

RECOMBINATION AND PARADOX

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1. Introduction

Consider a principle of modal plenitude according to which whatever could exist, does exist. Since an oak tree could have grown from an acorn from which no tree ever results, something exists which could have grown into an oak tree from the given acorn. Two philosophers who have recently developed the thought in different directions are David Lewis and Timothy Williamson. According to Lewis (1986), when we let our quantifiers range unrestrictedly over all there is, we quantify over merely possible objects as well as actual ones. If it's merely possible for an oak tree to grow from the given acorn, then some concrete universe, which is causally and spatiotemporally isolated from the actual world, contains, as a part, an oak tree originated from (a counterpart of) the acorn. Williamson (2013), in contrast, argues for all modal closures of the Barcan Formula:

$$(BF) \quad \Diamond \exists x A \rightarrow \exists x \Diamond A.^1$$

Since it's possible for something to be an oak tree originated from a given acorn, something is possibly an oak tree originated from the acorn. One difference between Lewis and Williamson is that the latter takes exception to the further claim that a possible oak tree need actually be an oak tree; a merely possible oak tree is best conceived as a non-concrete object, which could have nevertheless been a concrete oak tree. Despite important differences, Lewis and Williamson stand united by the thought that when appropriately unrestricted, our quantifiers range over a vast infinity of possibly concrete objects. In general, if there could have been at least κ concrete objects, then there are κ possible concrete objects. This profligate ontology may strike one as metaphysically extravagant, but another serious problem is that they provide an inhospitable environment for recombination principles.

1. A modal closure of a formula A is one that results from prefixing A with a finite sequence (which may be the null sequence) of universal quantifiers and necessity operators.

There is, first, an apparent tension between certain principles of recombination and the iterative conception of set, which motivates most of the axioms of Zermelo-Fraenkel set theory with choice and urelements (ZFCU). On the iterative conception, sets are formed in cumulative stages of an iterative hierarchy, which begins with an initial domain of *nonsets*, otherwise known as “urelements”. But while the iterative conception invites us to form a set of urelements at the very first stage of the iteration, modal plenitude yields far more urelements than can be collected into a set. Inspired by Nolan (1996), Sider (2009) and Menzel (2014) make the point by appeal to a consequence of unrestricted recombination:

(RC₁) For each cardinal number κ , it is possible that there are at least κ urelements.²

This principle enjoys general appeal, even if neither Lewis nor Williamson are officially committed to it. For to deny it is to posit an arbitrary and unmotivated constrain on the cardinality urelements could have. Now, if whatever could exist does exist, then, given (RC₁):

(U_∞) For each cardinal number κ , there are at least κ possible urelements.³

But it turns out that (U_∞) precludes the existence of a set of all urelements over ZFCU.⁴ To the extent to which such a set may be

2. The original principle of recombination comes from Lewis (1986) and states that given some objects, there is a world which contains any number of duplicates of any of them. Lewis eventually qualifies the principle to include the clause: “size and shape permitting”. Nolan (1996) argues that Lewis’ restriction is unmotivated and invites him to consider an unqualified form of recombination from which (RC₁) would be a consequence.

3. The argument involves auxiliary assumptions, e.g., no merely *possible* urelement is a set. Sider (2009) offers a careful regimentation of all the premises.

4. If A is the set of all urelements, then the cardinality of A will, in ZFCU, be given by some cardinal number κ . By (U_∞), there is a set of urelements of cardinality κ^+ , i.e., the successor of κ , which will have strictly larger cardinality than A . So, A cannot be a set of all urelements after all. More generally, no set of urelements contains all urelements, on pain of contradiction.

required by the iterative conception, we have a problem for modal plenitude.

There is, in addition, a conflict between certain principles of recombination and the Cantorian doctrine of the absolutely infinite. Georg Cantor, the founder of set theory, relied on a distinction between “transfinite multiplicities,” which are mathematically tractable and form sets, and “absolutely infinite multiplicities,” which are mathematically intractable and exemplify an unsurpassable magnitude, an “absolute maximum” (Cantor, 1932, p. 405):

The transfinite, with its wealth of arrangements and forms, points with necessity to an absolute, or the ‘true infinite’, whose magnitude is not subject to any increase or reduction, and for this reason it must be quantitatively conceived as an absolute maximum.

While Cantor’s distinction is mathematically fruitful and admittedly suggestive, it bears clarification.

One way to make the Cantorian thought precise is due to John von Neumann, who transformed it into a *maximality principle* according to which that a multiplicity forms a set if and only if it is not in one-one correspondence with all the objects there are.⁵

The problem, as developed by Hawthorne and Uzquiano (2011), concerns a recombination principle for angels conceived as objects capable of co-location in point-sized regions.⁶ In general:

(RC₂) For each cardinal κ , it is possible that there are at least κ angels.

5. This is Axiom IV.2 in an axiomatization developed in von Neumann (1925) and von Neumann (1928).

6. Without theological overtones, the problem can be formulated in terms of a recombination principle for objects capable of co-location, e.g., elementary particles like bosons. This requirement is meant to bypass Lewis’s qualification of the principle of recombination “size and shape permitting”: no matter how large κ may be, the existence of κ co-located objects will not pose stringent demands on the size or shape of spacetime.

But if whatever could exist does exist, then, given (RC_2) , we have:

(A_∞) For each cardinal number κ , there are at least κ possible angels.

In Cantorian jargon, the possible angels form an “absolutely infinite multiplicity” exemplifying an absolute quantitative maximum. But given reasonable assumptions, the operation of mereological fusion generates strictly more fusions of possible angels than there are possible angels.⁷ Hence:

$(A_{\infty+})$ There are strictly more fusions of possible angels than there are possible angels.

This means that there are at least two absolutely infinite magnitudes providing a clear-cut counterexample to von Neumann’s maximality interpretation of the Cantorian doctrine.

One may be tempted to dismiss the difficulty as a byproduct of the ontological profligacy of modal plenitude: when combined with it, attractive recombination principles commit one to a vast plethora of merely possible concrete objects, one which would be objectionable from the standpoint of more sober accounts of modal discourse.⁸ Consider, for example, an account of modal discourse governed by an abstractionist interpretation of the schema (\diamond) on which possible worlds are conceived as abstract objects of one sort or another:⁹

(\diamond) It is possible that A if, and only if, the proposition that A is true at some possible world w .

7. In Hawthorne and Uzquiano (2011), this claim is derived from the weaker assumption that no one object fuses two different multiplicities of possible angels and the principle of unrestricted composition according to which no matter what some objects may be, there is a fusion of them.

8. It is little help to deny that merely possible objects exist and then posit individuals essences that serve as surrogates for them. See the papers collected in Plantinga (2003).

9. I borrow the label ‘abstractionist’ from van Inwagen (1986). They stand in contrast with the modal realist interpretation of (\diamond) .

Possible worlds have been conceived as states of affairs (Plantinga (1976)), maximal consistent sets of propositions (Adams (1974) and Adams (1974)), total propositions, (Fine (1977)), and maximal ways the universe might have been (Kripke (1980) and Stalnaker (1976)). Whatever the choice, one may dismiss talk of merely possible objects as a colorful *façon de parler*: if no oak tree ever grows from an acorn, then nothing exists which could have developed into an oak tree from the given acorn. There is, after all, a difference between “it is possible that some oak tree might have developed from a given acorn”—in which the quantifier lies within the scope of the modal operator—and “some object could have developed from a given acorn”—which does, in fact, commit one to the existence of a possible oak tree. Now, without the extravagant commitment to extraordinarily large infinities of possibly concrete objects, they may hope to avoid the tension with the Cantorian doctrine of the absolutely infinite.

This would be a mistake. Lewis and Williamson are not alone in their predicament. We set out to argue that a parallel challenge is quite independent from modal plenitude. In particular, we’ll suggest that, given reasonable assumptions, a recombination principle in the spirit of (RC_1) and (RC_2) entails the existence of at least two absolutely infinite magnitudes. The challenge from recombination poses a distinctive difficulty for a large family of accounts of modal discourse, one that is importantly different from more traditional threats of paradox exemplified by modern descendants of Russell’s paradox of propositions and Kaplan’s paradox.¹⁰ They each require a different set of resources, and while there is little agreement as to how best to respond to them, common reactions to them are ineffectual when it comes to the challenge from recombination.

2. Methodological preliminaries

We introduce the framework for the problem from recombination.

10. Russell’s paradox comes from appendix B of Russell (1903), and Kaplan’s paradox is developed in Kaplan (1995).

2.1 ZFCU

The axioms of Zermelo-Fraenkel set theory with choice (ZFC) provide a standard axiomatization of *pure* set theory, where one takes the variables to be tacitly restricted to sets. In the language of ZFCU, we lift the restriction and introduce an additional primitive predicate, $Set(x)$, read: “ x is a set”.¹¹ This predicate is governed by the axiom:

$$(1) \quad \forall x \forall y (y \in x \rightarrow Set(x))$$

The axioms of ZFCU require minor modifications on the axioms of ZFC. One change with respect to ZFC is the relativization of the axiom of extensionality to apply to sets. Extensionality now reads:

Axiom of Extensionality:

$$(2) \quad \forall x \forall y ((Set(x) \wedge Set(y)) \rightarrow \forall z ((z \in x \leftrightarrow z \in y) \rightarrow x = y)).$$

Other axioms are counterparts of separation and pair, union, and power set axioms. The empty set axiom and the axiom of infinity state the unconditional existence of an empty set and an infinite set, and the axioms of replacement make sure that if some objects are in one-one correspondence with the elements of a set, then they themselves form a set. There are, in addition, axioms of choice and foundation. One version of the axiom of choice states that given a family of disjoint non-empty sets, there is a “choice set” containing exactly one element from each set in the family. The axiom of foundation requires sets to appear at cumulative ranks of a familiar cumulative hierarchy.

The axioms of ZFCU leave open whether a set contains all urelements, but it is not uncommon to assume that there is one.¹² In this

framework, one option is to supplement ZFCU with the axiom that there is a set of all urelements:

The Urelement Set Axiom:

$$(3) \quad \exists x (Set(x) \wedge \forall y (\neg Set(y) \rightarrow y \in x)).^{13}$$

There is, however, no mathematical difficulty associated with the development of ZFCU without the Urelement Set Axiom.¹⁴ Some axiomatizations of impure set theory explicitly rule out the existence of a set of urelements at the outset in order to make sure that they exist in great abundance.¹⁵

2.2 The iterative conception

The iterative conception of set motivates many of the axioms of ZFCU.¹⁶ On the iterative conception, sets are formed in stages of a cumulative hierarchy from an initial domain of urelements by iteration of the operation “set of”. In the words of Kurt Gödel in (Gödel, 1947, 180):

The concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation “set of”, and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly “naive” and uncritical working with this concept of set has so far proved self-consistent.

11. Impure set theory may be developed in different ways; Jech (2013), for example, expands the language of ZFC with a constant symbol, A , for the set of urelements and makes alternative modifications to the axioms of ZFC.

12. Jech (2013), for example, expands the language of ZFC with a constant, A , intended to denote a set of urelements.

13. McGee (1997) discusses the credentials of this axiom in detail.

14. In Barwise (1975), KPU is Kripke-Platek set theory with urelements and KPU⁺ is the theory augmented with the Urelement Set Axiom.

15. Two examples are given by Barwise and Moss (1996) and Friedman (2004).

16. According to Boolos (1989), neither choice nor replacement fall out of the iterative conception of set.

At the first stage, we obtain sets of urelements—which include the empty set. At subsequent stages—which may be successor or limit stages, we obtain sets from urelements and sets formed in earlier stages. This narrative makes vivid a familiar cumulative hierarchy:

$$\begin{aligned} U_0 &= U; \\ U_{\alpha+1} &= U_\alpha \cup \mathcal{P}(U_\alpha); \\ U_\lambda &= \bigcup_{\alpha < \lambda} U_\alpha, \end{aligned} \quad \text{for } \lambda \text{ a limit ordinal.}$$

On this model of the iterative conception, a set of urelements, U , is formed at the first stage of iteration. In general, if κ is a strongly inaccessible cardinal, $\langle \bigcup U_\kappa, \in \upharpoonright \bigcup U_\kappa \rangle$ yields a model of ZFCU + The Urelement Set Axiom in which *Set* is interpreted to apply to $\bigcup U_\kappa - U$. As expected, (U_∞) is falsified in models of this form.

We face a fork in the road. One option at this point is to make adjustments in the axioms of ZFCU in order to block the inconsistency of (U_∞) and the Urelement Set Axiom. The other option is to revert to a more liberal interpretation of the iterative conception on which the urelements are no longer required to form a set. Menzel (2014) has explored the prospects of the first option in great detail. His modification of ZFCU is intended to accommodate the existence of “wide sets”, which, while formed at low stages of the cumulative hierarchy, are not “mathematically determinable”. The axioms of replacement and power set, in particular, are appropriately restricted in order to accommodate them.¹⁷

17. Replacement is restricted to the claim that, roughly, only the range of an operation F on mathematically determinable sets determines a set. The axiom of power set is similarly stated as the claim that for every set a , there is a set which contains all and only the “mathematically determinable” subsets of a . The proposal requires two more axioms intended to enforce the existence of a partition of the universe into ranks and the thought that only mathematically determinable sets admit an increase in cardinality.

The second option is suggested by Ernst Zermelo’s characterization of *normal domains* as cumulative models of second-order ZFCU in Zermelo (1930). A normal domain is completely determined by two parameters: one is the cardinality of the *basis* of urelements, and the other is the *characteristic* of the model, i.e., the supremum of ordinal numbers represented in the model. If U is a basis and κ is the characteristic of a normal domain, then, in second-order ZFCU, we may recursively define the ranks:

$$\begin{aligned} U_0 &= U; \\ U_{\alpha+1} &= U_\alpha \cup \{x : x \subseteq U_\alpha\}; \\ U_\lambda &= \bigcup_{\alpha < \lambda} U_\alpha, \end{aligned} \quad \text{for } \lambda \text{ a limit ordinal.}$$

The *normal domain* determined by U and κ corresponds to a model, $\langle \bigcup U_\kappa, \in \upharpoonright \bigcup U_\kappa \rangle$, of ZFCU in which *Set* is interpreted in terms of $\bigcup U_\kappa - U$. In general, Zermelo proved that a subset X of a normal domain of characteristic κ is represented by a set in the model if, and only if $|X| < \kappa$. So, the Urelement Set Axiom is falsified in normal domains in which the cardinality of U is strictly larger than the characteristic κ of the model.¹⁸ When this is the case, in fact, none of the stages U_α are represented by a set in the model. There is, nonetheless, a clear sense in which sets are formed in stages of a cumulative hierarchy. The construction models the iteration of the “set of” operation, which turns “mathematically determinable” multiplicities into sets.¹⁹ So, (U_∞) is consistent with the core of the iterative conception.

18. This is Proposition 4 in Kanamori (2004).

19. A mathematically determinable multiplicity is one counted by a cardinal represented in the model.

2.3 *The absolutely infinite*

Zermelo often speaks of the set-theoretic universe as a merely potential succession of normal domains never reaching completion. Cantor, in contrast, conceived the set-theoretic universe as a completed “absolute infinity” that is beyond mathematical determination. He went on to make a distinction between “transfinite” or “consistent multiplicities,” capable of mathematical determination, and “absolutely infinite” or “inconsistent multiplicities”, which are mathematically intractable.

Cantor’s distinction is now mirrored by a distinction between sets and classes. What we learn from the set-theoretic antinomies is that some classes, e.g., the class of non-self-membered sets, fail to form a set. Sets have “elements”, but classes have “members”. Classes, like sets, are extensional in nature, but, unlike sets, they are never allowed to be members. When classes are reified as set-like objects, these differences become inexplicable. Perhaps one reason to tolerate them is that classes have important applications in set theory: they allow one to turn separation and replacement into single axioms; they play a crucial role in the motivation and formulation of certain large cardinal axioms; and they are often invoked in set-theoretic argumentation.²⁰ But whatever benefits classes are supposed to bring with them, talk of classes is hardly credible when conceived as set-like objects.

There is, however, no need to reify classes. No matter how convenient it may seem, talk of classes is best conceived as shorthand for talk that would have been expressed more perspicuously by metalinguistic ascent or by means of plural quantification over sets.²¹ This is the official position we adopt in the paper. The challenge from recombination

takes place within a two-sorted extension of Bernays-Gödel class theory with urelements (BGU), which is a predicative theory of classes over ZFCU.²² The language of BGU contains uppercase variables X, Y, Z, \dots for classes and lowercase variables x, y, z, \dots for urelements and sets. The other change with respect to ZFCU is that we let the \in be flanked by a set and a class variable, e.g., $x \in X$. Classes are primarily governed by two axioms:

Axiom of Class Extensionality:

$$(4) \quad \forall X \forall Y (\forall x (x \in X \leftrightarrow x \in Y) \rightarrow X = Y)$$

Predicative Class Comprehension:

$$(5) \quad \exists X \forall x (x \in X \leftrightarrow A), \text{ where } A \text{ has no bound class variables.}^{23}$$

Predicative class comprehension generates a class for each condition A that quantifies exclusively over urelements and sets. BGU incorporates the pair, union, and power set axioms of ZFCU as well as standard axioms of foundation, and infinity. Replacement becomes a single axiom from which separation follows.²⁴ The axiom of choice is not part of BGU. We will, in addition, consider an axiom of global choice (GC) according to which there is a functional relation F selecting exactly one element from each non-empty set, e.g., if x is non-empty, then $F(x) \in x$.

Given the vocabulary of classes, we are in a better position to regiment the Cantorian conception of absolute infinity. Cantor’s “transfinite multiplicities” correspond to infinite sets, whereas his “absolutely

20. Maddy (1983) discusses the role of classes in set theory.

21. The difference between the two strategies of paraphrase is not without significance. The metalinguistic interpretation in terms of satisfaction in line with Parsons (1974) affords an interpretation of a predicative theory of classes like Bernays-Gödel class theory (BG), whereas the plural interpretation helps one make sense of an impredicative theory of classes like Morse-Kelley class theory (MK). One way in which talk of predicative classes is innocuous is borne out by the fact that BG is a conservative extension of ZFC: a sentence A of the language of ZFC is a theorem of BG only if it is a theorem of ZFC.

22. Fraenkel et al. (1973) introduce two-sorted axiomatizations of the theory of classes. In general, Mendelson (1997) discusses a general procedure to move from one-sorted to two-sorted axiomatizations of the theory of classes.

23. As usual, we assume that the variable X doesn’t occur freely in the condition A . Likewise for subsequent comprehension principles.

24. Replacement states that if R is a functional relation, the range of R on a set x is itself a set. A relational class R is a class of ordered pairs. The domain, $dom(R)$, is $\{x : \exists y \langle x, y \rangle \in R\}$. The range of F , $ran(F)$, is $\{x : \exists y \langle y, x \rangle \in R\}$. F is a functional relation iff R is a relational class such that for all $x, y, z \in dom(F)$, if $\langle x, y \rangle \in F$ and $\langle x, z \rangle \in F$, then $y = z$.

infinite multiplicities" correspond to "proper classes" failing to form a set. The thought that absolutely infinite multiplicities represent an "absolute quantitative maximum" motivates, for Cantor, the hypothesis that A is a proper class if, and only if, the class Ω of all ordinals is injectable into A , which we abbreviate: $\Omega \preceq A$.²⁵ This is what Michael Hallett calls Cantor's *projection postulate* in (Hallett, 1986, p. 171):

Projection: A is a proper class iff $\Omega \preceq A$

Cantor's projection postulate gives only partial expression to the Cantorian thought that the absolutely infinite exemplifies an unsurpassable magnitude incapable of any further increase. However, John von Neumann offered his own interpretation of the Cantorian heuristic in terms of one-one correspondence or equinumerosity. In what follows, we often abbreviate the claim that a class A is equinumerous with another class B as: $A \sim B$:

Maximality: A is a proper class only if $A \sim V$.

Since proper classes are absolutely infinite, they are all equinumerous with the universal class V .

There is one important respect in which von Neumann's maximality principle outstrips Cantor's projection postulate: choice is independent from Projection but directly derivable from Maximality in BG.²⁶ However, choice is all that is required to derive Maximality from Projection in BG. It is not difficult to check that the equivalence between

Maximality and Projection plus choice carries over to the case of BGU plus the Urelement Set Axiom.²⁷

The absence of the Urelement Set Axiom severs the equivalence between Maximality and Projection plus choice over BGU.²⁸ But the Cantorian doctrine of absolute infinity fits well with the weaker hypothesis that the class U of all urelements is injectable into Ω : $U \preceq \Omega$. We know, by the Burali-Forti paradox, that Ω is absolutely infinite.²⁹ Let Cardinal Comparability (CC) be the hypothesis that if A and B are two classes, then one is injectable into the other.³⁰ In the presence of CC, the Cantorian doctrine that there is only one absolutely infinite magnitude suggests the existence of an injection of U into Ω . And, in fact, Maximality is provably equivalent to GC over BGU + $U \preceq \Omega$.³¹

3. Recombination and paradox

We have suggested that the real problem with modal plenitude is that it conflicts with the Cantorian doctrine of the absolutely infinite. We now argue that the difficulty arises quite independently from a plenitudinous commitment to merely possible concrete objects.

25. A functional class F is an injection of A into B iff $\text{dom}(F) = A$, $\text{ran}(F) \subseteq B$, and for all $x, y \in \text{dom}(F)$, $F(x) = F(y)$ iff $x = y$.

26. (Hallett, 1986, pp. 172-173) reports that Projection doesn't entail choice over BG. The axioms of choice, union, and replacement are all deductive consequences of Maximality in BG.

27. Proofs of this and related facts have been relegated to an appendix.

28. This related to Proposition 4 in Kanamori (2004) according to which Maximality is satisfied in a normal domain $\langle \bigcup U_\kappa, \in \upharpoonright \bigcup U_\kappa \rangle$ iff $|U| \leq \kappa$. To each normal domain model of ZFCU, there corresponds a model of BGU in which we let class variables range over the power set of the normal domain. Therefore, there are models of Projection and choice in which Maximality fails.

29. The crucial observation behind the Burali-Forti paradox is that the order-type of each set of ordinals less or equal to α well-ordered by $<$ is strictly greater than α . So, if Ω formed a set, then, since it is well-ordered by $<$, an ordinal would have to correspond to it greater than any ordinal in Ω .

30. See Linnebo (2010) for discussion.

31. It is not difficult to check that GWO entails CC, and as noted in the Appendix, CC turns out to be equivalent to GWO over BGU + Urelement Set Axiom. I do not know what is the status of CC in the absence of the latter axiom.

3.1 *How many concrete objects could there have been?*

Consider the question of how many concrete objects there could have been. Could there have been exactly two concrete objects in existence? Maybe not for a philosopher for whom the axioms of classical mereology amount to necessary generalizations over the concrete.³² But even if one is prepared to rule out two, or fourteen, or \aleph_0 as possible cardinalities for the concrete, it seems harder to deny that there could have been a strictly larger set of concrete objects in existence:

(RC₃) For every cardinal κ , there is a larger cardinal λ such that it is possible that there are exactly λ concrete objects.

This principle enjoys at least as much general appeal as its predecessors. For to deny it again is to hold that some cardinal number κ sets an absolute upper bound on the cardinality a set of concrete objects could have: necessarily, if the concrete objects compose a set, then it must have cardinality strictly less than κ .³³ However, it would certainly be incredible to claim that there could be at most a finite set of concrete objects in reality. Nor would it seem much better to claim that there is some transfinite cardinal, κ , such that there could be a set of exactly κ concrete objects in existence but not a set of more concrete objects. What could explain such an extraordinary constraint on what is possible for the realm of the concrete? It would be different of course if the constraint were to flow from the nature of the concrete, e.g., from abstract constraints on the structure of spacetime and the assumption that concrete objects must have spatiotemporal location. Indeed, some might even be encouraged by Lewis' own qualification of the official recombination principle in Lewis (1986) by means of a "size and shape

32. This consequence of classical mereology is discussed by Comesaña (2008). In fact, a model of classical mereology is a complete Boolean algebra without a zero element. But if κ is the cardinality of an infinite complete Boolean algebra, then $\kappa = \kappa^{\aleph_0}$, which immediately rules out cardinals of cofinality ω . See Koppelberg (1981).

33. The mere possibility of a proper class of concrete objects is not sufficient to satisfy the constraint set by (RC₃).

permitting" clause. Unfortunately, no such constraints seem relevant to the original question once we take seriously the possibility of cohabitation: if concrete objects could jointly cohabit a region of space, then it looks like no absolute upper bound on the cardinality of the concrete can be motivated purely by constraints on the structure of spacetime.

Before we introduce the puzzle, it may be helpful to note that the argument may be regimented in the framework of a modal extension of BGU. We would, first, extend the language of first-order BGU with a monadic predicate, C , read: "is concrete" and a modal operator, $\Box A$, read: "it is necessary that A "—and its dual, $\Diamond A$ (for $\neg\Box\neg A$), read: "it is possible that A ". The modal operator is governed by the axioms of the minimal propositional modal logic K .³⁴ The interaction of propositional modal logic and the axioms of classical quantificational logic immediately yield the Converse Barcan Formula:

(CBF) $\Box \forall x A \rightarrow \forall x \Box A$

and the Necessary Necessity of Existence:

(NNE) $\Box \forall x \Box \exists y x = y$.³⁵

Neither consequence is particularly attractive by the lights of an abstractionist interpretation of (\diamond). Fortunately, for the abstractionist, they are avoided, for example, if one restricts the axioms of quantificational logic in the manner suggested by Kripke (1963). So, the argument could be regimented against the background of a formalization of BGU in a free quantified modal logic.

34. This axiom system contains all instances of axiom K :

$$K(A \rightarrow B) \rightarrow (KA \rightarrow KB)$$

and the rule of necessitation:

From A , infer $\Box A$.

35. In addition to them, the Barcan Formula (BF) becomes provable as soon as one assumes axiom schema B : $A \rightarrow \Box \Diamond A$.

On to the argument now, call a cardinal κ a *live cardinal* if and only if there could be a set of exactly κ concrete objects. In symbols: “ $\diamond \exists s(\forall x(x \in s \leftrightarrow Cx) \wedge \text{card}(s) = \kappa)$ ”. We now record two important features of live cardinals to which we appeal later.

The first feature of live cardinals comes as a consequence of recombination. Since a cardinal is an initial von Neumann ordinal, given (RC_3) , we have that the live cardinals form an unbounded class of von Neumann ordinals. Therefore:

(C_∞) There is a proper class of live cardinals.

The second feature is based on familiar Cantorian considerations:

$(C_{\infty+})$ There are strictly more classes of live cardinals than there are live cardinals.

This is a direct consequence of the Cantorian lemma:

Cantorian Lemma: A class A has more subclasses than members.

The Cantorian lemma needs to be unpacked with care. It is supposed to generalize Cantor’s theorem that a set a has strictly more subsets than elements. However, when unfolded, the latter involves an explicitly comparison of cardinality between a set a and its powerset $\mathcal{P}a$. Unlike sets, classes are never members and there is never opportunity to collect all subclasses of a class A into another class. So, whatever the gloss, the Cantorian lemma had better not involve a comparison of cardinality between two classes. Fortunately, Bernays (1942) explained how to mimic the statement that a class has more subclasses than members within BG.

Bernays’ theorem encodes the claim that given a class A , there is “class-valued function” from A onto all subclasses of A . But this is only a first pass, since it is not clear how to make sense of a “class-valued function” in the framework of BG. Since classes are never members, they cannot appear in an ordered pair. No matter, Bernays simulates a “class-valued function” from A to subclasses of A by means of a binary

relational class R on A : in particular, we take R to map a member a of A to the class of members of A to which a is related, i.e., $\{x \in A : \langle a, x \rangle \in R\}$, which is itself a subclass of A . We may even write $R(a) = B$ to abbreviate: $\forall x(\langle a, x \rangle \in R \leftrightarrow x \in B)$. Bernays’ generalization of Cantor’s theorem is merely the observation that no binary relational class R on A can simulate a class-valued function from members of A onto subclasses of A : given a binary relational class R on A , there is, on pain of contradiction, some subclass D of A such that no member d of A is such that $R(d) = D$.

Proposition 3.1 (BG(U)) *No binary relational class R on a class A simulates a class-valued function from members of A onto subclasses of A .*

Proof Let R be a binary relation on A , and consider the class $D = \{x \in a : \langle x, x \rangle \notin R\}$, which exists by predicative class comprehension. Now, there is no member d of A such that $R(d) = D$. Otherwise, if a member d of A is such that $R(d) = D$, then $\langle d, d \rangle \in R$ iff $d \in D$ iff $\langle d, d \rangle \notin R$. ■

Notice that Bernays’ proof involves only an instance of predicative class comprehension, and it is therefore a theorem of BG(U). The result may seem perplexing considering the availability of interpretations of BG(U) on which there are no more classes than there are sets.³⁶ But while the proof of Bernays’ theorem does not require impredicative class comprehension, the link with the Cantorian lemma does presuppose it: if no relational class R simulates a map from members of A onto subclasses of A , then, given impredicative class comprehension, we know that no formula $\Phi(x, y)$ can simulate such a map.³⁷

36. If κ is a strongly inaccessible cardinal, we could take the set variables of the language to range over V_κ and we could let the class variables of the language to range over the set $Def(V_\kappa)$ of definable subsets of V_κ .

37. Many thanks are due to Øystein Linnebo for this observation. We discuss the link between Bernays’ theorem and the Cantorian lemma in more detail in section 5.2.

3.2 Modal comprehension principles

We focus on weak modal comprehension principles for propositions, but structurally analogous forms of comprehension for possible worlds, properties or states of affairs would have done as well. We have only chosen to look at the case of propositions in order to facilitate the comparison between the challenge from recombination and more traditional threats of paradox such as Russell's paradox of propositions or Kaplan's paradox. The main line of argument, however, can be transposed *mutatis mutandis* to other categories of abstract objects.

In what follows, we regiment a principle of modal comprehension for propositions in a three-sorted extension of BGU supplemented with third style of variable for propositions, p, q, \dots , a propositional truth predicate T , and a modal operator \Box .³⁸ For a first pass at a principle of modal propositional comprehension, consider:

$$\text{Comp} \quad \exists p \Box (Tp \leftrightarrow A).$$

The schema is designed to secure, for each sentence A , the existence of a proposition p which, necessarily, is *true* if, and only if, A obtains. On a possible worlds framework governed by (\diamond) , the schema makes sure that the truth conditions of the proposition p track the truth value of A across possible worlds. For example, if A is the statement that exactly three propositions are true, then *Comp* yields the existence of a proposition p , which, necessarily, is true if, and only if, exactly three propositions are true.

The impredicative nature of *Comp* is overkill for our purposes. So, we weaken the schema in one important respect: we need only assume *ultrapredicative* instances of *Comp* in which A contains no propositional variables at all. We thereby make sure to generate exclusively propositions concerned with strictly extensional matters such as the cardinality of the concrete. The restriction isn't motivated by concerns with

³⁸. It is a three-sorted language because we have three different styles of objectual variables for different categories of objects such as members of classes, classes, and propositions.

impredicativity but merely made for purposes of bookkeeping. The challenge from recombination requires only ultrapredicative instances of propositional comprehension:

$$\text{UComp} \quad \exists p \Box (Tp \leftrightarrow A), \text{ where } A \text{ contains no propositional variables.}$$

The ultrapredicative restriction of modal propositional comprehension is all we need in order to generate propositions concerned with such extensional matters such as the cardinality of the concrete. If A is the statement that there are exactly three concrete objects, then *UComp* yields the existence of a proposition p , which, necessarily, is true if, and only if, there are exactly three concrete objects.

3.3 Two absolutely infinite magnitudes

We now look at instances of *UComp* of the form:

$$(6) \quad \exists p \Box (Tp \leftrightarrow \exists x (x = \text{card}(C) \wedge x \in X)),$$

where $\exists x (x = \text{card}(C) \wedge x \in X)$ is an open formula in which the class variable X occurs free. Given a class of cardinals A , relative to an assignment α on which $\alpha(X) = A$, (6) guarantees the existence of a proposition p , which, necessarily, is true if, and only if, the concrete objects form a set whose cardinality is a member of A . In a possible worlds framework governed by (\diamond) , the truth conditions of the proposition tracks the truth value of the statement that the concrete objects form a set whose cardinality is a member of A across possible worlds.

- (7) For every class of cardinals A , there is a proposition p , which, necessarily, is true iff there is a set of concrete objects whose cardinality is a member of A .

To ease exposition, we write that p *tracks* a class of cardinals A if, and only if, necessarily, p is true if, and only if, the concrete objects form a set whose cardinality is a member of A . Notice that (7) only states that the class of cardinal numbers A is tracked by some proposition. This, however, is perfectly consistent with the hypothesis that one and the

same proposition can track two different classes of cardinal numbers. Consider a coarse-grained account of propositions on which one identifies necessarily equivalent propositions: if neither 2, nor 14, nor 22 are live cardinals, then *the* impossible proposition will track both the class $\{2, 14\}$ and the class of $\{14, 22\}$.

Matters are importantly different when we restrict attention to classes of *live cardinals*:

- (8) If two propositions p and q respectively track two classes of live cardinals A and B , then $p = q$ only if A and B are coextensive.

The reason is simple. If two classes of *live* cardinals A and B are not coextensive, then at least one *live* cardinal κ is a member of one but not the other. Without loss of generality, we may assume that $\kappa \in A$ but $\kappa \notin B$. Since κ is a live cardinal, it is possible that the concrete objects form a set of cardinality κ and not a set of cardinality λ . So, it is possible for p to obtain without q obtaining. This means that p and q are different propositions after all: identical propositions are necessarily equivalent!³⁹

We are led to the conclusion that there are strictly more propositions than there are live cardinals. The combination of (7) and (8) tells us that there are no fewer propositions than classes of live cardinals. But $(C_{\infty+})$ states that there are more classes of live cardinals than there are live cardinals. Since, by (C_{∞}) , there is an absolutely infinite class of live cardinals, there are at least two absolutely infinite magnitudes. It follows that von Neumann's interpretation of the Cantorian doctrine is violated again.

39. This falls out of a principle of indiscernibility of identicals for propositions.

4. The traditional threat of paradox

Given ultrapredicative instances of modal propositional comprehension, a principle of recombination like (RC_3) generates such a vast abundance of abstract objects that we are in a position to replicate the conflict with the Cantorian doctrine of the absolutely infinite. One may be tempted to respond that the difficulty pales in comparison with more familiar sources of anxiety with (\diamond) , which has been thought to lead to plain inconsistency, never mind a tension with the Cantorian doctrine of absolute infinity. Two problems that immediately come to mind are based on Russell's paradox of propositions and Kaplan's paradox. We now argue that each problem requires further resources beyond the minimal assumptions outlined above. Russell's paradox of propositions requires propositions to be sufficiently fine-grained, and Kaplan's paradox requires impredicative instances of modal propositional comprehension.⁴⁰

4.1 Russell's paradox of propositions

In Appendix B, Russell (1903) records the observation that the following three principles are inconsistent:

- (R1) For each class of propositions C , there is a proposition p_C , e.g., every proposition in C is true, which is associated with it.
- (R2) If C and D are two different classes of propositions, then p_C is different from p_D .
- (R3) There is a class of propositions R , which consists of all and only propositions p such that $p = p_C$ for some class of propositions C such that $p \notin C$.

40. Some of the material in this section overlaps with the discussion of Russell's paradox of propositions and Kaplan's paradox in Uzquiano (2015a).

The argument is impeccable.⁴¹ What is less clear is what to make of it. For (R2) has very little plausibility given a coarse-grained standard for the individuation of propositions. If necessarily equivalent propositions are in fact identical, then even if p and q are two different propositions, $\{p, \neg p\}$ and $\{q, \neg q\}$ will be two classes to which one and the same proposition corresponds, namely, *the impossible proposition*. And you may even take Russell's paradox of propositions to place a constraint on how fine-grained propositions can be.⁴²

The appeal to coarseness of grain may likewise be used to block other variants on Russell's argument against the coherence of (\diamond) .⁴³ To rehearse a version of the problem, notice that while proponents of (\diamond) often they differ as to how best to conceive of propositions, they often agree on two minimal constraints on worlds and propositions:

- (P1) Each proposition p has a negation $\neg p$.
 (P2) If p is a proposition and w is a possible world, then either p is true at w or $\neg p$ is true at w .

Call a class C of propositions *maximal* if, and only if, for each proposition p , either p is a member of C or $\neg p$ is a member of C . And call a class C of propositions *compossible* if, and only if, it is possible that every proposition in C is true. Given (P1) and (P2), (\diamond) leads to:

- (P3) If w is a possible world, then the propositions that are true at w form a maximal compossible class of propositions.

41. If R is the class of propositions described by (R3), let p_R be the proposition associated with it, which exists by (R1). First, observe that $p_R \in R$: otherwise, p_R would thereby meet a sufficient condition for membership in R . Second, since $p_R \in R$, by definition of R , $p_R = p_C$ for some class C to which p_R does not belong. Since $p_R \in R$, $C \neq R$, which contradicts (R2).

42. This is the line explored in Uzquiano (2015b).

43. (Divers, 2002, pp. 243-245) discusses the general style of argument.

The trouble again is that four principles are inconsistent:

- (M1) There is a maximal compossible class M of propositions.
 (M2) For each class C of propositions in M , there is a proposition p_C associated with C , e.g., the proposition that every proposition in C is true.
 (M3) If C and D are different classes of propositions in M , then the four propositions p_C , $\neg p_C$, p_D , and $\neg p_D$ are pairwise different.
 (M4) There is a class of all propositions p in M such that p is either p_C or $\neg p_C$ for some class C of propositions in M .

The argument mimics Russell's derivation of the contradiction.⁴⁴ But notice that (M3) is no more plausible than (R2) above: there is no reason to expect the relevant propositions to be pairwise different if, for example, one identifies necessarily equivalent propositions.

The crucial observation for our purposes is that the argument from recombination makes no appeal to the fineness of grain of propositions. When we argued that two propositions p and q track different live cardinals—or classes thereof—only if the live cardinals—or classes thereof—are different, we relied only on the indiscernibility of identical propositions and made no demands whatever on the standard for the individuation of propositions.

44. By (M2), consider the proposition p_R associated with the class R , which exists by (M1) and (M4). We argue that both p_R is a member of R and p_R is not a member of R . First, $p_R \in R$. Otherwise, if $p_R \notin R$, then p_R satisfies a sufficient condition for membership to R , whence $p_R \in R$. Now, that we know that $p_R \in R$, by definition of R , there is some class C of propositions in M such that p_R is either p_C or $\neg p_C$ but p_R is not a member of C . But since R and C differ by at least one member, namely, p_R , by (M3), p_R is different from p_C and $\neg p_C$. Contradiction.

4.2 Kaplan's paradox

Kaplan's paradox is supposed to raise a separate challenge for (\diamond) . Kaplan (1995) observed that (A) is unsatisfiable against the background of a possible worlds model theory for intensional logic:

$$(A) \quad \forall p \diamond (Qp \wedge \forall q (Qq \leftrightarrow (p = q))).^{45}$$

Kaplan seemed to conceive of this observation as a difficulty for the standard possible worlds model theory for intensional logic because logic alone, for him, should not adjudicate the status of (A). However, Kaplan's consideration has often been turned into a cardinality problem for (\diamond) : if QA is interpreted to mean: "it is queried whether A ", then if true, (A) would seem to require for each proposition p the possibility that p , and p alone, be queried. But since (\diamond) posits a possible world as a witness for each possibility, it requires at least as many possible worlds as propositions, which is impossible for familiar Cantorian reasons.

The real problem with (A), however, is that it is plainly inconsistent. To appreciate this, consider, first, a theorem of classical quantificational propositional logic pointed out by Prior (1961):

$$(B) \quad Q\forall p (Qp \rightarrow \neg p) \rightarrow \exists q (Qq \wedge p) \wedge \exists q (Qq \wedge \neg q).$$

Prior read QA to mean: "it is asserted by a Cretan that A ", but his observation carries over to the case at hand. Logic alone appears to rule out the scenario in which the proposition *all queried propositions are false*, and it alone, is queried.

45. A model generally interprets propositional quantification in terms of objectual quantification over sets of possible worlds in the model. Since $\diamond B$ is true at a world in the model just in case B is true at some world in the model, the satisfiability of (A) would require, for each set of possible worlds, a world to bear witness to the possibility that the corresponding set of worlds, and it alone, be in the extension of Q at a world. And this in turn is inconsistent with Cantor's theorem according to which there are strictly more sets of possible worlds than there are possible worlds in the model.

The inconsistency of (A) is not far behind. An application of necessitation in combination with the logic of propositional quantification delivers the negation of a simple consequence of (A^-) :

$$(A^-) \quad \forall p \diamond (Qp \wedge \forall q (Qq \leftrightarrow (p \leftrightarrow q))).^{46}$$

There are two main lines of response to Kaplan's paradox, but none of them help in the least with the problem of recombination. One of them is to reject the likes of (A) and (A^-) as unattainable for strictly logical reasons. Once we surrender such principles as inconsistent, it should be no surprise that the principles turn out to be unsatisfiable. This observation, however, is of no help whatever when it comes to the problem from recombination.

The other line of response involves a retreat from classical quantificational propositional logic to a weaker quantificational propositional logic motivated by a ramified vision of propositions on which they are classified into various orders in accordance to their subject matter. First, following Kaplan, we begin with propositions of level 0, which are concerned with strictly extensional matters such as different distributions of earth, wind, fire, and water. Then we form propositions of level 1, which are concerned with extensional matters and propositions of level 1. Then we form propositions of level 2, etc. Prior's theorem is then blocked by means of a restriction on universal instantiation: a propositional variable of a given level may only range over propositions of a lower level. Similar considerations now block the inconsistency of (A^-) .

Now, ramification undermines the motivation for impredicative modal propositional comprehension as stated by *Comp*, but it leaves ultrapredicative instances of modal propositional comprehension untouched. For if A is condition in which no propositional variables occur, then even by the lights of the ramified vision of propositions, A should determine a proposition of level 0 concerned with strictly extensional

46. This fact is discussed in Uzquiano (2015a).

matters. This means that the second line of response of Kaplan's paradox is powerless against ultrapredicative instances of modal propositional comprehension employed in the problem from recombination.

5. A tetralemma

What is to be learned from the challenge from recombination? We seem to face an uncomfortable tetralemma, once we realize that there is a conflict between four attractive thoughts:

- A. *Recombination*: For each cardinal number κ , there is a larger cardinal λ such that it is possible that there are exactly λ concrete objects.
- B. *Cantorian Lemma*: A class has more subclasses than members.
- C. *Maximality*: A is a proper class only if $A \sim V$.
- D. *Ultrapredicative Modal Comprehension*: To each ultrapredicative condition A , there corresponds a proposition p which, necessarily, is true iff A obtains.

We have only limited room for maneuver. One option is to take the difficulty to refute the principle of recombination. Two more options require us to reevaluate the evidence for the Cantorian lemma and von Neumann's interpretation of the Cantorian doctrine of the absolutely infinite. One last option is to take the problem to refute the ultrapredicative restriction on propositional comprehension. However, none of the attempted resolutions is cost-free.

5.1 Reject A: Pessimistic scepticism

One horn of the tetralemma turns the problem from recombination into a discovery in modal metaphysics. What we have learned from the conflict is that there is, in fact, *some* cardinal number κ such that it is strictly impossible for the concrete to compose a set of cardinality greater than κ . This places an absolute upper bound on the cardinality that a set of concrete objects could have, even if it's not one motivated

by reflection on the nature of the concrete. Nor do the Cantorian considerations given above help identify a specific cardinal κ as a cutoff on the series of possible cardinalities for a set of concrete objects. So, while, on this horn of the tetralemma, Cantorian considerations hand down to us a powerful reason to posit a cutoff, we are no closer to a reason to prefer one to another. Wherever the cutoff is located, we face the same threat of arbitrariness as before.

Maybe we should simply acknowledge that the location of the cutoff is just a brute modal fact for which there is no informative explanation to be had. If the cutoff is \aleph_{17} , then we need to come to terms with the fact that there *just* could not be a set of concrete objects of a greater cardinality. This is a stable position, but it comes at a cost. Once we posit unexplainable brute modal facts, we seem led to a form of pessimistic scepticism according to which our ability to gain modal knowledge is much more impoverished than one might have supposed.

5.2 Reject B: Bernays' theorem without the Cantorian lemma

The Cantorian lemma played a crucial role in the problem, and it may be identified as the weakest link by some philosophers. On the face of it, the Cantorian lemma is grounded on a theorem: Bernays' generalization of Cantor's theorem for classes. But one may still attempt to drive a wedge between the letter of Bernays' theorem and the claim that a class has more subclasses than members. In particular, one may think that whatever the moral of Bernays' theorem, it is still compatible with the existence of no more subclasses than members.

It is instructive to compare the situation with a second-order variation on Bernays' theorem:

$$(9) \quad \neg \exists R \forall X \exists x \forall y (Rxy \leftrightarrow Xy)$$

This theorem states that no relation maps objects onto the values of monadic second-order variables. And this is often glossed as the claim that there are more values of monadic second-order variables than

there are objects.⁴⁷ However, a close look at the proof reveals that it requires only instances of predicative second-order comprehension:

- (10) $\exists X \forall x (Xx \leftrightarrow A)$, provided A contains no bound second-order variables.

Now, let Predicative V (PV) be the theory axiomatized by predicative second-order logic and Frege's Axiom V:

- (V) $\forall X \forall Y (ext(X) = ext(Y) \leftrightarrow \forall x (Xx \leftrightarrow Yx))$,⁴⁸

where, as usual, $ext(X)$ is supposed to denote the extension of X . Not only is PV satisfiable, the range of monadic second-order variables is no larger than the domain of first-order variables. Indeed, it is not difficult to think of a binary condition that *simulates* a map of the latter onto the former: $y \in x$. There is, however, no conflict with (9). When suitably unpacked into primitive notation, $y \in x$ becomes: $\exists X (x = ext(X) \wedge Xy)$, which contains a bound occurrence of X . This means that predicative second-order comprehension does not, by itself, sanction the existence of a relation R corresponding to the condition $y \in x$. So, one may suggest that whatever the import of Bernays' theorem, it's not that there are fewer objects than values of monadic second-order variables.⁴⁹

The example may now be taken to suggest that we should revisit the import of Bernays' theorem. The mere fact that no relational class simulates a map of the latter onto the former is no conclusive reason for the Cantorian lemma: the fact that there are no more subclasses of A than members of A could still be witnessed by the existence of some

impredicative formula $\Phi(x, y)$, which can still simulate a map of the latter onto the former even if it doesn't determine a relational class.

One problem with this move is that it will be ineffectual, if you think that there is no real room between the existence of a formula $\Phi(x, y)$ of the sort mentioned above and the existence of a relational class generated by it. The gap can be closed, for example, by the impredicative class comprehension schema, which looks almost irresistible on the plural interpretation of class talk. This is not the place to argue for the intelligibility of impredicative class comprehension, but it is hard to see how the present predicament could, by itself, be the only reason to retreat to predicative class comprehension. So, the move will only be available to philosophers with independent grounds for advocating a predicative restriction on class comprehension principles.⁵⁰

5.3 *Reject C: More than one absolutely infinite magnitude*

Another horn of the tetralemma reverses the direction of discovery and allows for the mathematical theory of the infinite to be informed by modal metaphysics. The Cantorian doctrine of the absolutely infinite must be amended in order to accommodate more than one absolutely infinite magnitude.

We know that von Neumann's interpretation of the doctrine is equivalent, over BGU, to the combination of global choice and the claim that there are no more urelements than ordinal numbers, i.e., $U \preceq \Omega$.⁵¹ But maybe we could take the challenge from recombination to show that there are strictly more propositions than ordinals. And if propositions are urelements, then this refutes the hypothesis that U is injectable into Ω .

47. Shapiro (1991) offers just this gloss of the theorem.

48. The reader may consult chapter 2 of Burgess (2005) for some discussion of this axiom system and its models.

49. Many thanks are due to Øystein Linnebo for the example and helpful discussion of its significance in the present context.

50. Linnebo (2010), for example, proposes a parallel restriction on appropriately modalized versions of plural comprehension principles.

51. This is Proposition 2 in the Appendix.

But in the presence of Cardinal Comparability, the claim that $U \not\preceq \Omega$ appears to be in conflict with the spirit of the Cantorian doctrine. For if $U \not\preceq \Omega$, then $\Omega \preceq U$, which would suggest that U does in fact outstrip Ω . Now, Ω is certainly absolutely infinite, since to suppose otherwise yields a contradiction via the Burali-Forti paradox. So, if U outstrips Ω , then there are at least two absolutely infinite magnitudes.⁵² We have no choice but to amend the Cantorian doctrine to allow for more than one absolutely infinite magnitude.

The amendment comes at a cost. One motivation for the thesis that U is injectable into Ω relies not to the assumption that U must consist of just a few elementary particles or regions of spacetime but rather on the thought that Ω is an incredibly rich structure, one which is sufficiently rich and varied to contain representations of the order-types of arbitrary well-orderings of classes of urelements. If some subclass T of U is well-ordered by some relational class R , then one may use an injection of U into Ω in order to construct an isomorphism between T under R and some initial segment of Ω under $<$. If T under R is isomorphic to some proper initial segment of Ω , then it will be represented by $\langle \alpha, < \rangle$; otherwise, we will make do with Ω under $<$.

There is a second motivation, which proceeds from the assumption the Cantorian generation of ordinals proceeds completely unencumbered by the nature of U as far as it is conceivable. This is admittedly vague, but it should at least enable us to rule out as unintended a model M of BGU in which Ω^M —the interpretation of “ Ω ” in the model—is isomorphic to a proper initial segment of Ω^N —the interpretation of “ Ω ” in another model N of BGU. The intended models of BGU should let the generation of ordinals proceed *as far as possible*. So, in general, we should expect Ω not to be isomorphic to a proper initial

segment of Ω^M , where M is a model of BGU.⁵³ The claim that U is injectable into Ω makes sure that there is no model M of BGU in which Ω^M —the interpretation of “ Ω ” in the model—contains an isomorphic copy of Ω —the complete ordinal series—as a proper initial segment. So, there is a clear sense in which the generation of ordinals couldn’t have proceeded further than it did.

5.4 Reject D: Against reification

The final horn of the tetralemma involves the formulation of modal propositional comprehension, which could be rejected on the grounds that propositions should not be reified as individual objects. The problem with *Comp*[−] lies in the presupposition that propositions lie in the range of objectual variables.

There are at least two different strategies to consider, though only one of them holds some promise. The first strategy begins with the observation that propositions are often conceived as classes of possible worlds, e.g., Stalnaker (1976). So, if, in line with the official stance of the paper, we reject the reification of classes as individual objects, then there is no reason to think that propositions can be collected into classes in order to supply a counterexample to von Neumann’s maximality principle. The problem with this line of response is that it remains powerless when it comes to other forms of the challenge for separate categories of abstract object governed by appropriate modal comprehension principles.

52. There is anecdotal evidence to suggest that even when set theorists allow for a proper class of urelements, they never ever posit more urelements than ordinals. (For specific examples, the reader may consult Barwise and Moss (1996), Friedman (2004), and Rubin and Rubin (1985).) But this merely speaks to the extent to which the Cantorian conception of the absolutely infinite has been internalized by set theorists.

53. This is akin to (McGee, 1997, p. 54):

Completeness Principle: The pure sets are not isomorphic to a proper initial segment of the “pure sets” of any other model of second-order ZFCU.

This follows from the Urelement Set Axiom, but the argument carries over *mutatis mutandis* to the hypothesis that $U \preceq \Omega$.

To make the point vivid, consider the case of *ways the world might be*. Kripke (1980), for example, conceives of possible worlds as *states of the world*, which correspond to complete or maximally specific ways the world might be Stalnaker (1976). So conceived, ways the world might be are nothing but properties, which can be instantiated by the universe under various circumstances. In a suitable extension of BGU equipped with variables over ways and an instantiation predicate, I , we have a modal comprehension principle for *ways the world might be*:

$$(11) \quad \exists w \Box (Iw \leftrightarrow A).$$

This principle generates, for each condition A , a *way the world might be*, which, necessarily, is instantiated by the total universe iff A obtains. There is, for example, a way the world might be, which, necessarily, is instantiated iff snow is white. Indeed, this is, in fact, a (partial) way the total universe *is*. There is, likewise, a way the world might be, which, necessarily, is instantiated iff donkeys talk. The challenge from recombination may now be recreated by appeal only to ultrapredicative instances of (11) in which A is concerned with purely extensional matters. So, if ways the world might be are, in fact, individual objects in the range of our objectual variables, then we have made no progress whatever.

The more radical strategy is to draw a Fregean distinction between objectual quantification and quantification over predicate position. Even if the universe is an individual object in the range of objectual variables, *ways the universe might be* are not; instead, they are to be found in the range of predicate variables. In this respect, they are rather like *first-level* Fregean concepts over which predicate variables range.⁵⁴ Propositions could likewise be conceived as values of zero-place predicate variables governed by an appropriate form of modal comprehension:

54. They cannot quite be Fregean concepts because Frege went on to identify coextensive concepts.

$$(12) \quad \exists p \Box (p \leftrightarrow A).$$

The key difference between *Comp* and the present form of comprehension is that p is no longer treated as an objectual variable, but is rather a propositional variable occupying sentence position. While modal propositional comprehension still gives rise to a bewildering abundance of propositions, it would be misconceived to use this great wealth of propositions to mount a counterexample to von Neumann's maximality principle. Strictly speaking, only individual objects are members of classes. But the Cantorian doctrine of the absolutely infinite is exclusively concerned with classes of individual objects and has no bearing whatever when it comes to items in the range of predicate or sentence variables. More generally, the challenge from recombination only arises when we ignore the fundamental rift between the values of objectual variables and the values of predicate and sentence variables. While the quantifiers of BGU range over absolutely all objects, whether urelements or sets, propositions are neither and, as a result, they lie outside the scope of the framework.

We can, if we like, make sense of cardinality comparisons between items drawn from different ontological categories: there is still a clear sense in which there are strictly more propositions than ordinals.⁵⁵ But this is no more surprising than the realization that there are strictly more first-level Fregean concepts than there are objects, which is often taken to be the moral of Russell's paradox. We can, in fact, verify that there are in general strictly more Fregean concepts of level $n + 1$ than there are Fregean concepts of level n .⁵⁶

This line of response is not without costs. Possible worlds and propositions are standardly taken to lie in the range of objectual variables, and they are commonly collected into sets. But if possible worlds,

55. Think of an assignment of ordinals to propositions as given by a second-level binary predicate which takes a propositional variable and an individual variable as arguments: no assignment maps all propositions onto the ordinals.

56. In each case, we take an assignment of n th-level concepts to $n + 1$ th-level concepts to be given by a $n + 2$ th-level relational concept taking appropriate predicate variables as arguments.

for example, are identified with states of the world, which, in turn, lie in the range of first-level predicate variables, then apparent quantification over sets of worlds should be analyzed in terms of quantification over second-level predicates under which first-level predicates fall. The claim that the actual world state is a member of the set of world states in which snow is white would be akin to the claim that the actual world state falls under a certain second-order level concept, i.e., one under which a world-state falls iff necessarily, it is instantiated iff snow is white. Not that the translation is particularly difficult to achieve in this case, but it is not trivial to generalize the procedure to deal with other cases.⁵⁷

6. Conclusion

We have argued that a generally attractive recombination principle raises a distinctive challenge for different accounts of modality on which what is possible is explained in terms of what is the case at one world or another. The problem, in particular, requires a different set of resources from the ones generally employed in more traditional sources of anxiety with possible worlds and propositions. We have identified at least four different lines of response to the difficulty and while all seem costly, we have tentatively advocated the view that propositions—or possible worlds, for that matter—should not be conceived as objects.⁵⁸

57. The techniques developed in Linnebo and Rayo (2012) help with more general cases.

58. I'm grateful to audiences at OSU, Arizona State University, the University of Barcelona, UCLA, the University of St Andrews, and the Ranch Metaphysics Workshop, where I presented earlier versions of this paper. Special thanks are due to Kris McDaniel, Øystein Linnebo, and two anonymous referees for very helpful comments and discussion.

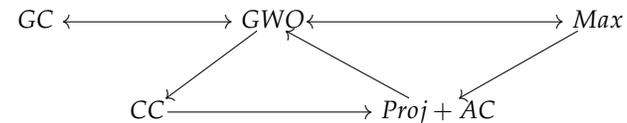
7. Appendix

Definition. Four class-theoretic choice principles.

- (i) GC (Global Choice): There is a global choice function F such that $F(x) \in x$ for each $x \neq \emptyset$.
- (ii) GWO (Global Well-Ordering): There is a global well-ordering R of V .
- (iii) Max (Maximality): A is a proper class only if $A \sim V$.
- (iv) Proj (Projection): A is a proper class only if $\Omega \preceq A$.
- (v) CC (Cardinal Comparability): Given two classes A and B , $A \preceq B$ or $B \preceq A$.

Proposition 1 We have the following entailments over BGU + the Urelement Set Axiom:

- (a) GWO is equivalent to GC.
- (b) GWO is equivalent to Max
- (c) Max entails Proj + Axiom of Choice
- (d) Proj + Axiom of Choice entails GWO
- (e) CC entails Proj + Axiom of Choice



Proof. By the Urelement Set Axiom, there is a set U of all the urelements. We now define the cumulative hierarchy by transfinite recur-

sion on the ordinals:

$$\begin{aligned} U_0 &= U; \\ U_{\alpha+1} &= U_\alpha \cup \mathcal{P}(U_\alpha); \\ U_\lambda &= \bigcup_{\alpha < \lambda} U_\alpha, \end{aligned} \quad \text{for } \lambda \text{ a limit ordinal.}$$

The axiom of foundation makes sure that for every object x —whether an urelement or a set—there is some ordinal α such that $x \in U_\alpha$. Define the rank of an object a , $\rho(a)$, to be the least α such that $x \in U_\alpha$.

- (a) See (Rubin and Rubin, 1963, Theorem 1.2S, and 2.3S).
- (b) This follows from an observation made in von Neumann (1928). See (Rubin and Rubin, 1963, Theorem 1.1S).
- (c) By the Burali-Forti paradox, Ω is a proper class, and, given *Max*, $\Omega \sim V$. So, if A is a proper class, then, given *Max* again, $A \sim V$, whence $\Omega \preceq A$. As for choice, given *Max*, we have, by (b), that some relational class R well-orders V . So, if m is a family of disjoint non-empty sets, we define a choice function f where $f(x)$ is the R -least element of x , for each element x of m .
- (d) The axiom of choice guarantees that each U_α is well-ordered by some relation R . So, we need now only consider the class $W = \{R : \exists \alpha \text{ } R \text{ well-orders } U_\alpha\}$. W is a proper class, since a well-ordering of U_α is different from a well-ordering of U_β whenever $\alpha \neq \beta$. By *Proj*, we have that $\Omega \preceq W$. Notice, in addition, that for each α , there is a set W_α of all well-orderings of U_α , which, by choice again, can be well-ordered. We now define a global well-ordering R of the universe: $\langle x, y \rangle \in R$ iff either $\rho(x) < \rho(y)$ or else if $\alpha = \rho(x) = \rho(y)$, x precedes y in the least well-ordering of U_α according to some well-ordering of W_α .
- (e) Let A be a proper class. Then, by CC, $A \preceq \Omega$ or $\Omega \preceq A$. We are done if $\Omega \preceq A$. Assume $A \preceq \Omega$. Since A is a proper class,

an injection from A into Ω must be cofinal in Ω . But now, we may map an ordinal α to the least member of A not assigned to an ordinal less than α in the order induced by the injection of A into Ω . This gives an injection of Ω into A . As for choice, notice that, by CC, every set is injectable into Ω . So, every set can be well-ordered. ■

Proposition 2. *Max* is equivalent to GC over $BGU + U \preceq \Omega$.

The crucial lemma is the observation that an injection between U and Ω gives rise to an injection between the universal class V and the class Π of pure sets. If U is a set, then, by choice, we are done. Assume that U is a proper class. By transfinite recursion on the ordinals, define a *class-valued* function R :

$$\begin{aligned} R(0) &= U; \\ R(\alpha + 1) &= R(\alpha) \cup \{x : x \subseteq R(\alpha)\}; \\ R(\lambda) &= \bigcup_{\alpha < \lambda} R(\alpha), \end{aligned} \quad \text{for } \lambda \text{ a limit ordinal.}$$

(Rubin and Rubin, 1985, p. xxiv) prove that for every object x , there is some ordinal α such that $x \in R(\alpha)$ from a class form of foundation.⁵⁹ Much like before, define the rank of an object a , $\rho(a)$, to be the least α such that $x \in R(\alpha)$. For each α , $V_\alpha \subseteq R(\alpha)$, where V_α is stage α in the hierarchy of pure sets.

Now, given an injection I of U into Ω , we define an injection J of V into the class Π of pure sets by induction on rank:

$$\begin{aligned} J(x) &= \langle I(x), \emptyset \rangle, \text{ if } x \text{ is an urelement.} \\ J(x) &= \langle \{J(y) : y \in x\}, \{\emptyset\} \rangle, \text{ if } x \text{ is a set.} \end{aligned}$$

59. This is Theorem 0.8 (b) in (Rubin and Rubin, 1985, p. xxiv). The class form of foundation states that every non-empty class A contains a set a as a member such that $a \cap A = \emptyset$.

So, if $A \subseteq V$, then A is injectable into the class of pure sets Π .

The arguments for (a) and (b) may be adapted to show that *Max* entails GC over $\text{BGU} + U \preceq \Omega$. To move from GC to *Max* in the present context, notice that if $A \subseteq V$ is a proper class, then, by the preceding lemma, A is injectable into Π , and, in particular, A is equinumerous with some $Q \subseteq \Pi$. By replacement, Q must itself be a proper class. As a special case of Proposition 1 when $U = \emptyset$ and $V = \Pi$, GC gives us that each of Q and $\Omega \sim \Pi$. So, $A \sim \Omega$ and all proper classes are equinumerous to each other.

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