Rapid and Robust Spin State Amplification

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(Received 11 November 2010; published 22 April 2011)

Electron and nuclear spins have been employed in many of the early demonstrations of quantum technology. However, applications in real world quantum technology are limited by the difficulty of measuring single spins. Here we show that it is possible to rapidly and robustly amplify a spin state using a lattice of ancillary spins. The model we employ corresponds to an extremely simple experimental system: a homogenous Ising-coupled spin lattice in one, two, or three dimensions, driven by a continuous microwave field. We establish that the process can operate at finite temperature (imperfect initial polarization) and under the effects of various forms of decoherence.

DOI: 10.1103/PhysRevLett.106.167204 PACS numbers: 75.10.Pq, 03.67.−a

The standard approach to implementing a quantum technology is to identify a physical system that can represent a qubit: it must exhibit two (or more) stable states, it should be manipulable through external fields and possess a long decoherence time. Provided that the system can controllably interact with other such systems, then it may be a strong candidate. Electron and nuclear spins, within suitable molecules or solid state structures, can meet these requirements. However, the drawback with spin qubits is that they have not been directly measured through a detection of the magnetic field they produce. The magnetic moment of a single electron spin is orders of magnitude too weak to be detected by standard ESR techniques, and even the most sensitive magnetometers still fall short of single spin measurement [1]—meanwhile the situation with nuclear spins is worse still. In a few special systems it is possible to convert the spin information into another degree of freedom. For example, a spin-dependent optical transition allows spin to photon conversion in some crystal defects [2–4], self-assembled semiconductor quantum dots [5,6], and trapped atoms held in a vacuum [7]. Alternatively, spin to charge conversion is an established technology in lithographic quantum dots [8]. However, the majority of otherwise promising spin systems do not have such a convenient property [9] and therefore cannot be measured directly.

One suggested solution is to ''amplify'' a single spin by using a set of ancillary spins that are (ideally) initialized to $|0\rangle$. We would look for a transformation of the form

$$|0\rangle|0\rangle^{\otimes n} \rightarrow |0\rangle|0\rangle^{\otimes n} \qquad |1\rangle|0\rangle^{\otimes n} \rightarrow |1\rangle|1\rangle^{\otimes n}, \tag{1}$$

the idea being that the n ancillary spins constitute a large enough set that state-of-the-art magnetic field sensing technologies can detect them. Note that the transformation need not be unitary or indeed even coherent: the intention is to make a measurement of the primary spin, it is not necessary to preserve any superposition (that is, we need not limit ourselves to transformations that take $\alpha|0\rangle|0\rangle^{\otimes n} + \beta|1\rangle|0\rangle^{\otimes n}$ to a state like $\alpha|0\rangle^{\otimes n+1} + \beta|1\rangle^{\otimes n+1}$).

This is a rather broadly defined transformation and there are a number of ways that one might perform it. Clearly one would like to find the method that is the least demanding experimentally. Previous authors have proposed schemes using a strictly 1D homogeneous lattice with continuous global driving [10], and an inhomogeneous 3D lattice with alternating timed electromagnetic (EM) pulses [11]. The former result has the advantage of simplicity, but the rate at which amplification occurs will inevitably be limited by the single dimension of the array; moreover, such a system must be highly vulnerable to imperfect initialization (i.e., finite temperature). Here we generalize to a homogeneous 2D square lattice, showing that a continuous global EM field can drive an amplification process that succeeds at finite temperatures (imperfect initialization of the ancilla spins) and in the presence of decoherence. By bringing the global EM field onto resonance with certain transitions, we are able to create a set of rules that govern locally how spins propagate over the lattice. We then look at the rate of increase in the total number of flipped spins as a measure of quality of the scheme. While our focus is on the 2D case, we are also able to predict the performance of the amplification protocol for a homogeneous 3D lattice with continuous driving.

The case of a 1D lattice has been studied in detail by Lee and Khitrin [10]. Before moving to the 2D spin lattice that will form the core of this Letter, we first recall how to simplify the description of this (semi-infinite) 1D spin chain, with nearest-neighbor Ising (ZZ) interactions. Under a microwave driving field of frequency ω, the Hamiltonian is given by

$$\mathcal{H} = \sum_{i=1}^{\infty} \epsilon_i \sigma_z^i + J_i \sigma_z^i \sigma_z^{i+1} + 2\Omega_i \sigma_x^i \cos(\omega t). \tag{2}$$

ϵ_i is the on-site Zeeman energy of spin i, and J_i is the magnitude of the coupling between spins i and $i+1$. Ω describes the coupling of spin i and the microwave field. In this case, spin $i = 1$ is the one whose state is supposed to

be amplified. If we assume that the chain is uniform, such that $\Omega_i = \Omega$, $\epsilon_i = \epsilon$, and $J_i = J$, then moving into a frame rotating at frequency ω, making a rotating wave approximation, and setting $\omega = \epsilon$ leads to

$$\mathcal{H} = \sum_{i=1}^{\infty} J\sigma_z^i\sigma_z^{i+1} + \Omega\sigma_x^i. \tag{3}$$

In order to understand the dynamics of the system, it is instructive to explicitly separate all terms that involve a particular spin k:

$$\mathcal{H} = J(\sigma_z^{k-1} + \sigma_z^{k+1})\sigma_z^k + \Omega\sigma_x^k + \sum_{i\neq\{k,k-1\}} \Omega\sigma_x^i + J\sigma_z^i\sigma_z^{i+1} + \Omega\sigma_x^{k-1}. \tag{4}$$

Choosing a driving field such that $\Omega \ll J$ means that spin k will only undergo resonant oscillations when the first term in Eq. (4) goes to zero—i.e., when the two spins neighboring spin k are oriented in opposite directions. In any other configuration the Ising coupling takes the spin k off resonance with the microwave and no appreciable dynamics are expected.

Let us now define a subset of states S that exist in the spin chain Hilbert space, $|n\rangle$, which have the first n spins of the chain in state $|\uparrow\rangle$ with the rest $|\downarrow\rangle$. If the rule we just derived holds exactly, these states define a closed subspace. We may then write a very simple isolated Hamiltonian for this subspace:

$$\mathcal{H}_S = \Omega \sum_{n=1}^{\infty} |n\rangle\langle n+1| + |n+1\rangle\langle n|. \tag{5}$$

With this simplification of the 1D Hamiltonian in mind, we progress now to a semi-infinite square spin lattice with nearest-neighbor ZZ interactions. For this case we have

$$\mathcal{H} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \epsilon\sigma_z^{i,j} + J\sigma_z^{i,j}\sigma_z^{i+1,j} + J\sigma_z^{i,j}\sigma_z^{i,j+1} + 2\Omega\sigma_x^{i,j}\cos(\omega t). \tag{6}$$

By again considering the terms affecting a particular spin in the main body of the lattice [$k(>1)$, $l(>1)$, say], we find for $\omega = \epsilon$ and after moving to a rotating frame and making the rotating wave approximation:

$$\mathcal{H} = J\sigma_z^{k,l}(\sigma_z^{k+1,l} + \sigma_z^{k,l+1} + \sigma_z^{k-1,l} + \sigma_z^{k,l-1}) + \cdots, \tag{7}$$

where we do not explicitly write out terms not involving spin (k, l). The microwave is now only resonant for spin (k, l) if it has two neighbor spins in each orientation. For a spin on the edge of the lattice there are an odd number of neighbors so resonance cannot be achieved. However, applying a second microwave with $\omega = \epsilon - J$ allows resonant flips on the edge if two neighbors are down and one up—and this second field has no effect on the bulk spins.

The spin to be measured is the corner spin ($i = j = 1$) and so would form part of a wider computational apparatus. We may therefore assume that it is a different species with a unique resonant frequency. The dynamics of the whole lattice may then be summarized by three rules (in order of precedence): (1) The corner (test) spin is fixed. (2) An edge spin can flip if it has one of its neighbors up and two down. (3) A body spin can flip if it has two of its neighbors up and two down. We begin by supposing all spins are initialized in the "down" state apart from the test spin, which is located in the upper left-hand corner of our lattice. We can describe this initial state by choosing two basis elements: $|0\rangle$ when the test spin is down, and $|1\rangle$ when the test spin is up. Using our heuristic rules we can see that these two states do not couple to each other—that $\langle 0|H|1\rangle = 0$. In fact, $|0\rangle$ does not couple to any other state, so if we start in the $|0\rangle$ state no amplification occurs, as desired.

We will now seek to construct a basis for the subspace containing our system evolution, by looking at states connected by our Hamiltonian. It will be convenient to represent these states on the nodes of a graph, using the edges to represent nonzero elements of the Hamiltonian.

Our starting point is the state $|1\rangle$, with just the corner spin "up." From this position our rules allow two possibilities: either the spin to the right of the corner flips or the spin below it flips (see Fig. 1). In each case the magnitude of the transition matrix element is Ω. As we continue this procedure, we notice that the states that arise for each excitation number can be characterized by a nonincreasing sequence of integers that represent the number of up spins in each column of the lattice (see Fig. 1). Such sequences can also be used to define partitions of an integer: ways of splitting an integer up into a sum of other integers, e.g., $3 = 3 = 2 + 1 = 1 + 1 + 1$. In fact, the states that arise are in one-to-one correspondence with such partitions; we call these states "partition states" and denote them with standard partition notation (see Fig. 1). The graph we have just described is depicted in Fig. 1, is known as "Young's lattice," and arises in areas of pure mathematics, such as the representation theory of the symmetric group and the theory of differential posets. We have drawn weights

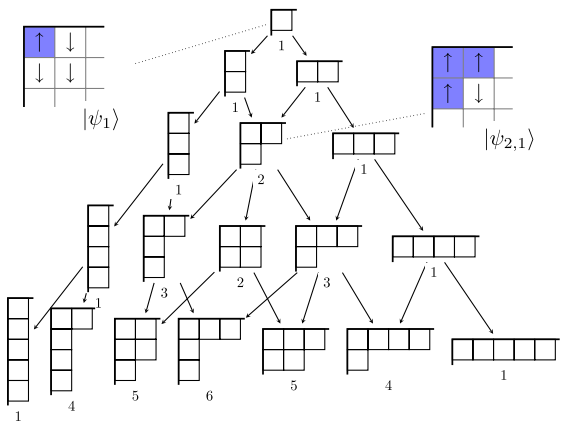

FIG. 1 (color online). Partition states arranged into a lattice. Edges represent a coupling through the Hamiltonian of strength Ω. Weights represent the number of different paths through the lattice to a given state.

beneath each state, recording the number of ways the state can be constructed. We will now further reduce the dimension of this basis by eliminating combinations of states which are inaccessible.

Starting with $|1\rangle$ we see that $\langle 1|H(\alpha_{1,1}|\psi_{1,1}\rangle + \alpha_2|\psi_2\rangle) = \Omega(\alpha_{1,1} + \alpha_2)$, so $|1\rangle$ does not couple to the two-excitation state $|\psi_{1,1}\rangle - |\psi_2\rangle$. We can eliminate this, leaving a single orthogonal, coupled state with two excitations: $|2\rangle := \frac{1}{\sqrt{2}}(|\psi_{1,1}\rangle + |\psi_2\rangle)$.

We may continue to build up coupled states with larger excitation numbers, and in fact we find that there is only a single coupled state in each case (i.e., we can always eliminate $k-1$ combinations of partition states with k excitations). To see this, first suppose we have the coupled state with k excitations, which by analogy with the 1D case we write as $|k\rangle$. We can write $|k\rangle = \frac{1}{N_k}\sum_{i\in P(k)} c_i|\psi_i\rangle$, where $P(k)$ is the set of partitions of the integer k and N_k a normalization factor. We want to construct the state $|k+1\rangle$ by eliminating the k-dimensional subspace with $k+1$ excitations, to which $|k\rangle$ does not couple.

Let $|\psi\rangle = \sum_{j\in P(k+1)} \alpha_j|\psi_j\rangle$ and consider the states $|\psi\rangle$ such that

$$0 = \langle k|H|\psi\rangle = \sum_{i\in P(k)} \sum_{j\in P(k+1)} c_i^* \alpha_j \langle \psi_i|H|\psi_j\rangle, \quad (8)$$

but $\langle \psi_i|H|\psi_j\rangle = \Omega$ if i is a *parent* of j (a state connected to j, in the lattice row above it), and 0 otherwise, so

$$0 = \langle k|H|\psi\rangle = \sum_{j\in P(k+1)} \alpha_j \sum_{i\in \text{ parents}(j)} c_i^*. \quad (9)$$

This is the equation of a hyperplane in $|P(k+1)|$ dimensions, defining the states that are not coupled to $|k\rangle$ through the Hamiltonian. There is a unique single state orthogonal to this hyperplane, $\beta_j = \sum_{i\in \text{ parents}(j)} c_i$, to which $|k\rangle$ couples. So the only state with $k+1$ up spins that $|k\rangle$ couples has coefficients proportional to β_j. After normalization, we call this state $|k+1\rangle$.

Unfortunately, there is no easy way to write down the partition states and weights for the nth row of the lattice. Fortunately, for our purposes, we only need to know that the states $|k\rangle$ exist and what the coupling between them is. To find this coupling, consider

$$\begin{aligned} g_{n-1,n} &= \langle n|H|n-1\rangle \\ &= \frac{1}{N_{n-1}N_n} \sum_{i\in P(n)} \sum_{j\in P(n-1)} c_i^* c_j \langle \psi_i|H|\psi_j\rangle \\ &= \frac{1}{N_{n-1}N_n} \Omega \sum_{i\in P(n)} c_i^* \sum_{j\in \text{ parents}(i)} c_j \\ &= \frac{1}{N_{n-1}N_n} \Omega \sum_{i\in P(n)} |c_i|^2 = \Omega \frac{N_n}{N_{n-1}}. \end{aligned} \quad (10)$$

To find the N_n we need the sum of the squares of the weights of partitions in a given row. A standard result about Young's lattice immediately gives us this sum: $n!$ [12].

Referring back to Eq. (10), and using $N_i = \sqrt{i!}$, we see that

$$\mathcal{H} = \Omega \sum_n \sqrt{n}(|n-1\rangle\langle n| + |n\rangle\langle n-1|). \quad (11)$$

In essence we have established a linear sequence of states, each coupled to the next analogously to the states on a 1D chain (5). However, each of our states is in fact a superposition of many configurations of the 2D array, and crucially the effective coupling from each state to the next increases along the sequence.

It has been shown (e.g., [13]) that a quantum state released at the end of a semi-inifinite chain of states, with constant couplings, will travel ballistically: the average position of the state along the chain is proportional to the time passed and inversely proportional to the coupling strength. Since, in the 1D case, the position is proportional to the number of spins that have flipped, we have that the total polarization will increase linearly with time.

We can establish the rate of propagation in the 2D case using the ansatz that the time taken to travel between two neighboring nodes is inversely proportional to the strength of the coupling between them. The total time is then $t_{2D} \propto \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \simeq n^{1/2}$. As in the one-dimensional case, the position along the chain corresponds to the number of spins that have flipped, and so we would expect the total polarization to be proportional to t^2. This prediction of a quadratic speedup of signal going from 1D to 2D is the central result of our Letter, and was confirmed by simple numerical simulations of Eq. (11) (Fig. 2).

Unfortunately, the mapping from 2D to 1D is not readily extendible to 3D. However, our results so far could have been anticipated using simple dimensional arguments; if one postulates that the rate of spin propagation is proportional to the boundary of the region, one can predict the correct scaling behavior. In 1D the boundary size is independent of the region size; no matter how many spins have flipped, it still has size one. The coupling strength between states $|n\rangle$ is constant. In the 2D case, the boundary size scales with the square root of the area, and the coupling goes with $\sqrt{n}$. In 3D, the boundary scales like the cube root of the volume squared, and so we expect the coupling to scale as $n^{2/3}$. Following similar logic to that used in the 2D case: $t_{3D} \propto \sum_{i=1}^{n} \frac{1}{i^{2/3}} \simeq n^{1/3}$, and so $n \sim t^3$.

We now consider the effect of decoherence. Much of the early work on continuous time quantum random walks looked at the speedup they afforded over their classical counterparts [14], but did not make any statement about the conditions under which we would expect the quantum walk to exhibit classical behavior, as we might expect in a regime of suitably heavy dephasing, say.

We begin by considering a collective noise operator: $L = \sum_n n|n\rangle\langle n|$. This represents noise that applies uniformly to the whole lattice: global fluctuations in the magnetic field, for example. As the effect of this noise depends only on the number of up spins, the system

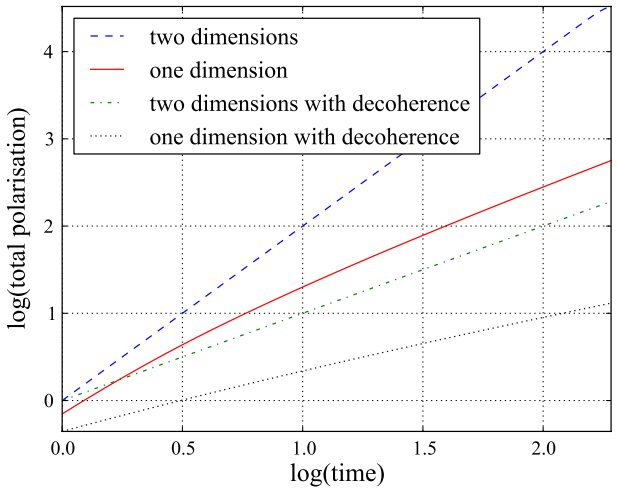

FIG. 2 (color online). Expected total polarization against time. Time in units of $1/\Omega$, dephasing rate $\Gamma = 1$. The gradient of the "one-dimension with decoherence" line tends to $1/2$ asymptotically.

remains in the reduced basis of number states calculated earlier, with only the coherences between these states affected.

Our starting point is the Lindblad master equation:

$$\dot{\rho} = i[\rho, H] + \tfrac{1}{2}\Gamma(2L\rho L^\dagger - L^\dagger L\rho - \rho L^\dagger L). \quad (12)$$

We proceed by splitting up the equation into diagonal and off-diagonal terms:

$$\dot{\rho}_{ii} = i\sum_{k=\pm i}(\rho_{ik}g_{ki} - \rho_{ki}g_{ik}) = -2\sum_{k=\pm i}\mathrm{Re}[\rho_{ik}g_{ki}], \quad (13)$$

$$\dot{\rho}_{ij} = i\Big(\sum_{k=\pm j}\rho_{ik}g_{kj} - \sum_{k=\pm i}\rho_{kj}g_{ik}\Big) - \Gamma\rho_{ij}, \quad (14)$$

where g_{ij} is the coupling between states i and j. In the limit of heavy dephasing ($\Gamma \gg g$), we have a process similar to adiabatic following, and we can make the approximation

$$\Gamma\rho_{ij} \approx i\Big(\sum_{k=\pm j}\rho_{ik}g_{kj} - \sum_{k=\pm i}\rho_{kj}g_{ik}\Big). \quad (15)$$

We consider the ρ_{ij} as a set of $\frac{n(n-1)}{2}$ variables and solve for them in terms of the ρ_{ii}. Neglecting terms that are second order in $\frac{g}{\Gamma}$ and substituting back into Eq. (13) gives

$$\dot{\rho}_{ii} = -\sum_{j=i\pm 1}\frac{2|g_{ij}|^2}{\Gamma}(\rho_{ii} - \rho_{jj}). \quad (16)$$

Our quantum chain formally reduces to a classical Markov chain on the same state space, with transition rates proportional to the coupling squared.

Although states with more up spins decohere more quickly, the decoherence rate Γ is not multiplied for higher states, as it is the *relative* decoherence rate between neighboring states which is of importance.

In one dimension $g_{ij} = 1$ and we are reduced to a simple random walk on a semi-infinite line. By analogy with simple diffusion we expect that the resulting distribution is roughly Gaussian, with the expected number of flipped spins going with $\sqrt{t}$: the rate of spin propagation drops from t to $\sqrt{t}$. This result was confirmed numerically (Fig. 2).

In the two-dimensional case $g_{ij} = \sqrt{j}$, $j = i + 1$: We get a random walk with increasing transition rates. Numerically (Fig. 2), we find that the rate of spin propagation drops from t^2 to t—still an encouraging scaling.

We also investigated the "individual noise" case, where the dephasing occurs independently on each site (see supplementary material [15]). Although the calculations differ, we see the same rate behavior as for the collective noise case.

Finally, we consider imperfect initial polarization (i.e., finite temperature)—a property exhibited by any real experimental system. As discussed in the supplementary material [15], a fortuitous consequence of the propagation rules is that our system is particularly robust against this source of error; below an initialization threshold of approximately 4%, it is extremely unlikely that a false positive will occur. This places our protocol well within experimental capabilities; for example, for an array placed in a standard W-band electron spin resonance system (100 GHz) and cooled using liquid 4He to 1.4 K, only 3.1% of electron spins will be in the "up" state.

We thank Gerard Milburn and John Morton for useful discussion. This work was supported by the EPSRC, the National Research Foundation and Ministry of Education, Singapore, and the Royal Society.

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