

## Transformation optics, isotropic chiral media and non-Riemannian geometry

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## Transformation optics, isotropic chiral media and non-Riemannian geometry

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**Abstract.** The geometrical interpretation of electromagnetism in transparent media (transformation optics) is extended to include chiral media that are isotropic but inhomogeneous. It was found that such media may be described through introducing the non-Riemannian geometrical property of torsion into the Maxwell equations, and it is shown how such an interpretation may be applied to the design of optical devices.

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## 1. Introduction

Several authors have noted that the free space Maxwell equations in a general space–time look like those in a certain kind of transparent material (see references cited in [1] and also [2, section 90]). For some time this was a curiosity, but recently this relationship has been put to work in reverse. *Transformation optics* [3–7] has been developed to use this formal analogy to design optical devices and even investigate analogues of astrophysical objects in the laboratory [8, 9].

So far the space–time geometry of transformation optics has been Riemannian—i.e. the properties of the geometry have been solely determined by a symmetric metric tensor,  $g_{\mu\nu}$ . Light propagation on a Riemannian space–time background can be intuitively understood in terms of rays following geodesics and the polarization undergoing parallel transport along each ray [3, 10]. Transformation optics uses this simple picture to *design* a space that acts on the optical field in a desired way and thereby determines the necessary material properties from geometrical quantities. Notably, this recipe has been applied to derive the material properties necessary for devices that conceal objects from the electromagnetic field [6, 11] and focus light intensity into a region that is smaller than what the diffraction limit leads us to expect [12, 13]. In this paper, the idea is to provide more variables for this design strategy to explore: in particular, to incorporate chiral media into transformation optics.

To motivate going beyond Riemannian geometry, consider the following counting argument. The term ‘transformation optics’ came from the initial use of co-ordinate transformations to arrive at material parameters—i.e. Euclidean geometry. Therefore, in the original sense, transformation optics works through the specification of *three* functions of position. The full Riemannian geometry has greater freedom, with a symmetric space–time metric containing *ten* independent functions of position, translating into nine independent material parameters<sup>1</sup>.

Recognizing that a geometry must affect electric and magnetic fields in the same way, this may not be the full story. Symmetric, impedance-matched permittivity and permeability tensors (i.e.  $\epsilon/\epsilon_0 = \mu/\mu_0$ ) represent six independent quantities, and the possibility of magneto-electric coupling, at least another six components. This leaves three real magneto-electric coupling parameters that cannot be represented within Riemannian geometry, but may well have a geometrical interpretation<sup>2</sup>. It is therefore worth investigating non-Riemannian extensions to transformation optics. In such a geometry there is at least one additional field (the space–time *torsion*,  $T_{\nu\sigma}^{\mu} = \Gamma_{\nu\sigma}^{\mu} - \Gamma_{\sigma\nu}^{\mu}$ ) that may be freely specified independent of the metric. Torsion has already been explored in other analogous systems: for example, the theory of sound waves propagating through superfluids, where non-Riemannian geometry has been used to describe the interaction with vorticity [14, 15].

In the following, it will be shown that if we couple a non-Riemannian geometrical background to the free space Maxwell equations in a certain way, then Maxwell’s equations can be interpreted as if in an inhomogeneous, isotropic, chiral medium described by both a Tellegen parameter,  $\chi$ , and a chirality parameter,  $\kappa$ . In the limit of geometrical optics, this coupling is shown to reproduce the usual geodesic and parallel transport equations, but in the presence of

<sup>1</sup> The conformal invariance of Maxwell’s equations (i.e. invariance under  $g_{\mu\nu} \rightarrow f(x^{\sigma})g_{\mu\nu}$ ) has the consequence that out of  $n$  independent space–time metric components there are  $n - 1$  independent material parameters.

<sup>2</sup> When considering complex entries in the  $\epsilon$ ,  $\mu$  and electric–magnetic coupling tensors, the number of under-described parameters is larger.

geometrical torsion: that is, optical activity is shown to have a geometrical interpretation in terms of the torsion tensor.

## 2. The relationship between electromagnetic fields in continuous media and Riemannian geometry

We begin by reviewing the existing theory of transformation optics [3, 7, 16].

Consider a space–time where the curvature tensor,  $R_{\nu\sigma\tau}^{\mu}$ , may not vanish and where the co-ordinates,  $x^{\mu}$ , are arbitrary. In this instance, we write down the free space Maxwell's equations using their usual four-dimensional (4D) form [2], along with the following convention from general relativity [17]: that ordinary partial derivatives be replaced by covariant ones,  $\partial_{\mu} \rightarrow \nabla_{\mu}$ , and that the permutation symbol be scaled by the volume element,  $e^{\mu\nu\sigma\tau} \rightarrow \epsilon^{\mu\nu\sigma\tau} = g^{-1/2} e^{\mu\nu\sigma\tau}$  [18],

$$\epsilon^{\mu\nu\sigma\tau} \nabla_{\nu} F_{\sigma\tau} = 0, \quad (1)$$

$$\nabla_{\mu} F^{\mu\nu} = 0. \quad (2)$$

The covariant derivative,  $\nabla_{\mu}$ , in (1) and (2) differs from an ordinary partial derivative by a quantity,  $\Gamma_{\nu\mu}^{\sigma}$ , known as the connection symbol [18]. Transformation optics works because, in the Riemannian case, the connection symbol in (1) and (2) plays the same algebraic role as the difference between the field equations in vacuum and in a polarizable medium. We can see this through explicitly identifying  $\Gamma_{\nu\mu}^{\sigma}$  in (1) and (2),

$$\epsilon^{\mu\nu\sigma\tau} [\partial_{\nu} F_{\sigma\tau} - \Gamma_{\sigma\nu}^{\alpha} F_{\alpha\tau} - \Gamma_{\tau\nu}^{\alpha} F_{\sigma\alpha}] = 0, \quad (3)$$

$$\partial_{\mu} F^{\mu\nu} + \Gamma_{\alpha\mu}^{\mu} F^{\alpha\nu} + \Gamma_{\alpha\mu}^{\nu} F^{\mu\alpha} = 0. \quad (4)$$

The assumptions of Riemannian geometry lead to a particular form of  $\Gamma_{\nu\sigma}^{\mu}$  that is known as a Christoffel symbol [18],  $\Gamma_{\nu\sigma}^{\mu} = \{\begin{smallmatrix} \mu \\ \nu\sigma \end{smallmatrix}\}$ , and depends only on the form of the metric tensor,  $g_{\mu\nu}$ ,

$$\{\begin{smallmatrix} \mu \\ \nu\sigma \end{smallmatrix}\} = \frac{1}{2} g^{\mu\tau} [\partial_{\sigma} g_{\nu\tau} + \partial_{\nu} g_{\tau\sigma} - \partial_{\tau} g_{\sigma\nu}]. \quad (5)$$

Due to the symmetry of the Christoffel symbol in its lower two indices, (3) is unaffected by the distinction between  $\nabla_{\mu}$  and  $\partial_{\mu}$ . *Therefore, the definition of the field tensor in terms of the vector potential is the same as if the co-ordinates were those of a Galilean system*<sup>3</sup>. Furthermore, the antisymmetry of the field tensor means that the final term to the left of the equals sign in (4) is also zero. We can now see that the only term that distinguishes the Maxwell equations (3) and (4) from a Galilean system is proportional to the trace of the Christoffel symbol [18],

$$\{\begin{smallmatrix} \mu \\ \alpha\mu \end{smallmatrix}\} = \frac{1}{2} g^{\mu\tau} \partial_{\alpha} g_{\mu\tau} = \frac{1}{2g} \partial_{\alpha} g,$$

where  $g = \det(g_{\mu\nu})$ . The free space Maxwell equations on a Riemannian background, (1) and (2), can thus be written in a form very similar to that of the free space Maxwell equations

<sup>3</sup> Throughout we use the term 'Galilean' in accordance with Landau and Lifshitz, i.e. to mean the system of co-ordinates where the metric equals  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  everywhere.

in Galilean space–time. All that is changed is the appearance of a factor of  $\sqrt{-g}$ , and the relationship between  $F^{\mu\nu}$  and  $F_{\mu\nu}$ ,<sup>4</sup>

$$e^{\mu\nu\sigma\tau} \partial_\nu F_{\sigma\tau} = 0, \quad (6)$$

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0. \quad (7)$$

There are two equivalent ways to understand (6) and (7): either, as we have done so far, in terms of an empty, possibly non-flat, space-time background, or equivalently in terms of a Galilean system containing a dielectric medium. For, if we consider the 4D Maxwell equations in the presence of a material medium, within a Galilean co-ordinate system and without any external sources, then Maxwell's equations can be written as [19]

$$e^{\mu\nu\sigma\tau} \partial_\nu F_{\sigma\tau} = 0, \quad (8)$$

$$\partial_\mu G^{\mu\nu} = 0. \quad (9)$$

In this case, there are two separate, but not independent, 'field tensors': the tensor,  $F_{\sigma\tau}$ , containing the physical fields,  $F_{\sigma\tau} = (\mathbf{E}/c, \mathbf{B})$ ; and the tensor,  $G^{\mu\nu}$ , containing the material fields,  $G^{\mu\nu} = (-c\mathbf{D}, \mathbf{H})$ , the vanishing divergence of which indicates that the material has no net charge and that no net current passes through any cross-section. The constitutive relationship between  $F_{\sigma\tau}$  and  $G^{\mu\nu}$  serves to indicate how the physical fields are influenced by the medium.

As the form of (6) and (7) is identical to that of (8) and (9), there is a correspondence between *each and every co-ordinate system* on any Riemannian background, and an equivalent material, described within a Galilean co-ordinate system,

$$G^{\mu\nu} \leftrightarrow \frac{\sqrt{-g}}{\mu_0} F^{\mu\nu}, \quad (10)$$

where the factor of  $\mu_0$  has been introduced so that the units agree with the physical interpretation of  $G^{\mu\nu}$ . The co-ordinates,  $x^\mu$ , on a general background are re-interpreted as a Galilean system,  $g_{\mu\nu} \leftrightarrow \eta_{\mu\nu}$ , with the contravariant field tensor appearing as a material field.

The constitutive relationship between the physical fields,  $F_{\mu\nu}$ , and the material fields,  $G^{\mu\nu}$ , is obtained from the relationship between covariant and contravariant indices on the Riemannian background,

$$F_{\mu\nu} = g_{\mu\sigma} g_{\nu\tau} F^{\sigma\tau} = g_{\mu\sigma} g_{\nu\tau} \frac{\mu_0}{\sqrt{-g}} G^{\sigma\tau}. \quad (11)$$

To make the interpretation more transparent, we introduce the usual 3D quantities:  $E_i = F_{0i}$ ,  $B_i = -\frac{1}{2} e^{ijk} F_{jk}$ ,  $cD_i = G^{i0}$  and  $H_i = -\frac{1}{2} e_{ijk} G^{jk}$ , and adopt dyadic notation, so that (11) becomes

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E} + \frac{1}{c^2} \mathbf{g} \times \mathbf{H}, \quad (12)$$

$$\mathbf{B} = \boldsymbol{\mu} \cdot \mathbf{H} - \frac{1}{c^2} \mathbf{g} \times \mathbf{E}, \quad (13)$$

<sup>4</sup> The minus sign is introduced into the square root of  $g$  for the sake of convention: physical space–time has the signature (1, 3) and therefore a negative value for the determinant,  $g$ . However, transformation optics is free to explore media that are equivalent to space–times with arbitrary signature (see e.g. [40]).

where  $\epsilon_{ij} = \frac{\epsilon_0 \sqrt{-g}}{g_{00}} \gamma_{ij}^{-1}$ ,  $\mu_{ij} = \frac{\mu_0 \sqrt{-g}}{g_{00}} \gamma_{ij}^{-1}$  and we have introduced the symbols  $\gamma_{ij} = (g_{0i} g_{0j} / g_{00} - g_{ij})$ ,  $\gamma_{ij}^{-1} = -g^{ij}$  and  $g_j = c g_{0j} / g_{00}$  [2]. The conclusion is, therefore, that each co-ordinate system, on every Riemannian background, can be considered to appear to the electromagnetic field as a continuous medium with equal relative permeability and permittivity tensors (an impedance-matched medium)— $\epsilon_{ij} / \epsilon_0 = \mu_{ij} / \mu_0$ —described within a Galilean system of co-ordinates.

Perhaps even more importantly, the reverse also holds; for a fixed frequency of the electromagnetic field, all transparent, impedance-matched media can be understood in terms of the vacuum Maxwell equations within a Riemannian geometry. However, this space–time co-ordinate system is not uniquely defined by the medium, due to the invariance of the Maxwell equations under conformal transformations.

### 3. Isotropic chiral media

Section 4 will describe a class of non-Riemannian geometries that are equivalent to inhomogeneous, isotropic, chiral media. Unfortunately, there seems to be no agreement regarding the form of the constitutive relations that should be used to describe such media. Although, in the frequency domain, these various constitutive relations can be shown to be *physically* equivalent [20, 21], the meaning of the individual material parameters is different for each constitutive relation. This will turn out to be important when we come to interpret chiral parameters in terms of geometrical quantities. Therefore, we spend this section distinguishing the constitutive relations before, in the next section, introducing the geometry.

Landau and Lifshitz determine the relationship between the material fields and the physical fields in optically active media in terms of antisymmetric complex components in the permittivity tensor,  $\epsilon_{ij} = \epsilon_{ji}^*$  [19]. This is equivalent to what is known as the Drude–Born constitutive relation [20, 21], where  $\mathbf{D}$  is coupled not only to  $\mathbf{E}$ , but also to  $\nabla \times \mathbf{E}$ . For a geometric interpretation, we must have constitutive relationships that are local and have symmetric coupling terms in  $\mathbf{D}$  and  $\mathbf{B}$  (or  $\mathbf{H}$ )<sup>5</sup>, and so we exclude this possibility. Instead, we describe isotropic chirality via the electric–magnetic coupling terms that were discussed in the introduction. The Tellegen constitutive relations are often used [20],

$$\mathbf{D} = \epsilon_T \cdot \mathbf{E} + \frac{1}{c} (\chi_T - i\kappa_T) \mathbf{H}, \quad (14)$$

$$\mathbf{B} = \mu_T \cdot \mathbf{H} + \frac{1}{c} (\chi_T + i\kappa_T) \mathbf{E}. \quad (15)$$

where  $\chi$  represents the Tellegen parameter [22] and  $\kappa$  the chiral parameter. The presence of the  $i$  indicates that, as long as  $\kappa \neq 0$ , (14) and (15) have meaning only in the frequency domain. While the existence of media with non-zero  $\kappa$  is unquestionable (the restrictions on  $\kappa$  have caused some controversy [23]), there is a history of debate regarding the reality of media with non-zero  $\chi$  [24–27].

We can observe that an isotropic chiral medium is a material where there is a linear coupling between one component of the electric (magnetic) polarization, say  $P_x$  ( $M_x$ ), and the

<sup>5</sup> The Drude–Born–Federov constitutive relations modify the Drude–Born relations to describe media with symmetric magneto-electric coupling (see e.g. [20, section 3.2]), coupling  $\mathbf{D}$  to  $\nabla \times \mathbf{E}$  as well as  $\mathbf{B}$  to  $\nabla \times \mathbf{H}$ . Clearly, these relationships are also non-local.

same component of the magnetic (electric) field. Therefore, for our purposes it is useful to see our initial suspicions confirmed and to note that it is not possible to have a totally spatially isotropic medium and have such a direct coupling described geometrically by (12) and (13), for in this case  $\mathbf{g}$  should vanish. Hence, optical activity in isotropic media cannot be understood in terms of the theory of transformation optics presented in section 2.

One common alternative form to (14) and (15) is the Boys–Post relation [20]

$$\mathbf{D} = \epsilon_P \cdot \mathbf{E} + \frac{1}{c} (\chi_P + i\kappa_P) \mathbf{B}, \quad (16)$$

$$\mathbf{H} = \mu_P^{-1} \cdot \mathbf{B} - \frac{1}{c} (\chi_P - i\kappa_P) \mathbf{E}. \quad (17)$$

In the frequency domain (where these relationships are defined), (14), (15), (16) and (17) are equivalent. However, it is important that in each case the meaning of the permittivity and permeability is different, as well as the interpretation of the chirality. Indeed, if we cast (16) and (17) into the form of (14) and (15), then we obtain

$$\begin{aligned} \mathbf{D} &= \left[ \epsilon_P + \frac{1}{c^2} (\chi_P^2 + \kappa_P^2) \mu_P \right] \cdot \mathbf{E} + \frac{1}{c} (\chi_P + i\kappa_P) \mu_P \cdot \mathbf{H}, \\ \mathbf{B} &= \mu_P \cdot \mathbf{H} + \frac{1}{c} (\chi_P - i\kappa_P) \mu_P \cdot \mathbf{E}, \end{aligned}$$

so that we can observe a correspondence in the coupling parameters

$$\begin{aligned} \epsilon_T &\leftrightarrow \epsilon_P + \frac{1}{c^2} (\chi_P^2 + \kappa_P^2) \mu_P, \\ \mu_T &\leftrightarrow \mu_P \\ \chi_T \pm i\kappa_T &\leftrightarrow \mu_P (\chi_P \pm i\kappa_P). \end{aligned} \quad (18)$$

Therefore, in general, when the magnetic susceptibility is anisotropic, whether the chirality is isotropic is relative to the interpretation. Furthermore, ‘impedance matching’ in the Post constitutive relations does not translate into impedance matching in the Tellegen interpretation. As noted in [20], true impedance matching must be done relative to the Tellegen parameters. However, in the limit of small chirality, the difference is negligible.

#### 4. Non-Riemannian geometry and isotropic chiral media

From the summary of the theory of transformation optics given in section 2, it is evident that there are two reasons why a Riemannian background has the same effect on the Maxwell equations as a material medium: (i) the definition of the field tensor in terms of the vector potential is unchanged by the co-ordinate system, so that the co-ordinates may be re-interpreted as if they were Galilean, and; (ii) the expression for the trace of the Christoffel symbols is such that (7) can be written as a total divergence. Consequently, when the co-ordinate system is re-interpreted as Galilean, the effect of the geometry is that of a charge neutral medium, with no net current passing through any cross-section.

Is it possible to fulfil both these conditions with a more general geometrical background? Let us explore more general forms of the connection,

$$\Gamma_{\nu\sigma}^{\mu} = \{\}_{\nu\sigma}^{\mu} + C_{\nu\sigma}^{\mu}. \quad (19)$$

It is instructive to write the non-Riemannian part of the connection,  $C_{\nu\sigma}^{\mu}$ , in terms of component tensors (see e.g. [17, section 3.3]),

$$C_{\nu\sigma}^{\mu} = K_{\nu\sigma}^{\mu} + H_{\nu\sigma}^{\mu},$$

where  $K_{\nu\sigma}^{\mu} = \frac{1}{2}g^{\alpha\mu} [T_{\alpha\nu\sigma} + T_{\nu\alpha\sigma} + T_{\sigma\alpha\nu}]$  is the contorsion tensor (recalling that the torsion,  $T_{\nu\sigma}^{\mu}$ , is the antisymmetric part of the connection), and  $H_{\nu\sigma}^{\mu}$  represents the non-metricity<sup>6</sup>. When  $H_{\nu\sigma}^{\mu} = 0$ , the connection satisfies the condition  $\nabla_{\mu}g_{\sigma\tau} = 0$  everywhere. In what follows, we assume that  $H_{\nu\sigma}^{\mu} = 0$ , so that  $C_{\nu\sigma}^{\mu} = K_{\nu\sigma}^{\mu}$ .

We must be careful how we write down Maxwell's equations in a geometry with torsion, for almost immediately we encounter problems if we add  $K_{\nu\sigma}^{\mu}$  into the connection. If we follow the usual 'partial derivative goes to covariant derivative' rule, then the antisymmetry in the lower indices interferes with the definition of the field tensor, (6),

$$e^{\mu\nu\sigma\tau} \nabla_{\nu} F_{\sigma\tau} = e^{\mu\nu\sigma\tau} [\partial_{\nu} F_{\sigma\tau} - T_{\sigma\nu}^{\rho} F_{\rho\tau}] \stackrel{?}{=} 0.$$

In itself this might not be a problem, did it not break the gauge invariance of the theory. For, if we perform a gauge transformation,  $A'_{\mu} = A_{\mu} + \nabla_{\mu}\varphi$ , the field tensor ends up depending upon  $\varphi$ :

$$\begin{aligned} F'_{\mu\nu} &= \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} + [\nabla_{\mu}, \nabla_{\nu}]\varphi, \\ &= \nabla_{\mu} A_{\nu} - \nabla_{\nu} A_{\mu} + T_{\mu\nu}^{\rho} \nabla_{\rho}\varphi. \end{aligned}$$

Therefore, we reach the conclusion that, with non-zero torsion, the 'partial derivative goes to covariant derivative' rule does not produce a gauge invariant theory. For a more extensive coverage of this issue, see [17, section 11.3] and [28]. Hence, we choose to keep the *definition* of the field tensor the same as in the Riemannian case,  $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ .

The second Maxwell equation does not relate to the definition of the field tensor, and we may suppose that the background geometry modifies this equation with terms including the contorsion tensor. Applying (19) to (4), we obtain

$$\nabla_{\mu} F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} F^{\mu\nu}) + \frac{1}{2} T^{\nu\sigma\tau} F_{\tau\sigma} = 0. \quad (20)$$

In order that this appears as a material medium—cf (9)—there is a minimal choice for the form of the torsion,

$$T^{\nu\sigma\tau} = \mu_0 \epsilon^{\nu\sigma\tau\mu} \partial_{\mu} \chi, \quad (21)$$

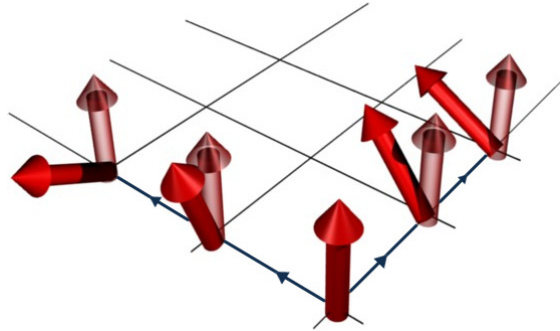
where  $\chi$  is an arbitrary single valued function of  $x^{\mu}$  (see figure 1 for the geometrical interpretation of (21)). The Maxwell equation associated with sources, (20), thus takes the following form

$$\partial_{\mu} \left( \sqrt{-g} F^{\mu\nu} + \frac{1}{2} \chi \mu_0 \epsilon^{\nu\sigma\tau\mu} F_{\tau\sigma} \right) = 0. \quad (22)$$

It is interesting that (22) has the same form of coupling to the torsion field as was obtained in [28] through microscopic considerations of the interaction between a space–time torsion field and the

<sup>6</sup> We should note that the covariant derivative of the Lévi–Civita symbol,  $\epsilon_{\mu\nu\sigma\tau}$ , vanishes when the trace of the connection equals the trace of the Christoffel symbol [3, section 20]; the contorsion tensor does not alter the trace of the connection.





**Figure 1.** A completely antisymmetric torsion field,  $T_{\mu\nu\sigma} = -T_{\mu\sigma\nu} = -T_{\nu\mu\sigma}$ , does not affect geodesics, and rotates vectors in the plane normal to the direction of parallel transport. In the figure, parallel transport is being performed along the blue lines. The transparent vectors show the case when torsion is not included, whereas the solid vectors show the case when we have a completely antisymmetric torsion field.

vacuum polarization associated with the quantized electromagnetic field. The divergence-less material field in (22) is

$$G^{\mu\nu} = \frac{\sqrt{-g}}{\mu_0} g^{\mu\sigma} g^{\nu\tau} F_{\sigma\tau} + \frac{1}{2} \chi e^{\mu\nu\sigma\tau} F_{\sigma\tau}. \quad (23)$$

Due to the appearance of the dual electromagnetic field tensor on the right-hand side of (23), the presence of the torsion (21) in the connection is equivalent to some coupling between alike components of the polarization (magnetization) and the magnetic (electric) field. To be explicit, we write (23) in terms of the usual fields,  $(\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H})$ .

Take the simplest case first, and consider an isotropic material,  $g_{\mu\nu} = \text{diag}(1, -n^2, -n^2, -n^2)$ . For this case, (23) becomes

$$\begin{aligned} \mathbf{D} &= \epsilon_0 n \mathbf{E} + \frac{\chi}{c} \mathbf{B}, \\ \mathbf{H} &= \frac{1}{\mu_0 n} \mathbf{B} - \frac{\chi}{c} \mathbf{E}. \end{aligned} \quad (24)$$

The addition of the torsion, (21), into the isotropic geometry may be understood in material terms as an isotropic chiral medium with a non-zero Tellegen parameter, which has equal relative permittivity and permeability as regards the Boys–Post prescription, (16) and (17),  $\mu_P/\mu_0 = \epsilon_P/\epsilon_0$ .

The next simplest case is where the medium is anisotropic, but where the magneto-electric coupling defined by the  $\mathbf{g}$  vector vanishes. Here, (23) gives

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\epsilon} \cdot \mathbf{E} + \frac{\chi}{c} \mathbf{B}, \\ \mathbf{H} &= \boldsymbol{\mu}^{-1} \cdot \mathbf{B} - \frac{\chi}{c} \mathbf{E}, \end{aligned} \quad (25)$$

where, as before,  $\epsilon_{ij} = \frac{\epsilon_0 \sqrt{-g}}{g_{00}} \gamma_{ij}^{-1}$  and  $\mu_{ij} = \frac{\mu_0 \sqrt{-g}}{g_{00}} \gamma_{ij}^{-1}$ . As the torsion, (21), is actually a pseudo-tensor, the  $\chi$  parameter changes sign under time reversal or spatial parity inversion: this is

consistent with a Tellegen medium, which is not time reversible. The constitutive relation is somewhat more complicated in the case of a space–time background with a non-vanishing  $\mathbf{g}$  vector as there is an interplay between the magneto–electric coupling due to the metric and that due to the torsion.

We conclude that the inhomogeneity of the Tellegen parameter is equivalent to space–time torsion in the covariant derivative of (4). Note that when the Tellegen parameter is homogeneous, it disappears from the geometry, which again becomes Riemannian: this is consistent with the known invariance of the Maxwell equations under transformations of the fields, where  $\mathbf{D} \rightarrow \mathbf{D} + \eta \mathbf{B}$  and  $\mathbf{H} \rightarrow \mathbf{H} - \eta \mathbf{E}$ , with uniform  $\eta$  (see e.g. [24, 29]).

It is evident that the more physically important magneto–electric parameter,  $\kappa$ , does not arise from the above modification to the space–time connection. The definition of this quantity, (16) and (17), anticipates that it can only arise in the frequency domain, which we now consider.

#### 4.1. The frequency domain

In the frequency domain, we consider a purely spatial geometry— $g_{00} = 1$ ,  $g_{0i} = g_{i0} = 0$ —and replace the time derivative with  $-i\omega$ . With this assumption, (20) is

$$\partial_i (\sqrt{-g} F^{iv}) + \frac{1}{2} \sqrt{-g} T^{\nu\sigma\mu} F_{\mu\sigma} = i \frac{\omega}{c} \sqrt{-g} F^{0v}, \quad (26)$$

where we have assumed, as in the previous section, that the additional torsion does not alter the trace of the connection symbol. One such set of components is as follows:

$$\begin{aligned} T^{0ij} &= \frac{i\mu_0}{\sqrt{-g}} e^{ijk} \partial_k \kappa = -T^{ij0}, \\ T^{ijk} &= -\frac{2\mu_0}{\sqrt{-g}} e^{ijk} \frac{\omega \kappa}{c}. \end{aligned} \quad (27)$$

Note that, despite appearances, (27) transforms as a pseudo-tensor under purely spatial co-ordinate transformations. We can decompose (26) into two equations, one for  $\nu = 0$ ,

$$\partial_i \left( \frac{\sqrt{-g}}{\mu_0} F^{i0} - i\kappa \frac{1}{2} e^{ijk} F_{jk} \right) = 0, \quad (28)$$

and the other for  $\nu = j$ ,

$$\partial_i \left( \frac{\sqrt{-g}}{\mu_0} F^{ij} - i\kappa e^{ijk} F_{0k} \right) = i \frac{\omega}{c} \left( \frac{\sqrt{-g}}{\mu_0} F^{0j} + i\kappa \frac{1}{2} e^{jik} F_{ik} \right). \quad (29)$$

From the form of (28) and (29), the torsion given by (27) defines the following divergence-less quantity:

$$\begin{aligned} G^{0i} &= \frac{\sqrt{-g}}{\mu_0} F^{0i} + i\kappa \frac{1}{2} e^{ijk} F_{jk}, \\ G^{ij} &= \frac{\sqrt{-g}}{\mu_0} F^{ij} - i\kappa e^{ijk} F_{0k}. \end{aligned}$$

In terms of a Galilean system containing a material medium, these are equivalent to the vector relationships

$$\begin{aligned} \mathbf{D} &= \boldsymbol{\epsilon} \cdot \mathbf{E} + i \frac{\kappa}{c} \mathbf{B}, \\ \mathbf{H} &= \boldsymbol{\mu}^{-1} \cdot \mathbf{B} + i \frac{\kappa}{c} \mathbf{E}, \end{aligned}$$

which define a material medium with a chiral parameter,  $\kappa$ , interpreted in the sense of a Boys–Post constitutive relationship. Combining this result with that of the previous section, a medium with both a chiral parameter and a Tellegen parameter may be defined via a torsion pseudo-tensor with the components

$$\begin{aligned} T^{ijk} &= -2\mu_0\epsilon^{ijk}\frac{\omega\kappa}{c}, \\ T^{ij0} &= \mu_0\epsilon^{ijk}\partial_k(\chi - i\kappa), \\ T^{0ij} &= \mu_0\epsilon^{ijk}\partial_k(\chi + i\kappa). \end{aligned} \quad (30)$$

As we are working in the frequency domain, we have assumed that the material parameters are independent of time. Note that when the material is uniform, the mixed time and space components of the  $T^{\alpha\beta\gamma}$  vanish, and we are left with only the spatial components of the torsion that are proportional to  $\kappa$ . Indeed, the spatial torsion ( $T^{123}$ ) is far and away the dominant part of the object, as it is also weighted by the factor  $\omega/c$ . This is particularly important in the limit of geometrical optics, which we now investigate.

## 5. Parallel transport and geometrical optics

Transformation optics is a geometrical theory that goes beyond ordinary geometrical optics: it is an exact mapping between geometries and materials. Yet geometrical optics encodes most of the intuitive content of the theory: rays follow geodesics, and polarization is parallel transported [3, 10]. Therefore, it is a minimal requirement that the theory of section 4 bear these intuitions out: does the geometrical optics of an isotropic chiral medium behave as a theory of rays on a background with torsion? Here we show that this is indeed the case. The approach of this section is like that of the last: the existing theory is briefly reviewed so that we can bring out the new features that come from the non-Riemannian modifications.

### 5.1. Riemannian media

The starting point of geometrical optics is the wave equation, which we now derive for the Riemannian case. As noted previously, in this case it is immaterial whether covariant or partial derivatives appear in the definition of the field tensor. Therefore, the 4D curl of (6) can be written as

$$\epsilon^{\alpha\beta\gamma\mu}\nabla_\gamma(\epsilon_{\mu\nu\sigma\tau}\nabla^\nu F^{\sigma\tau}) = 0. \quad (31)$$

As has been assumed throughout, the covariant derivative of the Lévi–Civita symbol is zero, and the usual formula  $\delta_{\nu\sigma\tau}^{\alpha\beta\gamma} = \epsilon^{\alpha\beta\gamma\mu}\epsilon_{\nu\sigma\tau\mu}$  (where  $\delta_{\nu\sigma\tau}^{\alpha\beta\gamma}$  is a  $3 \times 3$  determinant of the Kronecker deltas [18]) can be used to obtain the following result:

$$\nabla_\gamma\nabla^\alpha F^{\beta\gamma} + \nabla_\gamma\nabla^\gamma F^{\alpha\beta} + \nabla_\gamma\nabla^\beta F^{\gamma\alpha} = 0. \quad (32)$$

Applying (7) then gives the wave equation

$$\nabla_\gamma\nabla^\gamma F_{\alpha\beta} + [\nabla^\gamma, \nabla_\alpha]F_{\beta\gamma} + [\nabla^\gamma, \nabla_\beta]F_{\gamma\alpha} = 0, \quad (33)$$

where the commutators of the derivatives are proportional to the contraction of the Riemann curvature tensor,  $R_{\nu\sigma\tau}^\mu$ , against each index of the field tensor in turn: for example, for a four-vector,  $V^\mu$ ,  $[\nabla_\alpha, \nabla_\beta]V_\sigma = R_{\sigma\beta\alpha}^\tau V_\tau$ .

These curvature terms will not enter the approximation of geometrical optics, because they do not diverge as the wavelength goes to zero. To see this, we write down the field tensor in the form  $F_{\mu\nu} = f_{\mu\nu} e^{2\pi i S/\lambda}$ , where  $\lambda = 2\pi c/\omega$ , and take the limit of rapidly varying phase:  $\lambda \rightarrow 0$ , which is equivalent to assuming a length scale for the variation of the material properties that is much larger than the wavelength of the optical field.

Expanding (33) in powers of  $\lambda^{-1}$  and requiring that the coefficient of each power vanish separately (neglecting the zeroth order as  $\lambda \rightarrow 0$ ) yields the equations of geometrical optics

$$\lambda^{-2} : (\partial^\alpha S)(\partial_\alpha S) = 0, \quad (34)$$

$$\lambda^{-1} : (\partial_\gamma S)\nabla^\gamma f_{\mu\nu} = \frac{1}{2}f_{\mu\nu}\nabla^\gamma(\partial_\gamma S), \quad (35)$$

which, as expected, do not include the curvature tensor. The first of these relations, (34), is equivalent to the statement that rays follow geodesics. To see this, we operate on the left with the covariant derivative,  $\nabla_\beta$ , and recall that in a Riemannian space–time, the second-order derivatives of a scalar commute (the torsion is zero):  $[\nabla_\alpha, \nabla_\beta]\varphi = 0$ ,

$$(\partial^\alpha S)\nabla_\beta(\partial_\alpha S) = \frac{D^2 x_\beta}{ds^2} = 0, \quad (36)$$

where the gradient of the phase,  $\partial_\alpha S$ , is taken to equal the tangent vector to a curve,  $\partial_\alpha S = g_{\alpha\beta} dx^\beta/ds$ , and  $D \equiv dx^\alpha \nabla_\alpha$ . Equation (36) is precisely the rule for the geodesic motion of a material particle in a Riemannian space–time.

The second defining equation of geometrical optics, (35), illustrates how the polarization changes along a ray. We write the field tensor amplitude as a bivector that, in the frame we are considering, contains two unit three-vectors, multiplied by an amplitude,  $f_{\mu\nu} = u_{\mu\nu} F$ , where  $u_{\mu\nu} = (\mathbf{u}, \mathbf{v})$ :  $u_i(u^i)^i = 1$ ;  $v_i(v^i)^i = 1$ ;  $u_i k^i = v_i k^i = u_i v^i = 0$ . The tensor  $u_{\mu\nu}$  indicates the direction of the polarization, and (35) becomes

$$\frac{1}{2}\nabla_\gamma \left( \mathcal{F}^2 \frac{dx^\gamma}{ds} \right) u_{\mu\nu} + \mathcal{F}^2 \left( \frac{Du_{\mu\nu}}{ds} \right) = 0. \quad (37)$$

Both terms to the left of the equality in (37) must vanish separately if the unit polarization vectors defined within  $u_{\mu\nu}$  are to remain unit vectors at all points along a ray. Consequently,

$$\nabla_\gamma \left( \mathcal{F}^2 \frac{dx^\gamma}{ds} \right) = 0, \quad (38)$$

$$\frac{Du_{\mu\nu}}{ds} = 0. \quad (39)$$

Explicitly, (38) is equivalent to the continuity of the energy momentum along the ray, and (39) to the statement that the direction of the polarization is parallel transported along the ray. These are the essential results of geometrical optics in a Riemannian medium.

## 5.2. Non-Riemannian media

If we allow for the possibility of a space time with torsion as in section 4, then it is not immediately obvious whether the equations of geometrical optics will still carry the simple structure where rays follow geodesics, and polarization is parallel transported. Although the Maxwell equations remain identical in form to (6) and (7) throughout, this was achieved through

imposing the definition of the field tensor, and we should check that this imposition has not spoiled the geometrical interpretation (which is, after all, the point of transformation optics). We shall show that, due to the particular form of the torsion, (30), the geometrical optics limit is unaffected; rays still follow geodesics, and the polarization is still parallel transported.

The analogous situation to that represented by equation (31) contains an additional contribution that arises because of the use of partial, rather than covariant, derivatives to define the field tensor

$$\epsilon^{\alpha\beta\gamma\mu}\nabla_\gamma\epsilon_{\mu\nu\sigma\tau}(\nabla^\nu F^{\sigma\tau} - T^{\nu\sigma}F^{\alpha\tau}) = 0.$$

Following the same procedure as the one that led to (32) and expanding the contraction of the Lévi-Civita symbols,

$$\nabla_\gamma\nabla^\gamma F^{\alpha\beta} + [\nabla_\gamma, \nabla^\alpha]F^{\beta\gamma} + [\nabla_\gamma, \nabla^\beta]F^{\gamma\alpha}\nabla_\gamma(T_\rho^{\alpha\gamma}F^{\rho\beta} + T_\rho^{\beta\alpha}F^{\rho\gamma} + T_\rho^{\gamma\beta}F^{\rho\alpha}) = 0, \quad (40)$$

it is clear that there are terms in addition to the wave operator,  $\nabla_\gamma\nabla^\gamma F^{\alpha\beta}$ , that involve the contraction of geometric quantities not only against the field tensor, but also against *derivatives* of the field tensor. Such terms will remain in the equations of geometrical optics, and we must investigate them further.

The commutator of the derivatives of the field tensor,  $[\nabla_\gamma, \nabla^\alpha]F^{\beta\gamma}$ , now also contains terms involving the torsion tensor

$$[\nabla_\alpha, \nabla_\beta]F^{\mu\nu} = R_{\sigma\alpha\beta}^\mu F^{\sigma\nu} + R_{\sigma\alpha\beta}^\nu F^{\mu\sigma} + T_{\alpha\beta}^\rho\nabla_\rho F^{\mu\nu}. \quad (41)$$

Using (41) and applying (2), we can group the terms in addition to the wave operator in (40). The zeroth-order terms involving no derivatives of the fields are found to be

$$0^{\text{th}}: g^{\alpha\sigma}(R_{\rho\gamma\sigma}^\beta F^{\rho\gamma} + R_{\rho\gamma\sigma}^\gamma F^{\beta\rho}) - g^{\beta\sigma}(R_{\rho\gamma\sigma}^\alpha F^{\rho\gamma} + R_{\rho\gamma\sigma}^\gamma F^{\alpha\rho}) \\ + F^{\rho\beta}\nabla_\gamma T_\rho^{\alpha\gamma} - F^{\rho\alpha}\nabla_\gamma T_\rho^{\beta\gamma} + F^{\rho\gamma}\nabla_\gamma T_\rho^{\beta\alpha};$$

meanwhile, the terms that are first order in the derivatives of the field tensor are

$$1^{\text{st}}: (T^{\rho\alpha\gamma} - T^{\gamma\rho\alpha})\nabla_\gamma F_\rho^\beta - (T^{\rho\beta\gamma} - T^{\gamma\rho\beta})\nabla_\gamma F_\rho^\alpha. \quad (42)$$

For a general form of  $T^{\alpha\beta\gamma}$ , (42) is non-zero, and a simple geometric description will not apply. However, it is immediately clear that the description of Tellegen media given by (21) makes these first order terms vanish.

In treating (42) in the frequency domain, we assume, as in [30], that the optical activity of the chiral parameter,  $\kappa$ , is such that the polarization is only slightly changed over each optical cycle: i.e. that  $\omega\kappa/c$  does not diverge as  $\lambda \rightarrow 0$ . This means that the quantity  $\kappa$  is of order  $\lambda$ , and the non-zero part of (30) in the limit equals only the spatial part of the torsion,  $T^{ijk}$  (not including the Tellegen parameter, the contribution of which we have shown to equal zero), so that (42) also vanishes in this case. This proves that the limit of geometrical optics involves only the wave operator and zeroth-order terms, just as in the Riemannian case.

We have therefore established that, as in the usual situation presented in section 5.1, only the wave operator matters in the limit of geometrical optics,

$$\nabla_\gamma\nabla^\gamma F^{\mu\nu} = 0.$$

For slowly varying torsion and small curvature and torsion in comparison to  $1/\lambda$ , this is the equation obeyed by the exact solution to Maxwell's equations, to a good approximation. The geometrical understanding of the theory becomes more complicated when the torsion is rapidly

varying, just as it does when the curvature is large in the usual theory of transformation optics. So our formalism passes the first test.

Inserting the ansatz for the field tensor as in the previous section, we have again (34) and (35). However, the meaning of these equations is now slightly different. For, if we take (34) and attempt to derive the geodesic equation as before, then we find an additional term,

$$\begin{aligned} (\partial_\alpha S) \nabla_\beta (\partial^\alpha S) &= (\partial_\alpha S) \nabla^\alpha (\partial_\beta S) + (\partial_\alpha S) [\nabla_\beta, \nabla^\alpha] S \\ &= (\partial_\alpha S) \nabla^\alpha (\partial_\beta S) + T_{\sigma\beta\gamma} (\nabla^\sigma S) (\nabla^\gamma S). \end{aligned}$$

For the same reason that this limit works in the first place, namely the vanishing of the terms in (42), this additional contribution vanishes and we have

$$\frac{D^2 x^\beta}{ds^2} = \frac{d^2 x^\beta}{ds^2} + \Gamma_{\sigma\alpha}^\beta \frac{dx^\sigma}{ds} \frac{dx^\alpha}{ds} = 0. \quad (43)$$

This is the equation for an auto-parallel rather than a geodesic [31, section 10], as it contains the full connection and not only the Christoffel symbol: this equation formally determines the straightest line between two points and not the shortest. However, again due to the antisymmetry of (30) in all indices, the contribution of the contorsion is zero (i.e. chiral media are equivalent to a geometry with a  $K_{\nu\sigma}^\mu$  that is antisymmetric in the lower two indices), and *rays follow geodesics*.

The derivation of the equivalent of (37) is unaltered in this situation, and so (38) and (39) remain in the same form. First the propagation of energy–momentum

$$\nabla_\gamma \left( \mathcal{F}^2 \frac{dx^\gamma}{ds} \right) = \partial_\gamma \left( \mathcal{F}^2 \frac{dx^\gamma}{ds} \right) + \mathcal{F}^2 \Gamma^\gamma_{\alpha\gamma} \frac{dx^\alpha}{ds} = 0.$$

The contorsion that gave rise to the chirality did not alter the trace of the connection. Therefore, energy–momentum propagates relative to geodesics, as in the Riemannian case: this is consistent with the equivalence of (43) to geodesic motion. Meanwhile, the propagation of polarization along the ray *is* affected by the presence of torsion,

$$\frac{Du_{\mu\nu}}{ds} = \frac{du_{\mu\nu}}{ds} - \Gamma_{\mu\alpha}^\sigma u_{\sigma\nu} \frac{dx^\alpha}{ds} - \Gamma_{\nu\alpha}^\sigma u_{\mu\sigma} \frac{dx^\alpha}{ds} = 0. \quad (44)$$

So the formalism appears to be consistent with the idea that *weak chirality acts to rotate polarization during propagation* (cf figure 1).

*In summary:* for a chiral medium that rotates the polarization by a finite amount over a typical length scale, and where the change in the chiral parameter,  $\kappa$ , is not significant over a wavelength, the geometrical optics of chiral media requires that we add torsion into the connection. The form of the torsion, (30), is such that the propagation of a ray is unaffected—geodesics are equivalent to auto-parallels—whereas the parallel transport of the polarization is modified. This result holds also for Tellegen media, where  $\chi$  does not vary too rapidly.

## 6. Applications

### 6.1. A homogeneous, isotropic, chiral medium

The simplest test of this theory is to apply it to the simplest kind of chiral medium; one that is homogeneous. Note that throughout this section and the next, we implicitly work in the

frequency domain, despite using 4D notation. For an isotropic, homogeneous medium, the theory of sections 2 and 4 prescribes that the metric and torsion should be given by

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -n^2 & 0 & 0 \\ 0 & 0 & -n^2 & 0 \\ 0 & 0 & 0 & -n^2 \end{pmatrix}$$

and

$$T^{0ij} = T^{i0j} = 0; \quad T^{ijk} = -2\mu_0 \frac{\omega\kappa}{c} \epsilon^{ijk}.$$

Due to the assumed uniformity of  $n$ , the Christoffel symbols vanish, and the connection is equal to the contorsion tensor

$$\Gamma_{\nu\sigma}^{\mu} = \frac{1}{2} g^{\mu\alpha} [T_{\nu\alpha\sigma} + T_{\sigma\alpha\nu} + T_{\alpha\nu\sigma}] = \frac{1}{2} g^{\mu\alpha} T_{\alpha\nu\sigma}.$$

Therefore, (43) becomes

$$\frac{d^2 x^{\beta}}{ds^2} = 0. \quad (45)$$

Rays follow straight lines. Yet the polarization is changed along each ray, as is clear from (44). For instance, if we take the unit vector for the electric field,  $u_{0i} = u_i$ , then it changes according to

$$\frac{du_i}{ds} = -\frac{\alpha}{n} \epsilon_{ijk} k_j u_k, \quad (46)$$

where  $\alpha = \mu_0 \omega \kappa / c$  and  $k_j = dx_j / ds$ . Suppose that a ray travels along the  $x$ -axis,  $\mathbf{k} = (n, 0, 0)$ . In this case,  $\mathbf{u} = (0, u_y, u_z)$ , and (46) yields two coupled equations,

$$\begin{aligned} \frac{du_y}{ds} &= \alpha u_z, \\ \frac{du_z}{ds} &= -\alpha u_y. \end{aligned} \quad (47)$$

From the definition of  $dx^i / ds$ ,  $dx^i dx_i = ds^2$ , the line element on the ray is  $ds = n dx$ . Therefore the covariant unit vector,  $\mathbf{u}$ , has the following form:

$$\mathbf{u} = n (0, \sin(n\alpha x), \cos(n\alpha x)). \quad (48)$$

An almost identical calculation of the unit vector of the magnetic field,  $\mathbf{v}$ , shows that  $\mathbf{v}$  also satisfies (47). Applying the definition of the field tensor,

$$\mathbf{v} = n (0, -\cos(n\alpha x), \sin(n\alpha x)). \quad (49)$$

From (45), (48) and (49), the field in the material is therefore proportional to

$$\mathbf{E} = (0, \sin(n\alpha x), \cos(n\alpha x)) e^{i\frac{\omega}{c}(nx-ct)}, \quad (50)$$

$$\mathbf{B} = \frac{n}{c} (0, -\cos(n\alpha x), \sin(n\alpha x)) e^{i\frac{\omega}{c}(nx-ct)}. \quad (51)$$

In the weakly chiral limit in which we are working, (50) and (51) are the solutions to the Maxwell equations. To see this, we consider the wave equation that arises from the usual Maxwell equations ( $\mu/\mu_0 = \epsilon/\epsilon_0$ ) with the constitutive relations (16) and (17),

$$\nabla^2 \mathbf{E} + \frac{2\mu_0 n \kappa \omega}{c} \nabla \times \mathbf{E} + \frac{n^2 \omega^2}{c^2} \mathbf{E} = 0. \quad (52)$$

Substituting an electric field of the form  $\mathbf{u}(x) e^{in\frac{\omega}{c}x}$  into (52) yields,

$$\frac{\partial \mathbf{u}}{\partial x} + \frac{\mu_0 n \omega \kappa}{c} \hat{\mathbf{x}} \times \mathbf{u} = \frac{ic}{2n\omega} \left[ \frac{\partial^2 \mathbf{u}}{\partial x^2} + \frac{2\mu_0 n \kappa \omega}{c} \nabla \times \mathbf{u} \right]. \quad (53)$$

The right-hand side of (53) is proportional to  $\lambda$  times a quantity of the order of unity. Therefore, in the approximation of (50) and (51),

$$\frac{\partial \mathbf{u}}{\partial x} + \frac{\mu_0 n \omega \kappa}{c} \hat{\mathbf{x}} \times \mathbf{u} = 0,$$

which is identical to (47). This confirms that in the case of an isotropic, homogeneous, chiral medium, with  $\mu_0 \kappa \ll 1$ , non-Riemannian geometrical optics is equivalent to the solution of Maxwell's equations.

## 6.2. Maxwell's fish eye lens

In the formalism of transformation optics presented in section 2, geometry is implemented for the purpose of directing rays, and *polarization is a bystander*, responding in a way that is determined by the geodesics. However, one may wish to maintain a given polarization throughout a device or change it in some prespecified manner. Here we show, using the simplest example of a curved geometry for light, namely the Maxwell fish eye [3, 10, 32, 33], that torsion can be used to control the polarization of light without affecting the geodesics. We should note that chiral media have previously been considered for 'correcting' polarization in a variant of the planar fish eye [30].

The fish eye is a continuous medium in which the behaviour of light maps onto the free motion on the surface of a sphere (this may be a two-sphere or a three-sphere, depending on whether the medium is planar or truly 3D). It has received much attention recently, in part owing to its ability to periodically perfectly reconstruct an initial optical pulse as it propagates through the medium, as well as sub-diffraction resolution [13], [34–36] (see [37] for an interesting discussion of sub-wavelength focusing).

If the behaviour of light in terms of lab co-ordinates  $(x, y, z)$  corresponds to motion on the surface of a three-sphere (a hypersphere), the optical line element can be written in a form corresponding to an isotropic medium,

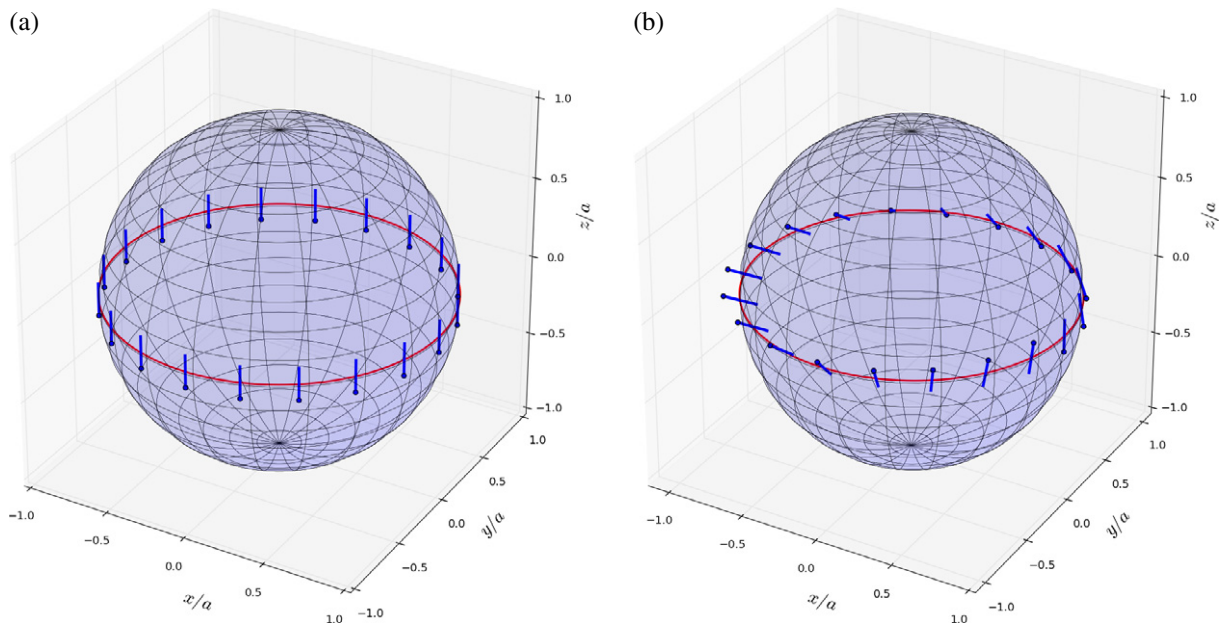
$$\begin{aligned} dl^2 &= n(r)^2 (dx^2 + dy^2 + dz^2) \\ &= a^2 \left[ d\Theta^2 + \sin^2(\Theta) (d\Phi^2 + \sin^2(\Phi) d\chi^2) \right], \end{aligned} \quad (54)$$

where  $r = \sqrt{x^2 + y^2 + z^2}$ , and  $a$  and  $(\Theta, \Phi, \chi)$  are the radius and angular co-ordinates of the equivalent three-sphere, respectively.

To motivate an expression for  $n(r)$ , we note that a spherical geometry has no boundary, whereas physical space has a 'boundary' at infinity. Therefore, as  $r \rightarrow \infty$  in physical space,  $n(r)$  should be such that the radius of any circle surrounding the origin, as experienced by a light ray,  $2\pi n(r)r$ , goes to zero. Also, the length of any optical path,  $\int_0^\infty n(r)dr$ , should equal  $\pi a$ , where  $a$  is the radius of the sphere. A refractive index fulfilling both these requirements is

$$n(r) = 2a \frac{d}{dr} \arctan(r/a) = \frac{2}{1 + (r/a)^2}. \quad (55)$$





**Figure 2.** In Maxwell's fish eye lens, the ray trajectories are the great circles of a three-sphere, as illustrated for the special case of  $r = a$  in (a) and (b). Panel (a) shows the usual situation, where the refractive index determines both the ray trajectory (red line) and the behaviour of the polarization (blue lines). Meanwhile, (b) shows that transformation optics with the inclusion of the geometrical torsion, (58) (i.e. isotropic chirality), allows for the manipulation of polarization without affecting the geodesics.

If we introduce spherical polar co-ordinates in the lab system,  $dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)$  in (54), then the observation that the angles  $\theta, \phi$  can, by symmetry, equal the corresponding angles on the three-sphere,  $\theta, \phi = \Theta, \Phi, \chi$ , leads via (55) to the identification  $r = a \tan(\Theta/2)$ . Performing this transformation of the radial co-ordinate in the first line of (54) yields the line element on the second line of (54), justifying (55).

The metric tensor associated with Maxwell's fish eye can be immediately written down from (54):

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 \sin^2(\Theta) & 0 \\ 0 & 0 & 0 & -a^2 \sin^2(\Theta) \sin^2(\Phi) \end{pmatrix}.$$

This corresponds to the non-zero Christoffel symbols:  $\left\{ \begin{smallmatrix} 1 \\ 22 \end{smallmatrix} \right\} = -\sin(\Theta) \cos(\Theta)$ ;  $\left\{ \begin{smallmatrix} 1 \\ 33 \end{smallmatrix} \right\} = -\sin(\Theta) \cos(\Theta) \sin^2(\Phi)$ ;  $\left\{ \begin{smallmatrix} 2 \\ 12 \end{smallmatrix} \right\} = \cot(\Theta)$ ;  $\left\{ \begin{smallmatrix} 2 \\ 33 \end{smallmatrix} \right\} = -\sin(\Phi) \cos(\Phi)$ ;  $\left\{ \begin{smallmatrix} 3 \\ 13 \end{smallmatrix} \right\} = \cot(\Theta)$ ; and  $\left\{ \begin{smallmatrix} 3 \\ 23 \end{smallmatrix} \right\} = \cot(\Phi)$ . These symbols determine the trajectories of the light rays, as well as the change in polarization during propagation. In particular, light rays with motion along the  $\Phi, \chi$  axes are confined to a two-sphere (of constant  $\Theta$ ) when  $\Theta = \pi/2$  ( $r = a$  in the lab system). As a visual example, let us examine the motion of light rays and transport of polarization in this specific case (see figure 2).

The geodesic equations (36) on the two-sphere defined by  $\Theta = \pi/2$  are

$$\begin{aligned}\frac{d^2\Theta}{ds^2} &= 0, \\ \frac{d^2\Phi}{ds^2} &= \sin(\Phi)\cos(\Phi)\frac{d\chi}{ds}\frac{d\chi}{ds}, \\ \frac{d^2\chi}{ds^2} &= -2\cot(\Phi)\frac{d\Phi}{ds}\frac{d\chi}{ds},\end{aligned}\tag{56}$$

so the rays remain confined to the two-sphere. The change in polarization is similarly obtained, through applying (39):

$$\begin{aligned}\frac{du_{01}}{ds} &= 0, \\ \frac{du_{02}}{ds} &= \cot(\Phi)u_{03}\frac{d\chi}{ds}, \\ \frac{du_{03}}{ds} &= \cot(\Phi)u_{03}\frac{d\Phi}{ds} - \sin(\Phi)\cos(\Phi)u_{02}\frac{d\chi}{ds}.\end{aligned}\tag{57}$$

Equations (56) can be integrated to give the ray trajectory

$$\frac{d\Phi}{d\chi} = \pm\sqrt{\sin^2(\Phi)(l/l_z)^2 - 1},$$

where  $l$  and  $l_z$  are the constants of integration in (56), and the sign of the derivative changes as the square root goes to zero. Noting the symmetry of the medium and setting  $\Phi = \pi/2$ ,  $l = l_z$ , it is clear that rays propagate along the great circles and polarization has a constant orientation along each ray (figure 2(a)). In Riemannian transformation optics, we can only manipulate the propagation of polarization along a ray by changing the geodesics. However, the additional geometrical property of torsion, outlined in sections 4 and 5.2, allows us to ‘twist’ the co-ordinate lines on the sphere so that the polarization is changed, while the geodesics is left unaltered.

For instance, if we wished to generate TE polarization at one point on the sphere (e.g.  $\Theta = \pi/2$ ,  $\Phi = \pi/2$ ,  $\chi = 0$ ) and have it arrive at the antipode ( $\Theta = \pi/2$ ,  $\Phi = \pi/2$ ,  $\chi = \pi$ ) with TM polarization, then we could add the following torsion into the connection (this picture is accurate as long as  $a \gg \lambda$ ):

$$T_{ijk} = a^2\epsilon_{ijk}.\tag{58}$$

The behaviour of the polarization on the surface of the two-sphere with the addition of (58) is shown in figure 2.

## 7. Conclusions

We have shown that non-Riemannian geometry can be introduced into the Maxwell equations to describe inhomogeneous media with isotropic chirality. If the definition of the field tensor is kept in the same gauge invariant form as in a Riemannian geometry, then we have shown that a background with a particular kind of torsion can reproduce the correct constitutive relations for such media. Furthermore, if the chirality produces only a small amount of optical activity over a single optical cycle and varies by only a small amount over a wavelength, then we have also

shown that geometrical optics behaves exactly as if on a background with a non-zero torsion. This formalism allows for a greater degree of control over polarization within the theory of transformation optics, and we have given an example of how torsion can be implemented to change polarization without changing the geodesics.

Any transformation-optics-based device can therefore perform an additional action on the polarization of light through a simple addition of torsion into the geometry. The resulting material parameters may be realized with a homogenization procedure such as that of [38], and this may allow for a reinvigorated exploration of potentially useful chiral devices. One suggestion could be to look at transformation optics designs for ‘sub-diffraction’ chiro-lenses. See, for example, [39] for a broad review of possible avenues that could be explored with chiral media.

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## References

- [1] Leonhardt U and Philbin T G 2006 *New J. Phys.* **8** 247
- [2] Landau L D and Lifshitz E M 2003 *The Classical Theory of Fields* (Oxford: Butterworth-Heinemann)
- [3] Leonhardt U and Philbin T G 2010 *Geometry and Light: The Science of Invisibility* (New York: Dover)
- [4] Chen H, Chan C T and Ping S 2010 *Nat. Mat.* **9** 387
- [5] Leonhardt U 2006 *Science* **312** 1777
- [6] Pendry J B, Schuring D and Smith D R 2006 *Science* **312** 1780
- [7] Cai W and Shalaev V 2010 *Optical Metamaterials* (London: Springer)
- [8] Philbin T G, Kuklewicz C, Robertson S, Hill S, König F and Leonhardt U 2008 *Science* **319** 1367
- [9] Belgiorno F, Cacciatori S L, Clerici M, Gorini V, Ortenzi G, Rizzi L, Rubino E, Sala V G and Faccio D 2010 *Phys. Rev. Lett.* **105** 203901
- [10] Born M and Wolf E 1999 *Principles of Optics* (Cambridge: Cambridge University Press)
- [11] Leonhardt U and Tyc T 2009 *Science* **323** 110
- [12] Pendry J B 2000 *Phys. Rev. Lett.* **85** 3966
- [13] Leonhardt U 2009 *New J. Phys.* **11** 093040
- [14] Garcia de Andrade L C 2004 *Phys. Rev. D* **70** 064004
- [15] Garcia de Andrade L C 2005 *Phys. Lett. A* **346** 327–9
- [16] Ward A J and Pendry J B 1996 *J. Mod. Opt.* **43** 773
- [17] Hurley D J and Vandyck M A 2000 *Geometry, Spinors and Applications* (Chichester: Springer)
- [18] Lovelock D and Rund H 1975 *Tensors, Differential Forms and Variational Principles* (New York: Dover)
- [19] Landau L D, Lifshitz E M and Pitaevskii L P 2004 *The Electrodynamics of Continuous Media* (Oxford: Butterworth-Heinemann)
- [20] Lakhtakia A 1994 *Beltrami Fields in Chiral Media* (Singapore: World Scientific)
- [21] Lindell I V, Sihvola A H, Tretyakov S A and Viitanen A J 1994 *Electromagnetic Waves in Chiral and Bi-Anisotropic Media* (London: Routledge)
- [22] Tellegen B D H 1948 *Philips Res. Rep.* **3** 81
- [23] Zhang C and Jun Cui T 2007 *Opt. Express* **15** 5114
- [24] Lakhtakia A 1994 *Int. J. Infrared Millim. Waves* **15** 1625
- [25] Lakhtakia A and Weiglhofer W S 1994 *IEEE Trans. Microwave Theory Tech.* **42** 1715
- [26] Raab R E and Sihvola A H 1997 *J. Phys. A: Math. Gen.* **30** 1335

- [27] Weiglhofer W S and Lakhtakia A 1998 *J. Phys. A: Math. Gen.* **31** 1113
- [28] de Sabbata V and Sivaram C 1994 *Spin and Torsion in Gravitation* (Singapore: World Scientific)
- [29] Sihvola A and Tretyakov S 2008 *Optik* **119** 247
- [30] Lindell I V, Sihvola A H, Viitanen A J and Tretyakov S A 1989 *Proc. 19th European Microwave Conf.* p 534
- [31] Kleinert H 2006 *Path Integrals in Quantum Mechanics, Statistics, Polymer Physics and Financial Markets* (Singapore: World Scientific)
- [32] Maxwell J C 1854 *Camb. Dublin Math. J.* **8** 188
- [33] Luneburg R K 1964 *Mathematical Theory of Optics* (Berkeley, CA: University of California Press)
- [34] Leonhardt U and Philbin T G 2010 *Phys. Rev. A* **81** 011804
- [35] Benítez P, Miñano J C and González J C 2010 *Opt. Express* **18** 7650
- [36] Miñano J C, Benítez P and González J C 2010 *New J. Phys.* **12** 123023
- [37] Guenneau S, Diatta A and McPhedran R C 2010 *J. Mod. Opt.* **57** 511
- [38] Guenneau S and Zolla F *Physica B* **394** 145
- [39] Cory H 1995 *J. Electromagn. Waves Appl.* **9** 805
- [40] Smolyaninov I I and Narimanov E E 2010 *Phys. Rev. Lett.* **105** 067402