 0$ sufficiently small, there exists a refinement $\tilde{E}_r\,(i\,(e)) \subset E^*$ and a refined FPR-operator $\widetilde{\mathcal{L}_w}$ which is of the form*

$$
\widetilde{\mathcal{L}_w} = \sum_{e\in E} \bigoplus_{\underline{f}\in \tilde{E}_r(i(e))} \left(D_{e,\underline{f}}\right)^w \cdot \Phi_{e,\underline{f}}.
$$

Here, $w \in \mathbb{C}$ and $D_{e,\underline{f}}$ is given for $z \in (X_{i(\underline{f})})_{\mathbb{C}}$ by $D_{e,\underline{f}}\,(z) := (\|(\phi_e')\|)_{\mathbb{C}}\,(z)$, and $\Phi_{e,\underline{f}} : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)\right)$ is given by $u\,(z) \mapsto u\,((\phi_e)_{\mathbb{C}}\,(z))$. Furthermore, we have

$$
\det(1-\mathcal{L}_w) = \det(1-\widetilde{\mathcal{L}_w}).
$$

Proof. Recall that the FPR-operator was defined by

$$
\mathcal{L}_w\,(u)\,(z) = \sum_{e\in E} \left((\|(\phi_e')\|)_{\mathbb{C}}\,(z)\right)^w u\,((\phi_e)_{\mathbb{C}}\,(z)) = \sum_{e\in E} \left(D_e\,(z)\right)^w \cdot \Phi_e\,(u)\,(z)\,.
$$

Let us concentrate on the composition operator Φ_e in one of these summands for the moment. We have that

$$\Phi_e : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right).$$

Recall that we defined $\mathcal{L}_w$ and all associated operators on functions which are defined on small neighbourhoods of the limit set. We now refine these neighbourhoods. For this, let $r > 0$ and define

$$E_r\left(i\left(e\right)\right) := \{\underline{f} \in E^* \mid \phi_{\underline{f}}\left(X_{i\left(\underline{f}\right)}\right) \subset X_{i(e)} \text{ and } \operatorname{diam}\left(\phi_{\underline{f}}\left(X_{i\left(\underline{f}\right)}\right)\right) \asymp r\}.$$

Then $\bigcup_{\underline{f}\in E_r(i(e))} \phi_{\underline{f}}\left(X_{i\left(\underline{f}\right)}\right)$ is a cover of $L(S) \cap X_{i(e)}$.
Therefore, without loss of generality, we can restrict $\mathcal{L}_w$ to the space $\mathcal{H}\left(\bigcup_{e\in E}\bigcup_{\underline{f}\in E_r(i(e))}\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i\left(\underline{f}\right)}\right)_{\mathbb{C}}\right)\right)$, which corresponds to restricting Φ_e to $\mathcal{H}\left(\bigcup_{\underline{f}\in E_r(i(e))}\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i\left(\underline{f}\right)}\right)_{\mathbb{C}}\right)\right)$. Since S satisfies SSC, we have for $f, g \in E_r\left(i\left(e\right)\right)$ that either $\phi_f\left(X_{i(f)}\right)$ and $\phi_g\left(X_{i(g)}\right)$ are disjoint, or one is a subset of the other. In the latter case, if $\phi_f\left(X_{i(f)}\right) \subset \phi_g\left(X_{i(g)}\right)$, we write $f < g$. Since the cardinality of $E_r\left(i\left(e\right)\right)$ is finite, this partial ordering allows us to determine maximal elements in $E_r\left(i\left(e\right)\right)$, and this allows us to define the set $\tilde{E}_r\left(i\left(e\right)\right)$ of maximal elements in $E_r\left(i\left(e\right)\right)$. We then have that $\bigcup_{\underline{f}\in\tilde{E}_r(i(e))} \phi_{\underline{f}}\left(X_{i\left(\underline{f}\right)}\right)$ is a cover of $L(S) \cap X_{i(e)}$ consisting of pairwise disjoint sets. Therefore, instead of considering the function space $\mathcal{H}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)$, we consider the direct sum $\bigoplus_{\underline{f}\in\tilde{E}_r(i(e))}\mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i\left(\underline{f}\right)}\right)_{\mathbb{C}}\right)\right)$. For this we have

$$\widetilde{\Phi_e} : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \bigoplus_{\underline{f}\in\tilde{E}_r(i(e))}\mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i\left(\underline{f}\right)}\right)_{\mathbb{C}}\right)\right),$$

given by $u\left(z\right) \mapsto u\left(\left(\phi_e\right)_{\mathbb{C}}\left(z\right)\right)$. Hence, we have for each $e \in E$ that

$$\begin{aligned}\left(\left(\|(\phi_e')\|\right)_{\mathbb{C}}(z)\right)^w \cdot \widetilde{\Phi_e}\left(u\right)\left(z\right) &= \bigoplus_{\underline{f}\in\tilde{E}_r(i(e))}\left(\left(\|(\phi_e')\|\right)_{\mathbb{C}}(z)\right)^w \cdot \Phi_{e,\underline{f}}\left(u\right)\left(z\right)\\ &= \bigoplus_{\underline{f}\in\tilde{E}_r(i(e))}\left(D_{e,\underline{f}}\left(z\right)\right)^w \cdot \Phi_{e,\underline{f}}\left(u\right)\left(z\right),\end{aligned}$$

with $\Phi_{e,\underline{f}} : \mathcal{H}\left((\phi_e)_{\mathbb{C}}\left(\left(X_{i(e)}\right)_{\mathbb{C}}\right)\right) \to \mathcal{H}\left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i\left(\underline{f}\right)}\right)_{\mathbb{C}}\right)\right)$ given by $u\left(z\right) \mapsto u\left(\left(\phi_e\right)_{\mathbb{C}}\left(z\right)\right)$.

In order to prove that $\det(1-\mathcal{L}_w) = \det(1-\widetilde{\mathcal{L}_w})$, recall that with the notation as in the proof of Proposition 2.2.9, we have that

$$\det(1-\mathcal{L}_w) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{(e_1,\ldots,e_n)\in E^n \\ i(e_1)=t(e_n)}} \frac{\left((D_{e_n} \circ \ldots \circ D_{e_1})\left(z^*_{\underline{e}}\right)\right)^w}{\det\left(1-(g_{\mathbb{C}})'\left(z^*_{\underline{e}}\right)\right)}.$$

Note that here the inner sum is actually taken over the unique fixed points of $\phi_{\underline{e}}$, $\underline{e} \in E^n$ and the summands are traces of weighted composition operators. Recall that these traces are given by evaluating certain expressions at the fixed points. Let us now compare these expressions with the corresponding expressions for $\widetilde{\mathcal{L}_w}$. Let $\underline{e} \in E^n$ be given and let $z^*_{\underline{e}}$ be the unique fixed point of $(\phi_{\underline{e}})_{\mathbb{C}}$. Recall that we have $z^*_{\underline{e}} \in L(S)$. Recall that we have $\widetilde{\Phi}_e = \bigoplus_{f\in\tilde{E}_r(i(e))} \Phi_{e,f}$. Note that there is a unique $f_1 \in \bigcup_{e\in E} \tilde{E}_r(i(e))$ such that $z^*_{\underline{e}} \in (\phi_f)_{\mathbb{C}}(X_{i(f)})$, since $\{\phi_{\underline{f}}(X_{i(\underline{f})})\}_{\underline{f}\in\tilde{E}_r(i(e))}$ is a cover of $L(S)\cap X_{i(e)}$ (and $\{X_{i(e)}\}_{e\in E}$ is a cover of $L(S)$) by pairwise disjoint sets. From this it is easy to see that there is a unique operator $\Phi_{e_n,f_n} \circ \ldots \circ \Phi_{e_1,f_1}$ with associated contraction having the unique fixed point $z^*_{\underline{e}}$ and $\underline{e} = (e_1,\ldots,e_n)$. Note that $\Phi_{e_n,f_n} \circ \ldots \circ \Phi_{e_1,f_1}$ is a composition operator. Applying the formula from Theorem 2.2.8, it is clear that its trace coincides with the trace of $\Phi_{\underline{e}}$. From this it follows that $\mathrm{Tr}(\mathcal{L}^n_w) = \mathrm{Tr}(\widetilde{\mathcal{L}_w}^n)$, for all $n \in \mathbb{N}$. Hence, we have that $\det(1-\mathcal{L}_w) = \det(1-\widetilde{\mathcal{L}_w})$. This completes the proof. □

2.3.2 An upper bound for the zeta function

The following lemma gives the key observation of this section. It will allow us to show that if the refinements in Lemma 2.3.1 are chosen appropriately, then $|D^w_{e,\underline{f}}(z)|$ can be bounded from above by some constant.

Lemma 2.3.2. *Let $e \in E$ be fixed and let $c > 0$ be given. Let $D_{e,\underline{f}}$ be as in Lemma 2.3.1, with $\underline{f} \in E_r(i(e))$ for some $r > 0$ sufficiently small. If for $w \in \mathbb{C}$ we have $\mathrm{Re}(w) \geq -c$ and $|\mathrm{Im}(w)| \asymp r^{-1}$, then $|(D_{e,\underline{f}}(z))^w| \ll 1$, for all $z \in \left(\phi_{\underline{f}}\right)_{\mathbb{C}}\left(\left(X_{i(\underline{f})}\right)_{\mathbb{C}}\right)$. In particular, $\|(D_{e,\underline{f}}(\cdot))^w\|_\infty \ll 1$.*

Proof. Although the following calculation is relatively straightforward, we present the details here for the sake of completeness. Here, $\mathrm{Arg}(z)$ denotes the number $0 \leq \mathrm{Arg}(z) < 2\pi$ with $z = |z| \cdot e^{\imath\,\mathrm{Arg}(z)}$. For two complex numbers $z, w \in \mathbb{C}$ we have that

$$\begin{aligned} |z^w| &= \left| \left(|z| \cdot \mathrm{e}^{(\imath\cdot\mathrm{Arg}(z))}\right)^w \right| \\ &\leq |z|^{\mathrm{Re}(w)} \cdot \left|\mathrm{e}^{(\imath\cdot w\cdot\mathrm{Arg}(z))}\right| \end{aligned}$$

$$
\begin{aligned}
&\le |z|^{\mathrm{Re}(w)} \cdot \left| \mathrm{e}^{(-\mathrm{Im}(w)\cdot \mathrm{Arg}(z))} \right| \\
&\le |z|^{\mathrm{Re}(w)} \cdot \mathrm{e}^{(|\mathrm{Im}(w)|\cdot |\mathrm{Arg}(z)|)}.
\end{aligned}
$$

Applying this inequality to the operator $D_{e,\underline{f}}$ given in Lemma 2.3.1, for e, $\underline{f}$, w and z as stated in the lemma, we obtain that

$$
|(D_{e,\underline{f}}(z))^w| \le \left| D_{e,\underline{f}}(z) \right|^{\mathrm{Re}(w)} \cdot \mathrm{e}^{(|\mathrm{Im}(w)|\cdot |\mathrm{Arg}(D^w_{e,\underline{f}}(z))|)}. \tag{9}
$$

Recall from the discussion preceding Definition 2.1.12 that $(\|\phi'_e\|)_{\mathbb{C}}$ was a holomorphic extension of $\|\phi'_e\|$ on a small neighbourhood $\left(X_{i(e)}\right)_{\mathbb{C}}$ and that $|(\|\phi'_e\|)_{\mathbb{C}}(z)| < 1$, for all $z \in \left(X_{i(e)}\right)_{\mathbb{C}}$. Therefore, for $\mathrm{Re}\,(w) \ge -c$ we have

$$
\left| \left| D_{e,\underline{f}}\,(z) \right|^{\mathrm{Re}(w)} \right| = \left| \, |(\|\phi'_e\|)_{\mathbb{C}}\,(z)|^{\mathrm{Re}(w)} \right| \le \left| \, |\,(\|\phi'_e\|)_{\mathbb{C}}\,(z)\,|^{-c} \right| \ll 1. \tag{10}
$$

Note that for $\mathrm{Im}(w) = 0$ this already proves the assertion in the theorem. Hence, assume $|\,\mathrm{Im}(w)| > 0$. Recall that by Definition 2.1.1 we have $|\,\|\phi'_e(x)\| - \|\phi'_e(y)\|\,| \le c|x-y|$. In other words, the maps $\|\phi'_e\|$ are Lipschitz continuous with a uniform Lipschitz constant $c > 0$. Hence, we can assume without loss of generality that the holomorphic extensions $(\|\phi'_e\|)_{\mathbb{C}}$ are also Lipschitz continuous with some uniform Lipschitz constant. Therefore, for all $z_1, z_2 \in \left(\phi_{\underline{f}}\right)_{\mathbb{C}} \left(\left(X_{i(\underline{f})} \right)_{\mathbb{C}} \right)$, we have that

$$
|(\|\phi'_e\|)_{\mathbb{C}}\,(z_1) - (\|\phi'_e\|)_{\mathbb{C}}\,(z_2)| \ll |z_1 - z_2| \le \mathrm{diam} \left(\left(\phi_{\underline{f}}\right)_{\mathbb{C}} \left(\left(X_{i(\underline{f})} \right)_{\mathbb{C}} \right) \right) \asymp r.
$$

Further recall from the discussion following Definition 2.1.1 that $\|\phi'_e\|$ is non-zero. In particular, by combining the bounded distortion property and the fact the S is finitely generated, we have that $\|\phi'_e(x)\|$ is uniformly bounded away from zero, for all $e \in E$ and all $x \in X_{i(e)}$. Hence, we can assume that $|\,(\|\phi'_e\|)_{\mathbb{C}}\,(z)|$ is bounded away from zero uniformly, say by some constant $c_0 > 0$, for all $e \in E$ and $z \in X_{\mathbb{C}}$. Combining these observations, we conclude that for all $z \in \left(\phi_{\underline{f}}\right)_{\mathbb{C}} \left(\left(X_{i(\underline{f})} \right)_{\mathbb{C}} \right)$ we have that

$$
|\,\mathrm{Arg}\left(D_{e,\underline{f}}\,(z) \right)| \;\le\; \arctan \left(\mathrm{diam} \left(\phi_{\underline{f}}\right)_{\mathbb{C}} \left(X_{i(\underline{f})} \right) / c_0 \right) \asymp r.
$$

Finally, for $|\,\mathrm{Im}\,(w)\,| > 0$ we can choose $r \asymp |\,\mathrm{Im}\,(w)\,|^{-1}$ and then the previous estimate implies that $|\,\mathrm{Im}\,(w)\,| \cdot |\,\mathrm{Arg}\left(D_{e,\underline{f}}\,(z) \right)| \ll 1$. Combining this with the inequalities (9) and (10), the lemma follows. □

2.3.3 Proof of Main Theorem 1

We are now ready to prove Main Theorem 1. Recall that the main statements of the theorem are as follows.

> *Let S be a finitely generated primitive conformal GDMS acting on $\mathbb{R}^m$ and satisfying the strong separation condition (SSC) and the nestedness condition (NC). For each $c > 0$ and $w \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) > -c, \ |\operatorname{Im}(z)| > 1\}$, we then have*
>
> $$\log |\zeta(w)| \ll \mathrm{e}^{\delta(S)\cdot\log(|\operatorname{Im}(w)|)}.$$
>
> *Moreover, for all $k > 0$ sufficiently large, we have*
>
> $$\operatorname{card}\left\{w \in Q_{k,k+1}^{-c,\infty} \mid \zeta(w) = 0\right\} \ll k^{\delta(S)}.$$

Let S be as stated in Main Theorem 1. For the first part of Main Theorem 1, let $w \in \{z \in \mathbb{C} \mid \operatorname{Re}(z) > -c, \ |\operatorname{Im}(z)| > 1\}$, for some $c > 0$. Combining Lemma 2.3.1 and equation (3) with the definition of the zeta function, we have for all $r > 0$ that

$$|\zeta(w)| \le \prod_{l=0}^{\infty}\left(1 + \sharp(E)\cdot \max_{e\in E, \underline{f}\in\tilde{E}_r(i(e))} \chi_{\left[\frac{l}{\sharp(E)\cdot\sharp\left(\tilde{E}_r(i(e))\right)}\right]}\left(\left(D_{e,\underline{f}}\right)^{w}\cdot\Phi_{e,\underline{f}}\right)\right).$$

By Lemma 2.3.2, we have that if $r^{-1} \asymp |\operatorname{Im}(w)|$ then there is some $c_2 > 0$ such that for every $\underline{f} \in E_r(i(e))$ we have $\|(D_{e,\underline{f}})^w\|_\infty < c_2$. Hence, let us choose r in this way, that is, let $r^{-1} \asymp |\operatorname{Im}(w)|$. This can be done, since we have seen before that ζ is independent of the choice of r. Combining this with the fact that $\chi_l(AB) \le \|A\|\chi_l(B)$, we have that

$$|\zeta(w)| \le \prod_{l=0}^{\infty}\left(1 + c_2\cdot\sharp(E)\cdot \max_{e\in E, \underline{f}\in\tilde{E}_r(i(e))} \chi_{\left[\frac{l}{\sharp(E)\cdot\sharp\left(\tilde{E}_r(i(e))\right)}\right]}\left(\Phi_{e,\underline{f}}\right)\right).$$

Note that since $\Phi_{e,\underline{f}}$ is a restriction of Φ_e, we have that $\chi_l\left(\Phi_{e,\underline{f}}\right) \le \chi_l(\Phi_e)$, for all $l \in \mathbb{N}\cup\{0\}$. Hence, we have

$$|\zeta(w)| \le \prod_{l=0}^{\infty}\left(1 + c_2\cdot\sharp(E)\cdot \max_{e\in E, \underline{f}\in\tilde{E}_r(i(e))} \chi_{\left[\frac{l}{\sharp(E)\cdot\sharp\left(\tilde{E}_r(i(e))\right)}\right]}(\Phi_e)\right).$$

Applying Lemma 2.2.5, we obtain the estimate

$$|\zeta(w)| \ll \mathrm{e}^{\sharp(E)\cdot\sharp\left(\tilde{E}_r(i(e))\right)}.$$

Taking the logarithm on both sides of the above inequality and recalling that $\sharp\left(\tilde{E}_r\left(i\left(e\right)\right)\right) \leq \sharp\left(E_r\left(i\left(e\right)\right)\right)$, it follows that

$$\log(|\zeta(w)|) \ll \sharp(E)\sharp(E_r\left(i\left(e\right)\right)) + const. \asymp const. + \sharp(E_r\left(i\left(e\right)\right)).$$

Applying Lemma 2.2.2, we then have

$$\log(|\zeta\left(w\right)|) \ll const. + r^{-\delta(S)}.$$

Since $r^{-1} \asymp |\operatorname{Im}(w)|$ we hence have $\log(|\zeta\left(w\right)|) \ll const. + |\operatorname{Im}(w)|^{\delta(S)}$. Note that $|\operatorname{Im}(w)|^{\delta(S)} + const. \ll |\operatorname{Im}(w)|^{\delta(S)}$, since $|\operatorname{Im}(w)| > 1$. From this the first part of Main Theorem 1 follows.

For the second part of the theorem, let us recall that the rectangle $Q_{c,d}^{a,b} \subset \mathbb{C}$ is defined for $a, b, c, d \in \mathbb{R}$ by

$$Q_{c,d}^{a,b} := \{z \in \mathbb{C} \mid a \leq \operatorname{Re}\left(z\right) \leq b,\ c \leq \operatorname{Im}\left(z\right) \leq d\}.$$

Moreover, let c be a fixed positive constant. We claim that for all sufficiently large $k \in \mathbb{R}$, the following upper bound for the growth of the number of resonances within the strip $Q_{k,k+1}^{-c,\infty}$ holds:

$$\operatorname{card}\left\{w \in Q_{k,k+1}^{-c,\infty} \mid \zeta\left(w\right) = 0\right\} \ll k^{\delta(S)}.$$

To show this, first recall that by Proposition 2.2.9, there exist two real constants $c_4, \widetilde{c_4}$ such that $|\zeta\left(z\right)| > \widetilde{c_4} > 0$ on the half-plane $\{z \in \mathbb{C} : \operatorname{Re}\left(z\right) \geq c_4\}$. Consequently, the strip $Q_{k,k+1}^{-c,\infty}$ can be replaced by the rectangle $Q_{k,k+1}^{-c,c_4}$. Let us now consider the ball $B_{c_5}\left(\imath k + c_4\right)$ such that $Q_{k,k+1}^{-c,c_4} \subset B_{c_5}\left(\imath k + c_4\right)$. In order to be able to apply certain standard techniques from complex analysis, let us normalise the situation as follows. With $\mathcal{Z} : \mathbb{C} \to \mathbb{C}$ given by $\mathcal{Z}\left(w\right) := \zeta\left(w + \imath k + c_4\right)$, we have that

$$\operatorname{card}\{w \in Q_{k,k+1}^{-c,c_4} \mid \zeta\left(w\right) = 0\} \leq \operatorname{card}\{w \in B_{c_5}\left(0\right) \mid \mathcal{Z}\left(w\right) = 0\}.$$

Let $n_{\mathcal{Z}}\left(t\right)$ denote the number of zeros of $\mathcal{Z}$ inside the ball $B_t\left(0\right)$, for t positive. Applying Jensen's Formula (see for example [93, 3.62(2)]), we obtain

$$\int_0^t \frac{n_{\mathcal{Z}}\left(x\right)}{x} dx = \frac{1}{2\pi} \int_0^{2\pi} \log|\mathcal{Z}\left(t \cdot \mathrm{e}^{\imath\theta}\right)| d\theta - \log|\mathcal{Z}\left(0\right)|.$$

Note that $|\mathcal{Z}\left(0\right)| = |\zeta\left(0 + \imath k + c_4\right)| = |\zeta\left(\imath k + c_4\right)| > \widetilde{c_4} > 0$. Therefore, we have $-\log|\mathcal{Z}\left(0\right)| < -\log\left(\widetilde{c_4}\right)$, and hence $-\log|\mathcal{Z}\left(0\right)|$ is finite. However, if $-\log|\mathcal{Z}\left(0\right)| > 0$, then there is at least one zero. Hence, for t sufficiently large, we have that

$$\int_0^t \frac{n_{\mathcal{Z}}\left(x\right)}{x} dx \ll \frac{1}{2\pi} \int_0^{2\pi} \log|\mathcal{Z}\left(t \cdot \mathrm{e}^{\imath\theta}\right)| d\theta.$$

Moreover, one immediately verifies that

$$n_{\mathcal{Z}}(t) \leq \frac{1}{\log 2}\int_t^{2t} \frac{n_{\mathcal{Z}}(x)}{x} dx \leq \frac{1}{\log 2}\int_0^{2t} \frac{n_{\mathcal{Z}}(x)}{x} dx.$$

Combining these two observations, it follows that

$$n_{\mathcal{Z}}(t) \ll \frac{1}{2\pi}\int_0^{2\pi} \log\left|\mathcal{Z}\left(2t\cdot \mathrm{e}^{\imath\theta}\right)\right| d\theta \ll \max_{\theta\in[0,2\pi]} \log\left|\mathcal{Z}\left(2t\cdot \mathrm{e}^{\imath\theta}\right)\right|.$$

This implies for $c_6 \geq c_5$ sufficiently large, that

$$n_{\mathcal{Z}}(c_6) = \mathrm{card}\{w \in B_{c_6}(0) \mid \mathcal{Z}(w) = 0\} \ll \max_{\theta\in[0,2\pi]} \log\left|\mathcal{Z}\left(2c_6\cdot \mathrm{e}^{\imath\theta}\right)\right|.$$

Also, by the definition of $\mathcal{Z}$, we have that

$$\max_{\theta\in[0,2\pi]} \log\left|\mathcal{Z}\left(2c_6\cdot \mathrm{e}^{\imath\theta}\right)\right| \leq \max_{w\in B_{2c_6}(\imath k+c_4)} \log|\zeta(w)|.$$

Furthermore, note that if $w \in B_{2c_6}(\imath k + c_4)$, then $\mathrm{Re}(w) \geq -2c_6 + c_4$ and hence,

$$B_{2c_6}(\imath k + c_4) \subset Q^{c_4-2c_6,\, c_4+2c_6}_{k-2c_6,\, k+2c_6}.$$

This shows that

$$\max\{\log|\zeta(w)| : w \in B_{2c_6}(\imath k + c_4)\} \leq \max\{\log|\zeta(w)| : w \in Q^{k-2c_6,\, k+2c_6}_{c_4-2c_6,\, c_4+2c_6}\}.$$

Combining these observations with the first part of Main Theorem 1, it now follows that for k sufficiently large, we have

$$\begin{aligned}\mathrm{card}\{w \in Q^{-c_4,\infty}_{k,k+1} : \zeta(w) = 0\} &\ll \max_{w\in B_{2c_6}(\imath k+c_4)} \log|\zeta(w)| \\ &\leq \max_{\substack{\mathrm{Re}(w)>c_4-2c_6,\\ \mathrm{Im}(w)\in[k-2c_6,k+2c_6]}} \log|\zeta(w)| \\ &\ll k^{\delta(S)}.\end{aligned}$$

This completes the proof of the second part of Main Theorem 1.

Remark: Note that the setting we have used in this part of the thesis is a special case of the setting used by Ruelle in [79] (see also [80] and [81]). More precisely, in [79] Ruelle considered more general transfer operators on exterior forms and has used arbitrary holomorphic functions instead of the very special functions $(\|\phi_e'\|)^w_{\mathbb{C}}$ which we have used in this part of the thesis. However, Ruelle studied transfer operators on the Banach-space of holomorphic functions with continuous extensions to the boundary of the domain of definition equipped with the uniform norm. In contrast, in this part of the thesis we have followed the approach of Guillopé *et al.* to study $\mathcal{L}_w$ on the Hilbert space of holomorphic L^2-functions. Note that the techniques we have used here are generalisations of the techniques used in [42] to GDMSs.

3 Geometry of infinitely generated function schemes

3.1 Introduction to infinitely generated pseudo GDMSs

The concept limit set has been studied for various types of function schemes. This includes schemes arising from fractal geometry, geodesic flows on manifolds and iterations of endomorphisms in the complex plane, to name but a few. It is presumably due to this diversity that one can find different definitions of the concept limit set. Usually, these definitions are very much adapted to the individual situation. Our definition will be based mainly upon symbolic dynamics. Note that this definition will allow us to introduce a canonical decomposition of the limit set, and this decomposition will be essential in what follows.
Throughout, let S be a function scheme coded by a countable alphabet $E(S)$, which can be finite or infinite. We assume that there exists a map π from the set $E^\infty(S)$ of admissible infinite words of the form $\underline{i} = (i_1, i_2, \ldots)$ into some complete topological space. Also, without loss of generality, we can assume that $E(S)$ is a subset of $\mathbb{N}$.

Definition 3.1.1. *For S as above we define*

$$\begin{array}{rcl} L_{ur}(S) & := & \pi\left(\{\underline{i} \in E^\infty(S) | \limsup\{i_k \mid k \in \mathbb{N}\} < \infty\}\right); \\ L_r(S) & := & \pi\left(\{\underline{i} \in E^\infty(S) | \liminf\{i_k \mid k \in \mathbb{N}\} < \infty\}\right); \\ L_d(S) & := & \pi\left(\{\underline{i} \in E^\infty(S) | \liminf\{i_k \mid k \in \mathbb{N}\} = \infty\}\right); \\ L_{dyn}(S) & := & L_d(S) \cup L_r(S); \\ L(S) & := & \overline{L_{dyn}(S)}; \\ L_J(S) & := & \overline{L_{dyn}(S)} \setminus L_{dyn}(S). \end{array}$$

Here, $\overline{L_{dyn}(S)}$ denotes the closure of $L_{dyn}(S)$ with respect to the topology of the topological space in which $L_{dyn}(S)$ is embedded.
We call $L(S)$ the limit set*, $L_{ur}(S)$ the* uniformly radial limit set*, $L_r(S)$ the* radial limit set*, $L_d(S)$ the* dissipative limit set*, $L_J(S)$ the* Jørgensen limit set *and $L_{dyn}(S)$ the* dynamical limit set*.*

These definitions are consistent with the definitions given in [85] for the case of Kleinian groups. In the case of IFSs for which I is finite, one immediately recovers the usual definition of the limit set (see for example [29, Chapter 12]), where one has $\pi(E^\infty(S)) = L(S) = L_{ur}(S)$. Note that in [63] the limit set for a GDMS was defined to be equal to $L_{dyn}(S)$, whereas in [87] the limit set was defined to be equal to $L(S)$.

We are mainly interested in limit sets for two types of function schemes. Namely, we are interested in limit sets of Kleinian groups and in limit sets of GDMSs. The advantage of GDMSs is that they represent a generalisation of IFSs, and that they are well suited to represent actions of finitely generated Kleinian groups of Schottky type (see for example [63, Example 5.1.5]).
Let us recall the following definition of a GDMS. Note that this definition differs from the one given by Mauldin and Urbański in [63]. In our definition, the direction of the edges is not reversed, that is, an edge starts at the initial vertex and ends at the terminal vertex.

Definition 3.1.2. *A graph directed Markov system (GDMS) S is an octuple $S := (V(S), E(S), i, t, A(S), \{X_v\}_{v\in V(S)}, \ell, \{\phi_e\}_{e\in E(S)})$ consisting of a non-empty finite set $V(S)$ of vertices, a non-empty countable set $E(S)$ of directed edges, two maps $i, t : E(S) \to V(S)$, and a $\operatorname{card} E \times \operatorname{card} E$-matrix $A(S)$ with entries in $\{0,1\}$, a collection of non-empty compact connected metric spaces $\{X_v\}_{v\in V} \subset \mathbb{R}^m$ of positive diameter, and injective contractions $\phi_e : X_{i(e)} \to X_{t(e)}$ with Lipschitz constants less than some given $\ell \in (0,1)$. Here, for each edge $e \in E(S)$ we have that $i(e)$ is the initial vertex of e and $t(e)$ is the terminal vertex of e.*
Moreover, if $E(S)$ is finite, then S is called finitely generated.

Additionally, we require that a GDMS satisfies the following *bounded distortion condition* (BDC).

(BDC) *There exist two constants $\underline{c}, \overline{c} > 0$ such that for all $e \in E(S)$ we have that there exists $x \in \phi_e(X_{i(e)})$ such that*

$$B(x, \underline{c} \cdot \operatorname{diam} \phi_e(X_{i(e)})) \subset \phi_e(X_{i(e)}) \subset B(x, \overline{c} \cdot \operatorname{diam} \phi_e(X_{i(e)})).$$

Here, $B(x,r)$ denotes the open m-ball of radius r centered at x. Furthermore, there are constants $\underline{c}, \overline{c} > 0$ (which might be different from the ones above), such that for all finite words $\underline{e} \in E^*(S)$ and all convex subsets $C \subset X_{i(\underline{e})}$ with $\operatorname{diam} C \neq 0$ we have that

$$\underline{c} \cdot \frac{\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})}{\operatorname{diam} X_{i(\underline{e})}} \leq \frac{\operatorname{diam} \phi_{\underline{e}}(C)}{\operatorname{diam} C} \leq \overline{c} \cdot \frac{\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})}{\operatorname{diam} X_{i(\underline{e})}}.$$

We remark that the condition BDC should not be confused with the bounded distortion property, which was considered in the first part of this thesis.
Also, we restrict the discussion of GDMSs to systems which satisfy the following *strong separation condition* (SSC).

(SSC) *For all $e, f \in E(S)$ with $e \neq f$ we have*

$$\phi_e(X_{i(e)}) \cap \phi_f(X_{i(f)}) = \emptyset.$$

It will turn out later (see Lemma 3.4.5), that GDMSs cannot be used to describe the dynamics of infinitely generated Kleinian groups of Schottky type. In order to describe this type of function scheme, we use the following notion from [87].

Definition 3.1.3. *A* pseudo GDMS *is a system which satisfies all the properties of a GDMS, except that the set of vertices is allowed to be a countable infinite set.*

As for GDMSs, we similarly restrict the discussion of pseudo GDMSs to those which satisfy SSC and BDP.
As already mentioned above, GDMSs are a generalisation of IFSs, which we introduce now.

Definition 3.1.4. *A GDMS* $(V(S), E(S), i, t, A(S), \{X_v\}_{v\in V(S)}, \ell, \{\phi_e\}_{e\in E(S)})$ *is called an* iterated function system (IFS) *if* $V(S)$ *consists of exactly one element and if the incidence matrix* $A(S)$ *consists only of ones, that is,* $A_{e,f} = 1$ *for all* $e, f \in E(S)$.

Once again, we restrict the discussion of IFSs to those which satisfy SSC and BDP.
Our main aim will be to investigate the Hausdorff dimension of the various types of limit set given in Definition 3.1.1 for the different types of function schemes defined above.
Let us recall some notation. For a pseudo GDMS $S = (V(S), E(S), i, t, A(S), \{X_v\}_{v\in V(S)}, \ell, \{\phi_e\}_{e\in E(S)})$, we define the set of admissible words of length $n \in \mathbb{N}$ by

$$E^n(S) := \Big\{(e_1, \ldots, e_n) \mid e_i \in E(S) \text{ and } A_{e_i,e_{i+1}} = 1, \text{ for all } i \geq 1\Big\}.$$

Similarly, we write $E^\infty(S)$ for the set of infinite admissible words. Moreover, the set $E^*(S)$ of all finite words is defined by $E^*(S) := \bigcup_{n\in\mathbb{N}} E^n(S)$.
Remark: Note that for $\underline{e} = (e_1, \ldots, e_n) \in E^*(S)$ we have

$$\phi_{\underline{e}} = \phi_{e_n} \circ \ldots \circ \phi_{e_1} : X_{i(e_1)} \to X_{t(e_n)}.$$

Also note that, due to the reversal of the edges, this notation differs from the notation in [63]. Here, $(f \circ g)(x) = f(g(x))$ represents the usual notation for composing maps. For $\underline{e} = (e_1, \ldots, e_n) \in E^*(S)$ and $\underline{f} = (f_1, \ldots, f_k) \in E^*(S)$ we use the notation

$$\underline{f}\underline{e} := (\underline{e}, \underline{f}) = (e_1, \ldots, e_n, f_1, \ldots, f_m).$$

For $\underline{e} \in E^*(S)$, we write $|\underline{e}|$ for the word length of $\underline{e}$, that is, $|\underline{e}| \in \mathbb{N}$ such that $\underline{e} \in E^{|\underline{e}|}(S)$. Furthermore, for $\underline{e} \in E^\infty(S)$ we define $\underline{e}_{|n} := (e_1, \ldots, e_n)$.

Note that throughout we always assume that our GDMSs are *finitely primitive* in the sense of Mauldin and Urbański [63]. That is, there exists a finite set of admissible words $W \subset E^*(S)$ of the same word length such that the following holds.

For each $\underline{e}, \underline{\hat{e}} \in E^*(S)$ there exists $\underline{w} \in W$ such that $\underline{e}\,\underline{w}\,\underline{\hat{e}} \in E^*(S)$.

We now give a geometric description of the limit set of a pseudo GDMS.

Lemma 3.1.5. *For a pseudo GDMS S we have*

$$L_{dyn}(S) = \bigcap_{n\in\mathbb{N}} \bigcup_{(e_1,\dots,e_n)\in E^n(S)} \phi_{e_n}\circ\ldots\circ\phi_{e_1}(X_{i(e_1)}).$$

Proof. Note that in Definition 3.1.1 we defined the limit set in terms of the natural coding by infinite words in the alphabet I. For a pseudo GDMS $(V(S), E(S), i, t, A(S), \{X_v\}_{v\in V(S)}, \ell, \{\phi_e\}_{e\in E(S)})$, the set of infinite words $E^\infty(S)$ admits a representation in the following way.
First note that for each $n\in\mathbb{N}$ we have

$$\bigcap_{i=1}^{n} \bigcup_{(e_1,\dots,e_i)\in E^i(S)} \phi_{e_i}\circ\ldots\circ\phi_{e_1}(X_{i(e_1)}) = \bigcup_{(e_1,\dots,e_n)\in E^n(S)} \phi_{e_n}\circ\ldots\circ\phi_{e_1}(X_{i(e_1)}).$$

Next, since we assume SSC, for each $n\in\mathbb{N}$ there is a bijection

$$\begin{array}{rcccl} \pi_n & : & E^n(S) & \rightarrow & \{\phi_{e_n}\circ\ldots\circ\phi_{e_1}(X_{i(e_1)}) | (e_1,\dots,e_n)\in E^n(S)\} \\ & & (e_1,\dots,e_n) & \mapsto & \phi_{e_n}\circ\ldots\circ\phi_{e_1}(X_{i(e_1)}). \end{array}$$

This gives rise to a map π from $E^\infty(S)$ to $L_{dyn}(S)$. □

For the investigations of the Hausdorff dimensions of the various types of limit sets of Definition 3.1.1 for pseudo GDMSs, we employ two series, namely the *distortion series* and the *side series*. The first series is well known in the theory of GDMSs, where it is sometimes referred to as the Poincaré series (see for instance [57]). As will become clear later, this name is in fact misleading. In the case of finitely generated GDMSs, the distortion series is closely related to the pressure function (see Lemma 3.2.1), and in the case of a linear IFS it gives rise to the well known Hutchinson formula (see Lemma 3.3.2). In contrast to this, the side series seems not to have been considered in the literature so far. In order to define these series for a pseudo GDMS $S := (V(S), E(S), i, t, A(S), \{X_v\}_{v\in V(S)}, \ell, \{\phi_e\}_{e\in E(S)})$ we require the following notation.

$$\|\phi'_{\underline{e}}\| := \frac{\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})}{\operatorname{diam} X_{i(\underline{e})}}, \text{ for each } \underline{e}\in E^*(S).$$

Definition 3.1.6. *The* distortion series *of a pseudo GDMS S is defined by*

$$\sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s.$$

The exponent of convergence of this series will be denoted by $\Lambda(S)$, and we refer to it as the distortion exponent *of S.*

Definition 3.1.7. *The* side series *of a pseudo GDMS S is defined by*

$$\sum_{\underline{e}\in E^*(S)} \left(\operatorname{diam}\left(\phi_{\underline{e}}\left(X_{i(\underline{e})}\right)\right)\right)^s.$$

The exponent of convergence of this series will be denoted by $\Delta(S)$, and we refer to it as the side exponent *of S.*

In analogy with the notations in the theory of Kleinian groups, we call $\delta(S) := \dim_{\mathrm{H}} L_{ur}(S)$ the *Poincaré exponent* of S.
Next, we give the definition of the pressure function.

Definition 3.1.8. *For a GDMS S, the* pressure function $P:\mathbb{R}\to\mathbb{R}\cup\{\infty\}$ *is defined by*

$$P(s) := \lim_{n\to\infty}\frac{1}{n}\log\left(\sum_{\underline{e}\in E^n(S)}\|\phi'_{\underline{e}}\|^s\right), \text{ for each } s\in\mathbb{R}.$$

We now show that this limit exists. First recall that, by the bounded distortion condition, there exists $c \geq 1$ such that $\frac{\operatorname{diam}\phi_{\underline{e}}(Y)}{\operatorname{diam} Y} \leq c\cdot\frac{\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})}{\operatorname{diam} X_{i(\underline{e})}}$, for all $Y\subset X(i(\underline{e}))$ with $\operatorname{diam} Y\neq 0$. Let us first show that the map given by

$$n\mapsto b_n := \log\sum_{\underline{e}\in E^n}\left(c\cdot\frac{\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})}{\operatorname{diam} X_{i(\underline{e})}}\right)^s$$

is subadditive. From this subadditivity we then deduce the existence of the limit $\lim_{n\to\infty} b_n/n$, and this will then guarantee the existence of the limit in the definition of the pressure function.

$$\begin{aligned}
\sum_{\underline{e}\in E^{n+m}(S)}(c\cdot\|\phi'_{\underline{e}}\|)^s &\leq \sum_{\underline{e}\in E^n(S)}\sum_{\underline{f}\in E^m(S)}\left(c\cdot\frac{\operatorname{diam}\phi_{\underline{e}}(\phi_{\underline{f}}(X_{i(\underline{f})}))}{\operatorname{diam} X_{i(\underline{f})}}\right)^s\\
&\leq \sum_{\underline{e}\in E^n(S)}\sum_{\underline{f}\in E^m(S)}\left(c\cdot\frac{\operatorname{diam}\phi_{\underline{e}}(\phi_{\underline{f}}(X_{i(\underline{f})}))}{\operatorname{diam} X_{i(\underline{f})}}\frac{\operatorname{diam}\phi_{\underline{f}}(X_{i(\underline{f})})}{\operatorname{diam}\phi_{\underline{f}}(X_{i(\underline{f})})}\right)^s\\
&= \sum_{\underline{e}\in E^n(S)}\sum_{\underline{f}\in E^m(S)}\left(\frac{\operatorname{diam}\phi_{\underline{e}}(\phi_{\underline{f}}(X_{i(\underline{f})}))}{\operatorname{diam}\phi_{\underline{f}}(X_{i(\underline{f})})}\right)^s(c\cdot\|\phi'_{\underline{f}}\|)^s
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{\underline{e}\in E^n(S)} \sum_{\underline{f}\in E^m(S)} \left(c \cdot \frac{\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})}{\operatorname{diam} X_{i(\underline{e})}} \right)^s (c \cdot \|\phi'_{\underline{f}}\|)^s \\
&\leq \sum_{\underline{e}\in E^n(S)} \sum_{\underline{f}\in E^m(S)} (c \cdot \|\phi'_{\underline{e}}\|)^s (c \cdot \|\phi'_{\underline{f}}\|)^s \\
&= \sum_{\underline{e}\in E^n(S)} (c \cdot \|\phi'_{\underline{e}}\|)^s \sum_{\underline{f}\in E^m(S)} (c \cdot \|\phi'_{\underline{f}}\|)^s .
\end{aligned}
$$

Hence, the map given by $n \mapsto b_n$ is subadditive. This implies that the limit $\lim_{n\to\infty} b_n/n$ exists (see [63, Lemma 2.1.1 and 2.1.2]), and hence the existence of the limit in the definition of the pressure function follows. Indeed, the latter follows since

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \sum_{\underline{e}\in E^n} \left(c \cdot \|\phi'_{\underline{e}}\|^s \right) &= \lim_{n\to\infty} \frac{1}{n} \left(s \cdot \log(c) + \log \sum_{\underline{e}\in E^n} \|\phi'_{\underline{e}}\|^s \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \log \sum_{\underline{e}\in E^n} \|\phi'_{\underline{e}}\|^s .
\end{aligned}
$$

3.2 Preliminaries and main results

3.2.1 Basic properties of pseudo GDMSs

In this section we establish some relationships between the concepts introduced in the previous section. We first investigate the relationship between the distortion series and the pressure function.

Lemma 3.2.1. *Let S be a finitely generated GDMS. We then have*

$$P(\Lambda(S)) = 0.$$

Proof. In order to relate the pressure function to the distortion series, let us define

$$a_n(s) := \exp\left(\frac{1}{n} \cdot \log \left(\sum_{\underline{e}\in E^n(S)} \|\phi'_{\underline{e}}\|^s \right) \right) \tag{11}$$

note that

$$
\begin{aligned}
\sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s &= \sum_{n\in\mathbb{N}} \sum_{\underline{e}\in E^n(S)} \|\phi'_{\underline{e}}\|^s \\
&= \sum_{n\in\mathbb{N}} \exp\left(\frac{n}{n} \cdot \log \left(\sum_{\underline{e}\in E^n(S)} \|\phi'_{\underline{e}}\|^s \right) \right) = \sum_{n\in\mathbb{N}} \left(a_n(s) \right)^n .
\end{aligned}
$$

Also, if $E(S)$ is finite, then $0 < a_n(s) < \infty$, for each $n \in \mathbb{N}$ and all $s \geq 0$. Since the exponential function and logarithmic function are continuous

and since the limit in the definition of the pressure function exists (possibly being $-\infty$), we have $\lim_{n\to\infty}\log(a_n(s)) = P(s)$. On the one hand, if $\lim_{n\to\infty}\log(a_n(s)) > 0$ then the $a_n(s)$ in (11) are greater than some constant $r > 1$, for all sufficiently large n. Therefore, the series in (11) diverges.
On the other hand, if $\lim_{n\to\infty}\log(a_n(s)) < 0$, all but finitely many of the $a_n(s)$ in (11) are less than some constant $r < 1$. Hence, in this situation, the sum in (11) is dominated by a convergent geometric series, and is therefore convergent. Combining these observations with the definition of $\Lambda(S)$, it follows that if $E(S)$ is finite, then $P(\Lambda(S)) = 0$. □

Remark: Note that if S is infinitely generated then the distortion series $\sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s$ does not necessarily converge for s equal to the zero of the pressure function (see for example [87], where this case has been studied).

Lemma 3.2.2. *For a pseudo GDMS S we have*

$$\Delta(S) \geq \dim_{\mathrm{H}} L_{dyn}(S).$$

Proof. Let $s < \dim_{\mathrm{H}} L_{dyn}(S)$ be given. Then, by definition of the Hausdorff dimension, we have that $\lim_{\epsilon\to 0}\inf_{\{U_i\}}\sum_{i\in\mathbb{N}}(\operatorname{diam} U_i)^s = \infty$, where U_i is an ϵ-cover. Furthermore, for all $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $\max_{\underline{e}\in E^n(S)} \operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})}) < \epsilon$. Hence $\bigcup_{\underline{e}\in E^n(S)} \phi_{\underline{e}}(X_{i(\underline{e})})$ is a covering of $L_{dyn}(S)$ with sets of diameter less than $\epsilon > 0$. Thus, we have

$$\begin{aligned}\inf_{\{U_i\}}\sum_{i\in\mathbb{N}}(\operatorname{diam} U_i)^s &\leq \sum_{\underline{e}\in E^n(S)}(\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})}))^s\\ &\leq \sum_{\underline{e}\in E^*(S)}(\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})}))^s.\end{aligned}$$

By letting ϵ tend to zero, this implies that $s \leq \Delta(S)$, and hence we obtain that $\dim_{\mathrm{H}} L_{dyn}(S) \leq \Delta(S)$. □

The following two lemmas describe the relationship between the side series and the distortion series.

Lemma 3.2.3. *For a pseudo GDMS S and $0 \leq s \leq m+1$ we have*

$$\sum_{\underline{e}\in E^*(S)}(\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})}))^s \ll \sum_{\underline{e}\in E^*(S)}\|\phi'_{\underline{e}}\|^s.$$

Proof. Firstly, recall that $\|\phi'_{\underline{e}}\| := \operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})/\operatorname{diam} X_{i(\underline{e})}$. Hence, $\|\phi'_{\underline{e}}\| \cdot \operatorname{diam} X_{i(\underline{e})} = \operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})$, and therefore,

$$\sum_{\underline{e}\in E^*(S)}\Big(\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})\Big)^s = \sum_{\underline{e}\in E^*(S)}\Big(\|\phi'_{\underline{e}}\|\cdot\operatorname{diam} X_{i(\underline{e})}\Big)^s$$

$$
\begin{aligned}
&\leq \max_{e\in E(S)} \Big(\operatorname{diam} X_{i(e)}\Big)^s \cdot \sum_{\underline{e}\in E^*(S)} \|\phi_{\underline{e}}\|^s \\
&\leq \max_{e\in E(S)} \Big(\operatorname{diam} X_{i(e)}\Big)^{m+1} \sum_{\underline{e}\in E^*(S)} \|\phi_{\underline{e}}\|^s.
\end{aligned}
$$

□

Lemma 3.2.4. *For a GDMS S and $0 \leq s \leq m+1$ we have*

$$\sum_{\underline{e}\in E^*(S)} (\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})}))^s \gg \sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s.$$

Proof. In a similar way to the proof of the previous lemma, we substitute $\|\phi'_{\underline{e}}\| \cdot \operatorname{diam} X_{i(\underline{e})} = \operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})$ into the side series as follows:

$$
\begin{aligned}
\sum_{\underline{e}\in E^*(S)} \Big(\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})\Big)^s &= \sum_{\underline{e}\in E^*(S)} \Big(\|\phi'_{\underline{e}}\| \cdot \operatorname{diam} X_{i(\underline{e})}\Big)^s \\
&\geq \inf_{e\in E(S)} (\operatorname{diam} X_{i(e)})^s \cdot \sum_{\underline{e}\in E^*(S)} \|\phi_{\underline{e}}\|^s \\
&\geq \min_{v\in V(S)} (\operatorname{diam} X_v)^s \cdot \sum_{\underline{e}\in E^*(S)} \|\phi_{\underline{e}}\|^s \\
&\geq \min\{1, \min_{v\in V(S)} (\operatorname{diam} X_v)^{m+1}\} \sum_{\underline{e}\in E^*(S)} \|\phi_{\underline{e}}\|^s.
\end{aligned}
$$

Here, we used the following property of a GDMS. Namely, we have

$$\inf_{e\in E(S)} (\operatorname{diam} X_{i(e)})^s \geq \inf_{v\in V(S)} (\operatorname{diam} X_{i(\underline{e})})^s = \min_{v\in V(S)} (\operatorname{diam} X_v)^s > 0,$$

which holds, since $V(S)$ is finite. □

Remark: Note that for the larger class of pseudo GDMSs with an infinite set of vertices we have that $\bigcup_{v\in V(S)} X_v$ is contained in a compact subset of $\mathbb{R}^m$. This implies that $\inf_{e\in E(S)} \operatorname{diam} X_{i(e)} = 0$, and this is precisely the reason why Lemma 3.2.4 does not hold for pseudo GDMSs in general.

Corollary 3.2.5. *For a GDMS S the side series and the distortion series are comparable, that is, for $0 \leq s \leq m+1$ we have*

$$\sum_{\underline{e}\in E^*(S)} (\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})}))^s \asymp \sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s.$$

In particular, we have $\Lambda(S) = \Delta(S)$.

Lemma 3.2.6. *For a finitely generated GDMS S, we have*

$$\Lambda(S) = \dim_{\mathrm{H}} L(S).$$

Proof. Let us first prove a kind of chain rule which we shall use frequently. Let $\underline{e}$ and $\underline{f}$ in $E^*(S)$ be such that the composition $\underline{e}\underline{f}$ also belongs to $E^*(S)$ and let $L_{\underline{e}}, L_{\underline{f}}$ denote the Lipschitz constants of $\phi_{\underline{e}}$ and $\phi_{\underline{f}}$ respectively. Note that for $x, y \in X_{i(\underline{f})}$ we have

$$\mathrm{dist}(\phi_{\underline{e}}(\phi_{\underline{f}}(x)), \phi_{\underline{e}}(\phi_{\underline{f}}(y))) \leq L_{\underline{e}} \cdot \mathrm{dist}(\phi_{\underline{f}}(x), \phi_{\underline{f}}(y)) \leq L_{\underline{e}} \cdot L_{\underline{f}} \cdot \mathrm{dist}(x, y).$$

Further, note that by BDC, there exists a positive constant c such that

$$\mathrm{dist}(\phi_{\underline{e}}(\phi_{\underline{f}}(x)), \phi_{\underline{e}}(\phi_{\underline{f}}(y))) \geq c \cdot L_{\underline{e}} \cdot \mathrm{dist}(\phi_{\underline{f}}(x), \phi_{\underline{f}}(y)) \geq c^2 \cdot L_{\underline{e}} \cdot L_{\underline{f}} \cdot \mathrm{dist}(x, y).$$

Therefore, we have $c^2 \cdot L_{\underline{e}} \cdot L_{\underline{f}} \leq \|(\phi_{\underline{e}} \circ \phi_{\underline{f}})'\| \leq L_{\underline{e}} \cdot L_{\underline{f}}$. Note that BDC also implies that $L_{\underline{e}} \asymp \|\phi'_{\underline{e}}\|$. Combining these observations, we have that $\|(\phi_{\underline{e}} \circ \phi_{\underline{f}})'\| \asymp \|\phi'_{\underline{e}}\| \cdot \|\phi'_{\underline{f}}\|$ or, more precisely, we have that there exist positive constants $\underline{c}$ and $\overline{c}$, with $\underline{c} \leq 1$ only depending on the system S, such that

$$\underline{c} \cdot \|\phi'_{\underline{e}}\| \cdot \|\phi'_{\underline{f}}\| \leq \|(\phi_{\underline{e}} \circ \phi_{\underline{f}})'\| \leq \overline{c} \cdot \|\phi'_{\underline{e}}\| \cdot \|\phi'_{\underline{f}}\|. \tag{12}$$

We are now going to define a probability measure supported on the limit set $L(S)$. This will be done analogously to the construction of the Patterson measure in the case of Fuchsian groups (see [68], see also [66], [88], [89]). For this, let $s > \Lambda(S)$ be given. For a Borel set $A \in \bigcup_{v \in V(S)} X_v$, let $\mu_s(A)$ be defined by

$$\mu_s(A) := \frac{\sum_{\{\underline{e} \in E^*(S) \mid \phi_{\underline{e}}(X_{i(\underline{e})}) \subset A\}} \|\phi'_{\underline{e}}\|^s}{\sum_{\underline{e} \in E^*(S)} \|\phi'_{\underline{e}}\|^s}.$$

Let μ denote a weak limit ($\lim^*$) of the family of measures $\{\mu_s\}$ as s tends to $\Lambda(S)$, that is $\mu = \lim^*_{s \to \Lambda(S)} \mu_s$ (a standard reference for weak convergence of measures is [12]).

Note that by (12), we have for each $\underline{e} \in E^*(S)$ and $0 \leq s \leq m+1$ that

$$\begin{aligned}
\sum_{\substack{\underline{f} \in E^*(S) \\ \phi_{\underline{f}}(X_{i(\underline{f})}) \subset \phi_{\underline{e}}(X_{i(\underline{e})})}} \|\phi'_{\underline{f}}\|^s &= \sum_{\underline{f} \in E^*(S)} \|(\phi_{\underline{f}} \circ \phi_{\underline{e}})'\|^s \\
&\asymp \sum_{\underline{f} \in E^*(S)} \|\phi'_{\underline{e}}\|^s \cdot \|\phi'_{\underline{f}}\|^s \\
&= \|\phi'_{\underline{e}}\|^s \sum_{\underline{f} \in E^*(S)} \|\phi'_{\underline{f}}\|^s.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_s(\phi_{\underline{e}}(X_{i(\underline{e})})) &= \frac{\sum_{\{\underline{f} \in E^*(S) \mid \phi_{\underline{f}}(X_{i(\underline{f})}) \subset \phi_{\underline{e}}(X_{i(\underline{e})})\}} \|\phi'_{\underline{f}}\|^s}{\sum_{\underline{f} \in E^*(S)} \|\phi'_{\underline{f}}\|^s} \\
&\asymp \|\phi'_{\underline{e}}\|^s \frac{\sum_{\underline{f} \in E^*(S)} \cdot \|\phi'_{\underline{f}}\|^s}{\sum_{\underline{f} \in E^*(S)} \cdot \|\phi'_{\underline{f}}\|^s} \\
&= \|\phi'_{\underline{e}}\|^s.
\end{aligned}$$

Let us now choose an open set A such that $\phi_{\underline{e}}(X_{i(\underline{e})}) \subset A$ and such that the closure $\overline{A}$ of A does not intersect any other $\phi_{\underline{f}}(X_{i(\underline{f})})$ for any $\underline{f} \in E^{|\underline{e}|}(S) \setminus \{\underline{e}\}$. This is possible since S is finite and satisfies SSC. For such an A we have that $\partial A \cap L(S) = \emptyset$ and hence A is a so-called "μ-continuity" set. Hence, we have $\lim^*_{s\to\delta} \mu_s(A) = \mu(A)$ (cf. [13][Theorem 2.1]). Thus, we can conclude that $\mu(\phi_{\underline{e}}(X_{i(\underline{e})})) \asymp \|\phi'_{\underline{e}}\|^{\delta}$.
In order to proceed, let us fix some notation. Let $\phi_{\min} := \min_{e\in E(S)} \|\phi'_e\|$ and recall that $\phi_{\max} := \max_{e\in E(S)} \|\phi'_e\|$. These are well defined, since S is finitely generated. Note that $0 < \phi_{\min} \le \phi_{\max} < \ell < 1$, with ℓ as given in the definition of S. Moreover, for each $k \in \mathbb{N}$ we define

$$A_k := \{\underline{e} \in E^*(S) \mid c^k(\phi_{\min})^k \ge \|\phi'_{\underline{e}}\| \ge c^{k+1}(\phi_{\min})^{k+1}\},$$

where $0 < c \le 1$ is equal to the constant $\underline{c}$ appearing in (12). In order to complete the proof of Lemma 3.2.6, we require the following three lemmas.

Lemma 3.2.7. *For each $k \in \mathbb{N}$ sufficiently large, we have that*

$$L(S) \subset \bigcup_{\underline{e}\in A_k} \phi_{\underline{e}}(X_{i(\underline{e})}).$$

Proof. Let $x \in L(S)$ be given. Since S satisfies SSC, we then have that for each $n \in \mathbb{N}$ there exists a unique $\underline{e}(n) \in E^n(S)$ such that $x \in \phi_{\underline{e}(n)}(X_{i(\underline{e}(n))})$. Note that since $\phi_{e(n+1)}(X_{i(e(n+1))}) \subset \phi_{e(n)}(X_{i(e(n))})$, we have that $e(n+1) = e(n)e_{n+1}$, for some $e_{n+1} \in E(S)$. By combining (12) with the fact that $\phi_{\min} \le \|\phi'_{e_{n+1}}\| \le \phi_{\max}$, we obtain with $\overline{c} > 0$ as in (12) that

$$c \cdot \phi_{\min} \cdot \|\phi'_{e(n)}\| \le \|\phi'_{e(n+1)}\| \le \overline{c} \cdot \phi_{\max} \cdot \|\phi'_{e_n}\|.$$

Recall that $c \cdot \phi_{\min} < 1$. Now let $k \in \mathbb{N}$ be chosen sufficiently large such that the set $\{n \in \mathbb{N} \mid \|\phi'_{e(n)}\| > c^k(\phi_{\min})^k\}$ is not empty. Then $n_k := \max\{n \in \mathbb{N} \mid \|\phi'_{e(n)}\| > c^k(\phi_{\min})^k\}$ is well defined, since $\lim_{n\to\infty} \|\phi'_{e(n)}\| = 0$. We then have $\|\phi'_{e(n_k+1)}\| \ge c \cdot \phi_{\min} \cdot \|\phi'_{e(n_k)}\| \ge c \cdot \phi_{\min} \cdot (c \cdot \phi_{\min})^k = c^{k+1}(\phi_{\min})^{k+1}$. Combining these observations, it follows that $e(n_k+1) \in A_k$, which then implies $L(S) \subset \bigcup_{\underline{e}\in A_k} \phi_{\underline{e}}(X_{i(\underline{e})})$. □

Lemma 3.2.8. *There exists a constant $c_S > 0$ such that for all $n \in \mathbb{N}$, $x \in L(S)$ and all $r > 0$ such that $(c\cdot\phi_{\min})^{n+1} \le r \le (c\cdot\phi_{\min})^n$, we have that*

$$1 \le \operatorname{card}\{\underline{f} \in A_n \mid \phi_{\underline{f}}(X_{i(\underline{f})}) \cap B(x,r) \ne \emptyset\} < c_S.$$

Proof. Let n, x and r be given as in the statement of the lemma. Let $A_n(x,r)$ be some subset of A_n such that $\phi_{\underline{e}}(X_{i(\underline{e})}) \cap B(x,r) \ne \emptyset$, for all $\underline{e} \in A_n(x,r)$, and if $\underline{e}, \underline{f} \in A_n(x,r)$ $(\underline{e} \ne \underline{f})$ then $\phi_{\underline{e}}(X_{i(\underline{e})}) \cap \phi_{\underline{f}}(X_{i(\underline{f})}) = \emptyset$.

The aim is to give an upper bound of the cardinality of $A_n(x,r)$. For this, note that $\phi_{\underline{e}}(X_{i(\underline{e})})$ contains an m-dimensional disc of radius comparable to $\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})})$. Furthermore, note that $\operatorname{diam}\phi_{\underline{e}}(X_{i(\underline{e})}) \asymp r$. Clearly, by comparing the volume of $B(x,r)$ and the lower bound for the volume of $\bigcup_{\underline{e}\in A_n(x,r)}\phi_{\underline{e}}(X_{i(\underline{e})})$ one immediately derives an upper bound for the cardinality of $A_n(x,r)$, which only depends on S. From this the lemma follows. □

Lemma 3.2.9. *Let $x \in L(S)$, and $r > 0$ sufficiently small. We then have*

$$\mu(B(x,r)) \ll r^{\Lambda(S)}.$$

Proof. Let x and r be given as in the statement of the lemma. Then there exists $k \in \mathbb{N}$ such that $(c\cdot\phi_{\min})^{k+1} \le r \le (c\cdot\phi_{\min})^{k}$. By Lemma 3.2.7, we have that $\bigcup_{\underline{e}\in A_k}\phi_{\underline{e}}(X_{i(\underline{e})})$ is a covering of $L(S)$. Let $\tilde{A}_k(x,r)$ be the subset of A_k of all elements in A_k such that for $\underline{e}\in A_k(x,r)$ we have that $\phi_{\underline{e}}(X_{i(\underline{e})}) \cap B(x,r) \neq \emptyset$ and $\phi_{\underline{e}}(X_{i(\underline{e})}) \not\subset \phi_{\underline{f}}(X_{i(\underline{f})})$ for all $\underline{f}\in A_k$ ($\underline{f}\neq\underline{e}$). By SSC we have that $\bigcup_{\underline{e}\in\tilde{A}_k(x,r)}\phi_{\underline{e}}(X_{i(\underline{e})})$ is a covering of $B(x,r)\cap L(S)$ of multiplicity 1. It now follows that

$$\begin{aligned}
\mu(B(x,r)) &\le \sum_{\underline{e}\in\tilde{A}_k(x,r)} \mu(\phi_{\underline{e}}(X_{i(\underline{e})})) \\
&\ll \sum_{\underline{e}\in\tilde{A}_k(x,r)} \|\phi'_{\underline{e}}\|^{\Lambda(S)} \\
&\le \left(c^k(\phi_{\min})^k\right)^{\Lambda(S)} \cdot \operatorname{card}\tilde{A}_k(x,r) \\
&\ll \left(c^{k+1}(\phi_{\min})^{k+1}\right)^{\Lambda(S)} \\
&\le r^{\Lambda(S)}.
\end{aligned}$$

Here we have used Lemma 3.2.8, which guarantees that the cardinality of $\tilde{A}_k(x,r)$ is bounded above by a universal constant. □

We are now in a position to complete the proof of Lemma 3.2.6. Combining Lemma 3.2.9 and Frostman's Lemma (see [62][Theorem 8.8]) we have that the $\Lambda(S)$-Hausdorff measure of $L(S)$ is positive. Hence, by the definition of the Hausdorff dimension, we have $\dim_{\mathrm{H}}(L(S)) \ge \Lambda(S)$. The lemma then follows by combining this observation with Lemma 3.2.2, Corollary 3.2.5 and the fact that S is finitely generated and therefore $L_{dyn}(S) = L(S)$. This completes the proof of Lemma 3.2.6. □

Lemma 3.2.10. *For a pseudo GDMS S we have*

$$\delta(S) = \lim_{n\to\infty}\Lambda(S_n).$$

Here, S_n refers to the finite subsystem of S whose set of edges is equal to $\{1,\dots,n\}$.

Proof. Clearly, we have $L_{ur}(S) = \bigcup_{n\in\mathbb{N}} L_{ur}(S_n)$ with $L_{ur}(S_n) \subset L_{ur}(S_{n+1})$. For the finitely generated systems S_n we have shown in Lemma 3.2.6 that $\dim_{\mathrm{H}} L(S_n) = \Lambda(S_n)$. Since the Hausdorff dimension is countably stable ([28, Section 2.2]), we can now complete the argument as follows.

$$\begin{aligned} \delta(S) &= \dim_{\mathrm{H}} L_{ur}(S) \;=\; \dim_{\mathrm{H}} \bigcup_{n\in\mathbb{N}} L_{ur}(S_n) \\ &= \sup_{n\in\mathbb{N}} \dim_{\mathrm{H}} L_{ur}(S_n) \;=\; \sup_{n\in\mathbb{N}} \Lambda(S_n) \;=\; \lim_{n\to\infty} \Lambda(S_n). \end{aligned}$$

□

We finish this section by investigating some relationships between the distortion exponent, the side exponent and the Poincaré exponent introduced in the previous section.

Proposition 3.2.11.

1. *Let S be a pseudo GDMS, then*

$$\delta(S) \le \Delta(S) \le \Lambda(S).$$

2. *Let S be a GDMS, then*

$$\delta(S) \le \Delta(S) = \Lambda(S).$$

3. *Let S be a finitely generated GDMS, then*

$$\delta(S) = \Delta(S) = \Lambda(S) = \dim_{\mathrm{H}} L(S).$$

Proof. To prove the assertion in *1*, note that, by Lemma 3.2.10, we have $\delta(S) = \dim_{\mathrm{H}} L_{ur}(S)$. Furthermore, since $L_{ur}(S) \subset L_{dyn}(S)$, we have $\dim_{\mathrm{H}} L_{ur}(S) \le \dim_{\mathrm{H}} L_{dyn}(S)$. Moreover, by Lemma 3.2.2, we have $\dim_{\mathrm{H}} L_{dyn}(S) \le \Delta(S)$. Combining these observations, it follows that

$$\delta(S) = \dim_{\mathrm{H}} L_{ur}(S) \le \dim_{\mathrm{H}} L_{dyn}(S) \le \Delta(S).$$

This proves the first inequality in *1*. For the second inequality, let $s < \Delta(S)$ be given. We then have that the side series $\sum_{\underline{e}\in E^*(S)} \left(\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})\right)^s$ diverges. Applying Lemma 3.2.3, we obtain $\sum_{\underline{e}\in E^*(S)} \left(\operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})\right)^s \ll \sum_{\underline{e}\in E^*(S)} \|\phi_{\underline{e}}\|^s$. Therefore, the series on the right hand side of the previous inequality diverges. This shows that $s \le \Lambda(S)$. Since $s < \Delta(S)$ was arbitrary, it follows that $\Delta(S) \le \Lambda(S)$.
For the proof of *2*, note that by Corollary 3.2.5 we have $\Delta(S) = \Lambda(S)$. Combining this with the statement in *1*, the proof of *2* follows.

For the proof of *3*, let S be a finitely generated GDMS. Recall that by Lemma 3.2.1 we have $\Lambda(S) = \dim_{\mathrm{H}} L(S)$. Furthermore, recall that in this situation we have that $L_{ur}(S) = L(S)$. Thus, the statement in *2* gives $\dim_{\mathrm{H}} L(S) = \delta(S) \leq \Delta(S) = \Lambda(S) = \dim_{\mathrm{H}} L(S)$, which completes the proof of the proposition. $\square$

REMARK: We would like to remark that Lemma 3.2.10 is an analogue of a result by Bishop and Jones in [14] (see also Theorem 3.4.7, where this will be discussed in greater detail).
In fact, we see in Lemma 3.4.15 that there are pseudo GDMSs for which $\delta(S) < \Delta(S)$. This will show that the Poincaré exponent and the distortion exponent are in general not equal.

3.2.2 On normal subsystems

In this section we introduce the notion of a normal subsystem of a GDMS. Our definition is motivated by normal subgroups of Kleinian groups of Schottky type. We then investigate these normal subsystems and give an analysis of their limit sets. For instance, we relate the Hausdorff dimension of the limit sets of such a system to the side exponent.
Let us begin by introducing the concepts "reduced" and "freely decomposable" for GDMSs.

Definition 3.2.12. *Let $S = (V(S), E(S), i, t, A(S), \{X_v\}_{v \in V(S)}, \ell, \{\phi_e\}_{e \in E(S)})$ be a GDMS. For a subset $V(G)$ of $V(S)$ we define the reduced GDMS G as follows.*

$$G := (V(G), E(G,G), i, t, A(G), \{X_v\}_{v \in V(G)}, \ell, \{\phi_e\}_{e \in E(G)})$$

Here $E(G,G) := \{e \in E(S) \mid i(e), t(e) \in V(G)\}$, and $A(G) := A(S)_{|E(G,G)}$ is the restriction of the incidence matrix of S to $E(G,G)$.

For ease of notation, in what follows we write $E(G)$ instead of $E(G,G)$.

Definition 3.2.13. *Let S be a GDMS, and let G and H be two subsystems of S. We say that S is* freely decomposable *into G and H if the following conditions are satisfied.*

1. *$V(S) = V(G) \sqcup V(H)$.*
2. *$E(S) = E(G) \sqcup E(H) \sqcup E(H,G) \sqcup E(G,H)$, where $E(G,H) := \{e \in E(S) \mid i(e) \in V(G), t(e) \in V(H)\}$ and $E(H,G) := \{e \in E(S) \mid t(e) \in V(G), i(e) \in V(H)\}$.*
3. *For every $v \in V(G)$ and every $w \in V(H)$, there exists a unique $e \in E(G,H)$ and a unique $f \in E(H,G)$ such that $i(e) = v$, $t(e) = w$, $i(f) = w$, and $t(f) = v$.*

4. *Let* $e \in E(G,H) \cup E(G,H)$ *and let* $f \in E(S)$*, then* $A_{e,f} = 1$ *if and only if* $i(e) = t(f)$*, and* $A_{f,e} = 1$ *if and only if* $i(f) = t(e)$*.*

If S *is freely decomposable into* G *and* H *we write* $S = G * H$*.*

For a pseudo GDMS S and a compact set $X \subset \{X_e\}_{e \in E(S)}$, we define the orbit of X by

$$\mathcal{O}_S(X) := X \cup \bigcup_{e \in E^*(S)} \phi_e(X \cap X_{i(e)}).$$

The following proposition has been obtained in [84]. For GDMSs, a similar statement can be found in [48].

Proposition 3.2.14. *For a finitely generated GDMS* S *and a compact set* $X \subset \bigcup_{v \in V(S)} X_v$*, which satisfies* $X \cap X_v \neq \emptyset$ *for all* $v \in V(S)$*, the following holds.*

$$\textit{If } X \cap \bigcup_{e \in E^*(S)} \phi_e(X_{i(e)}) = \emptyset, \quad \textit{then } \overline{\mathcal{O}_S(X)} \setminus \mathcal{O}_S(X) = L(S).$$

Proof. For ease of exposition, in the following let us skip the intersection with the domain of the map, that is, we just write $\phi_{\underline{e}}(X)$ instead of $\phi_{\underline{e}}(X \cap X_{i(\underline{e})})$. Recall that $\mathcal{O}_S(X) = X \cup \bigcup_{\underline{e} \in E^*(S)} \phi_{\underline{e}}(X)$. Note that for $n \in \mathbb{N} \setminus \{1\}$ we have

$$\bigcup_{\underline{e} \in E^n(S)} \phi_{\underline{e}}(X) = \bigcup_{e \in E(S)} \phi_e(X) \cup \bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^{(n-1)}(S) \\ e\underline{f} \in E^n(S)}} \phi_e(\phi_{\underline{f}}(X)),$$

where the latter union is taken only over such words $\underline{f}$ (of positive length) for which the composition with e forms admissible words of length n.
Hence, we have

$$\begin{aligned}
\mathcal{O}_S(X) &= X \cup \bigcup_{n \in \mathbb{N}} \bigcup_{\underline{e} \in E^n(S)} \phi_{\underline{e}}(X) \\
&= X \cup \bigcup_{e \in E(S)} \phi_e(X) \cup \bigcup_{\substack{n \in \mathbb{N} \\ n > 1}} \bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^{(n-1)}(S) \\ e\underline{f} \in E^n(S)}} \phi_e(\phi_{\underline{f}}(X)) \\
&= X \cup \bigcup_{e \in E(S)} \phi_e(X) \cup \bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ e\underline{f} \in E^*(S)}} \phi_e(\phi_{\underline{f}}(X)).
\end{aligned}$$

From this one sees that $\mathcal{O}_S(X)$ satisfies some kind of “self-similarity relation”. The important point, however, is that only admissible words occur.

Let us now assume that $x \in \overline{\mathcal{O}_S(X)} \setminus \mathcal{O}_S(X)$. We then have

$$\begin{aligned}
\overline{\mathcal{O}_S(X)} &= \overline{X \cup \bigcup_{e \in E(S)} \phi_e(X) \cup \bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ e\underline{f} \in E^*(S)}} \phi_e(\phi_{\underline{f}}(X))} \\
&= \overline{X} \cup \overline{\bigcup_{e \in E(S)} \phi_e(X)} \cup \overline{\bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ e\underline{f} \in E^*(S)}} \phi_e(\phi_{\underline{f}}(X))} \\
&= X \cup \bigcup_{e \in E(S)} \phi_e(X) \cup \overline{\bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ e\underline{f} \in E^*(S)}} \phi_e(\phi_{\underline{f}}(X))}.
\end{aligned}$$

Here, in the final equation we used that X is compact, that $E(S)$ is finite, and that the maps ϕ_e are continuous. Since x is not in $\mathcal{O}_S(X)$, we therefore have

$$\begin{aligned}
&\overline{\bigcup_{e \in E(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ e\underline{f} \in E^*(S)}} \phi_e(\phi_{\underline{f}}(X))} \\
= &\overline{\bigcup_{e \in E(S)} \bigcup_{\substack{f \in E(S) \\ ef \in E^2(S)}} \phi_e(\phi_f(X)) \cup \bigcup_{e \in E(S)} \bigcup_{\substack{f \in E(S) \\ ef \in E^2(S)}} \bigcup_{\substack{\underline{f} \in E^*(S) \\ ef\underline{f} \in E^*(S)}} \phi_e \circ \phi_f(\phi_{\underline{f}}(X))} \\
= &\overline{\bigcup_{\underline{e} \in E^2(S)} \phi_{\underline{e}}(X) \cup \bigcup_{\underline{e} \in E^2(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ \underline{e}\underline{f} \in E^*(S)}} \phi_{\underline{e}}(\phi_{\underline{f}}(X))} \\
= &\bigcup_{\underline{e} \in E^2(S)} \phi_{\underline{e}}(X) \cup \overline{\bigcup_{\underline{e} \in E^2(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ \underline{e}\underline{f} \in E^*(S)}} \phi_{\underline{e}}(\phi_{\underline{f}}(X))}.
\end{aligned}$$

Here, in the final equation we used that X is compact, that $E(S)$ (and therefore also $E^2(S)$) is finite, and that the maps under consideration are continuous. Since x is not in $\mathcal{O}_S(X)$, we therefore have that

$$x \in \overline{\bigcup_{\underline{e} \in E^2(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ \underline{e}\underline{f} \in E^*(S)}} \phi_{\underline{e}}(\phi_{\underline{f}}(X))}$$

Hence, by iteration we get

$$x \in \overline{\bigcup_{\underline{e} \in E^n(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ \underline{e}\underline{f} \in E^*(S)}} \phi_{\underline{e}}(\phi_{\underline{f}}(X))}, \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Finally, note that

$$\overline{\bigcup_{\underline{e} \in E^n(S)} \bigcup_{\substack{\underline{f} \in E^*(S) \\ \underline{e}\underline{f} \in E^*(S)}} \phi_{\underline{e}}(\phi_{\underline{f}}(X))} \subset \bigcup_{\underline{e} \in E^n(S)} \phi_{\underline{e}}(X_{i(\underline{e})}), \quad \text{for all } n \in \mathbb{N}. \tag{14}$$

Therefore, by combining (13) and (14), we now obtain that

$$x \in \bigcup_{\underline{e} \in E^n(S)} \phi_{\underline{e}}(X_{i(\underline{e})}), \quad \text{for all } n \in \mathbb{N}.$$

This implies that

$$x \in \bigcap_{n \in \mathbb{N}} \bigcup_{\underline{e} \in E^n(S)} \phi_{\underline{e}}(X_{i(\underline{e})}) \subset L_{dyn}(S),$$

and hence we have now shown that

$$\overline{\mathcal{O}_S(X)} \setminus \mathcal{O}_S(X) \subset L_{dyn}(S).$$

For the opposite inclusion we argue as follows. First we show that $L_{dyn}(S) \subset \overline{\mathcal{O}_S(X)}$, and then we proceed by showing that $L_{dyn}(S) \cap \mathcal{O}_S(X) = \emptyset$. Let $\xi \in L_{dyn}(S)$. Then there exists a sequence $(\underline{e}\,(n))_{n \in \mathbb{N}}$, such that $\underline{e}\,(n) \in E^n(S)$, for all $n \in \mathbb{N}$ and $\xi \in \bigcap_{n \in \mathbb{N}} \phi_{\underline{e}(n)} \left(X_{i(\underline{e}(n))} \right)$. Combining this with the fact that $X \cap X_{i(\underline{e})} \neq \emptyset$, we have for all $n \in \mathbb{N}$ that $\emptyset \neq \phi_{\underline{e}(n)}(X) \subset \phi_{\underline{e}(n)} \left(X_{i(\underline{e}(n))} \right)$. This implies that for all $n \in \mathbb{N}$ we have $\operatorname{dist}(\xi, \phi_{\underline{e}(n)}(X)) \leq \operatorname{diam} \phi_{\underline{e}(n)} \left(X_{i(\underline{e}(n))} \right) \leq \max_{e \in E(S)} \left(\operatorname{diam} X_{i(e)} \right) \cdot \ell^n \ll \ell^n$, where $\ell < 1$ depends only on S. Combining this observation with the fact that $\phi_{\underline{e}(n)}(X) \subset \mathcal{O}_S(X)$, we conclude that $\xi \in \overline{\mathcal{O}_S(X)}$ and, consequently, that $L_{dyn}(S) \subset \overline{\mathcal{O}_S(X)}$. Since $\bigcup_{e \in E(S)} \phi_e(X_{i(e)}) \cap X = \emptyset$ and $L_{dyn}(S) \subset \bigcup_{e \in E(S)} \phi_e(X_{i(e)})$, we have $L_{dyn}(S) \cap X = \emptyset$. Moreover, $\bigcup_{\underline{e} \in E^2(S)} \phi_{\underline{e}}(X_{i(\underline{e})}) \cap \bigcup_{e \in E(S)} \phi_e(X) = \emptyset$, since $\bigcup_{e \in E(S)} \phi_e(X_{i(e)}) \cap X = \emptyset$. Therefore, $L_{dyn}(S) \cap \bigcup_{e \in E(S)} \phi_e(X) = \emptyset$. Continuing in this way, we get that $L_{dyn}(S) \cap \mathcal{O}_S(X) = \emptyset$. By combining this with the fact that $L_{dyn}(S) \subset \overline{\mathcal{O}_S(X)}$, we derive the desired inclusion $L_{dyn}(S) \subset \overline{\mathcal{O}_S(X)} \setminus \mathcal{O}_S(X)$. Since $E(S)$ is finite, we have $L_{dyn}(S) = L(S)$, which then completes the proof of the proposition. □

Our next goal is to introduce the notion of a normal GDMS. We start with a freely decomposable GDMS $S = G * H$ and then construct a particular pseudo GDMS $N_{G,H}(S)$, which will depend on G and H. For ease of exposition, we restrict the discussion to a finitely generated S. We then show that for such a system $N_{G,H}(S)$ there is a canonical way to define the notion of a "quotient" $S/N_{G,H}(S)$.

Let us begin by giving the slightly technical construction of a normal subsystem $N := N_{G,H}(S)$.

Let $S = G * H$ be a finitely generated GDMS which is freely decomposable into the subsystems G and H. We define $\{X_v^N\}_{v \in V(N)}$ to be the set of connected components X_v^N of

$$\bigcup_{w \in V(H)} X_w \,\cup\, \mathcal{O}_G \left(\bigcup_{e \in E(H,G)} \phi_e \left(X_{i(e)} \right) \right). \tag{15}$$

That is,

$$\begin{aligned}\{X_v^N\}_{v\in V(N)} &:= \{X_w \mid w \in V(H)\} \cup \{\phi_e(X_{i(e)}) \mid e \in E(H,G)\} \\ &\quad \cup\{\phi_{\underline{g}}\phi_e(X_{i(e)}) \mid \underline{g} \in E^*(G), e \in E(H,G)\}.\end{aligned}$$

If $f \in E(H,G)$ and $v \in V(H)$, then $\phi_f : X_v \to \phi_f(X_v)$ is a bijection. Let $\phi_{f^{-1}}$ denote the inverse of ϕ_f. Note that $\phi_{f^{-1}}$ corresponds to the reversed edge f^{-1} which is obtained by reversing the direction of f.
Additionally, for each $v \in V(N)$ and for each $e \in E(G,H)$ such that $X_{i(e)} \cap X_v^N \neq \emptyset$, we define the restricted map

$$\psi_{e_v} := (\phi_e)\big|_{X_{i(e)}\cap X_v^N} : X_{i(e)} \cap X_v^N \to \phi_e\left(X_v^N\right).$$

Again, note that this map is a bijection. To each restricted map we associate a corresponding edge e_v, and the so obtained set of edges will be denoted by $E_r(N)$. The corresponding set of restricted maps will be denoted by $\{\psi_{e_v}\}_{e_v\in E_r(N)}$. For ease of exposition, from now on we usually write $e_r \in E_r(N)$.

Moreover, note that for $\underline{g} \in E^*(G)$, $f \in E(H,G)$, $h \in E(H)$ and $e_c := e_c(h,f,\underline{g})$, we find $\tilde{f} \in E(H,G)$ such that $i(\tilde{f}) = t(h)$ and $t(\tilde{f}) = i(\underline{g})$. This $\tilde{f}$ is uniquely determined by Definition 3.2.13. To each such e_c we then associate a contraction ψ_{e_c}, which is given by

$$\psi_{e_c} := \phi_{\underline{g}} \circ \phi_{\tilde{f}} \circ \phi_h \circ \phi_{f^{-1}} \circ \phi_{\underline{g}^{-1}} : \phi_{\underline{g}}\left(\phi_f(X_{i(f)})\right) \to \phi_{\underline{g}}\left(\phi_{\tilde{f}}\left(\phi_h(X_{i(f)})\right)\right).$$

We abbreviate the set of these contractions by $\{\psi_{e_c}\}_{e_c\in E_c(N)}$.
Furthermore, for $\underline{g} \in E^*(G)$, $f \in E(H,G)$ and $e_r \in E_r(N)$, let

$$\psi_{(e_r,f,\underline{g})} := \phi_{\underline{g}} \circ \phi_f \circ \psi_{e_r} : X^N_{i(e_r)} \to \phi_{\underline{g}}\left(\phi_f\left(\psi_{e_r}\left(X^N_{i(e_r)}\right)\right)\right),$$

and similarly,

$$\psi_{(e_r,f)} := \phi_f \circ \psi_{e_r} : X^N_{i(e_r)} \to \left(\phi_f\left(\psi_{e_r}\left(X^N_{i(e_r)}\right)\right)\right).$$

Let us define

$$\begin{aligned}\Psi(N) &:= \left\{\psi_{(e_r,f)} \mid e_r \in E_r(N),\, f \in E(H,G)\right\} \\ &\quad \cup\left\{\psi_{(e_r,f,\underline{g})} \;\middle|\; e_r \in E_r(N),\, f \in E(H,G),\, \underline{g} \in E^*(G)\right\}.\end{aligned}$$

For ease of exposition, let $\Psi(N) =: \{\psi_{e_t}\}_{e_t\in E_t(N)}$.

In (15) we defined the set of compact spaces for the normal subsystem N. The set of maps of N is now defined by

$$\{\psi_{e_N}\}_{e_N \in E(N)} := \{\phi_e\}_{e \in E(H)} \cup \{\psi_{e_r}\}_{e_r \in E_r(N)} \cup \{\psi_{e_c}\}_{e_c \in E_c(N)} \cup \{\psi_{e_t}\}_{e_t \in E_t(N)}. \tag{16}$$

It is clear that this also defines the set of edges $E(N)$ for N. Note that $E(N)$ can be decomposed as follows:

$$E(N) = E(H) \cup E_r(N) \cup E_c(N) \cup E_t(N).$$

For each of these four sets of edges, the two maps $i_N, t_N : E(N) \to V(N)$ are defined in the following way:

- For $e \in E(H)$, define $i_N(e) := v \in V(N)$ if $X_v^N = X_{i(e)}$. Likewise, let $t_N(e) := v \in V(N)$ if $X_v^N = X_{t(e)}$.
- For $e_r \in E_r(N)$, recall that the corresponding map is of the form

  $$\psi_{e_r} = (\phi_e)\big|_{X_{i(e)} \cap X_v^N} : X_v^N \to X_{t(e)},$$

  for some $v \in V(N)$. Define $i_N(e_r) := v \in V(N)$ if $X_v^N = \mathrm{Dom}(\psi_{e_r})$. Also, define $t_N(e_r) := v \in V(N)$ if $X_v^N = \mathrm{Im}(\psi_{e_r})$. (Here, Dom and Im denote the domain and the image respectively).
- For $e_c = (\underline{g}^{-1}, f^{-1}, h, \tilde{f}, \underline{g}) \in E_c(N)$, recall that there is the corresponding map

  $$\psi_{e_c} := \phi_{(\underline{g}^{-1}, f^{-1}, h, \tilde{f}, \underline{g})} : \phi_{\underline{g}}\left(\phi_f(X_{i(f)})\right) \to \phi_{\underline{g}}\left(\phi_{\tilde{f}}\left(\phi_h(X_{i(f)})\right)\right).$$

  Define $i_N(e_c) := v \in V(N)$ if $X_v^N = \phi_{\underline{g}}\left(\phi_f(X_{i(f)})\right)$, and define $t_N(e_c) := v \in V(N)$ if $X_v^N = \phi_{\underline{g}}\left(\phi_{\tilde{f}}\left(\phi_h(X_{i(f)})\right)\right)$.
- For $e_t := (e_r, f, \underline{g}) \in E_t(N)$, recall that there is the corresponding map

  $$\psi_{(e_r, f, \underline{g})} := \phi_{\underline{g}} \circ \phi_f \circ \psi_{e_r} : X_{i(e_r)}^N \to \phi_{\underline{g}}\left(\phi_f\left(\psi_{e_r}\left(X_{i(e_r)}^N\right)\right)\right).$$

  Here, $\phi_{\underline{g}}$ is allowed to be equal to the identity map $\mathbf{id}$. Define $i_N(e_t) := v \in V(N)$ if $X_v^N = X_{i_N(e_r)}^N$, and let $t_N(e_t) := v \in V(N)$ if $X_v^N = \phi_{\underline{g}}\left(\phi_f\left(\psi_{e_r}\left(X_{i(e_r)}^N\right)\right)\right)$.

Finally, we define the $(\mathrm{card}(E(N)) \times \mathrm{card}(E(N)))$-incidence matrix $A(N)$ for N. For $h, \tilde{h} \in E(H) \subset E(N)$, we define

$$A_{h,\tilde{h}}(N) := A_{h,\tilde{h}}(S).$$

Also, for edges $e_c = e_c(h, f, \underline{g}), e_{\tilde{c}} = e_{\tilde{c}}(\tilde{h}, \tilde{f}, \underline{\tilde{g}}) \in E_r(N)$ with $\underline{g}, \underline{\tilde{g}} \in E^*(G)$, $f, \tilde{f} \in E(H, G)$ and $h, \tilde{h} \in E(H)$, we define

$$A_{e_c(h,f,\underline{g}),e_{\tilde{c}}(\tilde{h},\tilde{f},\underline{\tilde{g}})}(N) := \begin{cases} A_{h,\tilde{h}}(S) & \text{if } i(\tilde{f}) = t(h) \text{ and } \underline{g} = \underline{\tilde{g}} \\ 0 & \text{otherwise} \end{cases} .$$

For the remaining entries of the incidence matrix, that is, for pairs $(e_N, \tilde{e}_N) \notin (E(H) \times E(H)) \cup (E_r(N) \times E_r(N))$, we define

$$A_{e_N,\tilde{e}_N}(N) := \begin{cases} 1 & \text{if } t_N(e_N) = i_N(\tilde{e}_N) \\ 0 & \text{otherwise} \end{cases} .$$

This finishes the definition of the edge incidence matrix $A(N)$, and hence completes the construction of N.

Definition 3.2.15. *With the notation as above, the pseudo GDMS*

$$N_{G,H}(S) := \left(V(N), E(N), i_N, t_N, A(N), \left\{ X_v^N \right\}_{v \in V(N)}, \ell, \{\psi_{e_N}\}_{e_N \in E(N)} \right)$$

will be referred to as a normal subsystem of S.

Remark: Note that, by construction, we have that $V(N)$ is countably infinite. However, there are cases in which the maps of a normal subsystem can be extended in such a way that one can reduce the set of vertices to a finite number. This is precisely the case when the system S corresponds to a "finite cover" of an IFS. In this case, N will be called a normal GDMS.

Remark: Note that, by construction, we clearly have that a normal subsystem of a GDMS is not finitely irreducible. In particular, there is no admissible word in which an edge $h \in E(H)$ is followed by an edge $e_N \in E(N) \setminus E(H)$. Instead, one could define the elements of E_t via maps $\psi_{(f',\underline{g'},e_r,f,\underline{g})}$ given by $\psi_{(f',\underline{g'},e_r,f,\underline{g})} := \phi_{\underline{g}} \circ \phi_f \circ \psi_{e_r} \circ \phi_{\underline{g'}} \circ \phi_{f'} : X_{i(f')} \to \phi_{\underline{g}} \left(\phi_f \left(\psi_{e_r} \left(\phi_{\underline{g'}} \left(\phi_{f'} \left(X_{i(f')} \right) \right) \right) \right) \right)$. With this alternative definition, the system N would be finitely irreducible and even finitely primitive. However, our analysis here does not require N to be finitely irreducible, and Definition 3.2.15 is more convenient for the proof of Theorem 3.2.16.

For ease of exposition, we also define for $\underline{g} \in E^*(G, G) \sqcup \{0\}$,

$$\phi_{\underline{g}} := \begin{cases} \phi_{\underline{g}} & \text{if} \quad \underline{g} \in E^*(G, G) \\ \mathbf{id} & \text{if} \quad \underline{g} = 0 \end{cases} .$$

Clearly, for every $e_N \in E(N)$, there exists a unique $\underline{e} \in E^*(S)$ such that $\psi_{e_N}(X_{i(e_N)}^N) = \phi_{\underline{e}}(X_{i(\underline{e})})$. Here, the uniqueness is an immediate consequence of our assumption that S satisfies SSC.

Theorem 3.2.16. *Let S be a finitely generated GDMS which is freely decomposable into G and H. Let $N = N_{G,H}(S)$ be a normal subsystem of S and let $\xi \in L_J(N)$. Then there exists $\underline{e}_N \in E^*(N) \sqcup \{0\}$, $e \in E(G,H)$, $\underline{g} \in E^*(G) \sqcup \{0\}$ and $f \in E(H,G)$ such that precisely one of the following cases occurs.*

$$(\mathbf{a})\ \ \xi \in L(G); \quad (\mathbf{b})\ \ \xi \in \phi_e\left(L(G)\right); \quad (\mathbf{c})\ \ \xi \in \psi_{\underline{e}_N}\phi_{\underline{g}}\phi_f\phi_e\left(L(G)\right).$$

For ease of exposition let

$$O_N(L(G)) := \mathcal{O}_N\left(\bigcup_{\substack{\underline{g}\in E^*(G)\sqcup\{0\}\\ f\in E(H,G)\\ e\in E(G,H)}} \phi_{\underline{g}}\phi_f\phi_e\left(L(G)\right)\right) \cup \bigcup_{e\in E(G,H)} \phi_e\left(L(G)\right) \cup L(G).$$

With this notation, the statement of the theorem is equivalent to

$$L_J(N) \subset O_N(L(G)).$$

Proof. Let $\xi \in L_J(N)$ be given. By definition of $L_J(N)$, we have that $\xi \in \overline{L_{dyn}(N)} \setminus L_{dyn}(N)$. We consider different covers of $L_{dyn}(N)$. We begin with the cover $\bigcup_{v\in V(N)} X_v^N$, and then subsequently consider the cover $\bigcup_{\underline{e}_N\in E^n(N)} \phi_{\underline{e}_N}\left(X^N_{\underline{e}_N}\right)$, for each $n \in \mathbb{N}$.

(a) Assume that

$$\xi \in \bigcup_{v\in V(S)} X_v \setminus \bigcup_{w\in V(N)} X_w^N.$$

Since $\xi \in L_J(N)$, we have

$$\xi \in \overline{\bigcup_{v\in V(N)} X_v^N} \setminus \bigcup_{w\in V(N)} X_w^N =: X_{(a)}.$$

Note that, since $V(H)$ is finite, we have that $\bigcup_{v\in V(H)} X_v$ is closed. Therefore, by (15), it follows that

$$X_{(a)} = \overline{\mathcal{O}_G\left(\bigcup_{f\in E(H,G)} \phi_f\left(X_{i(f)}\right)\right)} \setminus \mathcal{O}_G\left(\bigcup_{f\in E(H,G)} \phi_f\left(X_{i(f)}\right)\right).$$

Applying Proposition 3.2.14, we conclude that $X_{(a)} = L(G)$.

(b) Assume that

$$\xi \in \bigcup_{w\in V(N)} X_w^N \setminus \bigcup_{e_N\in E(N)} \psi_{e_N}\left(X^N_{i(e_N)}\right) =: X_{(b)}. \tag{17}$$

Clearly, we then have that $\xi \in X_w^N$, for some $w \in V(N)$. There are two cases to consider.

Case 1: There exists $v \in V(H)$ such that $\xi \in X_v = X_w^N$. Since $\xi \in L(S)$, there exists $e \in E(S)$ with $\xi \in \phi_e\left(X_{i(e)}\right)$. Note that

$$e \notin E(H) \subset E(N).$$

Indeed, assume by way of contradiction that $e \in E(H)$. Then $e \in E(N)$, and hence $\xi \in \phi_e\left(X_{i(e)}\right) = \phi_e\left(X_{i_N(e)}\right) \subset \bigcup_{e_N \in E(N)} \psi_{e_N}\left(X^N_{i(e_N)}\right)$. This is a contradiction to the statement in (17).
Hence, we can now assume that $e \notin E(H)$. Since $t(e) \in V(H)$, we have $e \in E(G,H)$. Therefore, (17) is equivalent to $\xi \in \phi_e\left(X_{i(e)}\right) \backslash \bigcup_{e_N \in E(N)} \psi_{e_N}\left(X^N_{i(e_N)}\right)$. Recall that $E_r(N) \subset E(N)$ and that a restriction ψ_{e_r} is of the form $\psi_{e_r} = (\phi_{\tilde{e}})_{|_{X^N_{\tilde{v}}}}$, for some $\tilde{e} \in E(G,H)$ with $X^N_{\tilde{v}} \subset X_{i(\tilde{e})}$. Therefore,

$$\xi \in \phi_e(X_{i(e)}) \backslash \bigcup_{\tilde{v} \in V(N)} \bigcup_{\tilde{e} \in E(G,H)} \phi_{\tilde{e}}(X_{i(\tilde{e})} \cap X^N_{\tilde{v}}) \subset \phi_e(X_{i(e)}) \backslash \bigcup_{\tilde{v} \in V(N)} \phi_e(X_{i(e)} \cap X^N_{\tilde{v}}).$$

Note that we have $\xi \in \phi_e(\overline{\bigcup_{\tilde{v} \in V(N)} X^N_{\tilde{v}} \cap X_{i(e)}})$, since $\xi \in \phi_e(X_{i(e)})$ and $\xi \in L_J(N) \subset \overline{L_{dyn}(N)}$. Combining these observations, it follows that

$$\xi \in \phi_e\left(\overline{\bigcup_{\tilde{v} \in V(N)} X^N_{\tilde{v}} \cap X_{i(e)}} \backslash \bigcup_{\tilde{v} \in V(N)} X_{i(e)} \cap X^N_{\tilde{v}}\right). \tag{18}$$

Finally, since $\phi_e : X_{i(e)} \to \phi\left(X_{i(e)}\right)$ is bijective, applying ϕ_e^{-1} to both sides of (18) gives

$$\phi_e^{-1}(\xi) \in \overline{\bigcup_{\tilde{v} \in V(N)} X^N_{\tilde{v}} \cap X_{i(e)}} \backslash \bigcup_{\tilde{v} \in V(N)} X^N_{\tilde{v}} \cap X_{i(e)} \subset X_{(a)}.$$

By what we have shown in the proof of (a), it now follows that $\phi_e^{-1}(\xi) \in L(G)$.

Case 2: Assume that there exists $v \in V(G)$ such that $\xi \in X_w^N \subset X_v$. Since $X_w^N \in \mathcal{O}_G\left(\left\{\phi_e\left(X_{i(e)}\right) \mid e \in E(H,G)\right\}\right)$, by definition of a normal subsystem, there exists $\underline{g} \in E^*(G)$ and $f \in E(H,G)$ such that $X_w^N = \phi_{\underline{g}} \circ \phi_f\left(X_{i(f)}\right)$. Therefore, (17) is equivalent to

$$\xi \in \phi_{\underline{g}} \circ \phi_f\left(X_{i(f)}\right) \backslash \bigcup_{e_N \in E(N)} \psi_{e_N}\left(X^N_{i_N(e_N)}\right).$$

Note that by definition of $N_{G,H}(S)$, the following holds. If $e_N \in E(N)$ is given such that $X^N_{t_N(e_N)} = X^N_w$, then $e_N \in E_t(N) \cup E_c(N)$. Hence, for any such e_N

the map ψ_{e_N} is either of the form $\phi_{(\underline{g}^{-1},\tilde{f}^{-1},h,f,\underline{g})}$ or of the form $\phi_{\underline{g}}\circ\phi_f\circ\psi_{e_r}$, for some $h\in E(H)$, $e_r\in E_r(N)$, and with $\tilde{f}\in E(H,G)$ uniquely determined by $i(\tilde{f})=i(h)$ and $t(\tilde{f})=t(f)$. Hence, we have $\phi_{(\underline{g}^{-1},\tilde{f}^{-1},h,f,\underline{g})}\left(\phi_{(\tilde{f},\underline{g})}\left(X_{i(\tilde{f})}\right)\right)=\phi_{(h,f,\underline{g})}\left(X_{i(h)}\right)$.

Using these observations, the statement in (17) can be simplified as follows:

$$\begin{aligned}\xi &\in \phi_{\underline{g}}\circ\phi_f\left(X_{i(f)}\right)\setminus\bigcup_{e_N\in E_c(N)\cup E_t(N)}\psi_{e_N}\left(X^N_{i_N(e_N)}\right)\\ &= \phi_{(f,\underline{g})}\left(X_{i(f)}\right)\setminus\left[\bigcup_{h\in E(H)}\phi_{(h,f,\underline{g})}\left(X_{i(h)}\right)\cup\bigcup_{e_r\in E_r(N)}\phi_{(f,\underline{g})}\psi_{e_r}\phi_{(f,\underline{g})}\left(X_{i(f)}\right)\right].\end{aligned}$$

Since $\phi_g\circ\phi_f: X_{i(f)}\to\phi_g\circ\phi_f\left(X_{i(f)}\right)$ is bijective, applying $\left(\phi_{(f,\underline{g})}\right)^{-1}$ to both sides of the equation above gives

$$\left(\phi_{(f,\underline{g})}\right)^{-1}(\xi) \in X_{i(f)}\setminus\left[\bigcup_{h\in E(H)}\phi_{(h,f,f^{-1})}\left(X_{i(h)}\right)\cup\bigcup_{e_r\in E_r(N)}\psi_{e_r}\left(\phi_{(f,\underline{g})}\left(X_{i(f)}\right)\right)\right].$$

Here, note that $\phi_{(h,f,f^{-1})}=\phi_h: X_{i(h)}\to X_{t(h)}$.

This shows that there exists $\tilde{v}\in V(H)$ such that $\left(\phi_{(f,\underline{g})}\right)^{-1}(\xi)\in X_{\tilde{v}}$. Therefore, we are now in the situation of Case 1, so there exists some $e\in E(G,H)$ such that

$$\phi_e^{-1}\phi^{-1}_{(f,\underline{g})}(\xi)=\phi_e^{-1}\circ\phi_f^{-1}\circ\phi_{\underline{g}}^{-1}(\xi)\in L(G).$$

This finishes the proof of (b).

(c) Assume that for some $n\in\mathbb{N}$ we have

$$\xi\in\bigcup_{\underline{e}_N\in E^n(N)}\psi_{\underline{e}_N}\left(X^N_{i_N(\underline{e}_N)}\right)\setminus\bigcup_{\underline{\tilde{e}}_N\in E^{n+1}(N)}\psi_{\underline{\tilde{e}}_N}\left(X^N_{i_N(\underline{\tilde{e}}_N)}\right). \tag{19}$$

Clearly, then there exists $(e_1,\dots,e_n)\in E^n(N)$ such that

$$\xi\in\psi_{(e_1,\dots,e_n)}\left(X^N_{i_N(e_1)}\right)\setminus\bigcup_{\underline{\tilde{e}}_N\in E^{n+1}(N)}\psi_{\underline{\tilde{e}}_N}\left(X^N_{i_N(\underline{\tilde{e}}_N)}\right).$$

Note that, since $\xi\in L(N)$ and since S satisfies SSC, this implies

$$\xi\in\psi_{(e_1,\dots,e_n)}\left(X^N_{i_N(e_1)}\right)\setminus\bigcup_{\underline{e}_N\in E^n(N)}\bigcup_{\substack{e_N\in E(N)\\ i_N(e_N)=t_N(\underline{e}_N)}}\psi_{e_N}\psi_{\underline{e}_N}\left(X^N_{i_N(\underline{e}_N)}\right).$$

Recall that $\psi_{(e_1,\dots,e_n)}: X^N_{i_N(e_1)}\to\psi_{(e_1,\dots,e_n)}\left(X^N_{i_N(e_1)}\right)$ is bijective. This implies that

$$\psi^{-1}_{(e_1,\dots,e_n)}(\xi) \in X^N_{i_N(e_1)}\setminus\bigcup_{\underline{e}_N\in E^n(N)}\bigcup_{\substack{e_N\in E(N)\\ i_N(e_N)=t_N(\underline{e}_N)}}\psi^{-1}_{(e_1,\dots,e_n)}\psi_{e_N}\psi_{\underline{e}_N}\left(X^N_{i_N(\underline{e}_N)}\right).$$

Now we have reduced the proof to the consideration of the following two cases.

(i) $\psi^{-1}_{(e_1,\dots,e_n)}(\xi)$ is contained in either $X_{(a)}$ or $X_{(b)}$.

(ii) Similar to the situation in (19), we have for some $k \in \mathbb{N}$,
$\psi^{-1}_{(e_1,\dots,e_n)}(\xi) \in \bigcup_{\underline{e}_N \in E^k(N)} \psi_{\underline{e}_N}\left(X^N_{i_N(\underline{e}_N)}\right) \setminus \bigcup_{e_N \in E(N)} \psi_{e_N}\psi_{\underline{e}_N}\left(X^N_{i_N(\underline{e}_N)}\right)$.

In case (i) one proceeds as in the proof of (a) and (b), and concludes that there exists $e \in E(G,H)$, $\underline{g} \in E^*(G) \sqcup \{0\}$ and $f \in E(H,G)$ such that

$$\phi_e^{-1} \circ \phi_f^{-1} \circ \phi_{\underline{g}}^{-1}\left(\psi^{-1}_{(e_1^N,\dots,e_n^N)}(\xi)\right) \in L(G).$$

In case (ii) we iterate this process of reduction. It is clear that after finitely many reductions this iteration will terminate in an element which is either in $X_{(a)}$ or in $X_{(b)}$, and we are again in the situation of (i). This completes the proof of (c), and hence of the theorem. □

Corollary 3.2.17. *With the notation as in Theorem 3.2.16, we have that*

$$L_J(N) = O_N(L(G)).$$

Proof. In order to prove the corollary, we show that $O_N(L(G)) \subset \overline{L_{dyn}(N)}$ and that $O_N(L(G)) \cap L_{dyn}(N) = \emptyset$. Combining these observations with the results obtained above will complete the proof.
Let us first show that $O_N(L(G)) \subset \overline{L_{dyn}(N)}$. For this, let $\xi \in L(G)$ be given and recall that $L(G) = \bigcap_{n\in\mathbb{N}} \bigcup_{g \in E^n(G)} \phi_{\underline{g}}(X_{i(\underline{g})})$. Hence, there exists a sequence $(\underline{g}\,(j))_{j\in\mathbb{N}}$ of words $\underline{g}\,(j) \in E^j(G)$ such that $\xi \in \phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right)$, for all $j \in \mathbb{N}$. Let $j \in \mathbb{N}$ be fixed, and note that we have that $\phi_{\underline{g}(j+1)}\left(X_{i\left(\underline{g}(j+1)\right)}\right) \subset \phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right)$. Recall that, by definition of a normal subsystem, we have that for $f \in E(H,G)$, $\underline{g} \in E^*(G)$, and $e_r \in E_r(N)$ there is a map $\phi_{\underline{g}} \circ \phi_f \circ \psi_{e_r}$ which belongs to the system N. We then clearly have, for some $f \in E(H,G)$ and $e_r \in E_r(N)$, that

$$R(\underline{g}\,(j)) := \phi_{\underline{g}(j)} \circ \phi_f \circ \psi_{e_r}(X_{i(e_r)}) \subset \phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right).$$

Note that we now have

$$\sup_{x \in R(\underline{g}(j))} \operatorname{dist}(x,\xi) \le \operatorname{diam}(\phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right)) \le \ell^j, \tag{20}$$

where $\ell \in (0,1)$ is the upper bound for the Lipschitz constants of the contractions in G, as given in the definition of a GDMS. Recall that $\underline{g}\,(j) \in E^j(G)$,

for each $j \in \mathbb{N}$. Note that for each $j \in \mathbb{N}$ we now have that there exists an edge $e_N \in E(N)$ corresponding to $\phi_{\underline{g}(j)} \circ \phi_f \circ \psi_{e_r}$, that is $\phi_{e_N} = \phi_{\underline{g}(j)} \circ \phi_f \circ \psi_{e_r}$. Now, for each $n \in \mathbb{N}$, we have that there exists a word $\underline{e}_N \in E^n(N)$ such that $\underline{e}_N = (\underline{\tilde{e}}_N, e_N)$, for some $\underline{\tilde{e}}_N \in E^{n-1}(N)$. Combining this with (20) for this $\underline{e}_N \in E^n(N)$, we have $\operatorname{dist}(\xi, \psi_{\underline{e}_N}(X_{i_N(\underline{e}_N)})) \leq \sup_{x \in R(\underline{g}(j))} \operatorname{dist}(x, \xi) \leq \operatorname{diam}(\phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right)) \leq \ell^j$.
Hence, for each $j \in \mathbb{N}$ and for each $n \in \mathbb{N}$ there is $\underline{e}_N \in E^n(N)$ such that

$$\operatorname{dist}(\xi, \psi_{\underline{e}_N}(X_{i_N(\underline{e}_N)})) \leq \ell^j.$$

Therefore, we have

$$\operatorname{dist}(\xi, \bigcap_{n \in \mathbb{N}} \bigcup_{e_N \in E^n(N)} \psi_{e_N}(X_{i_N(e_N)})) \leq \ell^j.$$

Since this holds for all $j \in \mathbb{N}$, it follows that $\xi \in \overline{L_{dyn}(N)}$. This implies that $L(G) \subset \overline{L_{dyn}(N)}$. Taking the orbit O_N on both sides gives $O_N(L(G)) \subset \overline{L_{dyn}(N)}$, since $O_N(\overline{L_{dyn}(N)}) = \overline{L_{dyn}(N)}$.
Now, we want to show that $O_N(L(G)) \cap L_{dyn}(N) = \emptyset$. Let $\xi \in L(G)$. Recall that we then have a sequence $(\underline{g}\,(j))_{j \in \mathbb{N}}$ of words $\underline{g}\,(j) \in E^j(G)$ such that $\xi \in \bigcap_{j \in \mathbb{N}} \phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right)$. By way of contradiction, let us now assume that $L(G) \cup L_{dyn}(N) \neq \emptyset$. Then, we have $\xi \in \bigcup_{e_N \in E(N)} \psi_{e_N}(X_{i_N(e_N)})$. Recall that the words in $E(N)$ correspond to finite words in $E^*(S) = E^*(G * H)$. This implies that there is a finite word $\underline{e} \in E^*(S)$ such that $\underline{e} \notin E^*(G)$ and $\xi \in \phi_{\underline{e}}(X_{i(\underline{e})})$. (To see that $\underline{e} \notin E^*(G)$ recall that by definition these do not occur in $E(N)$.) But then for $j = |\underline{e}|$ there is $\underline{g}\,(j)$ from the sequence $(\underline{g}\,(j))_{j \in \mathbb{N}}$ and we have

$$\phi_{\underline{g}(j)}\left(X_{i\left(\underline{g}(j)\right)}\right) \cap \phi_{\underline{e}}(X_{i(\underline{e})}) \neq \emptyset.$$

Since $|\underline{e}| = |\underline{g}\,(j)|$, this is a contradiction to the fact that S satisfied SSC. Hence, we have that $L(G) \cup L_{dyn}(N) = \emptyset$.
To complete the proof, assume that either $\xi \in \phi_e(L(G))$ or $\xi \in \psi_{\underline{e}_N} \circ \phi_{\underline{g}} \circ \phi_f \circ \phi_e(L(G))$, where the notation is as in Theorem 3.2.16. Recall from the proof of Theorem 3.2.16 that the maps $\psi_{\underline{e}_N}$, $\phi_{\underline{g}} \circ \phi_f$, ϕ_e are each bijections on their image and hence the assumption would imply that $(\phi_e)^{-1}(\xi) \in L(G)$ or $(\psi_{\underline{e}_N} \circ \phi_{\underline{g}} \circ \phi_f \circ \phi_e)^{-1}(\xi) \in L(G)$. Since both ϕ_e and $\psi_{\underline{e}_N} \circ \phi_{\underline{g}} \circ \phi_f \circ \phi_e$ correspond to a word in $E^*(S)$ of finite length, one can basically proceed as in the proof of Theorem 3.2.16 and conclude that $O_N(L(G)) \cap L_J(N) = \emptyset$.
Combining the facts that $O_N(L(G)) \subset \overline{L_{dyn}(N)}$ and $O_N(L(G)) \cap L_{dyn}(N) = \emptyset$, we have that $O_N(L(G)) \subset \overline{L_{dyn}(N)} \setminus L_{dyn}(N)$. Combining this with Theorem 3.2.16, the proof of the corollary follows. $\square$

The following corollary follows immediately.

Corollary 3.2.18. *Let $S = G * H$ be a pseudo GDMS, and let $N = N_{G,H}(S)$ be a normal subsystem of S. We then have*

$$\dim_{\mathrm{H}} L_J(N) = \dim_{\mathrm{H}} L(G).$$

Corollary 3.2.19. *Let S be a finitely generated GDMS, and let N be a normal subsystem of S. We then have*

$$L(N) = L(S).$$

Proof. By construction, it is clear that $L_{dyn}(N) \subset L(S)$. Hence, we have $L(N) \subset L(S)$, since $L(S)$ is a closed set. Therefore, it is enough to show that $L(S) \subset L(N)$. For this, let $\xi \in L(S)$. Then there exists a sequence $(\underline{e}(k))_{k\in\mathbb{N}}$ of words $\underline{e}(k) \in E^k(S)$ such that $\xi \in \phi_{\underline{e}(k)}\left(X_{i(\underline{e}(k))}\right)$, for all $k \in \mathbb{N}$. In particular, we have that $\phi_{\underline{e}(k+1)}\left(X_{i(\underline{e}(k+1))}\right) \subset \phi_{\underline{e}(k)}\left(X_{i(\underline{e}(k))}\right)$.
Let us first assume that there is some $j \in \mathbb{N}$ such that

$$\xi \in \bigcup_{\underline{e}_N \in E^j(N)} \psi_{\underline{e}_N}\left(X_{i_N\left(\underline{e}_N\right)}\right).$$

We call j maximal if $\xi \notin \bigcup_{\underline{e}_N \in E^{j+1}(N)} \psi_{\underline{e}_N}\left(X_{i_N\left(\underline{e}_N\right)}\right)$. Clearly, there is no maximal $j \in \mathbb{N}$ if and only if $\xi \in L_{dyn}(N)$. Hence, it remains to show that if there exists a maximal $j \in \mathbb{N}$, then $\xi \in L_J(N)$.
Let $j \in \mathbb{N}$ be maximal. Then there are words $\underline{e}_N(\xi) \in E^j(N)$ and $\underline{e(\xi)} \in E^*(S)$ such that $\xi \in \phi_{\underline{e(\xi)}}\left(X_{i\left(\underline{e(\xi)}\right)}\right) = \psi_{\underline{e}_N(\xi)}\left(X_{i\left(\underline{e}_N(\xi)\right)}\right)$. Since $|\underline{e(\xi)}| < \infty$, we have $\underline{e(\xi)} \in \{\underline{e}(k) \mid k \in \mathbb{N}\}$, since otherwise there would be two words $\underline{e(\xi)}$ and $\underline{e}(k)$ in $E^{|\underline{e(\xi)}|}(S)$ with $\phi_{\underline{e(\xi)}}\left(X_{i\left(\underline{e(\xi)}\right)}\right) \cap \phi_{\underline{e}(k)}\left(X_{i(\underline{e}(k))}\right) \neq \emptyset$, which would be a contradiction to S satisfying SSC.
Therefore, $\underline{e(\xi)} \in \{\underline{e}(k) \mid k \in \mathbb{N}\}$. Hence, for all $k \in \mathbb{N}$ and $n := |\underline{e(\xi)}| + k$, we have that $\underline{e}(n)$ is of the form $\underline{e}(n) = (e_1^{(n)}, \ldots, e_k^{(n)}, \underline{e(\xi)})$, and

$$\xi \in \phi_{\underline{e}(n)}\left(X_{i(\underline{e}(n))}\right) = \phi_{\underline{e(\xi)}} \circ \phi_{(e_1^{(n)},\ldots,e_k^{(n)})}(X_{i(e_1^{(n)})}).$$

Hence, we have for all $k \in \mathbb{N}$ that there exist $(e_1^{(n)}, \ldots, e_k^{(n)}) \in E^k(S)$ with

$$(\phi_{\underline{e(\xi)}})^{-1}(\xi) \in \phi_{(e_1^{(n)},\ldots,e_k^{(n)})}(X_{i(e_1^{(n)})}).$$

Since j is maximal, we in particular have that

$$(\phi_{\underline{e(\xi)}})^{-1}(\xi) \in X_{i(e_1^{(n)})} \setminus \bigcup_{e_N \in E(N)} \psi_{e_N}\left(X_{i_N(e_N)}\right).$$

Hence, it is enough to show that if $\xi \in L(S)$ is chosen such that $\xi \notin \bigcup_{e_N \in E(N)} \psi_{e_N}\left(X_{i_N(e_N)}\right)$, then $\xi \in L_J(N)$. In order to complete the proof, we need to consider the following two cases.
In what follows, let $(\underline{e}\,(n))_{n\in\mathbb{N}} \subset E^*(S)$ with $\xi \in \phi_{\underline{e}(n)}\left(X_{i(\underline{e}(n))}\right)$ for all $n \in \mathbb{N}$. Note that the latter exists, since $\xi \in L(S)$. Also, let $\underline{e}\,(n) = (e_1^{(n)}, \ldots, e_n^{(n)})$, as above.
Case 1: If $\xi \notin \bigcup_{v\in V(N)} X_v^N$, then recall that the X_v^N are of the form $X_v^N = \phi_{\underline{g}} \circ \phi_f \left(X_{i(f)}\right)$, for some $f \in E(H,G)$ and $g \in E^*(G)$. Hence, we have that $\xi \notin \bigcup_{\substack{\underline{g}\in E^*(G)\\ f\in E(H,G)}} \phi_{\underline{g}} \circ \phi_f \left(X_{i(f)}\right)$. Since $V(H) \subset V(N)$, we have that $t(\underline{e}\,(n)) = t(e_n^{(n)}) \in V(G)$, because otherwise $\xi \in \bigcup_{v\in V(H)} X_v \subset \bigcup_{v\in V(N)} X_v^N$. Note that if $i(e_n^{(n)}) \notin V(G)$, then $i(e_n^{(n)}) \in V(H)$. But this would imply

$$\phi_{\underline{e}(n)}\left(X_{i(\underline{e}(n))}\right) \subset \phi_{e_n(n)}(X_{i(e_n^{(n)})}) \subset \bigcup_{v\in V(H)} X_v \subset \bigcup_{v\in V(N)} X_v^N,$$

which leads to a contradiction. Hence, $i(e_n^{(n)}) \in V(G)$ and, by induction, we have that $\underline{e}\,(n) \in E^n(G)$. Since this holds for all $n \in \mathbb{N}$, we have that $\xi \in \bigcap_{n\in\mathbb{N}} \phi_{\underline{e}(n)}\left(X_{i(\underline{e}(n))}\right)$, and hence $\xi \in L(G) \subset L_J(N)$.
Case 2: If $\xi \in \bigcup_{v\in V(N)} X_v^N \setminus \bigcup_{e\in E(G,H)} \phi_e \left(X_{i(e)}\right)$, then $\xi \notin \bigcup_{v\in V(H)} X_v$. Indeed, suppose that $\xi \in X_v$ for some $v \in V(H)$. We then have $t(e_n^{(n)}) \in V(H)$. But we would then have $i(e_n^{(n)}) \in V(H)$, because otherwise $e_n^{(n)} \in E(G,H)$ and then $\phi_{\underline{e}(n)}\left(X_{i(\underline{e}(n))}\right) \subset \phi_{e_n^{(n)}}(X_{i(e_n^{(n)})}) \subset \bigcup_{e\in E(G,H)} \phi_e \left(X_{i(e)}\right)$ which leads to a contradiction. Hence, by induction, we would have $\underline{e}\,(n) \in E^n(H) \subset E^n(N)$, and hence $\xi \in L_{dyn}(N)$, which is a contradiction. Hence, we now have that $\xi \notin \bigcup_{v\in V(H)} X_v$. This implies that $\xi \in \bigcup_{\substack{\underline{g}\in E^*(G)\\ f\in E(H,G)}} \phi_{\underline{g}} \circ \phi_f \left(X_{i(f)}\right)$, and hence ξ belongs to at least one of the sets in this union, say $\xi \in \phi_{\underline{g}} \circ \phi_f \left(X_{i(f)}\right)$. We have for $n = |\underline{g}| + 1$ that $e_1^{(n)} = f$ and $(e_2^{(n)}, \ldots, e_n^{(n)}) = \underline{g}$. Note that $e^{(n+1)} = (e_1^{(n+1)}, e^{(n)})$, and hence $t(e_1^{(n+1)}) = i(f) \in V(H)$. It is now sufficent to consider the two cases.
(a) If $i(e_1^{(n+1)}) \in V(H)$, then we have that $e_1^{(n+1)} \in E(H) \subset E(N)$ and that there exists some $\mathrm{e}_c \in E_c(N) \subset E(N)$ such that

$$\psi_{e_c} \circ \phi_{\underline{g}} \circ \phi_{\tilde{f}} \left(X_{i(\tilde{f})}\right) = \phi_{\underline{e}(n+1)} \left(X_{i(\underline{e}(n+1))}\right).$$

In particular, $\psi_{e_c} : \phi_{\underline{g}} \circ \phi_{\tilde{f}} \left(X_{i(\tilde{f})}\right) \to \phi_{\underline{g}} \circ \phi_f \circ \phi_{e_1^{(n+1)}}(X_{i(e_1^{(n+1)})})$, where $\tilde{f} \in E(H,G)$ with $t(\tilde{f}) = i(\underline{g})$ and $i(\tilde{f}) = t(e_1^{(n+1)})$ is uniquely determined. Hence, $\xi \in \bigcup_{e_N\in E(N)} \psi_{e_N} \left(X_{i_N(e_N)}\right)$, which leads to a contradiction.

(b) If $i(e_1^{(n+1)}) \in V(G)$, then $e_1^{(n+1)} \in E(G,H)$. In this case we proceed by investigating $e_1^{(n+k)}$ for $k \in \mathbb{N}$ inductively. If $e_1^{(n+k)} \in E(G)$ for all $k \in \mathbb{N}$, then

$$(\phi_{e_1^{(n+1)}} \circ \phi_{\underline{g}} \circ \phi_f)^{-1}(\xi) \in \bigcap_{k \in \mathbb{N}} \phi_{(e_1^{(n+k)},\ldots,e_1^{(n+2)})}(X_{i(e_1^{(n+i)})}).$$

Since $(e_1^{(n+k)},\ldots,e_1^{(n+2)}) \in E^{k-1}(G)$, we have $(\phi_{e_1^{(n+1)}} \circ \phi_{\underline{g}} \circ \phi_f)^{-1}(\xi) \in L(G)$. Hence, $\xi \in L_J(N)$, as claimed.

Finally, suppose that there exists $j \in \mathbb{N}$ such that $e_1^{(n+j)} \in E(H,G)$ and $e_1^{(n+k)} \in E(G)$, for all $1 < k < j$. Note that $\phi_{(e_1^{(n+j)},e_1^{(n+j-1)},\ldots,e_1^{(n+1)})}$ is actually of the form $\phi_{(e_1^{(n+j)},(e_1^{(n+j-1)},\ldots,e_1^{(n+1)}))} = \phi_{\underline{\tilde{g}}} \circ \phi_{\tilde{f}}$, for some $\underline{\tilde{g}} \in E^*(G)$ and $\tilde{f} \in E(H,G)$. But then there is $v \in V(N)$ with $\phi_{\underline{\tilde{g}}} \circ \phi_{\tilde{f}}(X_{i(\tilde{f})}) = X_v^N$. Therefore, there exists $e_N \in E_t(N) \subset E(N)$ with $\psi_{e_N} : X_v^N \to \phi_{\underline{g}} \circ \phi_f \circ \phi_{e_r}(X_v^N)$ such that $\psi_{e_N}(X_v^N) = \phi_{e^{(n+j)}}(X_{i(e^{(n+j)})})$. This leads to a contradiction, since $\xi \in \phi_{e^{(n+j)}}(X_{i(e^{(n+j)})})$, and hence $\xi \in \bigcup_{e_N \in E(N)} \psi_{e_N}\left(X_{i_N(e_N)}\right)$.
This completes the proof.

□

We now extend Proposition 3.2.11 to normal subsystems.

Lemma 3.2.20. *Let S be a finitely generated GDMS, and let N be a normal subsystem of S. We then have*

$$\Delta(N) = \dim_{\mathrm{H}} L_{dyn}(N).$$

Remark: Note that in Lemma 3.2.20 the subsystem N does not necessarily have to be a GDMS.

Proof. Since N is a subsystem of S, we have by Lemma 3.2.3 and Lemma 3.2.5,

$$\begin{aligned} \sum_{\underline{e}_N \in E^*(N)} \left(\operatorname{diam}\left(\psi_{\underline{e}_N}\left(X^N_{i_N(\underline{e}_N)}\right)\right)\right)^s &\leq \sum_{\underline{e} \in E^*(S)} \left(\operatorname{diam}\left(\phi_{\underline{e}}\left(X^N_{i(\underline{e})}\right)\right)\right)^s \\ &\asymp \sum_{\underline{e} \in E^*(S)} \|\phi'_e\|^s. \end{aligned}$$

Since S is a finitely generated GDMS, we can apply Proposition 3.2.11(*3*), which gives that the exponent of convergence of the latter series is equal to $\delta(S) = \dim_{\mathrm{H}} L(S)$. Recall that in Theorem 3.2.16 it was shown that $L_J(N) \subset O_N(L(G))$. This implies that $\dim_{\mathrm{H}} L_J(N) \leq \dim_{\mathrm{H}} O_N(L(G)) = \dim_{\mathrm{H}} L(G) < \dim_{\mathrm{H}} L(S)$. Combining this with the fact that $L(N) = L(S)$, we conclude that $\dim_{\mathrm{H}} L_{dyn}(N) = \dim_{\mathrm{H}} L(S) = \delta(S)$. Therefore, we have that the series

$\sum_{\underline{e}\in E^*(S)} \|\phi'_e\|^{\epsilon+\dim_{\mathrm{H}} L_{dyn}(N)}$ converges, for all $\epsilon > 0$. Hence, the following series also converges:

$$\sum_{\underline{e}\in E^*(N)} \left(\operatorname{diam}\left(\phi_{\underline{e}}\left(X_{\underline{e}}^N\right)\right)\right)^{\epsilon+\dim_{\mathrm{H}} L_{dyn}(N)}.$$

This implies that $\Delta(N) \le \epsilon + \dim_{\mathrm{H}} L_{dyn}(N)$, for all $\epsilon > 0$. Hence, it follows that $\Delta(N) \le \dim_{\mathrm{H}} L_{dyn}(N)$. Combining this with Lemma 3.2.2 (where it was shown that $\Delta(N) \ge \dim_{\mathrm{H}} L_{dyn}(N)$), the proof of the lemma is complete. □

3.2.3 Proof of Main Theorem 2

In this section we prove Main Theorem 2. Recall that the main statements of this theorem are as follows.

> *For each $m \in \mathbb{N}$ and $d, j \in (0, m)$, there exists a GDMS S defined on $\mathbb{R}^m$ such that*
>
> $$\dim_{\mathrm{H}} L_J(S) = j \quad \textit{and} \quad \dim_{\mathrm{H}} L_{ur}(S) = d.$$
>
> *In particular, S can be chosen to be an IFS.*

In order to prove this theorem, we construct a GDMS having the required properties. We first need to describe how to add a map to a GDMS.

Definition 3.2.21. *Let $S = (V(S), E(S), i, t, A(S), \{X_v\}_{v\in V(S)}, \ell, \{\phi_e\}_{e\in E(S)})$ be a GDMS. Let ϕ be a map $\phi : X_w \to X_w$ which is Lipschitz continuous with Lipschitz constant $0 < \ell_\phi < 1$, for some $w \in V(S)$ such that $\phi(X_w) \cap \bigcup_{e\in E(S)} \phi_e\left(X_{i(e)}\right) = \emptyset$. The GDMS*

$$S \cup \{\phi\} := (V(S), E_\phi, i_\phi, t_\phi, A(S_\phi), \{X_v\}_{v\in V(S)}, \max\{\ell, \ell_\phi\}, \{\phi_e\}_{e\in E(S)} \sqcup \{\phi\})$$

obtained by adding ϕ to S is defined by the following:

- *$E_\phi := E(S) \sqcup e_\phi$, where e_ϕ is a new edge from w to w.*
- *The maps $i_\phi, t_\phi : E_\phi \to V(S)$ are given by*

$$i_\phi(\phi) = t_\phi(\phi) := w, (i_\phi)_{|_{E(S)}} := i \;\textit{ and }\; (t_\phi)_{|_{E(S)}} := t.$$

- *The transition matrix associated with $S \cup \{\phi\}$ is given by*

$$A(S_\phi) := \begin{pmatrix} & & \delta_{t(e_1),w} \\ & A(S) & \vdots \\ & & \delta_{t(e_q),w} \\ \delta_{i(e_1),w} & \dots\ \delta_{i(e_q),w} & \delta_{w,w} \end{pmatrix},$$

where δ_{v_1,v_2} denotes the Kronecker delta symbol, and $E(S) = \{e_1, \dots, e_q\}$.

Proof of Main Theorem 2

The idea of the proof is to construct a certain infinitely generated GDMS, which will be called T_∞. The construction will employ a nested inductive argument.

First, let $m \in \mathbb{N}$ and $d, j \in (0,m)$ be fixed. Then fix a strictly decreasing sequence $\left\{\overline{d}_n\right\}_{n\in\mathbb{N}}$ and a strictly increasing sequence $\{\underline{d}_n\}_{n\in\mathbb{N}\cup\{0\}}$ of positive real numbers such that $\lim_{n\to\infty}\overline{d}_n = \lim_{n\to\infty}\underline{d}_n = d$. Now choose a GDMS

$$T_0 := \left(V(T_0), E(T_0), i_{T_0}, t_{T_0}, A(T_0), \{X_v\}_{v\in V(T_0)}, \ell, \{\psi_l\}_{l\in E(T_0)}\right)$$

acting on $\mathbb{R}^m$, such that $\operatorname{card}(E(T_0)) < \infty$ and $\delta(T_0) < \underline{d}_0$. Throughout, let $w \in V(T_0)$ be fixed. Let $S := \left(\{w\}, E(S), i, t, A(S), X_w, \ell, \{\varphi_e\}_{e\in E(S)}\right)$ be a further GDMS acting on $\mathbb{R}^m$, such that

- $E(S)$ is finite;
- $\varphi_e(X_w) \cap \psi_l\left(X_{i(l)}\right) = \emptyset$, for all $e \in E(S)$, $l \in E(T_0)$;
- $\delta(S) = j$.

We now construct a family of GDMSs $\{T_n\}_{n\in\mathbb{N}\cup\{0\}}$ acting on $\mathbb{R}^m$ inductively as follows. The start of the induction is given by T_0. Let us assume that T_{n-1} has been constructed. We now construct T_n using the following finite inductive argument. For this, define $I_n := \{1, 2, \dots, \operatorname{card}(E^n(S))\}$, and fix a bijection $P_n : I_n \to E^n(S)$. Further choose a strictly decreasing sequence $\left\{\overline{d}_{n,k}\right\}_{k\in I_n}$ as well as a strictly increasing sequence $\left\{\underline{d}_{n,k}\right\}_{k\in I_n}$, such that $\overline{d}_{n-1} > \overline{d}_{n,k} > \overline{d}_n$ and $\underline{d}_{n-1} < \underline{d}_{n,k} < \underline{d}_n$, for all $k \in I_n$. Then define

$$\begin{aligned} T_{n-1} &:= T_{n,0} \\ &= \left(V(T_{n,0}), E(T_{n,0}), A(T_{n,0}), i_{T_{n,0}}, t_{T_{n,0}}, \{X_v\}_{v\in V(T_{n,0})}, \ell, \{\psi_l\}_{l\in E(T_{n,0})}\right). \end{aligned}$$

Clearly, for each $k \in I_n$ we can find an injective map $\phi_k : X_w \to X_w$ such that the following conditions are satisfied.

(a) The image $\phi_k(X_w)$ and the images of the maps in $\{\psi_l\}_{l\in E(T_{n,k-1})}$ are disjoint. That is, we have $\phi_k(X_w) \cap \psi_l\left(X_{i(l)}\right) = \emptyset$, for all $l \in E(T_{n,k-1})$.

(b) The map ϕ_k satisfies the bounded distortion condition (BDC) with the same constants as for the maps in $\{\psi_e\}_{e\in E(T_{n,k-1})}$.

(c) The image $\phi_k(X_w)$ and the images of the maps in $\{\varphi_{\underline{e}}\}_{\underline{e}\in E^n(S)}$ are disjoint. That is, we have $\phi_k(X_w) \cap \varphi_{\underline{e}}(X_{i(\underline{e})}) = \emptyset$, for all $\underline{e} \in E^n(S)$.

(d) The image $\phi_k(X_w)$ is 'near' to the imagees of the associated maps $\varphi_{P_n(k)}$. That is, we have $\operatorname{dist}\left(\phi_k(X_w), \varphi_{P_n(k)}(X_{i(P_n(k))})\right) \leq c \cdot \ell^n$.

(e) The distortion of ϕ_k is chosen suitably, such that

$$\underline{d}_{n,k} \leq \delta\left(T_{n,k-1} \cup \{\phi_k\}\right) \leq \overline{d}_{n,k}.$$

Here, ' dist ' denotes the Euclidean distance. We complete the finite inductive step by setting $T_{n,k} := T_{n,k-1} \cup \{\phi_k\}$, and the inductive step by setting $T_n := T_{n,\, c_n}$, where $c_n := \operatorname{card}(I_n)$. Finally, we let T_∞ denote the system obtained from the infinite induction for n tending to infinity. In other words, T_∞ is the system which contains precisely each system in $\{T_n\}_{n\in\mathbb{N}}$. Clearly, by condition (d), we have that the set of accumulation points of $\{\phi_e(X_{i(e)})\}_{e\in E(T_\infty)}$ is equal to $L(S)$. This implies that $\dim_{\mathrm{H}} L_J(T_\infty) = \dim_{\mathrm{H}} L(S)$. Furthermore, by definition, we have that $\delta(S) = j$. Also, since S is finitely generated, we can apply Proposition 3.2.11 (*3*), which gives $\delta(S) = \dim_{\mathrm{H}} L(S)$. Combining these observations, we conclude that $\dim_{\mathrm{H}} L_J(T_\infty) = j$. This gives the first equality stated in the theorem. For the second equality note that, by Lemma 3.2.10 and Proposition 3.2.11 (*3*), we have $\delta(T_\infty) = \lim_{n\to\infty} \delta(T_n)$. Since $\underline{d}_n \leq \delta(T_n) \leq \overline{d}_n$ for each $n \in \mathbb{N}$, we conclude that $d = \lim_{n\to\infty} \underline{d}_n \leq \lim_{n\to\infty} \delta(T_n) \leq \lim_{n\to\infty} \overline{d}_n = d$. Combining these observations, the second equality in the theorem follows. This finishes the proof of Main Theoerm 2.

3.3 Further properties of infinitely generated GDMSs

In this section we continue the investigation of the relationships between the Poincaré exponent, the side exponent, and the Hausdorff dimensions for the various types of limit sets. In particular, we show that for GDMSs the Poincaré exponent coincides with the Hausdorff dimension of the dynamical limit set. Let us begin with an elementary proof of the easiest of all cases, that of affine IFSs. Note that the proof of Lemma 3.3.3 is elementary and does not use the thermodynamic formalism.

Definition 3.3.1. *A GDMS S is called affine if all the maps ϕ_e in its definition are similarities. (Recall, a map $\phi : X \to X$ is called a similarity if* $\operatorname{dist}(\phi(x), \phi(y)) = \|\phi'\| \operatorname{dist}(x, y)$ *for all $x, y \in X$; where* 'dist ' *denotes once more the Euclidean distance.)*

The following lemma gives an analogy to the well known Hutchinson Formula (see for instance [28]).

Lemma 3.3.2. *For each affine IFS S, we have*

$$\sum_{e\in E(S)} \|\phi_e'\|^{\Delta(S)} = 1.$$

Proof. Clearly, we have $\operatorname{diam}(X)^s \cdot \sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s = \sum_{\underline{e}\in E^*(S)} (\operatorname{diam}\phi_{\underline{e}}(X))^s$, and therefore,

$$\begin{aligned}\sum_{\underline{e}\in E^*(S)} (\operatorname{diam}\phi_{\underline{e}}(X))^s &\asymp \sum_{\underline{e}\in E^*(S)} \|\phi'_{\underline{e}}\|^s = \sum_{n\in\mathbb{N}} \sum_{\underline{e}\in E^n(S)} \|\phi'_{\underline{e}}\|^s \\ &= \sum_{n\in\mathbb{N}} \sum_{e_1\in E(S)} \cdots \sum_{e_n\in E(S)} \|\phi'_{e_1}\|^s \cdot \ldots \cdot \|\phi'_{e_n}\|^s \\ &= \sum_{n\in\mathbb{N}} \prod_{k=1}^{n} \sum_{e\in E(S)} \|\phi'_e\|^s = \sum_{n\in\mathbb{N}} \left(\sum_{e\in E(S)} \|\phi'_e\|^s \right)^n .\end{aligned}$$

Obviously, here we have convergence if and only if $\sum_{e\in E(S)} \|\phi'_e\|^s < 1$. This proves the lemma. □

Lemma 3.3.3. *Let S be an affine IFS. We then have*

$$\delta(S) = \Delta(S) = \dim_{\mathrm{H}} L_{dyn}(S).$$

Proof. For $\operatorname{card}(E(S)) < \infty$, the lemma is a special case of Proposition 3.2.11 (*3*). So, let us assume that $\operatorname{card}(E(S)) = \infty$. First, recall that by definition we have $\delta(S) = \dim_{\mathrm{H}} L_{ur}(S)$. Let S_n be as in the definition given in Lemma 3.2.10. We then have $L_{ur}(S_n) \subset L_{ur}(S)$, for all $n \in \mathbb{N}$. Since the Hausdorff dimension is upper semi-continuous (see for example [28, Section 2.2]), this implies that $\delta(S_n) = \dim_{\mathrm{H}} L_{ur}(S_n) \leq \dim_{\mathrm{H}} L_{ur}(S) = \delta(S)$. By Proposition 3.2.11 (*3*), we have that $\delta(S_n) = \Delta(S_n)$. It follows by Lemma 3.3.2, that we have $\sum_{e\in E(S_n)} \|\phi'_e\|^{\delta(S_n)} = 1$, for each $n \in \mathbb{N}$. Furthermore, recall that Lemma 3.2.10 and Proposition 3.2.11 (*3*) give $\lim_{n\to\infty} \delta(S_n) = \delta(S)$. Combining these observations, we have that $\delta(S_n) < \delta(S_{n+1}) < \delta(S)$, for all $n \in \mathbb{N}$. Hence, we have that $\sum_{e\in E(S_n)} \|\phi'_e\|^{\delta(S)} < 1$, for all $n \in \mathbb{N}$. This implies that $\sum_{e\in E(S)} \|\phi'_e\|^{\delta(S)} \leq 1$. Hence, since $\sum_{e\in E(S)} \|\phi'_e\|^{\Delta(S)} = 1$, we conclude that $\delta(S) \geq \Delta(S)$. Combining the latter fact with Proposition 3.2.11 (*2*) (where it was shown that $\delta(S) \leq \Delta(S)$), the proof is complete.

□

Corollary 3.3.4. *For each affine IFS S we have*

$$\delta(S) = \Delta(S) = \Lambda(S) = \dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S) = \dim_{\mathrm{H}} L_{dyn}(S).$$

Proof. This follows immediately from Lemma 3.3.2, Proposition 3.2.11 and the fact that, by definition, $L_{ur}(S) \subset L_r(S) \subset L_{dyn}(S)$. □

Remark: Note that the proof of Lemma 3.3.3 only uses Lemma 3.3.2 and the results of Section 3.2. In contrast, the proof of the following theorem will be more involved.
Note that the result of Lemma 3.3.3 was essentially already obtained by Fernau in [33]. The following theorem extends this result to GDMSs.

Theorem 3.3.5. *Let S be a GDMS. We then have*

$$\delta(S) = \dim_{\mathrm{H}} L_{dyn}(S).$$

Proof. Recall the notation S_n from Lemma 3.2.10 for the finite subsystem of S whose set of edges is equal to $\{1, \dots, n\}$. Let P_S and P_{S_n} refer to the pressure function for the systems S and S_n respectively. Since S_n is finitely generated, the results of Lemma 3.2.1 and Lemma 3.2.6 give that $P_{S_n}(\delta(S_n)) = 0$. Combining this with the fact that the pressure function of a finitely generated GDMS T is always strictly decreasing on $\{t \geq 0 \mid P_T(t) < \infty\}$ (see for example [63, Proposition 4.2.8 (b)]), we have for all $n \in \mathbb{N}$ and for all $\epsilon > 0$ that $P_{S_n}(\delta(S_n) + \epsilon) < 0$. Since $P_S(t) = \sup_{n \in \mathbb{N}} P_{S_n}(t) = \lim_{n\to\infty} P_{S_n}(t)$ ([63, Theorem 2.1.5]), for all $t \in [0, \infty)$, it follows that $P_S(\delta(S) + \epsilon) \leq 0$.
Let $\theta := \inf\{t \in [0, \infty) \mid P_S(t) < \infty\}$. In [63, Proposition 4.2.8 (b)] it was shown that P_S is convex on (θ, ∞). For $t > \delta(S) + \epsilon$, we have that $P_S(t) < 0$, and hence $P_S(t) < P_S(t)/2$. This implies that there exists some $\epsilon_1 > 0$ such that $P_S(t) + \epsilon_1 < P_S(t)/2$. By definition of the pressure function we have that for all $\epsilon_2 > 0$ there exists $n_{\epsilon_2} \in \mathbb{N}$ such that for all $n > n_{\epsilon_2}$ we have

$$\frac{1}{n} \log \sum_{\underline{e} \in E^n(S)} \|\phi'_{\underline{e}}\|^t \leq P_S(t) + \epsilon_2.$$

Hence, by choosing $\epsilon_2 < \epsilon_1$, we have

$$\frac{1}{n} \log \sum_{\underline{e} \in E^n(S)} \|\phi'_{\underline{e}}\|^t \leq P_S(t)/2.$$

This implies that for $D := \max_{v \in V(S)} \operatorname{diam} X_v$ we have, for every $n \in \mathbb{N}$ sufficiently large, that

$$\sum_{\underline{e} \in E^n(S)} \operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})^t \leq D^t \cdot \sum_{\underline{e} \in E^n(S)} \|\phi'_{\underline{e}}\|^t \leq D^t \cdot \exp(nP_S(t)/2).$$

Note that $\bigcup_{\underline{e} \in E^n(S)} \operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})$ is a cover of $L_{dyn}(S)$ consisting of sets of diameter at most $\max_{\underline{e} \in E^n(S)} \operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})$. Since this maximum converges to zero for n tending to infinity, the right hand inequality gives an estimate for the Hausdorff dimension. Indeed, since $\lim_{n\to\infty} D^t \cdot \exp(nP_S(t)/2) = 0$, we have that $t \geq \dim_{\mathrm{H}} L_{dyn}(S)$. Hence, it follows that $\delta(S) \geq \dim_{\mathrm{H}} L_{dyn}(S)$. Combining this observation with the fact that $\delta(S) = \dim_{\mathrm{H}} L_{ur}(S) \leq \dim_{\mathrm{H}} L_{dyn}(S)$ completes the proof.

□

Remark: We would like to mention that [63, Proposition 4.2.8 (b)] is actually contained in a section dealing with conformal GDMSs. However, the proof there does not use conformality. It only uses the bounded distortion property, finite primitivity and the open set condition. Thus, it applies in our situation.

We finish this section by combining the results obtained in Theorem 3.3.5 and Proposition 3.2.11.

Corollary 3.3.6. *For an arbitrary GDMS S we have*

$$\dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S) = \dim_{\mathrm{H}} L_{dyn}(S) = \delta(S) \leq \Delta(S) = \Lambda(S).$$

3.3.1 Proof of Main Theorem 3

We are now ready to prove Main Theorem 3. Recall that the main statements of the theorem are as follows. Assuming that S and N are finitely primitive and satisfy BDC and SSC, the following hold.

1. *Let S be an IFS, then*

$$\dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S) = \dim_{\mathrm{H}} L_{dyn}(S) = \Delta(S) = \Lambda(S).$$

2. *Let S be a GDMS, then*

$$\dim_{\mathrm{H}} L_{ur}(S) = \dim_{\mathrm{H}} L_r(S) = \dim_{\mathrm{H}} L_{dyn}(S) \leq \Delta(S) = \Lambda(S).$$

3. *Let N be a normal subsystem of a finitely generated GDMS, then*

$$\dim_{\mathrm{H}} L_{ur}(N) = \delta(N) \leq \dim_{\mathrm{H}} L_{dyn}(N) = \Delta(N) \leq \Lambda(N).$$

4. *Let S be a pseudo GDMS, then*

$$\dim_{\mathrm{H}} L_{ur}(S) = \delta(S) \leq \dim_{\mathrm{H}} L_{dyn}(S) \leq \Delta(S) \leq \Lambda(S).$$

Proof of Main Theorem 3

The assertion in *1.* has been obtained in Corollary 3.3.4. Similarly, the assertion in *2.* has been obtained in Corollary 3.3.6. Moreover, the assertion in *3.* follows from Lemma 3.2.20, Proposition 3.2.11 (*1*) and the combination of the upper semi-continuity of the Hausdorff dimension with the fact that $L_{ur}(N) \subset L_{dyn}(N)$. Finally, the assertion in *4.* follows from Proposition 3.2.11 (*1*) and the combination of the upper semi-continuity of the Hausdorff dimension with the fact that $L_{ur}(N) \subset L_{dyn}(N)$.
This completes the proof of Main Theorem 3.

3.4 Applications to Kleinian groups of Schottky type

3.4.1 Preliminaries for Kleinian groups of Schottky type

In this section we consider Kleinian groups and investigate the relationships between the Poincaré exponent, the side exponent and Hausdorff dimensions of the various types of limit sets. Our main aim is to transfer the results obtained for GDMSs to this setting. Let us first recall some basic definitions and facts.

Definition 3.4.1. *The ball* $\mathbb{D}^{m+1} := \{x = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1} \mid \|x\| < 1\}$ *, when equipped with the metric given by* $ds^2 = dx^2/(1 - \|x\|^2)$ *, is called the* Poincaré model *of the* $(m+1)$ *-dimensional hyperbolic space. Its boundary will be denoted by* $\mathbb{S}^m$ *.*

Let B be a $(m+1)$-ball in $\mathbb{R}^{m+1}$ whose boundary ∂B is orthogonal to $\mathbb{S}^m$. Then $B \cap \mathbb{D}^{m+1} \neq \emptyset$ is a hyperbolic half-space. From now on we only consider hyperbolic half-spaces of this type. For each such $(m+1)$-dimensional hyperbolic half-space C there is a unique open Euclidean $(m+1)$-ball B_C whose boundary ∂B_C is orthogonal to $\mathbb{S}^m$ and for which we have that $B_C \cap \mathbb{D}^{m+1} = C$. Let $\mathrm{Ext}(B_C)$ and $\mathrm{Int}(B_C)$ refer to the exterior and interior of B_C. For a hyperbolic half-space C, we denote its hyperbolic boundary by ∂C, that is, $\partial C = \partial B_C \cap \mathbb{D}^{m+1}$, and its boundary in $\mathbb{S}^m$ by $\overline{\partial} C$, that is $\overline{\partial} C = \mathbb{S}^m \cap (B_C \cup \partial B_C)$. Let $\mathrm{diam}_E(C)$ denote the Euclidean diameter of B_C. We now give the definition of a Kleinian group of Schottky type.

Definition 3.4.2. *A group* Γ *acting on* $\mathbb{D}^{m+1}$ *will be called a Kleinian group of Schottky type if there exists a non-empty countable set* $\{C_i\}_{i \in I(\Gamma) \subset \mathbb{Z} \setminus \{0\}}$ *of pairwise disjoint* $(m+1)$ *-dimensional hyperbolic half-spaces and a set* $\{\gamma_i\}_{i \in I(\Gamma)}$ *of orientation preserving isometries of* $\mathbb{D}^{m+1}$ *such that the following hold.*

- *For each* C_i *there is a unique open Euclidean* $(m+1)$ *-ball* B_{C_i} *for which we have that* $B_{C_i} \cap \mathbb{D}^{m+1} = C_i$ *.*
- *For every* $i \in I(\Gamma)$ *we have that the map* γ_i *extends to a Lipschitz continuous map* g_i *(with the same Lipschitz constant as* γ_i *) which maps* $\mathrm{Ext}(B_{C_i})$ *onto* $\mathrm{Int}(B_{C_{-i}})$ *. Here, Lipschitz continous is meant with respect to the Euclidean metric.*
- *The group* Γ *is generated by* $\{\gamma_i\}_{i \in I(\Gamma)}$ *.*
- *There exists an* $\epsilon > 0$ *such that the following holds. For each* C_i *there exists finitely many* $C_j \in \{C_k\}_{k \in I(\Gamma)}$ *such that* $\mathrm{diam}_E(C_j) > \mathrm{diam}_E(C_i)$ *. For these* C_j *we then have* $B_{C_j} \cap (1+\epsilon) B_{C_i} = \emptyset$ *. Here,* $(1+\epsilon) B_{C_i}$ *refers to the Euclidean ball with centre equal to the centre of* B_{C_i} *and with diameter* $(1+\epsilon)\,\mathrm{diam}_E(C_i)$ *.*

With this notation let $D := \bigcap_{i \in I(\Gamma)} C_i^c$. Here, C_i^c denotes the complement of C_i in $\mathbb{D}^{m+1}$. Note that it was shown in [60] that D is a Dirichlet fundamental domain constructed with respect to the origin.

In other words, a group Γ will be called a Kleinian group of Schottky type if and only if Γ is a non-elementary free discrete subgroup of the group of orientation preserving isometries of the $(m+1)$-dimensional hyperbolic space. Furthermore, for ease of exposition, we always assume that Γ has only hyperbolic elements. For further details on Kleinian groups of Schottky type we refer to [60].
Let us quickly recall the following types of limit sets for a Kleinian group of Schottky type.

Definition 3.4.3. *Let Γ be a Kleinian group of Schottky type acting on $\mathbb{D}^{m+1}$. We then define the following types of limit sets of Γ (see e.g. [85]).*

- *For an arbitrary $x \in \mathbb{D}^{m+1}$, we have that the set $\bigcup_{\gamma \in \Gamma} \gamma(x)$ has accumulation points exclusively at the boundary $\partial \mathbb{D}^{m+1} = \mathbb{S}^m$ of hyperbolic space. The set $L(\Gamma)$ of these accumulation points is called the limit set of Γ. (Note that $L(\Gamma)$ is independent of the choice of x ([60][p. 22, D.3])).*

- *An element $x \in L(\Gamma)$ is called uniformly radial limit point if for some positive $c = c(x)$ we have that the ray from $0 \in \mathbb{D}^{m+1}$ to x is fully contained in $\bigcup_{\gamma \in \Gamma} b(\gamma(0), c)$. Here, $b(g(0), c)$ refers to the hyperbolic ball centred at $\gamma(0)$ of radius c. The set $L_{ur}(\Gamma)$ of uniformly radial limit points is called the uniformly radial limit set of Γ.*

- *An element $x \in L(\Gamma)$ is called Jørgensen limit point if and only if, for some Dirichlet domain D_z of Γ based at some point $z \in \mathbb{D}^{m+1}$, there exists $\gamma \in \Gamma$ such that $\gamma(D_z)$ contains the hyperbolic geodesic ray from $\gamma(z)$ to x. The set $L_J(\Gamma)$ of Jørgensen limit points is called the Jørgensen limit set of Γ.*

- *The dynamical limit set $L_{dyn}(\Gamma)$ is defined by*

$$L_{dyn}(\Gamma) := L(\Gamma) \setminus L_J(\Gamma).$$

Remark: For Kleinian groups of Schottky type our definitions of limit sets in terms of the coding given in Definition 3.1.1 are equivalent to our definitions given here.
In order to see that the latter geometrically defined uniformly radial limit set corresponds to the version in terms of the coding given in Definition 3.1.1, one proceeds as follows. First note that if $\Gamma_1 \subset \Gamma_2 \subset \ldots \subset \Gamma_k \subset \ldots$ is

an increasing sequence of subgroups of the Kleinian group $\Gamma = \bigcup_k \Gamma_k$, then $L_{ur}(\Gamma) = \bigcup_k L_{ur}(\Gamma_k)$. If Γ is a Kleinian group of Schottky type, then it is freely generated, say by generators $\gamma_1, \gamma_2, \ldots$. Hence, $\Gamma_k := \langle \gamma_i \mid i \leq k \rangle$ gives such an increasing sequence. For each of the finitely generated groups Γ_k one has that each limit point is coded by a unique infinite word (from the alphabet $\mathbb{N}$). Combining this observation with the fact that $L_{ur}(\Gamma_k) = L(\Gamma_k)$, it follows that $L_{ur}(\Gamma)$ can be symbolically discribed as stated in Definition 3.1.1.
In order to see that the Jørgensen limit set of a Kleinian group of Schottky type is contained in the set of limit points which do not have an infinite coding, let $x \in L_J(\Gamma)$ be fixed. By definition, we then have, for some Dirichlet domain D_z of Γ based at some point $z \in \mathbb{D}^{m+1}$, that there exists $\gamma \in \Gamma$ such that $\gamma(D_z)$ contains the hyperbolic geodesic ray from $\gamma(z)$ to x. Hence, the Euclidean distance from x to the set of sides of the Dirichlet domain D_z must be equal to zero. That is, x must be an accumulation point of sides of D_z, since a Kleinian group of Schottky type is by definition a free group generated by loxodromic elements (in particular, a Kleinian group of Schottky type has no parabolic elements). Note that if x is an accumulation point of sides of some Dirichlet domain D_z, then there exists a geodesic ray as above. Hence, $L_J(\Gamma)$ is equal to the Γ-orbit of the accumulation points of sides of D_z. Note that a word $i_1 i_2 \ldots$ can be interpreted as a coding obtained by listing fundamental domains in the Γ-orbit of D_z which are passed when one travels along the ray from 0 to x. In particular, this shows that a Jørgensen limit point x can only be coded by a finite word, since the ray from 0 to x intersects at most finitely many fundamental domains. In order to show that the set of limit points $x \in L(\Gamma)$ which do not have an infinite coding is contained in $L_J(\Gamma)$, we use the contra-positive method and proceed as follows. Assume that $x \notin L_J(\Gamma)$. For each $\gamma \in \Gamma$, we then have that the hyperbolic geodesic ray from $\gamma(z)$ to x is not completely contained in $\gamma(D_z)$. Now, if x would be coded by a finite word, then this would mean that the geodesic ray from 0 to x eventually stays in one of the image fundamental domains, say $g(D_z)$. By convexity of $g(D_z)$, it then follows that the geodesic ray from $g(z)$ to x is fully contained in $g(D_z)$. This is a contradiction, and hence shows that the geodesic ray from 0 to x must pass through infinitely many fundamental domains. This implies that there exists an infinite coding associated to x, and therefore, x is not contained in the set of limit points without infinite coding.
We have now shown that our definition of the Jørgensen limit set in terms of the coding is equivalent to our definition of the Jørgensen limit set given for Kleinian groups.

Let us begin by clarifying the relations between (pseudo) GDMSs and (infinitely generated) Kleinian groups of Schottky type.

Lemma 3.4.4. *The action of a finitely generated Kleinian group of Schottky type* Γ *can be represented by a GDMS* S_Γ *. In particular, we have that*

$$L(\Gamma) = L(S_\Gamma).$$

This fact is well known (see for example [63][Example 5.1.5]). However, for the sake of completeness, we recall the construction here.

Proof. Let Γ be a finitely generated Kleinian group of Schottky type. Without loss of generality we can assume that $I(\Gamma)$ from Definition 3.4.2 is equal to $\{1, \ldots, n\} \cup \{-1, \ldots, -n\}$. Let $V(S) := I(\Gamma)$, and for each $v \in V(S)$ set $X_v := B_{C_v}$. Let $E(S) := \{(v,w) \in V(S) \times V(S) \mid v \neq w\}$ be defined such that each (v,w) represents an edge from v to w. Clearly, the maps $i, t : E(S) \to V(S)$ are defined via $i(v,w) := v$ and $t(v,w) := w$. The map $\phi_{(v,w)} := (g_{-w})_{|B_{C_v}} : B_{C_v} \to B_{C_w}$ is well defined, for every $(v,w) \in E(S)$. Also, we define the incidence matrix $A(S)$ by

$$A_{(v,w),(\tilde{v},\tilde{w})} := \begin{cases} 1 & \text{if} \quad w = \tilde{v} \\ 0 & \text{if} \quad w \neq \tilde{v} \end{cases}$$

Therefore, the system $S_\Gamma := (V(S), E(S), i, t, A(S), \{X_v\}_{v \in V(S)}, \ell, \{\phi_e\}_{e \in E(S)})$ satisfies all conditions of a GDMS, except that the g_e might not be contractions and hence ℓ need not be smaller than 1. But since the Euclidean diameters of the sets $\phi_{\underline{e}}(X_{i(\underline{e})})$ tend to zero uniformly with respect to the length of $\underline{e} \in E^*(S)$, we have, after passing to a sufficiently high iterate of S_Γ, that the maps are uniformly contracting. Therefore, in the following we can assume without loss of generality that the maps of S_Γ are uniformly contracting.
Note that S_Γ satisfies SSC, since the C_i in Definition 3.4.2 were pairwise disjoint. Furthermore, S_Γ is finitely primitive by construction if and only if Γ is non-elementary. That is, Γ is generated by more than one element. The fact that S_Γ satisfies BDC follows directly from Koebe's distortion theorem ([19], [63][Theorem 4.1.1]) and the fact that isometries of $\mathbb{D}^{m+1}$ are conformal on $\mathbb{S}^m$ (see [60][B7, A4]), which implies that they map spheres to spheres. Hence, all the results for GDMSs obtained above are applicable.
Finally, it is well known that for the so obtained GDMS S_Γ we have $L(\Gamma) = L(S_\Gamma)$ ([63][Theorem 5.1.6]). $\square$

Lemma 3.4.5. *An infinitely generated Kleinian group* Γ *of Schottky type cannot be represented by a GDMS. However, there is always a finite index subgroup* $\tilde{\Gamma}$ *of* Γ *which can be represented by a pseudo GDMS* $S_{\tilde{\Gamma}}$ *.*

Since we are only interested in the geometry of limit sets and since limit sets are invariant under taking finite index subgroups, we shall always assume that Γ coincides with $\tilde{\Gamma}$.

Proof. Let Γ be an infinitely generated Kleinian group of Schottky type. Since Γ is of Schottky type, there is a pairing of the C_i as in Definition 3.4.2. Without loss of generality we can always assume that each generator γ_i is already a contraction. Recall that γ_i maps the exterior of $\mathrm{Ext}(C_i)$ into the interior of $\mathrm{Int}(C_{-i})$, for each generator γ_i of Γ. Likewise, γ_i is an expanding map which sends the interior of C_i to the exterior of C_{-i}. Recall that the maps ϕ_e of a GDMS are contractions by definition. In order to represent the action of Γ by a GDMS, we therefore have to restrict the action of γ_i to the complement C_i^c of C_i. Clearly, we have that $\gamma_i(C_i^c \cap C_{-i}^c) \cap (C_i^c \cap C_{-i}^c) = \emptyset$. Hence, for each generator γ_i of Γ there are at least two distinct compact sets $X_i \subset B_{C_i}^c$ and $X_{-i} \subset B_{C_{-i}}^c$ in the definition of the corresponding GDMS. On these compact sets the contractions ϕ_e will be defined by $\phi_{(i,j)} := (g_{-j})_{|X_i} : X_i \to X_j$. Using these observations, we now have to associate to each generator $\gamma_i \in \Gamma$ two distinct compact subsets X_i and X_{-i}, and for each pair of two distinct generators γ_i and γ_j there are two distinct compact subsets X_i and X_j as well. Hence, if Γ is generated by n elements, then the corresponding GDMS has at least $2n$ vertices. Finally, since an infinitely generated Kleinian group can be regarded as a limit of its finite subgroups, it follows that an infinitely generated Kleinan group of Schottky type cannot be represented by a GDMS. Finally, note that if Γ is infinitely generated, then there are infinitely many sets X_i and hence the set of vertices is infinite. Clearly, this situation is precisely mimiced by a pseudo GDMS. This finishes the proof of Lemma 3.4.5. $\square$

Remark: The proof of Lemma 3.4.5 clearly shows that if one wants to represent an infinitely generated Kleinian group of Schottky type Γ by means of a pseudo GDMS S_Γ, then one has to consider pseudo GDMSs. Note that for finitely as well as for infinitely generated Γ, we have by construction of S_Γ that $L_{dyn}(\Gamma) = L_{dyn}(S_\Gamma)$.

3.4.2 Results for Kleinian groups of Schottky type

Before we apply some of our results for (pseudo) GDMSs obtained above, let us first recall some facts which will be crucial for these applications.

Definition 3.4.6. *Let Γ be a Kleinian group acting on $\mathbb{D}^{m+1}$, and let $s \in \mathbb{R}$. The series*

$$\sum_{\gamma \in \Gamma} \mathrm{e}^{-sd(0,\gamma(0))}$$

will be called the Poincaré series *associated with Γ. The exponent of convergence of this series will be denoted by $\delta(\Gamma)$ and referred to as the* Poincaré exponent associated to the Kleinian group Γ.

REMARK: Recall that we already defined the Poincaré exponent for a GDMS S $\delta(S) := \dim_{\mathrm{H}} L_{ur}(S)$. The following theorem shows that the definition of $\delta(S)$ for GDMSs is compatible with the definition of $\delta(\Gamma)$ for a Kleinian group Γ, as given above.

Theorem 3.4.7 (Bishop, Jones)**.** *For each non-elementary Kleinian group Γ the Poincaré exponent $\delta(\Gamma)$ coincides with the Hausdorff dimension of the uniformly radial limit set.*

For a proof see [14]. A more detailed proof can be found in [86].

We define the *shadow map* $\pi : \mathbb{D}^{m+1} \to \mathbb{S}^m$ by $\pi(x) := \frac{x}{\|x\|}$, where $\|x\|$ denotes the Euclidean norm in $\mathbb{R}^{m+1}$. We let $\operatorname{diam}_0 \pi(U)$ denote the spherical diameter of $\pi(U)$ for a set $U \subset \mathbb{D}^{m+1}$. Moreover, for a non-elementary Kleinian group Γ without elliptic elements, we define the *side* S_γ associated to $\gamma \in \Gamma$ as follows. For $\gamma = \mathbf{id}$, we set $S_\Gamma := \emptyset$. For $\gamma \in \Gamma \setminus \mathbf{id}$, let D denote the Dirichlet fundamental domain of Γ from Definition 3.4.2, and define S_γ to be the unique $(m+1)$-spherical component of $\partial(\gamma(D))$ such that $S_\gamma \cap [0, \gamma(0)] \neq \emptyset$. Here, $[0, \gamma(0)]$ denotes the geodesic segment from 0 to $\gamma(0)$.
Let us now define the side series.

Definition 3.4.8. *Let Γ be a Kleinian group. The series*

$$\sum_{\gamma \in \Gamma} (\operatorname{diam}_0 \pi(S_\gamma))^s$$

will be called the side series *associated with Γ. Its exponent of convergence will be denoted by $\Delta(\Gamma)$ and referred to as the* side exponent *of Γ.*

Lemma 3.4.9. *Let Γ be a Kleinian group of Schottky type. Then we have that*

$$\Delta(\Gamma) \geq \dim_{\mathrm{H}} L_{dyn}(\Gamma)$$

.

Proof. Similar as in the proof of Lemma 3.2.2, note that for every $\epsilon > 0$ the set $\bigcup_{\gamma \in \Gamma} \pi(S_\gamma)$ contains a covering $\{U_i\}$ of $L_{dyn}(\Gamma)$ with sets of diameter less than ϵ. Therefore, for $s < \dim_{\mathrm{H}} L_{dyn}(\Gamma)$ we have

$$\inf_{\{U_i\}} \sum_{i \in \mathbb{N}} (\operatorname{diam}_0 U_i)^s \quad \leq \quad \sum_{\gamma \in \Gamma} (\operatorname{diam}_0 \pi(S_\gamma))^s ,$$

where the infimum is taken over all coverings $\{U_i\}$ of $L_{dyn}(\Gamma)$ of diameter less than ϵ. By letting ϵ tend to zero, this implies that $s \leq \Delta(\Gamma)$, and hence we obtain that $\dim_{\mathrm{H}} L_{dyn}(\Gamma) \leq \Delta(\Gamma)$. $\square$

Lemma 3.4.10. *Let* Γ *be a finitely generated Kleinian group, then we have*

$$\delta(\Gamma) = \Delta(\Gamma).$$

Proof. Let $\Gamma = \langle \gamma_1, \dots \gamma_n \rangle$, and define $d_{\max} := \max_{i \in \{1,\dots,n\}} d(0, \gamma_i(0))$. By the triangle inequality, we have for an irreducible word of the form $\gamma \circ \gamma_i$ that $d(0, \gamma(0)) \leq d(0, \gamma \circ \gamma_i(0)) \leq d(0, \gamma(0)) + d_{\max}$, for all $i \in \{1, \dots, n\}$. It is well known that $\operatorname{diam}_0 \pi(S_\gamma) \asymp e^{-d(0,S_\gamma)}$ (see [66]). Note that we have $d(0,\gamma(0)) \leq d(0, S_{(\gamma\circ\gamma_i)}) \leq d(0, (\gamma \circ \gamma_i)(0))$, for all $i \in \{1, \dots, n\}$. Therefore, for any $\gamma \in \Gamma \setminus \{\mathbf{id}\}$ we have

$$\operatorname{diam}_0 \pi(S_\gamma) \asymp e^{-d(0,\gamma(0))}.$$

From this the lemma follows immediately. □

Implications of our results

Let us now discuss the relation between the different types of limit sets of a Kleinian group of Schottky type and its associated pseudo GDMS.

Lemma 3.4.11. *Let* Γ *be a finitely generated Kleinian group of Schottky type, then we have*

$$\Delta(\Gamma) = \Delta(S_\Gamma).$$

Proof. We know from Lemma 3.4.4 that $L(S_\Gamma) = L(\Gamma)$. Furthermore, since S_Γ is finitely generated we have by Proposition 3.2.11(*3*) that $\dim_{\mathrm{H}} L(S_\Gamma) = \Delta(S_\Gamma)$. Combining this with Theorem 3.4.7, Lemma 3.4.10 and the fact that $L_{ur}(\Gamma) = L(\Gamma)$ completes the proof. □

Lemma 3.4.12. *Let* Γ *be an infinitely generated Kleinian group of Schottky type, then we have*

$$\Delta(\Gamma) \leq \Delta(S_\Gamma).$$

Proof. Note that for $\gamma \in \Gamma$ we have that $S_\gamma = \partial B_{C_i}$, for some $i \in I(\Gamma)$. Hence, there exists a word $\underline{e} \in E^*(S)$ such that $B_{C_i} = \phi_{\underline{e}}(X_{i(\underline{e})})$. Clearly, $\operatorname{diam}_0(\pi(S_\gamma)) \asymp \operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})$. Therefore, it follows that $\sum_{\gamma\in\Gamma} (\operatorname{diam}_0 \pi(S_\gamma))^s \ll \sum_{\underline{e}\in E^*(S)} \operatorname{diam} \phi_{\underline{e}}(X_{i(\underline{e})})^s$ and hence $\Delta(\Gamma) \leq \Delta(S_\Gamma)$. □

Corollary 3.4.13. *Let* Γ *be a Kleinian group of Schottky type, then we have*

$$L(\Gamma) = L(S_\Gamma).$$

Proof. Clearly, since $L_{dyn}(\Gamma) = L_{dyn}(S_\Gamma)$ holds in general, we always have that $L(\Gamma) = L(S_\Gamma)$. □

Lemma 3.4.14. *Let Γ, Γ_G and Γ_H be finitely generated Kleinian groups of Schottky type such that $\Gamma = \Gamma_G * \Gamma_H$ is the free product of Γ_G and Γ_H (see [70], [60]). Let $\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)$ denote the normal subgroup of Γ generated by Γ_H By setting $G := S_{\Gamma_G}$ and $H := S_{\Gamma_H}$ the GDMSs assocoiated to Γ_G and Γ_H respectivly, we then have:*

1. $\Delta(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)) = \dim_{\mathrm{H}} L_{dyn}(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma))$ *;*
2. $\delta(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)) = \delta(N_{G,H}(S_\Gamma))$ *;*
3. $L_J(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)) = L_J(N_{G,H}(S_\Gamma))$ *.*

Here, $N_{G,H}(S_\Gamma)$ denotes the normal subsystem of S_Γ defined in Definition 3.2.15.

Proof. Recall that $L_{dyn}(S_{\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)}) = L_{dyn}(N_{G,H}(S_\Gamma))$. By Lemma 3.2.20, we have $\Delta(N_{G,H}(S_\Gamma)) = \dim_{\mathrm{H}} L_{dyn}(N_{G,H}(S_\Gamma))$. Combining these observations with Lemma 3.4.12 and Lemma 3.4.9 completes the proof of the assertion in *1*.
In order to prove the assertion in *2*, we introduce the following notation. Let Γ_{H_0} and Γ_{G_0} denote the set of generators of Γ_H and Γ_G respectively. For each $n \in \mathbb{N}$, let S_n denote the GDMS associated with the finitely generated Kleinian group of Schottky type

$$\mathcal{N}_n := \langle \beta,\ \alpha_{i_1} \circ \ldots \circ \alpha_{i_k} \circ \beta \circ \alpha_{i_k}^{-1} \circ \ldots \circ \alpha_{i_1}^{-1} \mid k \in \{1, \ldots, n\},\ \begin{matrix} \alpha_{i_j} \in \Gamma_{G_0} \\ \beta \in \Gamma_{H_0} \end{matrix} \rangle.$$

By Lemma 3.4.11, we have that $L(S_n) = L(\mathcal{N}_n)$, for all $n \in \mathbb{N}$. Combining this with the definition of $L_{ur}(S_\Gamma)$, we obtain

$$L_{ur}(N_{G,H}(S_\Gamma)) = \bigcup_{n \in \mathbb{N}} L(S_n) = \bigcup_{n \in \mathbb{N}} L(\mathcal{N}_n) = L_{ur}(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)).$$

By combining this observation with the result of Bishop and Jones (Lemma 3.4.7) and the definition of $\delta(S_\Gamma)$, it follows that $\delta(N_{G,H}(S_\Gamma)) = \delta(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma))$. This completes the proof of the assertion in *2*.
Finally, note that the assertion in *3* follows immediately from the fact that $L_{dyn}(S_{\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma)}) = L_{dyn}(N_{G,H}(S_\Gamma))$. □

Let us finish this section by applying well known results of Kleinian groups to pseudo GDMSs.

Lemma 3.4.15. *There exists a pseudo generated GDMS S for which*

$$\dim_{\mathrm{H}} L_{ur}(S) < \dim_{\mathrm{H}} L_{dyn}(S).$$

Proof. Let us recall a result of Brooks in [16], who showed that if Γ is a finitely generated Kleinian group of Schottky type (with $\delta(\Gamma) > m/2$) and if $\mathcal{N}$ is a normal subgroup of Γ, then the first eigenvalues of the Laplacian on $\mathbb{D}^{m+1}/\Gamma$ and on $\mathbb{D}^{m+1}/\mathcal{N}$ do not coincide if and only if $\Gamma/\mathcal{N}$ is non-amenable. Furthermore, it is well known result that if $\delta(\Gamma) > m/2$, then the first eigenvalue of the Laplacian on $\mathbb{D}^{m+1}/\Gamma$ is equal to $\delta(\Gamma)(m-\delta(\Gamma))$ (see e.g. [68]). Combining these two facts one immediately verifies that $\delta(\mathcal{N}) < \delta(G)$. This implies that if $\mathcal{N}$ is a normal subgroup of Γ such that $\delta(\mathcal{N}) > m/2$, then we have that $\delta(\mathcal{N}) < \delta(\Gamma)$ if and only if $\Gamma/\mathcal{N}$ is non-amenable. (Since a free group with at least 2 generators is always non-amenable, all we require is that Γ_G is generated by at least two generators.) By combining Corollary 3.2.18, Lemma 3.4.11 and Lemma 3.4.14, we have that $\dim_{\mathrm{H}} L(\Gamma_G * \Gamma_H) > \dim_{\mathrm{H}} L(\Gamma_G)$ (see for example [70]) and $\dim_{\mathrm{H}} L(\Gamma_G) = \dim_{\mathrm{H}} L_J(\mathcal{N})$. This implies $\dim_{\mathrm{H}} L(\Gamma) = \dim_{\mathrm{H}} L_{dyn}(\mathcal{N}_{\Gamma_G,\Gamma_H}(\Gamma))$.
Combining this with Lemma 3.4.14, we have that $\dim_{\mathrm{H}} L(\Gamma) = \dim_{\mathrm{H}} L_{dyn}(N_{G,H}(S_\Gamma))$. Hence, by Lemma 3.2.20, we have that $\dim_{\mathrm{H}} L_{dyn}(N_{G,H}(S_\Gamma)) = \Delta(N_{G,H}(S_\Gamma))$. Applying Lemma 3.4.14, we hence have that if Γ_G is generated by at least two elements, then $\delta(N_{G,H}(S_\Gamma)) < \Delta(N_{G,H}(S_\Gamma))$. This shows that there exists a pseudo generated GDMS S for which $\dim_{\mathrm{H}} L_{ur}(S) < \dim_{\mathrm{H}} L_{dyn}(S)$. □

3.4.3 Proof of Main Theorem 4

We are now ready to prove Main Theorem 4. Recall that the main statements of the theorem are as follows.

> *For every $m \in \mathbb{N}$ and every $d, j \in (0, m)$, there exists a Kleinian group $\Gamma \subset \mathrm{Iso}(\mathbb{D}^{m+1})$ such that*
>
> $$\dim_{\mathrm{H}} L_{ur}(\Gamma) \leq d \quad \text{and} \quad \dim_{\mathrm{H}} L_J(S) = j.$$
>
> *In particular, Γ can be chosen to be of Schottky type.*

Let $m \in \mathbb{N}$ and $j, d \in (0, m)$ be fixed. We employ the same nested inductive argument as used in the proof of Main Theorem 2. The idea is to construct a infinitely generated Kleinian group of Schottky type Γ_∞.

Fix a strictly increasing sequence $\{\underline{d}_n\}_{n\in\mathbb{N}\cup\{0\}}$ of positive real numbers such that $\lim_{n\to\infty} \underline{d}_n = d$. Now choose a finitely generated Kleinian group of Schottky type $\Gamma_0 := \langle \gamma_1, \ldots, \gamma_l \rangle$ acting on $\mathbb{D}^{m+1}$ such that $\delta(\Gamma_0) < \underline{d}_0$. For $I(\Gamma_0) := \{1, \ldots, l\} \cup \{-1, \ldots, -l\}$ let $\{C_i\}_{i\in I(\Gamma_0)}$ denote the hyperbolic half-spaces associated with the generators of Γ_0 (as in Definition 3.4.2). We then have that $D = \bigcap_{i\in I(\Gamma_0)} C_i^c$ is a Dirichlet domain for Γ_0. Recall that $\overline{\partial} D$

denotes the intersection of $\mathbb{S}^m$ with the closure $\overline{D}$ of D. Choose a closed m-dimensional ball $X \subset \mathbb{S}^m$ which is contained in an open subset of $\overline{\partial} D$. Moreover, choose a GDMS

$$S := \Big(\{1\}, E(S), i, t, A(S), X, \ell, \{\varphi_e\}_{e\in E(S)}\Big)$$

acting on X such that $E(S)$ is finite, S satisfies SSC, and $\delta(S) = j$.
We now construct a family of Kleinian groups of Schottky type $\{\Gamma_n\}_{n\in\mathbb{N}\cup\{0\}}$ acting on $\mathbb{D}^{m+1}$ inductively. The start of the induction is given by Γ_0. Let us assume that Γ_{n-1} has been constructed. In order to construct Γ_n, define $I_n := \{1, 2, \ldots, \mathrm{card}(E^n(S))\}$, and fix a bijection $\pi : I_n \to E^n(S)$. Furthermore, choose a strictly increasing sequence $\left\{\underline{d}_{n,k}\right\}_{k\in I_n}$ with the property that $\underline{d}_{n-1} < \underline{d}_{n,k} < \underline{d}_n$, for all $k \in I_n$. Then define

$$\Gamma_{n,0} \quad := \Gamma_{n-1}.$$

This starts a finite induction as follows. Note that for each $k \in I_n$ we can find an isometry $\gamma \in \mathrm{Iso}(\mathbb{D}^{m+1})$ such that the following conditions are satisfied.

(0) There are hyperbolic half-spaces C_γ and $C_{-\gamma}$ such that $\gamma(C_\gamma) = C^c_{-\gamma}$, $\gamma(C^c_\gamma) = C_{-\gamma}$, and $\gamma(\partial C_\gamma) = \partial C_{-\gamma}$.

(1) The half-spaces $C_\gamma, C_{-\gamma}$ and $\{C_i\}_{i\in I(\Gamma_{n,k-1})}$ are pairwise disjoint.

(2) The free product $\Gamma_{n,k-1} * \langle\gamma\rangle$ is a Kleinian group of Schottky type.

(3) The spherical boundaries $\overline{\partial} C_\gamma$ and $\overline{\partial} C_{-\gamma}$ and the images of the maps in $\{\varphi_{\underline{e}}\}_{\underline{e}\in E^n(S)}$ are disjoint. That is, we have $\overline{\partial} C_{\pm\gamma} \cap \varphi_{\underline{e}}(X_{i(\underline{e})}) = \emptyset$, for all $\underline{e} \in E^n(S)$.

(4) We have that $\mathrm{dist}\left(B_{C_{\pm\gamma}}, \varphi_{\pi(k)}(X_{i(\pi(k))})\right) \leq c \cdot \ell^n$.

(5) The isometry γ is chosen such that

$$\delta\left(\Gamma_{n,k-1} * \langle\gamma\rangle\right) \leq \underline{d}_{n,k}.$$

In order to see how Condition (5) can be satisfied, we refer to [70] where this has been discussed in great detail. In essence, the idea is to choose a $\tilde{\gamma}$ satisfying the conditions (0) to (4), and then one sets $\gamma = h^n \circ \tilde{\gamma} \circ (h^{-1})^n$, for a suitable isometry $h \in \mathrm{Iso}(D^{m+1})$, and for $n \in \mathbb{N}$ sufficiently large.
We complete the finite inductive step by setting $\Gamma_{n,k} := \Gamma_{n,k-1} \cup \{\gamma\}$, as well as the inductive step by setting $\Gamma_n := \Gamma_{n,\, c_n}$, where $c_n := \mathrm{card}(I_n)$. Finally, we define $\Gamma_\infty := \bigcup_{n\in\mathbb{N}} \Gamma_n$.

Clearly, by construction (in particular condition (4) above), we have that the set of accumulation points of the set $\{\partial C_i : i \in I(\Gamma_\infty)\}$ of sides of $\partial D(\Gamma_\infty)$ in $\mathbb{S}^m$ is equal to $L(S)$. This implies that $\dim_{\mathrm{H}} L_J(\Gamma_\infty) = \dim_{\mathrm{H}} L(S)$. Furthermore, by choice of S, we have that $\delta(S) = j$. Also, since S is finitely generated, we can apply Proposition 3.2.11 (*3*) to obtain $\delta(S) = \dim_{\mathrm{H}} L(S)$. Combining these observations, we conclude that $\dim_{\mathrm{H}} L_J(S) = j$. This gives the equality stated in the theorem. For the inequality $\delta(\Gamma_\infty) \leq d$, note that by Lemma 3.4.7 and the definition of $L_{ur}(\Gamma_\infty)$, we have that $\delta(\Gamma_\infty) = \lim_{n\to\infty} \delta(\Gamma_n)$. Since $\delta(\Gamma_n) \leq \underline{d}_n$, for each $n \in \mathbb{N}$, we conclude that $\lim_{n\to\infty} \delta(\Gamma_n) \leq \lim_{n\to\infty} \underline{d}_n = d$. Combining these observations, the inequality in the theorem follows.

This completes the proof of Main Theorem 4.

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