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Abstract

This paper proposes a new methodology, the Domination Index, to evaluate non-income inequalities between social groups such as inequalities of educational attainment, occupational status, health or subjective well-being. The Domination Index does not require specific cardinalisation assumptions, but only uses the ordinal structure of these non-income variables. We approach from an axiomatic perspective and show that a set of desirable properties for a group inequality measure when the variable of interest is ordinal, characterizes the Domination Index up to a positive scalar transformation. Moreover we make use of the Domination Index to explore the relation between inequality and segregation and show how these two concepts are related theoretically.

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1 Introduction

Inequalities between groups are important determinants of social and economic well-being of societies. For more than a century now, economists have been interested in evaluating the extent of inequalities in order to understand: (i) how they change, by comparing them across time; (ii) why they change and what they change, by comparing them across societies with different characteristics, revealing their relation to other social and economical phenomena.

Income or wealth disparities between social groups are well-known, well-documented and deeply analyzed inequalities. However it has been recognized long ago that comparing levels of income is not sufficient on its own to assess differences in individual well-being. Atkinson makes the pioneering move in departing from the classical approach of measuring inequality as the dispersion of levels of income, first bringing in the idea of social-welfare based income inequality measurement (Atkinson, 1970) and second incorporating the differences in individual needs to the assessment of income inequality (Atkinson and Bourguignon, 1987). Sen, in a series of papers and books, explores the need for going beyond income inequality and shifts the focus to many other variables such as longevity, survival, literacy, fertility, employment status, that influence individual well-being “but not captured by the simple statistics of incomes and commodity holdings” (Sen, 1980, 1985, 1995, 1997). There are many other variables that jointly contribute to one’s quality of life, and hence whose uneven distribution between social groups is of interest. However they lack the attention and well-developed theoretical approach that income received.

Moreover for the treatment of these non-income variables such as education, health, occupational status or subjective well-being, we cannot generally apply the techniques developed for evaluating income inequality since these variables do not share a very important feature of income: they are not cardinal in nature. They are rather defined over categories that are not necessarily associated with cardinal values. However notice that although these categories do not convey any cardinal information, they are not completely unrelated either. In most of the cases, categories can be compared unambiguously. Everybody will agree that a college graduate’s educational attainment is higher than a secondary school drop out though we would not know by how much it is higher. Or it will be safe to claim that an individual that selects the score 3 as answer to the question of “Taking all things together, how happy would you say you are, on a scale from 1 to 10 where [1] means you are very unhappy and [10] means you are very happy?” has selected a lower happiness score than an individual with a score of 9.¹ But this would not necessarily imply that the second individual is three times happier

¹This is the Question 42 of the Second European Quality of Life Survey, 2007-2008. Questions of the same sort are found in population surveys such as United States General Social Survey or Euro-Barometer

than the first one. Since these variables are vaguely measurable, the methods designed for measuring income inequality are essentially futile. Representing these categories by making use of specific cardinalisations requires further assumptions if not result in misvaluations. Let us give a closer look to a pair of specific examples:

Example 1: Gender based occupational status inequality. The following table summarizes the gender distribution across occupational hierarchy within the class of the Management, business and financial occupations in United States.²

Table 1. Women Share of MBF Occupations, U.S., 2010

	Total employed	Percent women
MBF occupations	20,938	43%
Chief executives	1,505	25,5%
General and operations managers	1,007	29,9%
Managers	12,489	40,5%
Operations	5,937	54,9%

According to the US census data, in 2010, out of almost 21 million employees in management, business and financial occupations, 43% were women. Within this class, Chief executives are the ones with the highest status, followed by General and operations managers. Managers occupy the third position in the hierarchy and finally the last position is occupied by Operations employees. If the distributions of genders across these positions were completely equal, we would observe a women share of 43% in each position. However, the increasing women share going down the hierarchy signals an inequality in the distribution of genders. How do we treat this data? In order to assess gender inequality of occupational status consistently we need to take into account the hierarchy of positions. One could falsely argue that this hierarchy can be represented by the corresponding wage levels of the occupational statuses eliminating the need for going beyond wage inequality. However, as shown in different works (McLaughlin, 1978; England, 1979; Magnusson, 2009) the average wage of a female dominated job do not correspond to women’s occupational prestige for that status.

Example 2: Racial disparities in educational attainment. According to United States Census data in the year 1970, 43.2% of the White citizens and 27% of the Black citizens were high school graduates without a further degree, whereas 11,3% of the Whites

Survey Series.

²Civilian noninstitutionalized population 16 years old and over. Annual average of monthly figures. Figures are in millions. *Source:* U.S. Census Bureau, The 2012 Statistical Abstract, The National Data Book, Labor Force, Employment and Earnings Section, Table 616. Employed Civilians by Occupation, Sex, Race, and Hispanic Origin: 2010.

and 4.4% of the Blacks were college graduates or more. These numbers would roughly imply that back in '70s, in all higher categories of educational attainment, White citizens had more representation than Black citizens in relative terms. However in the year 2010, 57.3% of the Whites and 64.4% of the Blacks had high school diploma, while 30.3% of the Whites and 19.8% of the Blacks had college diplomas or even higher degrees, which definitely points to a decrease in the discrepancy in higher levels of educational attainment between race groups. But by how much? Or only by looking at these categories can we say that the inequality of educational attainment between race groups has declined from 1970 to 2010? Justified answers to these questions require to compare the entire distributions of Blacks and Whites across all educational attainment categories and a method to evaluate the difference in these distributions.³ Since the categories refer to the highest level of education attained, no obvious cardinal values are attached to them. In empirical works, this problem is often resolved by assigning the average number of years of schooling to educational attainment categories in order to make use of cardinal measures. Different countries, however, possess different educational cycles or countries make adjustments in their educational systems over time. Since cardinal measurement techniques are not robust to these changes, application of income inequality measures will cause misvaluations of inequality especially in cross-country comparisons (Meschi and Scervini, 2012).

Inequalities of health or subjective well-being are other examples of non-income variables that face the same difficulty of treatment. The data on health and subjective well-being are collected via nation-wide surveys held by the health or statistics authorities of the countries. For practical purposes these variables are either defined over ordered categories such as “poor, fair, good, excellent” or over a cardinal scale such as “1,2,3,4”, where 1 corresponding to “poor”, “2” to “fair” and so on.⁴ Allison and Foster (2004) show that application of cardinal measures of inequality over these categories results in incomparable levels of inequalities for different societies since these techniques are sensitive to scale changes.

As shown by the previous examples, measuring the extent of inequality in occupational status, educational attainment, health or subjective well-being is subject to restrictive assumptions or misvaluations caused by specific cardinalisations. There exists, therefore, a need for going beyond measurement of income inequality techniques and developing justified measurement methodologies for the evaluation of these non-income, social inequalities.

³Developed by UNESCO, the International Standard Classification of Education (ISCED) provides an internationally harmonized classification system for educational attainment.

⁴National Medical Expenditure Survey and National Health Interview Survey of United States, General Household Survey of United Kingdom, Swiss Health Survey and Survey on Health and Retirement in Europe make use of ordinal health categories.

In this study we suggest a methodology to evaluate social inequalities: the Domination Index. Given a society with ordered categories and two social groups, the Domination Index follows a very natural logic to compare the distributions of social groups over categories: It basically counts the number of times a group beats the other group in pairwise confrontations. Consider a pair of individuals where each of them is a member of a different group, say Women and Men. The woman beats, dominates the man if she is in a better category than him. That is what we define as a domination. Then, the total number of dominations by the group Women is the total number of times that a woman beats a man. The Domination Index evaluates inequality in terms of the difference in the number of dominations. It actually is equal to the absolute average difference in the number of dominations by groups.

On top of its conceptual simplicity, the Domination Index has several appealing properties. First of all, it has a very intuitive interpretation. Since it compares the average number of dominations, it actually gives out the ex-ante probability advantage of a group over the other. In other words, the Domination Index gives out the extra probability that on a random selection of a pair of individuals from different groups, the member of one group occupies a better category than the member of the other group.⁵ Second, it is efficient. It makes use of all the information available regarding the distributions of the social groups, and only of this information without going for further assumptions. Third, it is easy-to-use. Large samples of populations or long lists of categories do not create computational complexities. Fourth, it is well-founded. Our axiomatic analysis shows that it satisfies a set of reasonable properties. Moreover, it represents the only family that satisfies these properties. These characterizing properties are variations of classical notions such as a symmetry property that requires equal treatment to social groups; a monotonicity property that controls the change in inequality for very specific changes in the society and finally a decomposability and an additivity property that allow to concentrate in different parts of the society and express the overall inequality as an aggregation of the inequalities in these parts. For instance, in order to understand the specific structure of the inequality, one may want to focus on upper and lower parts of the society separately, where the upper part consists of the better positions and the lower part consists of the worse ones. Decomposability ensures that the inequality in the entire society can be expressed in terms of the inequalities in upper and lower parts. On the other hand, with the same purpose, one may want to identify the contributions of different sections of the social groups to the overall inequality. An additivity property ensures that the overall

⁵The Domination Index is closely related to Mann-Whitney's Statistic U and the Net Difference Index (Liebersohn, 1976). More on this can be found in the following review of literature.

inequality can be expressed in terms of the inequalities between different sections of the social groups. In a first theorem, we show that these properties yield us the Domination Index up to a positive scalar transformation.

The Domination Index is also instrumental to understand the connection of social inequalities with a related problem: segregation. Segregation is defined as the inequality in the distribution of groups over neither measurable nor comparable categories. The relation of between-group inequalities to segregation has been discussed in different literatures from both theoretical and empirical perspectives. Segregation simply captures the nominal difference of distributions without any regard to how relatively good or bad the distribution is. Inequality on the other hand involves an evaluation of the distributions. The difference of the distributions is assessed taking into account how beneficiary they are for the corresponding groups. Consider an imaginary building with the residents being from two different groups. A scenario such that one of the groups is occupying all the nicer flats with the view at the higher floors of the building, whereas all of the members of the other group living downstairs facing the facade of the building across the street will be maximally and as equally segregated as the scenario where all members of the first group are living in the odd numbered floors and all members of the second group are living in the even numbered floors, hence two groups are never sharing the same floor. However an inequality measure will label the first scenario more unequal than the second one. This certainly does not imply the dominance of one concept over the other but simply demonstrates that although closely related their focuses are different.

Clarifying the theoretical link between segregation and inequality, the Domination Index helps to understand the structure of the relation between these two concepts. We start by showing that for some societies the inequality between social groups measured by the Domination Index coincides with the level of segregation measured by a well-known segregation measure, the Gini Segregation Index. In other words, for a particular organization of the society, segregation is equal to the level of inequality between groups. This particular organization is the one in which the importance of each category reflects how uneven the distribution of groups in that category is. In other words, the order relation of the categories is in line with the relative distribution of groups across categories. If from the best to the worst category the ratio between the members of groups is always decreasing or increasing, i.e., if the ratio of the number of members of a group to the one of the other group is the highest in the best position, the second highest in the second best position and so on, then segregation in this society according to the Gini Segregation Index is equal to the inequality

measured by the Domination Index. We then show that this organization is actually the one that results in the maximum possible level of inequality for that society. Hence, level of segregation in general gives an upper bound for the level of inequality. These observations not only provide a theoretical contribution to the debate on the relation of segregation to inequality but also gives out the characterization of the Gini Segregation index as a by-product. We show that variants of the properties that characterize the Domination Index do characterize the Gini Segregation Index. As a second by-product, we consider an extension of our methodology to assess inequalities under incomplete information about the ordering of categories. We exploit the relation between segregation and inequality to provide a way to measure inequalities between groups when the categories are not completely ordered.

The organization of the paper is as follows: First in a subsection we present a review of related literature. Then, the following section introduces the basic set up and the Domination Index. We provide a set of properties and the foundational analysis of the Domination Index. In the third section, we explore the link between segregation and inequality with the help of the Domination Index. The fourth section is an extension of our model to incomplete information. The proofs of the theorems in general are left to an appendix.

1.1 Related Literature

Although works discussing evaluation of social inequalities have not developed in a comprehensive and systematic way, there are various related literatures that we refer to. The most deeply analyzed and well-developed is, not surprisingly, the literature of between-group income inequality. A major part of this literature analyzes the decomposition of income inequality to its within-group and between-group components. The measures that allow the overall inequality to be expressed as the sum of between-group and within-group inequalities are qualified as additively decomposable measures (Cowell, 1977, 1980; Shorrocks, 1980,1984). For this class of measures the between-group component of income inequality is simply found by assuming that each member of a social group receives that group's mean income. Then, comparison between-groups essentially becomes a comparison of group means. Bourguignon (1979) characterizes the family of decomposable functions that also satisfy other desirable properties and he shows that only two functions serve to this purpose: One of them is Theil's entropy measure (Theil, 1967) and the other is the mean logarithmic deviation, which is closely related to the Theil measure. Lasso de la Vega, Urrutia and Volij (2011) recently provide a characterization of the Theil measure by only making use of ordinal axioms. Other methods based on comparison of representative levels of income of groups

instead of mean income (Blackorby, Donaldson and Auersperg, 1981) or comparison of the observed between-group inequality with the maximum inequality that could occur (Elbers, Lanjouw, Mistiaen and Ozler, 2008) have been proposed as well.

A second major branch of group inequalities literature corresponds to segregation theories. The very first paper on this issue focuses on the residential segregation of race groups (Jahn, Schmidt and Schrag, 1947). Research on school segregation by ethno-race groups (Echenique, Fryer and Kaufman, 2006; Frankel and Volij, 2011) developed parallel to the research on residential segregation (Duncan and Duncan, 1955; Winship, 1977; Massey and Denton, 1988) as well as occupational segregation by gender and race (Deutsch, Fluckiger and Silber, 1994; Mora and Ruiz-Castillo, 2003; Chakravarty and Silber, 2007). Most of the literature on segregation is based on development and application of indices, that are generally adaptations of measures of income inequality. Hutchens (1991, 2001, 2004) and Echenique and Fryer (2005) provide axiomatic characterizations of indices that are relevant for all questions of segregation.

Research on the measurement of social inequalities is far from forming a well-developed, systematic literature but rather different pieces can be found as parts of different literatures. In a statistics spin off paper, Lieberman (1976) proposes the Net Difference Index to examine situations where two populations are to be compared with respect to a completely ordered characteristic such as age or years of schooling. Net Difference Index is based on Mann-Whitney's U Statistics (1947), which gives a non-parametric rank test that is used to determine if two samples are from the same population. The Statistics U is simply the number of times the observations from one sample precede the observations from the other sample when all of the observations are ordered into a single ranked series. The probability distribution tables of U are provided for testing the null hypothesis that two samples share the same distribution. The Statistics U is different from well-known Wilcoxon rank-sum statistics (Wilcoxon, 1945) in that U allows for different sample sizes. Another rank-based statistics Somer's D (Somer, 1962), which is essentially a measure of association for ordinal variables, is used by several sociologists in the measurement of gender-based inequality of occupational status (Blackburn, Brooks and Jarman, 2001).

Hutchens (2006) studies the question of gender-based occupational status inequality when the occupational status is determined by a prestige score. He provides a set of desirable properties both for cardinal and ordinal variables of prestige. Reardon (2009) discusses the inequality of an ordinal variable such as education or occupational status between social

groups and proposes to measure it in terms of the distances of the distributions of groups to a completely polarized distribution. He proposes desirable properties for a measure of this sort, and then introduces four different functions that satisfy those properties without going for further axiomatic analysis.

Allison and Foster (2004) discuss the measurement of health inequality using self reported health status data, which is based on ordinal categories attached to a scale. They argue that traditional measures are not applicable since they are not order preserving to scale changes and propose a partial ordering of health inequality that is invariant to scale changes. Based on Allison and Foster methodology, Naga and Yalcin (2008) propose a parametric family of indices that satisfy a basic normalization axiom. Dutta and Foster (2011) apply the same methodology to measure the inequality of happiness in US by using self reported subjective well-being data. They further use additive decomposition techniques to measure the group inequality of happiness between races, genders and regions. Kobus and Milos (2011) provides a characterization of a decomposable family of indices that respect Allison and Foster partial ordering. Kobus also proposes an extension of this ordering to evaluate multi-dimensional inequalities (Kobus 2011). In the measurement of inequality in educational attainment, although the data is collected over educational categories, the average number of years of schooling is assigned as a cardinal value to corresponding categories (Barro and Lee, 1993, 1996 and 2001; Thomas, Wang and Fan, 2001). This cardinalisation allows to use common inequality measures such as Gini coefficient and Theil indices while keeping the aforementioned problems of this procedure unsolved. Quite recently, Herrero and Villar (2012) propose a methodology to compare the educational achievements of different groups and provide different applications of this methodology in order to evaluate inequality of opportunities in education and health (Herrero, Mendez and Villar, 2012). Their methodology shares a similar statistical reference with the one proposed in this paper.

2 The Domination Index

A society is composed of individuals from different social groups distributed across ordered positions. We restrict our analysis to two social groups, namely Women and Men.⁶ Formally, a society is a pair of elements $(S, L_{\mathcal{I}})$, where \mathcal{I} denotes a finite set of I positions, S is a society matrix that shows the distribution of Women and Men over I positions and L is the order relation over \mathcal{I} . We assume that L is an exogenous total order (a complete, transitive and asymmetric binary relation), where for every i, j in \mathcal{I} , iLj is interpreted as social position i is

⁶An extension to multigroup case is immediate though, as suggested in the concluding remarks.

better than position j . A society matrix, $S = (S_W, S_M)$ is a positive real matrix of dimension $I \times 2$ with the first column, S_W describing the number of women in each position and the second column, S_M denoting the number of men. We denote by S_{iw} and S_{im} , the $i1^{st}$ and the $i2^{nd}$ elements of the matrix, the number of women and men in position i respectively, whereas S_w and S_m stand for total number of women and men, i.e.; $\sum_i S_{iw} = S_w$ and $\sum_i S_{im} = S_m$. Small letters denote the proportions of individuals, i.e.; s_{iw} denotes proportion of women in position i to total number of women in society and s_{im} denotes as of men. We consider S_{iw} and S_{im} to be nonnegative real numbers.⁷ We denote with C the space of all societies., i.e.; $C = \cup_{\mathcal{I}}(\mathbb{R}_+^{I \times 2} \times L^I)$, where $\mathbb{R}_+^{I \times 2}$ is the space of $I \times 2$ nonnegative real matrices and L^I stands for the space of total orders over \mathcal{I} .

We define social inequality as the inequality in the distributions of women and men across ordered positions. Then, a social inequality measure is a non-zero continuous function $H : C \rightarrow \mathbb{R}_+$ that attaches to each possible society $(S, L_{\mathcal{I}})$, a nonnegative real number that shows the amount of social inequality.

The Domination Index, D measures social inequality in terms of the number of times a group beats the other group in pairwise confrontations. Let us define a **domination** by a group as having a member in a better position than a counter-group member. Consider a woman in position i in a society $(S, L_{\mathcal{I}})$. Her position is better than all the men that are in worse positions than i , thus she creates $\sum_{j:iLj} S_{jm}$ dominations in total. Then, the total number of dominations by women is equal to $\sum_i (S_{iw} \sum_{j:iLj} S_{jm})$, where total number of dominations by men is $\sum_i (S_{im} \sum_{j:iLj} S_{jw})$. The absolute difference in average number of dominations by women and men gives us the Domination Index:

$$D(S, L_{\mathcal{I}}) = \left| \frac{\sum_i (S_{iw} \sum_{j:iLj} S_{jm}) - \sum_i (S_{im} \sum_{j:iLj} S_{jw})}{S_w S_m} \right| = \left| \sum_i (s_{iw} \sum_{j:iLj} s_{jm}) - \sum_i (s_{im} \sum_{j:iLj} s_{jw}) \right|$$

In a more compact form, it can equivalently be expressed as follows:

$$D(S, L_{\mathcal{I}}) = \left| \sum_i \sum_j c_{ij} s_{iw} s_{jm} \right| \text{ where } c_{ij} = \begin{cases} 1 & \text{if } iLj \\ 0 & \text{if } i = j \\ -1 & \text{if } jLi \end{cases}$$

This compact form notation highlights what D measures in essence. D actually gives out the ex-ante probability advantage between groups. Given a random pair of a woman and a man,

⁷This choice not only ensures the generalization of our results but also is the convention in group-inequalities literatures. As noted in Hutchens (2001), in some empirical applications part-time employees are treated as fractional employees.

the difference in probabilities of one individual beating the other is the ex-ante probability advantage of one group over the other, as shown in an immediate lemma:

Lemma 1 $D(S, L_{\mathcal{I}}) = |Pr(\text{Women beating Men}) - Pr(\text{Men beating Women})|$

Since at the essence of social inequalities lies the idea of having an advantageous or a disadvantageous position just because being a member of a social group, D does well in capturing this ex-ante probability advantage in evaluating social inequalities.

D takes values between 0 and 1, 0 being complete equality and 1 being maximum inequality. The main attraction of D depends on its simple structure and intuitive interpretation. It is very convenient and easy to apply to compare two distributions over ordered categories without making further cardinalisation assumptions. It is an efficient measure in the sense that it makes use of all the available information. Number of dominations by groups is the only relevant information of this setting and D evaluates social inequality in terms of it. Although being easy to use, efficient and intuitive is important for an inequality measure for practical purposes, it is never sufficient unless supported by the properties that summarize the behavior of the function. To understand how D behaves, we now introduce a set of properties that are not only satisfied by D but are also ‘reasonable’ properties for any social inequality measure H .

A first standard property is a symmetry property, that ensures equal treatment to groups. It simply requires that exchanging the distributions of groups should not change the amount of social inequality. Formally;

Symmetry for Groups (SYM): Consider two societies $(S, L_{\mathcal{I}})$ and $(S', L_{\mathcal{I}})$ with $S_W = S'_M$ and $S_M = S'_W$. Then $H(S, L_{\mathcal{I}}) = H(S', L_{\mathcal{I}})$.

The second property is about the relative character of the index. Inequality measures are usually differentiated according to their absolute or relative characters. For relative inequality measures what matters are the relative amount of individuals whereas absolute inequality measures do take into account absolute amounts. D is a relative inequality measure as ensured by the following property:

Scale Invariance (INV): Given $(S, L_{\mathcal{I}})$ and any $\alpha, \beta \in \mathbb{R}_{++}$, consider $(S', L_{\mathcal{I}})$ such that for all i , $S'_{iw} = \alpha S_{iw}$ and $S'_{im} = \beta S_{im}$. Then, $H(S, L_{\mathcal{I}}) = H(S', L_{\mathcal{I}})$.

For scale invariant functions what matters is the proportion of individuals in each position, not the absolute amounts. Especially for cross society analysis this is an important property, since otherwise larger populations would always imply higher inequality.⁸

Next we will introduce a monotonicity property that defines the behavior of the function for certain changes in the distributions. For some distributions of the society there exist some changes that clearly do not increase or do not decrease social inequality. For instance consider the following simple example of a society $(S, L_{\mathcal{I}})$ with 3 positions where all men occupy the best position and all women are grouped in the worst position: $S = \begin{pmatrix} 0 & 100 \\ 0 & 0 \\ 100 & 0 \end{pmatrix}$ and $1L2L3$. This is a society in which women and men are distributed in a maximum unequal way possible. Now consider addition of one women to the best position and one men to the worst position. The resulting society will be of the form: $S = \begin{pmatrix} 1 & 100 \\ 0 & 0 \\ 100 & 1 \end{pmatrix}$. One would not expect from a reasonable relative inequality measure to identify the resulting society with higher social inequality than the initial one. D evaluates the second society as less unequal than the first one. Now, consider exactly the opposite society $(S', L_{\mathcal{I}})$ such that $S' = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$ again with $1L2L3$. The same addition of one women to the best position and one man to the worst is now not an inequality decreasing change since it results in the society $S' = \begin{pmatrix} 101 & 0 \\ 0 & 101 \end{pmatrix}$. For any relative social inequality measure, inequality is still at its maximum. The monotonicity property will ensure that this particular addition will not increase inequality for societies like $(S, L_{\mathcal{I}})$ and it will not decrease inequality for societies like $(S', L_{\mathcal{I}})$.

We define a **women improving addition** to $(S, L_{\mathcal{I}})$, as a slight increase in the number of women in the best position and in the number of men in the worst position in S . Formally, a women improving addition to $(S, L_{\mathcal{I}})$ is the addition of an ε_W matrix of dimension $I \times 2$ that only possesses ε number of women in the best position and ε number of men in the worst position in S for ε small enough, all the other positions being empty.

We classify societies into two distinct types according to the reaction of the measured inequality to a women improving addition. If no women improving addition is resulting in a decrease in the social inequality, then we classify the society as of Women-type. On the contrary if any women improving addition is decreasing the social inequality, then the society is classified as of Men-type. Moreover, we define completely equal societies as of both Women-type and Men-type. Formally; given a society $(S, L_{\mathcal{I}})$ and a social inequality measure H , for any $\varepsilon \in \mathbb{R}_{++}$ in a δ neighborhood of 0, for δ small enough, S is said to be

⁸The absolute version of the index, i.e., $|\sum_i (S_{iw} \sum_{j:iL_j} S_{jm} - S_{im} \sum_{j:iL_j} S_{jw})|$ is actually characterized by similar properties but INV. The characterization result replaces DEC and SAD properties, that would be introduced soon, with non-weighted versions of them and it is available upon request.

of **W-type** if $H(S + \varepsilon_W, L_{\mathcal{I}}) \geq H(S, L_{\mathcal{I}})$ or $H(S, L_{\mathcal{I}}) = 0$. S is said to be of **M-type** if $H(S + \varepsilon_W, L_{\mathcal{I}}) < H(S, L_{\mathcal{I}})$ or $H(S, L_{\mathcal{I}}) = 0$. Notice that, by definition, not being a W -type matrix directly implies being an M -type matrix. Obviously which matrices are of which type will depend on the particular behavior of the specific functional form of H . But for some unambiguous distributions like the ones of S and S' given in the example above, all reasonably monotonic measures should agree on the effect of a women improving addition. For the society $(S, L_{\mathcal{I}})$, clearly no women improving addition should increase the relative inequality, implying that S is of M -type. On the contrary, for $(S', L_{\mathcal{I}})$, any women improving addition should not decrease inequality, implying that S' is of W -type. This reasoning is applied in those societies in which we can make use of the first-order stochastic dominance to compare women and men distribution.

For a society $(S, L_{\mathcal{I}})$ that has exactly the same number of women and men, we say that the distribution of women dominates the distribution of men if for any position there is always more women than men in total in the positions that are at least as good as that position. To put formally; for $(S, L_{\mathcal{I}})$ with $S_w = S_m$, S_W dominates S_M if for all $k \in \mathcal{I}$; $\sum_{i:iLk} S_{iw} \geq \sum_{i:iLk} S_{im}$. Symmetrically, we say that S_M dominates S_W if for all $k \in \mathcal{I}$; $\sum_{i:iLk} S_{im} \geq \sum_{i:iLk} S_{iw}$.

Monotonicity (MON): Given a society $(S, L_{\mathcal{I}})$ with $S_w = S_m$, (i) if S_W dominates S_M , then S is a W -type society matrix; (ii) if S_M dominates S_W , then S is a M -type society matrix.

For a society in which the women distribution dominates the distribution of men, there is always more women in better positions. The first part of MON ensures that these type of society matrices are of W -type and the second part is the symmetric counterpart. Note that MON also guarantees a zero level of inequality for equally distributed societies. The following lemma states that any monotone social inequality measure assigns a value 0 to an equally distributed society.

Lemma 2 *For any H that satisfies MON, for a society $(S, L_{\mathcal{I}})$ such that for any i , $S_{iw} = S_{im}$, we have $H(S, L_{\mathcal{I}}) = 0$.*

The properties that are introduced up to now, SYM, INV and MON are standard properties and are satisfied by many other functions in addition to D . The last two properties, however, will narrow down this class of functions extremely, up to a single family. Both properties are about the decomposability of overall inequality into the inequalities in different parts of the society. The concentration of social inequality in specific parts of a society is

not an uncommon phenomenon. For instance in explaining the structure of the gender-based inequality in the labor market the theories of “glass ceiling” or “sticky floor” supplement strong evidence for the unbalanced distribution of women and men in the upper and lower tails of the wage distribution.⁹ The following property, Decomposability allows to express overall inequality as an aggregation of inequalities in the upper part and the lower part of the society.

Given a society $(S, L_{\mathcal{I}})$, we define **an ordered division** of $(S, L_{\mathcal{I}})$ as a pair of societies $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$ such that: (i) \mathcal{I}^1 and \mathcal{I}^2 define a partition of \mathcal{I} such that for any i in \mathcal{I}^1 and any j in \mathcal{I}^2 we have iLj , (ii) for $k = 1, 2$, $iL_{\mathcal{I}^k}j$ if and only if iLj for any i, j in \mathcal{I}^k and (iii) for $k = 1, 2$, S^k is a $I^k \times 2$ society matrix such that each position possesses the same number of women and men in S and S^k . An ordered division of a society is basically a partition of the society respecting the order relation: there is the upper part that is composed of the better positions in the society and a lower part that is composed of the worse ones. D expresses the overall inequality as an aggregation of the inequalities in each of these parts and an interaction term between them that stems from the fact that all of the positions in the upper part are actually better than all the positions in the lower part. What we define as the interaction term is equal to the social inequality in a society with two positions. The first position is occupied by all the individuals of the upper part of the original society and the second position is occupied by all the individuals of the lower part. Formally; given an ordered division of a society $(S, L_{\mathcal{I}})$ as $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$, **the interaction society** $(S', L'_{\mathcal{I}'})$ is a society defined as the following: (i) It consists of two social positions: $\mathcal{I}' = 1, 2$ with $1L'2$ (ii) The first position contains all individuals of S^1 : $S'_{1w} = S^1_w$ and $S'_{1m} = S^1_m$ (iii) The second position contains those of S^2 : $S'_{2w} = S^2_w$ and $S'_{2m} = S^2_m$.

The decomposability property allows to decompose the total social inequality as the weighted sum of the inequalities in the upper part, lower part and the interaction society for any ordered division of a society, as long as all resulting society matrices are of the same type. The specific form of the weighting structure depends on the specific form of the measure. As D counts the number of dominations in pairwise confrontations, the weighting structure depends on the proportion of these pairwise confrontations in each part. Given an ordered division, we define a **population weight** as the proportion of pairs of women and men in each part to the overall number of pairs. Formally; for an ordered division of $(S, L_{\mathcal{I}})$ as $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$, the population weight of $(S^k, L_{\mathcal{I}^k})$ for $k = 1, 2$ is $\lambda_S^k = \frac{(S^k_w)(S^k_m)}{(S_w)(S_m)}$. Notice that the weight of the interaction society $(S', L'_{\mathcal{I}'})$ will be equal to 1 by this definition.

⁹See McDowell, Singell and Ziliak (1999), Blau and Kahn (2000), Baker (2003), de la Rica, Dolado and Llorens (2008)

Decomposability (DEC): For any ordered division of $(S, L_{\mathcal{I}})$ as $(S^1, L_{\mathcal{I}^1})$ and $(S^2, L_{\mathcal{I}^2})$ such that S^1, S^2 and S' are either all of W -type or all of M -type the following holds:

$$H(S, L_{\mathcal{I}}) = \lambda_{S^1} H(S^1, L_{\mathcal{I}^1}) + \lambda_{S^2} H(S^2, L_{\mathcal{I}^2}) + H(S', L'_{\mathcal{I}})$$

As the last property, we introduce Subgroup Additivity, that helps to deepen the analysis one step further by differentiating the effects of subgroups to overall social inequality. We define a subgroup as a subset of a social group. For instance Immigrant Women and Local Women refer to two subgroups of the social group Women. Subgroup Additivity will allow to identify how much of the overall inequality is between Men and Immigrant Women and how much of it is between Men and Local Women.

For a society $(S, L_{\mathcal{I}})$, a **partition into subgroups** is a pair of societies $(S', L_{\mathcal{I}}), (S'', L_{\mathcal{I}})$ such that for all i , either $S_{iw} = S'_{iw} = S''_{iw}$ and $S_{im} = S'_{im} + S''_{im}$ or $S_{im} = S'_{im} = S''_{im}$ and $S_{iw} = S'_{iw} + S''_{iw}$ holds. Hence, a partition of a society into subgroups results in a pair of societies that possess exactly the same distribution of one of the groups with the original society, and sum up to the original distribution of the other group. For the sake of simplicity we have defined a partition into subgroups only for two subgroups, but clearly repeated application of the partition will result in a partition into many subgroups.

Given a partition into subgroups, the Subgroup Additivity property allows to express the overall social inequality between Women and Men as a weighted aggregation of the inequalities between each subgroup and the other social group as long as all societies are of the same type. Similar to DEC, for a partition of a society $(S, L_{\mathcal{I}})$ into subgroups $(S', L_{\mathcal{I}})$ and $(S'', L_{\mathcal{I}})$, the population weight of the subgroups will be as $\lambda_{S'} = \frac{(S'_{iw})(S'_{im})}{(S_w)(S_m)} = \frac{(S'_{iw})}{(S_w)}$ and $\lambda_{S''} = \frac{(S''_{iw})(S''_{im})}{(S_w)(S_m)} = \frac{(S''_{im})}{(S_w)}$. But notice that this time the population weights of the subgroups add up to 1. Hence SAD expresses the overall inequality as a convex combination of the subsociety inequalities.

Subgroup Additivity (SAD): For any partition of a society $(S, L_{\mathcal{I}})$ into subgroups $(S', L_{\mathcal{I}})$ and $(S'', L_{\mathcal{I}})$ such that S' and S'' are both of W -type or M -type, the following holds:

$$H(S, L_{\mathcal{I}}) = \lambda_{S'} H(S', L_{\mathcal{I}}) + \lambda_{S''} H(S'', L_{\mathcal{I}})$$

With SAD, since the weights are nonnegative, the original society matrix S will necessarily be of the same type with S' and S'' . To see this notice that for S, S' and S'' with $H(S, L_{\mathcal{I}}) = \lambda_{S'} H(S', L_{\mathcal{I}}) + \lambda_{S''} H(S'', L_{\mathcal{I}})$, we have $H(S + \varepsilon_W, L_{\mathcal{I}}) = \lambda_{S'} H(S' + \frac{\varepsilon_W}{2}, L_{\mathcal{I}}) + \lambda_{S''} H(S'' + \frac{\varepsilon_W}{2}, L_{\mathcal{I}})$ for all ε in a δ neighborhood around 0, where δ is determined by the

smaller of the neighborhoods that are induced by S' and S'' . Then, the change in the overall inequality yielded by the addition of ε_W to S will be in the same direction with the changes in S' and S'' created by the addition of $\frac{\varepsilon_W}{2}$.

SAD is a strong property and as will be highlighted in the proof of the characterization result, it has an important role in determining the functional form of D . It actually implies INV property. In other words, any function that satisfies SAD is a relative inequality measure, as stated in the following proposition:

Proposition 1 *Any $H : C \rightarrow \mathbb{R}_+$ that satisfies SAD is Scale Invariant.*

We are now ready to introduce the main result of the paper. These properties listed not only are satisfied by D , but also they do characterize it up to a positive scalar transformation. As SAD implies INV, we do not include it as an additional axiom.

Theorem 1 *A social inequality function $H : C \rightarrow \mathbb{R}_+$ satisfies SYM, MON, DEC and SAD if and only if it is a positive scalar transformation of the Domination Index:*

$$D(S, L_{\mathcal{I}}) = \left| \sum_i (s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw}) \right|$$

The proof of the characterization can be summarized by the following steps: In Step 1, we consider a very specific type of a society and derive the functional form of H for it. We focus our attention to societies for which inequality is always favoring Women, i.e., for any subset of positions Women have a better distribution than Men. These would be the societies with strictly decreasing $\frac{s_{iw}}{s_{im}}$ ratios from the best to the worst position. We call them as Women-perfect societies. Since Women-perfectness allows for iterative application of DEC, together with INV and MON, we first show that for Women-perfect societies, overall inequality can be decomposed into the inequalities between the individuals of a position and all other individuals in worse positions. Hence, we remain with a collection of simpler hypothetical societies with two positions, where the first position of each hypothetical society possesses the individuals of an original position and the second position includes all individuals that are in worse positions than this one. In Step 2, we focus only to those 2×2 hypothetical societies. SAD ensures that the inequality of these societies is a function of the difference in number of dominations by groups. Then, aggregation of the inequalities of hypothetical societies yields the functional form as a positive multiple of the average number of dominations by Women net of average number of dominations by Men. Step 3 simply shows that by SYM, we arrive to the functional form of the index for societies for which inequality is always favoring Men, Men-perfect societies. In Step 4, we consider any W -type society and

associate it with a particular Women-perfect society. We do this by adding sufficient number of women to the original society. The functional form of the index for any W -type society appears from the difference of the inequalities of the Women-perfect society and the sub-society that includes the women that are added to the original society. Step 5 mimics Step 4 for any M -type society. Since any society is either W -type or M -type, we arrive to the index.

All of the characterizing properties are independent. SYM guarantees equal treatment to Women and Men. Hence, an asymmetric version of D that values dominations by Women and Men differently can be an example to a social inequality function that satisfies all of the other properties but SYM. For instance, $H(S, L_{\mathcal{I}}) = |\sum_i (2s_{iw} \sum_{j:iL_j} s_{jm} - s_{im} \sum_{j:iL_j} s_{jw})|$.

MON is responsible from the comparison between groups. The function that counts the total average number of dominations instead of the difference in average number of dominations will be an example to a function that only does not satisfy MON out of the stated properties, i.e., $H(S, L_{\mathcal{I}}) = |\sum_i (s_{iw} \sum_{j:iL_j} s_{jm} + s_{im} \sum_{j:iL_j} s_{jw})|$.

DEC ensures that individuals of each position are taken into account in relation to the relative order of the position. A function that only considers the dominations by some of the individuals will not satisfy DEC. An example that comply with MON, SYM and SAD will be a function that only counts the dominations by the individuals of the best position: $H(S, L_{\mathcal{I}}) = |s_{xw} \sum_{j:xL_j} s_{jm} - s_{xm} \sum_{j:xL_j} s_{jw}|$, where x denotes the best position according to L over \mathcal{I} .

Finally, SAD accounts for considering only the dominations between groups. A function that takes into account the dominations within groups will not satisfy SAD. For instance: $H(S, L_{\mathcal{I}}) = \frac{1}{s_w s_m} (|\sum_i (s_{iw} \sum_{j:iL_j} s_{jm} - s_{im} \sum_{j:iL_j} s_{jw})| + |\sum_i (s_{iw} \sum_{j:iL_j} s_{jw} - s_{im} \sum_{j:iL_j} s_{jm})|)$.

3 Segregation as Inequality

Segregation, in very general terms, is about how separated different groups of a society are. It is the degree to which social groups are distributed differently in the society. It has started to attract attention with the discrepancy in the distributions of different race groups across residential areas in United States at the first half of the 20th century and since then different types of segregation have been recognized, documented and analyzed both theoretically and empirically. A major part of the questions about segregation concerns the relation of segregation to inequalities. Spatial separation of social groups from each other has been detected

to be a cause and a consequence of the unequal levels of wellbeing between social groups. Higher levels of residential segregation by ethno-race groups is found to be responsible for low levels of education and occupation outcomes (Cutler and Glaeser, 1997; Echenique and Fryer, 2007). Lower racial inequality in terms of educational attainment is in turn shown to increase residential segregation (Bayer, Fang and McMillan, 2011). Segregation of black and non-black students into different schools has been blamed for substantial differences in achievement (Echenique and Fryer, 2007; Hanushek, Kain and Rivkin, 2009). An important share of wage inequality between women and men has been explained by gender-based occupational segregation (Macpherson and Hirsch, 1995; Peterson and Morgan, 1995; Blau and Kahn, 1997; Blau and Kahn, 2003). However not always the trend in wage inequality follows exactly the same pattern with occupational segregation (Preston, 1999). Hence there exists a close link between segregation and inequalities between groups, though the strength or the direction of this link is never clear.

Segregation is defined as a form of inequality. It is the inequality in the distributions of social groups across neither measurable nor comparable categories. Categories need not to be uncomparable by nature, but a comparison of these categories is not relevant to the question of segregation. For instance for residential segregation by race, neighborhood quality may very well define an ordering of neighborhoods. However the essence of the idea of residential segregation, “the degree to which two or more groups live separately from each other in different parts of the urban environment” (Massey and Denton, 1988) hinges on spatial difference in residential patterns. A quality ordering of the neighborhoods is not a part of the question of segregation itself, but brings in the notion of inequality. Segregation captures the nominal discrepancy of the distributions regardless of how good or bad the distributions are, whereas inequality involves an evaluation of the distributions. Hence the comparison of the categories do matter for inequality analysis in contrast to segregation. However, despite this conceptual distinction we can observe cases such that social groups are as segregated as unequally distributed. In other words, there exist societies in which segregation is actually the inequality between groups. Let us go back to the imaginary building example with residents from two different groups, given in the introduction. In the completely polarized society scenario, in which one of the groups is occupying the nicer flats with the view at the higher floors of the building, whereas all members of the other group are living downstairs facing the facade of the building across the street, both segregation and inequality between groups are at their maximum, hence equal for standardized measures of segregation and inequality. Moreover, this is not a unique example. One can find many other scenarios such that segregation coincides with the inequality. The Domination Index becomes helpful at

this point: It allows to identify the societies such that spatial inequality between groups captures the overall inequality, clarifying the structure of the relation between segregation and inequality.

In this section, we show that for particular societies the inequality between social groups measured by the Domination Index is equal to the segregation measured by a well-known segregation measure, Gini Segregation Index. Gini Segregation Index, G_S , is one of the oldest methods to measure segregation, suggested in the first paper on the subject (Jahn et. al., 1947). As the name suggests it shares the same underlying logic with Gini Inequality Index in measuring inequality as a normalized mean absolute difference between all pairs of components.¹⁰ Since the order relation L over \mathcal{I} is not an argument for segregation, a society is simply the society matrix over I positions that we denote as $S_{\mathcal{I}}$. Let S_{it} denote the total number of individuals in position i , q_i women share in position i and S_t and q the respective amounts for the whole society, i.e.; $S_{it} = S_{iw} + S_{im}$ and $q_i = S_{iw}/S_{it}$. A segregation index is simply a non-zero continuous function defined from $B = \cup_{\mathcal{I}} \mathbb{R}_+^{I \times 2}$ to \mathbb{R}_+ . Then, Gini Segregation Index is:

$$G_S(S_{\mathcal{I}}) = \sum_i \sum_j \frac{S_{it} S_{jt} |q_i - q_j|}{2S_t^2 q(1-q)}$$

G_S measures segregation in terms of the average difference in women shares of positions. It will be equal to zero only if the women shares of all positions are the same; for all i and j , $q_i = q_j$. This happens only if the proportion of women in each position, s_{iw} is equal to the proportion of men, s_{im} . Notice that in this case, as shown in Lemma 2 there will be zero social inequality as well, i.e., $D(S, L_{\mathcal{I}}) = 0$. Hence, no segregation implies no social inequality. On the other hand, G_S will take its maximum value as 1 under complete polarization, when all men occupy better positions than all women, or vice versa. In this case, D gives a value of 1, as well.

These two extreme cases with identical levels of segregation and inequality may seem as opposites. However they share an important property: In both of the cases the order of the positions are in line with the relative masses of groups occupying the positions. Going down in the hierarchy of positions, the ratio of number of women to number of men occupying a position follows a monotone path. Indeed this is a sufficient property to have equal levels of segregation and inequality. If the ratio of number of women to men is increasing or

¹⁰The general formulation of Gini inequality indices can be given as $\frac{1/T^2 \sum_{i=1}^n \sum_{j=1}^n |T_i - T_j|}{2T/n}$, where there are n components (individual, place, position) with component i possessing a T_i share of the T units (income, people) in total. In the context of income inequality it becomes $\frac{\sum_{i=1}^n \sum_{j=1}^n |y_i - y_j|}{2n \sum_{i=1}^n y_i}$, where y_i denotes the income level of individual i .

decreasing from the best to worse positions, then segregation of this society equals to the inequality between groups. Let r_i denote the proportion of number of women to number of men in position i , i.e.: $r_i = \frac{S_{iw}}{S_{im}}$. Then, we have the following proposition:

Proposition 2 *For any $(S, L_{\mathcal{I}})$ in C , we have $G_S(S_{\mathcal{I}}) = D(S, L_{\mathcal{I}})$ if and only if (i) $r_i \geq r_j$ for all i and j with iLj or (ii) $r_j \geq r_i$ for all i and j with iLj .*

The proof of the proposition follows fast from an alternative expression of Gini Segregation Index as $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}|$. This expression of G_S stresses out the relation between G_S and D , since D can alternatively be stated as: $D(S, L_{\mathcal{I}}) = |\sum_i \sum_{j:iLj} s_{iw}s_{jm} - s_{im}s_{jw}|$. Both of them evaluate the average difference in cross products of group shares, the amount $(s_{iw}s_{jm} - s_{im}s_{jw})$, for pairs of positions. However there are two main distinctions: (i) In case of inequality this amount refers to the difference in number of dominations by groups, hence D makes use of the order relation L and aggregates over for pairs of positions i and j with iLj . In case of segregation this amount is a measure of how differently distributed two groups over i and j , hence G_S aggregates it for any pair of positions without reference to an order relation. (ii) G_S is the summation of absolute values over pairs of positions; what matters is the nominal difference in distributions. For any pair of positions i, j , the contribution to the overall segregation is always nonnegative. D is the absolute value of a sum over positions. Inequality does not need to be in the same direction over all pairs of positions.

Thus, combining (i) and (ii), if the structure of the society is such that inequality is always favoring the same group, then D would be equal to G_S . This is possible only if the positions are ordered according to the relative masses of the groups occupying them. If from the best to the worst position r_i is always decreasing, then Women always have an advantageous distribution, i.e.; number of dominations by Women is larger than Men for any pair of positions. If, on the other hand, r_i is increasing from the best to the worst position, then Men always have an advantageous distribution.

Hence for a particular organization of the society, segregation is the inequality between groups. If the order of importance of the positions is reflected by the relative distribution of the groups, then segregation is actually responsible from the inequality.

This simple result not only helps to understand the theoretical link between segregation and inequality, but also provides a characterization of the Gini Segregation Index. We exploit the relation between D and G_S to adapt the characterizing properties of D : SYM and INV properties remain the same. MON property becomes redundant as there is no direction in segregation as opposed to inequality. However we need an additional property to fix the level

of no segregation to zero. A normalization property (NORM) requires that if the distribution of Women is exactly equal to the distribution of Men, then there is zero segregation. Notice that, this property was implied by MON in case of inequality. The DEC and SAD properties are the ones that require to be adapted with reference to the r_i ordering instead of the exogenous order of positions. We define r-DEC as the decomposability of overall segregation into two different segments of the society and an interaction term between them where the upper segment consists of the positions with higher r_i ratios with respect to the positions of the lower segment. Similarly, r-SAD ensures that overall segregation could be expressed as a weighted sum of the levels of segregation of the subsocieties if the r_i ordering of the positions is preserved for subsocieties with respect to the original society. As before, r-SAD implies INV. Formal definitions of the properties are introduced in the Appendix as well as the proof of the following theorem:

Theorem 2 *A segregation index $H : B \rightarrow \mathbb{R}_+$ satisfies SYM, NORM, r-DEC, r-SAD if and only if H is a positive scalar transformation of the Gini Segregation Index: $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}|$*

4 Extension: Inequality with Incomplete Information

The previous section has established that if the order of the positions of a society does not follow a particular pattern, then the level of inequality will be different than the level of segregation. The proof of this claim demonstrates that for any other organization of the society, inequality will actually be less severe than segregation. For the sake of completeness, let us state this observation formally:

Let $\mathcal{L}_{\mathcal{I}}$ denote the set of linear orderings of \mathcal{I} .

Corollary to Proposition 2 For any $(S_{\mathcal{I}})$ in B , the level of segregation measured by G_S is equal to the maximum level of inequality measured by D over all possible linear orderings of \mathcal{I} , i.e;

$$G_S(S_{\mathcal{I}}) = \max_{L_{\mathcal{I}} \in \mathcal{L}_{\mathcal{I}}} D(S, L_{\mathcal{I}})$$

When there is no information about the ordering of positions the only inequality between groups is due to segregation and is equal to the maximum group inequality over all possible linear orderings of \mathcal{I} . When the information is not null but not complete either, we could actually follow the same argumentation. For instance, in order to evaluate the inequality

of the distributions of gender groups in a firm hierarchy, one could encounter problems in ordering the positions completely. Firm hierarchies do not necessarily show a linear pattern, but they mostly follow a tree structure. This would mean that for some positions the order relation is clear, but not necessarily all positions are compared to each other. In other words, the order relation is incomplete. How could we measure the inequality of distributions if we have incomplete information about the ordering of the positions?

We propose to follow what is suggested by the previous observation and complete the missing information by considering all possible linear orderings of the positions and determining the maximum possible level of group inequality. When there is complete information about the ordering of the positions, the Domination Index makes use of all of the existing information. Under no information regarding the ordering of the positions, it would be safe to consider the maximum level of group inequality over all possible ways of completing the existing information since we have shown that this coincides with segregation. Then under incomplete information about the ordering of the positions a natural extension would be to consider all possible ways of completing it. The maximum level of group inequality over all possible completions would be qualified as the group inequality in that society.

Let $P_{\mathcal{I}}$ be a strict partial order over the set of positions \mathcal{I} . A society will be a pair of elements $(S, P_{\mathcal{I}})$, where S is the usual society matrix. Let $\mathcal{L}^{P_{\mathcal{I}}}$ denote the set of linear extensions of P over \mathcal{I} , i.e.; the set of complete, transitive and asymmetric binary relations over \mathcal{I} with for all $L_{\mathcal{I}}$ in $\mathcal{L}^{P_{\mathcal{I}}}$, iLj if iPj . Then, Maximum Group Inequality Index, M , will be a continuous function defined from the set of all possible societies to nonnegative real numbers in the following way:

$$M(S, P_{\mathcal{I}}) = \max_{L_{\mathcal{I}} \in \mathcal{L}^{P_{\mathcal{I}}}} D(S, L_{\mathcal{I}})$$

We know that M is a relative group inequality measure that takes values in $[0, 1]$ as well. If there is no missing information about the ordering of the positions, M is equal to D . If there is no ordering information available, then the only inequality between groups is due to segregation and that is completely captured by M , since it is equal to G_S for this case. In case of some missing information, M gives the maximum possible level of group inequality, which refers to the worst-case scenario of the society. If two positions remain uncomparated by the original ordering, this will be because of the fact that there is no unique universal way of ranking these positions; their ordering may change from time to time, society to society. Considering the worst-case scenario is consistent with a Rawlsian framework of welfare, apart from being a natural outcome of the structural relations between inequality and segregation.

4.1 An Empirical Exercise: Gender-based Occupational Inequality in Europe

In this section we provide a quick application of Maximum Inequality Index to assess the discrepancy in the distribution of genders across occupational groups in Europe. According to International Standard Classification of Occupations (ISCO), jobs are classified into occupational groups with respect to the skill level and skill specialization required to competently perform the tasks and duties of the occupations.¹¹ Figure 1 summarizes the gender distribution across 9 major occupational groups in 9 European countries by 2010.^{12 13}

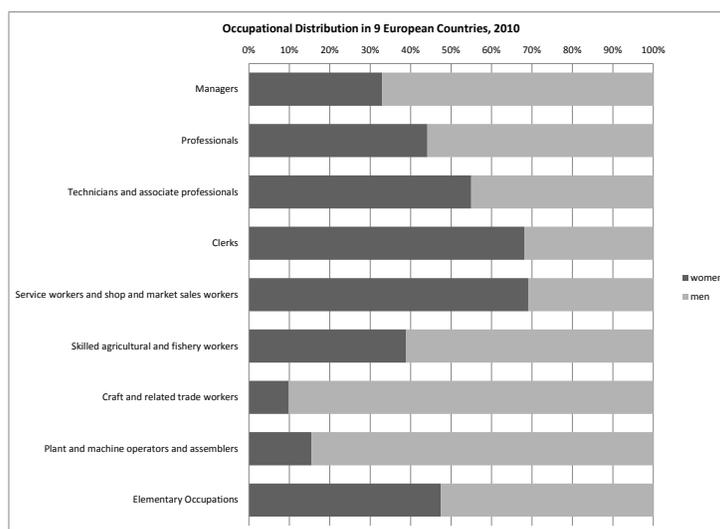


Figure 1: Occupational distribution of gender groups in Austria, Denmark, Greece, Germany, Iceland, Italy, Luxembourg, Portugal, United Kingdom in 2010

In order to evaluate the inequality between gender groups across these occupational categories we need to take into account welfare attributes of occupations such as income, working conditions or other socio-economic status indicators offered by the occupations. The hinge is that not necessarily all attributes are perfectly correlated. An occupation may have quite challenging working conditions, even resulting in health troubles, although offering a

¹¹We make use of the latest version of the classification system, ISCO08, which is published by International Labor Organization in 2008. (<http://www.ilo.org/public/english/bureau/stat/isco/isco08/index.htm>)

¹²According to ISCO08, there are 10 major occupational groups. We leave out category 0, "Armed forces occupation" due to data restrictions.

¹³The distribution data used in this exercise is taken from United Nations Economic Commission for Europe (UNECE) Statistical Division Database (<http://w3.unece.org/pxweb/>).

very high level of wage. Hence taking multi attributes into account, a linear ordering of occupations will not be possible. However, we can arrive to a partial ordering of occupations that will not contradict with any of the orderings suggested by each welfare attribute. Given this partial ordering, the Maximum Inequality Index will tell us the worst case scenario as the occupational inequality between gender groups.

Figure 2 shows the mean hourly wage pattern of occupational groups computed as 2010 European mean of total population and for women and men separately.¹⁴

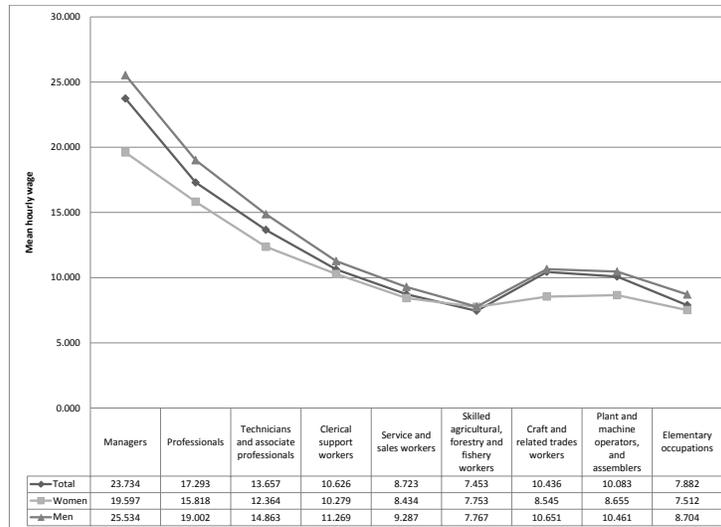


Figure 2: Mean hourly wage of occupational groups in Europe in 2010

Although wage levels for women and men differ, the order of occupational categories suggested by mean hourly wage of women, men and total population do coincide. The wage ordering of the occupations would be, in decreasing order, as the following: Managers, Professionals, Technicians and Associate Professionals, Clerical Support Workers, Craft and Related Trade Workers, Plant and Machine Operators and Assemblers, Service and Sales Workers, Elementary Occupations, Skilled Agricultural, Forestry and Fishery Workers.

The International Socio-Economic Index of occupational status (ISEI-08) is a scale designed for occupations using the required level of education and the earnings offered. It basically assigns an optimal score to each occupation that aims to minimize the direct effect of education on earnings and maximizing the indirect effect of education on earnings via

¹⁴The wage data is taken from the statistical database of Eurostat, Structure of earnings survey, 2010. (<http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/>)

occupation.¹⁵ The ISEI ordering of occupations computed according to ISCO08, with the corresponding ISEI08 scores in parentheses, gives us: Professionals (65), Managers (62), Technicians and Associate Professionals (51), Clerical Support Workers (41), Craft and Related Trade Workers (35), Plant and Machine Operators and Assemblers (32), Service and Sales Workers (31), Elementary Occupations (20), Skilled Agricultural, Forestry and Fishery Workers (18).

Finally, working conditions offered by the occupation is a significant welfare determinant, especially when health related outcomes are considered. In order to assess the working conditions of occupations we make use of five different variables related to work context: Cramped Work Space-Awkward Positions (How often does this job require working in cramped work spaces that requires getting into awkward positions?), Exposed to Hazardous Conditions (How often does this job require exposure to hazardous conditions?), Spend Time Making Repetitive Motions (How much does this job require making repetitive motions?), Deal With Unpleasant or Angry People (How frequently does the worker have to deal with unpleasant, angry, or discourteous individuals as part of the job requirements?), Lack of Decision Power (How much decision making freedom, without supervision, does the job offer?).¹⁶ Averaging the scores attached to each of these variables for each occupational category, we arrive to the work conditions ordering as the following: Managers (25.08), Professionals (26.61), Technicians and Associate Professionals (35.03), Clerical Support Workers (35.54), Service and Sales Workers (37.35), Skilled Agricultural, Forestry and Fishery Workers (38.73), Craft and Related Trade Workers (43.68), Elementary Occupations (43.83), Plant and Machine Operators and Assemblers (47.77).

The following figure corresponds to the partial ordering of occupations once we consider all three orderings together:

Below we present the gender-based occupational inequality in 9 European countries in 2000 and 2010, computed by the Maximum Inequality Index by making use of this partial ordering of occupations.

¹⁵For further reference: <http://www.harryganzeboom.nl/isco08/qa-isei-08.htm>

¹⁶Data related to these variables is taken from Occupational Information Network (ONET) database (<http://www.onetonline.org/>). ONET database contains information on a variety of standardized and occupation-specific descriptors and provides importance and levels of these descriptors for each occupation. It is based on the Standard Occupational Classification. In order to translate it to ISCO08 we make use of the crosswalk suggested by US Department of Labor, Bureau of Labor Statistics: <http://www.bls.gov/soc/soccrosswalks.htm>.

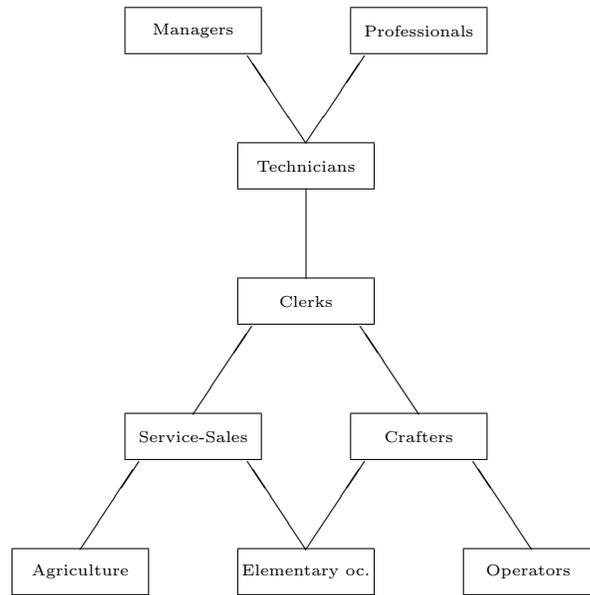


Figure 3: Order of Occupations according to Wage, Socio-Economic Status and Working Conditions

Table 2. Gender-based Occupational Inequality by Maximum Inequality Index

Country	2000	2010
Austria	0.1715	0.1928
Denmark	0.1958	0.1442
Germany	0.1499	0.1479
Greece	0.1993	0.2001
Iceland	0.2436	0.2704
Italy	0.2116	0.1577
Luxembourg	0.1676	0.1527
Portugal	0.1778	0.1821
United Kingdom	0.1867	0.1622

In 3 out of 9 countries, Austria, Greece and Iceland, occupational inequality has tended to increase in 10 years span. Let us note that, out of these 9 countries, only in Iceland, inequality is actually favoring women. In all countries but Iceland, men have a more advantageous distribution with respect to women.

5 Concluding Remarks

Unequal distribution of social groups across different levels of welfare is quite commonly observed. When we go beyond income inequality and consider non-cardinal welfare determining variables such as education, health, occupation or subjective well-being, we run short of well-developed inequality measurement techniques. This paper aimed to propose an intuitive and well-founded methodology to evaluate non-income inequalities between social groups without appealing to additional cardinalisation assumptions.

The Domination Index evaluates the discrepancy in group distributions as a function of the number of times a group beats the other group. We showed in a first result that a set of properties, a classical Symmetry property, a Monotonicity property and two decomposability properties characterize the Domination Index up to a positive scalar transformation. The Domination Index is instrumental in clarifying the intimate link between social inequalities and segregation. In a second result, we showed that segregation is actually the inequality for a very specific distribution of the society, where the organization coincides with the socially worst outcome. Furthermore, we exploited this theoretical link between segregation and inequality to propose a technique to evaluate inequalities where the information regarding the ordering of the categories is not necessarily complete and provided a simple empirical exercise to evaluate gender-based occupational inequality across nine European countries.

We provided the index for evaluation of inequalities between two social groups. However there are many real life cases that require a multi-group analysis. A natural way to extend the Domination Index to multi-group case is to consider an aggregation of the differences in pairwise dominations for any pair of groups. When there are more than two social groups, we first focus on pairs of groups and calculate for each pair the average difference in number of dominations, i.e., the Domination Index for two groups. Then, the average of these average differences would be a multi-group version of the Domination Index. Let us state this idea formally: Let \mathcal{G} be a set of social groups with cardinality G . Then a society with G groups and I positions will be a pair $(S, L_{\mathcal{I}})$ where S is a society matrix of dimension $I \times G$ and the multi-group Domination Index would be equal to $\frac{1}{2G} \sum_{M \in \mathcal{G}} \sum_{N \in \mathcal{G}} D(S_M, S_N)$, where S_M denotes the vector of group M in S as usual. Notice that this is again a relative inequality measure that takes values between 0 and 1.

The foundational analysis of the multi-group version of the Domination Index is a question of ongoing research as well as its relation to multi-group segregation indices. In addition, the algorithmic structure and behavior of the Maximum Group Inequality Index remain to be explored.

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6 Appendix

Extra Notation for the Proofs:

To denote a society $(S, L_{\mathcal{I}})$ with $1L2L \dots LI$, we use $S = (S_{1w}, S_{1m}; S_{2w}, S_{2m}; \dots; S_{lw}, S_{lm})$ or in a more compact form $S = (S_1; S_2; \dots; S_l)$ where each S_i is a row vector of dimension 2×1 such that $S_i = (S_{iw}, S_{im})$. Given $(S, L_{\mathcal{I}})$ with $1L2L \dots LI$, S_j^k is used to denote a row vector of dimension 2×1 where the first entry is the sum of all women in $(S, L_{\mathcal{I}})$ from position j to k and the second entry is the sum of all men in the same positions, i.e.; $S_j^k = \sum_{i=j}^k S_i = (\sum_{i=j}^k S_{iw}, \sum_{i=j}^k S_{im})$. We use $\sum_i S_{iw}$ to denote $\sum_{i \in \mathcal{I}} S_{iw}$ and $\sum_k^l S_{iw}$ to denote $\sum_{i=k}^l S_{iw}$.

Proof of Lemma 1: Immediate from the expression of D as: $D(S, L_{\mathcal{I}}) = |\sum_i (s_{iw} \sum_{j:iL_j} s_{jm}) - \sum_i (s_{im} \sum_{j:iL_j} s_{jw})|$. \square

Proof of Lemma 2: Consider any $(S, L_{\mathcal{I}})$ such that for any i , $S_{iw} = S_{im}$. Notice that since S_W dominates S_M , MON guarantees that S is of W -type. Similarly since S_M dominates S_W as well, S is of M -type. It is immediate to show that by definition, for any H , S is of both W -type and M -type if and only if $H(S, L_{\mathcal{I}}) = 0$. \square

Proof of Proposition 1: Consider any H that satisfies SAD and any society $(S, L_{\mathcal{I}}) = (S_W, S_M)$. (i) Let $\alpha \in \mathbb{N}_{++}$. By using induction, we will show that $H(\alpha S_W, S_M) = H(S_W, S_M)$. For $\alpha = 2$, SAD implies: $\frac{1}{2}H(S_W, S_M) + \frac{1}{2}H(S_W, S_M) = H(2S_W, S_M) = H(S_W, S_M)$. Now assume that the statement holds for $\alpha - 1$, i.e.: $H((\alpha - 1)S_W, S_M) = H(S_W, S_M)$. Since, $H((\alpha - 1)S_W, S_M)$ is of the same type with $H(S_W, S_M)$, by SAD: $\frac{\alpha-1}{\alpha}H((\alpha - 1)S_W, S_M) + \frac{1}{\alpha}H(S_W, S_M) = H(\alpha S_W, S_M)$, which implies by the inductive argument: $\frac{\alpha-1}{\alpha}H(S_W, S_M) + \frac{1}{\alpha}H(S_W, S_M) = H(S_W, S_M) = H(\alpha S_W, S_M)$ as claimed. (ii) Now consider $\alpha \in \mathbb{Q}_{++}$. Let $\alpha = \frac{p}{q}$ for some $p, q \in \mathbb{N}_{++}$. Then, repeated application of SAD ensures the following: $q \frac{p}{q} H(\frac{p}{q} S_W, S_M) = H(p S_W, S_M)$. Since for $p \in \mathbb{N}_{++}$ we have proved that $H(p S_W, S_M) = H(S_W, S_M)$, we arrive; $H(\frac{p}{q} S_W, S_M) = H(S_W, S_M)$ as claimed. (iii) Finally let $\alpha \in \mathbb{R}_{++}$. Since every irrational number can be expressed as the limit value of a sequence of rational numbers, let $\alpha = \lim q_i$ for some $q_i \in \mathbb{Q}_{++} \forall i$. Then, $H(\alpha S_W, S_M) = H(\lim q_i S_W, S_M) = \lim H(q_i S_W, S_M)$ by continuity of the function H . Since we have already showed that for any rational α the statement holds, we arrive: $H(\alpha S_W, S_M) = H(S_W, S_M)$, establishing that for any $\alpha \in \mathbb{R}_{++}$, $H(\alpha S_W, S_M) = H(S_W, S_M)$. Since the same argumentation could be made for the men distribution, we have proved that SAD implies INV. \square

Proof of Theorem 1 We omit the proof of necessary part. To prove the sufficiency part, first we introduce a lemma with three parts. We show that INV together with MON imply; (i) any society that has members from only one of the groups has zero inequality; (ii) any society that has members from both of the groups and only one position occupied by a strictly positive number of individuals has zero inequality; (iii) any society that has members from both of the groups and has women only in the best position or men only in the worst position is of W -type and any society that has men only in the best position and women only in the worst position is of M -type.

Lemma 3 Let $H : C \rightarrow \mathbb{R}_+$ satisfy INV and MON. For any $(S, L_{\mathcal{I}})$, (i) if $S_w = 0$ or $S_m = 0$, then $H(S, L_{\mathcal{I}}) = 0$; (ii) if there exists $i \in \mathcal{I}$ with $S_{iw} > 0$ and $S_{im} > 0$ and for all $j \neq i$, $S_{jw} = S_{jm} = 0$, then $H(S, L_{\mathcal{I}}) = 0$; (iii) if $S_w \neq 0 \neq S_m$ and for $i \in \mathcal{I}$ such that there does not exist any $j \in \mathcal{I}$ with $j L i$, $S_{iw} = S_w$ or for $k \in \mathcal{I}$ such that there does not exist any $j \in \mathcal{I}$ with $k L j$, $S_{km} = S_m$, then S is of W -type. Moreover if $S_{im} = S_m$ or $S_{kw} = S_w$, then S is of M -type.

Proof (i) Let $(S, L_{\mathcal{I}})$ be such that $S_w = 0$. Let $(S', L_{\mathcal{I}})$ be such that $S'_M = S_M$ and for some $\epsilon > 0$, $S'_W = \epsilon S_M$. By INV and Lemma 2, $H(S', L_{\mathcal{I}}) = 0$. By continuity of H , $\lim_{\epsilon \rightarrow 0} H(S', L_{\mathcal{I}}) = H(S, L_{\mathcal{I}}) = 0$. The same argument holds for any $(S, L_{\mathcal{I}})$ with $S_m = 0$. (ii) Let $(S, L_{\mathcal{I}})$ be as stated. The result is immediate from INV and Lemma 2. (iii) Let $(S, L_{\mathcal{I}})$ be such that for $i \in \mathcal{I}$ such that there does not exist $j \in \mathcal{I}$ with $j L i$, $S_{iw} = S_w > 0$ and $S_m > 0$. By INV $H(S_W, S_M) = H(\frac{S_W}{S_w}, \frac{S_M}{S_m}) = H(S'_W, S'_M)$. Since $S'_{iw} = 1$ and for any $j \in \mathcal{I} \setminus \{i\}$, $S'_{jw} = 0$, S'_W dominates S'_M . Thus, by MON, S' and S are of W -type. Similar arguments establish the result for the other society matrices defined in the statement of Lemma. \square

Now we start with the proof of Theorem 1. Let $(S, L_{\mathcal{I}}) \in C$. If $I = 1$, then by Lemma 3(ii), $H(S, L_{\mathcal{I}}) = 0$. Let $I \geq 2$. For notational simplicity, let us name the positions such that L over \mathcal{I} is as $1 L 2 L \dots L I$. Later we show that this holds without loss of generality.

In Step 1, we consider a very specific type of society and derive the functional form of H for it. We define a **W -perfect society matrix** as one with strictly positive number of women and men in each position and a strictly decreasing r_i ordering from the best to the worst position, i.e.; $\infty > r_1 > r_2 > \dots > r_I > 0$.

Step 1: Let S be a W -perfect society matrix. Then

$$H(S_1; \dots; S_I) = \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} H(S_i; S_{i+1}^I).^{17}$$

Step 1.1: S is of W -type.

By Proposition 1, H satisfies INV. Then, for $(S', L_{\mathcal{I}}) = (s_{1w}, s_{1m}; \dots; s_{Iw}, s_{Im})$, we have $H(S, L_{\mathcal{I}}) = H(S', L_{\mathcal{I}})$. Notice that S' is W -perfect and $S'_w = S'_m = 1$. We now show that S'_W dominates S'_M . Then, by MON, S' is of W -type, implying that S is of W -type.

By W -perfection, for any $k = 1, \dots, I$, and for $j = k + 1, \dots, I$

$$S'_{kw} S'_{jm} > S'_{jw} S'_{km}.$$

Thus, for each j , summing up these equations

$$S'_{kw} \sum_{k+1}^I S'_{jm} > S'_{km} \sum_{k+1}^I S'_{jw}. \quad (1)$$

Since (1) holds for each $k = 1, \dots, I - 1$, summing over all k

$$\begin{aligned} \sum_1^k (S'_{iw} \sum_{k+1}^I S'_{jm}) &> \sum_1^k (S'_{im} \sum_{k+1}^I S'_{jw}) \\ \sum_1^k S'_{iw} (1 - \sum_1^k S'_{jm}) &> \sum_1^k S'_{im} (1 - \sum_1^k S'_{jw}) \\ \sum_1^k S'_{iw} - \sum_1^k S'_{iw} \sum_1^k S'_{jm} &> \sum_1^k S'_{im} - \sum_1^k S'_{im} \sum_1^k S'_{jw} \\ \sum_1^k S'_{iw} &> \sum_1^k S'_{im}. \end{aligned}$$

Since $\sum_1^I S'_{iw} = \sum_1^I S'_{im} = 1$, it follows from MON that S' is of W -type. Then, S is of W -type as well. Hence any W -perfect S is of W -type.

As a direct implication of Step 1.1, for a W -perfect S , since for each $i = 1, \dots, I - 1$, $(S_i; \dots; S_I)$ is W -perfect, it is of W -type as well.

¹⁷Notice that for $i = I$, $H(S_I; S_{I+1}^I)$ does not exist. For the sake of simplicity we keep the notation this way.

Step 1.2: For any $i = 1, \dots, I - 1$, $(S_i; S_{i+1}^I)$ is of W -type.

Since for any $i \in \mathcal{I}$, $\infty > r_i > r_{i+1} > \dots > r_I > 0$, then

$$\begin{aligned} S_{iw} \sum_{i+1}^I S_{jm} &> S_{im} \sum_{i+1}^I S_{jw} \\ \frac{S_{iw}}{S_{im}} &> \frac{\sum_{i+1}^I S_{jw}}{\sum_{i+1}^I S_{jm}}. \end{aligned}$$

Then, $(S_i; S_{i+1}^I)$ is a W -perfect society matrix with 2 positions. By Step 1.1, for each $i = 1, \dots, I - 1$, $(S_i; S_{i+1}^I)$ is of W -type.

Step 1.3: $H(S_1; \dots; S_I) = \sum_i \frac{\sum_i S_{jw} \sum_i S_{jm}}{S_w S_m} H(S_i; S_{i+1}^I)$.

First consider the division of society $(S, L_{\mathcal{I}}) = (S_1; \dots; S_I)$ as S_1 and $(S_2; \dots; S_I)$. Since S is W -perfect, by Step 1.2, these societies and the interaction society are of W -type. Thus, by DEC

$$H(S_1; \dots; S_I) = \frac{S_{1w} S_{1m}}{S_w S_m} H(S_1) + \frac{\sum_2^I S_{iw} \sum_2^I S_{im}}{S_w S_m} H(S_2; \dots; S_I) + H(S_1; S_2^I). \quad (2)$$

By Lemma 3(ii), $H(S_1) = 0$. Now, consider the division of the society $(S_2; \dots; S_I)$ as S_2 and $(S_3; \dots; S_I)$. Again by W -perfection, DEC yields

$$H(S_2; \dots; S_I) = \frac{S_{2w} S_{2m}}{\sum_2^I S_{iw} \sum_2^I S_{im}} H(S_2) + \frac{\sum_3^I S_{iw} \sum_3^I S_{im}}{\sum_2^I S_{iw} \sum_2^I S_{im}} H(S_3; \dots; S_I) + H(S_2; S_3^I).$$

Since by Lemma 3(ii), $H(S_2) = 0$, substitution into (2) yields:

$$H(S_1; \dots; S_I) = \frac{\sum_3^I S_{iw} \sum_3^I S_{im}}{S_w S_m} H(S_3; \dots; S_I) + \frac{\sum_2^I S_{iw} \sum_2^I S_{im}}{S_w S_m} H(S_2; S_3^I) + H(S_1; S_2^I).$$

Iterative application of DEC and Lemma 3(ii) results in

$$H(S_1; \dots; S_I) = \sum_i \frac{\sum_i S_{jw} \sum_i S_{jm}}{S_w S_m} H(S_i; S_{i+1}^I), \quad (3)$$

concluding Step 1.¹⁸

¹⁸Notice that DEC allows to express overall inequality as a weighted sum of inequalities in 2×2 society matrices naming the best position as 1 and the other position as 2. This shows that neutrality of positions is implied by DEC, naming the positions as $1 \ L \ 2 \ L \ \dots \ L \ I$ is without loss of generality.

Step 2: For each society $(S, L_{\mathcal{I}}) \in C$ with a W -perfect society matrix S

$$H(S_1; \dots; S_I) = \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1).$$

Step 2.1: For any $i = 1, \dots, I - 1$

$$H(S_i; S_{i+1}^I) = \frac{S_{iw} \sum_{i+1}^I S_{jm} - S_{im} \sum_{i+1}^I S_{jw}}{\sum_i S_{jw} \sum_i S_{jm}} H(1, 0; 0, 1).$$

First notice that by INV, $H(S_{iw}, S_{im}; \sum_{i+1}^I S_{jw}, \sum_{i+1}^I S_{jm}) = H(s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$. For simplicity let us use the notation $H(a, b; c, d)$ instead of $H(s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$. By Step 1.2, $(a, b; c, d)$ is of W -type. Since $a + c = b + d = 1$, then $a > b$ and $d > c$. Since $\frac{b}{c} = 1 > \frac{c}{d}$, by Step 1.1, $(b, b; c, d)$ is of W -type and by Lemma 3(iii), $(a - b, b; 0, d)$ is of W -type. Then, by SAD

$$H(a, b; c, d) = \frac{b+c}{a+c} H(b, b; c, d) + \frac{a-b}{a+c} H(a-b, b; 0, d). \quad (4)$$

Moreover, since by Lemma 2, $H(b, b; c, c) = 0$, by definition $(b, b; c, c)$ is of W -type and by Lemma 3(iii), $(b, 0; c, d - c)$ is of W -type. Then, by SAD

$$H(b, b; c, d) = \frac{b+c}{b+d} H(b, b; c, c) + \frac{d-c}{b+d} H(b, 0; c, d - c). \quad (5)$$

Combining (4) and (5)

$$H(a, b, c, d) = \frac{(b+c)(d-c)}{(a+c)(b+d)} H(b, 0; c, d - c) + \frac{a-b}{a+c} H(a-b, b; 0, d).$$

Similarly, by Lemma 3(ii), $H(0, 0; c, d - c) = H(a - b, b; 0, 0) = 0$. Then, by definition $(0, 0; c, d - c)$ and $(a - b, b; 0, 0)$ are of W -type. By Lemma 3(iii), $(b, 0; 0, d - c)$ and $(a - b, 0; 0, d)$ are of W -type. Then, by SAD

$$\begin{aligned} H(b, 0; c, d - c) &= \frac{c}{b+c} H(0, 0; c, d - c) + \frac{b}{b+c} H(b, 0; 0, d - c), \\ H(a - b, b; 0, d) &= \frac{d}{b+d} H(a - b, 0; 0, d) + \frac{b}{b+d} H(a - b, b; 0, 0), \end{aligned}$$

resulting in

$$H(a, b, c, d) = \frac{b(d-c)}{(a+c)(b+d)} H(b, 0; 0, d - c) + \frac{(a-b)(d)}{(a+c)(b+d)} H(a - b, 0; 0, d).$$

Finally, by INV

$$\begin{aligned} H(a, b; c, d) &= \frac{b(d-c)}{(a+c)(b+d)} H(1, 0; 0, 1) + \frac{(a-b)d}{(a+c)(b+d)} H(1, 0; 0, 1) \\ &= \frac{ad-bc}{(a+c)(b+d)} H(1, 0; 0, 1). \end{aligned}$$

Going back to the original notation

$$H(S_i; S_{i+1}^I) = \frac{S_{iw} \sum_{j=i+1}^I S_{jm} - S_{im} \sum_{j=i+1}^I S_{jw}}{\sum_{j=i}^I S_{jw} \sum_{j=i}^I S_{jm}} H(1, 0; 0, 1). \quad (6)$$

Step 2.2: $H(S_1; \dots; S_I) = \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1)$.

Combining equation (3) and (6),

$$\begin{aligned} H(S_1; \dots; S_I) &= \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} \frac{S_{iw} \sum_{i+1}^I S_{jm} - S_{im} \sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw} \sum_i^I S_{jm}} H(1, 0; 0, 1) \\ &= \sum_i \frac{S_{iw} \sum_{i+1}^I S_{jm} - S_{im} \sum_{i+1}^I S_{jw}}{S_w S_m} H(1, 0; 0, 1) \\ &= \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1), \end{aligned} \quad (7)$$

concluding Step 2.

Now let us define an **M -perfect society matrix** as one with strictly positive number of women and men in each position and a strictly increasing r_i ordering, i.e.; $0 < r_1 < r_2 < \dots < r_I < \infty$.

Step 3: For each society $(S, L_{\mathcal{I}}) \in C$ with an M -perfect society matrix

$$H(S, L_{\mathcal{I}}) = - \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1).$$

Let $(S, L_{\mathcal{I}}) \in C$ be such that S is M -perfect. Let $(S', L_{\mathcal{I}}) \in C$ be such that $S'_W = S_M$

and $S'_M = S_W$. Thus S' is W -perfect. By SYM, $H(S', L_{\mathcal{I}}) = H(S, L_{\mathcal{I}})$. Then, by Step 2

$$\begin{aligned}
H(S, L_{\mathcal{I}}) &= H(S', L_{\mathcal{I}}) \\
&= \sum_i (s'_{iw} \sum_{i+1}^I s'_{jm} - s'_{im} \sum_{i+1}^I s'_{jw}) H(1, 0; 0, 1) \\
&= \sum_i (s_{im} \sum_{i+1}^I s_{jw} - s_{iw} \sum_{i+1}^I s_{jm}) H(1, 0; 0, 1) \\
&= - \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1).
\end{aligned}$$

Step 4: For each society $(S, L_{\mathcal{I}}) \in C$ with a W -type society matrix S

$$H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1).$$

Step 4.1: Let S be of W -type such that for all $i \in \mathcal{I}$, $S_{im} \neq 0$. There exist a W -perfect society matrix X with $X_M = S_M$ such that S' with $S'_W = S_W + X_W$ and $S'_M = S_M$ is W -perfect as well.

Let us denote the total number of women and men in S with S_t , i.e.; $S_t = S_w + S_m$. Let $(X, L_{\mathcal{I}}) \in C$ be such that $X_M = S_M$ and for any $i \in \mathcal{I}$, $X_{iw} = \sum_{k=2}^{I-i+1} (S_t)^k$. Then for any $i = 2, \dots, I$

$$\begin{aligned}
\frac{X_{(i-1)w}}{X_{(i-1)m}} &> \frac{X_{iw}}{X_{im}} \\
\frac{\sum_{k=2}^{I-i+2} (S_t)^k}{S_{(i-1)m}} &> \frac{\sum_{k=2}^{I-i+1} (S_t)^k}{S_{im}} \\
\frac{S_t \sum_{k=2}^{I-i+1} (S_t)^k}{S_{(i-1)m}} &> \frac{\sum_{k=2}^{I-i+1} (S_t)^k}{S_{im}} \\
\frac{S_t}{S_{(i-1)m}} &> \frac{1}{S_{im}}.
\end{aligned}$$

Thus, $\infty > r_{i-1} > r_i > 0$, establishing that X is W -perfect.

Now let $(S', L_{\mathcal{I}}) \in C$ be such that $S'_W = S_W + X_W$ and $S'_M = S_M$. For any $i = 2, \dots, I$,

$S'_{im} \neq 0$ and

$$\begin{aligned} \frac{S'_{(i-1)w}}{S'_{(i-1)m}} &> \frac{S'_{iw}}{S'_{im}} \\ \frac{S_{(i-1)w} + X_{(i-1)w}}{S_{(i-1)m}} &> \frac{S_{iw} + X_{iw}}{S_{im}} \\ \frac{S_{(i-1)w} + \sum_{k=2}^{I-i+2} (S_t)^k}{S_{(i-1)m}} &> \frac{S_{iw} + \sum_{k=2}^{I-i+1} (S_t)^k}{S_{im}} \\ S_{(i-1)w}S_{im} + S_{(i-1)w}S_{im} + \sum_{k=2}^{I-i+2} (S_t)^k S_{im} &> S_{iw}S_{(i-1)m} + \sum_{k=2}^{I-i+1} (S_t)^k S_{(i-1)m}. \end{aligned}$$

Since $\sum_{k=3}^{I-i+2} (S_t)^k S_{im} > \sum_{k=2}^{I-i+1} (S_t)^k S_{(i-1)m}$ and $(S_t)^2 S_{im} > S_{iw}S_{(i-1)m}$,

$$S_{(i-1)w}S_{im} + S_{(i-1)w}S_{im} + \sum_{k=3}^{I-i+2} (S_t)^k S_{im} + (S_t)^2 S_{im} > S_{iw}S_{(i-1)m} + \sum_{k=2}^{I-i+1} (S_t)^k S_{(i-1)m}.$$

Thus, $\infty > r'_{i-1} > r'_i > 0$, establishing that S' is W -perfect.

Step 4.2: For any W -type S such that for all $i \in \mathcal{I}$, $S_{im} \neq 0$, $H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1)$.

Since both S and X are of W -type, by SAD, $H(S', L_{\mathcal{I}}) = \frac{S_w}{S_w + X_w} H(S, L_{\mathcal{I}}) + \frac{X_w}{S_w + X_w} H(X, L_{\mathcal{I}})$. Since S' and X are W -perfect matrices, using the functional form of H for societies with W -perfect society matrices derived in Step 2.2,

$$\begin{aligned} H(S, L_{\mathcal{I}}) &= \frac{S_w + X_w}{S_w} H(S', L_{\mathcal{I}}) - \frac{X_w}{S_w} H(X, L_{\mathcal{I}}) \\ &= \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1). \end{aligned} \quad (8)$$

Step 4.3: Let S be of W -type. $H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1)$.

Let $(S', L_{\mathcal{I}}) \in C$ be such that for all $i \in \mathcal{I}$, $S'_{iw} = S_{iw}$, for all i with $S_{im} \neq 0$, $S'_{im} = S_{im}$ and for all i with $S_{im} = 0$, $S'_{im} = \varepsilon$ for some ε in a small neighborhood of 0. By Step 3 and Step 4.1:

$$H(S', L_{\mathcal{I}}) = \sum_i (s'_{iw} \sum_{i+1}^I s'_{jm} - s'_{im} \sum_{i+1}^I s'_{jw}) H(1, 0; 0, 1).$$

By continuity of H :

$$\begin{aligned}
H(S, L_{\mathcal{I}}) &= \lim_{\varepsilon \rightarrow 0} H(S', L_{\mathcal{I}}) \\
&= \lim_{\varepsilon \rightarrow 0} \sum_i (s'_{iw} \sum_{i+1}^I s'_{jm} - s'_{im} \sum_{i+1}^I s'_{jw}) H(1, 0; 0, 1) \\
&= \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1)
\end{aligned}$$

establishing the functional form for any W -type S .

Step 5: For each society $(S, L_{\mathcal{I}}) \in C$ with an M -type society matrix S

$$H(S, L_{\mathcal{I}}) = - \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1).$$

Symmetrically, now let S be not M -perfect but of M -type. Following the same technique in Step 4, one can establish the result.

Hence, for each society $(S, L_{\mathcal{I}}) \in C$ with a W -type society matrix S ,

$$H(S, L_{\mathcal{I}}) = \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1)$$

and with an M -type S

$$H(S, L_{\mathcal{I}}) = - \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1).$$

Since by construction each S is of W -type or of M -type, we have derived the functional form for all possible societies. At the beginning, we have assumed that $1 \ L \ 2 \ L \ \dots \ L \ I$. Then in general, for any $(S, L_{\mathcal{I}})$ in C

$$H(S, L_{\mathcal{I}}) = | \sum_i (s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw}) H(1, 0; 0, 1) |. \quad (9)$$

By definition, H is a nonzero function. Thus, $H(1, 0; 0, 1)$ is a strictly positive real number. \square

Proof of Proposition 2: It is straightforward to show that $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}|$.

First we will show that if (i) or (ii) holds, then $G_S(S_{\mathcal{I}}) = D(S, L_{\mathcal{I}})$. Notice that $(s_{iw}s_{jm} - s_{im}s_{jw}) > 0$ if and only if $r_i > r_j$ and $(s_{iw}s_{jm} - s_{im}s_{jw}) = 0$ if and only if $r_i = r_j$. Hence G_S can equivalently be expressed as:

$$\begin{aligned} G_S(S_{\mathcal{I}}) &= \frac{1}{2} \sum_i \sum_j |s_{iw}s_{jm} - s_{im}s_{jw}| \\ &= \frac{1}{2} \left(\sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) + \sum_i \sum_{j:r_i < r_j} -(s_{iw}s_{jm} - s_{im}s_{jw}) \right). \end{aligned}$$

Notice that $\sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) = \sum_i \sum_{j:r_i < r_j} -(s_{iw}s_{jm} - s_{im}s_{jw})$. Since for i, j with $r_i = r_j$, $(s_{iw}s_{jm} - s_{im}s_{jw}) = 0$, then:

$$G_S(S_{\mathcal{I}}) = \sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) = \sum_i \sum_{j:r_i \leq r_j} -(s_{iw}s_{jm} - s_{im}s_{jw}).$$

Now let us assume (i) holds. Since for any i , $(s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw}) \geq 0$, we have:

$$\begin{aligned} D(S, L_{\mathcal{I}}) &= \sum_i (s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw}) \\ &= \sum_i \sum_{j:iLj} (s_{iw}s_{jm} - s_{im}s_{jw}) \\ &= \sum_i \sum_{j:r_i \geq r_j} (s_{iw}s_{jm} - s_{im}s_{jw}) \end{aligned}$$

establishing the claim. Now assume that (ii) holds. Then, we have:

$$\begin{aligned} D(S, L_{\mathcal{I}}) &= \sum_i -(s_{iw} \sum_{j:iLj} s_{jm} - s_{im} \sum_{j:iLj} s_{jw}) \\ &= \sum_i \sum_{j:iLj} -(s_{iw}s_{jm} - s_{im}s_{jw}) \\ &= \sum_i \sum_{j:r_j \geq r_i} -(s_{iw}s_{jm} - s_{im}s_{jw}) \end{aligned}$$

as claimed.

Now we will show that if $G_S(S_{\mathcal{I}}) = D(S, L_{\mathcal{I}})$ then (i) or (ii) holds. First notice that $D(S, L_{\mathcal{I}}) = |\sum_i \sum_{j:iLj} d_{ij} a_{ij}| = |\sum_{ij:iLj} d_{ij} a_{ij}|$ where $a_{ij} = (s_{iw}s_{jm} - s_{im}s_{jw})$ and $d_{ij} = 1$ if $r_i \geq r_j$ and $d_{ij} = -1$ if $r_i < r_j$. Hence for any two positions i and j , since either iLj or jLi , if a_{ij} enters the sum, a_{ji} does not and for sure either a_{ij} or a_{ji} enters. If (i) holds,

then $d_{ij} = 1$ for all ij with iLj . Then, $D(S, L_{\mathcal{I}}) = \sum_{ij:iLj} |a_{ij}| = \sum_{ij:iLj} a_{ij} = G_S(S_{\mathcal{I}})$ as shown in the sufficiency part. If (ii) holds, then $d_{ij} = -1$ for all ij with iLj . Then, $D(S, L_{\mathcal{I}}) = |\sum_{ij:iLj} -|a_{ij}|| = \sum_{ij:iLj} |a_{ij}| = \sum_{ij:iLj} -a_{ij} = G_S(S_{\mathcal{I}})$ as shown in the sufficiency part. If neither (i) nor (ii) holds, then for some ij with iLj and $a_{ij} \neq 0$ we have $d_{ij} = 1$ and for some other ij with iLj we have $d_{ij} = -1$. Then $D(S, L_{\mathcal{I}}) < \sum_{ij:iLj} |a_{ij}| = G_S(S_{\mathcal{I}})$, concluding the proof. \square

Characterizing Properties of Gini Segregation Index

(SYM): For any $(S_{\mathcal{I}})$ and $(S'_{\mathcal{I}})$ with $S_W = S'_M$ and $S_M = S'_W$, $H(S_{\mathcal{I}}) = H(S'_{\mathcal{I}})$.

(INV): Given $(S_{\mathcal{I}})$ and any $\alpha, \beta \in \mathbb{R}_+$, for $(S'_{\mathcal{I}})$ such that for all i , $S'_{iw} = \alpha S_{iw}$ and $S'_{im} = \beta S_{im}$, $H(S_{\mathcal{I}}) = H(S'_{\mathcal{I}})$.

(NORM): For any $(S_{\mathcal{I}})$ such that for any i , $S_{iw} = S_{im}$, we have $H(S_{\mathcal{I}}) = 0$.

(r-DEC): For any r-ordered division of $S_{\mathcal{I}}$ as $S_{\mathcal{I}^1}^1$ and $S_{\mathcal{I}^2}^2$, the following holds: $H(S_{\mathcal{I}}) = \lambda_{S^1} H(S_{\mathcal{I}^1}^1) + \lambda_{S^2} H(S_{\mathcal{I}^2}^2) + H(S'_{\mathcal{I}'})$, where **an r-ordered division** of $S_{\mathcal{I}}$ is a pair of societies $S_{\mathcal{I}^1}^1$ and $S_{\mathcal{I}^2}^2$ such that: (i) \mathcal{I}^1 and \mathcal{I}^2 define a partition of \mathcal{I} such that for any i in \mathcal{I}^1 and any j in \mathcal{I}^2 we have $r_i \geq r_j$. (ii) for $k = 1, 2$, S^k is a $I^k \times 2$ society matrix such that each position i possess the same number of women and men in S and S^k ; $S'_{\mathcal{I}'}$ denotes **the interaction society**, which is a society of two positions, $\mathcal{I}' = 1, 2$, with $S'_{1w} = S_w^1$, $S'_{1m} = S_m^1$, $S'_{2w} = S_w^2$ and $S'_{2m} = S_m^2$; λ_{S^k} refers to the population weight of society part S^k .

(r-SAD): For any partition of a society $(S_{\mathcal{I}})$ into subsocieties $(S'_{\mathcal{I}'})$ and $(S''_{\mathcal{I}''})$ such that for any i and j , $r_i \geq r_j$ if and only if $r'_i \geq r'_j$ if and only if $r''_i \geq r''_j$ holds, the following holds: $H(S_{\mathcal{I}}) = \lambda_{S'} H(S'_{\mathcal{I}'}) + \lambda_{S''} H(S''_{\mathcal{I}''})$.

Proof of Theorem 2: We omit the necessary part. For sufficiency part we first introduce a couple of lemmas:

Lemma 4 *Any $H : B \rightarrow R_+$ that satisfies r-SAD is Scale Invariant.*

Proof of Lemma 4: The proof is exactly the same with the proof of Proposition 1 with the exception that r-SAD could be applied at any induction step since the r_i ordering of $(\alpha S_W, S_M)$ is the same with the one of (S_W, S_M) for any $\alpha > 0$. \square

Lemma 5 *Given H that satisfies INV and NORM, for any $(S_{\mathcal{I}})$ (i) if $S_w = 0$ or $S_m = 0$, then $H(S_{\mathcal{I}}) = 0$; (ii) if $\exists i \in \mathcal{I}$ with $S_{iw} > 0$ and $S_{im} > 0$ and for all $j \neq i$, $S_{jw} = S_{jm} = 0$, then $H(S, L_{\mathcal{I}}) = 0$.*

Proof of Lemma 5 See part (i) and (ii) of the Proof of Lemma 3. \square

Now consider any $S_{\mathcal{I}}$ in B . If $I = 1$, then by Lemma 5(ii), $H(S_{\mathcal{I}}) = 0$. Let $I \geq 2$. Let us name the positions such that $r_1 \geq r_2 \geq \dots \geq r_I$. The proof will closely follow the proof of Theorem 1.

Step 1: Consider the division of society $(S_{\mathcal{I}}) = (S_1, \dots, S_I)$ as (S_1) and (S_2, \dots, S_I) . Since $r_1 \geq r_j$ for all j in $\{2, 3, \dots, I\}$, r-DEC is applied:

$$H(S_1, \dots, S_I) = \frac{S_{1w}S_{1m}}{S_w S_m} H(S_1) + \frac{\sum_2^I S_{iw} \sum_2^I S_{im}}{S_w S_m} H(S_2, \dots, S_I) + H(S_1, S_2^I).$$

The rest of the iterative decomposition is the same as Step 1.3 of Theorem 1 with the exception that here r-DEC is applied thanks to the decreasing order of r_i ratios. Repeated application of r-DEC, and Lemma 5(ii) results in:

$$H(S_1, \dots, S_I) = \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} H(S_i, S_{i+1}^I) \quad (10)$$

concluding Step 1.

Step 2: In this step, similar to Step 2.1. of Theorem 1, first we focus on one component of the sum over positions derived in Step 1, $H(S_i; S_{i+1}^I)$ and show that for any i the following holds:

$$H(S_i, S_{i+1}^I) = \frac{S_{iw} \sum_{i+1}^I S_{jm} - S_{im} \sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw} \sum_i^I S_{jm}} H(1, 0; 0, 1).$$

First notice that since INV is ensured by Lemma 4, we have: $H(S_{iw}, S_{im}, \sum_{i+1}^I S_{jw}; \sum_{i+1}^I S_{jm}) = H(S_{iw}, S_{im}; \sum_{i+1}^I S_{jw}, \sum_{i+1}^I S_{jm})$. For simplicity let us use the notation $H(a, b; c, d)$ instead of $H(S_{iw}, S_{im}; \sum_{i+1}^I S_{jw}, \sum_{i+1}^I S_{jm})$. Notice that since $r_i \geq r_j$ for all $j \in \{i+1, \dots, I\}$, $r_i = \frac{a}{b} \geq \frac{\sum_{i+1}^I S_{jw}}{\sum_{i+1}^I S_{jm}} = \frac{c}{d}$. And since $a + c = b + d = 1$, then $a \geq b$ and $c \leq d$. If $a = b$, then $c = d$ yielding $H(a, b; c, d) = 0$. Now let $a > b$ and $c < d$. Let $b \neq 0$. Then for $X = \frac{ad-bc}{d}$, r-SAD results in:

$$H(a, b; c, d) = \frac{a + c - X}{a + c} H(a - X, b; c, d) + \frac{X}{a + c} H(X, b; 0, d).$$

Notice that this is admissible since $\frac{a-X}{b} = \frac{c}{d}$ and $\frac{X}{b} > \frac{0}{d}$. As by INV and NORM, $H(a - X, b; c, d) = 0$, we have:

$$H(a, b; c, d) = \frac{X}{a + c} H(X, b; 0, d). \quad (11)$$

Now notice that for all ϵ in $(0, d)$, the following holds:

$$H(X, b; 0, d) = \frac{d - \epsilon}{b + d} H(X, 0; 0, d - \epsilon) + \frac{b + \epsilon}{b + d} H(X, b; 0, \epsilon)$$

and hence:

$$\lim_{\epsilon \rightarrow 0} H(X, b; 0, d) = \lim_{\epsilon \rightarrow 0} \left(\frac{d - \epsilon}{b + d} H(X, 0; 0, d - \epsilon) \right) + \lim_{\epsilon \rightarrow 0} \left(\frac{b + \epsilon}{b + d} H(X, b; 0, \epsilon) \right).$$

Then by continuity of H :

$$H(X, b; 0, d) = \frac{d}{b + d} H(X, 0; 0, d) + \frac{b}{b + d} H(X, b; 0, 0).$$

Since, $H(X, b; 0, 0) = 0$ by NORM, combining with (11), we arrive:

$$H(a, b; c, d) = \frac{X}{a + c} \frac{d}{b + d} H(X, 0; 0, d).$$

By INV,

$$H(a, b; c, d) = \frac{X}{a + c} \frac{d}{b + d} H(1, 0; 0, 1).$$

And finally for values of $X = \frac{ad - bc}{d}$ and $(a, b; c, d) = (s_{iw}, s_{im}; \sum_{i+1}^I s_{jw}, \sum_{i+1}^I s_{jm})$, we arrive:

$$H(S_i; S_{i+1}^I) = \frac{S_{iw} \sum_{i+1}^I S_{jm} - S_{im} \sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw} \sum_i^I S_{jm}} H(1, 0; 0, 1). \quad (12)$$

Notice that for $b = 0$, we would have the same by continuity.

Step 3: Combining the results of Step 1 and Step 2, (10) and (12) we arrive;

$$\begin{aligned} H(S_{\mathcal{I}}) &= \sum_i \frac{\sum_i^I S_{jw} \sum_i^I S_{jm}}{S_w S_m} \frac{S_{iw} \sum_{i+1}^I S_{jm} - S_{im} \sum_{i+1}^I S_{jw}}{\sum_i^I S_{jw} \sum_i^I S_{jm}} H(1, 0; 0, 1) \\ &= \sum_i (s_{iw} \sum_{i+1}^I s_{jm} - s_{im} \sum_{i+1}^I s_{jw}) H(1, 0; 0, 1). \end{aligned}$$

We have named the positions as, $r_1 \geq r_2 \geq \dots \geq r_I$. Then, in general we have:

$$\begin{aligned} H(S_{\mathcal{I}}) &= \sum_i (s_{iw} \sum_{j:r_i \geq r_j} s_{jm} - s_{im} \sum_{j:r_i \geq r_j} s_{jw}) H(1, 0; 0, 1) \\ &= \sum_i \sum_{j:r_i \geq r_j} (s_{iw} s_{jm} - s_{im} s_{jw}) H(1, 0; 0, 1). \end{aligned}$$

Since $G_S(S_{\mathcal{I}}) = \frac{1}{2} \sum_i \sum_j |s_{iw} s_{jm} - s_{im} s_{jw}|$ could equivalently be expressed as $\sum_i \sum_{j:r_i \geq r_j} (s_{iw} s_{jm} - s_{im} s_{jw})$ and $H(1, 0; 0, 1)$ is a strictly positive constant, we establish the result. \square