University of
St Andrews

School of Economics & Finance Discussion Papers

Comparative Statics of Asset Prices: the effect of other assets' risk

Theodoros M. Diasakos

Comparative Statics of Asset Prices
The Effects of Other Assets' Risk

Theodoros M. Diasakos[*]

August 6, 2013

Abstract

Currently, financial economics is unable to predict changes in asset prices with respect to changes in the underlying risk factors, even when an asset's dividend is independent of a given factor. This paper takes steps towards addressing this issue by highlighting a crucial component of wealth effects on asset prices hitherto ignored by the literature. Changes in wealth do not only alter an agent's risk aversion, but also her perceived "riskiness" of a security. The latter enhances significantly the extent to which market-clearing leads to endogenously-generated correlation across asset prices, over and above that induced by correlation between payoffs, giving the appearance of "contagion."

Keywords: General Equilibrium Asset-Pricing, Geometric Brownian Motion, Contagion.
JEL Classification Numbers: G12

1   Introduction

Whether or not and exactly how the prices of different assets are related to one another is of fundamental importance in financial economics. During the 2008 crisis, for instance, one of the most pressing questions was whether emerging markets' securities would be able to shield investors from the losses of western banking stocks, associated with the US subprime debacle. Alas, at the moment, financial economic theory cannot really answer this at any significant extent of generality. For it has not studied yet analytically and in a general model how the equilibrium price of a given asset depends upon the individual risk factors that contribute to the dividend of another asset. In fact, it has not done so even for the limiting case in which the two dividend processes are independent. Of course, there is an extensive literature that attempts to study such relations using specific models, simulations, or empirical methods, but it lacks the rigorousness of general theoretical analysis.

The present paper makes steps towards filling this gap. Using the theoretical back-drop of representative agent general equilibrium asset-pricing, it conducts a comparative statics analysis of the price of the typical security with respect to the current realization of the typical underlying risk factor. As it turns out, market-clearing generically leads to endogenously-generated correlations across asset prices, over and above those induced by correlations between the respective dividends. Given this, an obvious aim of study is to determine the direction of asset-price movements, especially when the security's dividend is independent of the given risk factor.

We refer to this situation as "contagion" and show that it may or may not be detrimental for a risk-averse investor.[1] We identify fairly general settings of economic primitives under which asset prices in emerging markets would fall in the aftermath of the 2008 crisis, even though their dividends do not depend at all upon shocks in the US housing sector (see Propositions 2, 6, and 10). We also find settings, however, under which these asset prices would increase, so that contagion would provide a means for diversification (see Propositions 3, 7, and 11 but also Propositions 2, 6, and 10 under IARA).

As is well-known, in a representative-agent financial equilibrium, the price of the typical security is the current expectation of its future dividends valued at the agent's marginal rate of substitution between consumption at the dividend-collection date and the present. This asset-pricing framework has been used extensively in the literature but the dynamics

---

[1]The literature on contagion has focused mostly on the propagation of shocks across national or regional stock markets. One of its peculiarities is that, although there is fairly widespread agreement about the contagion events themselves, there is no consensus on exactly what constitutes contagion or how it should be defined. One preferred definition is the propagation of shocks in excess of that which can be explained by fundamentals. Another (often referred to as "shift-contagion") looks for changes in how shocks are propagated between normal and crisis periods. Yet another labels contagion the transmission of shocks through specific channels, such as herding or irrational investor behavior. And an even broader definition identifies contagion as any linkage mechanism that causes markets or asset prices to move together. The main reason for this prolificness is that each definition seems to run in its own difficulties when it comes to empirical identification. My focus being strictly theoretical, I will be referring to contagion having in mind the first definition.

of the equilibrium pricing process with respect to the underlying risk process have not been thus far investigated - not analytically and, hence, not to a satisfactory degree of generality with respect to the economic primitives. In the sequel, we identify three different effects the current realization of a given risk factor may have upon the equilibrium price of a given security.

The first two effects are well-known. Other things being equal, an increase in the current realization of a risk factor raises the expected dividend if the latter is positively correlated with the former. This is an improvement in first-order stochastic dominance terms which, due to her non-satiation, renders the representative agent more willing to hold the corresponding security. Given fixed net supply, this exerts upwards pressure on the security's equilibrium price, giving rise to the *dividend effect*. Yet, the increase in the expected dividend raises also the agent's expected wealth. Due to her risk-aversion, this reduces the expected marginal utility of wealth, exerting downward pressure on the security's price. This is the *risk-aversion effect*, responsible for the fact that the price of a security need not increase when its dividend increases.

However, changes in the realization of the risk factor are related to the security's price also via the correlation between the dividend and the marginal utility of wealth. This is the *asset-riskiness effect* and may well be the main driver of contagion in asset prices. As shown in Section 3, there are settings in which, following an increase in the agent's wealth, the corresponding decrease in its marginal utility is small when the dividend realization is large and vice versa. That is, the dividend becomes less positively correlated with wealth and the agent perceives the security as less "risky." As a result, she demands more of it and, in the face of fixed net supply, pushes up the security's equilibrium price.

Of course, marginal utilities are not observable in practice and securities are priced with respect to a numeraire so that what really matters is the relative price between two securities. To study analytically the comparative statics of the typical relative price, we restrict attention to a pure exchange economy in which the the typical dividend is proportional to a geometric Brownian motion. This specification has been used widely in theoretical as well as empirical studies because it allows the equilibrium asset prices to be identified either in closed form or as solutions to well-known stochastic differential equations.

Yet, describing their evolution with respect to the typical underlying risk factor, the typical standard Brownian motion, is not straightforward. In fact, despite the absence of jumps, extreme events or other irregularities in the underlying risk process, the relative price dynamics are rich and complex. The present analysis attests to the richness by establishing that, as a norm, the typical relative price will be correlated with the typical Brownian factor. When the relative dividend is correlated with the Brownian factor, the relative price varies always monotonically (see Propositions 1, 5, and 9). However, the relative price will be correlated with the factor even when the relative dividend is independent of that factor. Section 4 attests to the complexity of the underlying dynamics by indicating how difficult

it is to pin down settings in which the sign of this correlation is identifiable.

More precisely, we identify conditions on the factor loadings of the geometric Brownian motions and on the agent's endowment process so that a given relative price will vary monotonically with the realization of a given Brownian motion. This allows us to sign each and every entry in the dispersion matrix of relative prices, even when none of the two dividends is correlated with the Brownian motion. When the representative agent exhibits decreasing absolute risk aversion (DARA), the monotone relation is positive (see Propositions 2, 6, and 10). And it obtains even under the setting in which all dividends in the model are independent of one another while the agent's endowment is deterministic - admittedly, the most inhospitable economic environment for cross-correlations in asset prices.

By contrast, under increasing (IARA) or constant (CARA) absolute risk aversion, the monotone relation is negative (see Propositions 3, 7, and 11). This contradicts the rather widely-held view that, under CARA, changes in the agent's wealth should not matter for the relative prices. For it establishes that the typical relative price will not respond to a Brownian realization which does not affect either dividend, irrespective of the specifications for the dividend and endowment processes, if and only if the Brownian motion in question and those correlated with the two dividends affect the agent's wealth through independent channels (see Propositions 4, 8, and 12).

Needless to say, the possibility for correlation in asset prices when there is no common factor in cash flows is well-known. But it has not been demonstrated before analytically in a general equilibrium model.[2] And this is important because the very presence of contagion renders general equilibrium a sine qva non approach towards understanding asset-price dynamics. To this end, restricting attention to the representative agent and the monotonicity of her risk aversion coefficient (rather than a particular utility specification) allows for limited loss of generality.[3] More importantly perhaps, the investigation being analytical not only facilitates economic intuition but also highlights that the direction of contagion depends fundamentally on the assumed specification for the dividend and endowment processes as

---

[2]Contagion is noted, for example, in Raimondo [62] as well as Anderson and Raimondo [6] but no formula is given for the cross-derivative. Kodres and Pritsker [43], Kyle and Xiong [44], but also Lagunoff and Schreft [45] show that contagion can obtain as a wealth effect in rational expectations equilibria. These are not general equilibrium models, however, as some market participants are not rational (the former two models require the presence of noise traders, the latter of irrational ones). Contagion equilibria arise as well in Aliprantis et al. [1] within the context of a monetary model where players engage in strategic, non price-taking behavior.

[3]In fact, our analysis carries through for a general pricing kernel as long as this can be a function of the economy's total wealth. The latter restriction imposes no loss of generality when the financial market under study is dynamically complete (which, for the informational and trading structures under study here, has been characterized by Diasakos [23]). In this case, the coefficient of risk tolerance of the representative agent is the sum of the individual ones (see, for instance, Theorem 4 in Wilson [68]). For the representative agent to exhibit DARA (IARA), therefore, it suffices that the coefficient of absolute risk aversion of each individual is non-increasing (non-decreasing) with at least one individual exhibiting DARA (IARA). Similarly, the representative agent will exhibit CARA if every individual in the economy does so. By contrast, she will not exhibit for example CRRA unless all individuals are identical regarding preferences as well as endowments.

well as on the representative agent's attitudes towards risk.

On the empirical side, the literature finds contagion to be pervasive.[4] There is also ample evidence that conditional correlations across asset prices and returns are stochastic, and of a magnitude that cannot be explained by covariances between their respective payoffs alone.[5] With respect to such findings, the present analysis provides theoretical justification for excess asset-price comovements within the context of general equilibrium asset-pricing. In fact, by assuming constant covariances between asset payoffs, it indicates that such comovements ought to be generic phenomena, even when due to market-clearing alone.

Regarding the asset-price dynamics per se, the studies that are closest to the present are Cochrane et al. [18] and Martin [49]. Both papers investigate a pure-exchange, infinite-horizon, representative-agent economy. In Cochrane et al. [18], the agent has log-utility for instantaneous consumption and access to the dividend stream of at most two Lucas trees, each following a geometric Brownian motion. The paper provides closed-form solutions for a large collection of variables of interest (asset-prices, expected returns, market-betas, and return correlations), but they are given with respect to the dividend-share (the share of total output due to a tree's dividend) rather than the underlying Brownian process, while the corresponding dynamics are examined numerically rather than analytically. In fact, the solution method depends fundamentally upon the dividend-share being the unique state variable, in a way that makes it applicable only to log-utility and at most two trees. By contrast, Martin's [49] approach extends to power utility and many trees, whose dividend streams may follow geometric Brownian motions with normally-distributed jumps. This paper offers also closed-form solutions for absolute prices, expected returns, and bond-yields, but again in terms of a state-vector which is not the underlying stochastic process (it depicts instead the relative sizes of the dividends), while the corresponding dynamics are investigated again through calibrations.

Both papers draw attention to the fact that there is significant price comovement even between assets whose dividends are independent. In the case of two trees, when one has a positive dividend shock, its dividend becomes a larger share of a now larger total consumption. As a result, investors want to rebalance by spreading some of their larger wealth across both trees. Yet, in the face of fixed net supply, they cannot collectively rebalance, so asset prices must adjust. Typically, the price of the tree with the positive shock rises whereas the risk premium of the other falls. And the latter can happen only via an increase in the price of the second asset if the two dividend streams are independent. Indeed, being

---

[4]See, for instance, Shiller [66] or Forbes and Rigobon [30] regarding contagion across national or regional stock markets. Gropp and Moerman [35] identify within-country contagion among large European bank stocks while Pindyck and Rotemberg [61] find evidence of excess correlation in asset price comovements.

[5]Moskowitz [54] finds evidence that risk-premia are better represented by covariances with the implied market portfolio than by own-variances. Andersen and Lund [4], on the other hand, suggest that U.S. risk-free short-term interest rates can be consistently estimated as stochastic-volatility diffusions. On stochastic second moments of returns, see also Andersen et al. [3]-[2], Longin and Solnik [46] or Schwert and Seguin [64].

now a smaller part of total consumption means that the second asset becomes less positively correlated with consumption, inducing investors to demand more of it and, thus, force its price to rise.

In both papers, however, this is what happens typically, not always, which is most evident when the second asset is a zero-coupon bond.[6] Given its smaller dividend-share, it is still true that investors want to spread their larger wealth across both trees, which should raise the price of the bond. Alas, the interest rate also changes and this may more than offset the rebalancing desire (see Figure 9 in Cochrane et al. [18]). Yet, as shown by the present analytical approach, this ambiguity in the characterization of asset-price dynamics can be eliminated. Both papers attempt to relate a given asset price to the current realization of a given dimension of the underlying risk process via a state-variable, be it the dividend-share or the relative size of the dividends, whose own change cannot be isolated to come from that dimension alone. By contrast, we look at asset-price dynamics with respect to the underlying risk process directly. As it turns out, there are settings of economic primitives under which the results are not ambiguous at all. In fact, for the two-assets example above, this is true not only when the representative agent exhibits constant relative risk aversion (CRRA) but more generally DARA (see Proposition 6).

The remainder of the paper evolves as follows. The next section describes the model under study and its relation to the pertinent literature. Section 3 investigates the comparative statics of equilibrium relative prices under the benchmark scenario for illustrative purposes: when the time horizon is finite and the securities pay only lump-sum dividends on the terminal date. Section 4 identifies specific settings of economic primitives under which contagion dynamics can be unambiguously foretold. Section 5 extends the analysis to the case in which the securities pay also dividend flows during the trading interval while the horizon may be infinite. Section 6 concludes while the Appendix contains the proofs and supporting technical material.

2   Setup and Related Literature

In a financial market where trading occurs over a time-interval $\mathcal{T} \subseteq \mathbb{R}_+$ and the informational structure is given by a standard Brownian motion, well-known no-arbitrage conditions ensure that the securities' prices are the current expectations of their future dividends valued at some pricing kernel, a strictly-positive one-dimensional Ito process. In what follows, the underlying standard Brownian process will be K-dimensional ($K \in \mathbb{N}^*$), defined on a complete probability space $(\Omega, \mathcal{F}, \mu)$, and depicted as $\beta : \Omega \times \mathcal{T} \mapsto \mathbb{R}^K$ or $\beta_k : \Omega \times \mathcal{T} \mapsto \mathbb{R}$ with $k \in \mathcal{K} = \{1, \ldots, K\}$ for the typical dimension. As usual, this is meant to fully de-

---

[6]Indeed, the relation between an asset's risk-premium and the dividend-share does not depend only on the "cash-flow beta" intuition given above. It depends also on "valuation-beta," the tendency of the price-dividend ratio to change with the market and, thus, total consumption. And this relation is not always positive. There are ranges of dividend-share values where the price of the second asset falls in the given example (see Figure 3 in Cochrane et al. [18] and Figure 7(a) in Martin [49]).

scribe the exogenous financial risk in the sense that the collection of the sample paths $\{\beta(\omega, t) : t \in [0, T]\}_{\omega \in \Omega}$ specifies all the distinguishable events.

The trading structure will consist of $N + 1$ securities ($N \in \mathbb{N}$), indexed by $n \in \mathcal{N} = \{0, \ldots, N\}$, which are traded continuously over $\mathcal{T}$. The public informational endowment will be taken to be the filtration $\{\mathcal{F}_t : t \in \mathcal{T}\}$ that is generated by β. Hence, the dividend process of each security will be a function of the process $\mathcal{I} = \{t, \beta(\omega, t)\}_{(\omega, t) \in \Omega \times \mathcal{T}}$, which is adapted to the given filtration. It will be instructive, moreover, to distinguish between two different forms the dividend process may take. Along the Brownian path $\{\beta(\omega, t)\}_{t \in \mathcal{T}}$, the typical security may be paying the dividend flow $g_n(\mathcal{I}(\omega, \cdot))$ while, if the time-horizon is finite ($\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}_{++}$), also the lump sum $G_n(\mathcal{I}(\omega, T))$ on the terminal date. Then, the current price of the typical security is given by

$$
\begin{aligned}
P_n(\mathcal{I}(\omega, t)) & = \mathbb{E}_\mu \left[\frac{u'(W(\mathcal{I}(\omega, T))) G_j(\mathcal{I}(\omega, T))}{\widetilde{u}'\left(\widetilde{W}(\mathcal{I}(\omega, t))\right)} | \mathcal{F}_t \right] \\
& + \mathbb{E}_\mu \left[\int_t^T \frac{\widetilde{u}'\left(\widetilde{W}(\mathcal{I}(\omega, s))\right) g_j(\mathcal{I}(\omega, s))}{\widetilde{u}'\left(\widetilde{W}(\mathcal{I}(\omega, t))\right)} \mathrm{d}s | \mathcal{F}_t \right]
\end{aligned}
\tag{1}
$$

taking the representative agent as having von-Neumann Morgenstern utility functions $\widetilde{u}, u : \mathbb{R}_{++} \mapsto \mathbb{R}$, which are both twice continuously-differentiable, strictly-increasing, and concave everywhere in their respective instantaneous wealth domains:

$$
\begin{aligned}
\widetilde{W}(\mathcal{I}(\omega, t)) & = \widetilde{\rho}(\mathcal{I}(\omega, t)) + \sum_{n=0}^N g_n(\mathcal{I}(\omega, t)) \\
W(\mathcal{I}(\omega, T)) & = \rho(\mathcal{I}(\omega, T)) + \sum_{n=0}^N G_n(\mathcal{I}(\omega, T))
\end{aligned}
$$

with $\widetilde{\rho} : \Omega \times \mathcal{T} \mapsto \mathbb{R}_+$ and $\rho : \Omega \times \{T\} \mapsto \mathbb{R}_+$ being continuous endowment functions and under the proviso that, in the infinite-horizon case ($\mathcal{T} = \mathbb{R}_{++}$), only the second term on the right-hand side above applies (with $T = \infty$).

The essential premise that lies underneath the asset-pricing equation above (and, thus, also behind the analysis that follows) is that utilities, dividends, endowments, and wealth are allowed to be time- as well as state-dependent, as long as this obtains through the realizations of the process $\mathcal{I}(\cdot)$. As an approach towards equilibrium asset-pricing theory, this has been the building block for much of the seminal literature. The starting point has been to assume that agents have identical preferences. This has been the launching pad of two related strands of the literature. The first restricts attention to what is essentially the continuous-time analogue of the static (one-period) model: the setting in which the time-horizon is finite and securities pay only lump-sum dividends on the terminal date. The resulting asset-pricing process is given by the first term on the right-hand side of (1) - as in,

for example, Bick [9]-[10], He and Leland [37], Raimondo [62], or Anderson and Raimondo [6].

The second approach has been to allow for securities that pay also dividend flows during the time interval while the time-horizon may be infinite. Perhaps the most well-known paper in this strand is Cox et al. [20], the continuous-time analogue of the famous model in Lucas [47], enhanced to include production.[7] In Cox et al. [20], the asset-pricing formula takes the same form as (1), which can be found also in Cochrane et al. [18], Martin [49], Merton [51]-[52], or Wang [67] (whose pricing formula derives actually from Example 3 in Duffie and Skiadas [24]).

Even when the economy consists of agents with heterogenous preferences, the pricing kernel remains a linear function of the equilibrium marginal utilities (the Negishi weights are constant) if the equilibrium allocation is Pareto-optimal (i.e., the market is effectively complete). And again, also in this case, the pricing formula retains the same basic form as in (1) - see, for instance, Anderson and Raimondo [5], Hugonnier et al. [39], Basak and Cuoco [8], Duffie and Zame [25] (see Theorem 1 and the subsequent discussion in Section 5), Dumas [22], Karatzas et al. [42] (see Corollary 10.4), or Riedel [63] (see Theorem 2.1). Clearly, in the context of financial equilibrium, the pricing process under consideration here is general, at least as long as the financial market is effectively complete.[8]

Of course, in an equilibrium model, one must choose also a numeraire. Yet, since the underlying informational structure is a filtration, the choice of numeraire here is essentially arbitrary because the equilibrium market-clearing condition will depend only on the relative prices of the traded entities, and will do so node (ω, t) by node (ω, s), for $s \neq t$. As a consequence, it is without loss of generality to normalize such that the price of one of the traded entities (typically, one of the commodities) is 1 at all $(\omega, t) \in \Omega \times \mathcal{T}$.

It is also typical in continuous-time models to assume that one of the traded securities (say the zeroth one) is a money-market account, an instantaneously risk-free asset. Alas, as emphasized in Anderson and Raimondo [5], this is an endogenous assumption because it restricts directly the market value of this security. Instead, to render the zeroth security

---

[7]Allowing for time- but not state-dependent preferences, the representative agent of Cox et al. [20] seeks to maximize the current expectation of the future utility flow, $\mathbb{E}_\mu \left[\int_t^T \widetilde{u} \left(W \left(\mathcal{I} \left(\omega, s \right) \right), s \right) \mathrm{d}s | \mathcal{F}_t \right]$ and the equilibrium relative price of any real asset will be given by the second term in (1). I am referring to the last term of equation (38) in Cox et al. [20]. This term prices real assets, claims that pay $\delta \left(W \left(s \right), Y \left(s \right), s \right)$ units of consumption at time s when the realization of the stochastic process is $Y \left(s \right)$. By contrast, the first two terms allow for the pricing of general financial assets, including options and futures. These are claims that pay $\Theta \left(W \left(\tau \right), Y \left(\tau \right) \right)$ if some underlying variables do not leave a certain region before the maturity date τ and $\Psi \left(W \left(s \right), Y \left(s \right), s \right)$ every time s they do, otherwise. Notice that $J \left(W \left(s \right), Y \left(s \right), s \right)$ is the agent's equilibrium indirect utility at time s, given the realization $Y \left(s \right)$. It depends on the date s and the state variable Y as the authors allow for the direct utility to be time- and state-dependent. As I establish in the sequel, all of my results remain valid in the face of the former dependence. The latter is beyond my current scope.

[8]In fact, regarding the dividend specification the present paper will examine, [23] shows that, as long as the dispersion matrix of the securities' dividends is non-degenerate, the market will meet the stronger condition of dynamic completeness.

instantaneously risk-free, one can simply divide all prices in the model by that of the zeroth security. When one does so, it is now the price of the latter security that is 1 at all $(\omega, t) \in \Omega \times \mathcal{T}$ and, most importantly, this is without any loss of generality (see Anderson and Raimondo [5] for a more detailed discussion).

What matter then are the relative prices of the remaining securities, the typical one being

$$p_n\left(\mathcal{I}\left(\omega, t\right)\right) = \frac{P_n\left(\mathcal{I}\left(\omega, t\right)\right)}{P_0\left(\mathcal{I}\left(\omega, t\right)\right)} \qquad n \in \mathcal{N} \setminus \{0\}$$

Hence, in what follows, our focus will be on the derivative of the typical relative price with respect to changes in the current realization of the typical Brownian component:

$$\frac{\partial p_n\left(\mathcal{I}\left(\omega, t\right)\right)}{\partial \beta_k\left(\omega, t\right)} = \frac{\frac{\partial P_n\left(\mathcal{I}\left(\omega, t\right)\right)}{\partial \beta_k\left(\omega, t\right)} - p_n\left(\mathcal{I}\left(\omega, t\right)\right)\frac{\partial P_0\left(\mathcal{I}\left(\omega, t\right)\right)}{\partial \beta_k\left(\omega, t\right)}}{P_0\left(\mathcal{I}\left(\omega, t\right)\right)} \quad (n, k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K} \tag{2}$$

Needless to say, these relative price dynamics cannot be readily identified from the information process available in the economy. To be able to examine them analytically, I will restrict attention to the dividend processes[9]

$$g_n\left(\mathcal{I}\left(\omega, t\right)\right) = a_n\left(t\right) e^{\widetilde{\sigma}_n^\intercal \beta\left(\omega, t\right)} \tag{3}$$

$$G_n\left(\mathcal{I}\left(\omega, T\right)\right) = A_n\left(T\right) e^{\sigma_n^\intercal \beta\left(\omega, T\right)} \tag{4}$$

where $a_n, A_n : \mathcal{T} \mapsto \mathbb{R}_{++}$ are deterministic supply functions while the instantaneous dispersion vectors $\sigma_n, \widetilde{\sigma}_n \in \mathbb{R}^K$ are constant.

Here, the typical terminal dividend is proportional to a K-dimensional geometric Brownian motion as long as $\sigma_n \neq \mathbf{0}$ and $A_n\left(t\right) = \xi\left(t\right) e^{\mu_n t}$ for some $\mu_n \in \mathbb{R}^*$ and some deterministic supply function $\xi : [0, T] \mapsto \mathbb{R}_{++}$ (with $\int_0^T \xi\left(s\right) \mathrm{d}s < \infty$). By contrast, $\sigma_n = \mathbf{0}$ renders the dividend riskless. It may be now a money-market account if $A_n\left(t\right) = e^{\int_0^t \mu_n(s)\mathrm{d}s}$ for some deterministic function $\mu_n : [0, T] \mapsto \mathbb{R}_{++}$ (with $\int_0^T \mu_n\left(s\right)\mathrm{d}s < \infty$) or a zero-coupon bond if $A_n\left(t\right) = 1$ on $[0, T]$. Similarly, the typical intermediate dividend is proportional to a K-dimensional geometric Brownian motion if $\widetilde{\sigma}_n \neq \mathbf{0}$ and $a_n\left(t\right) = \widetilde{\xi}\left(t\right) e^{\widetilde{\mu}_n t}$ for some $\widetilde{\mu}_n \in \mathbb{R}^*$ and some function $\widetilde{\xi} : \mathcal{T} \mapsto \mathbb{R}_{++}$ (with $\int_{\mathcal{T}} \widetilde{\xi}\left(s\right)\mathrm{d}s < \infty$). When $\sigma_n = \mathbf{0}$, on the other hand,

---

[9]Let $\mathrm{d}Y = \mu\mathrm{d}t + \Sigma\mathrm{d}\beta$ be an N-dimensional Ito process and $V \subseteq \mathbb{R}^N$ an open set such that $Y\left(\omega, t\right) \in V \; \forall \left(\omega, t\right) \in \Omega \times [0, T]$ almost surely. Consider now a twice-differentiable function $f : V \mapsto \mathbb{R}$ (such as any price in the model). By Ito's lemma, and not displaying the dependence upon (ω, t), $\mathrm{d}f\left(Y\right) = \left[f_Y\left(Y\right)\mu + \frac{1}{2}\mathrm{tr}\left(\Sigma^\intercal f_{YY}\left(Y\right)\Sigma\right)\right]\mathrm{d}t + f_Y\left(Y\right)\Sigma\mathrm{d}\beta$ where $f_Y = \left(\frac{\partial f}{\partial Y_1}, \ldots, \frac{\partial f}{\partial Y_N}\right)$ and $f_{YY} = \left(\frac{\partial^2 f}{\partial Y_i \partial Y_j}\right)_{i,j=1}^N$ denote the gradient vector (in row form) and the Hessian matrix of f, respectively. If one fixes time, the sensitivity of f with respect to changes in the realization of the underlying risk factors is given by $\mathrm{d}f\left(Y\right) = \sum_{n=1}^N \sum_{k=1}^K \frac{\partial f(Y)}{\partial Y_n}\sigma_{nk}\mathrm{d}\beta_k$. In particular, restricting attention to changes in the kth risk source only, $\frac{\partial f(Y)}{\partial \beta_k} = \sigma_k^\intercal f_Y\left(Y\right)$ where σ_k is the kth column of Σ. In our case, we have $\mathrm{d}\ln G_n = \left(\mu_n - \frac{\sigma_n^\intercal \sigma_n}{2}\right)\mathrm{d}t + \sigma_n^\intercal\mathrm{d}\beta$ so that the sensitivity of p_n is given by $\mathrm{d}p_n\left(X\right) = \sum_{n=1}^N \sum_{k=1}^K \frac{\partial p_n(X)}{\partial X_n}\sigma_{nk}\mathrm{d}\beta_k$. That is, $\frac{\partial p_n(X)}{\partial \beta_k} = \sigma_k^\intercal p_{n_X}\left(X\right)$ where σ_k is the kth column of the $N \times K$ matrix whose typical row is $\sigma_n^\intercal$.

the dividend may be a money-market account, if $a_n(t) = e^{\int_0^t \widetilde{\mu}_n(s)\mathrm{d}s}$ for some deterministic function $\widetilde{\mu}_n : \mathcal{T} \mapsto \mathbb{R}_{++}$ (with $\int_{\mathcal{T}} \widetilde{\mu}_n(s)\,\mathrm{d}s < \infty$) or an annuity if $a_n(t) = 1$ on $\mathcal{T}$.

As dividend specification, this has been the main building block of the theoretical (see, for instance, Bick [10], Cochrane et al. [18], Constandinides and Zariphopoulou [19], Merton [51]-[52], Oksendal and Sulem [55], Raimondo [62], or Anderson and Raimondo [6]) as well as applied finance literature (see Martens and van Dijk [48], Wong [69], Instefjord [41], Gerber and Shiu [32]-[33], Gatheral and Schied [31], Browne [12], Biger and Hull [11]). Recently, moreover, it has started featuring prominently also in applied macro- and micro-economic studies (see Postali and Picchetti [58], Farhi and Panageas [28],Epaulard and Pommeret [26], Hadjiliadis [36], Hull [40], He [38], Candenillas and Zapatero [13], Capozza and Kazarian [14], Ericsson [27], Mella-Baral and Perraudin [50], Oren [56], Pennings [57], Promislow and Young [59], Maratha and Ryan [60], Schmidli [65], Milevsky [53], Fleten et al. [29], Deng et al. [21], Carey and Zilberman [15]).

3   Price Dynamics with Lump-sum Dividends

To fix ideas and facilitate intuitive reasoning, let us begin by considering a finite time-horizon ($\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}_{++}$) and assuming that the securities pay only lump-sum dividends on the terminal date. That is, the dividend process of every security $n \in \mathcal{N}$ is given by (4) while $g_n(\cdot) = 0$ a.e. on $\Omega \times [0, T]$. In this case, only the first term on the right-hand side of (1) applies (as in, for example, Bick [9]-[10], Anderson and Raimondo [6], Raimondo [62], He and Leland [37]). Letting then $\beta(\omega, T) - \beta(\omega, t) = \sqrt{T - t}\mathbf{x}$ with $\mathbf{x} \sim N(\mathbf{0}, \mathrm{I}_K)$, we may write

$$
\begin{aligned}
P_n(\mathcal{I}(\omega, t)) &= \mathrm{Cov}_{\mathbf{x}}\left[u'(W(T, \mathcal{I}(\omega, t), \mathbf{x})), G_n(T, \mathcal{I}(t), \mathbf{x})\right] \\
&+ \mathbb{E}_{\mathbf{x}}\left[u'(W(T, \mathcal{I}(\omega, t), \mathbf{x}))\right]\mathbb{E}_{\mathbf{x}}\left[G_n(T, \mathcal{I}(\omega, t), \mathbf{x})\right]
\end{aligned}
\tag{5}
$$

$$
\begin{aligned}
\frac{\partial P_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} &= \frac{\partial \mathrm{Cov}_{\mathbf{x}}\left[u'(W(T, \mathcal{I}(\omega, t), \mathbf{x})), G_n(T, \mathcal{I}(t), \mathbf{x})\right]}{\partial \beta_k(\omega, t)} \\
&+ \mathbb{E}_{\mathbf{x}}\left[G_n(W(T, \mathcal{I}(\omega, t), \mathbf{x}))\right]\mathbb{E}_{\mathbf{x}}\left[\frac{\partial u'(W(T, \mathcal{I}(\omega, t), \mathbf{x}))}{\partial \beta_k(\omega, t)}\right] \\
&+ \mathbb{E}_{\mathbf{x}}\left[u'(W(T, \mathcal{I}(\omega, t), \mathbf{x}))\right]\mathbb{E}_{\mathbf{x}}\left[\frac{\partial G_n(T, \mathcal{I}(\omega, t), \mathbf{x})}{\partial \beta_k(\omega, t)}\right]
\end{aligned}
\tag{6}
$$

3.1   The Dividend and Risk-Aversion Effects

Other things remaining equal, a change $\mathrm{d}\beta_k(\omega, t)$ in the kth component of $\beta(\omega, t)$ alters by $\sigma_{nk}\mathrm{d}\beta_k(\omega, t)$ the $\mathcal{F}_t$-conditional drift, $\mu_n T + \sigma_n^{\mathsf{T}}\beta(\omega, t)$, of the underlying stochastic process that determines the dividend.[10]   Its $\mathcal{F}_t$-conditional expectation changes by

---

[10] "Other things remaining equal" (or similar expressions) refer henceforth to the current realizations of the remaining $K - 1$ sources of uncertainty, $\{\beta_m(\omega, t)\}_{m \in \{1,\ldots,K\}\setminus\{k\}}$.

$\sigma_{nk}\mathbb{E}_{\mathbf{x}}\left[G_n\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right]\mathrm{d}\beta_k\left(\omega,t\right)$. Suppose now that $\beta_k\left(\omega,t\right)$ increases. If $\sigma_{nk}>0$ ($\sigma_{nk}<0$), the currently expected dividend will be higher (lower). Due to non-satiation ($u'\left(\cdot\right)>0$), this increases (decreases) the willingness of the representative agent to hold the nth security. As she must, however, continue to hold the net supply in equilibrium, the security's price must rise (fall). It does so by $\sigma_{nk}\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[G_n\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right]\mathrm{d}\beta_k\left(\omega,t\right)$, which is the change in the expected dividend measured in terms of marginal utility. This is the *dividend effect* of $\mathrm{d}\beta_k\left(\omega,t\right)$ on the typical equilibrium price, depicted by the third term on the right-hand side of (6).

Of course, for any realization $\sqrt{T-t}\mathbf{x}$, $\mathrm{d}\beta_k\left(\omega,t\right)$ reveals information that alters by $\sigma_{n'k}\mathbb{E}_{\mathbf{x}}\left[G_{n'}\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right]\mathrm{d}\beta_k\left(\omega,t\right)$ the expected dividend of any security $n'\in\mathcal{N}$. These changes along with $\mathrm{d}\rho\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)$, the change in the terminal-period endowment, give the corresponding change in the $\mathcal{F}_t$-conditional terminal-period wealth. Ceteris paribus, the representative agent's risk aversion ($u''\left(\cdot\right)<0$) induces an opposite change in marginal utility and, thus, also in the equilibrium prices. The *risk-aversion effect* of $\mathrm{d}\beta_k\left(\omega,t\right)$ on the absolute price of the typical security is given by the second term on the right-hand side of (6). Needless to say, the direction of this effect is the same for each and every security in the model.[11]

3.2 The Asset-Riskiness Effect

Given $\sqrt{T-t}\mathbf{x}$, the extent to which $\mathrm{d}\beta_k\left(\omega,t\right)$ alters a security's price by changing the marginal utility of terminal wealth depends on the future realization of the security's dividend. Similarly, the extent to which $\mathrm{d}\beta_k\left(\omega,t\right)$ alters a security's price via a change in its dividend depends on the marginal utility of the future realization of wealth. Which is to say that changes in $\beta_k\left(\omega,t\right)$ affect the equilibrium price of the typical security through changes in the correlation between its dividend and the marginal utility of wealth. This is the *asset-riskiness effect* of $\mathrm{d}\beta_k\left(\omega,t\right)$ on the typical equilibrium price, depicted by the first term on the right-hand side of equation (6).

To investigate this effect formally, we will deploy the following notation. For an arbitrary $(n,k)\in\mathcal{N}\times\mathcal{K}$, let

$$K_n=\{k\in\mathcal{K}:\sigma_{nk}\neq 0\}\quad\text{and}\quad N_m=\{n\in\mathcal{N}:\sigma_{nm}\neq 0\}$$

be, respectively, the collection of Brownian components that are correlated with the nth dividend and that of the dividends that are correlated with the mth Brownian component. We will also adopt the convention that, if there is a security with riskless dividend, it will be designated as the zero one. Given this and to avoid redundancies, assume that $N_m\neq\varnothing$

---

[11]Equally obviously, between any two securities, the relative magnitude of the risk-aversion effect is precisely the ratio of their currently-expected terminal dividends. Indeed, the latter is nothing but the proportionality constant needed to convert units of one security into units of the other, in terms of $\mathcal{F}_t$-conditional expected terminal wealth.

$\forall m \in \mathcal{K}$ and $K_n \neq \varnothing$ $\forall n \in \mathcal{N} \setminus \{0\}$ with $\sigma_n \neq \sigma_{n'}$ for any two different $n, n' \in \mathcal{N}$. It will be instructive also to define

$$N_M = \left\{ n' \in \mathcal{N} : \exists m \in K_n, \sigma_{n'm} \neq 0 \right\}$$

where $M = |K_n| < K$.

An example when identification is straightforward

To understand the mechanics of the asset-riskiness effect, it is instructive to consider a setting in which the kth Brownian component is not correlated with the nth dividend ($k \notin K_n$) while those components that are correlated with this dividend ($\beta_m(\omega, t)$ with $m \in K_n$) affect the terminal wealth (i) only through dividends ($\frac{\partial \rho(\mathcal{I}(\omega,T))}{\partial \beta_m(\omega,T)} = 0$ if $m \in K_n$), and (ii) through dividends that are not correlated with the remainder of the Brownian process ($K_n \cap K_{n'} \neq \varnothing$ only if $K_{n'} \subseteq K_n$). That is, consider the wealth specification[12]

$$
\begin{aligned}
W(T, \mathcal{I}(\omega, t), \mathbf{x}) &= \rho(T, \mathcal{I}(\omega, t), \mathbf{y}) \\
&+ \sum_{n'' \in \mathcal{N} \setminus N_M} G_{n'}(T, \mathcal{I}(\omega, t), \mathbf{y}) + \sum_{n' \in N_M} G_{n'}(T, \mathcal{I}(\omega, t), \mathbf{z}) \\
&:= W_{-M}(T, \mathcal{I}(\omega, t), \mathbf{y}) + W_M(T, \mathcal{I}(\omega, t), \mathbf{z})
\end{aligned}
\tag{7}
$$

where $\mathbf{x} = (\mathbf{z}, \mathbf{y}) \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} I_M & \mathbb{O}_{M \times (K-M)} \\ \mathbb{O}^{\mathsf{T}} & I_{K-M} \end{bmatrix} \right)$.

In this case, $\frac{\partial W(T, \mathcal{I}(\omega,t), \mathbf{x})}{\partial \beta_k(\omega,t)} = \frac{\partial W_{-M}(T, \mathcal{I}(\omega,t), \mathbf{z})}{\partial \beta_k(\omega,t)}$ so that the first term on the right-hand side of (6) can be written out as follows

$$
\begin{aligned}
&\mathrm{Cov}_{\mathbf{x}} \left[u''(W(T, \mathcal{I}(\omega, t), (\mathbf{z}, \mathbf{y}))) \frac{\partial W_{-M}(T, \mathcal{I}(\omega, t), \mathbf{y})}{\partial \beta_k(\omega, t)}, G_n(T, \mathcal{I}(\omega, t), \mathbf{z}) \right] \\
&= \int_{\mathbb{R}^{K-M}} \left(\begin{array}{c} \int_{\mathbb{R}^M} u''(W(T, \mathcal{I}(\omega, t), (\mathbf{z}, \mathbf{y}))) G_n(T, \mathcal{I}(\omega, t), \mathbf{z}) \, \mathrm{d}\Phi(\mathbf{z}) - \\ \int_{\mathbb{R}^M} u''(W(T, \mathcal{I}(\omega, t), (\mathbf{z}, \mathbf{y}))) \, \mathrm{d}\Phi(\mathbf{z}) \int_{\mathbb{R}^M} G_n(T, \mathcal{I}(\omega, t), \mathbf{z}) \, \mathrm{d}\Phi(\mathbf{z}) \end{array} \right) \\
&\quad \times \frac{\partial W_{-M}(T, \mathcal{I}(\omega, t), \mathbf{y})}{\partial \beta_k(\omega, t)} \mathrm{d}\Phi(\mathbf{y}) \\
&= \int_{\mathbb{R}^{K-M}} \mathrm{Cov}_{\mathbf{z}} \left[u''(W(T, \mathcal{I}(\omega, t), (\mathbf{z}, \mathbf{y}))), G_n(T, \mathcal{I}(\omega, t), \mathbf{z}) \right] \frac{\partial W_{-M}(T, \mathcal{I}(\omega, t), \mathbf{y})}{\partial \beta_k(\omega, t)} \mathrm{d}\Phi(\mathbf{y})
\end{aligned}
\tag{8}
$$

In this setting, conditional on the realization $\mathbf{y}$, the terminal wealth $W(\cdot)$ is strictly comonotonic in $\mathbf{z}$ with $G_n(\cdot)$. Under non-increasing absolute risk aversion (NIARA), so is $u''(\cdot)$ which implies in turn that the covariance within the last integral above is strictly positive (see Appendix B).[13] Clearly, the sign of the asset-riskiness effect of $\mathrm{d}\beta_k(\omega, t)$ on $P_n(\omega, t)$

---

[12]It is without loss of generality to take the first M indices in $\mathcal{K}$ to be the set $K_n \setminus \{k\}$. In what follows, $\mathbf{x}_M \in \mathbb{R}^M$ depicts the realizations of the Brownian increments $\{\beta_m(T) - \beta_m(t)\}_{m \in \{1,\dots,M\}}$, while $\mathbf{x}_{-M} \in \mathbb{R}^{K-M}$ will refer to those of the increments $\{\beta_k(T) - \beta_k(t)\}_{k \in \{M+1,\dots,K\}}$, and $\mathbf{x}_{-(M,k)} \in \mathbb{R}^{K-M-1}$ to the realizations of $\{\beta_{k'}(T) - \beta_{k'}(t)\}_{k' \in \{M+1,\dots,K\} \setminus \{k\}}$.

[13]The coefficient of absolute risk-aversion is the function $r_A : \mathbb{R}_+ \to \mathbb{R}_{++}$ defined by $r_A(\cdot) = -u''(\cdot)/u'(\cdot)$.

will be given by the sign of $\frac{\partial W_{-M}(T,\mathcal{I}(\omega,t),\mathbf{y})}{\partial \beta_k(\omega,t)}$, as long as this sign remains unchanged on $\mathbb{R}^{K-M}$.

To see why this ought to be so, let for instance $\frac{\partial W_{-M}(T,\mathcal{I}(\omega,t),\mathbf{y})}{\partial \beta_k(\omega,t)} > 0 \ \forall \mathbf{y} \in \mathbb{R}^{K-M}$. Then, an increase in $\beta_k(\omega,t)$ raises the $\mathcal{F}_t$-conditional terminal wealth, reducing its marginal utility. Under NIARA, though, the decrease in the marginal utility is smaller when the nth dividend is large and larger when it is small. And, due to risk aversion, this means that the increase in $\beta_k(\omega,t)$ makes the terminal wealth less positively correlated with the nth dividend. This diminishes the representative agent's perceived "riskiness" of the nth security, inducing her to demand more of it and (in the face of fixed supply) raise its price in equilibrium. Observe also that the risk-aversion effect of $\mathrm{d}\beta_k(\omega,t)$ on $P_n(\omega,t)$ will have the opposite sign of $\frac{\partial W_{-M}(T,\mathcal{I}(\omega,t),\mathbf{y})}{\partial \beta_k(\omega,t)}$. In this example, the asset-riskness and risk-aversion effects push the security's price in opposite directions.

An example when it is not

The preceding illustration relies heavily on the fact that the kth Brownian component is not correlated with the nth dividend. By contrast, when $\sigma_{nk} \neq 0$, the mechanics of the asset-riskness effect become more complicated. Given a change $\mathrm{d}\beta_k(\omega,t)$, the new wealth realization will be $W(T,\mathcal{I}(\omega,t),\mathbf{x}) + \mathrm{d}W(T,\mathcal{I}(\omega,t),\mathbf{x})$ while the new covariance of its marginal utility with the dividend will be given by

$$e^{\sigma_{nk}\mathrm{d}\beta_k(\omega,t)}\mathrm{Cov}_{\mathbf{x}}\left[u'\left(W(T,\mathcal{I}(\omega,t),\mathbf{x}) + \mathrm{d}W(T,\mathcal{I}(\omega,t),\mathbf{x})\right), G_n(T,\mathcal{I}(\omega,t),\mathbf{x})\right]$$

Obviously, what happens to the perceived "riskness" of the security is determined now not only by the covariance above, but also by the term $e^{\sigma_{nk}\mathrm{d}\beta_k(\omega,t)}$.

Suppose, for example, that $W(T,\mathcal{I}(\omega,t),\mathbf{x})$ and $G_n(T,\mathcal{I}(\omega,t),\mathbf{x})$ are again strictly comonotonic in $\mathbf{x}$. As before, $u'(\cdot)$ is strictly countermonotonic and, thus, negatively corre-lated with the dividend. Let, however, $\sigma_{nk}\mathrm{d}\beta_k(\omega,t) > 0$ so that we have the drift of the nth dividend is now higher. Even though the change in terminal wealth renders its marginal utility less negatively correlated with the nth dividend, as $e^{\sigma_{nk}\mathrm{d}\beta_k(\omega,t)} > 1$, the increase in the latter's drift might be sufficient to make their covariance more negative overall. In this case, and in sharp contrast to the preceding example, the perceived "riskiness" of the nth security increases with $\beta_k(\omega,t)$, exerting a downward pressure on its equilibrium price. The asset-riskness and risk-aversion effects push now the price in the same direction, either opposing the dividend effect.

3.3   The Combined Effect on Relative Prices

Turning now to the typical relative price, its equilibrium dynamics with respect to $\beta_k(\omega,t)$ are given by (2), which may be re-written in terms of percentage changes in the two absolute

---

It is non-increasing $(r'_A(\cdot) \leq 0)$ only if $u'''(\cdot) \geq -u''(\cdot)r_A(\cdot) > 0$.

prices:

$$\frac{1}{p_n\left(\mathcal{I}\left(\omega,t\right)\right)}\frac{\partial p_n\left(\mathcal{I}\left(\omega,t\right)\right)}{\partial\beta_k\left(\omega,t\right)}=\frac{1}{P_n\left(\mathcal{I}\left(\omega,t\right)\right)}\frac{\partial P_n\left(\mathcal{I}\left(\omega,t\right)\right)}{\partial\beta_k\left(\omega,t\right)}-\frac{1}{P_0\left(\mathcal{I}\left(\omega,t\right)\right)}\frac{\partial P_0\left(\mathcal{I}\left(\omega,t\right)\right)}{\partial\beta_k\left(\omega,t\right)}\quad(9)$$

The resulting relation is in general complex enough to preclude predictions using only economic intuition.[14] The risk-aversion effect, by pushing the two absolute prices in the same direction, has an ambiguous effect on the relative price. And to further complicate things, it pushes each absolute price always in the opposite direction of the dividend effect while, as we know from the preceding discussion, it may pull it in either direction relative to the asset-riskiness effect.

An example when asset-riskiness is dominant

A concrete example of such equilibrium price dynamics is provided by the setting in which the agent exhibits DARA while the dividends of the nth and n'th securities $(n'\in\mathcal{N}\setminus\{0,n\})$ are correlated with the mth and the kth Brownian motions $(m\neq k)$, respectively, with either correlation being exclusive. In addition, these Brownian motions do not affect other components of the terminal-period wealth $(\sigma_n=\sigma_{nm}\mathbf{e}_m,\ \sigma_{n'}=\sigma_{n'k}\mathbf{e}_k,\ \frac{\partial\rho(\mathcal{I}(\omega,T))}{\partial\beta_k(\omega,T)}=0=\frac{\partial\rho(\mathcal{I}(\omega,T))}{\partial\beta_m(\omega,T)}$, and $\sigma_{n''k}=0=\sigma_{n''m}$ for all $n''\in\mathcal{N}\setminus\{n,n'\})$.[15] The corresponding terminal-period wealth specification is a special case of (7) with $M=1$

$$\begin{aligned}W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)&=&W_{-(k,m)}\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}_{-(k,m)}\right)\quad(10)\\&+&G_{n'}\left(T,\mathcal{I}\left(\omega,t\right),x_k\right)+G_n\left(T,\mathcal{I}\left(\omega,t\right),x_m\right)\end{aligned}$$

In this case (see Proposition 2), the nth relative price is increasing (decreasing) in the realization $\beta_k\left(\omega,t\right)$ if $\sigma_{n'k}>0$ $(\sigma_{n'k}<0)$. Observe, however, that (6) and (9) give the risk-aversion effect on the relative price as

$$\begin{aligned}&\frac{\mathbb{E}_\mathbf{x}\left[G_n\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right]}{P_0\left(\mathcal{I}\left(\omega,t\right)\right)}\mathbb{E}_\mathbf{x}\left[u''\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\frac{\partial W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)}{\partial\beta_k\left(\omega,t\right)}\right]\\&-\frac{p_n\left(\mathcal{I}\left(\omega,t\right)\right)}{P_0\left(\mathcal{I}\left(\omega,t\right)\right)}\mathbb{E}_\mathbf{x}\left[G_0\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right]\mathbb{E}_\mathbf{x}\left[u''\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\frac{\partial W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)}{\partial\beta_k\left(\omega,t\right)}\right]\end{aligned}$$

Suppose now that the dividend of the zero security is riskless. Then, $P_0\left(\mathcal{I}\left(\omega,t\right)\right)=G_0\left(T\right)\mathbb{E}_\mathbf{x}\left[u'\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\right]$ and, by Lemma A.2 in the Appendix, the expression above

---

[14]Observe that, albeit the typical dividend follows a geometric Brownian motion, the typical relative price will not do so apart from a special case. For the setting in which there is one risky security and a zero-coupon bond while the agent has no endowment (other than the net supply of the stock), Bick [10] shows that the relative price follows a geometric Brownian motion in equilibrium if and only if the agent exhibits CRRA. For general dimensions of the Brownian process and many risky assets, this is confirmed in Raimondo [62] (see Remark 1 and Example 1).

[15]As usual, $\mathbf{e}_m\in\mathbb{R}^K$ denotes the vector with 1 at its mth entry and zeroes elsewhere.

simplifies to[16]

$$(\mathbb{E}_{\mathbf{x}}\left[G_n\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right] - G_0\left(T\right)p_n\left(\mathcal{I}\left(\omega,t\right)\right))\frac{1}{G_0\left(T\right)}\frac{\frac{\partial P_0(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)}}{P_0\left(\mathcal{I}\left(\omega,t\right)\right)} \tag{11}$$

$$= \frac{\mathbb{E}_{\mathbf{x}}\left[G_n\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right]}{G_0\left(T\right)}\left(1 - \frac{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}+\sqrt{T-t}\sigma_n\right)\right)\right]}{\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\right]}\right)\frac{\frac{\partial P_0(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)}}{P_0\left(\mathcal{I}\left(\omega,t\right)\right)}$$

Under (10), however, we have

$$\begin{aligned}
W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}+\sqrt{T-t}\sigma_n\right) &= W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right) \\
&+ G_n\left(T,\mathcal{I}\left(\omega,t\right),x_m+\sqrt{T-t}\sigma_{nm}\right) - G_n\left(T,\mathcal{I}\left(\omega,t\right),x_m\right) \\
&= W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right) + \left(e^{(T-t)\sigma_{nm}^2}-1\right)G_n\left(T,\mathcal{I}\left(\omega,t\right),x_m\right)
\end{aligned}$$

Clearly, the bracketed term on the right-hand side of (11) is strictly positive and, thus, the risk-aversion effect pulls the nth relative price in the same direction as either absolute price. And since $\sigma_{n'k}\frac{\partial W(T,\mathcal{I}(\omega,t),\mathbf{x})}{\partial\beta_k(\omega,t)} > 0$, the absolute prices move contrary to (in) the direction of $d\beta_k\left(\omega,t\right)$ when $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$). In other words, the risk-aversion effect on the relative price is negative (positive) if $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$). Clearly, the monotonicity in the relative price dynamics is due to the fact that the asset-riskness effect dominates the risk-aversion one.

An example when risk aversion is dominant

The direction and importance of the asset-riskness effect for the relative price dynamics depends also on the agent's attitude towards risk. To illustrate, consider the setting in which the agent exhibits CARA, the zero dividend is again riskless, while the mth Brownian motion affects both the nth and n'th terminal dividends.[17] Let also the former dividend be independent of any other Brownian component and the latter be correlated also with but only with the kth Brownian motion. This Brownian dimension in turn affects no other component of wealth ($\sigma_n = \sigma_{nm}\mathbf{e}_m$, $\sigma_{n'} = \sigma_{n'm}\mathbf{e}_m + \sigma_{n'k}\mathbf{e}_k$, $\frac{\partial\rho(\mathcal{I}(\omega,T))}{\partial\beta_k(\omega,T)} = 0$, and $\sigma_{n''k} = 0$ for all $n'' \in \mathcal{N}_0 \setminus \{n,n'\}$). The corresponding wealth specification is given by

$$\begin{aligned}
W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right) &= W_{-k}\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}_{-k}\right) \tag{12} \\
&+ G_n\left(T,\mathcal{I}\left(\omega,t\right),x_m\right) + G_{n'}\left(T,\mathcal{I}\left(\omega,t\right),(x_k,x_m)\right)
\end{aligned}$$

In this setting (see Proposition 3), as long as $\sigma_{nm}\sigma_{n'm} > 0$, a rise in $\beta_k\left(t\right)$ decreases (increases) the nth relative price if $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$). To analyze this result in terms

---

[16] Under the specification in (4), $G_n\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right) = e^{\mu_n T+\sigma_n^{\mathsf{T}}\left(\beta(\omega,t)+\sqrt{T-t}\mathbf{x}\right)}$. Lemma A.2 then gives $\mathbb{E}_{\mathbf{x}}\left[G_n\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\right] = e^{\mu_n T+\sigma_n^{\mathsf{T}}\left(\beta(\omega,t)+\frac{(T-t)\sigma_n}{2}\right)}$. And applying it once more, $P_n\left(\mathcal{I}\left(\omega,t\right)\right) = \mathbb{E}_{\mathbf{x}}\left[G_n\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}\right)\right)\right]\mathbb{E}_{\mathbf{x}}\left[u'\left(W\left(T,\mathcal{I}\left(\omega,t\right),\mathbf{x}+\sqrt{T-t}\sigma_n\right)\right)\right]$.

[17] Recall that $u: \mathbb{R}_{++} \mapsto \mathbb{R}$ exhibits CARA if it is given as $u\left(w\right) = \gamma e^{\alpha w}$ for some $\gamma,\alpha < 0$. In particular, we have $u''\left(w\right) = -r_A u'\left(w\right)$ with r_A a constant.

of the asset-riskness and risk-aversion effects, we need to determine the direction of the latter. To this end, let us restrict attention further to the case in which the mth Brownian motion affects no other component of the terminal-period wealth but the two dividends ($\frac{\partial \rho(\mathcal{I}(\omega,T))}{\partial \beta_m(\omega,T)} = 0$ and $\sigma_{n''m} = 0$ for all $n'' \in \mathcal{N} \setminus \{n, n'\}$). The subcase of (12) in question

$$W\left(T, \mathcal{I}\left(\omega, t\right), \mathbf{x}\right) \tag{13}$$
$$= W_{-(k,m)}\left(T, \mathcal{I}\left(\omega, t\right), \mathbf{x}_{-(k,m)}\right) + G_{n'}\left(T, \mathcal{I}\left(\omega, t\right), (x_k, x_m)\right) + G_n\left(T, \mathcal{I}\left(\omega, t\right), x_m\right)$$

gives

$$W\left(T, \mathcal{I}\left(\omega, t\right), x_m + \sqrt{T-t}\sigma_{nm}\right) - W\left(T, \mathcal{I}\left(\omega, t\right), \mathbf{x}\right)$$
$$= \left(e^{(T-t)\sigma_{n'm}\sigma_{nm}} - 1\right) G_{n'}\left(T, \mathcal{I}\left(\omega, t\right), (x_m, x_k)\right) + \left(e^{(T-t)\sigma_{nm}^2} - 1\right) G_n\left(T, \mathcal{I}\left(\omega, t\right), x_m\right)$$

so that $W\left(T, \mathcal{I}\left(\omega, t\right), x_m + \sqrt{T-t}\sigma_{nm}\right) > W\left(T, \mathcal{I}\left(\omega, t\right), \mathbf{x}\right)$ if $\sigma_{nm}\sigma_{n'm} > 0$. As before, by (11), the risk-aversion on the relative price operates in the same direction as it does on either of the absolute prices. And again as before, $\sigma_{n'k}\frac{\partial W(T,\mathcal{I}(\omega,t),\mathbf{x})}{\partial \beta_k(\omega,t)} > 0$ means that the absolute prices move contrary to (in) the direction of $\mathrm{d}\beta_k\left(\omega, t\right)$ when $\sigma_{n'k} > 0$ ($\sigma_{n'k} < 0$). As opposed to the preceding DARA example, the relative price dynamics are determined here by the risk-aversion effect.

Identifying the combined effect

The preceding observations attest to the richness of the dynamics under study. And this arises even though, in either example above, the relative dividend

$$\overline{G}_n\left(\mathcal{I}\left(\omega, T\right)\right) := \frac{G_n\left(\mathcal{I}\left(\omega, T\right)\right)}{G_0\left(\mathcal{I}\left(\omega, T\right)\right)} = \frac{A_n\left(T\right)}{A_0\left(T\right)} e^{\overline{\sigma}_n^{\mathsf{T}}\beta(\omega,T)}$$

where

$$\overline{\sigma}_n := (\overline{\sigma}_{nk})_{(n,k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K}} = (\sigma_{nk} - \sigma_{0k})_{(n,k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K}}$$

is not correlated with the kth Brownian component ($\sigma_{nk} = 0 = \sigma_{0k}$). When it is, the combination of the three potentially contradicting effects that drive the equilibrium price dynamics ought to produce even more complexity.

Let for instance the terminal wealth be increasing in the current realization of the kth Brownian motion ($\frac{\partial W(\mathcal{I}(\omega,T))}{\partial \beta_k(\omega,T)} > 0$, which would obtain for example if $\sigma_{nk} > 0$ and $\frac{\partial \rho(\mathcal{I}(\omega,T))}{\partial \beta_k(\omega,T)}, \sigma_{n',k} \geq 0 \ \forall n' \in \mathcal{N} \setminus \{n\}$). If $\sigma_{nk} > 0$, an increase in $\beta_k\left(\omega, t\right)$ raises its $\mathcal{F}_t$-conditional expectation of the nth dividend, pushing $P_n\left(\mathcal{I}\left(\omega, t\right)\right)$ upwards through the own-dividend effect. Yet, it increases also the agent's terminal wealth and, if for instance she exhibits DARA, this exerts negative risk-aversion effects on either of $P_0\left(\mathcal{I}\left(\omega, t\right)\right)$ and $P_n\left(\mathcal{I}\left(\omega, t\right)\right)$. Finally, as pointed out in the previous subsection, the asset-riskness effect on

15

either absolute price may go in either direction.

Nevertheless, and somewhat surprisingly, it turns out that the relation between the typical relative price and the typical Brownian component is always monotone as long as the relative dividend varies with the Brownian motion in question.

Proposition 1 *Let $\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of every security $n \in \mathcal{N}$ is given by (4) and $g_n(\cdot) = 0$ a.e. on $\Omega \times [0, T]$. Then,*

$$\forall (n, k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K} : \ \overline{\sigma}_{nk} \neq 0 \Rightarrow \overline{\sigma}_{nk} \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} > 0$$

Proof. See Appendix C. Keep in mind also that the proof remains valid if $u'(W(\mathcal{I}(\omega, T)))$ and $\widetilde{u}'\left(\widetilde{W}(\mathcal{I}(\omega, \cdot))\right)$ are replaced, respectively, by general pricing kernels $M : \mathcal{I}(\omega, T) \mapsto \mathbb{R}_{++}$ and $m : \mathcal{I}(\omega, \cdot) \mapsto \mathbb{R}_{++}$. ∎

An intuitive exegesis of this result is straightforward when the nth relative dividend is correlated with only one Brownian component and this relation is exclusive. The corresponding wealth specification is another special case of (7) with $M = 1$, and would obtain if $\overline{\sigma}_n = \overline{\sigma}_{nm}\mathbf{e}_m$, $\frac{\partial \rho(\mathcal{I}(\omega, T))}{\partial \beta_m(\omega, T)} = 0$, and $\sigma_{n'm} = 0$ $\forall n' \in \mathcal{N} \setminus \{0, n\}$. In this setting, let $\beta_m(\omega, t)$ change by $\mathrm{d}\beta_m(\omega, t)$. For any realization x_m, the terminal wealth changes now only through the nth dividend, the new value of which is

$$
\begin{aligned}
&\overline{G}_n\left(T, \beta_m(\omega, t) + \mathrm{d}\beta_m(\omega, t) + \sqrt{T - t}x_m\right) \\
=\ & e^{(\mu_n - \mu_0)T + \overline{\sigma}_{nm}\left(\beta_m(\omega, t) + \mathrm{d}\beta_m(\omega, t) + \sqrt{T - t}x_m\right)} \\
=\ & e^{\overline{\sigma}_{nm}\mathrm{d}\beta_m(\omega, t)}\overline{G}_n(T, \mathcal{I}(\omega, t), x_m)
\end{aligned}
$$

Moreover, since the representative agent is everywhere non-satiated $(u'(\cdot) > 0)$ and any other component of her terminal wealth remains unaffected by $\mathrm{d}\beta_m(\omega, t)$, her preferences change in the direction of First-order Stochastic Dominance (FSD).

Suppose, more specifically, that $\beta_m(\omega, t)$ increases (decreases). If $\overline{\sigma}_{nm} > 0$, the new relative dividend dominates (is dominated by) the old in the sense of FSD. The agent is now more (less) willing to hold the nth security relative to the zero one and, facing their respective fixed supplies, pushes up (down) the nth relative price. If $\overline{\sigma}_{nm} < 0$, on the other hand, the new relative dividend is dominated by (dominates) the old in terms of FSD so that now the agent finds the nth security less (more) attractive relative to the zero one. In either case, therefore, $\overline{\sigma}_{nm} \frac{\partial p_n(\omega, t)}{\partial \beta_m(\omega, t)} > 0$.[18]

---

[18]Put differently, when $\overline{\sigma}_{nm} > 0$ ($\overline{\sigma}_{nm} < 0$), going from the old to the new relative dividend takes us in the opposite (same) direction as Proposition 1 in Gollier [34], the factor being $e^{\overline{\sigma}_{nm}\mathrm{d}\beta_m(\omega, t)}$. For any risk-averse individual, $\mathrm{d}\beta_m(\omega, t)$ increases (reduces) the optimal demand for the nth security relative to the zero one and, consequently, the nth equilibrium relative price. Of course, Gollier studies probability distributions whose supports are closed intervals but this restriction is inconsequential in the present context.

4   Contagion with Lump-sum Dividends

We just established that the typical relative price varies monotonically with the current realization of the typical Brownian component, when the relative dividend is correlated with this component ($\bar{\sigma}_{nk} \neq 0$). Next, we examine the case when it is not ($\bar{\sigma}_{nk} = 0$). As it turns out, apart from a very special case, the typical relative price will be correlated with the typical Brownian component even when the relative dividend is not, a phenomenon which I will refer to as contagion.

My aim in what follows will be to identify conditions on the economic primitives that suffice for $p_n(\mathcal{I}(\omega,t))$ to vary with $\beta_k(\omega,t)$, and monotonically so. To this end, I will present some results which, in conjunction with Proposition 1, completely describe the comparative statics of the corresponding economy. Namely, they sign every entry in the dispersion matrix of relative prices, the Jacobian

$$J_p(\mathcal{I}(\omega,t)) = \left[\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)}\right]_{(n,k) \in \mathcal{N}\setminus\{0\}\times\mathcal{K}}$$

Contagion under DARA

The characteristics of contagion due to market-clearing depend on (i) the terminal-wealth specification with respect to terminal realization of the Brownian process, and (ii) the functional form of the agent's utility function (in particular, her attitude towards risk). To demonstrate the prevalence of contagion, I will progressively stack the cards against it, starting with the hypothesis that the terminal endowment is correlated neither with the kth nor with any Brownian dimension which affects the nth dividend (condition (i) below) while the dividends that vary with the kth Brownian dimension do so in the same direction (condition (iv) below), and are not correlated with any of the dimensions that affect the nth dividend (condition (ii) below). In addition, the factor loadings on each of the Brownian motions that do affect the nth dividend are proportional across the dividends with which they are correlated (condition (iii) below). When the agent exhibits DARA, these restrictions suffice for monotone relative price contagion.[19]

Proposition 2 *Let $\mathcal{T} = [0,T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $n \in \mathcal{N}$ is given by (4) and $g_n(\cdot) = 0$ a.e. on $\Omega \times [0,T]$. Suppose also that $u(\cdot)$ exhibits DARA while the following conditions apply.*

(i) $\sigma_{nk} = 0 = \sigma_{0k}$ *and* $\frac{\partial \rho(\mathcal{I}(\omega,T))}{\partial \beta_{k'}(\omega,T)} = 0$ $\forall k' \in K_n \cup \{k\}$.

(ii) $\forall n' \in \mathcal{N} \setminus \{n\}$: $K_{n'} \cap K_n \neq \varnothing$ *only if* $k \notin K_{n'}$

---

[19] As obviously $n \in N_M$, condition (iii) precludes the existence of any $m \in K_n$ s.t. $\sigma_{nm} = \sigma_{0m}$. It also implies that $\sigma_{n'm} \neq 0$ for some $m \in K_n$ iff $\sigma_{n'm} \neq 0$ across K_n; hence, the notation $N_M = \cup_{m \in K_n} N_m$. In addition, as long as $0 \in N_M$, it dictates that $\sigma_{0m} = \lambda_0(\sigma_{nm} - \sigma_{0m})$ and $\sigma_{nm} = \lambda_n(\sigma_{nm} - \sigma_{0m})$ for some $\lambda_0, \lambda_n \in \mathbb{R}^*$. That is, $\sigma_{nm} = \left(\frac{1+\lambda_0}{\lambda_0}\right)\sigma_{0m}$ and $\sigma_{0m} = \left(\frac{\lambda_n-1}{\lambda_n}\right)\sigma_{nm}$, which together require that $\lambda_n\lambda_0 = (\lambda_n - 1)(1 + \lambda_0)$. By contrast, if $\sigma_{0m} = 0$ for some $m \in K_n$, then $(\lambda_0, \lambda_n) = (0, 1)$.

(iii.a) $\forall n' \in N_M := \cup_{m \in K_n} N_m,\ \exists \lambda_{n'} \in \mathbb{R}^*$ *s.t.* $\sigma_{n'm} = \lambda_{n'} \overline{\sigma}_{nm}\ \forall m \times K_n.$

(iii.b) $\lambda_n \lambda_{n'} > 0\ \forall n' \in N_M.$

(iv) $\sigma_{n''k} \sigma_{n'''k} > 0\ \forall n'', n''' \in N_k.$

Then

$$\frac{\sigma_{n''k} \sigma_{n'm}}{\overline{\sigma}_{nm}} \frac{\partial p_n \left(\mathcal{I}(\omega, t) \right)}{\partial \beta_k (\omega, t)} > 0 \quad \text{if } \left(n', n'', m \right) \in N_M \times N_k \times K_n$$

Proof. See Appendix C. Notice also that the proof remains valid when the agent exhibits instead IARA, in which case the statement is exactly the same but for reversing the last inequality above. ∎

This result refers to the following terminal-wealth specification

$$
\begin{aligned}
W \left(\mathcal{I}(\omega, T) \right) &= \rho \left(T, \beta_{-(M,k)}(\omega, T) \right) \\
&+ \sum_{n'' \notin N_M} e^{\mu_{n''} T + \sum_{k \in \mathcal{K} \setminus \overline{K}_n} \sigma_{n''k} \beta_k(\omega, T)} \\
&+ \sum_{n' \in N_M} e^{\sum_{m \in K_n} \lambda_{n'} \overline{\sigma}_{nm} \beta_m(\omega, T)} e^{\mu_{n'} T + \sum_{k' \in \mathcal{K} \setminus (K_n \cup \{k\})} \sigma_{n'k'} \beta_{k'}(\omega, T)}
\end{aligned}
\tag{14}
$$

although it is perhaps more instructive for our purposes to look at the special case in which the nth dividend is correlated with only one Brownian component ($K_n = \{m\}$ for some $m \in \mathcal{K} \setminus \{k\}$). Now, the terminal wealth specification is given by

$$
\begin{aligned}
W \left(\mathcal{I}(\omega, T) \right) &= \rho \left(T, \beta_{-(m,k)}(\omega, T) \right) \\
&+ \sum_{n'' \in N_k} e^{\mu_{n''} T + \sum_{k \in \mathcal{K} \setminus \{m\}} \sigma_{n''k} \beta_k(\omega, T)} \\
&+ \sum_{n' \in N_m} e^{\lambda_{n'} \overline{\sigma}_{nm} \beta_m(\omega, T)} e^{\mu_{n'} T + \sum_{k' \in \mathcal{K} \setminus \{m,k\}} \sigma_{n'k'} \beta_{k'}(\omega, T)}
\end{aligned}
\tag{15}
$$

and the covariance matrix Σ_1 depicts a relevant situation regarding the factor loadings of all but the zero security. In the corresponding economy, the first security is correlated only with the first Brownian component, a macro-risk factor affecting all securities (but possibly the zero one). Proposition 2 dictates that $\lambda \sigma_{2k} \frac{\partial p_1 (\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)} > 0$ for $k \in \{2,3\}$, as long as $\alpha > 0$, $\sigma_{01} \neq \sigma_{11}$, $\sigma_{0k} = 0$, the terminal-period endowment is independent of the first and the kth Brownian components, and $\gamma_k > 0$.

$$
\Sigma_1 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \lambda \overline{\sigma}_{11} & \sigma_{22} & \sigma_{23} \\ \alpha \lambda \overline{\sigma}_{11} & \gamma_2 \sigma_{22} & \gamma_3 \sigma_{23} \end{pmatrix} \quad \Sigma_2 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & \sigma_{23} \\ 0 & \gamma_2 \sigma_{22} & \gamma_3 \sigma_{23} \end{pmatrix}
$$

Recall now that Proposition 1 applies to all but the second and third entries of the Jacobian of relative prices. Namely, we have $\sigma_{nk} \frac{\partial p_n (\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)} > 0$ for $(n, k) \in \{(1,1)\} \cup$

$(\{2,3\} \times \{2,3\})$ and $\alpha^{n-2}\lambda\bar{\sigma}_{11}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_1(\omega,t)} > 0$ for $n \in \{2,3\}$. In conjunction, therefore, with the preceding result we may sign the entire matrix. All relations between relative prices and Brownian components will be then monotone.

A particular case of the economic setting just described is the one depicted by the covariance matrix Σ_2, where the first Brownian component represents a risk-factor for which the first security is an exclusive "bet." In this case, Proposition 2 gives $\frac{\sigma_{2k}\sigma_{11}}{\bar{\sigma}_{11}}\frac{\partial p_1(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$ as well as $\frac{\sigma_{11}\gamma_k}{\sigma_{2k}}\frac{\partial p_k(\mathcal{I}(\omega,t))}{\partial\beta_1(\omega,t)} > 0$ for $k \in \{2,3\}$. This example brings us forward in our quest to stack the cards as much as possible against contagion. For contagion obtains now even though the nth dividend is correlated with only one Brownian component, which in turn affects only the nth relative dividend. Indeed, when $K_n = \{m\}$ and $N_m \in \{\{n\},\{0,n\}\}$ for some $m \in \mathcal{K} \setminus \{k\}$, Proposition 2 gives $\frac{\sigma_{n'k}\sigma_{nm}}{\bar{\sigma}_{nm}}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0 \ \forall n' \in N_k$.

This is the culminant result in our attempt to demonstrate the prevalence of contagion. For it applies even when the underlying setting is the most restrictive against cross-correlations in relative prices. I am referring of course to the terminal wealth specification in (10) which restricts the kth and mth Brownian components to be correlated exclusively with the n'th and nth dividends. Under this requirement, both sets K_n and N_k are singletons ($K_n = \{m\}$, $N_m = \{n\}$, and $N_k = \{n'\}$ for some $m \in \mathcal{K} \setminus \{k\}$ and $n' \in \mathcal{N} \setminus \{0,n\}$) and the result reads $\sigma_{n'k}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$.

$$\Sigma_3 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \quad \Sigma_4 = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \lambda\bar{\sigma}_{11} & \sigma_{22} & 0 \\ \alpha\lambda\bar{\sigma}_{11} & 0 & \sigma_{33} \end{pmatrix}$$

And this applies even for the case in which the matrix of factor loadings is diagonal, as in the example depicted by the covariance matrix Σ_3. Here, the claim is valid for any security n and any Brownian motion $k \neq n$ as long as the dividend of the zero security and terminal-period endowment are correlated with neither the nth nor the kth Brownian component. In particular, if these two wealth components are deterministic, the corresponding terminal-wealth specification

$$W(\mathcal{I}(\omega,T)) = \rho(T) + G_0(T) + \sum_{k=1}^{K} G_k(T, \beta_k(\omega,T)) \tag{16}$$

is such that each and every relative price varies monotonically with each and every Brownian dimension. Once again, in conjunction with Proposition 1, we can sign the entire Jacobian matrix of the relative price process: $\sigma_{kk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0 \ \forall (n,k) \in \mathcal{K}$.

Remark. Notice that Proposition 2 applies also under the specification

$$
\begin{aligned}
W\left(\mathcal{I}\left(\omega,T\right)\right) &= \rho_1\left(T,\beta_{-m}\left(\omega,T\right)\right) + \rho_2\left(T,\beta_m\left(\omega,T\right)\right) \\
&\quad + \sum_{n''\in\cup_{k\in\mathcal{K}\backslash\{m\}}N_k} G_{n''}\left(T,\beta_{-m}\left(\omega,T\right)\right) + \sum_{n'\in N_m} G_{n'}\left(T,\beta_m\left(\omega,T\right)\right) \\
&= W_1\left(T,\beta_{-m}\left(\omega,T\right)\right) + W_2\left(T,\beta_m\left(\omega,T\right)\right)
\end{aligned} \tag{17}
$$

for some continuous functions $\rho_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_+$ and $\rho_2 : \mathbb{R} \mapsto \mathbb{R}_+$, as long as $\bar{\sigma}_{nm} \neq 0$, $\lambda_k \frac{\partial W_1(T,\beta_{-m}(\omega,T))}{\partial\beta_k(\omega,T)} > 0$ and $\lambda_m \frac{\partial W_2(T,\beta_m(\omega,T))}{\partial\beta_m(\omega,T)} > 0$ for some $\lambda_k, \lambda_m \in \mathbb{R}^*$ and for all $\beta\left(\omega,T\right) \in \mathbb{R}^K$. It is straightforward to reproduce the proof in the Appendix for this setting and verify that $\bar{\sigma}_{nm}\lambda_m\lambda_k \frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$.

Contagion under CARA

Cross-correlations will generally be nonzero even when the representative agent exhibits CARA. And here too, there are settings of economic primitives under which the relative price $p_n\left(\mathcal{I}\left(\omega,t\right)\right)$ remains monotone in $\beta_k\left(\omega,t\right)$ even though $k \notin K_n$. As before, I will demonstrate the prevalence of contagion due to market-clearing by progressively stacking the cards against contagion.

To this end, consider first the same setting as in Proposition 2 but for the fact that now the terminal wealth endowment needs to be uncorrelated only with the kth Brownian motion (condition (i) below), while any dividend which is correlated with the kth Brownian motion is correlated also with each and every of the Brownian dimensions that affect the nth dividend (condition (ii) below), and the signs of the respective factor loadings satisfy condition (iv) below.[20]

Proposition 3 *Let $\mathcal{T} = [0,T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $n \in \{0,\ldots,N\}$ is given by (4) and $g_n\left(\cdot\right) = 0$ a.e. on $\Omega \times [0,T]$. Suppose also that $u\left(\cdot\right)$ exhibits CARA while the following conditions apply.*

(i) $\sigma_{nk} = 0 = \sigma_{0k}$ and $\frac{\partial\rho(\mathcal{I}(\omega,T))}{\partial\beta_k(\omega,T)} = 0$.

(ii) $\forall n' \in \mathcal{N} \backslash \{n\}$: $k \in K_{n'}$ only if $K_n \subseteq K_{n'}$.

(iii) $\forall n' \in N_M, \exists\lambda_{n'} \in \mathbb{R}^*$: $\sigma_{n'm} = \lambda_{n'}\bar{\sigma}_{nm} \; \forall m \in K_n$.

(iv) $\frac{\sigma_{n'm}\sigma_{n'k}}{\sigma_{n''m}\sigma_{n''k}} > 0 \; \forall (n',n'',m) \in N_k^2 \times K_n$.

Then

$$
\frac{\sigma_{n'k}\sigma_{n'm}}{\bar{\sigma}_{nm}}\frac{\partial p_n\left(\mathcal{I}\left(\omega,t\right)\right)}{\partial\beta_k\left(\omega,t\right)} < 0 \quad \text{if } \left(n',m\right) \in N_k \times K_n
$$

---

[20]Even though not directly relevant for the type of contagion under study here, it should be pointed out that Proposition 3 applies also under the less restrictive assumption $\sigma_{nk} = \sigma_{0k}$ in condition (i). When $\sigma_{nk} = \sigma_{0k}$ but $\sigma_{nk}\sigma_{0k} \neq 0$, condition (iii) dictates that $\lambda_n\lambda_0 > 0$ (recall the preceding footnote), so that either $(\lambda_0, \lambda_n) \in \mathbb{R}_+^* \times (1,\infty)$ or $(\lambda_0, \lambda_n) \in (-\infty,-1) \times \mathbb{R}_-^*$.

Proof. See Appendix C. ∎

This result refers to the following terminal-wealth specification

$$
\begin{aligned}
W\left(\mathcal{I}\left(\omega,T\right)\right) \;=\; & \rho\left(T,\beta_{-k}\left(\omega,T\right)\right) \\
& + \sum_{n'\in N_k} e^{\sum_{m\in K_n}\lambda_{n'}\bar{\sigma}_{nm}\beta_m(\omega,T)}e^{\mu_{n'}T+\sum_{k'\in\mathcal{K}\setminus K_n}\sigma_{n'k'}\beta_{k'}(\omega,T)} \\
& + \sum_{n'\notin N_k} e^{\mu_{n'}T+\sum_{k'\in\mathcal{K}\setminus\{k\}}\sigma_{n'k'}\beta_{k'}(\omega,T)}
\end{aligned}
\tag{18}
$$

To illustrate, consider again the dispersion matrix Σ_1. Under CARA, we can determine the dynamics of the first relative price with respect to changes in the current realization of one of the non-macroeconomic risk-factors ($k \in \{2,3\}$), as long as $\sigma_{01} \neq \sigma_{11}$, the zero dividend and the terminal-period endowment are not correlated with this factor, and $\alpha\gamma_k > 0$. In this case, Proposition 3 requires that $\lambda\sigma_{2k}\frac{\partial p_1(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} < 0$.

A special case that constrains this economic setup against cross-correlations more is depicted by the matrix Σ_4. In this example, the kth Brownian component ($k \in \{2,3\}$) affects only the one terminal dividend which is correlated also with the first Brownian motion. As long as $\sigma_{01} \neq \sigma_{11}$ and the zero dividend and the terminal-period endowment are not correlated with this factor, we ought to have $\alpha^{k-2}\lambda\sigma_{kk}\frac{\partial p_1(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} < 0$.

The special case of no contagion

At first glance, Proposition 3 might seem puzzling as it contradicts the rather commonly-held view that, under CARA, changes in wealth that are independent of an asset's payoff should not matter for its equilibrium relative price. This assertion stems from the well-known fact that, under CARA, changes in wealth that do not affect the risk premium of an asset should leave its relative price unchanged. Even in the absence of own-dividend effects ($\sigma_{nk} = 0 = \sigma_{0k}$), however, this type of wealth changes follow from changes in the current information $\beta_k(\omega,t)$ only if the asset-riskiness effect on the relative price exactly cancels out the risk-aversion one. And, under CARA, the latter relation obtains whenever the kth Brownian component affects the terminal wealth independently from the Brownian dimensions which are correlated with the nth relative dividend.

Proposition 4 *Let $\mathcal{T} = [0,T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of each security $n \in \{0,\ldots,N\}$ is given by (4) and $g_n(\cdot) = 0$ a.e. on $\Omega \times [0,T]$. Suppose also that $u(\cdot)$ exhibits CARA while the following conditions apply.*

(i) *$\sigma_{nk} = 0 = \sigma_{0k}$ and $\rho\left(\mathcal{I}\left(\omega,T\right)\right) = \rho_1\left(\mathcal{I}\left(\omega,T\right)\right) + \rho_2\left(\mathcal{I}\left(\omega,T\right)\right)$, for some continuous functions $\rho_1,\rho_2 : \mathbb{R}^K \mapsto \mathbb{R}_+$, such that $\frac{\partial\rho_1(\mathcal{I}(\omega,T))}{\partial\beta_k(\omega,T)} = 0 = \frac{\partial\rho_2(\mathcal{I}(\omega,T))}{\partial\beta_m(\omega,T)}$ $\forall m \in K_n \cup K_0$.*

(ii) *$N_m \cap N_k = \varnothing$ $\forall m \in K_n \cup K_0$.*

Then

$$\frac{\partial p_n \left(\mathcal{I}\left(\omega, t\right)\right)}{\partial \beta_k \left(\omega, t\right)} = 0$$

Proof. See Appendix C. ∎

Regarding the corresponding wealth specification, consider the partition $\{K_L, \mathcal{K} \setminus K_L\}$ such that $k \notin K_L \supseteq K_n \cup K_0$. Let then

$$
\begin{aligned}
&W\left(T, \mathcal{I}\left(\omega, t\right), \mathbf{x}\right) \\
&= \rho_1 \left(\left(T, \beta_L\left(\omega, t\right) + \sqrt{T-t}\mathbf{x}_L\right)\right) + \rho_2 \left(\left(T, \beta_{-L}\left(\omega, t\right) + \sqrt{T-t}\mathbf{x}_{-L}\right)\right) \\
&\quad + \sum_{n \in \cup_{m \in K_L} N_m} G_n \left(\left(T, \beta_L\left(\omega, t\right) + \sqrt{T-t}\mathbf{x}_L\right)\right) \\
&\quad + \sum_{n' \in \cup_{m \in \mathcal{K} \setminus K_L} N_m} G_{n'} \left(\left(T, \beta_{-L}\left(\omega, t\right) + \sqrt{T-t}\mathbf{x}_{-L}\right)\right) \quad (19) \\
&= W_1 \left(\left(T, \beta_L\left(\omega, t\right) + \sqrt{T-t}\mathbf{x}_L\right)\right) + W_2 \left(\left(T, \beta_{-L}\left(\omega, t\right) + \sqrt{T-t}\mathbf{x}_{-L}\right)\right)
\end{aligned}
$$

which embeds (7) - thus, also (17) - as well as (18) and (16). For the latter formulation, moreover, notice that we remain within the realm of the specification above if the endowment functional is replaced by

$$\rho\left(\mathcal{I}\left(\omega, T\right)\right) = \sum_{k=1}^{K} \rho_k \left(T, \beta_k\left(\omega, T\right)\right)$$

for some continuous functions $\rho_k : \mathbb{R} \to \mathbb{R}_+$. Observe also that yet another subcase is the following

$$
\begin{aligned}
W\left(\mathcal{I}\left(\omega, T\right)\right) &= \rho_1\left(T, \beta_{-k}\left(\omega, T\right)\right) + \rho_2\left(T, \beta_k\left(\omega, T\right)\right) \\
&\quad + \sum_{n \notin N_k} G_n\left(T, \beta_{-k}\left(\omega, T\right)\right) + G_{n'}\left(T, \beta_k\left(\omega, T\right)\right) \quad (20)
\end{aligned}
$$

for some continuous functions $\rho_1 : \mathbb{R}^{K-1} \mapsto \mathbb{R}_+$ and $\rho_2 : \mathbb{R} \mapsto \mathbb{R}_+$.

Of course, Proposition 4 appears to support the premise that, under CARA, changes in wealth that are independent of an asset's payoff should not matter for its relative price. Yet, the fact that the relative price is not correlated with the kth Brownian dimension is neither because the relative dividend itself is uncorrelated nor due to CARA alone. The result depends also, and fundamentally so, upon the separability between the channels through which the kth and any of the Brownian motions that affect the nth dividend operate in (19). Indeed, we do know (recall Proposition 3) that, without separability, the relative price may well be correlated with the kth Brownian dimension even though both the CARA as well as the $\bar{\sigma}_{nk} = 0$ assumption are maintained.

To see what is so special about the underlying separability in (19), recall the fundamental pricing relation in (5). Since $k \notin K_L \supseteq K_n \cup K_0$, we have

$$
\begin{aligned}
P_j \left(\mathcal{I}(\omega, t) \right) &= \mathbb{E}_{\mathbf{x}} \left[\gamma e^{\alpha W(\mathcal{I}(\omega, T))} G_j \left(T, \beta_M(\omega, T) \right) \right] \\
&= \mathbb{E}_{\mathbf{x}_{-L}} \left[e^{\alpha W_2(T, \beta_{-L}(\omega, T))} \right] \mathbb{E}_{\mathbf{x}_L} \left[e^{\alpha W_1(T, \beta_L(\omega, T))} G_j \left(T, \beta_M(\omega, T) \right) \right]
\end{aligned}
$$

for $j = 0, n$. That is,

$$
\frac{\partial P_j \left(\mathcal{I}(\omega, t) \right)}{\partial \beta_k(\omega, t)} = P_j \left(\mathcal{I}(\omega, t) \right) \frac{\mathbb{E}_{\mathbf{x}_{-L}} \left[\alpha \gamma e^{\alpha W_2(T, \beta_{-L}(\omega, T))} \frac{\partial W_2(T, \beta_{-L}(\omega, T))}{\partial \beta_k(\omega, T)} \right]}{\mathbb{E}_{\mathbf{x}_{-L}} \left[e^{\alpha \gamma W_2(T, \beta_{-L}(\omega, T))} \right]}
$$

and, by (9), the relative price cannot be correlated with the kth Brownian dimension.[21]

Most probably, the erroneously crude intuition behind the "zero cross-correlations under CARA" premise stems from the multitude of examples in the financial economics literature that take the agent's wealth to be linearly-dependent upon asset payoffs. Although rendering discrete-time models analytically tractable and elegant, the linearity assumption obscures our grasp of the interaction between the asset-riskiness and risk-aversion effects on relative equilibrium prices. Indeed, letting $W_2 \left(\mathcal{I}(\omega, T) \right) = W_0 \left(T, \beta_{-(L,k)}(\omega, T) \right) + \lambda_k \beta_k(\omega, T)$, for some continuous function $W_0 : \mathbb{R}^{K-L-1} \to \mathbb{R}_+$ and $\lambda_k \in \mathbb{R}^*$, falls well within the setting just analyzed. In this sense, the linearity assumption forces the asset-riskiness and risk-aversion effects on the relative equilibrium price to cancel each other out. And this is so irrespectively of the correlations between the various other elements of the agent's wealth.[22]

5   Price Dynamics with Dividend Flows

We now turn to the setting in which the securities pay also dividend flows during $\mathcal{T}$. If they do so without offering lump sums, the asset-pricing equation is given by the second term on the right-hand side of (1) (as in, for instance, Cochrane et al. [18], Martin [49], or Farhi and Panageas [28]). In this case, [23] (see Section 3.2) shows that the equilibrium price dynamics are given by

$$
P_n \left(\mathcal{I}(\omega, t) \right) = \int_t^T P_{n,s} \left(\mathcal{I}(\omega, t) \right) \mathrm{d}s \qquad \frac{\partial P_n \left(\mathcal{I}(\omega, t) \right)}{\partial \beta_k(\omega, t)} = \int_t^T \frac{\partial P_{n,s} \left(\mathcal{I}(\omega, t) \right)}{\partial \beta_k(\omega, t)} \mathrm{d}s
$$

and

$$
P_0 \left(\mathcal{I}(\omega, t) \right)^2 \frac{\partial p_n \left(\mathcal{I}(\omega, t) \right)}{\partial \beta_k(\omega, t)} = \int_t^T P_{0,s} \left(\mathcal{I}(\omega, t) \right)^2 \frac{\partial p_{n,s} \left(\mathcal{I}(\omega, t) \right)}{\partial \beta_k(\omega, t)} \mathrm{d}s
$$

---

[21]It is trivial to check that this result obtains also when $u(\cdot)$ is quadratic.

[22]Needless to say, given that the $\mathcal{F}_t$-conditional future realizations $\beta_k(T) - \beta_k(t)$ are normally-distributed here, the linearity assumption requires also unlimited liability, an unrealistically strong condition (implying that the agent may lose more than everything with positive probability). Yet, this drawback is well-known.

for $(n, k) \in \mathcal{N} \times \mathcal{K}$, with $P_{n,s}(\cdot)$ and $p_{n,s}(\cdot)$ being the respective prices in the analysis of the preceding subsection taking s to be the terminal date. Clearly, our previous results can be re-stated now as follows.[23]

Proposition 5 *Let $\mathcal{T} \subseteq \mathbb{R}_{++}$ and suppose that the dividend process of every $n \in \mathcal{N}$ is given by (3) and $G_n(\cdot) = 0$ a.e. on $\Omega \times \{T\}$. Then,*

$$\forall (n, k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K}: \ (\widetilde{\sigma}_{nk} - \widetilde{\sigma}_{0k}) \neq 0 \Rightarrow (\widetilde{\sigma}_{nk} - \widetilde{\sigma}_{0k}) \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} > 0$$

Proposition 6 *Let $\mathcal{T} \subseteq \mathbb{R}_{++}$ and suppose that the dividend process of every $n \in \mathcal{N}$ is given by (3) and $G_n(\cdot) = 0$ a.e. on $\Omega \times \{T\}$. Suppose also that $\widetilde{u}(\cdot)$ exhibits DARA while the following conditions apply.*

(i) $\widetilde{\sigma}_{nk} = 0 = \widetilde{\sigma}_{0k}$ and $\frac{\partial \widetilde{\rho}(\mathcal{I}(\omega,t))}{\partial \beta_{k'}(\omega,t)} = 0 \ \forall k' \in \widetilde{K}_n \cup \{k\}$.

(ii) $\forall n' \in \mathcal{N} \setminus \{n\}: \ \widetilde{K}_{n'} \cap \widetilde{K}_n \neq \varnothing$ only if $k \notin \widetilde{K}_{n'}$

(iii.a) $\forall n' \in \widetilde{N}_M := \cup_{m \in \widetilde{K}_n} \widetilde{N}_m, \ \exists \widetilde{\lambda}_{n'} \in \mathbb{R}^*$ s.t. $\widetilde{\sigma}_{n'm} = \widetilde{\lambda}_{n'}(\widetilde{\sigma}_{nm} - \widetilde{\sigma}_{0m}) \ \forall m \times \widetilde{K}_n$.

(iii.b) $\widetilde{\lambda}_n \widetilde{\lambda}_{n'} > 0 \ \forall n' \in \widetilde{N}_M$.

(iv) $\widetilde{\sigma}_{n''k} \widetilde{\sigma}_{n'''k} > 0 \ \forall n'', n''' \in \widetilde{N}_k$.

Then

$$\frac{\widetilde{\sigma}_{n''k} \widetilde{\sigma}_{n'm}}{\widetilde{\sigma}_{nm} - \widetilde{\sigma}_{0m}} \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} > 0 \quad \text{if } (n', n'', m) \in \widetilde{N}_M \times \widetilde{N}_k \times \widetilde{K}_n$$

Proposition 7 *Let $\mathcal{T} \subseteq \mathbb{R}_{++}$ and suppose that the dividend process of every security $n \in \mathcal{N}$ is given by (3) and $G_n(\cdot) = 0$ a.e. on $\Omega \times \{T\}$. Suppose also that $\widetilde{u}(\cdot)$ exhibits CARA while the following conditions apply.*

(i) $\widetilde{\sigma}_{nk} = 0 = \widetilde{\sigma}_{0k}$ and $\frac{\partial \widetilde{\rho}(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)} = 0$.

(ii) $\forall n' \in \mathcal{N} \setminus \{n\}: \ k \in \widetilde{K}_{n'}$ only if $\widetilde{K}_n \subseteq \widetilde{K}_{n'}$.

(iii) $\forall n' \in \widetilde{N}_M, \ \exists \widetilde{\lambda}_{n'} \in \mathbb{R}^*$ s.t. $\widetilde{\sigma}_{n'm} = \widetilde{\lambda}_{n'}(\widetilde{\sigma}_{nm} - \widetilde{\sigma}_{0m}) \ \forall m \times \widetilde{K}_n$.

(iv) $\left(\frac{\widetilde{\sigma}_{n'm}}{\widetilde{\sigma}_{n''m}}\right)\left(\frac{\widetilde{\sigma}_{n'k}}{\widetilde{\sigma}_{n''k}}\right) > 0 \ \forall (n', n'', m) \in \widetilde{N}_k^2 \times \widetilde{K}_n$.

Then

$$\widetilde{\sigma}_{n'k}\left(\frac{\widetilde{\sigma}_{n'm}}{\widetilde{\sigma}_{nm} - \widetilde{\sigma}_{0m}}\right)\frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} < 0 \quad \text{if } (n', m) \in \widetilde{N}_k \times \widetilde{K}_n$$

---

[23]The notation here is the direct analogue of the one used thus far. For $(n, k) \in \mathcal{N} \times \mathcal{K}$, we let $\widetilde{K}_n = \{k \in \mathcal{K} : \sigma_{nk} \neq 0\}$ and $N_m = \{n \in \mathcal{N} : \sigma_{nm} \neq 0\}$. We also assume that $\widetilde{N}_m \neq \varnothing \ \forall m \in \mathcal{K}$ and $\widetilde{K}_n \neq \varnothing$ $\forall n \in \mathcal{N} \setminus \{0\}$ with $\widetilde{\sigma}_n \neq \widetilde{\sigma}_{n'}$ for any two different $n, n' \in \mathcal{N}$. Moreover, $\widetilde{N}_M = \{n' \in \mathcal{N} : \exists m \in K_n, \widetilde{\sigma}_{n'm} \neq 0\}$.

Proposition 8 *Let $\mathcal{T} \subseteq \mathbb{R}_{++}$ and suppose that the dividend process of every security $n \in \mathcal{N}$ is given by (3) and $G_n(\cdot) = 0$ a.e. on $\Omega \times \{T\}$. Suppose also that $\widetilde{u}(\cdot)$ exhibits CARA while the following conditions apply.*

(i) $\widetilde{\sigma}_{nk} = 0 = \widetilde{\sigma}_{0k}$ and $\widetilde{\rho}(\mathcal{I}(\omega, t)) = \widetilde{\rho}_1(\mathcal{I}(\omega, t)) + \widetilde{\rho}_2(\mathcal{I}(\omega, t))$, for some continuous functions $\widetilde{\rho}_1, \widetilde{\rho}_2 : \mathbb{R}^K \mapsto \mathbb{R}_+$, such that $\frac{\partial \widetilde{\rho}_1(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)} = 0 = \frac{\partial \widetilde{\rho}_2(\mathcal{I}(\omega,t))}{\partial \beta_m(\omega,t)} \ \forall m \in \widetilde{K}_n \cup \widetilde{K}_0$.

(ii) $\widetilde{N}_m \cap \widetilde{N}_k = \varnothing \ \forall m \in \widetilde{K}_n \cup \widetilde{K}_0$.

Then

$$\frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} = 0$$

Needless to say, the entire analysis in the preceding section remains valid as is but for the replacement of the respective factor loadings.

Lump-sums and Flows

It remains to examine the case in which the securities may pay both dividend flows during the time-interval as well as lump sums on the terminal date. This presupposes a finite time-horizon and requires that both terms on the right-hand side of (1) apply (as in, for example, Anderson and Raimondo [5] or Cox et al. [20]). Thus, the partial derivative of interest $\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)}$ becomes the sum of two terms, the two partial derivatives analyzed in the preceding two subsections which I will denote here by $\frac{\partial p_{n1}(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)}$ and $\frac{\partial p_{n2}(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)}$, respectively. Applying the respective analysis to each term, an appropriate re-statement of our result is again immediate.[24]

The only statements that do not follow from our analysis thus far are parts II of Proposition 9 and I.B of Propositions 10 and 11. With respect to these results, notice first that $a_j(\cdot)$ being analytic suffices for $g_j(\cdot)$ to be so given that the exponential function is analytic and so is the product of two analytic functions. Observe also that the current setting can be embedded in the one of Anderson and Raimondo [5]. The latter paper established that, for arbitrary $\omega \in \Omega$ and $(n, k) \in \mathcal{N} \times \mathcal{K}$, as a function $[0, T] \times \mathbb{R}^K \mapsto \mathbb{R}$, the partial derivative $\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial \beta_k(\omega,t)}$ is continuous on this domain and analytic on $(0, T) \times \mathbb{R}^K$.[25] This means that, for an arbitrary $v \in \mathbb{R}$, the set

$$S_{nk}(v) = \left\{ \mathcal{I}(\omega, t) \in (0, T) \times \mathbb{R}^K : v \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} = 0 \right\}$$

---

[24]Notice that the analysis applies also when the agent exhibits IARA. In this case, Propositions 2, 6, and 10 are stated in exactly the same way but for reversing the respective last inequalities.

[25]We are referring to the analysis in Appendices B and D of Anderson and Raimondo [5]. Continuity requires that $(uG_j)(\cdot)$ satisfies some mild boundedness condition on $\{T\} \times \mathbb{R}^K$ (see pp. 888-889 and assumption (1) in Anderson and Raimondo [5]). Analyticity, on the other hand, requires that $\widetilde{u}$ and g_j are both analytic on $(0, T) \times \mathbb{R}^K$.

has positive Lebesgue measure only if $v\left(\frac{\partial p_{n1}(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} + \frac{\partial p_{n2}(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)}\right) = 0$ *everywhere* on $(0,T) \times \mathbb{R}^K$.[26]

As $t \to T$, however, on the one hand, the second term in (1) vanishes in the limit and so does $\frac{\partial p_{n2}(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)}$. On the other, the first term in (1) tends to $P_j(\mathcal{I}(\omega,T))$. That is, $\frac{\partial p_{n1}(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)}$ approaches $\frac{\partial p_n(\mathcal{I}(\omega,T))}{\partial\beta_k(\omega,T)}$. Letting, therefore, v being one of $\overline{\sigma}_{nk}$, $\frac{\sigma_{n''k}\sigma_{n'm}}{\overline{\sigma}_{nm}}$, or $\frac{\sigma_{n'k}\sigma_{n'm}}{\overline{\sigma}_{nm}}$ as required below, means that $v\frac{\partial p_n(\mathcal{I}(\omega,T))}{\partial\beta_k(\omega,T)} \neq 0$, its actual sign given by Propositions 1-3, respectively. But then, continuity ensures the existence of a neighborhood $V_{\mathcal{I}(\omega,T)}$ of $\mathcal{I}(\omega,T)$ such that $v\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} \neq 0$ everywhere in $V_{\mathcal{I}(\omega,T)}$. And since $V_{\mathcal{I}(\omega,T)} \cap \left((0,T) \times \mathbb{R}^K\right) \neq \varnothing$, analyticity requires that $S_{nk}(v)$ is null set.

Proposition 9 *Let $\mathcal{T} = [0,T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of every $n \in \mathcal{N}$ is given by (4)-(3). Then,*

(I) $\forall (n,k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K}$: $\overline{\overline{\sigma}}_{nk}\overline{\sigma}_{nk} > 0$ *only if* $\overline{\sigma}_{nk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$.

(II) $\forall (n,k) \in \mathcal{N} \setminus \{0\} \times \mathcal{K}$: $\overline{\overline{\sigma}}_{nk} = 0 \neq \overline{\sigma}_{nk}$ *while both functions $\widetilde{u}(\cdot)$ and $a_n(\cdot)$ are analytic on $(0,T) \times \mathbb{R}^K$, only if $\overline{\sigma}_{nk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} \neq 0$ a.e. on $[0,T] \times \mathbb{R}^K$.*

Proposition 10 *Let $\mathcal{T} = [0,T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of every $n \in \mathcal{N}$ is given by (4)-(3). Suppose also that both $\widetilde{u}(\cdot)$ and $u(\cdot)$ exhibit DARA. Then,*

(I.A) $\overline{\overline{\sigma}}_{nk} \neq 0$, *conditions (i)-(iv) in Proposition 2 apply, and $\frac{\sigma_{n''k}\sigma_{n'm}}{\overline{\sigma}_{nm}\overline{\overline{\sigma}}_{nk}} > 0$ for $(n',n'',m) \in N_M \times N_k \times K_n$, only if $\overline{\overline{\sigma}}_{nk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$.*

(I.B) $\overline{\overline{\sigma}}_{nk} = 0$, *conditions (i)-(iv) in Proposition 2 apply, and both functions $\widetilde{u}(\cdot)$ and $a_n(\cdot)$ are analytic on $(0,T) \times \mathbb{R}^K$, only if $\frac{\sigma_{n''k}\sigma_{n'm}}{\overline{\sigma}_{nm}}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} \neq 0$ for $(n',n'',m) \in N_M \times N_k \times K_n$ a.e. on $[0,T] \times \mathbb{R}^K$.*

(II) $\overline{\sigma}_{nk} \neq 0$, *conditions (i)-(iv) in Proposition 6 apply, and $\frac{\widetilde{\sigma}_{n''k}\widetilde{\sigma}_{n'm}}{\overline{\widetilde{\sigma}}_{nm}\overline{\sigma}_{nk}} > 0$ for $(n',n'',m) \in \widetilde{N}_M \times \widetilde{N}_k \times \widetilde{K}_n$, only if $\overline{\sigma}_{nk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$.*

(III) *conditions (i)-(iv) in either Proposition 2 and 6 apply and $\frac{\widetilde{\sigma}_{n''k}\widetilde{\sigma}_{n'm}}{\overline{\widetilde{\sigma}}_{nm}}\frac{\sigma_{n''k}\sigma_{n'm}}{\overline{\sigma}_{nm}} > 0$ for $(n',n'',m) \in \left(\widetilde{N}_M \times \widetilde{N}_k \times \widetilde{K}_n\right) \cap (N_M \times N_k \times K_n)$, only if*

$$\frac{\sigma_{n''k}\sigma_{n'm}}{\overline{\sigma}_{nm}}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} > 0$$

Proposition 11 *Let $\mathcal{T} = [0,T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of every $n \in \mathcal{N}$ is given by (4)-(3). Suppose also that both $\widetilde{u}(\cdot)$ and $u(\cdot)$ exhibit CARA. Then,*

---

[26]See Theorem B.3 in Anderson and Raimondo [5]

(I.A) $\overline{\overline{\sigma}}_{nk} \neq 0$, conditions (i)-(iv) in Proposition 3 apply, and $\frac{\sigma_{n'k}\sigma_{n'm}}{\overline{\sigma}_{nm}\overline{\overline{\sigma}}_{nk}} > 0$ for $(n', m) \in N_k \times K_n$, only if $\overline{\overline{\sigma}}_{nk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} < 0$.

(I.B) $\overline{\overline{\sigma}}_{nk} = 0$, conditions (i)-(iv) in Proposition 3 apply, and both functions $\widetilde{u}(\cdot)$ and $a_n(\cdot)$ are analytic on $(0, T) \times \mathbb{R}^K$, only if $\frac{\sigma_{n'k}\sigma_{n'm}}{\overline{\sigma}_{nm}}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} \neq 0$ for $(n', m) \in N_k \times K_n$ a.e. on $[0, T] \times \mathbb{R}^K$.

(II) $\overline{\sigma}_{nk} \neq 0$, conditions (i)-(iv) in Proposition 7 apply, and $\frac{\widetilde{\sigma}_{n'k}\widetilde{\sigma}_{n'm}}{\overline{\widetilde{\sigma}}_{nm}\overline{\sigma}_{nk}} > 0$ for $(n', m) \in \widetilde{N}_k \times \widetilde{K}_n$, only if $\overline{\sigma}_{nk}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} < 0$.

(III) conditions (i)-(iv) in either Proposition 3 and 7 apply and $\frac{\widetilde{\sigma}_{n'k}\widetilde{\sigma}_{n'm}}{\overline{\widetilde{\sigma}}_{nm}}\frac{\sigma_{n'k}\sigma_{n'm}}{\overline{\sigma}_{nm}} > 0$ for $(n', m) \in \left(\widetilde{N}_k \times \widetilde{K}_n\right) \cap (N_k \times K_n)$, only if

$$\frac{\sigma_{n'k}\sigma_{n'm}}{\overline{\sigma}_{nm}}\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} < 0$$

Proposition 12 *Let $\mathcal{T} = [0, T]$ for some $T \in \mathbb{R}_{++}$ and suppose that the dividend process of every $n \in \mathcal{N}$ is given by (4)-(3). Suppose also that both $\widetilde{u}(\cdot)$ and $u(\cdot)$ exhibit CARA while conditions (i)-(ii) in either Proposition 4 and 8 apply. Then, $\frac{\partial p_n(\mathcal{I}(\omega,t))}{\partial\beta_k(\omega,t)} = 0$.*

6   Concluding Remarks

These results contribute towards our ability for economic analysis and prediction, even in the cases when the representative agent's portfolio policy is well-known. To illustrate, let the zero security be a zero-coupon bond (or an annuity in terms of dividend flows) and the agent exhibit CRRA. Suppose also that currently she is investing \$150 (\$1 representing one unit of consumption) in the stock market, of which \$100 are placed on risky securities and the remainder on the bond. Following a negative shock that reduces the risky part of her invested wealth to \$85, it is well known that she will want to adjust her portfolio so that her invested wealth remains split between stocks and bond in the original 2:1 ratio. She will seek, that is, to invest \$90 on stocks and \$45 on the bond. Since the securities are in fixed supply, their prices must adjust but is not clear how. Obviously, the price of at least one stock (since each is in positive net supply) must fall whereas that of the bond (as it is in zero net supply and the agent is risk averse) must rise. But which one is this stock and what happens to the other stocks' relative prices?

The preceding analysis sheds light on questions of this kind by looking at the economic mechanism that determines how the relative price of the typical security responds to such shocks. It highlights two separate channels through which shocks to current wealth affect asset prices: by changing the agent's risk aversion but also altering her perception of the security's "riskiness." The dynamics of the former mechanism are well-known and straightforward. Those of the latter, however, have not hitherto been analyzed to any significant

extent of generality.

Even when the dividend process is the geometric Brownian motion, without jumps, rare events or other irregularities, the asset-price dynamics with respect to the underlying fundamental risk are complex. In fact, they are so to the extent that assertions about the direction of asset-price movements cannot be easily made, except for particular situations, even when the dividend is independent from the risk source under study. To make this point, my strategy has been to find specifications for the economic primitives under which the sign of the correlation between the relative price of the typical security and the typical Brownian motion remains unambiguous throughout the stochastic domain.

By establishing that, as a norm, asset prices are correlated with an underlying risk source even when payoffs are not, my findings indicate that asset-price dynamics are much richer than one is led to expect at first glance, armed with basic economic intuition. By showing, on the other hand, that it is by no means straightforward to identify settings in which the sign of this correlation remains constant, they attest to the complexity of these dynamics. Together, richness and complexity suggest a tumultuous financial world, even in the benchmark model of the present paper. They have also significant implications for empirical asset-pricing. In particular, for the large body of work that focusses on partial-equilibrium analysis, treating a small number of securities in isolation from the rest of the market or modeling the equilibrium price process of an asset as a relation that depends only on those risk sources that directly affect its payoff.

Of course, my results do not extend beyond state-independent utility functions for the representative agent. Yet, within the context of general equilibrium analysis, this restriction should not be taken at face value. One of the reasons that state-dependence appears natural in some models is because they are partial equilibrium studies. If a significant portion of household wealth is held on an asset that is not included in the model, changes in the value of this asset induce wealth effects that alter the agents' willingness to hold those assets the model does include. As a consequence, value changes in the omitted asset seem to be instances of state-dependent felicity. In a general equilibrium model, however, all relevant assets are included by definition. This kind of state-dependence, therefore, would disappear and the fact that the utility function is exogenously specified comes without loss of generality.

In this sense, it is important that my results apply on the entire family of state-independent utility functions that are monotone in risk-aversion. For, as long as the financial market is dynamically complete, they encompass a wide array of economies that have many agents with heterogenous preferences. In fact, the real limitation of my analysis lies in the dividend specification. Even though it has been used widely in continuous-time finance, it does nonetheless constrain the scope of my results given that my proofs, at some point or another, all exploit the symmetry of the normal distribution.

References

[1] Aliprantis C.D., Camera G., and D. Puzzello (2007): "Contagion Equilibria in a Monetary Model," *Econometrica*, **75**:277-82.

[2] Andersen T.G., Bollerslev T., Diebold F.X., and P. Labys (2003): "Modeling and Forecasting Realized Volatility," *Econometrica*, **71**:579-625.

[3] Andersen T.G., Bollerslev T., Diebold F.X., and H. Ebens (2001): "The Distribution of Realized Stock Return Volatility," *Journal of Financial Economics*, **61**:43-76.

[4] Andersen T.G. and J. Lund (1997): "Estimating Continuous-time Stochastic Volatility Models of the Short-term Interest Rate," *Journal of Econometrics*, **77**:343-77.

[5] Anderson R.M. and R.C. Raimondo (2008): "Equilibrium in Continuous-Time Financial Markets: Endogenously Dynamically Complete Markets," *Econometrica*, **76**:841-907.

[6] Anderson R.M. and R.C. Raimondo (2005): "Market Clearing and Derivative Pricing," *Economic Theory*, **25**:21-34.

[7] Bansal R., Khatchatrian V., and A. Yaron (2005): "Interpretable Asset Markets?," *European Economic Review*, **49**:531-60.

[8] Basak S. and D. Cuoco (1998): "An Equilibiurm Model with Restricted Stock Market Participation," *Review of Financial Studies*, **11**:309-41.

[9] Bick A. (1990): "On Viable Diffusion Price Processes of the Market Portfolio," *Journal of Finance*, **45**:673-89.

[10] Bick A. (1987): "On the Consistency of the Black-Scholes Model with a General Equilibrium Framework," *Journal of Financial and Quantitative Analysis* **22**:259-75.

[11] Biger N. and J. Hull (1983): "The Valuation of Currency Options," *Financial Management*, **12**:24-28.

[12] Browne S. (1995): "Optimal Investment Policies for a Firm with a Random Risk Process: Exponential Utility and Minimizing the Probability of Ruin," *Mathematics of OPerations Research*, **20**:937-58.

[13] Candenillas A. and F. Zapatero (1999): "Optimal Central Bank Intervention in the Foreign Exchange Market," *Journal of Economic Theory*, **87**:218-42.

[14] Capozza D.R. and D. Kazarian (1998): "The Conditional Probability of Mortgage Default," *Real Estate Economics*, **26**:259-89.

[15] Carey J.M. and D. Zilberman (2002): "A Model of Investment under Uncertainty: Modern Irrigation Technology and Emerging Markets in Water," *American Journal of Agricultural Economics*, **84**:171-83.

[16] Chateauneuf A., Kast R., and A. Lapied: "Market Preferences Revealed by Prices: Non-linear Pricing in Slack Markets", in Munier B. and M.J. Machina (ed.) *Models and Experiments in Risk and Rationality*, Kluwer Academic Publishers, Dordrecht (1994)

[17] Chen N., Roll R., and S.A. Ross (1986): "Economic Forces and the Stock Market: Testing the APT and Alternative Pricing Theories," *Journal of Business*, **59**:383-403.

[18] Cochrane J.H., Longstaff F.A., and P. Santa-Clara (2008): "Two-Trees," *Review of Financial Studies*, **21**(1):347-85.

[19] Constandinides G.M. and T. Zariphopoulou (1999): "Bounds on Prices of Contingent Claims in an Intertemporal Economy with Proportional Transaction Costs and General Preferences," *Finance and Stochastics*, **3**:345-69.

[20] Cox J.C., Ingersoll J.E. Jr., and S.A. Ross (1985): "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, **53**:363-84.

[21] Deng S-J., Johnson B., and A. Sogomonian (2001): "Exotic Electicty Options and the Valuation of Electicity Generation and Transmission Assets," *Decision Support Systems*, **30**:383-92.

[22] Dumas B. (1989): "Two-Person Dynamic Equilibrium in a Capital Market," *Review of Financial Studies*, **2**:157-88.

[23] Diasakos T.M. (2012): "A Simple Characterization of Dynamic Completeness in Continuous Time," Carlo Alberto Notebook #211.

[24] Duffie D. and C. Skiadas (1994): "Continuous-time Security Pricing: a Utility Gradient Approach," *Journal of Mathematical Economics*, **23**:107-31.

[25] Duffie D. and W. Zaime (1989): "The Consumption-Based CAPM," *Econometrica*, **57**:1279-97.

[26] Epaulard A. and A. Pommeret (2003): "Optimally Eating a Stochastic Cake: a Recursive Utility Approach," *Resource and Energy Economics*, **25**:129-39.

[27] Ericsson J. (2006): "Liquidity and Credit Risk," *Journal of Finance*, **61**:2219-50.

[28] Farhi E. and S. Panageas (2007): "Saving and Investing for Early Retirement: a Theoretical Analysis," *Journal of Financial Economics*, **83**:87-121.

[29] Fleten S-E., Maribu K.M., and I. Wangersteen (2007): "Optimal Investment Strategies in Decentralized Renewable Power Generation under Uncertainty," *Energy*, **32**:803-15.

[30] Forbes K.J. and R. Rigobon (2001): "Contagion in Latin America: Definitions, Measurement, and Policy Implications" *Economia*, **1**:1-46.

[31] Gatheral J. and A. Schied (2011): "Optimal Trade Execution under Geometric Brownian Motion in the Almgren and Chriss Framework," *International Journal of theoretical and Applied Finance* **14**:353-68.

[32] Gerber H.U. and E.S.W. Shiu (2004): "Optimal Dividends: Analysis with Brownian Motion," *North American Actuarial Journal*, **8**:1-20.

[33] Gerber H.U. and E.S.W. Shiu (2003): "Geometric Brownian Motion Models for Assets and Liabilities: From Pension Funding to Optimal Dividends," *North American Actuarial Journal*, **7**:37-103.

[34] Gollier C. (1995): "The Comparative Statics of Risk Revisited," *Journal of Economic Theory*, **66**:522-35.

[35] Gropp R. and G. Moerman (2004): "Measurement of Contagion in Banks' Equity Prices," *Journal of International Money and Finance*, **23**:405-59.

[36] Hadjiliadis O. (2006): "Drawdowns Preceding Rallies in the Brownian Motion Model," *Quantitative Finance*, **6**:403-9.

[37] He H. and H. Leland (1993): "On Equilibrium Asset Price Processes," *Review of Financial Studies* **6**: 593-617.

[38] He Z. (2009): "Optimal Executive Compensation when Firm Size Follows Geometric Brownian Motion," *Review of Financial Studies*, **22**:859-92.

[39] Hugonnier J., Malamud S., and E. Trubowitz: "Endogenous Completeness of Diffusion Driven Equilibrium Markets," *Econometrica*, forthcoming.

[40] Hull J. (2004): "How to Value Employee Stock Options," *Financial Analysts Journal*, **60**:114-19.

[41] Instefjord N. (2005): "Risk and Hedging: Do Credit derivatives Increase Bank Risk?," *Journal of Banking and Finance*, **29**:333-45.

[42] Karatzas I., Lehoczky J.P., and S.E. Shreve (1990): "Existence and Uniquence of Multi-Agent Equilibrium in a Stochastic Dynamic Consumption/Investement Model," *Mathematics of Operations Research*, **15**:80-128.

[43] Kodres L.E. and M. Pritsker (2002): "A Rational Expectations Model of Financial Contagion," *Journal of Finance*, **57**:769-99.

[44] Kyle A.S. and W. Xiong (2001): "Contagion as a Wealth Effect," *Journal of Finance*, **56**:1401-40.

[45] Lagunoff R. and S. L. Schreft (1999): "Financial fragility with Rational and Irrational Exuberance," *Journal of Money, Credit and Banking*, **31**:531-60.

[46] Longin F. and B. Solnik (1995): "Is the Correlation in International Equity Returns Constant: 1960-1990," *Journal of International Money and Finance*, 14:3-26.

[47] Lucas R.E., Jr. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, **46**:1429-45.

[48] Martens M. and D. van Dijk (2007): "Measuring Volatility in the Realized Range," *Journal of Econometrics*, **138**:181-207.

[49] Martin I."The Lucas Orchard" Harvard University, Ph.D. Thesis (2007)

[50] Mella-Baral P. and W. Perraudin (1997): "Strategic Debt Service," *Journal of Finance*, **52**:531-56.

[51] Merton R.C. (1990): "A Complete-Markets General Equilibrium Theory of Finance in Continuous Time" Ch. 16 in R.C. Merton *Continuous-Time Finance*, Basil Blackwell, Cambridge, MA

[52] Merton R.C. (1971): "Optimal Consumption and Portfolio Rules in a Continuous-Time Model," *Journal of Economic Theory*, **3**:373-413.

[53] Milevsky M.A. (1999): "Time Diversification, Safety-First and Risk," *Review of Quantitative Finance and Accounting*, **12**:271-81.

[54] Moskowitz T. (2003): "An Analysis of Covariance Risk and Pricing Anomalies," *Review of Financial Studies*, **16**:417-57.

[55] Oksendal B. and A. Sulem (2002): "Optimal Consumption and Portfolio with both Fixed and Proportional Transaction Costs," *SIAM Journal of Control and Optimization*, **40**:1765-90.

[56] Oren S.S. (1985): "Intergrating Real and Financial Contracts in Demand-side Electricity Contracts," *Decision Support Systems*, **30**:279-88.

[57] Pennings E. (2000): "Taxes and Stimuli under Uncertainty," *European Economic Review*, **44**:383-91.

[58] Postali F.A.S. and P. Picchetti (2006): "Geometric Brownian Motion and structural breaks in oil prices: A quantitative analysia," *Energu Economics*, **28**:506-22.

[59] Promislow S.D. and V.A. Young (2005): "Minimizing the Probability of Ruin when Claims Follow Borwnian Motion with drift," *North American Actuarial Journal*, **9**:109-28.

[60] Maratha R.R. and S.M. Ryan (2007): "On the Validity of the Geometric Brownian Motion Assumption," *The Engineer Economist*, **50**:159-92.

[61] Pindyck R. and J. Rotemberg (1993): "The Comovement of Stock Prices," *Quarterly Journal of Economics*, 108:1073-104.

[62] Raimondo R.C. (2005): "Market Clearing, Utility Functions and Security Prices," *Economic Theory*, **25**:265-85.

[63] Riedel F. (2001): "Existence of Arrow-Radner Equilibrium with Endogenously Complete Markets under Incomplete Information," *Journal of Economic Theory*, **97**:109-22.

[64] Schwert W. G. and P. J. Seguin (1990): "Hetreoschedasticity in Stock Returns," *Journal of Finance*, **45**:1129-55

[65] Schmidli H., (2001): "Optimal Proportional Reinsurance Policies in a Dynamic Setting," *Scandinavian Actuarial Journal*, **1**:55-68.

[66] Shiller R.J.(1989): "Comovements in Stock Prices and Comovements in Dividends," *The Journal of Finance*, **44**:719-29.

[67] Wang S. "The Integrability Problem of Asset Prices," *Journal of Economic Theory* **59**(1993), 199-213.

[68] Wilson R. "The Theory of Syndicates," *Econometrica* **36**(1968), 119-32.

[69] Wong K.P. (2007): "The Efect of Uncertainty on Investment Timing in a Real Options Model," *Journal of Economic Dynamics and Control*, **31**:2152-67.

Appendices

A   Preliminaries

The following result is borrowed from Diasakos [23] (see Lemma 4 in his Appendix) while its antecedent is well-known.

Lemma A.1 *Let $S \subseteq \mathbb{R}^K$ be of non-zero Lebesgue measure. Suppose also that the functions $f : S^2 \mapsto \mathbb{R}_+$ and $g : S^2 \mapsto \mathbb{R}$ are such that*

  (i) $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$ a.e. on S^2,[27]

  (ii) $h(\mathbf{x}, \mathbf{y}) + h(\mathbf{y}, \mathbf{x}) \geq 0$ a.e. on S^2, and

---

[27]As usual, almost everywhere is meant to indicate validity throughout S^2 modulo subsets of Lebesgue-measure zero.

33

(iii) $(fh)(\cdot)$ *is Lebesgue-integrable over* S^2.

Then

$$\int_{S^2} f(\mathbf{x}, \mathbf{y}) \, h(\mathbf{x}, \mathbf{y}) \, d(\mathbf{x}, \mathbf{y}) \geq 0$$

with strict inequality iff $f(\mathbf{x}, \mathbf{y}) [h(\mathbf{x}, \mathbf{y}) + h(\mathbf{y}, \mathbf{x})] \neq 0$ *on a subset of* S^2 *of positive Lebesgue measure.*

Lemma A.2 *Let* $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, I_K)$, $\theta \in \mathbb{R}^K$, *and* $h : \mathbb{R}^K \to \mathbb{R}$ *s.t.* $\mathbb{E}_{\mathbf{x}}\left[e^{\theta^{\mathsf{T}}\mathbf{x}} h(\mathbf{x})\right]$ *is well-defined. Then* $\mathbb{E}_{\mathbf{x}}\left[e^{\theta^{\mathsf{T}}\mathbf{x}} h(\mathbf{x})\right] = e^{\frac{\theta^{\mathsf{T}}\theta}{2}} \mathbb{E}_{\mathbf{x}}\left[h(\mathbf{x} + \theta)\right]$.

Lemma A.3 *Let the random vector* $\mathbf{x} \in \mathbb{R}^K$ *and the function* $g : \mathbb{R}^K \mapsto \mathbb{R}$ *be s.t.* $\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})]$ *and* $\mathbb{E}_{\mathbf{x}}[x_k g(\mathbf{x})]$ *are well-defined, with* $\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] \neq 0$. *Suppose also that* $f : \mathbb{R} \mapsto \mathbb{R}$ *is given by* $f(y_k) = \mathbb{E}_{\mathbf{x}}[(y_k - x_k) g(\mathbf{x})]$. *Then,*

$$\exists y_k^* \in \mathbb{R} : \ (y_k - y_k^*) f(y_k) \mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] > 0 \quad \forall y_k \in \mathbb{R} \setminus \{y_k^*\}$$

Proof. Given that $\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] \neq 0$, we can write

$$f(y_k) = \mathbb{E}_{\mathbf{x}}[g(\mathbf{x})] \left(y_k - \frac{\mathbb{E}_{\mathbf{x}}[x_k g(\mathbf{x})]}{\mathbb{E}_{\mathbf{x}}[g(\mathbf{x})]} \right)$$

and it suffices to define $y_k^* = \mathbb{E}_{\mathbf{x}}[x_k g(\mathbf{x})] / \mathbb{E}_{\mathbf{x}}[g(\mathbf{x})]$. $\blacksquare$

B Comonotonicity and Covariance

For a set S and an algebra σ on S, let $B(S, \mathbb{R})$ be the set of bounded σ-measurable functions $S \mapsto \mathbb{R}$. Two random variables $f, g \in B(S, \mathbb{R})$ are said to be <u>comonotonic</u> if

$$[f(\mathbf{x}) - f(\mathbf{y})][g(\mathbf{x}) - g(\mathbf{y})] \geq 0 \quad \forall \mathbf{x}, \mathbf{y} \in S$$

and <u>strictly comonotonic</u> if this inequality is strict whenever $\mathbf{x} \neq \mathbf{y}$. The following result is borrowed from Chateauneuf et al. [16]. I present the relevant for my argument "only if" part of the proof.

Lemma B.1 *Let* $f, g \in B(S, \mathbb{R})$. *The following are equivalent.*

f, g are (strictly) comonotonic.

$Cov_\pi[f, g] (>) \geq 0$ for any prob. measure π on (S, σ).

Proof. Let $f, g \in B(S, \mathbb{R})$ be comonotonic and π a probability measure on (S, σ). Then, we have

$$
\begin{aligned}
2\mathrm{Cov}_\pi [f, g] &= 2\left(\mathbb{E}_\pi [fg] - \mathbb{E}_\pi [f]\mathbb{E}_\pi [g]\right) \\
&= 2\left(\int_S f(\mathbf{x}) g(\mathbf{x}) \, \mathrm{d}\pi(\mathbf{x}) - \int_S f(\mathbf{y}) \, \mathrm{d}\pi(\mathbf{y}) \int_S g(\mathbf{x}) \, \mathrm{d}\pi(\mathbf{x})\right) \\
&= \int_S f(\mathbf{x}) g(\mathbf{x}) \, \mathrm{d}\pi(\mathbf{x}) + \int_S f(\mathbf{y}) g(\mathbf{y}) \, \mathrm{d}\pi(\mathbf{y}) \\
&\quad - \int_S f(\mathbf{y}) \, \mathrm{d}\pi(\mathbf{y}) \int_S g(\mathbf{x}) \, \mathrm{d}\pi(\mathbf{x}) - \int_S f(\mathbf{x}) \, \mathrm{d}\pi(\mathbf{x}) \int_S g(\mathbf{y}) \, \mathrm{d}\pi(\mathbf{y}) \\
&= \int_{S \times S} [f(\mathbf{x}) - f(\mathbf{y})] [g(\mathbf{x}) - g(\mathbf{y})] \, \mathrm{d}\pi(\mathbf{x}) \, \mathrm{d}\pi(\mathbf{y}) \geq 0
\end{aligned}
$$

where the third equality is due to a change in the variables of integration. The validity of the claim when the comonotonicity is strict is obvious. ∎

Regarding the application of this result in the main text, notice that f and g need not be bounded there. The boundedness condition guarantees that the integrals above exist for *any* prob. measure π on (S, σ). In the analysis of the asset-riskness effect, I fix $((\omega, t), \mathbf{y}) \in \Omega \times \mathcal{T} \times \mathbb{R}^{K-M}$ and take $g = G_n$ as well as $f : \mathbb{R}^M \mapsto \mathbb{R}_{--}$ with $f(\mathbf{z}) = u''(W(\mathcal{I}(\omega, t), \mathbf{z}, \mathbf{y}))$ for $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, I_M)$. The relevant expectations are well-defined even though $G_n(\cdot)$ and $u''(\cdot)$ are, respectively, not and not necessarily bounded. The strict comonotonicity between them is due to non-increasing absolute risk aversion. This implies that $u'''(\cdot) > 0$ and the result follows since, other things being equal, the terminal wealth in (7) is strictly increasing in the realization of the nth dividend.

C   Proofs of the Results in the Text

This section presents the proofs for the various results in the paper. To keep notation simple, whenever possible, I will only display the Brownian increment $\beta(\omega, T) - \beta(\omega, t)$ part of the process $\mathcal{I}(\omega, T)$ in the relevant functional arguments. In addition, even though not shown again for notational parsimony, all expectations are supposed to be conditional on the current filtration $\mathcal{F}_t$.

Proof of Proposition 1

By Claim 1 in Diasakos [23], for any $(n, \mathbf{v}) \in \mathcal{N} \setminus \{0\} \times \mathbb{R}^K$, we have

$$
\begin{aligned}
&P_0 (\mathcal{I}(\omega, t))^2 \sum_{k=1}^{k} v_k \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} \\
&= \mathbb{E}_{(\mathbf{x}, \mathbf{y})} \left[\begin{array}{c} f_0\left(T, \beta(\omega, t) + \sqrt{T-t}\mathbf{x}\right) f_0\left(T, \beta(\omega, t) + \sqrt{T-t}\mathbf{y}\right) \\ \mathbf{v}^\mathsf{T}(\mathbf{x} - \mathbf{y}) \overline{G}_n\left(T, \beta(\omega, t) + \sqrt{T-t}\mathbf{x}\right) \end{array} \right]
\end{aligned} \tag{21}
$$

where the functions $\overline{G}_n, f_0 : \mathcal{I}(\omega, t) \times \mathbb{R}^K \mapsto \mathbb{R}_+$ are given by $\overline{G}_n(\cdot) = G_n(\cdot)/G_0(\cdot)$ and $f_0(\cdot) = u'(W(\cdot))G_0(\cdot)$.[28]

For any $\sigma \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, however, the family $\left\{H_{(\rho,\sigma)} = \left\{\mathbf{x} \in \mathbb{R}^K : \sigma^\mathsf{T}\mathbf{x} = \rho\right\}\right\}_{\rho \in \mathbb{R}}$ of hyperplanes spans the space $\mathbb{R}^K$. Hence, given $\mathbf{v} \in \mathbb{R}^K \setminus \{\mathbf{0}\}$, we may write

$$
\begin{aligned}
H_{(0,\mathbf{v})} &= \cup_{\rho \in \mathbb{R}} H_{(\rho,\sigma)} \cap H_{(0,\mathbf{v})} \\
&= \cup_{\rho \in \mathbb{R}} \left\{\mathbf{x} \in \mathbb{R}^K : \begin{pmatrix} \sigma^\mathsf{T}\mathbf{x} \\ \mathbf{v}^\mathsf{T}\mathbf{x} \end{pmatrix} = \begin{pmatrix} \rho \\ 0 \end{pmatrix}\right\} := \cup_{\rho \in \mathbb{R}} H_\rho^{(\sigma,\mathbf{v})}
\end{aligned}
$$

Consider, moreover, the line $L(\widetilde{\mathbf{x}}, \mathbf{v}) = \left\{\mathbf{x} \in \mathbb{R}^K : \mathbf{x} = \widetilde{\mathbf{x}} + r\mathbf{v},\ r \in \mathbb{R}\right\}$ passing through $\widetilde{\mathbf{x}} \in H_{(0,\mathbf{v})}$ and parallel to $\mathbf{v}$ (see Figure 1). Since $\mathbf{v}$ and $H_{(0,\mathbf{v})}$ are not collinear, we have[29]
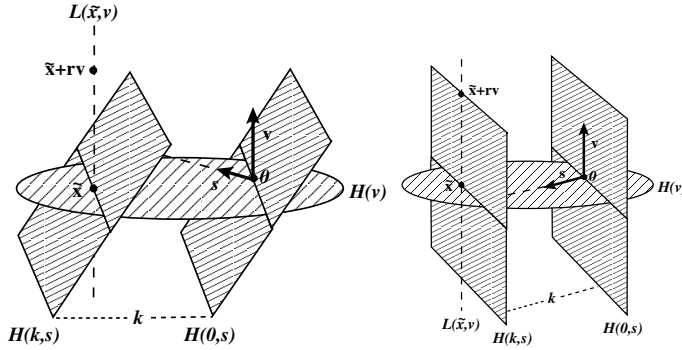

Figure 1: Spanning the space using $\mathbf{v}$ and σ

$$
\mathbb{R}^K = \cup_{\widetilde{\mathbf{x}} \in H_{(0,\mathbf{v})}} L(\widetilde{\mathbf{x}}, \mathbf{v}) = \cup_{\rho \in \mathbb{R}} \cup_{\widetilde{\mathbf{x}} \in H_\rho^{(\sigma,\mathbf{v})}} L(\widetilde{\mathbf{x}}, \mathbf{v})
$$

For any $\mathbf{x} \in L(\widetilde{\mathbf{x}}, \mathbf{v})$, however, the fact $\widetilde{\mathbf{x}} \in H_\rho^{(\sigma,\mathbf{v})} \subseteq H_{(0,\mathbf{v})}$ implies that $\mathbf{v}^\mathsf{T}\mathbf{x} = \mathbf{v}^\mathsf{T}\widetilde{\mathbf{x}} + r\mathbf{v}^\mathsf{T}\mathbf{v} = r\mathbf{v}^\mathsf{T}\mathbf{v}$ while $\mathbf{x}^\mathsf{T}\mathbf{x} = \widetilde{\mathbf{x}}^\mathsf{T}\widetilde{\mathbf{x}} + r^2\mathbf{v}^\mathsf{T}\mathbf{v} + 2r\widetilde{\mathbf{x}}^\mathsf{T}\mathbf{v} = \widetilde{\mathbf{x}}^\mathsf{T}\widetilde{\mathbf{x}} + r^2\mathbf{v}^\mathsf{T}\mathbf{v}$. Hence, (21) gives (suppressing the non-relevant functional arguments)

$$
\frac{P_0^2}{\mathbf{v}^\mathsf{T}\mathbf{v}} \sum_{k=1}^K v_k \frac{\partial p_n(\mathcal{I}(\omega, t))}{\partial \beta_k(\omega, t)} = \int_{\mathbb{R}^2} S(\rho, \rho')\, \mathrm{d}\rho \mathrm{d}\rho' \tag{22}
$$

with $S : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$
S(\rho, \rho') = \int_{\left(H_\rho^{(\sigma,\mathbf{v})} \times H_{\rho'}^{(\sigma,\mathbf{v})}\right) \cap V_0(\omega,t)^2} \int_{\mathbb{R}^2} (r - r')\, \overline{G}_n(\widetilde{\mathbf{x}} + r\mathbf{v}) F_0(\widetilde{\mathbf{x}}, r) F_0(\widetilde{\mathbf{y}}, r)\, \mathrm{d}r \mathrm{d}r' \mathrm{d}\widetilde{\mathbf{x}} \mathrm{d}\widetilde{\mathbf{y}}
$$

---

[28]Regarding Claim 1 in [23], observe that, under the specification in (4), the set $V_0(T, \mathcal{I}(\omega, t)) = \{\mathbf{x} \in \mathbb{R}^K : G_0(T, \beta(\omega, t) + \sqrt{T - t}\mathbf{x}) \neq 0\}$ covers the entire space $\mathbb{R}^K$.

[29]Let $\{\mathbf{v}_k\}_{k=1}^{K-1}$ be a basis for $H_{(0,\mathbf{v})}$. As the hyperplane is not collinear with $\mathbf{v}$, it follows that $\{\mathbf{v}, \mathbf{v}_1, \ldots, \mathbf{v}_{K-1}\}$ is a basis of $\mathbb{R}^K$. That is, any $\mathbf{x} \in \mathbb{R}^K$ can be written uniquely as $\mathbf{x} = \sum_{k=1}^{K-1} r_k \mathbf{v}_k + r\mathbf{v}$ for some $(r, r_1, \ldots, r_{K-1}) \in \mathbb{R}^K$. Equivalently, $\mathbf{x} = \widetilde{\mathbf{x}} + r\mathbf{v}$ for a unique $\widetilde{\mathbf{x}} = \sum_{k=1}^{K-1} r_k \mathbf{v}_k \in H_{(0,\mathbf{v})}$.
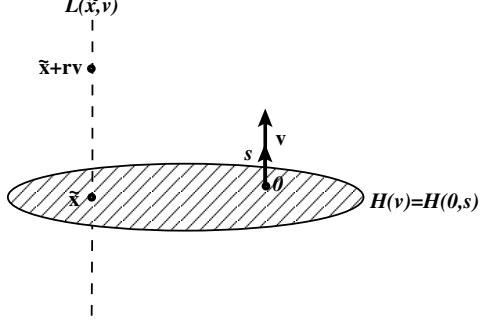
Figure 2: Spanning the space when $\mathbf{v}$ and σ are parallel

and, given $\mathcal{I}(\omega, t)$, $F_0 : \mathbb{R}^{k+1} \mapsto \mathbb{R}_{++}$ defined as

$$F_0\left(\widetilde{\mathbf{x}}, r\right) = (2\pi)^{-K/2} f_0\left(\widetilde{\mathbf{x}} + r\mathbf{v}\right) e^{-\frac{\widetilde{\mathbf{x}}^\mathsf{T}\widetilde{\mathbf{x}} + r^2 \mathbf{v}^\mathsf{T}\mathbf{v}}{2}}$$

It should be noted, of course, that the above relation applies in general but for when σ and $\mathbf{v}$ are collinear. In this case, the hyperplanes $H_{(0,\mathbf{v})}$, $H_{(0,\sigma)}$, and $H_0^{(\sigma,\mathbf{v})}$ all coincide (see Figure 2) so that

$$\mathbb{R}^K = \cup_{\widetilde{\mathbf{x}} \in H_0^{(\sigma,\mathbf{v})}} L\left(\widetilde{\mathbf{x}}, \mathbf{v}\right)$$

and, thus,

$$\frac{P_0^2}{\mathbf{v}^\mathsf{T}\mathbf{v}} \sum_{k=1}^{K} v_k \frac{\partial p_n\left(\mathcal{I}(\omega,t)\right)}{\partial \beta_k(\omega,t)} = S(0,0) \tag{23}$$

$$= \int_{\left(H_{(0,\sigma)} \cap V_0(\omega,t)\right)^2} \int_{\mathbb{R}^2} \left(r - r'\right) \overline{G}_n\left(\widetilde{\mathbf{x}} + r\mathbf{v}\right) F_0\left(\widetilde{\mathbf{x}}, r\right) F_0\left(\widetilde{\mathbf{y}}, r\right) \mathrm{d}r\mathrm{d}r'\mathrm{d}\widetilde{\mathbf{x}}\mathrm{d}\widetilde{\mathbf{y}}$$

Yet, the typical relative dividend is given as

$$\overline{G}_n\left(\mathcal{I}(\omega,T)\right) = \overline{A}_n(T) e^{\overline{\sigma}_n^\mathsf{T} \beta(\omega,T)} \quad n \in \mathcal{N} \setminus \{0\}$$

where $\overline{A}_n(\cdot) := A_n(\cdot)/A_0(\cdot)$. At any $(\rho, \rho') \in \mathbb{R}^2$, therefore, we have

$$\frac{S(\rho, \rho')}{\overline{A}_n(T) e^{\overline{\sigma}_n^\mathsf{T} \beta(\omega,t)}}$$

$$= \int_{H_\rho^{(\sigma,\mathbf{v})} \times H_{\rho'}^{(\sigma,\mathbf{v})}} \int_{\mathbb{R}^2} \left(r - r'\right) e^{\overline{\sigma}_n^\mathsf{T}(\widetilde{\mathbf{x}} + r\mathbf{v})} F_0\left(\widetilde{\mathbf{x}}, r\right) F_0\left(\widetilde{\mathbf{y}}, r\right) \mathrm{d}r\mathrm{d}r'\mathrm{d}\widetilde{\mathbf{x}}\mathrm{d}\widetilde{\mathbf{y}}$$

$$= e^\rho \int_{H_\rho^{(\sigma,\mathbf{v})} \times H_{\rho'}^{(\sigma,\mathbf{v})}} \int_{\mathbb{R}^2} \left(r - r'\right) e^{r\overline{\sigma}_n^\mathsf{T}\mathbf{v}} F_0\left(\widetilde{\mathbf{x}}, r\right) F_0\left(\widetilde{\mathbf{y}}, r\right) \mathrm{d}r\mathrm{d}r'\mathrm{d}\widetilde{\mathbf{x}}\mathrm{d}\widetilde{\mathbf{y}}$$

37

Let now $\mathbf{v} = (\sigma_{nk} - \sigma_{0k})\,\mathbf{e}_k$. Since $\bar{\sigma}_n^{\mathsf{T}}\mathbf{v} = (\sigma_{nk} - \sigma_{0k})^2$, the last integrand above reads $e^{(\sigma_{nk}-\sigma_{0k})^2 r}\,(r - r')\,F_0\,(\widetilde{\mathbf{x}}, r)\,F_0\,(\widetilde{\mathbf{y}}, r')$. And as

$$e^{(\sigma_{nk}-\sigma_{0k})^2 r}r\,(r - r')\,F_0\,(\widetilde{\mathbf{x}}, r)\,F_0\,(\widetilde{\mathbf{y}}, r') + e^{(\sigma_{nk}-\sigma_{0k})^2 r'}r'\,(r' - r)\,F_0\,(\widetilde{\mathbf{y}}, r')\,F_0\,(\widetilde{\mathbf{x}}, r)$$
$$= \left(e^{(\sigma_{nk}-\sigma_{0k})^2 r} - e^{(\sigma_{nk}-\sigma_{0k})^2 r'}\right)(r - r')\,F_0\,(\widetilde{\mathbf{x}}, r)\,F_0\,(\widetilde{\mathbf{y}}, r')$$

is strictly positive everywhere on $\mathbb{R}^{2K} \times \mathbb{R}^2$ but for $\mathbb{R}^{2K} \times \{(r, r') \in \mathbb{R}^2 : r = r'\}$, a null-set, Lemma A.1 implies that $S\,(\rho, \rho') > 0\ \forall\,(\rho, \rho') \in \mathbb{R}^2$. In either of the two possible cases depicted by (22)-(23), the required result follows immediately from (21).

Proofs of Propositions 2-4

Letting $\mathbf{v} = \mathbf{e}_k$ in (21) gives

$$\frac{\partial p_n\,(\mathcal{I}\,(\omega, t))}{\partial \beta_k\,(\omega, t)} = \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[\begin{array}{c} u'\,(T, \beta\,(\omega, t) + \sqrt{T - t}\mathbf{x})\,u'\,(T, \beta\,(\omega, t) + \sqrt{T - t}\mathbf{y}) \\ (y_k - x_k) \\ G_n\,(T, \beta\,(\omega, t) + \sqrt{T - t}\mathbf{y})\,G_0\,(T, \beta\,(\omega, t) + \sqrt{T - t}\mathbf{x}) \end{array}\right]$$

Observe, however, that

$$\mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'\,(W\,(\mathbf{y}))\,G_0\,(\mathbf{y})\,(y_k - x_k)\,u'\,(W\,(\mathbf{x}))\,G_0\,(\mathbf{x})\right] = 0$$

which, after renaming the variables $\mathbf{y}_M \in \mathbb{R}^M$, can be written also as

$$\mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{(\mathbf{z}_M, y_k)}\left[\begin{array}{c} u'\,(W\,(\mathbf{y}_{-M}, \mathbf{z}_M))\,G_0\,(\mathbf{y}_{-M}, \mathbf{z}_M) \\ \mathbb{E}_{\mathbf{x}}\,[(y_k - x_k)\,u'\,(W\,(\mathbf{x}))\,G_0\,(\mathbf{x})] \end{array}\,|\mathbf{y}_{-(M,k)}\right]\right] = 0 \tag{24}$$

Hence,

$$\frac{\partial p_n\,(\mathcal{I}\,(\omega, t))}{\partial \beta_k\,(\omega, t)}$$
$$= \mathbb{E}_{(\mathbf{x},\mathbf{y})}\left[u'\,(W\,(\mathbf{y}))\,G_n\,(\mathbf{y})\,(y_k - x_k)\,u'\,(W\,(\mathbf{x}))\,G_0\,(\mathbf{x})\right]$$
$$\quad - \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{(\mathbf{z}_M, y_k)}\left[\begin{array}{c} u'\,(W\,(\mathbf{y}_{-M}, \mathbf{z}_M))\,G_0\,(\mathbf{y}_{-M}, \mathbf{z}_M) \\ \mathbb{E}_{\mathbf{x}}\,[(y_k - x_k)\,u'\,(W\,(\mathbf{x}))\,G_0\,(\mathbf{x})] \end{array}\,|\mathbf{y}_{-(M,k)}\right]\right]$$
$$= \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\left(\begin{array}{c} \mathbb{E}_{\mathbf{y}_M}\,[u'\,(W\,(\mathbf{y}))\,G_n\,(\mathbf{y})\,|\mathbf{y}_{-(M,k)}] \\ -\mathbb{E}_{\mathbf{z}_M}\,[u'\,(W\,(\mathbf{y}_{-M}, \mathbf{z}_M))\,G_0\,(\mathbf{y}_{-M}, \mathbf{z}_M)\,|\mathbf{y}_{-(M,k)}] \\ \mathbb{E}_{\mathbf{x}}\,[(y_k - x_k)\,u'\,(W\,(\mathbf{x}))\,G_0\,(\mathbf{x})\,|y_k] \end{array}\right)\right]\right]$$
$$= \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\begin{array}{c} \left(\frac{\mathbb{E}_{\mathbf{y}_M}[u'(W(\mathbf{y}))G_n(\mathbf{y})|\mathbf{y}_{-(M,k)}]}{\mathbb{E}_{\mathbf{z}_M}[u'(W(\mathbf{y}_{-M},\mathbf{z}_M))G_0(\mathbf{y}_{-M},\mathbf{z}_M)|\mathbf{y}_{-(M,k)}]} - 1\right) \\ \mathbb{E}_{\mathbf{z}_M}\,[u'\,(W\,(\mathbf{y}_{-M}, \mathbf{z}_M))\,G_0\,(\mathbf{y}_{-M}, \mathbf{z}_M)\,|\mathbf{y}_{-(M,k)}] \\ \mathbb{E}_{\mathbf{x}}\,[(y_k - x_k)\,u'\,(W\,(\mathbf{x}))\,G_0\,(\mathbf{x})\,|y_k] \end{array}\right]\right]$$

38

Recall, however, the dividend specification in (4). Clearly, the derivative of interest is proportional to the quantity

$$
\begin{aligned}
\delta_{nk} &= \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\left(\frac{e^{\sqrt{T-t}\sigma_n^{\mathsf{T}}\mathbf{y}_{-(M,k)}}\mathbb{E}_{\mathbf{y}_M}\left[u'(W(\mathbf{y}))e^{\sqrt{T-t}\sigma_n^{\mathsf{T}}\mathbf{y}_M}|\mathbf{y}_{-(M,k)}\right]}{e^{\sqrt{T-t}\sigma_0^{\mathsf{T}}\mathbf{y}_{-(M,k)}}\mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M},\mathbf{z}_M))e^{\sqrt{T-t}\sigma_0^{\mathsf{T}}\mathbf{z}_M}|\mathbf{y}_{-(M,k)}\right]}-1\right)\frac{\mathbb{E}_{\mathbf{z}_M}\left[u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)G_0\left(\mathbf{y}_{-M},\mathbf{z}_M\right)|\mathbf{y}_{-(M,k)}\right]}{\mathbb{E}_{\mathbf{x}}\left[(y_k-x_k)\,u'\left(W\left(\mathbf{x}\right)\right)G_0\left(\mathbf{x}\right)|y_k\right]}\right]\right]\\
&= \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\left(\frac{e^{\sqrt{T-t}\left(\sigma_n^{\mathsf{T}}\mathbf{y}_{-(M,k)}-\sigma_0^{\mathsf{T}}\mathbf{z}_{-(M,k)}\right)}\mathbb{E}_{\mathbf{y}_M}\left[u'(W(\mathbf{y}))e^{\sqrt{T-t}\sigma_n^{\mathsf{T}}\mathbf{y}_M}|\mathbf{y}_{-(M,k)}\right]}{\mathbb{E}_{\mathbf{z}_M}\left[u'(W(\mathbf{y}_{-M},\mathbf{z}_M))e^{\sqrt{T-t}\sigma_0^{\mathsf{T}}\mathbf{z}_M}|\mathbf{y}_{-(M,k)}\right]}-1\right)\frac{\mathbb{E}_{\mathbf{z}_M}\left[u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)G_0\left(\mathbf{y}_{-M},\mathbf{z}_M\right)|\mathbf{y}_{-(M,k)}\right]}{\mathbb{E}_{\mathbf{x}}\left[(y_k-x_k)\,u'\left(W\left(\mathbf{x}\right)\right)G_0\left(\mathbf{x}\right)|y_k\right]}\right]\right]
\end{aligned}
$$

the first equality using that $\sigma_{nk}=0=\sigma_{0k}$.[30]

Proposition 3

Fix an arbitrary point $\mathbf{y}_{-(M,k)}\in\mathbb{R}^{K-M-1}$. I will show that the function $g:\mathbb{R}\mapsto\mathbb{R}$ given by

$$
g\left(y_k\right)=\frac{e^{\sqrt{T-t}\left(\sigma_n^{\mathsf{T}}\mathbf{y}_{-(M,k)}-\sigma_0^{\mathsf{T}}\mathbf{z}_{-(M,k)}\right)}\mathbb{E}_{\mathbf{y}_M}\left[u'\left(W\left(\mathbf{y}\right)\right)e^{\sqrt{T-t}\sigma_n^{\mathsf{T}}\mathbf{y}_M}|\mathbf{y}_{-(M,k)}\right]}{\mathbb{E}_{\mathbf{z}_M}\left[u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)e^{\sqrt{T-t}\sigma_0^{\mathsf{T}}\mathbf{z}_M}|\mathbf{y}_{-(M,k)}\right]}-1
$$

is monotone under the conditions of the proposition. To this end, consider any $y_k\in\mathbb{R}$. Clearly, $g'\left(y_k\right)$ has the same sign as the quantity

$$
\begin{aligned}
I\left(y_k\right) &= \mathbb{E}_{(\mathbf{y}_M,\mathbf{z}_M)}\left[\left(e^{\sqrt{T-t}\sum_{m\in K_n}(\sigma_{nm}y_m+\sigma_{0m}z_m)}\begin{array}{c}u''\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)\frac{\partial W(\mathbf{y})}{\partial y_k}\\-u'\left(W\left(\mathbf{y}\right)\right)u''\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)\frac{\partial W(\mathbf{y}_{-M},\mathbf{z}_M)}{\partial y_k}\end{array}\right)|\mathbf{y}_{-(M,k)}\right]\\
&= r_A\mathbb{E}_{(\mathbf{y}_M,\mathbf{z}_M)}\left[e^{\sqrt{T-t}\sum_{m\in K_n}(\sigma_{nm}y_m+\sigma_{0m}z_m)}u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)\left(\frac{\partial W(\mathbf{y}_{-M},\mathbf{z}_M)}{\partial y_k}-\frac{\partial W(\mathbf{y})}{\partial y_k}\right)|\mathbf{y}_{-(M,k)}\right]
\end{aligned}
$$

Recall now the wealth specification in (18). We must have

$$
\frac{\partial W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)}{\partial y_k}
$$

$$
=\sqrt{T-t}\sum_{n'\in N_k}\sigma_{n'k}e^{\mu_{n'}T+\sigma_{n'}\beta+\sqrt{T-t}\left(\sum_{m\in K_n}\sigma_{n'm}z_m+\sum_{k'\in\left(K_{n'}\setminus K_n\right)\cup\{k\}}\sigma_{n'k'}y_{k'}\right)}
$$

---

[30]In fact, $\sigma_{nk}=\sigma_{0k}$ is what we really need here. And one may well use the latter restriction to state condition (i) of Proposition 3. For nothing would change in the proof that follows if we were to replace $\sum_{m\in K_n}$ by $\sum_{m\in K_n\setminus\{k\}}$.

where $K_n \subseteq K_{n'}$, if $\sigma_{n'k} \neq 0$, is due to condition (ii). That is,

$$I\left(y_k\right) = \sqrt{T-t}\, tr_A \sum_{n' \in N_k} \left(\begin{array}{c} \sigma_{n'k} e^{\mu_{n'}T + \sigma_{n'}\beta + \sqrt{T-t}\sum_{k' \in \left(K_{n'}\backslash K_n\right)\cup\{k\}} \sigma_{n'k'}y_{k'}} \\ \mathbb{E}_{\left(\mathbf{y}_M,\mathbf{z}_M\right)}\left[u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)h_{n'}\left(\mathbf{y}_M,\mathbf{z}_M\right)|\mathbf{y}_{-(M,k)}\right] \end{array} \right)$$

with $h_{n'} : \mathbb{R}^{2M} \mapsto \mathbb{R}$ defined by

$$
\begin{aligned}
h_{n'}\left(\mathbf{y}_M,\mathbf{z}_M\right) &= e^{\sqrt{T-t}\sum_{m \in K_n}\left(\sigma_{nm}y_m + \sigma_{0m}z_m\right)} \\
&\quad \left(e^{\sqrt{T-t}\sum_{m \in K_n}\sigma_{n'm}z_m} - e^{\sqrt{T-t}\sum_{m \in K_n}\sigma_{n'm}y_m}\right) \\
&= e^{\sqrt{T-t}\sum_{m \in K_n}\overline{\sigma}_{nm}y_m} e^{\sqrt{T-t}\sum_{m \in K_n}\sigma_{0m}\left(y_m + z_m\right)} \\
&\quad \left(e^{\sqrt{T-t}\lambda_{n'}\sum_{m \in K_n}\overline{\sigma}_{nm}z_m} - e^{\sqrt{T-t}\lambda_{n'}\sum_{m \in K_n}\overline{\sigma}_{nm}y_m}\right)
\end{aligned}
$$

the last equality due to condition (iii). Observe now that

$$
\begin{aligned}
&h_{n'}\left(\mathbf{y}_M,\mathbf{z}_M\right) + h_{n'}\left(\mathbf{z}_M,\mathbf{y}_M\right) \\
={}& e^{\sqrt{T-t}\sum_{m \in K_n}\sigma_{0m}\left(y_m + z_m\right)} \\
&\left(e^{\sqrt{T-t}\sum_{m \in K_n}\overline{\sigma}_{nm}y_m} - e^{\sqrt{T-t}\sum_{m \in K_n}\overline{\sigma}_{nm}z_m}\right) \\
&\left(e^{\sqrt{T-t}\lambda_{n'}\sum_{m \in K_n}\overline{\sigma}_{nm}z_m} - e^{\sqrt{T-t}\lambda_{n'}\sum_{m \in K_n}\overline{\sigma}_{nm}y_m}\right)
\end{aligned}
$$

Clearly, $\lambda_{n'} > 0$ for the typical $n' \in N_M$ implies $h_{n'}\left(\mathbf{y}_M,\mathbf{z}_M\right) + h_{n'}\left(\mathbf{z}_M,\mathbf{y}_M\right) \leq 0$ on $\mathbb{R}^{2M}$, with equality only on the zero-measure subset $\left\{\left(\mathbf{y}_M,\mathbf{z}_M\right) \in \mathbb{R}^{2M} : \sum_{m \in K_n}\sigma_{nk'}\left(y_{k'} - z_{k'}\right) = 0\right\}$. In this case, Lemma A.1 ensures that the corresponding expectation in the sum of $I\left(y_k\right)$ is negative.[31] By contrast, it is positive if $\lambda_{n'} < 0$. It follows then that the typical term in the sum of $I\left(y_k\right)$ has the opposite sign from the quantity $\lambda_{n'}\sigma_{n'k}$. To sign $I\left(y_k\right)$, therefore, it suffices that the latter quantity maintains the same sign across N_M. This is guaranteed by condition (iv) while condition (ii), by implying that $N_M \subseteq N_k$, ensures it as meaningful statement.

Next, define the function $g^* : \mathbb{R} \mapsto \mathbb{R}$ by

$$g^*\left(y_k\right) = \mathbb{E}_{\mathbf{z}_M}\left[u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)G_0\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right]\mathbb{E}_{\mathbf{x}}\left[\left(y_k - x_k\right)u'\left(W\left(\mathbf{x}\right)\right)G_0\left(\mathbf{x}\right)\right]$$

Since $u'\left(\cdot\right) > 0$, by Lemma A.3, there exists some $y_k^* \in \mathbb{R}$ with $\left(y_k - y_k^*\right)g^*\left(y_k\right) > 0$ $\forall y_k \in \mathbb{R} \backslash \{y_k^*\}$. Let now $\lambda_{n'}\sigma_{n'k} > 0$ for the typical $n' \in N_M \cap N_k$. Since $\lambda_{n'}\sigma_{n'k}g\left(\cdot\right)$ is

---

[31]To use the lemma here, let $g := h_{n'}$ and define $f : \mathbb{R}^{2M} \mapsto \mathbb{R}_{++}$ by $f\left(\mathbf{y}_M,\mathbf{z}_M\right) = u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_M\right)\right)e^{-\frac{\mathbf{y}_M\mathbf{y}_M + \mathbf{z}_M\mathbf{z}_M}{2}}$.

strictly decreasing on $\mathbb{R}$, we have

$$\mathbb{E}_{y_k}\left[\lambda_{n'}\sigma_{n'k}g\left(y_k\right)g^*\left(y_k\right)\right]$$

$$< \int_{-\infty}^{y_k^0}\lambda_{n'}\sigma_{n'k}g\left(y_k^*\right)g^*\left(y_k\right)\mathrm{d}\Phi\left(y_k\right) + \int_{y_k^*}^{+\infty}\lambda_{n'}\sigma_{n'k}g\left(y_k^*\right)g^*\left(y_k\right)\mathrm{d}\Phi\left(y_k\right)$$

$$= \lambda_{n'}\sigma_{n'k}g\left(y_k^*\right)\mathbb{E}_{y_k}\left[g^*\left(y_k\right)\right]$$

and, thus,

$$\lambda_{n'}\sigma_{n'k}\delta_{nk} = \mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[\lambda_{n'}\sigma_{n'k}g\left(y_k\right)g^*\left(y_k\right)\right]\right]$$

$$< \lambda_{n'}\sigma_{n'k}g\left(y_k^*\right)\mathbb{E}_{\mathbf{y}_{-(M,k)}}\left[\mathbb{E}_{y_k}\left[g^*\left(y_k\right)\right]\right] = 0$$

the last equality following from (24). The claim now follows immediately from the definition of $\lambda_{n'}$.

Proposition 4

Observe first that nothing in the preceding proof would change if the restriction $\sigma_{nk} = 0 = \sigma_{0k}$ were to be relaxed to $\sigma_{nk} = \sigma_{0k}$. Consider now the terminal wealth specification $W\left(\mathbf{x}\right) = W_1\left(\mathbf{x}_L\right) + W_2\left(\mathbf{x}_{-L}\right)$ for some continuous functions $W_1 : \mathbb{R}^L \mapsto \mathbb{R}_{++}$ and $W_2 : \mathbb{R}^{K-L} \mapsto \mathbb{R}_{++}$. As long as $K_n \cap \cup_{n' \in N_k} K_{n'} = \varnothing$, it is without loss of generality to take $\mathbf{x}_M$ and x_k as the projections of $\mathbf{x}_L$ and $\mathbf{x}_{-L}$ on $\mathbb{R}^M$ and $\mathbb{R}$ (along the kth dimension), respectively. But then

$$\frac{\partial W\left(\mathbf{y}\right)}{\partial y_k} = \frac{\partial W_2\left(\mathbf{y}_{-L}\right)}{\partial y_k} = \frac{\partial W\left(\mathbf{y}_{-L}, \mathbf{z}_L\right)}{\partial y_k}$$

and proceeding as before results in $I\left(\cdot\right)$ being the zero function on $\mathbb{R}$. That is, $g\left(\cdot\right)$ is a constant function and, by Lemma A.1, $\delta_{nk} = 0$.

Proposition 2

This proof proceeds in the same fashion as that of Proposition 3. Fixing an arbitrary $\mathbf{y}_{-(M,k)} \in \mathbb{R}^{K-M-1}$, we have now

$$I\left(y_k\right) = \mathbb{E}_{\left(\mathbf{y}_M, \mathbf{z}_M\right)}\left[\begin{array}{c} u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M}, \mathbf{z}_M\right)\right)e^{\sqrt{T-t}\sum_{m \in K_n}\left(\sigma_{nm}y_m + \sigma_{0m}z_m\right)} \\ \left[r_A\left(W\left(\mathbf{y}_{-M}, \mathbf{z}_M\right)\right)\frac{\partial W\left(\mathbf{y}_{-M}, \mathbf{z}_M\right)}{\partial y_k} - r_A\left(W\left(\mathbf{y}\right)\right)\frac{\partial W\left(\mathbf{y}\right)}{\partial y_k}\right] \end{array}\right]$$

Under (14), moreover, we have

$$\frac{\partial W\left(\mathbf{y}\right)}{\partial y_k} = \sqrt{T-t}\sum_{n'' \in N_k}\sigma_{n''k}e^{\mu_{n''}T + \sigma_{n''}^{\mathsf{T}}\beta + \sqrt{T-t}\sum_{k' \in K \setminus K_n}\sigma_{n''k'}y_{k'}}$$

or

$$\frac{I\left(y_{k}\right)}{\sqrt{T-t}} \;=\; \mathbb{E}_{\left(\mathbf{y}_{M},\mathbf{z}_{M}\right)}\left[u'\left(W\left(\mathbf{y}\right)\right)u'\left(W\left(\mathbf{y}_{-M},\mathbf{z}_{M}\right)\right)h\left(\mathbf{y}_{-M},\mathbf{z}_{M}\right)\right]$$

$$\sum_{n''\in N_{k}}\sigma_{n''k}e^{\mu_{n''}T+\sigma_{n''}^{\mathsf{T}}\beta+\sqrt{T-t}\sum_{k'\in K\setminus K_{n}}\sigma_{n''k'}y_{k'}}$$

with $h:\mathbb{R}^{K+M}\mapsto\mathbb{R}$ given by

$$h\left(\mathbf{y}_{-M},\left(\mathbf{y}_{M},\mathbf{z}_{M}\right)\right) \;=\; e^{\sqrt{T-t}\sum_{m\in K_{n}}\sigma_{0m}\left(y_{m}+z_{m}\right)}$$

$$\left[r_{A}\left(W\left(\mathbf{y}_{-M},\mathbf{z}_{M}\right)\right)-r_{A}\left(W\left(\mathbf{y}\right)\right)\right]e^{\sqrt{T-t}\sum_{m\in K_{n}}\overline{\sigma}_{nm}y_{m}}$$

Recall now condition (iii). As before, we may deploy the notation $N_{M}=\left\{n'\in\mathcal{N}:\sigma_{n'm}\neq0\;\forall m\in\overline{K}_{n}\right\}$. Here, however, condition (ii) implies that $N_{M}\cap N_{k}=\varnothing$ and the wealth specification in (14) may be written also as follows

$$W\left(\mathbf{y}\right) \;=\; \rho\left(\mathbf{y}_{-(M,k)}\right)+W_{1}\left(\mathbf{y}_{-M}\right)$$

$$+\sum_{n'\in N_{M}}e^{\sqrt{T-t}\lambda_{n'}\overline{\sigma}_{n}^{\mathsf{T}}\mathbf{y}_{M}}e^{\mu_{n'}T+\sigma_{n'}^{\mathsf{T}}\beta+\sqrt{T-t}\sum_{k'\in\mathcal{K}\setminus(K_{n}\cup\{k\})}\sigma_{n'k'}y_{k'}}$$

Obviously, letting $\lambda_{n'}>0$ for the typical $n'\in N_{M}$, we ought to have $\partial W\left(\mathbf{y}\right)/\partial e^{\sqrt{T-t}\sum_{m\in K_{n}}\overline{\sigma}_{nm}y_{m}}>0$. And, under DARA, $\partial r_{A}\left(W\left(\mathbf{y}\right)\right)/\partial e^{\sqrt{T-t}\sum_{m\in K_{n}}\overline{\sigma}_{m}y_{m}}<0$. But then

$$h\left(\mathbf{y}_{-M},\left(\mathbf{y}_{M},\mathbf{z}_{M}\right)\right)+h\left(\mathbf{y}_{-M},\left(\mathbf{z}_{M},\mathbf{y}_{M}\right)\right)$$
$$=\;e^{\sqrt{T-t}\sum_{m\in K_{n}}\sigma_{0m}\left(y_{m}+z_{m}\right)}$$
$$\left[r_{A}\left(W\left(\mathbf{y}_{-M},\mathbf{z}_{M}\right)\right)-r_{A}\left(W\left(\mathbf{y}\right)\right)\right]\left(e^{\sqrt{T-t}\sum_{m\in K_{n}}\overline{\sigma}_{nm}y_{m}}-e^{\sqrt{T-t}\sum_{m\in K_{n}}\overline{\sigma}_{nm}z_{m}}\right)$$

is non-negative on $\mathbb{R}^{2M}$, being zero only on the zero-measure set consisting of the vectors $\left(\mathbf{y}_{M},\mathbf{z}_{M}\right):\sum_{m\in K_{n}}\overline{\sigma}_{nm}\left(y_{m}-z_{m}\right)=0$. By Lemma A.1, then, the last expectation above is positive and, thus, condition (iv) ensures that $I\left(y_{k}\right)$ has the same sign as $\sigma_{n''k}$ for the typical $n''\in N_{k}$. By the same argument now as in the last part of the proof of Proposition 3, since $\sigma_{n''k}g\left(\cdot\right)$ is strictly increasing on $\mathbb{R}$, it cannot but be $\sigma_{n''k}\delta_{nk}>0$. Equally obviously, it must be $\sigma_{n''k}\delta_{nk}<0$ when $\lambda_{n'}<0$ for the typical $n'\in N_{M}$.