Cyclotron maser radiation from inhomogeneous plasmas

R. A. Cairns, I. Vorgul, R. Bingham, K. Ronald, D. C. Speirs, S. L. McConville, K. M. Gillespie, R. Bryson, A. D. R. Phelps, B. J. Kellett, A. W. Cross, C. W. Roberston, C. G. Whyte, and W. He

Citation: Physics of Plasmas (1994-present) **18**, 022902 (2011); doi: 10.1063/1.3551697
View online: http://dx.doi.org/10.1063/1.3551697
View Table of Contents: http://scitation.aip.org/content/aip/journal/pop/18/2?ver=pdfcov
Published by the AIP Publishing

Cyclotron maser radiation from inhomogeneous plasmas

R. A. Cairns,[1] I. Vorgul,[1] R. Bingham,[2,3] K. Ronald,[3] D. C. Speirs,[3] S. L. McConville,[3] K. M. Gillespie,[3] R. Bryson,[3] A. D. R. Phelps,[3] B. J. Kellett,[2] A. W. Cross,[3] C. W. Roberston,[3] C. G. Whyte,[3] and W. He[3]
[1]*School of Mathematics and Statistics, University of St. Andrews, St. Andrews, Fife KY16 9SS, United Kingdom*
[2]*STFC Rutherford Appleton Laboratory, Chilton, Didcot, Oxon OX11 0QX, United Kingdom*
[3]*Department of Physics, University of Strathclyde, Glasgow G4 0NG, United Kingdom*

(Received 20 July 2010; accepted 6 January 2011; published online 16 February 2011)

Cyclotron maser instabilities are important in space, astrophysical, and laboratory plasmas. While extensive work has been done on these instabilities, most of it deals with homogeneous plasmas with uniform magnetic fields while in practice, of course, the systems are generally inhomogeneous. Here we expand on our previous work [R. A. Cairns, I. Vorgul, and R. Bingham, Phys. Rev. Lett. **101**, 215003 (2008)] in which we showed that localized regions of instability can exist in an inhomogeneous plasma and that the way in which waves propagate away from this region is not necessarily obvious from the homogeneous plasma dispersion relation. While we consider only a simple ring distribution in velocity space, because of its tractability, the ideas may point toward understanding the behavior in the presence of more realistic distributions. The main object of the present work is to move away from consideration of the local dispersion relation and show how global growing eigenmodes can be constructed. © *2011 American Institute of Physics*.
[doi:10.1063/1.3551697]

I. INTRODUCTION

Cyclotron maser instability[1] is important in the laboratory for the generation of high power radiation in devices like gyrotrons and is also thought to play a crucial role in the generation of auroral kilometric radiation (AKR) and similar electromagnetic wave emissions from other astrophysical objects including planets,[2–6] stars,[7–10] brown dwarfs,[11] and astrophysical shocks.[12] In previous work we have analyzed the instability produced by a horseshoe shaped distribution in electron velocity space and suggested that it is a likely source of AKR.[2,7,13] A laboratory experiment and simulations in which this type of distribution was produced showed that it did indeed generate waves with many of the properties of AKR.[14–19] In the simulations and in the laboratory experiment an electron beam with initial finite pitch spread $\alpha = v_\perp / v_z$ was injected into an increasing axial magnetic field. Through conservation of magnetic moment, a progressive magnetic mirroring of the beam resulted in the conversion of axial momentum into perpendicular momentum yielding a horseshoe shaped velocity distribution. Such a process has been observed[5] to occur in the AKR source region. The electrons are accelerated into the increasing magnetic field of the terrestrial auroral magnetosphere.[20] The efficiency of conversion of electron energy into rf energy in both the experiment and simulations is typically 1%, which compares well with the observed natural AKR generation efficiency. The experimental work and simulations have strong relevance to the Interrelationship between Plasma Experiments in Laboratory and Space (IPELS) community, with results having been presented at the associated biennial conference and published in an IPELS special issue of Plasma Physics and Controlled Fusion (PPCF).[14,18] Both theory and experiment indicate that the radiation is generated in the extraordinary (X) mode below the local cyclotron frequency. If we look at the cold plasma dispersion curves for the X mode perpendicular or near perpendicular to the magnetic field, then the branch which propagates below the cyclotron frequency does not connect to the branch which propagates in the vacuum. However, it is clear that the AKR does escape the region where it is generated since it is observed outside planetary magnetospheres. This is a problem which has attracted a number of suggested solutions over the years,[6] but no definitive answer. In a previous paper,[21] we considered a plasma with a particularly simple and analytically tractable distribution, a ring in velocity space, and showed that in a slab geometry in which this unstable distribution was confined to a local region, there are unstable eigenmodes and that radiation then propagates away in both directions from the slab, regardless of whether the field outside the slab is greater or less than that inside. We also gave some preliminary analysis of the behavior in a smoothly varying inhomogeneity. The purpose of the present paper is to expand on this work and show that with a suitable magnetic field configuration it is possible for the wave to grow in a localized region. The basic result underlying our work is that if a suitable complex frequency is chosen, with positive growth rate, then at a localized point in a magnetic field gradient forward and backward waves couple and there is partial reflection. If two such points exist in the magnetic field profile then a standing wave can be set up in the region between them. This allows the existence of temporally growing eigenmodes of the system.

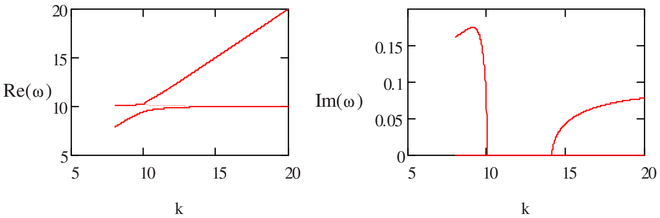

FIG. 1. (Color online) Real and imaginary parts of ω (scaled to ω_{pe}) as a function of k (scaled to ω_{pe}/c) for $\gamma=1.02$.

II. THEORY

We look at an electron distribution of the simple form $f(p_\parallel,p_\perp)=(n_0/2\pi p_\perp)\delta(p_\parallel)\delta(p_\perp-p_0)$ in momentum space. The dispersion relation for this can be found by substituting it into the standard expression for the dielectric tensor in a plasma, including relativistic effects which are important. For propagation perpendicular to a steady magnetic field we get the relation

$$k^2=\omega^2\left(\varepsilon_\parallel-\frac{\varepsilon_\perp^2}{\varepsilon_\parallel}\right), \qquad (1)$$

with

$$\varepsilon_\parallel=1-\frac{1}{2\gamma}\frac{\omega_{pe}^2}{\omega(\omega-\Omega)}+\frac{T}{4\gamma^2}\frac{\omega_{pe}^2}{(\omega-\Omega)^2},$$

$$\varepsilon_\perp=1-\varepsilon_\parallel,$$

$$\gamma=\left(1+\frac{p_0^2}{m_e^2c^2}\right)^{1/2},$$

$$T=\frac{p_0^2}{m_e^2c^2},$$

$$\Omega=\frac{\omega_{ce}}{\gamma},$$

with $\omega_{ce}=eB/m_e$, the electron cyclotron frequency. In deriving this it has been assumed that the wave frequency is close to ω_{ce} so that only one of the infinite series of terms in the dispersion relation is important and that the wavelength is much greater than the electron Larmor radius, so that in the Bessel functions which appear we can make the approximation $J_1(z)\approx z/2$. Typical dispersion curves arising from this relation are shown in Fig. 1 for $\Omega=10$ and $\gamma=1.02$. The unstable modes correspond to the almost horizontal branches of the real part. This is clearly a very simple distribution unlikely to occur in naturally occurring plasmas. Strangeway[22] carried out a detailed study of the effects of thermal spread and a background plasma on the ring distribution. However, our objective here is to show how to go beyond a consideration of the local dispersion relation and demonstrate the existence of growing eigenmodes in an inhomogeneous system. To develop the methodology for this it is convenient to start with this analytically tractable distribution.

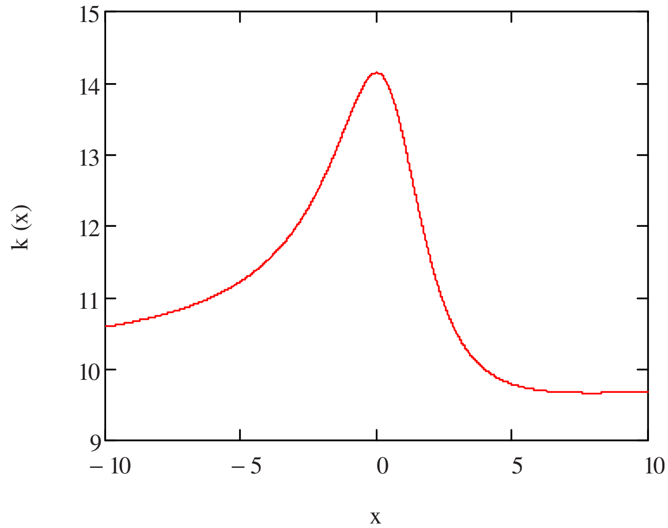

FIG. 2. (Color online) The wavenumber k with $\omega=10$ and $\Omega=10-0.05x$.

Now we consider a nonuniform plasma with the magnetic field as a function of x, the coordinate along k perpendicular to the magnetic field. From Eq. (1) it is clear that the local value of k^2 is uniquely defined at each value of x (which appears in the dispersion relation through an x dependence of Ω) and that if ω is real then so is k^2. This means that the wave either propagates without growth or damping or is evanescent if k^2 is negative. In the neighborhood of the cyclotron resonance, k^2 is positive and varies smoothly with x, as shown in Fig. 2, so that a wave with real ω propagates through this region and the instability does not go over into a spatial amplification.

If, however, we give ω an imaginary part comparable to the growth rate found in the homogeneous case, then we get the behavior shown in Fig. 3, where k comes close to passing through zero and varies rapidly in the region of cyclotron resonance. The result of this is that the left and right propagating waves no longer pass through the resonance region

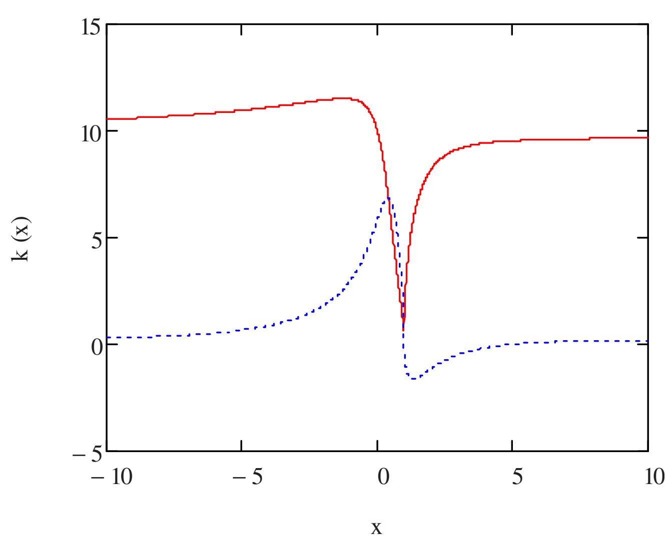

FIG. 3. (Color online) Real (full) and imaginary (dotted) parts of k with $\omega=10+0.13i$ and $\Omega=10-0.05x$.

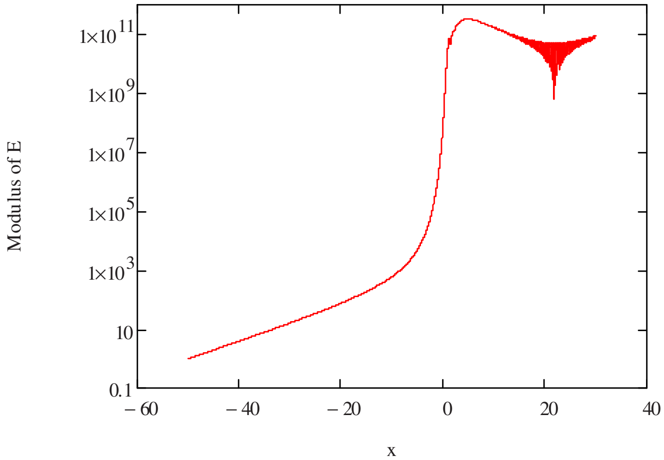

FIG. 4. (Color online) $|E|$ as a function of x from the differential equation.

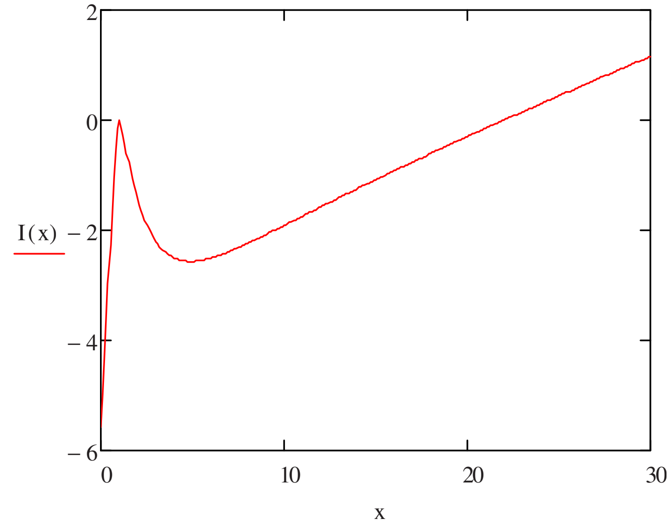

FIG. 5. (Color online) $I(x)$ as a function of x.

independently, but interact so that a purely outgoing wave on one side connects to a mixture of incoming and outgoing waves on the other side. Previously we suggested that it is possible, for a suitable choice of the growth rate, for a mode to grow locally around the cyclotron resonance and to produce outward propagating waves on both sides, as needed for a physically realistic solution. However, as we shall show, when there is just one position of cyclotron resonance there is always an incoming wave on one side, although its amplitude may be small until we get to some considerable distance from the resonance. What is needed for a solution which really does have only outgoing waves at infinity is a field configuration in which the wave passes through resonance twice.

We first elaborate on what happens with a simple linearly varying field with $\Omega=10-0.05x$, as in the plots above. If we choose the value of ω correctly then we get the sort of behavior shown in Fig. 3, but with the real and imaginary parts of k passing through zero together (which is not the case in Fig. 3). A value for which this happens, at least to a very good approximation, is $10+0.1298i$. In finding this the real part has been chosen to be 10, but with a different value of the real part the roots come together at a different point in the magnetic field profile, with a different imaginary part. Corresponding to the dispersion relation (1), we can construct a differential equation for the wave electric field which takes the form

$$\frac{d^2E}{dx^2}=-k^2(x)E, \tag{2}$$

and whose solution with the boundary condition of an outgoing wave on the left (i.e., the high field side) is as shown in Fig. 4.

Far from the resonance region around $x=0$ the solution of the dispersion relation is close to $k=\omega$ (in scaled units) and so there is straightforward exponential growth or decay with the wave amplitude decreasing along the direction of propagation. This is not damping, just a consequence of the fact that the wave source is growing in time. So, in Fig. 4 we see an outgoing wave on the left, while on the right the solution for small x is dominated by a wave propagating to the left, but eventually an incoming wave becomes evident and dominates for large enough x. Previously, we attributed this incoming wave to not having found exactly the correct value of ω or to numerical error, but now we will argue that it would be expected to appear.

If we look at the neighborhood of the zero of k^2 then locally we can consider its variation with x to be linear, although it does not take real values. Nevertheless, the local behavior is described by Airy's equation, just as for a cut-off in which k^2 just goes from positive to negative. Regardless of the complication of having a complex coefficient in Airy's equation, the way in which the solution connects through zero is the same, in that a solution going as either $\exp[\pm i\int k(x)dx]$ on one side connects to a superposition of equal amplitudes of the two waves on the other side. If we look at the behavior, which is close to that shown in Fig. 3, we see that far to the left k is approximately constant and, having set up an outgoing wave on this side we get the amplitude increasing exponentially with x. Near $x=0$ the imaginary part of k becomes large and there is a very rapid growth of the amplitude. On the other side of the resonance, the outgoing wave, which corresponds to the positive $\mathrm{Re}(k)$ root plotted in the diagram, initially has $\mathrm{Im}(k)$ negative and continues to grow until $\mathrm{Im}(k)$ changes sign and the amplitude falls. However, as we pass through $k=0$ we expect to pick up an equal amplitude incoming wave. Initially this has $\mathrm{Im}(k)$ large and positive and falls to a very small level until $\mathrm{Im}(k)$ changes sign and it starts to grow, eventually becoming comparable in amplitude to the outgoing wave and producing the oscillations around $x=20$ then becoming dominant for large x. To check this interpretation we have looked at a WKB approximation starting with the assumption of equal amplitudes at the $k=0$ point which is at $x_0=0.975$. In Fig. 5 we show

$$I(x)=\int_{x_0}^{x}\mathrm{Im}[k(y)]dy.$$

This takes the value zero for x a little above 20, so that at this point the incoming and outgoing waves have the same

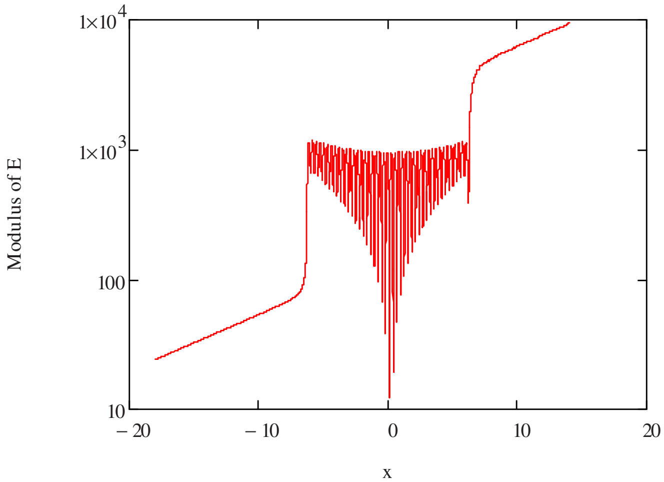

FIG. 6. (Color online) Numerical solution for $\omega=11+0.0995i$ with an incoming wave to the right and an outgoing wave to the left.

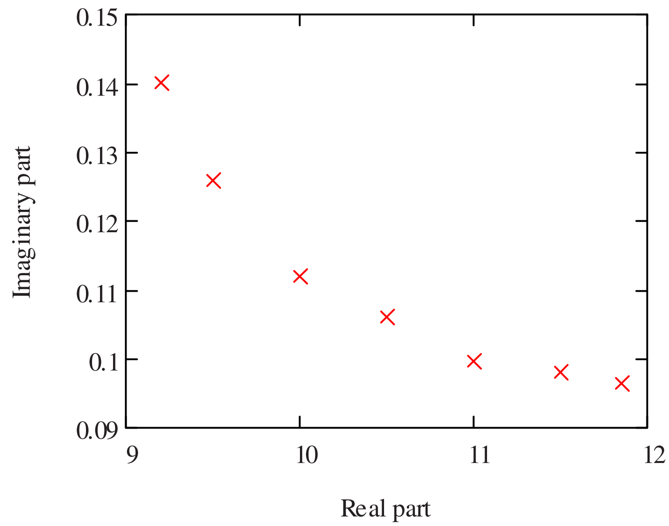

FIG. 7. (Color online) Real and imaginary parts of complex eigenvalues.

amplitude. It is clear that this agrees with the behavior found numerically. The connection of an outgoing wave on one side to incoming and outgoing waves of comparable amplitude on the other side of the resonance does not depend on the wavenumber passing through zero exactly. The sort of behavior shown in Fig. 3 where the real part comes close to zero or, sometimes, dips below zero is sufficient.

Since a solution with only outgoing waves at infinity is impossible with a simple linear field gradient, we now turn to the question of when it is possible. The answer, clearly, is to have a profile with two resonance positions such that there are left and right propagating waves between the two and they match to purely outgoing waves on the outside. This requires that not only the amplitude but the phase matches up correctly, giving rise to the sort of eigenvalue problem to be expected for a system of this sort. We will look at a situation where there are many wavelengths between the resonance points and the eigenvalues are consequently closely spaced. Rather than look at the detailed structure of these eigenvalues we will simply look for an approximate matching of the amplitude, a much simpler problem which should give the overall trend of the eigenvalues. Specifically we look at a profile with $\Omega(x)=9+0.5x^2$. With waves of frequency of around 10 that we have looked at, there will then be two resonances across which we can expect outgoing waves in the high field region outside to connect to waves propagating in both directions in the central region where the field is lower. In Fig. 6 we show a numerical solution with $\omega=11+0.0995i$. While this is evidently not an exact eigenvalue of the problem, it is clear that the amplitudes of the incoming and outgoing waves come very close to matching at the center so that we are not far away from an eigenvalue. The behavior of the solution is very sensitive to the choice of ω, so that this will give a good approximation to an eigenvalue.

In Fig. 7 we show the trend of the eigenvalues for a number of values estimated in this way. As pointed out before, the exact values would be expected to be a closely spaced series following this line. The growth rates found are, as might be expected, a little below the maximum value found for a homogeneous plasma. The lowest real frequency we have found is 9.2 and it does not seem possible to find solutions with a value much below this. Similar behavior is found for parallel propagation, although in this case the behavior across the resonance is the reverse of that found for perpendicular incidence. The result is that instability occurs in regions of high field with small amplitude waves propagating away into the surrounding low field region.

If we go back to our consideration of a linear magnetic field gradient we can note that there may be other possible ways for a localized instability to occur. From Fig. 6 it can be seen that for the parameters presented there the ratio of the outgoing wave amplitude to the incoming wave amplitude reaches a maximum value of around e^5, which corresponds to an intensity ratio in excess of 10^4. So, if there was some sort of discontinuity in the plasma parameters which produced a reflection of more than 10^{-4} of the incident energy flux, perhaps a steep density jump, then this would, again, open up the possibility of having a resonant cavity with right and left traveling waves between the cyclotron resonance and the reflection point connecting to outgoing waves on each side.

III. CONCLUSION

We have analyzed a simple cyclotron maser instability in an inhomogeneous plasma and have shown that it has properties which are not obvious from the dispersion relation for a homogeneous system. Convective growth of a wave with real frequency passing through a cyclotron resonance does not seem to occur. On the other hand, if there is a suitable magnetic field profile a temporally growing wave can be reflected between two regions of cyclotron resonance, with some of the energy leaking away into the surrounding plasma. The possibility of obtaining instability with only one resonance region and some sort of reflecting layer in the plasma is also pointed out. The ring distribution has been chosen because of its analytically tractable nature, the main point of the present analysis being to show how to go beyond a consideration of the local dispersion relation and show that global growing eigenmodes exist in a suitable inhomogeneous system. In the future we hope to attempt a similar analysis with more realistic distributions of more direct relevance to problems in space and astrophysics.

ACKNOWLEDGMENTS

R.B. would like to thank the STFC Centre for Fundamental Physics for support. This work was supported by EPSRC Grant No. EP/G04239X.

[1]K. R. Chu, Rev. Mod. Phys. **76**, 489 (2004).
[2]R. Bingham and R. A. Cairns, Phys. Plasmas **7**, 3089 (2000).
[3]R. Bingham, B. J. Kellett, R. A. Cairns, I. Vorgul, A. D. R. Phelps, K. Ronald, and D. C. Speirs, Contrib. Plasma Phys. **44**, 382 (2004).
[4]P. L. Pritchett, R. J. Strangeway, R. E. Ergun, and C. W. Carlson, J. Geophys. Res., [Space Phys.] **107**, 1437, doi:10.1029/2002JA009403 (2002).
[5]G. T. Delory, R. E. Ergun, C. W. Carlson, L. Muschietti, C. C. Chaston, W. Peria, J. P. McFadden, and R. Strangeway, Geophys. Res. Lett. **25**, 2069, doi:10.1029/98GL00705 (1998).
[6]R. E. Ergun, C. W. Carlson, J. P. McFadden, G. T. Delory, R. J. Strangeway, and P. L. Pritchett, Astrophys. J. **538**, 456 (2000).
[7]R. Bingham, R. A. Cairns, and B. J. Kellett, Astron. Astrophys. **370**, 1000 (2001).
[8]C. Trigilio, P. Leto, G. Umana, C. Buemi, and F. Leone, Mon. Not. R. Astron. Soc. **384**, 1437 (2008).
[9]B. J. Kellet, V. Graffignino, R. Bingham, T. W. B. Muxlow, and A. G. Gunn, "CU Virginis–The First Stellar Pulsar," arXiv:astro-ph/0701214.
[10]D. B. Melrose and G. A. Dulk, Astrophys. J. **259**, 844 (1982).
[11]G. Hallinan, S. Bourke, C. Lane, A. Antonova, R. T. Zavala, W. F. Brisken, R. P. Boyle, F. J. Vrba, J. G. Doyle, and A. Golden, Astrophys. J. **663**, L25 (2007).
[12]R. Bingham, B. J. Kellett, R. A. Cairns, J. Tonge, and J. T. Mendonca, Astrophys. J. **595**, 279 (2003).
[13]I. Vorgul, R. A. Cairns, and R. Bingham, Phys. Plasmas **12**, 122903 (2005).
[14]D. C. Speirs, S. L. McConville, K. M. Gillespie, K. Ronald, A. D. R. Phelps, A. W. Cross, R. Bingham, C. W. Robertson, C. G. Whyte, I. Vorgul, R. A. Cairns, and B. J. Kellett, Plasma Phys. Controlled Fusion **50**, 074011 (2008).
[15]K. Ronald, D. C. Speirs, S. L. McConville, A. D. R. Phelps, C. W. Robertson, C. G. Whyte, W. He, K. M. Gillespie, A. W. Cross, and R. Bingham, Phys. Plasmas **15**, 056503 (2008).
[16]K. M. Gillespie, D. C. Speirs, K. Ronald, S. L. McConville, A. D. R. Phelps, R. Bingham, A. W. Cross, C. W. Robertson, C. G. Whyte, W. He, I. Vorgul, R. A. Cairns, and B. J. Kellett, Plasma Phys. Controlled Fusion **50**, 124038 (2008).
[17]D. C. Speirs, K. Ronald, S. L. McConville, K. M. Gillespie, A. D. R. Phelps, A. W. Cross, R. Bingham, C. W. Robertson, C. G. Whyte, W. He, I. Vorgul, R. A. Cairns, and B. J. Kellett, Phys. Plasmas **17**, 056501 (2010).
[18]S. L. McConville, D. C. Speirs, K. Ronald, A. D. R. Phelps, A. W. Cross, R. Bingham, C. W. Robertson, C. G. Whyte, W. He, K. M. Gillespie, I. Vorgul, R. A. Cairns, and B. J. Kellett, Plasma Phys. Controlled Fusion **50**, 074010 (2008).
[19]K. Ronald, S. L. McConville, D. C. Speirs, A. D. R. Phelps, C. W. Robertson, C. G. Whyte, K. M. Gillespie, A. W. Cross, and R. Bingham, Plasma Sources Sci. Technol. **17**, 035011 (2008).
[20]R. E. Ergun, L. Andersson, D. Main, Y. J. Su, D. L. Newman, M. V. Goldman, C. W. Carlson, A. J. Hull, J. P. McFadden, and F. S. Mozer, J. Geophys. Res., [Space Phys.] **109**, A12220, doi:10.1029/2004JA010545 (2004).
[21]R. A. Cairns, I. Vorgul, and R. Bingham, Phys. Rev. Lett. **101**, 215003 (2008).
[22]R. J. Strangeway, J. Geophys. Res., [Space Phys.] **91**, 3152, doi:10.1029/JA091iA03p03152 (1986).