TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 365, Number 11, November 2013, Pages 5967–5997
S 0002-9947(2013)05823-0
Article electronically published on June 19, 2013

FREE PRODUCTS IN R. THOMPSON'S GROUP V

COLLIN BLEAK AND OLGA SALAZAR-DÍAZ

Abstract. We investigate some product structures in R. Thompson's group V, primarily by studying the topological dynamics associated with V's action on the Cantor set $\mathfrak{C}$. We draw attention to the class $\mathcal{D}_{(V,\mathfrak{C})}$ of groups which have embeddings as *demonstrative subgroups of* V whose class can be used to assist in forming various products. Note that $\mathcal{D}_{(V,\mathfrak{C})}$ contains all finite groups, the free group on two generators, and $\mathbf{Q}/\mathbf{Z}$, and is closed under passing to subgroups and under taking direct products of any member by any finite member. If $G \leq V$ and $H \in \mathcal{D}_{(V,\mathfrak{C})}$, then $G \wr H$ embeds into V. Finally, if G, $H \in \mathcal{D}_{(V,\mathfrak{C})}$, then $G * H$ embeds in V.

Using a dynamical approach, we also show the perhaps surprising result that $Z^2 * Z$ does not embed in V, even though V has many embedded copies of Z^2 and has many embedded copies of free products of various pairs of its subgroups.

1. Introduction

We study aspects of the subgroup structure of R. Thompson's group V, realized here as a group of homeomorphisms of the Cantor set $\mathfrak{C}$. In particular, we study a class $\mathcal{D}_{(V,\mathfrak{C})}$ of groups corresponding to particular subgroups of V which can be used under specific conditions to construct restricted wreath products and free products, and these products will then embed in V. Thus, our first results show that V contains many restricted wreath products, and many free products, of various isomorphism classes of its non-trivial subgroups.

In this paper, we also show that although V contains many embedded copies of both $\mathbf{Z}$ and $\mathbf{Z}^2$, V does not contain any embedded copy of $\mathbf{Z}^2 * \mathbf{Z}$. We believe this non-embedding theorem is in some sense a consequence of the fact that $\mathbf{Z}^2 \notin \mathcal{D}_{(V,\mathfrak{C})}$, although in this note we only see that $\mathbf{Z}^2 \notin \mathcal{D}_{(V,\mathfrak{C})}$ as a corollary to our non-embedding result (using the fact that $\mathbf{Z} \in \mathcal{D}_{(V,\mathfrak{C})}$ and if G, $H \in \mathcal{D}_{(V,\mathfrak{C})}$, then $G * H$ embeds in V).

The class $\mathcal{D}_{(V,\mathfrak{C})}$ consists of all groups isomorphic to a member of the set $\dot{\mathcal{D}}_{(V,\mathfrak{C})}$, which is a set of subgroups of V where each such subgroup acts on $\mathfrak{C}$ in such a way as to exemplify a particular dynamical property. We would be interested in reading a proof of our main non-embedding result which only used algebraic methods based on a presentation of V; it seems to us that such a proof would be quite difficult.

1.1. **The class $co\mathscr{C}\mathscr{F}$.** The initial motivation for this note springs from the work of Holt, Röver, Rees and Thomas, who introduce and study the class $co\mathscr{C}\mathscr{F}$ of

Received by the editors November 13, 2009 and, in revised form, February 22, 2011 and March 7, 2012.

2010 *Mathematics Subject Classification.* Primary 20F65, 37C85, 20E07, 20E32.

Key words and phrases. Thompson's group V, subgroup structure, group actions, demonstrative groups, $co\mathscr{C}\mathscr{F}$ groups.

groups which have context free co-word problem in [19]. In that article they show that $co\mathscr{C}\mathscr{F}$ is closed under direct products, standard restricted wreath products where the top group is in the subclass of groups with context-free word problem (these are the virtually free groups by Muller and Schupp, [15, 25, 26]), passing to finitely generated subgroups, and passing to finite index over-groups. They further conjecture in [19] that $co\mathscr{C}\mathscr{F}$ is not closed under free product, and currently, a lead candidate for demonstrating that conjecture is $\mathbf{Z}^2 * \mathbf{Z}$.

The papers [5, 6] include proofs that $\mathbf{Z} \wr (\mathbf{Z}^2)$ fails to embed in the R. Thompson groups F and T. Also, the article [24] of Lehnert and Schweitzer shows that any finitely generated subgroup of V is a $co\mathscr{C}\mathscr{F}$ group, while Holt, Rees, Röver, and Thomas conjecture that if $C \wr T$ is a $co\mathscr{C}\mathscr{F}$ group, then T is a group with context-free word problem, i.e., a virtually free group [19]. Meanwhile, Brin in [10] gathers techniques which enable one to detect when a group of homeomorphisms has sufficient dynamical conditions to conclude that the group is a standard restricted wreath product of groups. From our own point of view, as $\mathbf{Z}^2 \notin \mathcal{D}_{(V,\mathfrak{C})}$, it is apparent that one cannot directly apply Brin's "pre-wreath structure" technology to find a copy of $\mathbf{Z} \wr (\mathbf{Z}^2)$ in V. These facts cause us to conjecture that $\mathbf{Z} \wr \mathbf{Z}^2$ does not embed into V, effectively re-stating a special case of the Holt, Rees, Röver, and Thomas conjecture mentioned earlier in this paragraph.

In Lehnert's dissertation, there is a conjecture that a group $QAut(\mathcal{T}_{2,c})$ is a universal $co\mathscr{C}\mathscr{F}$ group; i.e., $QAut(\mathcal{T}_{2,c})$ is a $co\mathscr{C}\mathscr{F}$ group so that every group in $co\mathscr{C}\mathscr{F}$ embeds into $QAut(\mathcal{T}_{2,c})$. It is of interest to the authors of this note to fully understand the connections between the group V and the class $co\mathscr{C}\mathscr{F}$, as the theory of $QAut(\mathcal{T}_{2,c})$ is strongly linked with that of V.

1.2. **Local realization and demonstrative subgroups.** The nascent theory of groups acting with local realization and of demonstrative subgroups, as introduced in this note, seems applicable in the wider domain of automorphism groups of topological structures. It is our hope that some of these ideas may see use in a wider context, particularly in the development of the theory of groups where McCleary-Rubin (see, e.g., [28]) type theorems apply.

Given any positive integer s, the paper [6] has an argument that shows that $Z \wr Z^s$ embeds in $\mathrm{PL}_o(I^s)$, the group of orientation preserving piecewise-linear homeomorphisms of the s-dimensional cube I^s. That argument goes through with almost no change if one replaces I^s with $\mathfrak{C}^s$ and $\mathrm{PL}_o(I^s)$ with Brin's higher-dimensional R. Thompson group sV. One point of this discussion is that $\mathbf{Z}^s$ is isomorphic with demonstrative subgroups of both $\mathrm{PL}_o(I^s)$ and of sV. On the other hand, we expect that $\mathbf{Z}^s$ does not embed as a demonstrative subgroup of either $\mathrm{PL}_o(I^{s-1})$ or of $(s-1)V$ for any index $s \in \{2, 3, \ldots\}$.

1.3. **Another non-embedding result, and alternative technologies.** There is another non-embedding result for V which is of interest in this context. Higman in [18] uses his *semi-normal forms* to study the dynamics of automorphism groups $G_{n,r}$ acting on specific algebras (where $n \geq 2$ and r are positive integers). Semi-normal forms can help detect infinite orbits under these actions, and other dynamical properties of elements of the groups $G_{n,r}$. In any case, $V = G_{2,1}$ in Higman's notation, and Higman shows in [18], using semi-normal forms, that $\mathrm{GL}(3, Z)$ does not embed into $G_{n,r}$ for any indices n and r.

In our investigations below, we make use of flow graphs as developed in [7]. Flow graphs are essentially a version of train tracks for the R. Thompson family of groups. In various guises, such train track objects have shown themselves to be of great value in the study of the R. Thompson family of groups [13, 3, 4, 7] from the dynamical perspective. It may be the case that the flow graph technology can be developed from Higman's semi-normal forms, however, we follow [7] and work from Brin's revealing pair technology as developed in [9, 29]. It is not clear whether Higman's semi-normal forms and Brin's revealing pairs formally contain the same information, but there is certainly a large overlap (indeed, while working on the simultaneous conjugacy problem in V, Nathan Barker in [2] replicates, in slightly less detail, many of the results in [7], but by using only Higman's semi-normal forms).

1.4. **Recognition.** The authors offer thanks to Daniel Farley for his participation in early phases of this project. We also offer thanks to Mark Sapir both for interesting discussions with regards to the family of R. Thompson groups and for suggesting we discover the full statement of Theorem 1.7 (in an early draft, we maintained data sufficient only to ensure a proof of Theorem 1.5). Finally, we offer thanks to Nathan Corwin for carefully reading a draft of this article and suggesting some changes.

Separately, we would like to thank Étienne Ghys, Vlad Sergiescu, and Takashi Tsuboi for enlightening conversations regarding the history of the discoveries by various parties of the connections between the Thompson groups T and V, and the group $PSL(2,\mathbf{Z})$.

On one final note, the authors would like to draw attention to the Java software package which Roman Kogan wrote at the NSF funded Cornell Summer 2008 Mathematics Research Experience for Undergraduates. His software provides a convenient interface for calculating and storing products, inverses, conjugates and commutators in the higher-dimensional Thompson groups nV created by Brin in [9] (and thus, in the R. Thompson groups $F < T < V$ as well). While this software was not used to prove any statement within this article, it was often used to verify the efficacy of constructions.

1.5. **Statements of main results.** Below is a partial description of some of the isomorphism types of the demonstrative subgroups of V.

Lemma 1.1. *Let $\mathcal{A}$ denote the smallest class of groups so that*

(1) *$\mathcal{A}$ contains all finite groups,*
(2) *$Z \in \mathcal{A}$,*
(3) *$Q/Z \in \mathcal{A}$, and*
(4) *$\mathcal{A}$ is closed under*
    (a) *isomorphism,*
    (b) *passing to a subgroup, and*
    (c) *taking the direct product of any finite member with any member.*

If $G \in \mathcal{A}$, then $G \in \mathcal{D}_{(V,\mathfrak{C})}$, and so there is an embedding $\phi : G \to V$ with $G\phi \in \dot{\mathcal{D}}_{(V,\mathfrak{C})}$.

We can now state our first and second theorems.

Consider the following theorem regarding wreath products embedding in V. For many cases of top group H and any subgroup $G \leq V$, there are embeddings of

$G \wr H$ into V, as a consequence of Claas Röver's discussion of his Lemma 2.5 in [27]. Determining the exact overlap of Röver's Lemma 2.5 and our statement below seems non-trivial. Röver indicates his Lemma 2.5 can be proven through the use of his Lemma 3.5, which itself is derived as an application of E. Scott's theory of expansible groups from [30].

Theorem 1.2. *If $G \leq V$ and $H \in \mathcal{D}_{(V,\mathfrak{C})}$, then the restricted wreath product $G \wr H$ embeds in V.*

(Note: in the above theorem the group H is the top group so that $G \wr H \cong (\bigoplus_H G) \rtimes H$.)

Naturally, the theorem above has the following corollary, which also appears as Lemma 2.5 in Röver's dissertation.

Corollary 1.3. *Let $G \leq V$, and H be a finite group. If E is an extension of G by H, then E embeds in V.*

Note in particular that the above implies V contains copies of all of the countable virtually free groups. As in Röver's dissertation, the corollary above follows from a result of Kaloujnine and Krasner (see [21, 22, 23]), as every finite extension E of a subgroup G of V by a finite group H will appear as a subgroup of $G \wr H$, which appears as a subgroup of V since H finite implies $H \in \mathcal{D}_{(V,\mathfrak{C})}$. Of course, if $H \in \mathcal{D}_{(V,\mathfrak{C})}$ is infinite, then we cannot make the same claims towards extensions; the Kaloujnine-Krasner result applies to unrestricted wreath products.

We also have the following embedding result which discusses embedding certain free products of subgroups of V into V.

Theorem 1.4. *If K_1, $K_2 \in \mathcal{D}_{(V,\mathfrak{C})}$, then the group $K_1 * K_2$ embeds in V.*

The fact that $\Gamma \cong \mathbf{Z}_2 * \mathbf{Z}_3$ embeds into T and hence V has been well understood by experts for some time and seems to have originated with Thurston from around 1975 (see [14], section 7). Ghys and Sergiescu [17] generated their own proof when they studied smooth representations of T for the paper [16], and Tsuboi in [31] effectively states a full proof of this fact. Finally, Röver shows in Corollary 2.13 of his dissertation that any finite iterated free product of finite groups embeds in V [27].

On the other hand, our chief result is likely the following theorem, which shows that while free products of groups in the isomorphism classes of V's subgroups may often be found in V, one cannot choose these subgroups indiscriminately.

Theorem 1.5. *The group $Z^2 * Z$ does not embed in V.*

In fact, we prove a stronger result about relations which must exist in the image of any homomorphism from $Z^2 * Z$ into V, and the above theorem is then a consequence of the more complex theorem stated below. In order to state that theorem, we need a further definition.

Given symbols x and y, denote by $[x, y]$ the commutator expression $x^{-1}y^{-1}xy$. Now given symbols a, b, and c, we say an expression W is an (a, b, c)-*commutator* if and only if there are integers $n > 0$, x_i, y_i, and z_i with $|x_i| + |y_i| \neq 0$ and $z_i \neq 0$ for all indices $1 \leq i \leq n$ so that

$$W = [a^{x_1}b^{y_1}, [a^{x_2}b^{y_2}, \ldots [a^{x_{n-1}}b^{y_{n-1}}, [a^{x_n}b^{y_n}, c^{z_n}]^{z_{n-1}}]^{z_{n-2}} \ldots]^{z_1}].$$

Note that if we replace a, b, and c by elements in a group G, then we will also call the element h that results as the consequence of evaluating the expression W in G an *(a,b,c)-commutator in G*.

One sees by a simple induction argument that given a, b, and $c \in G$ for some group G, the set of (a,b,c)-commutators in G has the property that if one considers z, an (a,b,t)-commutator, where t is an (a,b,c)-commutator, then z is also an (a,b,c)-commutator.

In subsection 4.1, we prove the following lemma, which is an essential tool in our proof of Theorem 1.7.

Lemma 1.6. *Let $Z^2 * Z$ be given by the presentation $\langle a, b, c \mid [a, b]\rangle$, and let t be an (a,b,c)-commutator in $Z^2 * Z$. Then the subgroup $\langle a, b, t^k\rangle$ factors as $\langle a, b\rangle * \langle t^k\rangle \cong Z^2 * Z$ for any non-zero integer k.*

We are now ready to state our non-embedding result in its full generality.

Theorem 1.7. *Suppose we are given a homomorphism $\phi : \mathbf{Z}^2 * \mathbf{Z} \to V$, where V is R. Thompson's group V, and let $\mathbf{Z}^2 * \mathbf{Z}$ be given by the presentation $\langle a, b, c \mid [a, b]\rangle$. Then, setting $\alpha = a\phi$, $\beta = b\phi$, and $\gamma = c\phi$, there are integers k, x, y, and n with $n \in \{1, 2, 3, 6\}$ and $|x| + |y| \neq 0$ so that there is an (α, β, γ)-commutator θ in V with $[\theta^k, (\theta^k)^{\alpha^x\beta^y}]^n = 1_V$.*

Theorem 1.5 now follows as a corollary, since if ϕ were an embedding, and if α, β, γ, θ, k, x, y, and n had values which satisfied the conditions in the statement above, then the element in V given as the product expression

$$[\theta^k, (\theta^k)^{\alpha^x\beta^y}]^n$$

cannot evaluate as the identity (see the technical lemmas of subsection 4.1).

1.6. **Questions and conjectures.** We state the following questions.

Question 1. Is it true that if G and H are non-trivial subgroups of V, with $|G| \geq 3$ and $\langle G, H\rangle \cong G * H$ in V, then there are non-empty sets P_G and P_H in $\mathfrak{C}$ with $P_G \not\subset P_H$ so that for any non-trivial elements $g \in G$ and $h \in H$ we have $P_H g \subset P_G$ and $P_G h \subset P_H$?

Colloquially, must every free product of groups in V arise from a Ping-Pong in V?

A positive answer to the question below will force a positive answer to the question above (see subsection 4). The other direction is not clear.

Question 2. Suppose that G, $H \leq V$ so that $G * H$ embeds in V. Is it the case that there are isomorphic copies of G and H in $\mathcal{D}_{(V,\mathfrak{C})}$?

Question 3. Let $\mathcal{B}$ represent the set of isomorphism classes of groups in $\mathcal{D}_{(V,\mathfrak{C})}$. Can one find a universal description of this set (e.g., along the lines of the description of the class $\mathcal{A}$ of Lemma 1.1)?

A positive answer to both of the last two questions would provide a universal description of the class of free products available in V.

Finally, we formally state the conjecture mentioned earlier in the introduction.

Conjecture 4. *The group $\mathbf{Z} \wr (\mathbf{Z}^2) \cong (\bigoplus_{\mathbf{Z}^2} \mathbf{Z}) \rtimes \mathbf{Z}^2$ fails to embed in R. Thompson's group V.*

2. Basic definitions and constructions

In this section, we define the terminology of the paper and state some easy or known facts that will be useful to us in our arguments.

2.1. The Cantor set.

We define language and notation describing the Cantor set and various aspects and subsets thereof.

Fix the discrete space $X = \{0, 1\}$, which we will call our *alphabet* X. Let X^* represent the free monoid on X (so that X^* consists of all finite strings over X under the product of concatenation). Define the infinite rooted, directed tree $\mathcal{T}_2$ as follows. The set of nodes of $\mathcal{T}_2$ is the set X^*. If u, $v \in X^*$ we have an edge from u to v if and only if $ux = v$ for some $x \in X$.

Given u, $v \in X^*$, we say that v *is a descendent of* u or that u *is an ancestor of* v if and only if there is $w \in X^*$ so that $uw = v$. In this case we may also say that u *is a prefix of* v and w *is a suffix of* v. We say u *and* v *are unrelated nodes* if and only if neither u nor v is a prefix of the other node.

In this article, if $T \leq \mathcal{T}_2$ is a sub-tree of $\mathcal{T}_2$, then we draw T "upside down" so that the root of T is drawn at the top and so that T "opens downward". Given a node $u \in \mathcal{T}_2$, we draw the descendent $u0$ below and to the left of u, and $u1$ below and to the right of u.

Corresponding to the standard identification of $\{0, 1\}^\omega$ (under the product topology) with the standard middle-third Cantor set $\mathfrak{C}$, we consider $\mathfrak{C}$ to be the boundary of the tree $\mathcal{T}_2$. We then extend this identification, identifying $\mathfrak{C}$ with the set of infinite directed paths in $\mathcal{T}_2$ starting at the root. We call these paths the *infinite descending paths in* $\mathcal{T}_2$.

We will say $x \in \{0, 1\}^\omega$ underlies a node n of $\mathcal{T}_2$ if n is a node of $\mathcal{T}_2$ and the descending path corresponding to x passes through n. This occurs if and only if n is a prefix of the infinite string describing x. We will call the set of such points the *Cantor set underlying* n and denote this set by $\mathfrak{C}_n$ (noting that $\{0, 1\}^\omega \cong \mathfrak{C}_n \cong \mathfrak{C}$ for any node n).

For a finite collection S of points in $\mathfrak{C}$, we will specify that *a neighbourhood* U *of* S is precisely given as a finite union of the Cantor sets underlying a finite set $\mathcal{N}$ of nodes of $\mathcal{T}_2$, where we further require that for each point $s \in S$, there is precisely one node $n_s \in \mathcal{N}$ so that s underlies the node n_s. Given a set $S \subset \mathfrak{C}$, we will call any open subset $U \subset \mathfrak{C}$ containing S a *general open neighbourhood of* S. We shall require the use of a general open neighbourhood only one time in this note.

2.2. R. Thompson's group V.

The group V can be thought of as a specific collection of self-homeomorphisms of $\mathfrak{C}$, under the operation of composition (that is, $V < Aut(\mathfrak{C})$). We first discuss our notation relating to $Aut(\mathfrak{C})$ and then focus on details relevant for V.

Throughout this note, we will have elements of $Aut(\mathfrak{C})$ act on $\mathfrak{C}$ on the right, so that if $x \in \mathfrak{C}$, and u is any homeomorphism $u : \mathfrak{C} \to \mathfrak{C}$, we denote by xu the image of x under the action of u. If $X \subset \mathfrak{C}$, then we also denote by Xu the set $\{xu | x \in X\}$. Following these conventions, if v is also an element of $Aut(\mathfrak{C})$, then the symbols u^v and $[u, v]$ will be defined by the equations $u^v = v^{-1}uv$ and $[u, v] = u^{-1}v^{-1}uv = (v^{-1})^u \cdot v = u^{-1} \cdot u^v$.

We also define $\operatorname{Supp}(u) = \{x \in \mathfrak{C} \mid xu \neq x\}$. We now have the following standard lemma from permutation group theory, which we use freely in the remainder.

Lemma 2.1. *Let* $u, v \in Aut(\mathfrak{C})$*; then* $\operatorname{Supp}(u^v) = \operatorname{Supp}(u)v$*.*

For $x \in \mathfrak{C}$ and for $u \in Aut(\mathfrak{C})$ we define the orbit of x under the action of $\langle u \rangle$ to be the set

$$\mathcal{O}_{(x,u)} = \left\{ xu^k \mid k \in Z \right\}.$$

If this set does not have cardinality one, then we say that *the orbit of x under $\langle u \rangle$ is non-trivial*. Other language to this effect is to be interpreted in the obvious fashion and should cause the reader no confusion.

Let us now focus on the group V. Each element u of V can be represented non-uniquely as a triple such as (D, R, σ), where D and R are finite binary sub-trees of $\mathcal{T}_2$ rooted at the root of $\mathcal{T}_2$ with a fixed number of leaves and where σ is a particular bijection between the leaves of D and the leaves of R. We will use the notation $u \sim (D, R, \sigma)$ when we wish to denote the fact that (D, R, σ) represents u. An example of an element of V as represented by such a triple $P = (D, R, \sigma)$ is given in the diagram below Corollary 2.3, where we describe σ by labelling corresponding leaves from the two trees with the same number.

If $u \in V$ and $u \sim (D, R, \sigma)$, we may refer to (D, R, σ) simply as a *tree-pair representing* u (or by similar language), and we drop explicit mention of the bijection unless we need it in the course of events.

Given a node w of the infinite tree $\mathcal{T}_2$, we will say a homeomorphism $\theta : \mathfrak{C} \to \mathfrak{C}$ *is affine over the Cantor set $\mathfrak{C}_w$ underlying w* when there is an affine map $L : \mathbf{R} \to \mathbf{R}$ so that for all $x \in \mathfrak{C}_w$ we have $xL = x\theta$. In this case, if $x \in \mathfrak{C}_w$, then we will say that the *slope of θ at x is xL'*, the derivative of L at x.

The method which translates a tree-pair such as $P = (D, R, \sigma)$ into a homeomorphism of the Cantor set is as follows. We say P *represents* $u \in V$ if and only if u is the unique homeomorphism of the Cantor set $\mathfrak{C}$ to itself which is orientation preserving and affine when restricted and co-restricted respectively to the Cantor sets underlying the leaves l of D and $l\sigma$ of R, for each leaf l of D. Thus, in this discussion, D represents the domain and R represents the range. Since tree pairs determine how the elements they represent move points in the Cantor set, we might refer to them as *rules*, using language such as "The rule P determines that ..." or other similar phrases.

We note that given any $v \in V$, there is an induced "action" on almost all of the nodes of the infinite binary tree $\mathcal{T}_2$, in the following sense. Suppose m is a descendent of a leaf l of the domain tree of a representative tree pair for v. Then the node m will be mapped to a node n of $\mathcal{T}_2$; that is, the Cantor set underlying m will be carried affinely and bijectively to the Cantor set underlying n. We thus say $mv = n$ when we are thinking of the induced action of v on nodes of $\mathcal{T}_2$. We describe this action with language such as *the node m is carried to n by v*, or by similar language. (Note that nodes above the leaves of the domain tree will very likely get "split" by v so that there is always a finite subset of $\mathcal{T}$ where this sort of induced action makes no sense.)

2.3. **Revealing pairs.** We will make use of Brin's revealing pair technology (see [9]) in order to define interesting subsets of the Cantor set $\mathfrak{C}$ for specific elements of V. The general argument can be given without using revealing pairs, but they provide a useful context for our discussion. Here we will define everything which is required for this note. Lemmas and corollaries appearing in this subsection before Lemma 2.4 can all either be found in [9] or in [29], or are easy consequences of the results found therein.

For the interested reader, in [7] there is a detailed discussion, with an example, explaining how to "read" a general revealing pair so as to understand the dynamics of the action of the element it represents upon the Cantor set. For the remainder of this subsection, let us assume we have a tree pair $P = (D, R, \sigma)$.

We consider the *common tree* $C = D \cap R$, which is the finite subtree of the rooted tree $\mathcal{T}_2$ consisting of the nodes and edges common to both D and R when they are thought of as rooted subtrees of $\mathcal{T}_2$, both rooted at the root of $\mathcal{T}_2$. With this point of view, each leaf of C is either the root of a non-empty subtree of D, the root of a non-empty subtree of R, or a leaf of both D and R. We call each maximal subtree of D which is rooted at a leaf of C a *component of* $D \backslash R$, and each maximal subtree of R which is rooted at a leaf of C a *component of* $R \backslash D$. (Note that here we are using the phrase "component of $D \backslash R$" to represent the phrase "topological component of the closure of $D - R$ in D", where we think of D as a 1-complex.) Each leaf of C which is both a leaf of D and of R will be called a *neutral leaf* of P.

The tree pair P is called a *revealing pair* if it satisfies two conditions. The first condition is that for each component X of $D \backslash R$, X has a leaf r_X which, as a node of $\mathcal{T}_2$, and under iteration of the action of v on the nodes of $\mathcal{T}_2$, travels through the neutral leaves of C until it is finally mapped to the root of X (r_X is unique for X and is called the *repelling leaf of* X or the *repeller of* X). The second condition is similar; if n_Y is the root of a component Y of $R \backslash D$, then iteration of the action of v on the nodes of $\mathcal{T}_2$ must have n_Y travel through the neutral leaves of C until it finally maps to a leaf l_Y of Y which is a decendent of n_Y (the leaf l_y is called the *attracting leaf of* Y or the *attractor of* Y). By the discussion preceding Lemma 10.2 of [9], each element in V has a revealing pair representative (and as before, this is not unique).

Let $v \in V$. An easy consequence of Proposition 10.1 of [9] is that there is a minimal non-negative power k so that v^k acts on $\mathfrak{C}$ with no non-trivial finite orbits. Set $u = v^k$ and fix the symbols v and u throughout the remainder of this subsection and through the next subsection as always representing elements of V, related and having properties as described earlier in this paragraph.

Let us now assume (and through to the end of the next subsection) that the pair $P = (D, R, \sigma)$ is actually a revealing pair representing u. We obtain a list of useful, obvious results, which we leave to the reader to verify. The phrase "maps to" in this following lemma is referring to the action of w on nodes of $\mathcal{T}_2$.

Lemma 2.2. *Suppose $w \in V$ so that w admits no non-trivial finite orbits in its action on the Cantor set $\mathfrak{C}$. Suppose further that a revealing pair $P_w = (D_w, R_w, \sigma_w)$ represents w.*

(1) *Any repeller r_X of a component X of $D_w \backslash R_w$ always maps to the root of X by the rule P_w.*
(2) *The root n_Y of any component Y of $R_w \backslash D_w$ always maps to the attractor l_Y of Y by the rule P_w.*
(3) *The map w restricted to any Cantor set underlying a node r_X or n_Y as above is affine with slope not equal to one.*
(4) *Every point in $\mathfrak{C}$ which is fixed by w and which does not underlie a node r_X or n_Y as above lies under a neutral leaf n of the pair P_w upon which w must act as the identity.*

We continue our discussion with the element u constructed previously.

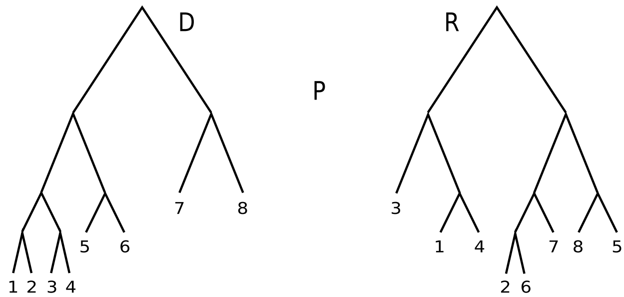

FIGURE 1. The element u.

By an application of the standard Contraction Lemma, we observe that if a leaf l of D is mapped above or below itself in $\mathcal{T}_2$ by the rule P, then there will be a unique fixed point in the Cantor set underlying l (if l maps above itself, consider the inverse map u^{-1} in order to force a contraction). Fixed points underlying repellers of D will be called *repelling fixed points of* u, and fixed points underlying attractors in components of $D\backslash R$ will be called *attracting fixed points of* u.

Corollary 2.3. *Suppose $w \in V$ so that w admits no non-trivial finite orbits in its action on the Cantor set $\mathfrak{C}$. Suppose further that a revealing pair $P_w = (D_w, R_w, \sigma_w)$ represents w.*

For each repeller r_x of a component X of $D_w\backslash R_w$ there is a unique repelling fixed point p_x underlying it, and for each attractor l_Y of $R_w\backslash D_w$ there is a unique attracting fixed point underlying it.

Figure 1 illustrates a possible tree-pair for our element u. This particular tree-pair indicates that the element u has one repelling fixed point (under 0010) and two attracting fixed points (under 10 and 11 respectively).

We call the repelling and attracting fixed points under the action of u the *important points of* u and denote the set of such as $I(u)$.

Conventions:

Throughout the remainder of this paper, if we discuss the important points of an element w of V, it is to be understood that w does not admit finite non-trivial orbits in its action on $\mathfrak{C}$. We also modify the standard concepts of "basin of attraction" and "basin of repulsion" slightly in our context. Given a revealing pair $Q = (S, T, \theta)$ representing w, the Cantor set underlying each root of a component of $S\backslash T$ represents a *repelling basin for* w, and the Cantor set underlying a root of a component of $T\backslash S$ represents an *attracting basin for* w.

2.4. **Flow graphs and components of support.** Now returning to our tree pair P, let $\{E_i\}_{i=1}^n$ represent the components of $D\backslash R$ and let $\{F_j\}_{j=1}^m$ represent the components of $R\backslash D$. For each i, E_i contains a repelling leaf, as defined above, and all other leaves of E_i are called *sources*. Likewise, for each component F_j, there is an attracting leaf, as defined above, and all other leaves are called *sinks*. For each source leaf s_0, there is a path $s_0 = n_0, n_1, \ldots, n_t = s_k$ through neutral leaves $n_1, \ldots, n_{t-1}$ of C, which then visits a sink s_k, so that u^p will throw the Cantor set underlying $s_0 = n_0$ onto the Cantor set underlying n_p for all indices $0 \leq p \leq t$. We call this path the *source-sink chain* $s_0 - s_k$ *for* P. We can now make a bi-partite graph whose vertices are labelled by the repelling and attracting basins and whose edges correspond with and are labelled by source-sink chains connecting repelling basins to attracting basins. We call this graph the *flow graph for P*. (In the general context, the flow graph derived from a revealing pair representing an element of V has other information appended to it, as in [7, 2]. Here, we only define the portions of the standard flow graph required to support the discussion in the remainder of this note.)

A flow graph for our example above is as diagrammed in Figure 2.

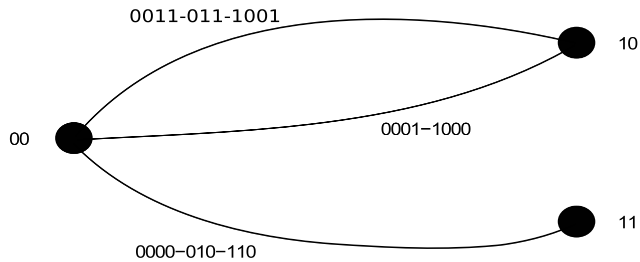

FIGURE 2. A flow graph for u.

Let X now represent a connected component of the flow graph for P. We can form a set, the *Cantor set underlying X*, by taking the union of the Cantor sets underlying the attracting and repelling basins of X, and underlying the neutral leaf nodes occuring in the source-sink chain labels of the edges of X. This union is immediately independent of the revealing pair representing u, so we will also call it a *component of support of u*. We will define the *component support of u*, denoted $\overline{\mathrm{Supp}(u)}$, as the union of the components of support of u. We note that $\overline{\mathrm{Supp}(u)}$ actually is the topological closure of the support of u, since it consists of the support of u together with the important points of u.

In particular, we have a useful lemma, which is easy to see if the above is understood.

Lemma 2.4. *Let $p \in \mathfrak{C}$ and g, $h \in V$ so that they admit no finite non-trivial orbits, with $p \in I(g) \cap I(h)$. Then, there is a node n in $\mathcal{T}_2$ so that*

(1) *p underlies n,*
(2) *$\mathfrak{C}_n \subset \overline{\mathrm{Supp}(g)} \cap \overline{\mathrm{Supp}(h)}$, and*
(3) *the commutator $[g, h]$ acts as the identity over $\mathfrak{C}_n$.*

We now state a collection of points which follow easily from the work in [7], including their proofs for completeness.

Lemma 2.5. *Let g, $h \in V$ so that*

a. *both g and h are non-trivial,*
b. *the commutator $[g, h] = 1$, and*
c. *neither g nor h admit finite non-trivial orbits.*

Then we have the following statements.

(1) *The sets of components of support of g and of h are both non-empty.*
(2) *The following chain of equalities holds:*

$$\mathrm{I}(g) \cap \mathrm{I}(h) = \mathrm{I}(g) \cap \overline{\mathrm{Supp}(h)} = \mathrm{I}(h) \cap \overline{\mathrm{Supp}(g)}.$$

(3) *Let X denote a component of support for g and Y denote a component of support for h. If $X \cap Y \neq \emptyset$, then*
  i. $X \subset \overline{\mathrm{Supp}(h)}$ *and* $Y \subset \overline{\mathrm{Supp}(g)}$, *and in fact,*
  ii. $X = Y$.

(4) *Let $\mathcal{C}_g$ be the components of support of g which non-trivially intersect the components of support of h. Let $\mathcal{C}_h$ be the components of support of h which non-trivially intersect the components of support of g. Under these circumstances, we have*

$$\overline{\mathrm{Supp}(g)} \cap \overline{\mathrm{Supp}(h)} = \bigcup_{A \in \mathcal{C}_g} A = \bigcup_{B \in \mathcal{C}_h} B.$$

Proof. Point 1) follows from the non-triviality of g and h, and the fact that neither of these elements are torsion.

To prove point 2), we show that if $x \in \mathrm{I}(g) \cap \overline{\mathrm{Supp}(h)}$, then $x \in \mathrm{I}(h)$.

If x is not an important point of h, then g must have infinitely many important points, since the full orbit of x under that action of h consists of important points for the functions $g^{(h^n)} = g$.

Point 3.*i.*) can now be shown as follows.

Suppose $x \in X \cap Y$. By Lemma 2.4, we can choose x so that x is not an important point of g or of h.

Let $x_- = \lim_{n\to\infty} xg^{-n}$ and $x_+ = \lim_{n\to\infty} xg^n$. It is immediate that these limits exist and that $x_-, x_+ \in \mathrm{I}(g)$.

We further see that both x_- and x_+ are important points for h by applying 2). For example, if x_+ is not in $\overline{\mathrm{Supp}(h)}$, then there is $n \in N$ with $xg^n h = xg^n \neq xhg^n$.

Now, as X underlies a connected component $\mathcal{C}$ of the flow graph of g, the set of important points of g in X is actually contained in the component support of h. Therefore, some neighbourhood of these points in X is actually contained in the component support of h. It now follows that all of X must be contained in the component support of h and, by symmetry, all of Y must be contained in the component support of g.

We leave point 4) to the reader, as it easily follows from 3.*i.*).

We now show 3.*ii.*), assuming 1), 2) and 3.*i*), and 4).

We note that $(X \cap Y)\backslash(\mathrm{I}(g) \cup \mathrm{I}(h))$ is not empty, and we choose some point x in this set. Now define $x_+ = \lim_{n\to\infty} xg^n$ as before.

Suppose there is $m \in N$ so that $xh^m \notin X$.

Since x_+ is an important point of h as well as g, if S is large enough, then $x_S = xg^S$ will be close enough to x_+ such that m applications of h to x_S will result in a point still in a basin of attraction of g containing x_+ (recall that such a basin is a Cantor set underlying a root of a complimentary component of $R_g\backslash D_g$ for some representative revealing pair $P_g = (D_g, R_g, \sigma_g)$ for g). In particular, we see that $xh^m g^S$ is not in X while $xg^S h^m \in X$.

Our result now follows from the connectivity of the component of the flow graph of g over X, the connectivity of the component of the flow graph of h over Y, the fact that given any important point p in $X \cap Y$, there is a neighbourhood N_p of that point which is fully contained in both X and Y (so that all orbits in X under g which enter N_p must be fully contained in Y and all orbits in Y under h which enter N_p must stay in X), and the fact that every point in X or Y limits to the important points of g and h under repeated applications of g or h. $\square$

The following lemma is reminiscent of a similar result by Brin and Squier for elements of Thompson's group F (or actually, for $\mathrm{PL}_o(I)$) in [11], and the proof is philosophically the same (although the details here are slightly more complicated due to the presence of extra attractors and repellers). This lemma combines very powerfully with conclusion 3.*ii*) of Lemma 2.5.

Lemma 2.6. *Suppose that g, $h \in V$ so that they admit no finite non-trivial orbits. Suppose further that g and h have a common component of support X and that the actions of g and h commute over X. Then there are non-trivial powers m and n so that $g^m = h^n$ over X.*

Proof. Let us suppose g is represented by a revealing pair $P_g = (D_g, R_g, \sigma_g)$ so that given an important point q of g, the phrase "the basin containing q" is well defined (i.e., the Cantor set underlying the root of the complimentary component of $D_g \backslash R_g$ or of $R_g \backslash D_g$ which has a repelling or attracting leaf as a node over q).

Fix p, an important point of both g and h in X. Since g and h are affine in a neighbourhood N_p of p, there are non-trivial powers m and n so that $g^m = h^n$ on N_p. Now the element $g^m h^{-n}$ is trivial on N_p. For each $x \in \mathrm{I}(g) \cap X$ where $g^m h^{-n}$ is fixed on some neighbourhood of x, let N_x be such a neighbourhood, and let $\mathcal{N}$ be the union of these neighbourhoods.

We now have that $\mathcal{N}$ is actually a neighbourhood of $\mathrm{I}(g) \cap X = \mathrm{I}(h) \cap X$. Otherwise, there is a pair $r, a \in \mathrm{I}(g) \cap X$ with a an attracting fixed point of g and r a repelling fixed point of g, where one of $\{a, r\}$ is in $\mathcal{N}$ and the other is not in $\mathcal{N}$, and where there is a source-sink chain from the basin B_r containing r to the basin B_a containing a (this follows from the connectivity of the component of the flow graph of g over X). In the case where $a \in \mathcal{N}$, there is $x \in \mathrm{Supp}(g^m h^{-n}) \cap B_r$ and a positive power k so that $xg^k \in \mathcal{N} \cap B_a$. In the case where $r \in \mathcal{N}$, there is $x \in \mathrm{Supp}(g^m h^{-n}) \cap B_a$ so that a negative power k has $xg^k \in \mathcal{N} \cap B_r$. In either of these cases, Lemma 2.1 now shows that $(g^m h^{-n})^{g^k}$ has support where $g^m h^{-n}$ acts as the identity, which is impossible since g commutes with g and with h.

Now again by Lemma 2.1, $g^m h^{-n}$ cannot have any support in X. Otherwise, for any $x \in X \cap \mathrm{Supp}(g^m h^{-n})$, there is a positive power k of g so that xg^k is near to an attractor of g in X, in particular, k can be taken large enough so that $xg^k \in \mathcal{N}$. But then $(g^m h^{-n})^{g^k}$ has support in $\mathcal{N}$. □

3. Demonstrative subgroups and embedding theorems

We first develop a basic tool which assists us in finding embeddings of some products of subgroups of V into V.

3.1. The definition and some basic properties of demonstrative groups.

It is now time to give the definition of demonstrative subgroups, as mentioned in the introduction. Suppose that a group H acts on a space Y. We say that a subgroup $G \leq H$ is a *demonstrative subgroup of H over Y* if and only if there is an open set $U \subset Y$ so that for any two elements g_1, $g_2 \in G$ we have $Ug_1 \cap Ug_2 \neq \emptyset$ if and only if $g_1 = g_2$. In this case, we write $G \in \dot{\mathcal{D}}_{(H,Y)}$ and we say U is a demonstration set for G. If Y is understood, we may simply say G is a *demonstrative subgroup of* H. We denote by $\mathcal{D}_{(H,Y)}$ the full class of groups which are isomorphic to a group in $\dot{\mathcal{D}}_{(H,Y)}$.

We call any node $p \in \mathcal{T}_2$ which has the set underlying p as a demonstration set for a demonstrative subgroup $G \in \dot{\mathcal{D}}_{(V,\mathfrak{C})}$ a *demonstration node for* G.

There is a collection of obvious facts about demonstrative subgroups of V, whose facts, in turn, demonstrate why we are interested in these groups.

Lemma 3.1. *Let $G \leq V$. If G is a demonstrative group, then there is a node n of $\mathcal{T}_2$ so that for all non-trivial $g \in G$,*

$$\mathfrak{C}_n g \cap \mathfrak{C}_n = \emptyset.$$

Proof. Let $G \in \dot{\mathcal{D}}_{(V,\mathfrak{C})}$, and suppose U is a demonstration set for G. The U can be written as a union of basic open sets in the topology of $\mathfrak{C}$, thus U contains the Cantor set $\mathfrak{C}_n$ underlying some node $n \in \mathcal{T}_2$. Now, as $Ug_1 \cap Ug_2 = \emptyset$ whenever g_1, $g_2 \in G$ with $g_1 \neq g_2$, we must have $\mathfrak{C}_n g \cap \mathfrak{C}_n = \emptyset$ for all non-trivial $g \in G$. $\square$

Thus, if G is a demonstrative group, with node d as a demonstration node for G, then $\{dg | g \in G\} = \mathscr{D}_G$ is a set node of the infinite binary tree, in bijective correspondence with G. In this case G acts on $\mathscr{D}_G$ on the right as G acts on itself via right multiplication and, if $n, m \in \mathscr{D}_G$, then $m \neq n$ implies that m and n are not related.

We now show that the set of demonstrative groups is closed under some nice operations.

Lemma 3.2. *Suppose G and H are demonstrative groups and G is finite. Suppose further that m serves as a demonstration node for G and n serves as a demonstration node for H. Then*

(1) *Given any subgroup $K \leq H$, we have that K is demonstrative and that m serves as a demonstration node for K and*
(2) *there is a demonstrative group L with $L \cong G \times H$ with the node mn serving as a demonstration node for L.*

Proof. The first point is immediate from the definition of a demonstrative group.

The second point requires a bit more care. Let m be a demonstration node for G. For each element $g \in G$, let $P_g = (D_g, R_g, \sigma_g)$ be a revealing pair for g which has m as a neutral leaf. Since G is finite, the orbit of m in $\mathcal{T}_2$ under the action of G is a finite collection $\mathcal{O}_m$ of nodes of $\mathcal{T}_2$.

We note Lemma 3.1 guarantees that $g_1 \neq g_2 \in G$ implies $\mathfrak{C}_m g_1 \cap \mathfrak{C}_m g_2 = \emptyset$.

Let n be a demonstration node for H. We will find H_m, an embedded copy of H in V, which is demonstrative with demonstration node mn (concatenate the names of the nodes m and n in $\mathcal{T}_2$) and where $\langle G, H_m \rangle \cong G \times H$ is also demonstrative with demonstration node mn.

We build H_m as follows. For every element $h \in H$, let $P_h = (D_h, R_h, \sigma_h)$ be a representative tree-pair for h which has n as a neutral leaf, in accordance with the definition of H being demonstrative with demonstration node n. We now build the tree-pair $P_{\overline{h}} = (D_{\overline{h}}, R_{\overline{h}}, \sigma_{\overline{h}})$ representing $\overline{h}$, the element of H_m which will be the image of h under the embedding $H \to V$ sending H to H_m.

Let T be a finite binary subtree in $\mathcal{T}_2$ with its root the root of $\mathcal{T}_2$ which contains every node in $\mathcal{O}_m$ (there are infinitely many such nodes if $\mathcal{O}_m$ does not form the set of leaves for a finite binary subtree of $\mathcal{T}_2$ with its root the root of $\mathcal{T}_2$). Define $D_{\overline{h}}$ to be the extension of the tree T, where we append the tree D_h to each of the leaves of T in $\mathcal{O}_m$. Define $R_{\overline{h}}$ to be the extension of T we get when we append the tree R_h to each leaf of T in $\mathcal{O}_m$. Use the identity permutation on the leaves of T not in $\mathcal{O}_m$, and for a particular node $l \in \mathcal{O}_m$, use the corresponding bijection for h on the leaves under l in the domain and range trees $D_{\overline{h}}$ and $R_{\overline{h}}$.

It is immediate from construction that this produces a set of elements H_m so that $\langle H_m \rangle \cong H$.

The node mn is demonstrative for H_m. It is also demonstrative for G. Let A be a tree which has n as a node. Now, for each element g of G, simply append A to every leaf in the finite cycle of neutral leaves containing m for the tree pair P_g in order to make a new revealing tree pair for g which now has the node mn as a neutral leaf (extending the bijection in the obvious fashion).

Since H_m acts in the same fashion under every node in $\mathcal{O}_m$, we see that the elements of G and the elements of H_m commute and that $\mathfrak{C}_{mn}\alpha \cap \mathfrak{C}_{mn} = \emptyset$ for any non-trivial $\alpha \in \langle G, H_m \rangle$. $\square$

Note that the assumption that G is finite cannot be removed from the proof of Lemma 3.2; the homeomorphisms of $\mathfrak{C}$ created in the process of realizing the non-trivial elements of H_m would not be in V.

Lemma 3.3. *There are demonstrative embeddings of*

(1) *every finite group,*
(2) Z*, and*
(3) Q/Z

in V*. Furthermore, for each such group, we can find a demonstrative embedding with demonstration node "0".*

Proof. We first show that for any fixed positive natural number n, the symmetric group on n letters S_n has a demonstrative representation with demonstration node "0".

Let T be a binary tree with $n!$ leaves, which includes "0" as a leaf, with a secondary labelling on leaves using the elements of S_n in some order (with the node "0" being labelled with the identity element). Represent each element α of S_n by the tree-pair (T, T), and use the bijection which sends the node labelled m to the node labelled with the group element $m \cdot \alpha$. It is immediate that this is a faithful representation of S_n and that every node is moved by every non-trivial element of S_n. In particular, the node "0" labelled by the identity element in our initial labelling of T works as well as any other leaf of T as our demonstration node.

Passing to a subgroup of a demonstrative group G with demonstration node m produces a demonstrative group with demonstration node m by Lemma 3.2, so every finite group has a demonstrative representation with node "0" as a demonstration node.

Figure 3 represents a revealing pair for an element $g \in V$ that generates an infinite cyclic group with demonstration node "0".

Finally, the embedding of Q/Z in V as given in Proposition 5.6 of [8] is demonstrative. In Figure 2 of that article, either of the quarters "01" and "11" can serve as demonstration nodes for the embedded image of Q/Z in V. More generally, the

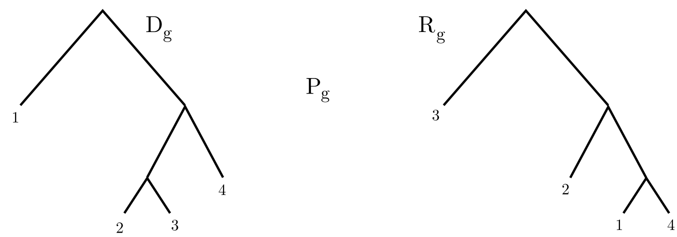

FIGURE 3. An element g.

node for "$J_{2,1}$" given in the embedding described in that article (whose node is chosen by the reader) can serve as the demonstration node for that embedding. Now consider the affine conjugate demonstrative embedding of Q/Z in V, determined by conjugating the demonstrative embedding above by the affine real map $x \mapsto 1/3x+2/3$. This produces a conjugate image of Q/Z in V which has the "same action" as our original embedding, but only on the right hand side of the tree $\mathcal{T}_2$ (and it acts as the identity on the left hand side). Conjugate the new version of Q/Z in V by the unique element of V which precisely switches the two nodes "0" and "101" in order to produce a demonstrative copy of Q/Z with demonstration node "0". (Here we have been careful to make our demonstration node become "0" using two conjugations in a fashion which will guarantee that "0" will never be "split" by the "action" of the conjugate version of our initial demonstrative group on the nodes of the infinite tree.) □

In fact, by essentially the argument given above for the case of Q/Z, if we have a demonstrative group $\widehat{G}$, there is a version G of $\widehat{G}$ with demonstration node "0". We therefore have the following lemma.

Lemma 3.4. *Suppose G is a demonstrative subgroup of V. There is an isomorphic copy G_0 of G in V with demonstration node "0" and another isomorphic copy G_1 of G with demonstration node "1".*

The following is an immediate consequence of Lemma 3.2, Lemma 3.3, the definition of the demonstrative groups, and the definition of the class of groups $\mathcal{A}$.

Corollary 3.5. *If G is a group in the class $\mathcal{A}$, then there is a demonstrative group K so that $G \cong K$.*

3.2. **Demonstrative groups, wreath products, and extensions.** Let H be a group acting faithfully on a space Y. We say H *acts with local realization* if given any open set $U \subset Y$, we can find a subgroup $H_U \leq H$ so that $H_U \cong H$ and $\mathrm{Supp}(H_U) \subset U$. Note that H acting faithfully guarantees us that any such H_U will also act faithfully.

Groups acting with local realization appear to be more common than one might initially expect. For example, in the usual representations of R. Thompson's groups F, T, and V as groups of homeomorphisms of the unit interval $[0,1]$, the circle S^1, or the Cantor set $\mathfrak{C}$ respectively (as given, say, in Cannon, Floyd, and Parry's survey [14]), we have that both F and V act with local realization, while T does not.

In any case, the following proposition now follows directly from the definitions, and we obtain Theorem 1.2 as an immediate corollary.

Proposition 3.6. *Suppose H acts on a space Y with local realization and that $G \leq H$ is a demonstrative subgroup of H with demonstration set U. Then the standard restricted wreath product $H \wr G$ embeds in H.*

Proof. Suppose $H_U \leq H$ so that $H_U \cong H$ and $\mathrm{Supp}(H_U) \subset U$, where U is a demonstration set for a demonstrative group $G \in \dot{\mathcal{D}}_{(H,Y)}$. For any two elements $g_1, g_2 \in G$ with $g_1 \neq g_2$, we see that the groups $H_U^{g_1}$ and $H_U^{g_2}$ are both congruent to H and have disjoint supports in Y (and hence, $[H_U^{g_1}, H_U^{g_2}] = \{1\}$). It is therefore immediate that $\langle H_U, G\rangle \cong H \wr G$. □

We note in passing that it is not easy to find a non-trivial wreath product as a demonstrative subgroup; it is difficult to find an open set which will move entirely off itself under the action of all possible non-trivial elements in the base group.

Of course, recapping a point in the introduction, if $G \leq V$ and H is a finite group, then the above proposition assures us that $G \wr H$ embeds in V. Now as a consequence of the work of Kaloujnine and Krasner in [21, 22, 23], we see that any finite extension of any subgroup of V will embed in V. That is, we have shown Corollary 1.3.

3.3. **Free products which are embeddable in V.** In this subsection we prove Theorem 1.4. Our main constructive tool is the standard Ping-Pong Lemma of Fricke and Klein [20].

3.3.1. *Free product recognition.* We give the version of the Ping-Pong Lemma essentially as it appears in [1].

Lemma 3.7 (Ping Pong Lemma)**.** *Let G be a group acting on a set X and let H_1, H_2 be two subgroups of G such that $|H_1| \geq 3$ and $|H_2| \geq 2$. Suppose there exist two non-empty subsets X_1 and X_2 of X such that the following hold:*

- *X_1 is not contained in X_2,*
- *for every $h_1 \in H_1$, $h_1 \neq 1$ we have $h_1(X_2) \subset X_1$,*
- *for every $h_2 \in H_2$, $h_2 \neq 1$ we have $h_2(X_1) \subset X_2$.*

Then the subgroup $H = \langle H_1, H_2\rangle$ of G is isomorphic to the free product of H_1 and H_2:

$$H \cong H_1 * H_2.$$

3.3.2. *Free products of demonstrative subgroups of V embed in V.* In order to make use of the Ping-Pong Lemma, we first define a set of subgroups of V which are easy to use as factors in free product decompositions.

We are now ready to prove our first primary result.

Proof of Theorem 1.4. By Corollary 3.5, we need only show that given two demonstrative groups, we can find a copy of their free product in V.

In general, this will follow easily from the Ping-Pong Lemma. We will need a separate argument for $Z_2 * Z_2$.

Let G and H be non-trivial demonstrative groups, not both isomorphic with Z_2. Let $X_1 = \mathfrak{C}_1$ and $X_2 = \mathfrak{C}_0$, that is, the right and left halves of the Cantor set, respectively.

Let G_0 and H_1 be the copies of G and H with demonstration nodes "0" and "1", respectively (as in Lemma 3.4).

We note that any non-trivial element of G_0 takes the entire left half of the Cantor set $\mathfrak{C}_0$ into the right half $\mathfrak{C}_1$. Similarly, any non-trivial element of H_1 takes the entire right half of the Cantor set into the left half of the Cantor set. Therefore, by the Ping-Pong Lemma, the group $\langle G_0, H_1\rangle \cong G_0 * H_1 \cong G * H$.

Now let $G = \langle g\rangle$ and $H = \langle h\rangle$, where g and h are represented by the tree-pairs $P_g = (D_g, R_g)$ and $P_h = (D_h, R_h)$, as in Figure 4. Both g and h are order two, so $G \cong Z_2 \cong H$. However, direct calculation shows that gh has infinite order. In particular, $\langle G, H\rangle \cong Z_2 * Z_2$. □

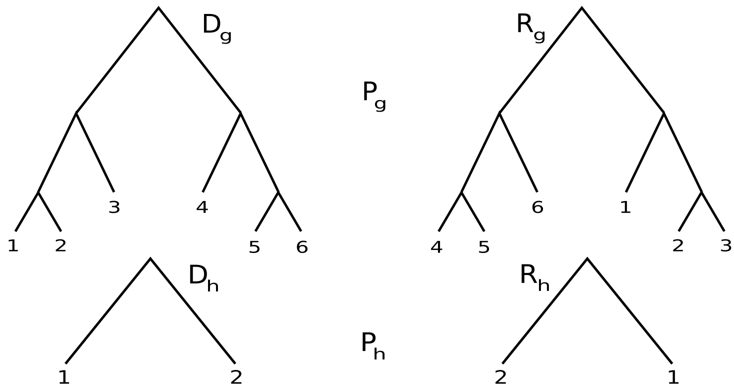

FIGURE 4. Generators for $Z_2 * Z_2$ embedded in V.

4. NON-EMBEDDING RESULTS

We now begin to prove our primary non-embedding result, Theorem 1.5.

After we develop some algebraic processes with controlled dynamical impact, we will carry out a direct computation which will show that any purported embedding of $Z^2 * Z$ in V has non-trivial kernel.

4.1. **Some commutators in $Z^2 * Z$.** Let Y be a non-empty set. Let Y^{-1} be a set disjoint from Y in bijective correspondence with Y. If $\tau : Y \to Y^{-1}$ is the bijection, for each $a \in Y$, denote by a^{-1} the element $a\tau$. If $z \in Y^{-1}$, denote by z^{-1} the element $z\tau^{-1}$. We will call Y the alphabet, and for any element $a \in (Y \cup Y^{-1})$, we will call a a letter. We will call any finite string of letters a word in Y. If $w = w_1w_2\cdots w_k$ is a word in Y, then we will denote by w^{-1} the word $w_k^{-1}w_{k-1}^{-1}\cdots w_1^{-1}$. For any integer n, define the expression w^n as n successive occurrences of the word w if $n \geq 0$ and $-n$ successive occurrences of the word w^{-1} if $n < 0$.

Now let a, b, and c be words in Y. We will say that a word w in Y is an (a, b, c)-*commutator* if there are minimal integers $n > 0$, x_i, y_i, and z_i with $|x_i| + |y_i| \neq 0$ and $z_i \neq 0$ for all indices $1 \leq i \leq n$ so that

$$w = [a^{x_1}b^{y_1}, [a^{x_2}b^{y_2}, \ldots [a^{x_{n-1}}b^{y_{n-1}}, [a^{x_n}b^{y_n}, c^{z_n}]^{z_{n-1}}]^{z_{n-2}} \ldots]^{z_1}].$$

Note that in this paper, the commutator bracket $[u, v]$ will always represent the expression $u^{-1}v^{-1}uv$, as before for general elements of V.

The following is immediate from the definition of an (a, b, c)-commutator.

Lemma 4.1. *Suppose Y is an alphabet with a, b, and c words in Y, and let t be an (a, b, c)-commutator. If k is a non-zero integer, and w is an (a, b, t^k)-commutator, then w is an (a, b, c)-commutator.*

In the next lemma, we abuse notation by treating words in the alphabet $\{a,b,c\}$ as elements of the group $Z^2 * Z$ given by the presentation $\langle a,b,c \mid [a,b]\rangle$ (recall that by our definition, words in an alphabet also include "inverse" letters). Given words $w_1, w_2, \ldots, w_j$, in the alphabet $\{a,b,c\}$, we denote by $\langle w_1, w_2, \ldots, w_j\rangle$ the subgroup of $Z^2 * Z$ generated by the elements represented by these words. Henceforth, we will confuse words in an alphabet with group elements when it seems unlikely to cause confusion.

Lemma 4.2. *Let i, j, and k be integers, and define $t = [a^ib^j, c^k]$. If $|i| + |j| \neq 0$ and $k \neq 0$, then $\langle a,b,t\rangle$ factors as $\langle a,b\rangle * \langle t\rangle \cong Z^2 * Z$.*

Proof. Let

$$w = A_0T_0A_1T_1 \ldots A_nT_n, \tag{1}$$

where $A_p \in \langle a,b\rangle$ and $T_p \in \langle t\rangle$ for all valid indices p, where $Y_p \neq 1$ for $Y \in \{A,T\}$ except possibly for A_0 or T_n and where if $n = 0$, one of A_0, T_0 is non-trivial in $\langle a,b\rangle$ and $\langle t\rangle$ respectively. We say w is given in form (1).

We will have our lemma if we can show that w does not represent a trivial element in $Z^2 * Z$.

We proceed by induction on n. We will use the phrase *resultant form* for any expression written as

$$\prod_{p=0}^{m} a^{x_p}b^{y_p}c^{z_p},$$

where m is an integer with $0 \leq m$, $|x_p| + |y_p| \neq 0$ if $p > 0$ and where $z_p \neq 0$ if $p < m$. We note that in all such forms, the resulting expression cannot represent the trivial element in $Z^2 * Z$ unless $m = x_0 = y_0 = z_0 = 0$.

We shall prove our result by showing that if w is given as in (1), where we further have that $T_n \neq 1$, then w will have resultant expression ending with one of the two forms below:

$$c^{-k}a^{fi}b^{fj}c^k, \tag{2a}$$

$$c^{-k}a^{-fi}b^{-fj}c^ka^ib^j \tag{2b}$$

(where in both forms f is a positive integer). In both cases w cannot be trivial in $Z^2 * Z$.

The more general result then follows; elements of the form a^xb^y are not trivial in $Z^2 * Z$ if $|x| + |y| \neq 0$, and any element of $Z^2 * Z$ with a resultant expression ending with either of the forms in (2) cannot be made trivial by a post-multiplication by a string a^xb^y for integer values of x and y.

We now begin our induction.

(1) Suppose $w = A_0T_0$.

In this base case, if A_0 is trivial, the resultant form is t^z for some $z \neq 0$. This expression ends with one of the two forms in (2) (the resultant form depends on the sign of z). In a similar fashion, if both A_0 and T_0 are not trivial in $\langle a,b\rangle$ and $\langle t\rangle$ respectively, we then have

$$w = A_0T_0 = a^xb^yt^z = a^xb^y(a^{-i}b^{-j}c^{-k}a^ib^jc^k)^z.$$

If $z < 0$, this word admits no simple cancellations in the group $Z^2 * Z$ and is thus already effectively in the resultant non-trivial form in $Z^2 * Z$ (formally, we will need to re-arrange the orders of some a's and b's after expanding the negative power z). In any case, the word will end in the form $c^{-k}a^{-i}b^{-j}c^k a^i b^j$, which is in form (2b).

If $z > 0$, then w simplifies as

$$w = a^{x-i}b^{y-j}c^{-k}a^i b^j c^k (a^{-i}b^{-j}c^{-k}a^i b^j c^k)^{z-1},$$

which again is a non-trivial resultant form in $Z^2 * Z$, even if the leading a^{x-i} and b^{y-j} terms are trivial. In particular, this word ends in form (2a).

(2) Suppose now that n is some positive integer, and we know by induction that for any expression of the form

$$\prod_{p=0}^{m} A_p t^{s_p}$$

(where $n > m \geq 0$, $A_p \neq 1$ for all $p > 0$, and $s_p \neq 0$ for all p) the resultant form of our word ends in one of the two forms in (2).

We now show that w has resultant form in (2).

To begin, we note that w can be expressed as

$$w = rA_n t^{s_n},$$

where r is expressed in resultant form and ends in one of the two forms in (2) (by our induction hypothesis), where $s_n \neq 0$ and where A_n is not trivial in $\langle a, b\rangle$.

There are now two cases in our analysis, each of which splits into two further subcases.

2.(a) Suppose r ends with the form $c^{-k}a^{fi}b^{fj}c^k$, where $f > 0$, $|i|+|j| \neq 0$, and $k \neq 0$.

If $s_n < 0$ we obtain

$$w = \ldots c^{-k}a^{fi}b^{fj}c^k \cdot A_n \cdot c^{-k}a^{-i}b^{-j}c^k a^i b^j \cdot \ldots \cdot c^{-k}a^{-i}b^{-j}c^k a^i b^j,$$

where there are $-s_n$ copies of $c^{-k}a^{-i}b^{-j}c^k a^i b^j$ at the end of this expression. In this case there is absolutely no internal cancelling, and the expression as written above is in resultant form (recall that A_n is not trivial in $\langle a, b\rangle$). This expression is in form (2b).

If $s_n > 0$ we obtain

$$w = \ldots c^{-k}a^{fi}b^{fj}\mathbf{c^k A_n} \cdot \mathbf{a^{-i}b^{-j}c^{-k}}a^i b^j c^k \cdot \ldots \cdot a^{-i}b^{-j}c^{-k}a^i b^j c^k,$$

which either fails to cancel the A_n expression with $a^{-i}b^{-j}$, in which case we obtain form (2a), or which has A_n cancel with $a^{-i}b^{-j}$, in which case the bold substring above cancels so that we obtain the form

$$\begin{aligned} w &= \ldots c^{-k}a^{fi}b^{fj} \cdot a^i b^j c^k \cdot \ldots \cdot a^{-i}b^{-j}c^{-k}a^i b^j c^k \\ &= \ldots c^{-k}a^{(f+1)i}b^{(f+1)j}c^k \cdot a^{-i}b^{-j}c^{-k}a^i b^j c^k \cdot \ldots \cdot a^{-i}b^{-j}c^{-k}a^i b^j c^k, \end{aligned}$$

where we again obtain form (2b), even in the case where $s_n = 1$.

2.(b) Suppose instead that r ends in the form $c^{-k}a^{-fi}b^{-fj}c^k a^i b^j$ where $f > 0$, $|i| + |j| \neq 0$, and $k \neq 0$.

If $s_n < 0$ we obtain

$$w = \ldots c^{-k}a^{-fi}b^{-fj}c^k a^i b^j \cdot A_n \cdot c^{-k}a^{-i}b^{-j}c^k a^i b^j \cdot \ldots \cdot c^{-k}a^{-i}b^{-j}c^k a^i b^j.$$

In the above there are $-s_n$ total occurrences of $c^{-k}a^{-i}b^{-j}c^k a^i b^j$ at the end of the expression.

If $a^ib^j \cdot A_n$ is trivial in $\langle a, b\rangle$, then the expression reduces to the resultant form

$$w = \ldots c^{-k}a^{-(f+1)i}b^{-(f+1)j}c^ka^ib^j \cdot \ldots \cdot c^{-k}a^{-i}b^{-j}c^ka^ib^j,$$

which is in form (2b), even when $s_n = -1$. If $a^ib^j \cdot A_n$ is not trivial in $\langle a, b\rangle$, then there is less reduction, and again we obtain an expression in form (2b).

If $s_n > 0$ we obtain

$$w = \ldots c^{-k}a^{-fi}b^{-fj}c^k\mathbf{a^ib^j} \cdot \mathbf{A_n} \cdot \mathbf{a^{-i}b^{-j}}c^{-k}a^ib^jc^k \cdot \ldots \cdot a^{-i}b^{-j}c^{-k}a^ib^jc^k.$$

In this expression there are s_n copies of the sub-expression $a^{-i}b^{-j}c^{-k}a^ib^jc^k$ at the end. The bold sub-expression $a^ib^j \cdot A_n \cdot a^{-i}b^{-j}$ resolves to A_n, which is non-trivial, and there can be no further reductions, so that we have a form where no 'c' sub-expressions can cancel, and we obtain an expression in form (2a).

Thus, in all cases, our expression for w ends in one of the forms in (2). $\square$

Lemma 1.6 from the introduction now follows from the previous two lemmas.

4.2. $Z^2 * Z$ **cannot embed in** V**.** Throughout the remainder of the paper, we will assume $G = \langle a, b, c \mid [a, b]\rangle \cong Z^2 * Z$ and that $\phi : G \to V$ is a homomorphism. Define $\alpha = a\phi$, $\beta = b\phi$, and $\gamma = c\phi$.

We now find positive integral powers k, r, s, and t, an $(\alpha^r, \beta^s, \gamma^t)$-commutator θ, and further non-zero integral powers x and y so that $\omega = [\theta^k, (\theta^k)^{\alpha^x\beta^y}]$ has order 1, 2, 3, or 6. This will demonstrate Theorem 1.7, since an $(\alpha^r, \beta^s, \gamma^t)$-commutator is also an (α, β, γ)-commutator if r, s, and t are not zero.

Note that it is an easy consequence of Lemma 1.6 that if ϕ were an embedding, then any such ω would have to be an element of infinite order in V. Thus, this process will also demonstrate Theorem 1.5.

In the constructions below, we quickly fix the values of the integers r, s, and t referred to above. Then, we examine many different $(\alpha^r, \beta^s, \gamma^t)$-commutators as prospective candidates for the θ we seek. We will call each of these θ, but we will repeatedly replace a current version of θ with a new and improved version. If we find one such θ which evaluates as the identity, then we choose that one as our final θ, and the claimed result follows (ω will then be the identity as well). If we discover a candidate θ with finite order $p \neq 1$, then $[\alpha^r\beta^s, \theta^p]$ will be an $(\alpha^r, \beta^s, \gamma^t)$-commutator which evaluates as the identity in V and we will switch to this commutator as our new θ, and again we will obtain our desired result. Thus, in the discussion below, we will make the simplifying assumption that all $(\alpha^r, \beta^s, \gamma^t)$-commutators we encounter are of infinite order. (It is easy to check if an element of V is of finite order. Any revealing pair for the element will have domain and range trees identical, and in turn the property of an element of V having a representative tree-pair with equivalent domain and range trees implies the element has finite order. Furthermore, note that finding a revealing pair for an element of V (given any initial representative tree pair) is an easy process. Burillo, Cleary, Stein and Taback's paper [12] provides the first proof that an element of V has finite order only if it admits a tree-pair representation with identical domain and range trees.)

4.2.1. *Removing finite periodic orbits.* If any of α, β, or γ are trivial or of finite order, then it is easy to pick positive values for r, s, t so that there is some $(\alpha^r, \beta^s, \gamma^t)$-commutator which is the identity in V. In this case, we obtain the conclusion of Theorem 1.7 with little effort. Thus, we will assume that each of α, β, and γ has infinite order.

Now, by Proposition 10.1 of [9], one can find positive integers r, s, and t so that $\mu = \alpha^r$, $\nu = \beta^s$, and $\rho = \gamma^t$ do not admit non-trivial finite orbits. We fix our choices for these integers for the remainder of this article.

In this subsubsection we will also develop some notation for various sets and quantities which will be important in the remainder.

By Lemma 2.5 we see that μ and ν have various components of support, some of which may be common to both elements. In particular, we can give names to various sets as follows. Let $\{A_i\}_{i=1}^m$ represent the components of support of μ which are disjoint from the components of support of ν. Let $\{B_j\}_{j=1}^n$ represent the components of support of ν which are disjoint from the components of support of μ. Finally, let $\{C_k\}_{k=1}^p$ represent the common components of support of μ and ν.

For each component of support C_k, fix non-trivial integers m_k and n_k so that $\mu^{m_k}\nu^{n_k}$ has trivial action over C_k. These integers exist by Lemma 2.6.

Set our initial θ as $\theta := [\mu\nu, \rho]^j$, where j is chosen as the minimal positive integer so that θ admits no non-trivial finite orbits. Our strategy will be to repeatedly improve θ by replacing it with powers of carefully chosen (μ, ν, θ)-commutators. This process will steadily improve the interaction between dynamical aspects of the action of the group $\langle \mu, \nu \rangle$ on $\mathfrak{C}$ and dynamical aspects of the action of the group $\langle \theta \rangle$ on $\mathfrak{C}$, until we can claim the conclusion of Theorem 1.7.

4.2.2. *Improving θ so that $I(\theta) \cap (I(\mu) \cup I(\nu)) = \emptyset$.* Our goal in this subsubsection is to arrange the components of support of the final version of θ so that the set of important points of θ is disjoint from the sets of important points of both μ and ν.

Let $S = I(\theta) \cap (I(\mu) \cup I(\nu))$. If S is non-empty, then define the (μ, ν, θ)-commutator

$$\tau := [\mu\nu, \theta].$$

Otherwise, we already have the goal of the subsubsection and we can pass to the next subsubsection without modifying θ.

If $x \in S$, then either x is an important point of $\mu\nu$ or $\mu\nu$ acts as the identity in a neighbourhood of x (μ and ν may act as local inverses in a neighbourhood of x). In either case, τ will act as the identity in a neighbourhood of x (either by invoking Lemma 2.4 or simply, if $\mu\nu$ acts trivially near x, then the commutator resolves as $\theta^{-1}\theta$ in a neighbourhood of x).

Improve θ by replacing it with τ^j, where j is the least positive integer so that the new θ admits no non-trivial finite orbits.

We now should mention a useful lemma.

Lemma 4.3. *If y is an important point of μ or ν and if ρ acts as the identity in some neighbourhood $\mathcal{M}_y$ of y, then any (μ, ν, ρ)-commutator τ will act as the identity in some neighbourhood $\mathcal{N}_y$ of y.*

Proof. We show that if p, q are integers with $|p| + |q| > 0$, $z \neq 0$ is an integer, and $\tau = [\mu^p\nu^q, \rho^z]$, then y has a neighbourhood $\mathcal{N}_y$ so that τ acts as the identity on $\mathcal{N}_y$. The general lemma then follows by an easy induction.

Let p, q, z and τ be as in the previous paragraph. We have

$$\tau = (\rho^{-1})^{\mu^p\nu^q} \cdot \rho.$$

In particular, the support of τ is contained in the union $\mathrm{Supp}(\rho)\cup$ $\mathrm{Supp}(\rho)\mu^p\nu^q$. Let $\mathcal{M}_y$ be a neighbourhood of y disjoint from the action of ρ, and let m be the node of $\mathcal{T}_2$ corresponding to $\mathcal{M}_y$. Pass deeply into $\mathcal{T}_2$ beneath the node m to find a node n which has y underlying n, and so that $\mu^p\nu^q$ acts affinely over the Cantor set $\mathcal{N}_y$ underlying n, and so that $\mathcal{N}_y\mu^{-p}\nu^{-q} \subset \mathcal{M}_y$. It is immediate that such a node n exists. As $\mathrm{Supp}(\rho)$ lies outside of $\mathcal{M}_y$, the action of $\mu^p\nu^q$ cannot throw the support of ρ into $\mathcal{N}_y$. □

If our new θ has new important points in common with the important points of μ or the important points of ν, then return to the beginning of this subsubsection, observing that by the lemma we have just proven, any further versions of θ that we create will act as the identity in a neighbourhood of the set S which we defined at the beginning of this section.

This process must stop, since the important points of μ and ν are finite in number. The final version θ that we have created has no important points in common with the important points of μ or the important points of ν.

Proceed to the next subsubsection.

4.2.3. *Improving θ so that* $\mathrm{Supp}(\theta) \cap (I(\mu) \cup I(\nu)) = \emptyset$. We may suppose immediately that $I(\theta) \cap (I(\mu) \cup I(\nu)) = \emptyset$, or we could not have entered this subsubsection by following the directions in our process.

If $\mathrm{Supp}(\theta)$ does not contain any of the important points of μ or ν, then proceed to the next subsubsection; otherwise suppose $x \in \mathrm{Supp}(\theta) \cap (I(\mu) \cup I(\nu))$ and continue in this subsection.

By point (2) of Lemma 2.5 we see that x cannot be in the support of either μ or ν.

We must carefully consider two cases, depending on the dynamics of μ near $y = x\theta^{-1}$. Let us suppose first that y is disjoint from the support of μ. In this case, define $\tau = [\mu, \theta] = (\mu^{-1}\theta^{-1}\mu\theta)$. We observe the following:

$$x\tau = x\mu^{-1} \cdot \theta^{-1}\mu\theta = x\theta^{-1}\mu\theta = y\mu\theta = y\theta = x.$$

So x is fixed by τ.

Furthermore, the action of $\langle\tau\rangle$ on some neighbourhood of x will be as the trivial group if the slope of μ in small neighbourhoods of y is the same as the slope of μ in small neighbourhoods of x.

Now suppose instead that y is not disjoint from the support of μ. Then either y is in a common component of support C_k of μ and ν, or y is in a component of support A_k for μ for some index k. In the first case we can use Corollary 2.6 to find p, q non-trivial integers so that $\mu^p\nu^q$ acts trivially over C_k, while in the second case, take $p = 0$ and $q = 1$ so that $\mu^p\nu^q$ is trivial over A_k. In either case, define $\tau = [\mu^p\nu^q, \theta]$. Now the action of $\langle\tau\rangle$ will fix x.

Now replace θ by τ^j, where j is chosen as the least positive integer so that τ^j admits no non-trivial finite orbits. If the new θ has any important points in common with the important points of μ or ν, return to the previous subsubsection. Otherwise, if there are any further important points of μ or ν in the support of γ, then return to the beginning of this subsubsection. Finally, if none of the important points of μ or ν are in the component support of θ, we can proceed to the next subsubsection.

Note that if we had $\mathrm{Supp}(\theta) \cap (I(\mu) \cup I(\nu)) \neq \emptyset$ at the beginning of this subsubsection, then the cardinality of that intersection is reduced by at least one by the process here (no new points of $I(\mu) \cup I(\nu)$ enter the support of θ under this process by Lemma 4.3). Further, again by Lemma 4.3, this subsubsection will only force the process to return to the previous subsubsection a finite number of times, and the cardinality of $\mathrm{Supp}(\theta) \cap (I(\mu) \cup I(\nu))$ does not increase when we apply the process of the previous section. Therefore, the process given to this point eventually passes to the next subsubsection, and at that time none of the important points of μ or of ν are in the component support of θ.

4.2.4. *Finding a torsion element in the image of ϕ.* We can now assume that there is a neighbourhood of the important points of μ and ν so that θ acts trivially over this neighbourhood.

We first note the effects of conjugating θ by powers of μ and ν, assuming the details of our current dynamical situation with respect to important points and supports.

Lemma 4.4. *Suppose δ, ϵ, and τ are elements in V with the properties that*

(1) *δ and ϵ commute,*
(2) *none of δ, ϵ or τ admit non-trivial finite orbits in their action on the Cantor set,*
(3) *the support of τ is disjoint from a neighbourhood of the important points of δ and ϵ.*

Then there are infinitely many pairs of non-zero integers x and y so that

$$\mathrm{Supp}(\tau^{\delta^x\epsilon^y}) \cap \mathrm{Supp}(\tau) \cap (\mathrm{Supp}(\delta) \cup \mathrm{Supp}(\epsilon)) = \emptyset.$$

Proof. The lemma follows immediately from the observation that there are non-zero integers i and j so that $\delta^i\epsilon^j$ is non-trivial over every component of common support of δ and ϵ (since there are only finitely many such components), and thus over every component of support of δ and ϵ. Now for large enough integers n, setting $x = ni$ and $y = nj$ produces integers so that $\delta^x\epsilon^y$ throws the support of τ entirely off of itself within the supports of δ and ϵ (the resulting support within the support of δ and ϵ will be near to the important points of δ and ϵ). ☐

Thus we may use conjugation of θ by powers of μ and ν to find non-trivial elements in $\langle \mu, \nu, \theta \rangle$ whose actions *within the supports of μ and ν* are disjoint from the action of θ.

Let x and y be a pair of non-zero integers as guaranteed by Lemma 4.4, where μ plays the role of δ, ν plays the role of ϵ, and θ plays the role of τ, so that

$$\mathrm{Supp}(\theta^{\mu^x\nu^y}) \cap \mathrm{Supp}(\theta) \cap (\mathrm{Supp}(\mu) \cup \mathrm{Supp}(\nu)) = \emptyset.$$

Now set, for each positive index j, $\omega_j = [\theta^j, (\theta^j)^{\mu^x\nu^y}]$.

Note that the component support of ω_j is contained in the set $\Gamma = \overline{\mathrm{Supp}(\theta^j)} \cup \overline{\mathrm{Supp}((\theta^j)^{\mu^x\nu^y})} = \overline{\mathrm{Supp}(\theta)} \cup \overline{\mathrm{Supp}((\theta)^{\mu^x\nu^y})}$. On the next page, we will find a particular value k for j so that one can demonstrate by direct calculation that every

$x \in \Gamma$ must travel, under the action of $\langle \omega_k \rangle$, along an orbit of length either one, two, or three. This implies that the order of ω_k is one, two, three, or six, our desired result.

The remainder of this subsubsection verifies the calculation mentioned in the previous paragraph. The discussion is highly technical. We first carefully define fifteen disjoint subsets of the potential support of ω_j. Then, we analyze the flow of points in these sets under the action of $\langle \omega_k \rangle$ for a particular value of k which we will determine shortly after defining the relevant sets.

<u>Defining sets and notation</u>:
To simplify notation below, set $\zeta = \theta^{\mu^x \nu^y}$, noting that $\zeta^j = (\theta^j)^{\mu^x \nu^y}$. Now define

$$\mathcal{I} := \mathrm{I}(\theta) \cup \mathrm{I}(\zeta).$$

We will focus on the set $\mathcal{I}$ since the important points of θ and ζ are also the important points of θ^j and ζ^j for any positive integer j. Note that while the important points of θ^j and ζ^j cannot be important points of ω_j, the dynamical analysis of the action of ω_j is greatly assisted by paying careful attention to a neighbourhood of the points in $\mathcal{I}$.

The set $\mathcal{I}$ decomposes as a disjoint union of six sets, some of which might be empty:

$$\mathcal{I} = R_c \cup R_\theta \cup R_\zeta \cup A_c \cup A_\theta \cup A_\zeta.$$

Here, R_c is the set of repelling fixed points of θ and ζ which lie outside of the support of $\mu^x \nu^y$. We note in passing that this is the same set for both θ and ζ (the subscript c is to denote the phrase "common for θ and ζ").

Similarly, A_c is the set of attracting fixed points of θ and ζ which lie outside the support of $\mu^x \nu^y$.

The sets R_θ and A_θ are respectively the sets of repelling and attracting fixed points of θ which happen to lie in the support of $\mu^x \nu^y$.

The sets R_ζ and A_ζ are respectively the sets of repelling and attracting fixed points of ζ which happen to lie in the support of $\mu^x \nu^y$. One sees immediately that $R_\zeta = R_\theta \mu^x \nu^y$ and $A_\zeta = A_\zeta \mu^x \nu^y$.

For each $x \in R_c$, let n_x denote a node of $\mathcal{T}_2$ so that $x \in \mathfrak{C}_{n_x} \subset \overline{\mathrm{Supp}(\theta)}$ and so that $\mathfrak{C}_{n_x} \cap \mathrm{Supp}(\mu^x \nu^y) = \emptyset$. Similarly, for each $y \in A_c$, let $\hat{n}_y$ denote a node of $\mathcal{T}_2$ so that $y \in \mathfrak{C}_{\hat{n}_y} \subset \overline{\mathrm{Supp}(\theta)}$ and so that $\mathfrak{C}_{\hat{n}_y} \cap \mathrm{Supp}(\mu^x \nu^y) = \emptyset$.

For each $x \in R_\theta$, let n_x denote a node of $\mathcal{T}_2$ so that $x \in \mathfrak{C}_{n_x} \subset \overline{\mathrm{Supp}(\theta)}$ and so that $\mathfrak{C}_{n_x} \cap \mathrm{Supp}(\mu^x \nu^y) = \mathfrak{C}_{n_x}$. Similarly, for each $y \in A_\theta$, let $\hat{n}_y$ denote a node of $\mathcal{T}_2$ so that $y \in \mathfrak{C}_{\hat{n}_y} \subset \overline{\mathrm{Supp}(\theta)}$ and so that $\mathfrak{C}_{\hat{n}_y} \cap \mathrm{Supp}(\mu^x \nu^y) = \mathfrak{C}_{\hat{n}_y}$.

For each $x \in R_\zeta$, let n_x denote a node of $\mathcal{T}_2$ so that $x \in \mathfrak{C}_{n_x} \subset \overline{\mathrm{Supp}(\zeta)}$ and so that $\mathfrak{C}_{n_x} \cap \mathrm{Supp}(\mu^x \nu^y) = \mathfrak{C}_{n_x}$. Similarly, for each $y \in A_\zeta$, let $\hat{n}_y$ denote a node of $\mathcal{T}_2$ so that $y \in \mathfrak{C}_{\hat{n}_y} \subset \overline{\mathrm{Supp}(\zeta)}$ and so that $\mathfrak{C}_{\hat{n}_y} \cap \mathrm{Supp}(\mu^x \nu^y) = \mathfrak{C}_{\hat{n}_y}$.

All of the nodes n_x and $\hat{n}_y$ chosen above can be selected in such a fashion as to have all of the various properties mentioned above, as well as the further property that given any distinct pair of nodes, the underlying Cantor sets of the nodes are disjoint. We assume that the nodes have been chosen in such a fashion. The reader may verify that such choices can be made.

Let $R = R_c \cup R_\theta \cup R_\zeta$, and let $A = A_c \cup A_\theta \cup A_\zeta$, and set

$$\begin{aligned}\mathcal{N}_r &= \bigcup_{x\in R} \mathfrak{C}_{n_x},\\ \widehat{\mathcal{N}}_a &= \bigcup_{y\in A} \mathfrak{C}_{\hat{n}_y}.\end{aligned}$$

The set $\mathcal{N}_r$ is a neighbourhood of the repelling fixed points of θ^j and ζ^j (this is independent of j positive), which we will use often in our calculations below. A modified version of $\widehat{\mathcal{N}}_a$ will play the role of a similar neighbourhood around the attractors (we will have to pass to a generalized neighbourhood, as mentioned in the section on definitions pertaining to the Cantor set).

The integer k of Theorem 1.7:

We specify the value of the variable k as the least integer $j > 0$ so that $\mathcal{N}_r\theta^j \cup \mathcal{N}_r\zeta^j \cup \widehat{\mathcal{N}}_a = \mathrm{Supp}(\theta) \cup \mathrm{Supp}(\zeta)$. Further set $\omega = \omega_k$. To simplify notation throughout, denote θ^k by $\hat{\theta}$ below, and likewise denote ζ^k by $\hat{\zeta}$. We now have

$$\mathcal{N}_r\hat{\theta} \cup \mathcal{N}_r\hat{\zeta} \cup \widehat{\mathcal{N}}_a = \mathrm{Supp}(\hat{\theta}) \cup \mathrm{Supp}(\hat{\zeta}).$$

For each $x \in R$, set $U_x = \mathfrak{C}_{n_x}$. For each $y \in A$, set $V_y = \mathfrak{C}_{\hat{n}_y}\backslash(\mathcal{N}_r\hat{\theta} \cup \mathcal{N}_r\hat{\zeta})$. Given $x \in R_c$, R_θ, or R_ζ respectively, we may denote U_x by U_x^c, U_x^t, or U_x^z for clarity. Similarly, given $y \in A_c$, A_θ, or A_ζ, we may denote V_y by V_y^c, V_y^t, or V_y^z respectively.

Set $\mathcal{N}_a = \bigcup_{y\in A} V_y$. This is a generalized neighbourhood of the attracting fixed points of $\hat{\theta}$ and $\hat{\zeta}$ lying entirely in the component supports of $\hat{\theta}$ and $\hat{\zeta}$.

Denote by M the set $(\mathrm{Supp}(\hat{\theta}) \cup \mathrm{Supp}(\hat{\zeta}))\backslash(\mathcal{N}_r \cup \mathcal{N}_a)$. This is the set of potential support of ω away from the neighbourhoods of the attracting and repelling fixed points of $\hat{\theta}$ and $\hat{\zeta}$. (The set M is the "middle" of the potential support of ω.) The set M decomposes as a disjoint union of sets $M_c = M\backslash\mathrm{Supp}(\mu^x\nu^y)$, $M_t = M \cap \mathrm{Supp}(\hat{\theta}) \cap \mathrm{Supp}(\mu^x\nu^y)$, and $M_z = M \cap \mathrm{Supp}(\hat{\zeta}) \cap \mathrm{Supp}(\mu^x\nu^y)$.

We now use M to further decompose the sets U_x and V_y for $x \in R$ and $y \in A$.

Given $x \in U_x$ set

$$U_{x,1} = \left\{x \in U_x \mid x\hat{\theta} \in M \text{ or } x\hat{\zeta} \in M\right\}$$

and set $U_{x,2} = U_x\backslash U_{x,1}$. (At least two applications of $\hat{\theta}$ or $\hat{\zeta}$ are required for a point in $U_{x,2}$ to leave U_x.) Extend this notation in the obvious fashion so that $U_{x,i}^c$, $U_{x,i}^t$, and $U_{x,i}^z$ are well defined for $i = 1$ or $i = 2$. We will not need larger values of i (i loosely represents minimal "escape time" from U_x under the actions of $\hat{\theta}$ or $\hat{\zeta}$) for the analysis to follow.

Finally, and in a similar fashion, given $y \in R$, define

$$V_{y,1} = V_y \cap (M\hat{\zeta} \cup M\hat{\theta})$$

and set $V_{y,2} = V_y\backslash V_{y,1}$. Extend this notation in like fashion so that the sets $V_{y,i}^c$, $V_{y,i}^t$, and $V_{y,i}^z$ are well defined for indices $i = 1$ and $i = 2$.

We have now defined the fifteen types of (pairwise disjoint) subsets of the potential support of ω which we will require in our analysis of the action of ω on $\mathfrak{C}$. In particular, these are $U_{x,i}^c$, $U_{x,i}^t$, $U_{x,i}^z$, $V_{x,i}^c$, $V_{x,i}^t$, $V_{x,i}^z$, M_c, M_t, and M_z, where $i = 1$ or $i = 2$.

Figure 5 provides an informal guide to the arrangement of the fifteen sets.

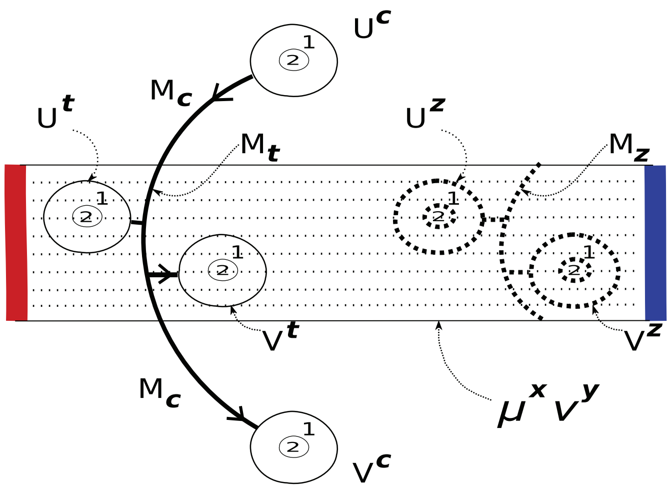

FIGURE 5. Fifteen sets for tracking dynamics.

Analysis of dynamics:
We are now ready to prove the following lemma.

Lemma 4.5. *The element* ω *defined above has order* 1, 2, 3 *or* 6.

Proof. The support of ω is contained in the union of the support of $\hat{\theta}$ and the support of $\hat{\zeta}$. In particular, we need to trace the orbits of the points in each of the fifteen sets defined in the paragraph just before the "Analysis of dynamics" header.

That effort is simplified by the fact that the action of $\langle\omega\rangle$ in many of those sets is trivial (as one might expect, given that ω is a commutator).

We perform the orbit calculations for some of the sets before confirming the statement in the last paragraph, in order to acquaint the reader with a method of orbit calculation.

In the diagram below, if there are several arrows leaving a node, this represents the fact that a point may move to distinct locations depending on further subdivisions within the fifteen sets. It often occurs that a previous choice at a branch makes later choices invalid. In our first calculation, we will draw such invalid possibilities with a "dotted" arrow. Arrows are sometimes decorated with strings to help explain the dynamics.

If $x \in R$ in the diagrams below, we will drop the occurrence of x from the names of the repulsive sets $U^c_{x,1}$, $U^c_{x,2}$, $U^t_{x,1}$, $U^t_{x,2}$, $U^z_{x,1}$, and $U^z_{x,1}$, writing instead names such as U^c_1. We will treat the attractive V sets in a similar fashion. This should lead to no confusion.

The reader may be assisted in following the calculations below by recalling that $M_z = M_t\mu^x\nu^y$, $U^z_* = U^t_*\mu^x\nu^y$, $V^z_* = V^t_*\mu^x\nu^y$, and that these sets are "parallel" in some sense due to the relationship $\mathcal{O}(p,\hat{\zeta}) = \mathcal{O}(p,\hat{\theta})\mu^x\nu^y$ for $p \in \mathfrak{C}$. The reader should also observe that $\hat{\theta}|_{V^c_*} = \hat{\zeta}|_{V^c_*}$ and $\hat{\theta}^{-1}|_{U^c_*} = \hat{\zeta}^{-1}|_{U^c_*}$ for $* = 1$ or $* = 2$.

We now begin to trace orbits.

Assume $x_0 \in M_c$.

$$\begin{tikzcd}
x_0 \in M_c \arrow[r, "\hat\theta^{-1}"] \arrow[ddr, "\hat\theta^{-1}"] & x_1 \in U_1^c \arrow[r, "\hat\zeta^{-1}"] & x_2 \in U_2^c \arrow[r, "\hat\theta"] & x_1 \arrow[r, "\hat\zeta"] \arrow[dr, dotted, "\hat\zeta"] & x_0 \\
 & & & & M_z \\
 & x_3 \in U_1^t \arrow[r, "\hat\zeta^{-1}", "="'] & x_3 \arrow[r, "\hat\theta"] & x_0 \arrow[r, "\hat\zeta"] \arrow[dr, "\hat\zeta", "=\hat\theta"'] & x_4 \in V_1^z \\
 & & & & x_5 \in V_1^c
\end{tikzcd}$$

$$\begin{tikzcd}
x_4 \arrow[r, "\hat\theta^{-1}", "="'] & x_4 \arrow[r, "\hat\zeta^{-1}"] & x_0 \arrow[r, "\hat\theta"] \arrow[dr, dotted, "\hat\theta"] & x_6 \in V_1^t \arrow[r, "\hat\zeta", "="'] & x_6 \\
 & & & V_1^c &
\end{tikzcd}$$

$$\begin{tikzcd}
x_5 \arrow[r, "\hat\theta^{-1}"] \arrow[dr, dotted, "\hat\theta^{-1}"] & x_0 \arrow[r, "\hat\zeta^{-1}"] & x_7 \in U_1^z \arrow[r, "\hat\theta", "="'] & x_7 \arrow[r, "\hat\zeta"] & x_0 \\
 & M_t & & &
\end{tikzcd}$$

$$\begin{tikzcd}
x_6 \arrow[r, "\hat\theta^{-1}"] & x_0 \arrow[r, "\hat\zeta^{-1}"] & x_8 \in U_1^z \arrow[r, "\hat\theta", "="'] & x_8 \arrow[r, "\hat\zeta"] & x_0
\end{tikzcd}$$

In this example, each diagram component represents the possibilities of one application of ω. Under the action of $\langle\omega\rangle$, the point x_0 had potential orbits of length one ($\{x_0\}$), of length two ($\{x_0, x_5\}$), and of length three ($\{x_0, x_4, x_6\}$).

We now free the symbols x_i used in the above calculation, so that we can enter into similar calculations below with different values for the x_i. Whenever we are about to trace the orbits of one of the fifteen sets, we will assume the variables x_i are unbound and available.

In each of the calculations below, we will no longer draw explicit "dotted arrows" for potential branches which cannot actually occur given previous information within the calculation.

Suppose $x_0 \in M_z$:

$$\begin{tikzcd}
x_0 \in M_z \arrow[r, "\hat{\theta}^{-1}", "="'] & x_0 \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_1 \in U_1^z \arrow[r, "\hat{\theta}", "="'] & x_1 \arrow[r, "\hat{\zeta}"] & x_0 \\
 & & x_2 \in U_1^c \arrow[r, "\hat{\theta}"] & x_3 \in M_t \arrow[r, "\hat{\zeta}", "="'] & x_3
\end{tikzcd}$$

$$\begin{tikzcd}
x_3 \arrow[r, "\hat{\theta}^{-1}"] & x_2 \arrow[r, "\hat{\zeta}^{-1}"] & x_3 \in U_2^c \arrow[r, "\hat{\theta}"] & x_2 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

We thus see that the orbits of points in M_z are either trivial or of length two under the action of $\langle \omega \rangle$.

Suppose now that $x_0 \in M_t$:

$$\begin{tikzcd}
x_0 \in M_t \arrow[r, "\hat{\theta}^{-1}"] \arrow[dr, "\hat{\theta}^{-1}"] & x_1 \in U_1^t \arrow[r, "\hat{\zeta}^{-1}", "="'] & x_1 \arrow[r, "\hat{\theta}"] & x_0 \arrow[r, "\hat{\zeta}", "="'] & x_0 \\
 & x_2 \in U_1^c \arrow[r, "\hat{\zeta}^{-1}"] & x_3 \in U_2^c \arrow[r, "\hat{\theta}"] & x_2 \arrow[r, "\hat{\zeta}"] & x_4 \in M_z
\end{tikzcd}$$

$$\begin{tikzcd}
x_4 \arrow[r, "\hat{\theta}^{-1}", "="'] & x_4 \arrow[r, "\hat{\zeta}^{-1}"] & x_2 \arrow[r, "\hat{\theta}"] & x_0 \arrow[r, "\hat{\zeta}", "="'] & x_0
\end{tikzcd}$$

So the points in M_t also have only trivial orbits or orbits of length two under the action of $\langle \omega \rangle$.

Armed with these previous examples the reader should now be able to check that if an initial point p has $p \in U_{\#}^{*}$, then for any valid values of the symbols $\#$ and $*$, ω fixes p. For $p \in V_2^*$ with $*$ not "c", it is also easy to see that $p\omega = p$. Thus, we have seen that our lemma is supported over the sets M_c, M_t, M_z, U_1^c, U_2^c, U_1^t, U_2^t, U_1^z, U_2^z, V_2^t and V_2^z. There remain four sets to check.

Suppose now that $x_0 \in V_2^c$:

$$\begin{tikzcd}
x_0 \in V_2^c \arrow[r, "\hat{\theta}^{-1}"] \arrow[ddr, "\hat{\theta}-1"] & x_1 \in V_1^c \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_2 \in M_c \arrow[r, "\hat{\theta}"] & x_1 \arrow[r, "\hat{\zeta}"] & x_0 \\
 & & x_3 \in M_z \arrow[r, "\hat{\theta}", "="'] & x_3 \arrow[r, "\hat{\zeta}"] & x_1 \\
 & x_4 \in V_2^c \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_5 \in V_1^c \arrow[r, "\hat{\theta}"] & x_4 \arrow[r, "\hat{\zeta}"] & x_0 \\
 & & x_6 \in V_2^c \arrow[r, "\hat{\theta}"] & x_4 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

$$\begin{tikzcd}
x_1 \arrow[r, "\hat{\theta}^{-1}"] & x_7 \in M_t \arrow[r, "\hat{\zeta}^{-1}", "="'] & x_7 \arrow[r, "\hat{\theta}"] & x_1 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

In particular, the orbits in $\mathfrak{C}$ under the action of $\langle \omega \rangle$ which actually intersect V_2^c are either trivial or of length two.

Suppose now $x_0 \in V_1^c$:

$$\begin{tikzcd}
x_0 \in V_1^c \arrow[r, "\hat{\theta}^{-1}"] \arrow[ddr, "\hat{\theta}-1"] & x_1 \in V_1^c \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_2 \in M_c \arrow[r, "\hat{\theta}"] & x_1 \arrow[r, "\hat{\zeta}"] & x_0 \\
 & & x_3 \in M_z \arrow[r, "\hat{\theta}", "="'] & x_3 \arrow[r, "\hat{\zeta}"] & x_1 \\
 & x_4 \in V_2^c \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_5 \in V_1^c \arrow[r, "\hat{\theta}"] & x_4 \arrow[r, "\hat{\zeta}"] & x_0 \\
 & & x_6 \in V_2^c \arrow[r, "\hat{\theta}"] & x_4 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

$$\begin{tikzcd}
x_1 \arrow[r, "\hat{\theta}^{-1}"] & x_7 \in M_t \arrow[r, "\hat{\zeta}^{-1}", "="'] & x_7 \arrow[r, "\hat{\theta}"] & x_1 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

Thus, every orbit under the action of $\langle\omega\rangle$ which intersects V_1^c is either trivial or of order two.

Suppose now $x_0 \in V_1^z$:

$$\begin{tikzcd}
x_0 \in V_1^z \arrow[r, "\hat{\theta}^{-1}", "="'] & x_0 \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_1 \in M_z \arrow[r, "\hat{\theta}", "="'] & x_1 \arrow[r, "\hat{\zeta}"] & x_0 \\
 & & x_2 \in M_c \arrow[r, "\hat{\theta}"] & x_3 \in V_1^t \arrow[r, "\hat{\zeta}", "="'] & x_3
\end{tikzcd}$$

$$\begin{tikzcd}
x_3 \arrow[r, "\hat{\theta}^{-1}"] & x_2 \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_4 \in U_1^z \arrow[r, "\hat{\theta}", "="'] & x_4 \arrow[r, "\hat{\zeta}"] & x_2 \\
 & & x_5 \in U_1^c \arrow[r, "\hat{\theta}"] & x_2 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

$$\begin{tikzcd}
x_2 \arrow[r, "\hat{\theta}^{-1}"] & x_6 \in U_1^t \arrow[r, "\hat{\zeta}^{-1}", "="'] & x_6 \arrow[r, "\hat{\theta}"] & x_2 \arrow[r, "\hat{\zeta}"] & x_0
\end{tikzcd}$$

In this case, it is possible for points in V_1^z to be in orbits of order one, two, or three under the action of $\langle\omega\rangle$.

Finally, suppose $x_0 \in V_1^t$:

$$\begin{tikzcd}
x_0 \in V_1^t \arrow[r, "\hat{\theta}^{-1}"] \arrow[dr, "\hat{\theta}^{-1}"] & x_1 \in M_t \arrow[r, "\hat{\zeta}^{-1}", "=" '] & x_1 \arrow[r, "\hat{\theta}"] & x_0 \arrow[r, "\hat{\zeta}", "=" '] & x_0 \\
& x_2 \in M_c \arrow[r, "\hat{\zeta}^{-1}"] \arrow[dr, "\hat{\zeta}^{-1}"] & x_3 \in U_1^z \arrow[r, "\hat{\theta}", "=" '] & x_3 \arrow[r, "\hat{\zeta}"] & x_2 \\
& & x_4 \in U_1^c \arrow[r, "\hat{\theta}"] & x_2 \arrow[r, "\hat{\zeta}"] & x_5 = x_0\mu^x\nu^y \in V_1^z
\end{tikzcd}$$

$$x_2 \xrightarrow{\hat{\theta}^{-1}} x_6 \in U_1^t \xrightarrow[=]{\hat{\zeta}^{-1}} x_6 \xrightarrow{\hat{\theta}} x_2 \xrightarrow{\hat{\zeta}} x_5 = x_0\mu^x\nu^y \in V_1^z$$

$$x_5 \xrightarrow[=]{\hat{\theta}^{-1}} x_5 \xrightarrow{\hat{\zeta}^{-1}} x_2 \xrightarrow{\hat{\theta}} x_0 \xrightarrow[=]{\hat{\zeta}} x_0$$

Hence, under the action of $\langle\omega\rangle$, we see V_1^t is also a set where orbits in $\mathfrak{C}$ which interesect V_1^t may be of order one, two, or three.

We have now shown that if $p \in \mathfrak{C}$, then the cardinality of $\mathcal{O}(p,\omega)$ is one, two, or three. We conclude that the order of ω divides six. $\square$

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School of Mathematics and Statistics, Mathematical Institute, University of St. Andrews, North Haugh, St. Andrews, Fife KY16 9SS, Scotland
E-mail address: collin@mcs.st-and.ac.uk

Escuela de Matemáticas, Universidad Nacional de Colombia, Medellín, Colombia
E-mail address: opsalaza@yahoo.com