FINITENESS CONDITIONS FOR UNIONS OF SEMIGROUPS

Nabilah Hani Abu-Ghazalh

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

2013

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University of
St Andrews

600
YEARS

Thesis submitted to the University of St Andrews
for the degree of Doctor of Philosophy
March, 2013

بسم الله الرحمن الرحيم

CONTENTS

LIST OF TABLES

DECLARATION

I, Nabilah Hani Abu-Ghazalh, hereby certify that this thesis, which is approximately 26.000 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree. I was admitted as a research student in January 2009 and as a candidate for the degree of Doctor of Philosophy in September 2009; the higher study for which this is a record was carried out in the University of St Andrews between 2009 and 2012.

Signature: Name: Nabilah Abu-Ghazalh Date

I, Nik Ruškuc, hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree.

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ACKNOWLEDGMENTS

It is my pleasure to thank the many people who made this thesis possible.

From the formative, early stage of this thesis, to the final draft, I owe an enormous debt of gratitude to my supervisor Professor Nik Ruškuc. His careful guidance, sound advice, clear explanation and his unlimited encouragement and help were invaluable, especially in regard to the most difficult problem for me with this thesis (Residual finiteness, Chapter 8). He provided me many ideas, directions, and open questions, all of which allowed this thesis to be possible. I will not forget his constant attempts to understand my ”unclear and brief” but right explanations.

Special thanks to Dr. Yann Peresse, whose careful reading, kind guidance, and helpful suggestions improved not just the language of the thesis but also the structure of some chapters in the thesis.

The Ministry of Higher Education in Saudi Arabia (Princess Nora Bint Abdul Rahman University in Riyadh), the Saudi Embassy and the Saudi Cultural Bureau in London deserve my sincerest thanks. Their generous financial support and assistance have meant more to me than I could ever express.

My friends in Saudi and UK were sources of laughter, joy, and support. Thank you to everyone of you, especially my childhood friend Asma Al-Thumairi, who in every call reminded me of how many years were left. Now I can say to you that there are no more years, just days. Asma, your tears when I left Saudi is just describe how much of a loyal person you are. Aimilia Symyrli, my first office-mate in St Andrews, I am very pleased that, in many cases, my friendship with you has extended well beyond our shared time in the office. Your kind words in supporting me when I missed my family, when I stuck mathematically,. . . , are very much appreciated. You both are such real friends.

I feel incredibly grateful to my father Hani Abu-Ghazalh and my mother Jamilah Al-Qahtani, whose liberal attitude, never-ending support, unconditional love, strength, and generosity, leave me speechless. There is nothing I can give you in return, but I hope that this thesis at least makes you proud.

I am deeply grateful to My brother Hamad and my sisters Norah, Kholoud, Sarah and Rabab. You have all inspired me, helped me to remain focused and finish my studies, especially with all the love you give to my kids Danah and Naif. Sometimes I feel that Danah loves you (in particular Kholoud) more than she loves me.

My ideal husband Mohammed Al-Qahtani has been always the driving force and guiding light in my life, who patiently standing beside me during all these years. The person who says "just concentrate on your studies and leave everything else to me" cannot be described. I am so blessed to have such an amazing husband.

I will say thank you to all these wonderful people, and I will say sorry to my angels Danah and Naif for not being a very good mother, as I have spent too much time on this thesis. Nonetheless, I know very well that you are the greatest thesis in my life.

نبيلة هاني أبوغزالة

The last thank goes to my internal examiner Dr. James Mitchell and my external examiner Prof. Petter Higgins for the comments and corrections they suggested and for encouraging me to publish some chapters of the thesis.

Nabilah Abu-Ghazalh

ABSTRACT

In this thesis we prove the following:

- The semigroup which is a disjoint union of two or three copies of a group is a Clifford semigroup, Rees matrix semigroup or a combination between a Rees matrix semigroup and a group. Furthermore, the semigroup which is a disjoint union of finitely many copies of a finitely presented (residually finite) group is finitely presented (residually finite) semigroup.

- The constructions of the semigroup which is a disjoint union of two copies of the free monogenic semigroup are parallel to the constructions of the semigroup which is a disjoint union of two copies of a group, i.e. such a semigroup is Clifford (strong semilattice of groups) or Rees matrix semigroup. However, the semigroup which is a disjoint union of three copies of the free monogenic semigroup is not just a strong semillatice of semigroups, Rees matrix semigroup or combination between a Rees matrix semigroup and a semigroup, but there are two more semigroups which do not arise from the constructions of the semigroup which is a disjoint union of three copies of a group. We also classify semigroups which are disjoint unions of two or three copies of the free monogenic semigroup. There are three types of semigroups which are unions of two copies of the free monogenic semigroup and nine types of semigroups which are unions of three copies of the free monogenic semigroup. For each type of such semigroups we exhibit a presentation defining semigroups of this type.

- The semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup is finitely presented, residually finite, hopfian, has soluble word problem and has soluble subsemigroup membership problem.

PREFACE

Unlike the 'classical' algebraic structures, such as groups and rings, it is well-known that a semigroup may decompose into a disjoint union of subsemigroups. Indeed many structural theories of semigroups have such decompositions at their core. For example:

- Every completely simple semigroup is isomorphic to a Rees matrix semigroup over a group G, and is thus a disjoint union of copies of G; see Theorem 1.5.4.

- Every Clifford semigroup is isomorphic to a strong semilattice of groups, and is thus a disjoint union of its maximal subgroups; see Theorem 1.5.6.

- Every commutative semigroup is a disjoint union (indeed a semilattice) of archimedean commutative semigroups; see ([Gri95], Theorem 4.2.2).

It is therefore natural to ask how properties of a semigroup S which can be decomposed into a disjoint union of subsemigroups $S = T_1 \sqcup \cdots \sqcup T_n$ depend on properties of the T_i. For instance, it is obvious that if all T_i are finitely generated then so is S (Proposition 1.2.1). Araujo et al. [ABF$^+$01] discuss finite presentability in this context, and show that there exists a non-finitely presented semigroup which is a disjoint union of two finitely presented subsemigroups. On the other hand, it can be shown that in many special instances finite presentability of the T_i implies finite presentability of S. For example, this is the case when all T_i are groups (i.e. when S is a completely regular semigroup with finitely many idempotents; see Corollary 1.5.3; this follows from ([Rus99],Theorem 4.1). Further such instances are discussed in [ABF$^+$01].
Turning to the finiteness condition of residual finiteness, we have a similar landscape. It is easy to construct a non-residually finite semigroup which is a disjoint

union of two residually finite subsemigroups. One such example, consisting of a free group and a zero semigroup, can be found in ([GR10], Example 5.6).
On the other hand, it follows from Golubov [Gol75] that if all T_i are residually finite groups then S is residually finite as well.

The thesis is divided into two parts. The finiteness conditions as finite presentability and residual finiteness for the semigroup which can be decomposed into small numbers (two or three) copies of a (semi)group, and the constructions of such semigroups, are considered in the first part. The finiteness conditions as finite presentability, residual finiteness, word problem, membership problem and hopficity for the semigroup which can be decomposed into any number of copies of the free monogenic semigroup, are considered in the second part.

Preliminary and basic materials are presented in the next chapter. The first two results are given in Chapter 2, which play a significant role in proving some theorems about rectangular band semigroups in Chapter 7. Chapter 2 has many counterexamples of some provided questions related to finite presentability. The ideas of these examples are based on finding a semigroup which is a direct product of a finite semigroup and an infinite finitely presented semigroup as in [AR00] and [RRW98].

Semigroups which are disjoint unions of groups are considered in Chapter 3. Working on such semigroups is more flexible than disjoint unions of semigroups since we have well-known theorems in [How95] and [CP61] which provide us with some constructions as Clifford and Rees matrix semigroups. We initially start with the semigroup which is a disjoint union of two copies of a group which implies that such a semigroup is completely regular since every element in the semigroup lies in a subgroup, so by knowing the size of the semilattice, we can classify this semigroup as in Theorem 3.2.2. Analogously, we classify the semigroups which are disjoint unions of three copies of a group as in Theorem 3.3.2. After classifying these semigroups we simply prove the finite presentability and residual finiteness for these semigroups in Section 3.4 by [Rus99] and [Gol75].

In Chapter 4 we study the semigroups which are disjoint unions of two copies of the free monogenic semigroup. We find interestingly that such semigroups behave in parallel with the semigroups which are disjoint unions of two copies of the infinite cyclic group and it has three constructions, strong semilattice of semigroups and Rees matrix semigroup of two types (Theorem 4.3.1) which is the same constructions that have been obtained in the group case (Theorem 3.2.2).

The semigroups which are disjoint unions of three copies of the free monogenic semigroup is the subject of Chapter 5. After classifying these semigroups and proving that there are just nine types, each type of such a semigroup is defined by a certain presentation see Table 5.1, we end up with a result which says that the construction of such semigroups is not just a strong semillatice of semigroups, Rees matrix semigroup or a combination between a Rees matrix semigroup and a semigroup -the constructions of three copies of the infinite cyclic group- but there are some others which do not lie under any of the mentioned constructions (Remark 5.4.1). Chapters 4 and 5 have been submitted to publication [AGMRed]

The last type of semigroups in this thesis which are disjoint unions of small number (two copies) of a semigroup are presented in Chapter 6. We classify -after some long proofs- balanced semigroups which are disjoint unions of two copies of the free semigroup of rank two. Although, a strong condition is provided on these semigroups, they are quite tricky to be classified. The main theorem in this chapter is Theorem 6.4.2 which states clearly that there are just six balanced semigroups, each of which has a certain finite presentation. This implies that every balanced semigroup is finitely presented. In Section 6.5, we prove that such semigroups are residually finite since the relations in each presentation preserve length and this is the end of Part I.

In Part II of the thesis we consider semigroups which are disjoint unions of finitely many copies of the free monogenic semigroup (i.e. natural numbers under addition). We show that even though there is no general structural theory for such semigroups, which would yield positive results of the above type 'for free', they nonetheless display the same behaviour as unions of groups.

We start Part II with a rectangular bands of finitely presented semigroups, which is a nice case of disjoint unions of semigroups. A general result of finite presentability is provided by Theorem 2.3.1 as we have mentioned before. In addition, there is a theorem for residual finiteness but in the case that the rectangular band is just blocks of copies of the free monogenic semigroup.

The strongest results of this thesis are in Chapters 8 and 9 which have appeared in the work of Abu-Ghazalh and Ruškuc ([AGR13], [AGRin]). There are 6 open problems which have been proved in these two chapters. and it has a strong result on finite presentability. The finiteness conditions theorems for the semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup are Theorems (8.2.1, 8.3.4, 8.4.2, 9.2.2, 9.3.9) which state respectively that every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup is finitely presented, residually finite, hopfian, has a soluble word problem and has a soluble subsemigroup membership problem. Moreover, we have Theorem 8.5.2 on commutative semigroups which are disjoint unions of finitely many copies of the free monogenic semigroup, says that every commutative semigroup is a finite disjoint union of copies of the free monogenic semigroup if and only if it is a strong semilattice of copies of the free monogenic semigroup.

CHAPTER

ONE

SEMIGROUP THEORY PRELIMINARIES

1.1 Introduction

In this chapter all the basic semigroup theory needed to understand the results of the thesis are covered and mostly taken from [Hig92], [How95], [Rus95], [Har96], [RT98] and [Gal05]. For further background information on semigroup theory see [CP61], [Lal79], [Hig92], [Gri95] or [How95].

1.2 Basics and monogenic semigroups

A *semigroup* is a set S equipped with an associative binary operation, usually called multiplication, which means $(xy)z = x(yz)$ for all $x, y, z \in S$. For instance, the set of natural number $\mathbb{N}$ forms a semigroup under addition.

A *subsemigroup* of S is a subset $T \subseteq S$ which is closed under the same multiplication, that is, if $(\forall x, y \in T)\ xy \in T$. We abbreviate T is a subsemigroup of S as $T \leq S$. A *proper subsemigroup* of S is a subsemigroup which is not equal to S. Suppose that S is a semigroup and $T_i \leq S$ for all $i \in I$. Then

$$\cap_{i \in I} T_i \neq \emptyset \Longrightarrow \cap_{i \in I} T_i \leq S.$$

Let A be a non-empty subset of S. The intersection of all the subsemigroups of S that contain A is non-empty and is a subsemigroup of S. We use $\langle A \rangle$ to denote this subsemigroup and call it the subsemigroup of S generated by the set A. Equivalently, the subsemigroup $\langle A \rangle$ can also be described as the set of all elements in S

that can be written as finite products of elements of A.

Proposition 1.2.1 ([ABF$^+$01], Proposition 3.1)**.** *Let S be a semigroup which is a disjoint union of a family $(S_i)_{i\in I}$ of its subsemigroups. If each S_i is generated by a set X_i $(i \in I)$, then S is generated by $\sqcup_{i\in I} X_i$.*

If $A = \{a_1, a_2, \ldots, a_n\}$ then we shall write $\langle A \rangle$ as $\langle a_1, a_2, \ldots, a_n \rangle$ and if $A = \{a\}$, a singleton set, and $\langle a \rangle = \{a, a^2, a^3, \ldots\}$, then we refer to $\langle a \rangle$ as the *monogenic* subsemigroup of S generated by the element a. The *order* of the element a is defined, as in group theory, as the order of the subsemigroup $\langle a \rangle$. If S is a semigroup in which there exists an element a such that $S = \langle a \rangle$, then S is said to be a *monogenic* semigroup as in [How95] and a *cyclic* semigroup as in [CP61].

Assume that a is an element of a semigroup S and $\langle a \rangle = \{a, a^2, a^3 \ldots\}$, if there is no repetition in the list $a, a^2, a^3, \ldots$, which means

$$a^m = a^n \Rightarrow m = n,$$

then the semigroup $\langle a \rangle$ is isomorphic to the semigroup $\mathbb{N}$ (natural numbers with respect to addition). Note that we use the notations N and $\mathbb{N}$ to mean, respectively, the monogenic semigroup and the natural number semigroup throughout the thesis and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

We say that $e \in S$ is an identity element (or identity, usually denoted 1) of S if $es = se = s$ for all $s \in S$. If $e \in S$ then S is a monoid and e is the only identity in S. For any semigroup S we may form a monoid S^1 as follows. If S is not a monoid then, we adjoin the symbol 1 to S, we let $S^1 = S \cup \{1\}$ and we extend the multiplication on S by defining $1s = s1 = s$ for all $s \in S^1$. If S is a monoid then $S^1 = S$.

1.3 Relations, congruences and homomorphisms

Let X and Y be two non-empty sets. Then the set $X \times Y$ is defined as

$$\{(x, y) : x \in X, y \in Y\}.$$

A *binary relation* on X is a subset $R \subseteq X \times X$. We call the relation

$$1_X = \{(x, x) \ : \ x \in X\}$$

the *diagonal* relation on X and

$$R^{-1} = \{(x, y) \in X \times X \ : \ (y, x) \in R\}$$

the *converse* of R. If R and Q are two binary relations on X then we may define the composition $R \circ Q$ as

$$\{(x, z) : (\exists y \in X)((x, y) \in R, (y, z) \in Q)\}.$$

Let R be a binary relation on X. Then, R is *reflexive* if xRx, for all $x \in X$. R is *symmetric* if xRy implies yRx, for all $x, y \in X$. R is *anti-symmetric* if xRy and yRx implies that $x = y$. R is *transitive* if xRy and yRz implies xRz. If R is reflexive, anti-symmetric and transitive then it is an *order relation*. If R is reflexive, symmetric and transitive then it is an *equivalence relation*.

Let R be a reflexive relation on X. Then we have

$$R \subseteq R \circ R \subseteq R \circ R \circ R \subseteq \cdots,$$

which we can write in simpler notation as

$$R \subseteq R^2 \subseteq R^3 \subseteq \cdots$$

The relation

$$R^\infty = \bigcup \{R^n \ : \ n \geq 1\} \tag{1.1}$$

is called the transitive closure of the relation R.

Lemma 1.3.1 ([How95], Lemma 1.4.8)**.** *For every reflixive relation R on a set X, the relation R^∞ defined by* (1.1) *is the smallest transitive relation on X containing R.*

Now we can introduce the relation R^e which is the *smallest equivalence on X containing R* and it is defined as

$$R^e = [R \cup R^{-1} \cup 1_X]^\infty.$$

The number of the equivalence classes is called the *index* of R in X and is denoted by $[X : R]$. An equivalence relation R on a semigroup S is a *right congruence* if xRy implies $xzRyz$ for all $x, y, z \in S$. *Left congruences* are defined analogously and a relation is a *congruence* if it is both a left congruence and a right congruence. For an arbitrary relation R on a semigroup S we define

$$R^c = \{(xay, xby) \ : \ x, y \in S^1, \ (a,b) \in R\}.$$

Thus, the smallest congruence generated by R is $(R^c)^e$, see ([How95], Proposition 1.5.8).

Let S and T be semigroups. A *homomorphism* from S into T is a mapping $\phi : S \to T$ which satisfies $\phi(s_1 s_2) = \phi(s_1)\phi(s_2)$ for all $s_1, s_2 \in S$. An *epimorphism* is a surjective homomorphism, a *monomorphism* is an injective homomorphism and an *isomorphism* is a bijective homomorphism. We say that T is a homomorphic image of S if there is an epimorphism $\phi : S \to T$. If $\phi : S \to T$ is a homomorphism then the *kernel*

$$\ker(\phi) = \{(s_1, s_2) \mid s_1, s_2 \in S, \phi(s_1) = \phi(s_2)\}$$

of ϕ is a congruence on S and $S/\ker(\phi) \cong \mathrm{Im}\phi$. We say that S and T are *isomorphic* if there is an isomorphism $\phi : S \to T$.

Theorem 1.3.2 ([How95], Theorem 1.5.3)**.** *Let ρ be a congruence on a semigroup S, and let $\phi : S \to T$ be a homomorphism such that $\rho \subseteq \ker\phi$. Then there is a unique homomorphism $\beta : S/\rho \to T$ such that $\mathrm{im}\beta = \mathrm{im}\phi$.*

An *anti-homomorphism* from the semigroup S into the semigroup T is a mapping $\phi : S \to T$ which satisfies $\phi(s_1 s_2) = \phi(s_2)\phi(s_1)$ for all $s_1, s_2 \in S$. An *anti-isomorphism* is a bijective anti-homomorphism. We say S is *anti-isomorphic* to T if ϕ is anti-isomorphism.

An *endomorphism* is a homomorphism $S \to S$ and an *automorphism* is an isomorphism $S \to S$.

Let S be a semigroup, and let T is a subsemigroup of S. Let π be the relation

$$(T \times T) \cup \big((S \backslash T) \times (S \backslash T)\big).$$

Let $\Sigma_r(\pi)$, $\Sigma_l(\pi)$, respectively, be the largest right congruence and largest left congruence contained in π as follows:

$$\Sigma_r(\pi) = \{(x,y) \in S \times S : (xs, ys) \in \pi \ \text{ for all } \ s \in S^1\},$$

$$\Sigma_l(\pi) = \{(x,y) \in S \times S : (sx, sy) \in \pi \ \text{ for all } \ s \in S^1\}.$$

And then the largest congruence which contained in π is

$$\Sigma(\pi) = \{(x,y) \in S \times S : (s_1xs_2, s_1ys_2) \in \pi \text{ for all } s_1, s_2 \in S^1\}.$$

Notice that

$$\Sigma_r(\Sigma_l(\pi)) = \Sigma_l(\Sigma_r(\pi)) = \Sigma(\pi).$$

We shall call $\Sigma_r(\pi)$, $\Sigma_l(\pi)$, respectively, the *right, left syntactic congruence*, and $\Sigma(\pi)$ the *syntactic congruence*.

Theorem 1.3.3 ([RT98], Theorem 2.4)**.** *Let S be a semigroup. If ρ is a right congruence of finite index in S, then the congruence $\Sigma(\rho)$ also has finite index in S. In particular, for any equivalence relation π on S, $[S : \Sigma_r(\pi)]$ is finite if and only if $[S : \Sigma(\pi)]$ is finite if and only if $[S : \Sigma_l(\pi)]$ is finite.*

1.4 Ideals and Green relations

A *right ideal* R of a semigroup S is a non-empty subset of S such that $r \in R$ and $s \in S$ imply $rs \in R$. Equivalently, $RS \subseteq R$. Analogously, a *left ideal* of S is a non-empty subset L satisfying $SL \subseteq L$. A *two sided ideal* I is a subset which is both a left and right ideal. It satisfies $IS \cup SI \subseteq I$. Any of these sets is called proper if it does not equal to S.

We say that a (left, right, two-sided) ideal I of a semigroup S is *minimal* if it contains no other (left, right, two-sided) ideal of S. The existence of a minimal (left, right) ideal is not necessary, we may find a semigroup with no minimal (left, right) ideal and a semigroup with several minimal left and right ideals or with several minimal left (right) ideals. However, there is no semigroup with more than one minimal two-sided ideal because if I_1 and I_2 are minimal two-sided ideals in S then $I_1I_2 \subseteq I_1 \cap I_2$. Notice that I_1I_2 is an ideal that would otherwisw be strictly contained in each of the (distinct) ideals I_1 and I_2.

Let S be a semigroup. Define the following relations on S:

$$x\mathcal{L}y \iff \exists\, s, s' \in S^1 \ : \ x = sy \text{ and } y = s'x,$$

$$x\mathcal{R}y \iff \exists\, s, s' \in S^1 \ : \ x = ys \text{ and } y = xs',$$

$$x\mathcal{J}y \iff \exists\, s, s', t, t' \in S^1 \ : \ x = sys' \text{ and } y = txt'.$$

The relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are equivalence relations and furthermore the relations

$\mathcal{L}$ and $\mathcal{R}$ are respectively a right congruence and a left congruence. The relations $\mathcal{L}$ and $\mathcal{R}$ commute i.e: $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ and then we have the equivalence relation $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. $\mathcal{D}$ is the smallest equivalence relation on S which contains both $\mathcal{L}$ and $\mathcal{R}$. Let $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and then $\mathcal{H}$ is the largest equivalence relation of S that is contained in both of $\mathcal{L}$ and $\mathcal{R}$. The inclusion diagram of the Greens relations is given below.

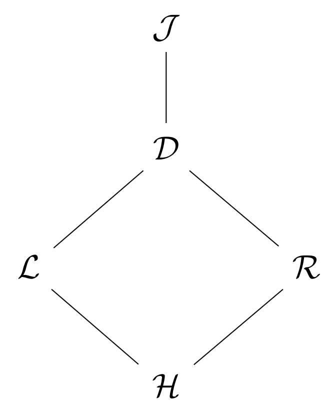

Figure 1.1: Inclusion diagram of the Green's relations

1.5 Classes and constructions

We say that the semigroup S is *commutative* if for every $a, b \in S \ : \ ab = ba$. The element $e \in S$ is *idempotent* if $e^2 = e$. A semigroup where all elements are idempotents is called a *band*.

Let E be a semigroup of idempotents. We define a relation $\leq$ on E by

$$(\forall x, y \in E)\ x \leq y \iff xy = yx = x.$$

It is an easy exercise to show that $\leq$ is a partial order (reflexive, anti-symmetric and transitive) relation on E. If $x \leq y$ in S we say x is *under* y and y is *over* x. We say the element b of a partially ordered set X is a *lower bound* of a subset Y of X if $b \leq y$ for every y in Y. The lower bound b of Y is called a *greatest lower bound* or *meet* of Y if $b \geq c$ for every lower bound c in Y. *Upper bound* and *least upper bound* or *join* are defined analogously.

If S is a commutative semigroup and all its elements are idempotents, then S is called a *semilattice*. Hence, in this case, for all $x, y \in S : x^2 = x$ and $xy = yx$.

The element $x \in S$ is *regular* if there exists $y \in S$ such that $xyx = x$. We say that S is a *regular semigroup* if all of its elements are regular and we say that S is *simple* if it has no proper ideals. The semigroup S is *completely simple* if it is simple and contains a primitive idempotent (that is, an idempotent which is minimal within the set of all idempotents of S).

Theorem 1.5.1 ([How95], Theorem 3.3.2)**.** *Let S be a simple semigroup (without zero). Then S is completely simple if and only if S contains at least one minimal left ideal and at least one minimal right ideal.*

Lemma 1.5.2 ([CP67], Lemma 8.13)**.** *let S be a simple semigroup containing a minimal left (right) ideal. Then S is the disjoint union of its minimal left (right) ideals.*

We say that S is *completely regular* if for every $x \in S$ there exists $y \in S$ such that $xyx = x$ and $xy = yx$.

Corollary 1.5.3 ([How95], Corollary 2.2.6)**.** *If e is an idempotent in a semigroup S, then H_e is a subgroup of S. No $\mathcal{H}-$class in S can contain more than one idempotent.*

Let G be a group and let I and Λ be non-empty sets. Let P be a matrix, indexed by Λ and I, respectively, with entries from G. We denote this by $P = (p_{\lambda i})_{\lambda\in\Lambda, i\in I}$. Let S be the semigroup with elements $(I \times G \times \Lambda)$ and multiplication defined as $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu)$ for all $i, j \in I,\ g, h \in G$ and $\lambda, \mu \in \Lambda$. Then S is a *Rees matrix semigroup of the type* $|\Lambda| \times |I|$ with respect to G, I, Λ and P. We write $S = \mathcal{M}[G; I, \Lambda; P]$. The Rees matrix construction $\mathcal{M}[S; I, \Lambda; P]$ works even if S is just a semigroup.

The *near Rees matrix semigroup (NRMS)* construction is defined as follows, at first we have a semigroup S, and then we adjoin the identity element to form S^1 and we have the $|\Lambda| \times |I|$ matrix P with the identity entries. Thus if

$$\bar{S} = \mathcal{M}\left[S^1; I, \Lambda; P\right]$$

and

$$S = \{(i, s, \lambda) : s \neq 0,\ s \in S\} \leq \bar{S}$$

then S is a NRMS and we will call it briefly Rees matrix semigroup of the type $|\Lambda| \times |I|$.

Theorem 1.5.4 ([Lal79], Theorem 2.7)**.** *A semigroup S is completely simple if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}[G; I; \Lambda; P]$ where the matrix P is normal, i.e $p_{i1} = p_{1\lambda} = 1_G$ for all $i \in I$ and all $\lambda \in \Lambda$.*

A semigroup S is said to be a *semilattice of semigroups T_α, $\alpha \in Y$* if Y is a semilattice, S=$\sqcup_{\alpha \in Y} T_\alpha$ with multiplication defined naturally within each T_α and globally obeying the rule $T_\alpha T_\beta \subseteq T_{\alpha\beta}$. The set of natural numbers $\mathbb{N}$ forms a semilattice under the natural partial order $\leq$. *Strong semilattices of semigroups T_α* if, in addition, there is a family of homomorphisms $\{\phi_{\alpha,\beta} : \alpha, \beta \in Y, \alpha \geq \beta\}$ where each $\phi_{\alpha,\beta}$ maps from T_α to T_β, $\phi_{\alpha,\alpha}$ is the identity mapping on T_α and $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$. The semigroup has elements $\sqcup_{\alpha \in Y} T_\alpha$ and multiplication is defined, for $x \in T_\alpha$ and $y \in T_\beta$, as $xy = \phi_{\alpha,\alpha\beta}(x)\phi_{\beta,\alpha\beta}(y)$. We denote this semigroup as

$$\mathcal{S}\big[Y; \{T_\alpha \ : \ \alpha \in Y\}; \{\phi_{\alpha,\beta} \ : \ \alpha, \beta \in Y, \ \alpha \geq \beta\}\big].$$

For instance, if Y is small, e.g $Y = \{0, 1\}$, we may write S as $\mathcal{S}\big[Y; \{T_0, T_1\}; \{\phi_{1,0}\}\big]$.

Theorem 1.5.5 ([How95], Theorem 4.1.3)**.** *Every completely regular semigroup is a semilattice of completely simple semigroups.*

A *Clifford semigroup* is defined as a completely regular semigroup in which the idempotents are central (c is *central* if $cs = cs$ $\forall s \in S$). Strong semilattices of semigroups provide one of the main tools for the structure theory of semigroups as in the following theorem.

Theorem 1.5.6 ([How95], Theorem 4.2.1)**.** *Let S be a semigroup with set E of idempotents. Then the following statements are equivalent:*

(1) S is a Clifford semigroup;

(2) S is a semilattice of groups;

(3) S is a strong semilattice of groups;

(4) S is regular, and the idempotents of S are central.

1.6 Free semigroups and presentations

An *alphabet* A is a non-empty, but finite, set of *symbols* or *letters*. Any finite sequence of letters is a *word* (or a string) *over* A. The set of all words over A, with at least one letter, is denoted by A^+. For clarity, we shall often write $u \equiv v$, if the words u and v are the same (letter by letter). The set A^+ is a semigroup, the *word semigroup* over A, when the product is defined as the *concatenation* of words, that is, the product of the words $w_1 \equiv a_1a_2\cdots a_n$, $w_2 \equiv b_1b_2\cdots b_m (a_i, b_i \in A)$ is the word $w_1.w_2 = w_1w_2 \equiv a_1a_2\cdots a_nb_1b_2\cdots b_m$. When we join the empty word 1 (which has no letters) to A^+, we have the *word monoid* $A^*, A^* = A^+ \cup \{1\}$. Clearly, $1.w = w = w.1$ for all words $w \in A^*$. For example, let $A = \{a, b\}$ be a *binary alphabet*. Then $a, b, aa, ab, ba, bb, aaa, aab, \cdots$ are words in A^+. Now, $ab.bab \equiv abbab$. As usual, w^k means the concatenation of w with itself k times and so for instance, $v \equiv ab^3(ba)^2 \equiv abbbbaba \equiv ab^4aba$.

Let S be a semigroup. A subset $A \subseteq S$ *generates S freely*, if $S = \langle A \rangle$ and every mapping $\psi : A \to P$ (where P is any semigroup) can be extended to a unique homomorphism $\phi : S \to P$ such that $\phi \restriction_A = \psi$. Here we say that ϕ is a *homomorphic extension* of the mapping ψ. If S is freely generated by some subset, then S is a *free semigroup*.

Example. $(\mathbb{N}, +)$ is free since the generating set is $A = \{1\}$ and if $\psi : A \to P$ is a homomorphism, and we define $\phi : \mathbb{N} \to P$ by $\phi(n) = \psi(1)^n$. Now $\phi \restriction_A = \psi$ and ϕ is a homomorphism:

$$\phi(n + m) = \psi(1)^{n+m} = \psi(1)^n \,.\, \psi(1)^m = \phi(n) \,.\, \phi(m).$$

Proposition 1.6.1 ([Rus95], Proposition 1.1)**.** *Let A be a set, and let S be any semigroup. Then any mapping $\psi : A \to S$ can be extended in a unique way to a homomorphism $\phi : A^+ \to S$, and A^+ is determined up to isomorphism by these properties.*

Thus, for any alphabet A, A^+ is a free semigroup, and it is freely generated by A.

Proposition 1.6.2 ([Rus95], Proposition 2.1)**.** *Let $\langle A|\ R\rangle$ be a presentation, let S be the semigroup defined by this presentation, and let T be a semigroup satisfying R. Then T is a natural homomorphic image of S.*

Since A is a generating set for S, the identity mapping on A induces an epimorphism $\pi : A^+ \to S$. The kernel $\ker(\pi)$ is a congruence on S; if $R \subseteq A^+ \times A^+$ is a generating set for this congruence we say that $\langle A|R\rangle$ is a presentation for S. We

say that S *satisfies a relation* $(u, v) \in A^+ \times A^+$ if $\pi(u) = \pi(v)$; we write $u = v$ in this case. Suppose we are given a set $R \subseteq A^+ \times A^+$ and two words $u, v \in A^+$. We say that the relation $u = v$ is a *consequence* of R if there exist words $w_1, \cdots, w_k$ ($k \geq 1$) such that for each $i = 1, \cdots, k-1$ we can write $w_i \equiv \alpha_i u_i \beta_i$ and $w_{i+1} \equiv \alpha_i v_i \beta_i$ where $(u_i, v_i) \in R$ or $(v_i, u_i) \in R$.
It is well known that the following are equivalent:

(P1) $\langle A \mid R \rangle$ is a presentation for S.

(P2) S satisfies all relations from R, and every relation that S satisfies is a consequence of R.

(P3) There exists a set $W \subseteq A^+$ such that π maps W bijectively onto S, and for every $u \in A^+$ there exists $w \in W$ such that $u = w$ is a consequence of R.

(P1)⇔(P2) is ([Lal79], Proposition 1.4.2). (P2)⇒(P3) is proved by choosing a single preimage for every $s \in S$, and letting the resulting set be W. (P3)⇒(P2) is obvious. The set W in (P3) is referred to as a set of *normal forms* for elements of S. We say that S is *finitely presented* if A and R were chosen to be finite. For further information see [Rus95].

1.7 Residual finiteness

A semigroup S is said to be *residually finite* if for any two distinct elements $s, t \in S$ there exists a homomorphism ϕ from S into a finite semigroup such that $\phi(s) \neq \phi(t)$. It is well-known that the following are equivalent:

(RF1) S is residually finite.

(RF2) There exists a congruence ρ of *finite index* (i.e. with only finitely many equivalence classes) such that $(s, t) \notin \rho$.

(RF3) There exists a right congruence ρ of finite index such that $(s, t) \notin \rho$.

(RF1)⇔(RF2) is an immediate consequence of the connection between homomorphisms and congruences via kernels. (RF2)⇒(RF3) is trivial. (RF3)⇒(RF2) follows from the fact that for a right congruence ρ of finite index, the largest two-sided congruence contained in ρ also has finite index; see Theorem 1.3.3, it is sufficient to prove residual finiteness by the existence of a right congruence of finite index (RF3).

Residual finiteness is one of the more important finiteness conditions (the properties of semigroups which all finite semigroups have). Every finitely presented residually finite semigroup has solvable word problem ([Eva69], Theorem 2) and so it is closely connected with algorithmic problems.

Part I

DISJOINT UNIONS OF A SMALL NUMBER OF COPIES OF A (SEMI)GROUP

CHAPTER

TWO

GENERAL DISJOINT UNIONS OF TWO SEMIGROUPS

2.1 Introduction

In this chapter, most of the examples are based on the paper [AR00], Araújo and Ruškuc investigated finite generation and finite presentability of the direct product $S \times T$ of a finite semigroup S and an infinite semigroup T. So first we introduce the necessary definitions and theorems. We use Gap in this Chapter and Chapter 5 to prove that the given set which defined by a multiplication table is basically a semigroup.

Definition 2.1.1. A (finite) semigroup S *preserves finite generation (resp., finite presentability)* in direct products if it satisfies the following property: for every infinite semigroup T, the direct product $S \times T$ is finitely generated (resp., finitely presented) if and only if T is finitely generated (resp., finitely presented). Also, we say that S *destroys finite generation* (resp., *finite presentability*) in direct products if $S \times T$ is not finitely generated (resp., finitely presented) for some infinite semigroup T.

In addition, two relations $\prec_r$ and $\prec_l$ have been defined on an arbitrary semigroup S as follows:

$$t \prec_r s \iff s = t \text{ or } (\exists x \in S)(sx = t),$$

$$t \prec_l s \iff s = t \text{ or } (\exists x \in S)(xs = t).$$

These relations are pre-orders (reflexive and transitive). And then the sets of all maximal elements with respect to $\prec_r$ and $\prec_l$ were denoted by

$$\Re(S) = \{s \in S \,:\, (\forall t \in S)(s \prec_r t \Rightarrow t \prec_r s\},$$

$$\mathcal{L}(S) = \{s \in S \ : \ (\forall t \in S)(s \prec_l t \Rightarrow t \prec_l s\}.$$

For an arbitrary $s \in S$, defined a graph $\Gamma(s)$ as follows. The set of vertices is

$$\{(\alpha, \mu, \omega) \ : \ \alpha \in \Re(S),\ \mu \in S,\ \omega \in \mathcal{L}(S),\ \alpha\mu\omega = s\}.$$

Two vertices $(\alpha_1, \mu_1, \omega_1)$ and $(\alpha_2, \mu_2, \omega_2)$ are joined by an edge if

$$(\alpha_1 = \alpha_2 \quad \& \quad \mu_1\omega_1 = \mu_2\omega_2) \ \text{ or } \ (\alpha_1\mu_1 = \alpha_2\mu_2 \quad \& \quad \omega_1 = \omega_2).$$

We say that an element $e \in S$ is a *relative left (resp., right) identity* for an element $s \in S$ if $es = s$ $(resp.,\ se = s)$. If e is a *relative left (resp., right) identity* for every $s \in S$ we say that e is a *left (resp,. right) identity*.

Theorem 2.1.2 ([AR00], Theorem 1.2)**.** *The following two conditions are equivalent for a finite semigroup S:*

(i) S preserves finite presentability in direct products.

(ii) $S^2 = S$ *and all the graphs* $\Gamma(s)$ $(s \in S)$ *are connected.*

Corollary 2.1.3 ([AR00])**.** *If S satisfies one of the following conditions:*

(i) Every element of S has a relative left identity and a relative right identity,

(ii) S has a left identity or a right identity,

then S preserves finite presentability in direct products.

We address the following question when we start to look at the semigroups which can be decomposed into two subsemigroups. Let S be a semigroup which is a disjoint union of two subsemigroups A, B, where A and B are finitely presented. Is S finitely presented? The answer to this question is negative by the following counterexample.

Example([ABF$^+$01], Example 3.4). Let T be the semigroup defined by the following multiplication table:

	a	b	c	0
a	a	a	c	0
b	b	b	c	0
c	0	0	0	0
0	0	0	0	0

and let U be any finitely presented infinite semigroup. Then we have the semigroup $S = T \times U$ and the two subsemigroups $S_1 = \{a\} \times U$ and $S_2 = \{b, c, 0\} \times U$. It is clear that $S = S_1 \sqcup S_2$. The semigroup S_1 is finitely presented since it is isomorphic to U, and S_2 is finitely presented because the subsemigroup $Q = \{b, c, 0\}$ of T has a left identity b and then by Corollary 2.1.3, Q preserves finite presentability in S_2. However, S is not finitely presented since

$$\Gamma(c) = \{(a, a, c), (a, b, c), (b, a, c), (b, b, c)\},$$

and the two vertices $(a, b, c), (b, a, c)$ are not connected since $c = c$ and $ab \neq ba$.

Consequently, we try to add some conditions to these subsemigroups to obtain a finitely presented semigroup. For instance, if we define a syntactic congruence on S and each class of this congruence is finitely presented, does this lead to a positive answer (S is finitely presented)? or, if we add another condition that is related to (left,right) ideals, is this enough for S to be finitely presented? The answer to all these questions is in the following two sections.

2.2 Syntactic congruence condition

Question. Suppose that S is a semigroup which is a disjoint union of two subsemigroups A, B and let π be the relation $(A \times A) \cup (B \times B)$. Let $\Sigma_r(\pi)$, $\Sigma_l(\pi)$, respectively, be the right syntactic congruence and left syntactic congruence contained in π as follows:

$$\Sigma_r(\pi) = \{(x, y) \in S \times S : (xs, ys) \in \pi \text{ for all } s \in S^1\},$$

$$\Sigma_l(\pi) = \{(x, y) \in S \times S : (sx, sy) \in \pi \text{ for all } s \in S^1\}.$$

And then the the syntactic congruence which contained in π is

$$\Sigma(\pi) = \{(x, y) \in S \times S : (s_1 x s_2, s_1 y s_2) \in \pi \text{ for all } s_1, s_2 \in S^1\},$$

where

$$\Sigma_r(\Sigma_l(\pi)) = \Sigma_l(\Sigma_r(\pi)) = \Sigma(\pi).$$

i) Are the $\Sigma-$classes necessarily subsemigroups?

ii) If all the $\Sigma-$classes were finitely presented subsemigroups, is S necessarily finitely presented semigroup?

The answer to these two questions is negative by the following counterexamples.
Counterexample to *i*. Let S be the semigroup defined by the following multiplication table:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	2	0	0	5
3	0	3	4	3	4	3
4	0	4	4	0	0	3
5	0	5	2	5	2	5

Hence the semigroup $S = A \sqcup B$ where $A = \{0,3,4\}$ and $B = \{1,2,5\}$. Then

$$\pi = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(0,3),(3,0),(0,4),(4,0),(3,4),(4,3), (1,2),(2,1),(1,5),(5,1),(2,5),(5,2)\},$$

$$\Sigma_r(\pi) = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5),(0,3),(3,0),(0,4),(4,0),(3,4), (4,3),(1,2),(2,1)\},$$

$$\Sigma_l\Sigma_r(\pi) = \{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\},$$

and thus $4/\Sigma_l\Sigma_r = \{4\}$ which is not a semigroup.

Counterexample to *ii*. Let T be the semigroup defined by the following multiplication table:

	a	b	c	d
a	a	a	a	d
b	a	b	a	d
c	a	a	c	d
d	d	d	d	d

and let U be any finitely presented infinite semigroup. Then we have the semigroup $S = T \times U$ and the two subsemigroups $S_1 = \{a,b,c\} \times U$ and $S_2 = \{d\} \times U$. It is clear that $S = S_1 \sqcup S_2$. Now we define the relation $\pi = (S_1 \times S_1) \cup (S_2 \times S_2)$

and then we have the following:

$$\begin{aligned}\pi = \Sigma_r(\pi) = \Sigma_l(\Sigma_r(\pi)) = \bigcup_{u_i,u_j\in U} \; & \{((a,u_i),(a,u_j)),((b,u_i),(b,u_j)),((c,u_i),(c,u_j)),\\ & ((d,u_i),(d,u_j)),((a,u_i),(b,u_j)),((b,u_i),(a,u_j)),\\ & ((a,u_i),(c,u_j)),((c,u_i),(a,u_j)),((b,u_i),(c,u_j)),\\ & ((c,u_i),(b,u_j))\}.\end{aligned}$$

Now we have precisely two classes

$$(a,u_i)/\Sigma_l\Sigma_r = \bigcup_{u_i,u_j\in U}\{(a,u_i,),(b,u_j),(c,u_j)\} \cong \{a,b,c\}\times U,$$

$$(d,u_i)/\Sigma_l\Sigma_r = \bigcup_{u_i\in U}\{(d,u_i)\} \cong \{d\}\times U.$$

It follows that $(a,u_i)/\Sigma_l\Sigma_r = S_1$ and $(d,u_i)/\Sigma_l\Sigma_r = S_2$ are finitely presented since every element in the subsemigroup $Q = \{a,b,c\}$ of T, has a relative left identity and a relative right identity and by Corollary 2.1.3, Q preserves finite presentability in S_1, and S_2 is isomorphic to U. However S is not finitely presented because of the following, $\Re(S) = \{b,c\} = \mathcal{L}(S)$ and then

$$\begin{aligned}\Gamma(a) = \; & \{(b,a,b),(b,a,c),(b,b,c),(b,c,b),(b,c,c),(c,a,b),(c,a,c),(c,b,b),(c,b,c),\\ & (c,c,b)\},\end{aligned}$$

but there is no edge which joins the two vertices $(b,c,b),(b,c,c)$ where $b = b, cb \neq c^2$. Therefore, $\Gamma(a)$ is not connected and hence T destroys finite presentability in S.

2.3 (Left,Right) ideals conditions

Now we list the ideals cases and there are four cases two of which give us interesting theorems. We start with the cases which give us negative results. So the question is as follows:

Question. If the semigroup S is a disjoint union of the two finitely presented subsemigroups A, B and one of the following holds,

i) A is a left ideal and B is a right ideal in S.

ii) A is a left (right) ideal in S and no assumption on B.

iii) A and B are left (right) ideals in S.

iv) A is an ideal in S.

then is S finitely presented?
Firstly, notice that in (i) we have $BA \subseteq A$ as A is a left ideal and $BA \subseteq B$ as B is a right ideal. Therefore, $A \cap B \neq \emptyset$ which is not our case. In (ii) the answer is negative by the following counterexample ([AR00], Example 3.2). Let T be the semigroup defined by the following multiplication table:

	a	b	c	0
a	0	0	0	0
b	a	b	b	0
c	a	c	c	0
0	0	0	0	0

and let U be any finitely presented infinite semigroup. Then we have the semigroup $S = T \times U$ and the two subsemigroups of S, $S_1 = \{a, c, 0\} \times U$ and $S_2 = \{b\} \times U$. Notice that S_1 is a right ideal in S $(S_1 S_2 \subseteq S_1)$. It follows that S_1 and S_2 are finitely presented since in S_1 we have the subsemigroup $Q = \{a, c, 0\}$ of T, which has a left identity c and by Corollary 2.1.3, Q preserves finite presentability. Also S_2 is isomorphic to U. However S is not finitely presented because of the following, $\Re(S) = \{b, c\}$ and $\mathcal{L}(S) = \{a, b, c\}$ and then

$$\Gamma(a) = \{(b, b, a), (b, c, a), (c, b, a), (c, c, a)\},$$

but there is no edge which joins the two vertices (b, c, a) and (c, b, a) where $a = a, bc \neq cb$. Therefore, $\Gamma(a)$ is not connected and hence T destroys finite presentability. Analogously if S_1 is a left ideal.

Now the answer to $(iii), (iv)$ is in the two following theorems which prove some other theorems about rectangular band semigroups in Chapter 7.

Theorem 2.3.1. *Suppose that S is a semigroup which is the disjoint union of two subsemigroups A, B. Assume that A, B are finitely presented left (right) ideals in S. Then S is finitely presented.*

PROOF. Suppose that A and B are defined by the presentations $\langle X, R\rangle$ and $\langle Y, Q\rangle$ respectively and they are left ideals in S. S is generated by the set $X \cup Y$ by Proposition 1.2.1. For every $x \in X, y \in Y$ the product $yx \in A$ and the product $xy \in B$. Therefore, there exist words $\beta_{xy} \in Y^+, \gamma_{yx} \in X^+$ such that $xy = \beta_{xy}$, $yx = \gamma_{yx}$ hold in S. Now we claim that S has the presentation

$$\langle X \cup Y | R, Q, xy = \beta_{xy}, yx = \gamma_{yx}\rangle. \tag{2.1}$$

It is obvious that S satisfies all the relations in the presentation (2.1). Let u, v be two words in $(X \cup Y)^+$ such that $u = v$ holds in S. We want to show that $u = v$ can be deduced from the relations in the presentation (2.1). First notice that $u = v$ holds in A or B. Suppose that $u = v$ holds in A. Notice that there is no word in A that ends with $y \in Y$ because A is a left ideal in S. If u, v do not have any letters from Y, then $u, v \in X^+$ and so $u = v$ holds in A. Thus $u = v$ is a consequence of R. Now if u, v have letters from Y then we use the relation $yx = \gamma_{yx}$ to eliminate all occurrences of letters of Y from u, v. We get two words $\bar{u}, \bar{v} \in X^+$ such that $u = \bar{u}$, $v = \bar{v}$ are consequences of our presentations (2.1) and $\bar{u}, \bar{v}$ represent the same element from A. Thus $\bar{u} = \bar{v}$ is a consequence of R. Hence $u = v$ is a consequence of our presentation. Analogously if $u = v$ holds in B. Similarly, if A and B are right ideals in S. □

Theorem 2.3.2. *Suppose that S is a semigroup which is the disjoint union of two subsemigroups A, B. Assume that A, B are finitely presented and A is an ideal in S. Then S is finitely presented.*

PROOF. Suppose that A and B are defined by the presentations $\langle X, R \rangle$ and $\langle Y, Q \rangle$ respectively. S is generated by the set $X \cup Y$ by Proposition 1.2.1. For every $x \in X, y \in Y$ the products $xy, yx \in A$. Therefore, there exist words β_{xy}, $\beta_{yx} \in X^+$ such that $xy = \beta_{xy}$, $yx = \beta_{yx}$ hold in S. Now we claim that S has the presentation

$$\langle X \cup Y | R, Q, xy = \beta_{xy}, yx = \beta_{yx} \rangle. \tag{2.2}$$

It is obvious that S satisfies all the relations in the presentation (2.2). Let u, v be two words in $(X \cup Y)^+$ such that $u = v$ holds in S. We want to show that $u = v$ can be deduced from the relations in the presentation (2.2). First notice that $u = v$ holds in A or B. Firstly suppose that $u = v$ holds in B. Then $u, v \in Y^+$ since A is an ideal. Hence $u = v$ is a consequence of Q. Secondly suppose that $u = v$ holds in A. If u, v do not have any letters from Y then $u = v$ is a consequence from R. Now if u, v have letters from Y then we use the relations $xy = \beta_{xy}, yx = \beta_{yx}$ to eliminate all occurrences of letters of Y from u, v. We get two words $\bar{u}, \bar{v} \in X^+$ such that $u = \bar{u}$, $v = \bar{v}$ are consequences of the presentations (2.2) and $\bar{u}, \bar{v}$ represent the same element from A. Thus $\bar{u} = \bar{v}$ is a consequence of R. Hence $u = v$ is a consequence of the presentation (2.2). □

After the two previous theorems we came across the following question, which is a weaker condition than Theorem 2.3.1 but with a negative result. Let S be a semigroup which is the disjoint union of two subsemigroups A, B and let $ABA \subseteq A$, $BAB \subseteq B$. If A, B are finitely presented, is S finitely presented? The answer is negative by the following counterexample.

Counterexample. Let T be the semigroup defined by the following multiplication table:

	a	b	c	d	e	f	g	h
a	a	a	a	a	a	a	a	a
b	a	a	a	a	a	b	c	d
c	a	b	c	d	d	d	d	d
d	d	d	d	d	d	d	d	d
e	e	e	e	e	e	e	e	e
f	e	e	e	e	e	f	g	h
g	e	f	g	h	h	h	h	h
h	h	h	h	h	h	h	h	h

and let U be any finitely presented infinite semigroup. Then we have the semigroup $S = T \times U$ and the two subsemigroups $S_1 = \{a, b, c, d\} \times U$ and $S_2 = \{e, f, g, h\} \times U$. It is clear that $S = S_1 \sqcup S_2$. It follows that S_1 and S_2 are finitely presented since the subsemigroup $Q_1 = \{a, b, c, d\}$ of T, has a left identity c and the subsemigroup $Q_2 = \{e, f, g, h\}$ of T, has a left identity f and by Corollary 2.1.3, Q_1, Q_2 preserve finite presentability in S_1, S_2 respectively. Notice that $S_1S_2S_1 \subseteq S_1$ and $S_2S_1S_2 \subseteq S_2$. In spite of all this, S is not finitely presented because of the following:

$$\Re(S) = \{b, c, f, g\} = \mathcal{L}(S),$$

and the graph $\Gamma(a)$ has the two non-connected vertices $(b, f, b), (b, a, c)$ where $b = b, fb \neq ac$. Therefore, $\Gamma(a)$ is not connected and hence T destroys finite presentability.

CHAPTER

THREE

CLASSIFICATION OF DISJOINT UNIONS OF TWO AND THREE COPIES OF A GROUP

3.1 Introduction

In this chapter we consider semigroups which can be decomposed into two or three groups G. Notice that dealing with disjoint unions of groups is much easier than disjoint unions of semigroups since in this case we have well-known theorems [CP61] and [How95] which help to classify such semigroups. For instance every completely regular semigroup is a semilattice of completely simple semigroups (Theorem 1.5.5), and if the semigroup S is a finite disjoint union of copies of a group, then S is a completely regular semigroup. So by knowing the size of the semilattice, we can classify S and study the finiteness conditions such as finite presentability and residual finiteness for S. We use the notation $\sqcup$ to indicate to disjoint union of (semi)groups.

3.2 Unions of two copies of a group

Theorem 3.2.1. *Let S_α and S_β be two copies of a group. For every semigroup S, which is a disjoint union of S_α and S_β, one of the following cases must hold:*

i) One of S_α or S_β is an ideal in S.

ii) Each S_α and S_β is a left ideal in S.

iii) Each S_α and S_β is a right ideal in S.

PROOF. Since S is a completely regular semigroup, S is a semilattice Y of completely simple semigroups by Theorem 1.5.5. Furthermore, S is a disjoint union of just two groups. Thus, the semilattice Y must have at most two elements. Therefore, we have the following two cases:

Case1: Y has two elements α, β, that means $\alpha\beta = \beta\alpha = \beta$ and that leads us to $S_\alpha S_\beta = S_\beta S_\alpha \subseteq S_\beta$. Hence, S_β is an ideal in S.

Case2: Y has just one element, that means S is a completely simple semigroup and it has minimal left (right) ideals by Theorem 1.5.1. Then by Lemma 1.5.2, S is a disjoint union of all minimal left (right) ideals which are S_α and S_β. □

Theorem 3.2.2. *Every semigroup which is a disjoint union of two copies of a group has one of the following forms:*

i) *Clifford semigroup (Strong semilattice of groups).*

ii) *Rees matrix semigroup over a group of the type* 1×2.

iii) *Rees matrix semigroup over a group of the type* 2×1.

PROOF. We consider the two cases of the proof of Theorem 3.2.1.

In case 1, in which the semilattice $|Y| = 2$, S is a semilattice of two groups and this implies that S is a Clifford semigroup $\mathcal{S}[Y; \{S_\alpha, S_\beta\}; \phi_{\alpha,\beta}]$ by Theorem 1.5.6. In case 2, in which the semilattice $|Y| = 1$, S is a completely simple semigroup and that means S is isomorphic to a Rees matrix semigroup $\mathcal{M}[G;\ I, \Lambda; P]$ of a particular shape by Theorem 1.5.4. Now, we have $G \cong S_\alpha \cong S_\beta$ and $|I| \cdot |\Lambda|$ copies of G. So, $|I| \cdot |\Lambda| = 2$. In addition, P is a $|\Lambda| \times |I|$ matrix and then it consists of one row and two columns as $P = [1\ 1]$ or two rows and one column as $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and then we have Rees matrix semigroups of types 1×2 and 2×1. □

3.3 Unions of three copies of a group

Theorem 3.3.1. *Let S_α, S_β and S_γ be three copies of a group. For every semigroup S, which is a disjoint union of S_α, S_β and S_γ, one of the following cases must hold (up to a permutation of S_α, S_β and S_γ):*

i) S_α is an ideal in S and $S_\beta S_\gamma, S_\gamma S_\beta \subseteq S_\alpha$.

ii) S_α is an ideal in S, $S_\beta \sqcup S_\gamma \leq S$ and S_β is an ideal in $S_\beta \sqcup S_\gamma$.

iii) S_α *is an ideal in* S, $S_\beta \sqcup S_\gamma \leq S$ *and* S_β, S_γ *are left ideals in* $S_\beta \sqcup S_\gamma$.

iv) S_α *is an ideal in* S, $S_\beta \sqcup S_\gamma \leq S$ *and* S_β, S_γ *are right ideals in* $S_\beta \sqcup S_\gamma$

v) $S_\beta \sqcup S_\gamma$ *is an ideal in* S *and* S_β, S_γ *are left ideals in* $S_\beta \sqcup S_\gamma$.

vi) $S_\beta \sqcup S_\gamma$ *is an ideal in* S *and* S_β, S_γ *are right ideals in* $S_\beta \sqcup S_\gamma$.

vii) S_α, S_β *and* S_γ *are left ideals in* S.

viii) S_α, S_β *and* S_γ *are right ideals in* S.

PROOF. The proof is similar to the proof of Theorem 3.2.1. Since S is a completely regular semigroup, S is a semilattice Y of completely simple semigroups by Theorem 1.5.5. Furthermore, S is a disjoint union of just three groups. Thus, the semilattice Y must have at most three elements. Therefore, we have three cases as follows:
Case 1: Y has just one element. Then S is a completely simple semigroup and it has minimal left (right) ideals by Theorem 1.5.1. Then by Lemma 1.5.2, S is a disjoint union of all minimal left (right) ideals which are S_α, S_β and S_γ, which gives cases $(vii), (viii)$.
Case 2: Y has two elements. We split this case into four cases as follows:

1) S_α is an ideal in S where S_β, S_γ are left ideals in $S_\beta \sqcup S_\gamma$.

2) S_α is an ideal in S where S_β, S_γ are right ideals in $S_\beta \sqcup S_\gamma$.

3) $S_\beta \sqcup S_\gamma$ is an ideal in S where S_β, S_γ are left ideals (three cases).

4) $S_\beta \sqcup S_\gamma$ is an ideal in S where S_β, S_γ are right ideals (three cases), and this proves $(iii), (iv), (v), (vi)$.

Case 3: Y has three elements α, β and γ. It follows that

1) $\alpha\beta = \alpha\gamma = \beta\gamma = \alpha$ which means $\beta \geq \alpha$, $\gamma \geq \alpha$. Thus S_α is an ideal in $S_\alpha \sqcup S_\beta \sqcup S_\gamma$. This proves (i).

2) $\alpha\beta = \beta\alpha = \alpha$, $\beta\gamma = \gamma\beta = \beta$, $\alpha\gamma = \gamma\alpha = \alpha$ which means $\gamma \geq \beta \geq \alpha$. Thus S_β is an ideal in $S_\beta \sqcup S_\gamma$ and S_α is an ideal in $S_\alpha \sqcup S_\beta \sqcup S_\gamma$. This proves (ii).

Now we present some methods of constructions of semigroups which are disjoint unions of three copies of a group, by using the structure theory of Clifford semigroups and Rees matrix semigroups. Let V denote the partial order with elements $\{\alpha, \beta, \gamma\}$ defined by $\{(\alpha, \beta), (\alpha, \gamma)\} \cup \Delta$ where Δ is the diagonal relation on

$\{\alpha, \beta, \gamma\}$. Let I denote the linear order with three elements $\{\alpha, \beta, \gamma\}$ defined by $\{(\alpha, \beta), (\alpha, \gamma), (\beta, \gamma)\}$. Let $\top$ and $\perp$ denote the linear orderers with two elements as it is shown below.

Constructions:

1. Clifford semigroup of the type V. For this type we have the following:

   $i)$ Three groups $G_\alpha, G_\beta, G_\gamma$.

   $ii)$ Two homomorphisms $\phi_{\beta,\alpha} : G_\beta \to G_\alpha,\ \phi_{\gamma,\alpha} : G_\gamma \to G_\alpha$.

   The semigroup is then constructed as $\mathfrak{C}\left[V; \{G_\alpha, G_\beta, G_\gamma\}; \{\phi_{\beta,\alpha}, \phi_{\gamma,\alpha}\}\right]$.

2. Clifford semigroup of the type I. For this type we have the following:

   $i)$ Three groups $G_\alpha, G_\beta, G_\gamma$.

   $ii)$ Three homomorphisms $\phi_{\gamma,\alpha} : G_\gamma \to G_\alpha,\ \phi_{\gamma,\beta} : G_\gamma \to G_\beta,\ \phi_{\beta,\alpha} : G_\beta \to G_\alpha$.

   The semigroup is then constructed as $\mathfrak{C}\left[I; \{G_\alpha, G_\beta, G_\gamma\}; \{\phi_{\gamma,\alpha}, \phi_{\gamma,\beta}, \phi_{\beta,\alpha}\}\right]$.

3. Combination semigroup S of the type $\top$. For this type we have the following:

   $i)$ A group G.

   $ii)$ A Rees matrix semigroup R of the type 2×1 over G $\left(\mathcal{M}\left[G;\ I, \Lambda; P\right]\right)$ where $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

   $iii)$ A homomorphism ϕ from R into G with $\phi(1, x, 1) = \phi(1, x, 2) = \varphi(x)$ where φ is a homomorphism from G into G.

   The semigroup is then constructed as $\mathfrak{C}[\top; \{R, G\}; \phi_{R,G}]$.

4. Combination semigroup S of the type $\perp$. For this type we have the following:

   $i)$ A group G.

   $ii)$ A Rees matrix semigroup R of the type 2×1 over G $\left(\mathcal{M}\left[G;\ I, \Lambda; P\right]\right)$ where $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and R is an ideal in S.

   The semigroup is then a semilattice of a group G and a Rees matrix semigroup R. Furthermore, there is a special case of this type and in this case we have the following:

i) A group G.

ii) A Rees matrix semigroup R of the type 2×1 over G $\left(\mathcal{M}\left[G;\ I, \Lambda; P\right]\right)$ where $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and R is an ideal in S.

iii) A homomorphism ϕ from G into R with $\phi(g) = (1, \varphi(g), 1)$ where φ is a homomorphism from G into G .

The semigroup is then a strong semilattice of a group G and a Rees matrix semigroup R which is constructed as $\mathfrak{C}[\perp; \{G, R\}; \phi_{G,R}]$.

5. Combination semigroup of types $\vdash$ and $\dashv$.
These two combination semigroups of types $\vdash$ and $\dashv$ are anti-isomorphic to the combination semigroups of types $\top$ and $\perp$ respectively, by taking the Rees matrix semigroup of the type 1×2 over G $\left(\mathcal{M}\left[G;\ I,\ \Lambda;\ [1\ 1]\right]\right)$.

Theorem 3.3.2. *Every semigroup which is a disjoint union of three copies of a group is one of the following types:*
Type 1: Rees matrix semigroup construction.

i) Rees matrix semigroup of the type 1×3.

ii) Rees matrix semigroup of the type 3×1.

Type 2: Combination semigroup construction.

i) Combination semigroup of the type $\top$.

ii) Combination semigroup of the type $\perp$.

iii) Combination semigroup of the type $\vdash$.

iv) Combination semigroup of the type $\dashv$.

Type 3: Clifford semigroup construction.

i) Clifford semigroup of the type V.

ii) Clifford semigroup of the type I

PROOF. Consider the three cases in the proof of Theorem 3.3.1. In case 1, in which the semilattice $|Y| = 1$, S is a completely simple semigroup and that means S is isomorphic to a Rees matrix semigroup $\mathcal{M}\left[G; I, \Lambda; P\right]$ of a particular shape by

Theorem 1.5.4. Now, we have $G \cong S_\alpha \cong S_\beta \cong S_\gamma$ and $|I| \cdot |\Lambda|$ copies of G. So, $|I| \cdot |\Lambda| = 3$. In addition, P is a $|\Lambda| \times |I|$ matrix and so it consists of one row and three columns as $P = [1\ 1\ 1]$ or three rows and one column as $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Thus we have Rees matrix semigroups of types 1×3 or 3×1.

In case 2, in which the semilattice $|Y| = 2$, S is a semilattice of a Rees matrix semigroup $S_\delta = S_\beta \sqcup S_\gamma$ and a group S_α and then we have the semilattice $Y = \{\alpha, \delta\}$. As a consequence of this semilattice, we have four cases. Case 2_1 is when $\alpha\delta = \delta\alpha = \alpha$, which means that S_α is an ideal in S and S_δ is a Rees matrix semigroup of the type 2×1. Case 2_2 is the same of case 2_1 but the Rees matrix semigroup S_δ is of the type 1×2. Case 2_3 is when $\alpha\delta = \delta\alpha = \delta$, which means that the Rees matrix semigroup S_δ of the type 2×1 is an ideal in S. Case 2_4 is the same of case 2_3 but the Rees matrix semigroup S_δ is of the type 1×2. We will prove the theorem for case 2_1 and case 2_3 and the proof of case 2_2 and case 2_4 is analogous. In case 2_1 we have a homomorphism ϕ from the Rees matrix semigroup $S_\beta \sqcup S_\gamma$ into the group S_α. Let φ_1 and φ_2 be two homomorphisms from S_α to S_α with

$$\phi(1, x, 1) = \varphi_1(x),\ \phi(1, x, 2) = \varphi_2(x)$$

and then

$$\phi(1, x, 1)\phi(1, x^{-1}, 2) = \varphi_1(x)\varphi_2(x^{-1})$$

and

$$\phi(1, 1, 2) = \varphi_2(1) = 1.$$

So

$$\varphi_1(x)\varphi_2(x^{-1}) = 1$$

and similarly $\varphi_2(x^{-1})\varphi_1(x) = 1$. Hence, $\varphi_1(x) = \varphi_2(x)$. Thus S is a combination semigroup of the type $\top$. In case 2_3 we have a Rees matrix semigroup of the type 2×1 which is an ideal in S and we have a group S_α. Notice that in this type, either the group acts on the block of the Rees matrix semigroup by swopping them around, for more details see Section 5.6 case 6, or there exists a homomorphism ϕ from S_α into $S_\beta \sqcup S_\gamma$ and because of the fact that the homomorphic image of a group is a group. Then ϕ must map S_α into one of the two groups S_β or S_γ. Hence, S is a combination semigroup of the type $\perp$.

In case 3, in which the semilattice $|Y| = 3$, S is a strong semilattice of three groups

S_α, S_β and S_γ and that implies S is a Clifford semigroup of two types V and I by Theorem 1.5.6. □

3.4 Finiteness conditions for disjoint unions of groups

Corollary 3.4.1. *Every semigroup which is a disjoint union of finitely many finitely presented groups is finitely presented.*

PROOF. The semigroup S which is a disjoint union of groups is regular with finitely many $\mathcal{H}-$classes (the subgroups themselves) by Corollary 1.5.3. Thus S has finitely many $\mathcal{R}-$classes and $\mathcal{L}-$classes and the result follows from ([Rus99], Theorem 4.1). □

Corollary 3.4.2. *Every semigroup which is a disjoint union of finitely many residually finite groups is residually finite.*

PROOF. Since S is regular and all subgroups are residually finite and the set of idempotents of each principal factor of S is finite, S is residually finite by [Gol75]. □

CHAPTER

FOUR

CLASSIFICATION OF DISJOINT UNIONS OF TWO COPIES OF THE FREE MONOGENIC SEMIGROUP

4.1 Introduction

Let S be a semigroup which is a disjoint union of two copies of the free monogenic semigroup A and B in which A, B are generated by a and b respectively. That means all elements in S are of the form a^i or b^j where $i, j \in \mathbb{N}$. Hence, the product of any two elements in S is again a power of a or b, and because of the disjointness, we have a crucial property which is $a^i \neq b^j$ for every $i, j \in \mathbb{N}$. Using this property together with the fact that S is associative and does not have an element of finite order, will play a significant role in some of our proofs. Notice that this observation applies to every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup.

In this chapter we start with a description of S and then we provide some properties of this semigroup, we prove that there are an infinite number of such semigroups, and then we give a construction based on Clifford and Rees matrix semigroups see [CP61] and [How95].

Lemma 4.1.1. *Let A be a set. Let $\Re$ be a set of relations on A^+. Let ρ be the smallest congruence on A^+ generated by $\Re$. Let $\varphi : A \to \mathbb{N}_0$ be a mapping and let $\psi : A^+ \to \mathbb{N}_0$ be the unique homomorphism determined by φ. If $\rho \subseteq \ker \psi$ and $a \in A$ such that $\varphi(a) \neq 0$ then $\langle a/\rho \rangle$ is an infinite subsemigroup of A^+/ρ.*

PROOF. Since $\rho \subseteq \ker \psi$ we have that the mapping $\overline{\psi} : A^+/\rho \to \mathbb{N}_0$ defined by $\overline{\psi}(w/\rho) = \psi(w)$ is a homomorphism by Theorem 1.3.2. Since a homomorphism maps elements of finite order to elements of finite order, and since 0 is the only element of finite order in $\mathbb{N}_0$, it follows from $\overline{\psi}(a/\rho) = \psi(a) = \phi(a) \neq 0$ that a/ρ must have infinite order in A^+/ρ. $\square$

Lemma 4.1.2. *Suppose that S' is a semigroup which is a disjoint union of n copies of the free monogenic semigroup $A_i = \langle a_i \rangle$; $i = \{1, 2, \ldots, n\}$ and S' is defined by the presentation*

$$\langle a_1, a_2, \ldots, a_n \mid a_i a_j = a_k^l \rangle, \tag{4.1}$$

where $i, j, k \in \{1, 2, \ldots, n\}$. Then if S is a semigroup which is a disjoint union of n copies of the free monogenic semigroup and S is satisfied relations in the presentation (4.1) *then $S \cong S'$.*

PROOF. If S is satisfied relations in the presentation then S is a homomorphic image of S' by Proposition 1.6.2. If it is a proper homomorphic image then there is without loss of generality a_i^k and a_i^l or a_i^k and a_j^l in S' such that $a_i^k = a_i^l$ or $a_i^k = a_j^l$ which contradicts with the fact that there is no element in A_i is of finite order or contradicts with $A_i \cap A_j = \emptyset$. Thus $S' \cong S$. $\square$

4.2 Classification of semigroups which are disjoint unions of two copies of the free monogenic semigroup

Lemma 4.2.1. *Let A and B be free monogenic semigroups generated by a and b respectively. Suppose that S is a semigroup which is a disjoint union of A and B. Then:*

i) If $ab = a^k$ and $ba = a^l$ in S, then we must have $l = k$.

ii) If $ab = a^k$ and $ba = b^l$ in S, then we must have $l = k = 2$.

PROOF. (i) We start with aba and note that:

$$a(ba) = aa^l = a^{l+1}$$

and also

$$(ab)a = a^k a = a^{k+1}.$$

Thus, since we have an associative operation, $l = k$.

Corollary 4.2.2. *Let A and B be free monogenic semigroups generated by a and b respectively. Suppose that S is a semigroup which is a disjoint union of A and B. Then A is an ideal in S or A and B are left(right) ideals in S.*

PROOF. Directly by Lemma 4.2.1. □

Now we prove the second case (ii), we start with aba as well. Thus we have

$$a(ba) = ab^l = a^k b^{l-1} = \underbrace{a^{k-1} \cdots a^{k-1}}_{l-1} a^k = a^{(k-1)(l-1)} a^k = a^{l(k-1)+1}$$

and

$$(ab)a = a^k a = a^{k+1}.$$

Thus a^{k+1} must be equal to $a^{l(k-1)+1}$. That implies $k = l(k-1)$ and then the only values that satisfy this is when $k = l = 2$. □

Lemma 4.2.1 gives us the necessary conditions to obtain a semigroup which is a disjoint union of two copies of the free monogenic semigroup, and this leads us to the following classification.

Theorem 4.2.3. *Every semigroup S is a disjoint union of two copies of the free monogenic semigroup $\langle c \rangle$ and $\langle d \rangle$ if and only if S is isomorphic to the semigroup which is defined by one of the following presentations:*

i) $\langle a, b \mid ab = ba = a^k,\ k \geq 1 \rangle$;

ii) $\langle a, b \mid ab = a^2,\ ba = b^2 \rangle$;

iii) $\langle a, b \mid ab = b^2,\ ba = a^2 \rangle$.

PROOF. We divide the proof into two parts. . In the first part we show that each semigroup S which is a disjoint union of two copies of the free monogenic semigroup is actually defined by one of these presentations. In the second part we show that each semigroup S' which is defined by one of the presentations i, ii, or iii, is a disjoint union of two copies of the free monogenic semigroup

Part 1. ($\Rightarrow$) Since S is a disjoint union of two copies of the free monogenic semigroup $\langle c \rangle, \langle d \rangle$, S satisfies one of the relations in Theorem 4.2.3. Then S is a homomorphic image of S' by Proposition 1.6.2. Hence $S \cong S'$ by Lemma 4.1.2.

Part 2. ($\Leftarrow$) Firstly, any element of the semigroup S' with the presentation (i)

$$\langle a, b | ab = a^k, ba = a^k,\ k \geq 1 \rangle$$

is a power of a, b or a product of a's and b's but if there was at least one a in the product then this product will be a power of a. Thus,

$$\langle a, b | ab = a^k, ba = a^k,\ k \geq 1 \rangle = A \cup B,$$

where $A = \{a^i : i \in \mathbb{N}\}$ and $B = \{b^j : j \in \mathbb{N}\}$. Since there is no relation which can be applied to a power of b, we have $A \cap B = \emptyset$ and B is infinite. We now prove that A is infinite. Let $\varphi : \{a, b\} \to \mathbb{N}_0$ be a mapping with $\varphi(a) = 1$, $\varphi(b) = k - 1$ where $k \in \mathbb{N}$ and then there exists a homomorphism $\psi : \{a, b\}^+ \to \mathbb{N}_0$ since $\{a, b\}^+$ is the free semigroup on $\{a, b\}$ and ψ uniquely determined by the images of the generators a and b by Proposition 1.6.1. Observe that

$$\psi(ab) = \psi(a) + \psi(b) = 1 + k - 1 = k = \psi(a^k),$$

and similarly $\psi(ba) = \psi(a^k)$, from which we deduce that

$$\Re = \{ab = ba = a^k\} \subseteq \ker \psi.$$

Hence, by Lemma 4.1.1 all elements of the semigroup S' are distinct and then $S' = A \sqcup B$ as required.

Secondly, any element of the semigroup S' with the presentation (ii)

$$\langle a, b | ab = a^2, ba = b^2 \rangle$$

is a power of a, b or a product of a's and b's but observe that any word starts with a will be a power of a and any word starts with b will be a power of b. That implies any word in A must start with a and any word in B must start with b. Thus,

$$\langle a, b | ab = a^2, ba = b^2 \rangle = A \cup B,$$

where $A = \{a^i : i \in \mathbb{N}\}$ and $B = \{b^j : j \in \mathbb{N}\}$ and $A \cap B = \emptyset$. Then the proof is similar to the proof of (i) but the difference is in the definition of the mapping φ since we define $\varphi : \{a, b\} \to \mathbb{N}$ as $\varphi(a) = 2$ and $\varphi(b) = 2$ and then there exists a homomorphism $\psi : \{a, b\}^+ \to \mathbb{N}$ since $\{a, b\}^+$ is the free semigroup on $\{a, b\}$ and ψ uniquely determined by the images of the generators a and b by Proposition 1.6.1. Observe that

$$\psi(ab) = \psi(a) + \psi(b) = 2 + 2 = 4 = \psi(a^2),$$

and similarly $\psi(ba) = \psi(b^2)$, from which we deduce that

$$\Re = \{ab = a^2, ba = b^2\} \subseteq \ker \psi.$$

Hence, by Lemma 4.1.1 all elements of the semigroup $A \sqcup B$ are distinct and then $S' = A \sqcup B$ as required. Similarly with the semigroup S' which is defined by the presentation (iii). $\square$

Now, we have an infinite number of such semigroups which are defined by the presentations $\langle a, b \mid ab = ba = a^k,\ k \geq 1 \rangle$. Hence, the next step is to prove that all of these semigroups are non-isomorphic.

Lemma 4.2.4. *Suppose that $S_1 = A \sqcup B$ and $S_2 = A \sqcup B$. Suppose that $S_1 \cong S_2$ via the isomorphism $\varphi : S_1 \to S_2$. Then*

i) If $\varphi(a) = a^t$ where $t \in \mathbb{N}$ then we must have $t = 1$.

ii) If $\varphi(a) = b^t$ where $t \in \mathbb{N}$ then $\varphi(b) = a^u$ and $t = u = 1$.

PROOF. Let $\varphi^{-1} : S_2 \to S_1$ be the inverse of φ. (i) First note that $\varphi^{-1}(a) = a^u$ where $u \in \mathbb{N}$ because if we say that $\varphi^{-1}(a) = b^u$ then

$$a = \varphi^{-1}\Big(\varphi(a)\Big) = \varphi^{-1}(a^t) = \Big(\varphi^{-1}(a)\Big)^t = b^{ut},$$

a contradiction. So, $a = \varphi^{-1}\Big(\varphi(a)\Big) = \varphi^{-1}(a^t) = a^{ut}$ and that implies $ut = 1$. Thus both of u and t are equal to 1.

(ii) Now if $\varphi(a) = b^t$ and $\varphi(b) = b^u$ then from (i), $u = 1$ and then $a = \varphi^{-1}(b^t) = b^t$, a contradiction. Thus $\varphi(b) = a^u$, suppose that $t > 1$ or $u > 1$ and then there is no $x \in S_1$ such that $\varphi(x) = b^{t-1}$ or b^{u-1}, a contradiction. Therefore $t = u = 1$. $\square$

Proposition 4.2.5. *All semigroups which are defined by presentations*

$$\langle a, b \mid ab = ba = a^k,\ k \geq 1 \rangle$$

are non-isomorphic.

PROOF. For $k \geq 1$, let $S_k = A \sqcup B$ such that $ab = ba = a^k$. Thus S_k is the semigroup which is defined by the presentation

$$\langle a, b \mid ab = ba = a^k,\ k \geq 1 \rangle,$$

by Theorem 4.2.3. Suppose $S_k \cong S_l$ for some k, l with $k \neq l$. So there is an isomorphism $\varphi : S_k \to S_l$. Clearly, A is the only ideal which is isomorphic to $\mathbb{N}$ in both of S_k and S_l because the presentation says that any word which has a letter a will be in A and hence each ideal must consist of elements from just A or from A and B, but the latter has at least two generators which implies that it is not isomorphic to $\mathbb{N}$. Consequently, $\varphi(a) = a^u$ and by Lemma 4.2.4 we must have $u = 1$. Thus $\varphi(b) = b^t$ and by Lemma 4.2.4 we must have $t = 1$. But $\varphi(ab) = \varphi(a^k) = a^k$ and $\varphi(a)\varphi(b) = ab = a^l$ which contradicts the assumption. □

4.3 Constructions

Corollary 4.2.2 says that, in order to construct a semigroup which is a disjoint union of two copies of the free monogenic semigroup, we must have one of three cases. The first case is when one of the two copies is an ideal in S, the second case is when the two copies are left ideals in S and the last case is when the two copies are right ideals in S. This leads us to well-known constructions in semigroup theory which are Clifford and Rees matrix semigroups. This is explained in the next theorem.

Theorem 4.3.1. *Every semigroup which is a disjoint union of two copies of the free monogenic semigroup has one of the following forms:*

$i)$ *Strong semilattice of free monogenic semigroups;*

$ii)$ *Rees matrix semigroup over a free monogenic semigroup of the type* 1×2;

$iii)$ *Rees matrix semigroup over a free monogenic semigroup of the type* 2×1.

PROOF. Let S be a semigroup that is a disjoint union of two copies of the free monogenic semigroup $S_\alpha = \langle a \rangle$ and $S_\beta = \langle b \rangle$. Then we have two cases. The first case is when one of the two copies is an ideal in S (Theorem 4.2.3(i)). Suppose that S_α is an ideal in $S_\alpha \sqcup S_\beta$ with $ab = ba = a^i$ ($i \in \mathbb{N}$). So, there is a semilattice $Y = \{\alpha, \beta\}$ where to each element $\alpha \in Y$ we assign a semigroup S_α such that $\alpha\beta = \alpha$ if $S_\alpha S_\beta \subseteq S_\alpha$ and to a pair of elements $\alpha,\ \beta$ we assign a map $\phi_{\alpha,\beta}$ of S_α into S_β with $\phi_{\beta,\alpha}(b) = a^{i-1}$. Also there are two identity maps, $\phi_{\alpha,\alpha}$ on S_α and $\phi_{\beta,\beta}$ on S_β. Clearly, $\phi_{\alpha,\beta}\phi_{\beta,\beta} = \phi_{\alpha,\beta}$ if $\alpha \geq \beta$ and $\phi_{\alpha,\alpha}\phi_{\alpha,\beta} = \phi_{\alpha,\beta}$ if $\alpha \geq \beta$. Now, notice that

$$\begin{aligned} a^x b^y &= a^{x-1} a^i b^{y-1} \\ &= a^{x-1} a^{i-1} a^i b^{y-2} \\ &= a^{x-1} a^{(y-2)(i-1)} a^i b^{y-(y-1)} \\ &= a^{x-1} a^{(y-2)(i-1)} a^{i-1} a^i \\ &= a^{iy-y+x}, \end{aligned}$$

and

$$\phi_{\alpha,\alpha}(a^x)\phi_{\beta,\alpha}(b^y) = a^x a^{y(i-1)} = a^{iy-y+x} = a^x b^y.$$

And similarly with $b^x a^y$. Thus, we can define a multiplication on S by

$$a^x b^y = \phi_{\alpha,\alpha}(a^x)\phi_{\beta,\alpha}(b^y),$$

and then we get a semigroup S which is a strong semilattice of free monogenic semigroups. The above argument works for every $i > 1$ and a small modification is needed for the case $i = 1$. This based on the same idea as near Rees matrix semigroup, but this time for strong semilattice of semigroups. More precisely, we adjoin the identity 1_β to S_β and let $\bar{S} = S_\alpha \sqcup S_\beta^{1_\beta}$. Next define $\phi_{\alpha,\beta}(a) = 1_\beta$ to form $\bar{S}$ into a strong semilattice of semigroups with multiplication $ab = ba = b$. Finally let $S = S_\alpha \sqcup S_\beta \leq \bar{S}$.

There are infinite number of these homomorphisms where each i indicates to a certain semigroup but surely all these kind of semigroups are non-isomorphic by Proposition 4.2.5.

The second case is if S_α and S_β were left ideals (Theorem 4.2.3 (iii)). Now we want to construct all potential Rees matrix semigroups over $S_\alpha^{1_\alpha}$. Therefore, we can choose $I = \{1\}$, $\Lambda = \{1,2\}$ and P to be the $|\Lambda| \times |I|$ matrix over $S_\alpha^{1_\alpha}$ as $P = \begin{bmatrix} 1_\alpha \\ 1_\alpha \end{bmatrix}$. Thus, $\bar{S} \cong \mathcal{M}\left[S_\alpha^{1_\alpha}; \{1\}, \{1,2\}; P\right]$ is a semigroup. Now, it is clear for us that S is isomorphic to $\{(i, a^t, \lambda) : t \neq 0\} \leq \bar{S}$, because there exists a mapping ψ from S into $\{(i, a^t, \lambda) : t \neq 0\}$ with $\psi(a) = (1, a, 1)$, $\psi(b) = (1, a, 2)$. Obviously, ψ is injective and surjective, so it remains to verify that ψ is a homomorphism as follows:

$$\psi(ab) = \psi(b^2) = (1, a^2, 2) \ , \ \psi(a)\psi(b) = (1, a, 1)(1, a, 2) = (1, a^2, 2).$$

Hence,

$$S \cong \{(i, a^t, \lambda) : t \neq 0\} \cong \{(i, b^t, \lambda) : t \neq 0\}.$$

Analogously, if S_α and S_β are right ideals (Theorem 4.2.3 (ii)) then

$$\bar{S} \cong \mathcal{M}\left[S_\alpha^{1_\alpha}; \{1,2\}, \{1\}; [1_\alpha \, 1_\alpha]\right] \cong \mathcal{M}\left[S_\beta^{1_\beta}; \{1,2\}, \{1\}; [1_\beta \, 1_\beta]\right].$$

As required. □

4.4 Comparison between disjoint unions of two copies of the free monogenic semigroup and two copies of the infinite cyclic group

Consider the semigroup $S = S_\alpha \sqcup S_\beta$ where $S_\alpha = \langle a, a^{-1} | aa^{-1} = a^{-1}a = 1_\alpha \rangle$ and $S_\beta = \langle b, b^{-1} | bb^{-1} = b^{-1}b = 1_\beta \rangle$ are two copies of the infinite cyclic group. By Theorems 3.2.1, 3.2.2, we have two cases. The first case is when S_β is an ideal in S. Then we have two groups S_α , S_β with $ab = b^i = ba$ and that because if $ba = b^j$ then $b(ab) = b^{i+1}$ and $(ba)b = b^{j+1}$ which implies that $i = j$. Similarly $a^{-1}b^{-1} = b^j = b^{-1}a^{-1}$, $ab^{-1} = b^k = b^{-1}a$, $a^{-1}b = b^l = ba^{-1}$. Thus, there is a semilattice $Y = \{\alpha, \beta\}$ where to each element $\alpha \in Y$ we assign a group S_α such that $\alpha\beta = \beta\alpha = \beta$ if $S_\alpha S_\beta \subseteq S_\beta$, $S_\beta S_\alpha \subseteq S_\beta$. There is also a pair of elements α, β in which we assign a map $\phi_{\alpha,\beta}$ from S_α into S_β with $\phi_{\alpha,\beta}(a) = b^{i-1}$, $\phi_{\alpha,\beta}$. Also there are two identity maps $\phi_{\alpha,\alpha}$, $\phi_{\beta,\beta}$ on S_α and S_β respectively. Clearly, $\phi_{\alpha,\alpha}\phi_{\alpha,\beta} = \phi_{\alpha,\beta}$ if $\alpha \geq \beta$, $\phi_{\alpha,\beta}\phi_{\beta,\beta} = \phi_{\alpha,\beta}$ if $\alpha \geq \beta$. Now we consider i as a positive number and if i is a negative number then it is proved similarly. Notice that

$$a^x b^y = a^{x-1} b^i b^{y-1} = b^{x(i-1)+y} \ , \ \phi_{\alpha,\beta}(a^x)\phi_{\beta,\beta}(b^y) = b^{x(i-1)+y}.$$

Analogously with the relation $b^y a^x$, $a^{-x}b^{-y}$, $b^{-y}a^{-x}$, $a^x b^{-y}$, $b^{-y}a^x$, $a^{-x}b^y$, $b^y a^{-x}$. Thus S is a strong semilattice of groups and so S is a Clifford semigroup by Theorem 1.5.6.

The second case is when S_α, S_β are left ideals in S. Then $S = S_\alpha \sqcup S_\beta$ is a completely simple semigroup and that means S is isomorphic to a Rees matrix semigroup of a particular shape by Theorem 1.5.4. Now we want to construct all such Rees matrix semigroups. In fact, we have two left ideals S_α, S_β. So,

$$S_\alpha = L_1 = S1_\alpha,\ S_\beta = L_2 = S1_\beta \text{ and } R_1 = 1_\alpha S = 1_\beta S = S.$$

Clearly, L_1, L_2 are $L-$classes and R_1 is $R-$class. Hence $I = \{1\}$, $\Lambda = \{1,2\}$ and since P is a $|\Lambda| \times |I|$ matrix, it consists of two rows and one column as $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Observe that $R_1 \cap L_1 = H_{11}$, $R_1 \cap L_2 = H_{12}$. Therefore,

$$H_{11} = L_1 = S_\alpha,$$

$$H_{12} = L_2 = S_\beta,$$

and so

$$S = M\big[S_\alpha; \{1\}, \{1,2\}; P\big] \text{ or } S = M\big[S_\beta; \{1\}, \{1,2\}; P\big].$$

Notice that the two Rees matrix semigroups are isomorphic and that because both of them have the same sets I, Λ and the two semigroups S_α, S_β are isomorphic. Analogously, if S_α, S_β are right ideals.

Remark 4.4.1. Clearly, in each of the above constructions we can substitute the free cyclic group by the free monogenic semigroup, to obtain unions of two copies of the free monogenic semigroup of these two types. In a different way, applying all the above constructions on a free monogenic semigroup, we get a collection of semigroups which are disjoint unions of two copies of the free monogenic semigroup. We can show this directly by a comparison between these constructions and Theorem 4.3.1.

CHAPTER

FIVE

CLASSIFICATION OF DISJOINT UNIONS OF THREE COPIES OF THE FREE MONOGENIC SEMIGROUP

5.1 Introduction

Let S be a semigroup such that $S = A \sqcup B \sqcup C$ where A, B and C are free monogenic semigroups generated by a, b and c respectively with

$$ab = t_1^i,\ ba = t_2^j,\ ac = t_3^k,\ ca = t_4^l,\ bc = t_5^p,\ cb = t_6^q,$$

where $t_1, t_2, t_3, t_4, t_5, t_6 \in \{a, b, c\}$ and $i, j, k, l, p, q \in \mathbb{N}$. Then we say S is of the *type* $(t_1, t_2, t_3, t_4, t_5, t_6)$. Use $*$ to indicate to t_i where $i \in \{1, 2, 3, 4, 5, 6\}$. Therefore, potentially there are 729 types, but in this chapter we will see that this number can be reduced to 9. For each of these types we exhibit a presentation defining semigroups of this type, see Main Theorem and Table 5.1. We use this to make the following interesting observation. We saw in Theorem 4.3.1 that the disjoint union of two copies of the free monogenic semigroup is either a strong semilattice of semigroups or a Rees matrix semigroup, and this parallels the case when S is a disjoint union of two copies of a group. But for three copies of the free monogenic semigroup we show that there are not only strong semilattices of semigroups, Rees matrix semigroups and a combination of these two constructions paralleling with the group case, but also we have some other semigroups which do not arise from any of these constructions.

Main Theorem. *Up to isomorphism and anti-isomorphism, every semigroup S is a disjoint union of three copies of the free monogenic semigroup $D = \langle d \rangle$, $G = \langle g \rangle$ and $H = \langle h \rangle$ if and only if S is isomorphic to the semigroup which is defined by one of the following presentations:*

(1) $\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = a^p, cb = a^p \rangle$,
where $i + k - p = 2$ *and* $i, k, p \in \mathbb{N}$.

(2) $\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = b^p, cb = b^p \rangle$,
where $i + k + p - ip = 2$ *and* $i, k, p \in \mathbb{N}$.

(3) $\langle a, b, c | ab = a^i, ba = a^i, ac = a^i, ca = a^i, bc = c^2, cb = b^2 \rangle$,
where $i \in \mathbb{N}$.

(4) $\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i \rangle$,
where $i \in \mathbb{N}$.

(5) $\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = c^i \rangle$,
where $i \in \mathbb{N}$.

(6) $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = b^2 \rangle$,

(7) $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \rangle$,

(8) $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2 \rangle$,

(9) $\langle a, b, c | ab = b^2, ba = a^2, ac = b^i, ca = a^i, bc = a^i, cb = b^i \rangle$,
where $i \in \mathbb{N}$.

This entire chapter is devoted to proving this theorem. We proceed as follows. We have 6 products ab, ba, ac, ca, bc, cb. So firstly, we consider the different possibilities for the first two products, ab and ba. We start with $ab = a^i$ and $ba = a^j$, and then we figure out what all the possible semigroups that can be obtained from this possibility. We do the same with the possibility

$$ab = b^i, \; ba = a^j,$$

and these two possibilities, up to isomorphism and anti-isomorphism, cover all the cases as we show in this chapter.

Now we summarize the results of this chapter in the following table:

2-letter prefix	Types	Semigroups
$(a,a,*,*,*,*)$	(a,a,a,a,a,a)	$\langle a,b,c \mid ab=a^i, ba=a^i, ac=a^k, ca=a^k, bc=a^p, cb=a^p\rangle$
	(a,a,a,a,b,b)	$\langle a,b,c \mid ab=a^i, ba=a^i, ac=a^k, ca=a^k, bc=b^p, cb=b^p$
	(a,a,a,a,c,b)	$\langle a,b,c \mid ab=a^i, ba=a^i, ac=a^i, ca=a^i, bc=c^2, cb=b^2\rangle$
	(a,a,c,a,c,a)	$\langle a,b,c \mid ab=a^i, ba=a^i, ac=c^2, ca=a^2, bc=c^i, cb=a^i\rangle$
	(a,a,c,a,c,c)	$\langle a,b,c \mid ab=a^i, ba=a^i, ac=c^2, ca=a^2, bc=c^i, cb=c^i\rangle$
$(b,a,*,*,*,*)$	(b,a,c,a,c,b)	$\langle a,b,c \mid ab=b^2, ba=a^2, ac=c^2, ca=a^2, bc=c^2, cb=b^2\rangle$
	(b,a,c,b,c,a)	$\langle a,b,c \mid ab=b^2, ba=a^2, ac=c^2, ca=b^2, bc=c^2, cb=a^2\rangle$
	(b,a,c,a,c,a)	$\langle a,b,c \mid ab=b^2, ba=a^2, ac=c^2, ca=a^2, bc=c^2, cb=a^2\rangle$
	(b,a,b,a,a,b)	$\langle a,b,c \mid ab=b^2, ba=a^2, ac=b^i, ca=a^i, bc=a^i, cb=b^i\rangle$

Table 5.1: The nine types of semigroups which are disjoint unions of three copies of the free monogenic semigroup (up to isomorphism and anti-isomorphism)

Lemma 5.1.1. *Suppose S is a semigroup generated by the set A, and suppose that for every $x, y \in A$, $i \in \mathbb{N}$ there exist $z \in A$ and $j \in \mathbb{N}$ such that $x^i y = z^j$. Then for every $w \in A^+$ there exist $t \in A$ and $l \in \mathbb{N}$ such that $w = t^l$.*

PROOF. Suppose that the word $w \in A^+$ is of length n and we want to show that w is a power of a generator. We prove this by induction on n. It is obvious that the statement holds if $n = 1$. Assume that the statement holds for every $k \leq n$. Now if $n = k+1$ we have $w = w'x$ where $|w'| = k$ and by the inductive hypothesis $w' = t^l$ and then $t^l x = r^p$ by the assumption. Thus

$$w = w'x = t^l x = r^p,$$

where $r \in A$, $p \in \mathbb{N}$ as required. □

Definition 5.1.2. We say that the type $(x_1, x_2, x_3, x_4, x_5, x_6)$ is (anti)isomorphic to the type $(y_1, y_2, y_3, y_4, y_5, y_6)$ if and only if every semigroup of the type $(x_1, x_2, x_3, x_4, x_5, x_6)$ is (anti)isomorphic to a semigroup of the type $(y_1, y_2, y_3, y_4, y_5, y_6)$ and vice versa.

Lemma 5.1.3. *If S is a semigroup which is a disjoint union of three copies of the free monogenic semigroup $A = \langle a \rangle$, $B = \langle b \rangle$ and $C = \langle c \rangle$ then one of $A \sqcup B$, $A \sqcup C$, $B \sqcup C$ is a subemigroup of S.*

PROOF. Suppose that none of the mentioned sets is a subsemigroup of S. Thus ab or ba is in C and ac or ca is in B and bc or cb is in A. This gives us the following 8 types:

$$(c,*,b,*,a,*),\ (c,*,*,b,a,*),\ (c,*,*,b,*,a),\ (c,*,b,*,*,a),\ (*,c,b,*,a,*),$$

$$(*,c,*,b,a,*),\ (*,c,*,b,*,a),\ (*,c,b,*,*,a).$$

It suffices to show, in each of these 8 cases that some power of a, say, equals a power of b or c and this appears clearly in the following table:

i	$(c,*,b,*,a,*)$	$a(bc) = aa^p = a^{p+1}$	$(ab)c = c^i c = c^{i+1}$
ii	$(c,*,*,b,a,*)$	$c(ab) = cc^i = c^{i+1} =$	$(ca)b = b^l b = b^{l+1}$
iii	$(c,*,*,b,*,a)$	$c(ab) = cc^i = c^{i+1} =$	$(ca)b = b^l b = b^{l+1}$
iv	$(c,*,b,*,*,a)$	$a(cb) = aa^q = a^{q+1}$	$(ac)b = b^k b = b^{k+1}$
v	$(*,c,b,*,a,*)$	$b(ac) = bb^k = b^{k+1}$	$(ba)c = c^j c = c^{j+1}$
vi	$(*,c,*,b,a,*)$	$b(ca) = bb^l = b^{l+1}$	$(bc)a = a^p a = a^{p+1}$
vii	$(*,c,*,b,*,a)$	$c(ba) = cc^j = c^{j+1}$	$(cb)a = a^q a = a^{q+1}$
$viii$	$(*,c,b,*,*,a)$	$b(ac) = bb^k = b^{k+1}$	$(ba)c = c^j c = c^{j+1}$

Table 5.2: General forbidden types

which contradicts with the fact that S is a semigroup. □

Lemma 5.1.4. *There is no semigroup of the type (b,a,b,c,a,c).*

PROOF. Assume that S is a semigroup of the type (b,a,b,c,a,c). Then as we know $i = j = 2$ by Lemma 4.2.1 and we have

$$c(ab) = cb^2 = c^q b = c^{q-1}c^q = c^{2q-1},$$

$$(ca)b = c^l b = c^{l-1}c^q = c^{q+l-1}.$$

Thus, $2q - 1 = q + l - 1$ and then $q = l$. Also we have

$$c(bc) = ca^p = c^l a^{p-1} = c^{l-1}c^l a^{p-2} = c^{lp-p+1},$$

$$(cb)c = c^q c = c^{q+1}.$$

Hence, $q = p(l-1) = l$ and then $p = l = 2$. Furthermore,

$$c(ac) = cb^k = c^q b^{k-1} = c^{q-1} c^q b^{k-2} = c^{qk-k+1},$$

$$(ca)c = c^l c = c^{l+1}.$$

Then we have $l = k(q-1) = q$ and thus $k = q = 2$. Eventually, we have $i = j = k = l = p = q = 2$ but this contradicts with

$$a(cb) = ac^2 = b^2 c = ba^2 = a^2 a = a^3,$$

$$(ac)b = b^2 b = b^3.$$

Therefore, there is no semigroup of the type (b, a, b, c, a, c). □

5.2 Classification of semigroups which are disjoint unions of three copies of the free monogenic semigroup

Theorem 5.2.1. *Up to isomorphism and anti-isomorphism, every semigroup S is a disjoint union of three copies of the free monogenic semigroup $D = \langle d \rangle$, $G = \langle g \rangle$ and $H = \langle h \rangle$ if and only if S is isomorphic to the semigroup which is defined by one of the following presentations:*

(1) $\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = a^p, cb = a^p \rangle$,
where $i + k - p = 2$ *and* $i, k, p \in \mathbb{N}$.

(2) $\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = b^p, cb = b^p \rangle$,
where $i + k + p - ip = 2$ *and* $i, k, p \in \mathbb{N}$.

(3) $\langle a, b, c | ab = a^i, ba = a^i, ac = a^i, ca = a^i, bc = c^2, cb = b^2 \rangle$,
where $i \in \mathbb{N}$.

(4) $\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i \rangle$,
where $i \in \mathbb{N}$.

(5) $\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = c^i \rangle$,
where $i \in \mathbb{N}$.

(6) $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = b^2 \rangle$,

(7) $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \rangle$,

(8) $\langle a,b,c|ab=b^2, ba=a^2, ac=c^2, ca=a^2, bc=c^2, cb=a^2\rangle$,

(9) $\langle a,b,c|ab=b^2, ba=a^2, ac=b^i, ca=a^i, bc=a^i, cb=b^i\rangle$,
where $i\in\mathbb{N}$.

PROOF. We devide the proof into three parts. In part 1 we show that that there are just nine possible types. In part 2 we show that each semigroup S which is a disjoint union of three copies of the free monogenic semigroup is actually defined by one of these presentations. In part 3 we show that each semigroup S' which is defined by one of the presentations in the theorem, is a disjoint union of three copies of the free monogenic semigroup.

Part 1. It follows by Lemma 5.1.3, we may assume without loss of generality $A\sqcup B$ is a subsemigroup of S. Hence we only need to consider tuples of the form $(a,a,*,*,*,*)$ and $(b,a,*,*,*,*)$. We show that the tuples, up to isomorphism and anti-isomorphism, (a,a,a,a,a,a), (a,a,a,a,b,b), (a,a,a,a,c,b), (a,a,c,a,c,a), (a,a,c,a,c,c), (b,a,c,a,c,b), (b,a,c,b,c,a), (b,a,c,a,c,a), (b,a,b,a,a,b) are the only 9 possible tuples.
The following table shows the forbidden types in the family $(a,a,*,*,*,*)$:

i	$(a,a,*,b,a,*)$	$b(ca) = bb^l = b^{l+1}$	$(bc)a = a^p a = a^{p+1}$
ii	$(a,a,b,*,*,*)$	$b(ac) = bb^k = b^{k+1}$	$(ba)c = a^j c = a^{j-1}b^k = a^{j-2}a^i b^{k-1} = a^{j-2+i+(i-1)(k-1)} = a^{ik-k+j-1}$
iii	$(a,a,*,b,*,*)$	$c(ab) = ca^i = b^l a^{i-1} = b^{l-1}a^j a^{i-2} = a^{lj-l+i-1}$	$(ca)b = b^l b = b^{l+1}$
iv	$(a,a,c,*,a,*)$	$a(bc) = aa^p = a^{p+1}$	$(ab)c = a^i c = a^{i-1}c^k = a^{i-2}c^k c^{k-1} = c^{ki-i+1}$
v	$(a,a,*,a,a,c)$	$c(bc) = ca^p = a^l a^{p-1} = a^{l+p-1}$	$(cb)c = c^q c = c^{q+1}$
vi	$(a,a,*,c,b,*)$	$b(ca) = bc^l = b^p c^{l-1} = b^{p-1}b^p c^{l-2} = b^{pl-l+1}$	$(bc)a = b^p a = b^{p-1}a^j = b^{p-2}a^j a^{j-1} = a^{jp-p+1}$
vii	$(a,a,a,*,c,a)$	$c(bc) = cc^p = c^{p+1}$	$(cb)c = a^q c = a^{q-1}a^k = a^{q+k-1}$
viii	$(a,a,c,*,*,b)$	$a(cb) = ab^q = a^i b^{q-1} = a^{i-1}a^i b^{q-2} = a^{iq-q+1}$	$(ac)b = c^k b = c^{k-1}b^q = c^{k-2}b^q b^{q-1} = b^{kq-k+1}$
ix	$(a,a,*,*,a,b)$	$b(cb) = bb^q = b^{q+1}$	$(bc)b = a^p b = a^{p-1}a^i = a^{i+p-1}$
x	$(a,a,*,*,b,a)$	$b(cb) = ba^q = a^j a^{q-1} = a^{j+q-1}$	$(bc)b = b^p b = b^{p+1}$
xi	$(a,a,*,c,*,b)$	$c(ab) = ca^i = c^l a^{i-1} = c^{l-1}c^l a^{i-2} = c^{li-i+1}$	$(ca)b = c^l b = c^{l-1}b^q = b^{ql-l+1}$
*xii**	$(a,a,*,c,c,a)$	$b(cb) = ba^q = a^j a^{q-1} = a^{j+q-1}$	$(bc)b = c^p b = c^{p-1}a^q = c^{p-2}c^l a^{q-1} = c^{p-2+l+(l-1)(q-1)} = c^{p+lq-q-1}$
xiii	(a,a,a,c,a,a)	$c(bc) = ca^p = c^l a^{p-1} = c^{l-1}c^l a^{p-2} = c^{lp-p+1}$	$(cb)c = a^q c = a^{q-1}a^k = a^{q+k-1}$
xiv	(a,a,b,c,a,a)	$c(bc) = ca^p = c^l a^{p-1} = c^{l-1}c^l a^{p-1} = c^{lp-p+1}$	$(cb)c = a^q c = a^{q-1}b^k = a^{q-2}a^i b^{k-1} = a^{ki-k+q-1}$
xv	$(a,a,c,*,b,*)$	$b(ac) = bc^k = b^p c^{k-1} = b^{p-1}b^p c^{k-2} = b^{pk-k+1}$	$(ba)c = a^j c = a^{j-1}c^k = a^{j-2}c^k c^{k-1} = c^{kj-j+1}$

Table 5.3: Forbidden types in the family $(a,a,*,*,*,*)$

Notice that in (xii^*) if $p = 1$ then $q = j + q - 1$ and thus $j = 1$. So we have $ba = a$, $ca = c^l$, $bc = c$, $cb = a^q$. But $c(ba) = ca = c^l$ and $(cb)a = a^q a = a^{q+1}$, a contradiction.

Now suppose that S is a semigroup of type $(a, a, t_3, t_4, t_5, t_6)$. Clearly, $t_4 \neq b$ by Table 5.3(*iii*) and then $t_4 = a$ or c, if $t_4 = c$ then we have the type (a, a, t_3, c, t_5, t_6) and thus $t_6 \neq b$ by Table 5.3(xi). So t_6 could be equal to a and hence we have the type (a, a, t_3, c, t_5, a) in which $t_5 \neq b$ nor c by Table 5.3(x, xii) respectively and also if $t_5 = a$ then $t_3 \neq a$, b nor c by Table 5.3($xiii, ii, iv$) respectively and that implies $t_6 = c$. Therefore, we reached the type (a, a, t_3, c, t_5, c) where $t_3 \neq b$ by Table 5.3(ii) Thus, if $t_3 = c$ then $t_5 = c$ by Table 5.3(iv, vi) and if $t_3 = a$ then we have the type (a, a, a, c, t_5, c). In this type $t_5 = a$ or c by Table 5.3(vi). Now, if $t_4 = a$ then $t_3 = a$ or c by Table 5.3(ii). So if $t_3 = a$ then we have the type (a, a, a, a, t_5, t_6). Here we need to add another relation. So if $t_5 = a$ then $t_6 = a$ by Table 5.3(ix, v) and if $t_5 = b$ then $t_6 = b$ or c by Table 5.3(x) also if $t_5 = c$ then $t_6 \in \{c, b\}$ by Table 5.3(vii). Eventually, if $t_3 = c$ then we have the type (a, a, c, a, t_5, t_6). So $t_5 \neq a$ nor b by Table 5.3(iv, xv). So $t_5 = c$ and then $t_6 = c$ or a by Table 5.3($viii$). Thus we have ten types (a, a, a, a, a, a), (a, a, a, a, b, b), (a, a, a, a, c, c), (a, a, a, a, b, c), (a, a, a, a, c, b), (a, a, c, c, c, c), (a, a, c, a, c, a), (a, a, c, a, c, c), (a, a, a, c, a, c), (a, a, a, c, c, c). But there are just five types up to isomorphism and anti-isomorphism since we can just replace a, say, by b or c. And they are (a, a, a, a, a, a), (a, a, a, a, b, b), (a, a, a, a, c, b), (a, a, c, a, c, a), (a, a, c, a, c, c).

Now we show the forbidden types in the family $(b, a, *, *, *, *)$ in the following table:

i	$(b,a,a,*,*,c)$	$a(cb) = ac^q = a^k c^{q-1} =$ $a^{k+(k-1)(q-1)} = a^{kq-q+1}$	$(ac)b = a^k b =$ $a^{k-1}b^i = b^{ik-k+1}$
ii	$(b,a,a,*,*,a)$	$a(cb) = aa^q = a^{q+1}$	$(ac)b = a^k b =$ $a^{k-1}b^i = b^{ik-k+1}$
iii	$(b,a,b,*,*,a)$	$a(cb) = aa^q = a^{q+1}$	$(ac)b = b^k b = b^{k+1}$
iv	$(b,a,*,c,b,*)$	$b(ca) = bc^l = b^p c^{l-1} =$ $b^{p+(p-1)(l-1)} = b^{pl-l+1}$	$(bc)a = b^p a = b^{p-1}a^j =$ $b^{p-2}a^j a^{j-1} = a^{jp-p+1}$
v^*	$(b,a,*,c,*,a)$	$c(ab) = cb^i = a^q b^{i-1} =$ $a^{q-1}b^i b^{i-2} = b^{iq-q+i-1}$	$(ca)b = c^l b = c^{l-1}a^q =$ $c^{l-2}c^l a^{q-1} = c^{lq-q+l-1}$
vi	$(b,a,*,c,*,b)$	$c(ba) = ca^j =$ $c^l a^{j-1} = c^{lj-j+1}$	$(cb)a = b^q a = b^{q-1}a^j =$ $a^{j+(j-1)(q-1)} = a^{jq-q+1}$
vii	$(b,a,c,*,b,*)$	$b(ac) = bc^k = b^p c^{k-1} =$ $b^{p-1}b^p c^{k-2} = b^{pk-k+1}$	$(ba)c = a^j c = a^{j-1}c^k =$ $a^{j-2}c^k c^{k-1} = c^{kj-j+1}$
$viii$	$(b,a,*,a,*,c)$	$c(ab) = cb^i = c^q b^{i-1} =$ $c^{q-1}c^q b^{i-2} = c^{qi-i+1}$	$(ca)b = a^l b = a^{l-1}b^i =$ $a^{l-2}b^i b^{i-1} = b^{il-l+1}$
ix^*	$(b,a,*,b,*,c)$	$c(ba) = ca^j = b^l a^{j-1} =$ $b^{l-1}a^j a^{j-2} = a^{jl-l+j-1}$	$(cb)a = c^q a = c^{q-1}b^l =$ $c^{q-2}c^q b^{l-1} = c^{ql-l+q-1}$
x	$(b,a,b,b,*,*)$	$a(ca) = ab^l =$ $b^i b^{l-1} = b^{i+l-1}$	$(ac)a = b^k a = b^{k-1}a^j =$ $b^{k-2}a^j a^{j-1} = a^{jk-k+1}$
xi	$(b,a,b,c,c,*)$	$a(ca) = ac^l = b^k c^{l-1} =$ $b^{k-1}c^p c^{l-2} = c^{pk-k+l-1}$	$(ac)a = b^k a = b^{k-1}a^j =$ $b^{k-2}a^j a^{j-1} = a^{jk-k+1}$
xii	$(b,a,*,*,a,a)$	$b(cb) = ba^q =$ $a^j a^{q-1} = a^{q+j-1}$	$(bc)b = a^p b = a^{p-1}b^i =$ $a^{p-2}b^i b^{i-1} = b^{ip-p+1}$
$xiii$	(b,a,a,b,t_5,t_6)	$a(ca) = ab^l = b^i b^{l-1}$ $= b^{i+l-1}$	$(ac)a = a^k a = a^{k+1}$
xiv	(b,a,c,c,a,t_6)	$b(ca) = bc^l = a^p c^{l-1} =$ $a^{p-1}c^k c^{l-2} = c^{kp+l-p-1}$	$(bc)a = a^p a = a^{p+1}$
xv	(b,t_2,a,t_4,c,t_6)	$a(bc) = ac^p = a^k c^{p-1} =$ $a^{k-1}a^k c^{p-2} = a^{kp-p+1}$	$(ab)c = b^i c = b^{i-1}c^p =$ $b^{i-2}c^p c^{p-1} = c^{pi-i+1}$
xvi	$(b,a,*,b,a,*)$	$b(ca) = bb^l = b^{l+1}$	$(bc)a = a^p a = a^{p+1}$
$xvii$	(b,a,b,c,a,c)	see Lemma 5.1.4	

Table 5.4: Forbidden types in the family $(b,a,*,*,*,*)$

Notice that in (v^*) if $i = l = 1$ then we have so far $ab = b,\ ca = c,\ cb = a^q$ and then

$$ba = a^j \Rightarrow a(ba) \neq (ab)a;$$

$$ba = b^j \Rightarrow c(ba) \neq (cb)a;$$

$$ba = c^j \Rightarrow b(ab) \neq (ba)b.$$

Similarly, in (ix^*) if $j = q = 1$ then we have so far $ba = a,\ ca = b^l\ cb = c$ and then

$$ab = a^i \Rightarrow i = 1 \text{ by the associativity on } aba \Rightarrow c(ab) \neq (ca)b;$$

$$ab = b^i \Rightarrow b(ab) \neq b(ab);$$

$$ab = c^i \Rightarrow a(ba) \neq (ab)a.$$

Suppose that S is a semigroup of the type $(b, a, t_3, t_4, t_5, t_6)$. Firstly, let $t_3 = a$. Then $t_4 \neq b$ by Table 5.4($xiii$). Also if $t_4 = c$ then we cannot decide whether the operation is associative or not. So we assume that $t_6 = a$, b or c but all of these values are rejected by Table 5.4(v, vi, i) respectively and thus $t_4 \neq c$. So surely $t_4 = a$. Now we have the type (b, a, a, a, t_5, t_6) but $t_6 \neq a$ nor c by Table 5.4(ii, i) respectively and hence $t_6 = b$. Then $t_5 = a$ or b by Table 5.4(xv).

Secondly, let $t_3 = b$ and then we have the type (b, a, b, t_4, t_5, t_6). Here if $t_4 = c$ then $t_5 \neq c$ by Table 5.4(xi). Thus if $t_5 = a$ then $t_6 \notin \{a, b, c\}$ by Table 5.4($iii, vi, xvii$) respectively, that implies that $t_5 = b$. Now we have the type (b, a, b, c, b, t_6) which is rejected by Table 5.4(iv) and thus t_4 must be equal to a or b. Let $t_4 = b$. Then we have the type (b, a, b, b, t_5, t_6) which is rejected by Table 5.4(x) and hence $t_4 = a$ and then we have the type (b, a, b, a, t_5, t_6). However $t_6 \notin \{a, c\}$ by Table 5.4($iii, viii$) respectively. So we reached the type (b, a, b, a, t_5, b) where $t_5 \in \{a, b, c\}$.

Thirdly, if $t_3 = c$ and if $t_4 = c$ then $t_5 \neq a$ nor b by Table 5.4(xiv, iv) respectively. Thus $t_5 = c$. So we have the type (b, a, c, c, c, t_6) and by Table 5.4(v, vi), $t_6 \neq a$ nor b. So $t_6 = c$. Now if we have $t_4 = a$ then $t_5 \neq b$ by Table 5.4(vii). Thus $t_5 = a$ or c. If $t_5 = a$ then we have the type (b, a, c, a, a, t_6), here $t_6 \neq a$ nor c by Table 5.4($xii, viii$) and hence $t_6 = b$. Now if $t_5 = c$ then we have the type (b, a, c, a, c, t_6) where $t_6 \neq c$ by Table 5.4($viii$). Thus $t_6 \in \{a, b\}$. If $t_4 = b$ then we have the type (b, a, c, b, t_5, t_6), here $t_5 \neq b$ nor a by Table 5.4(vii, xvi) respectively that implies $t_5 = c$ and then we have the type (b, a, c, b, c, t_6) where $t_6 \neq c$ by Table 5.4(ix) and then $t_6 \in \{a, b\}$.

Consequently, we have eleven types (b,a,a,a,a,b), (b,a,a,a,b,b), (b,a,b,a,a,b), (b,a,b,a,b,b), (b,a,b,a,c,b), (b,a,c,a,a,b), (b,a,c,a,c,b), (b,a,c,b,c,b), (b,a,c,c,c,c), (b,a,c,a,c,a), (b,a,c,b,c,a). But there are just four types up to isomorphism and anti-isomorphism and they are (b,a,c,a,c,b), (b,a,c,b,c,a), (b,a,c,a,c,a), (b,a,b,a,a,b).

So in total we just have nine types (a,a,a,a,a,a), (a,a,a,a,b,b), (a,a,a,a,c,b), (a,a,c,a,c,a), (a,a,c,a,c,c) (b,a,c,a,c,b), (b,a,c,b,c,a), (b,a,c,a,c,a), (b,a,b,a,a,b) in corresponding to the two families $(a,a,*,*,*,*)$ and $(b,a,*,*,*,*)$.

Part 2. Suppose that S is a semigroup which is a disjoint union of three copies of the free monogenic semigroup $\langle d\rangle$, $\langle g\rangle$ and $\langle h\rangle$. Thus S is a type of the nine types which are (d,d,d,d,d,d), (d,d,d,d,g,g), (d,d,d,d,h,g), (d,d,h,d,h,d), (d,d,h,d,h,h), (g,d,h,d,h,g), (g,d,h,g,h,d), (g,d,h,d,h,d), (g,d,g,d,d,g). Therefore, if S of the type (d,d,d,d,d,d) then S has the relations

$$dg = d^i, gd = d^j, dh = d^k, hd = d^l, gh = d^p, hg = d^q$$

and then $i = j$ and $k = l$ by Lemma 4.2.1. Since S is a semigroup, the operation must be associative. Thus we have

$$g(hg) = gd^q = d^j d^{q-1} = d^{j+q-1},$$

$$(gh)g = d^p g = d^{p-1} d^i = d^{i+p-1}.$$

And this clearly implies that $p = q$. Furthermore,

$$d(gh) = dd^p = d^{p+1},$$

$$(dg)h = d^i h = d^{i-1} d^k = d^{i+k-1}.$$

So $i + k - 1 = p + 1$ and hence $i + k - p = 2$.

If S is of the type (d,d,d,d,g,g) then S has the relations

$$dg = d^i, gd = d^j, dh = d^k, hd = d^l, gh = g^p, hg = g^q,$$

and then $i = j$, $k = l$, $p = q$ by Lemma 4.2.1. Since S is a semigroup, the operation must be associative.

$$d(gh) = dg^p = d^i g^{p-1} = d^{i-1} d^i g^{p-2} = d^{ip-p+1},$$

$$(dg)h = d^i h = d^{i-1} d^k = d^{k+i-1}.$$

Then $k + i - 1 = ip - p + 1$ which implies $k + p - ip + i = 2$. Thus $i = j, k = l, p = q$ with $k + p - ip + i = 2$.

If S is of the type (d, d, d, d, h, g) then S has the relations

$$dg = d^i, gd = d^j, dh = d^k, hd = d^l, gh = h^p, hg = g^q$$

and then $i = j$, $k = l$ and $p = q = 2$ by Lemma 4.2.1. Since S is a semigroup,

$$h(gd) = hd^j = d^l d^{j-1} = d^{l+j-1},$$

$$(hg)d = g^2 d = gd^j = d^j d^{j-1} = d^{2j-1}.$$

Then $l + j - 1 = 2j - 1$. Thus $l = j$ and then $i = j = k = l$ and $p = q = 2$.

If S is of the type (d, d, h, d, h, d) then S has the relations

$$dg = d^i, gd = d^j, dh = h^k, hd = d^l, gh = h^p, hg = d^q,$$

then $i = j$ and $k = l = 2$ by Lemma 4.2.1. Since S is a semigroup, the operation must be associative. We have

$$g(hg) = gd^q = d^i d^{q-1} = d^{i+q-1},$$

$$(gh)g = h^p g = h^{p-1} d^q = d^{p+q-1}.$$

Thus, $i + q - 1 = p + q - 1$ which implies $i = p$. Also,

$$h(gh) = hh^p = h^{p+1},$$

$$(hg)h = d^q h = h^{q+1}.$$

Thus, $p = q$. Thus $i = j = p = q$ and $k = l = 2$.

If S is of the type (d, d, h, d, h, h) then S has the relations

$$dg = d^i, gd = d^j, dh = h^k, hd = d^l, gh = h^p, hg = h^q$$

then $i = j$, $k = l = 2$ and $p = q$ by Lemma 4.2.1. Since S is a semigroup then the operation must be associative. We have

$$g(hd) = gd^2 = d^i d = d^{i+1},$$

$$(gh)d = h^p d = h^{p-1}d^2 = h^{p+1}.$$

Thus, $i + 1 = p + 1$ and then $i = p$. Hence, $i = j = p = q$ and $k = l = 2$.

If S is of the type (g, d, h, d, h, g) then S has the relations

$$dg = g^i, gd = d^j, dh = h^k, hd = d^l, gh = h^p, hg = g^q$$

then $i = j = k = l = p = q = 2$ by Lemma 4.2.1.

If S is of the type (g, d, h, g, h, d) then S has the relations

$$dg = g^i, gd = d^j, dh = h^k, hd = g^l, gh = h^p, hg = d^q$$

then $i = j = 2$ by Lemma 4.2.1. Since S is a semigroup, the operation must be associative. We have

$$h(dh) = hh^k = h^{k+1},$$

$$(hd)h = g^l h = g^{l-1}h^p = h^{pl-l+1}$$

and then $k + 1 = pl - l + 1$ and thus $k = pl - l$. Also

$$h(gh) = hh^p = h^{p+1},$$

$$(hg)h = d^q h = d^{q-1}h^k = h^{kq-q+1}.$$

Then $p + 1 = kp - q + 1$. Hence, $p = kp - q$. And

$$d(gh) = dh^p = h^k h^{p-1} = h^{k+p-1},$$

$$(dg)h = g^2 h = gh^p = h^p h^{p-1} = h^{2p-1}.$$

Therefore, $2p - 1 = k + p - 1$ and then we get $k = p$. Now by using the same process on $h(dg)$, $h(gd)$ and $g(dh)$ we obtain that $l = q$, $k = p$. Hence

$$p = l(p - 1) \Longrightarrow p = l = 2$$

then we have $i = j = k = l = p = q = 2$.

If S is of the type (g, d, h, d, h, h, d) then S has the relations

$$dg = g^i, gd = d^j, dh = h^k, hd = d^l, gh = h^p, hg = d^q$$

then $i = j = k = l = 2$ by Lemma 4.2.1. Since S is a semigroup, the operation must be associative. We have

$$g(hg) = gd^q = d^{q+1},$$

$$(gh)g = h^p g = h^{p-1} d^q = d^{p+q-1}.$$

So it is easy to see $q + 1 = p + q - 1$ and then $p = 2$. And also

$$h(gh) = hh^2 = h^3,$$

$$(hg)h = d^q h = h^{q+1}$$

which implies $q + 1 = 3$ and then $q = 2$. Thus $i = j = k = l = p = q = 2$.

If S is of the type (g, d, g, d, d, g) then S has the relations

$$dg = g^i, gd = d^j, dh = g^k, hd = d^l, gh = d^p, hg = g^q$$

then $i = j = 2$ by Lemma 4.2.1. Since S is a semigroup, the operation must be associative. We have

$$h(dh) = hg^k = g^q g^{k-1} = g^{q+k-1},$$

$$(hd)h = d^l h = d^{l-1} g^k = g^{k+l-1}.$$

And then $q + k - 1 = k + l - 1$ which implies $q = l$. Also

$$d(hd) = dd^l = d^{l+1},$$

$$(dh)d = g^k d = d^{k+1}.$$

Then $k = l$. In addition

$$g(hg) = gg^q = g^{q+1},$$

$$(gh)g = d^p g = g^{p+1}.$$

Hence $p = q$. Thus $k = l = p = q$ and $i = j = 2$.

Therefore, S satisfies the relations in one of the presentations in the theorem and by Lemma 4.1.2, S is isomorphic to S' where S' is a semigroup which is defined by one of the presentations in the theorem.

Part 3. We start with the semigroup S' which is defined by the presentation (1)

$$\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = a^p, cb = a^p \rangle.$$

Each element of this semigroup is either a power of a, b, c or a product of them, but by the relations mentioned above, the product of any two different kind of elements gives a^i. So these elements are definitely of the form a^i. Thus

$$\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = a^p, cb = a^p \rangle =$$

$$\{a^i : i \in \mathbb{N}\} \cup \{b^i : i \in \mathbb{N}\} \cup \{c^i : i \in \mathbb{N}\} =$$

$$A \cup B \cup C.$$

Now as a consequence of not having a relation which can be applied to a power of b or c, $A \cap B$=$A \cap C = B \cap C = \emptyset$ and thus A, B and C are pairwise disjoint and B, C are infinite but in A we have to prove that, since there is a possibility of repetitions. Let $\varphi : \{a, b, c\} \to \mathbb{N}_0$ be a mapping with $\varphi(a) = 1$, $\varphi(b) = i - 1$, $\varphi(c) = k - 1$ where i, $k \in \mathbb{N}$ and then there exists a homomorphism $\psi : \{a, b, c\}^+ \to \mathbb{N}_0$ such that $\psi \restriction_{\{a,b,c\}} = \varphi$ by Proposition 1.6.1. Thus

$$\psi(ab) = \psi(a) + \psi(b) = 1 + i - 1 = i = \psi(a^i),$$

$$\psi(ac) = \psi(a) + \psi(c) = 1 + k - 1 = k = \psi(a^k),$$

$$\psi(bc) = \psi(b) + \psi(c) = i - 1 + k - 1 = i + k - 2 = p = \psi(a^p).$$

And similarly $\psi(ba)$, $\psi(ca)$, $\psi(cb)$. That implies

$$\Re = \{ab = ba = a^i, ac = ca = a^k, bc = cb = a^p\} \subseteq \ker \psi.$$

Hence, by Lemma 4.1.1 all elements of this semigroup are distinct and and then S' with presentation (1) is a disjoint union of three copies of the free monogenic semigroup.

The semigroup S' which is defined by the presentation (2)

$$\langle a, b, c | ab = a^i, ba = a^i, ac = a^k, ca = a^k, bc = b^p, cb = b^p \rangle$$

is a disjoint union of three copies of the free monogenic semigroup $A = \langle a \rangle,\ B = \langle b \rangle, C = \langle c \rangle$ because each of $A \cup B,\ B \cup C,\ A \cup C$ is a subsemigroup of S' which is defined by one of the presentations in the Theorem 4.2.3. Thus $S' = A \sqcup B \sqcup C$.

Similarly the semigroup S' which is defined by the presentation (3)

$$\langle a, b, c | ab = a^i, ba = a^i, ac = a^i, ca = a^i, bc = c^2, cb = b^2 \rangle$$

is a disjoint union of three copies of the free monogenic semigroup $A = \langle a \rangle,\ B = \langle b \rangle, C = \langle c \rangle$ because each of $A \cup B,\ B \cup C, A \cup C$ is a subsemigroup of S' which is defined by one of the presentations in the Theorem 4.2.3. Thus $S' = A \sqcup B \sqcup C$.

The semigroup S' which is defined by the presentation (4)

$$\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i \rangle,$$

is a disjoint union of three copies of the free monogenic semigroup because of the following:
Firstly we show that $x^j y = z^k$ where x, y, z are in $\{a, b, c\}$ in the following table:

x	y	$x^j y$
a	a	a^{j+1}
b	a	$b^{j-1}a^i = b^{j-2}a^i a^{i-1} = \cdots = a^{ij-j+1}$
c	a	$c^{j-1}a^2 = \cdots = a^{j+1}$
a	b	$a^{j-1}a^i = a^{i+j-1}$
b	b	b^{j+1}
c	b	$c^{j-1}a^i = \cdots = a^{i+j-1}$
a	c	$a^{j-1}c^2 = \cdots = c^{j+1}$
b	c	$b^{j-1}c^i = b^{j-2}c^i c^{i-1} = \cdots = c^{ij-j+1}$
c	c	c^{j+1}

Table 5.5: The multiplication of x^j and y where $x,\ y \in \{a, b, c\}$ in the presentation $\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i \rangle$

So by the above table and Lemma 5.1.1, each element of our presentation is of the form a^i, b^i or c^i. Thus

$$\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i \rangle =$$

$$\{a^i : i \in \mathbb{N}\} \cup \{b^i : i \in \mathbb{N}\} \cup \{c^i : i \in \mathbb{N}\} =$$

$$A \cup B \cup C.$$

Now we want to prove that $A,\ B$ and C are pairwise disjoint. Let T be the semigroup with multiplication table:

	e	f	g
e	e	e	g
f	e	f	g
g	e	e	g

We define a mapping $\varphi : \{a, b, c\} \to T$ with $\varphi(a) = e$, $\varphi(b) = f$, $\varphi(c) = g$. Thus, there exists a homomorphism $\psi : \{a, b, c\}^+ \to T$. It is obvious that ψ preserves our relation as follows:

$$\psi(ab) = \psi(a)\psi(b) = ef = e = \psi(a^i),$$

$$\psi(ba) = \psi(b)\psi(a) = fe = e = \psi(a^i),$$

$$\psi(ac) = \psi(a)\psi(c) = eg = g = \psi(c^2),$$

$$\psi(ca) = \psi(c)\psi(a) = ge = e = \psi(a^2),$$
$$\psi(bc) = \psi(b)\psi(c) = fg = g = \psi(c^i),$$
$$\psi(cb) = \psi(c)\psi(b) = gf = e = \psi(a^i).$$

Thus, ψ is a homomorphism from S into T and therefore $A \cap B = A \cap C = B \cap C = \emptyset$. So A, B and C are a pairwise disjoint. The next step is to prove that A, B and C are infinite. Define a mapping $\varphi : \{a, b, c\} \to \mathbb{N}_0$ with $\varphi(a) = 1$, $\varphi(b) = i - 1$ and $\varphi(c) = 1$. Then there exists a homomorphism $\psi : \{a, b, c\}^+ \to \mathbb{N}_0$ in which $\{a, b, c\}^+$ is the free semigroup on $\{a, b, c\}$ and ψ uniquely determined by the images of the generators a, b and c by Proposition 1.6.1. It is obvious that ψ preserves the all above relations and based on this we deduce that

$$\Re = \{ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = a^i\} \subseteq \ker \psi.$$

Hence, by Lemma 4.1.1 all elements of semigroup $A \sqcup B \sqcup C$ are distinct as required.

The semigroup S' which is defined by the presentation (5)

$$\langle a, b, c | ab = a^i, ba = a^i, ac = c^2, ca = a^2, bc = c^i, cb = c^i \rangle,$$

is a disjoint union of three copies of the free monogenic semigroup $A = \langle a \rangle$, $B = \langle b \rangle, C = \langle c \rangle$ because each of $A \cup B$, $B \cup C, A \cup C$ is a subsemigroup of S' which is defined by one of the presentations in the Theorem 4.2.3. Thus $S' = A \sqcup B \sqcup C$.

Analogously, the semigroup S' which is defined by the presentation (6)

$$\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = b^2 \rangle,$$

is a disjoint union of three copies of the free monogenic semigroup $A = \langle a \rangle$, $B = \langle b \rangle, C = \langle c \rangle$ because each of $A \cup B$, $B \cup C, A \cup C$ is a subsemigroup of S' which is defined by one of the presentations in the Theorem 4.2.3. Thus $S' = A \sqcup B \sqcup C$.

The semigroup S' which is defined by the presentation (7)

$$\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \rangle,$$

is a disjoint union of three copies of the free monogenic semigroup because of the following:
We show that $x^i y = z^j$ where x, y, z are in $\{a, b, c\}$ in the following table:

x	y	$x^i y$
a	a	a^{i+1}
b	a	$b^{i-1}c^2 = b^{i-2}a^2c = \cdots = c^{i+1}$
c	a	$c^{i-1}b^2 = c^{i-2}a^2b = c^{i-2}b^3 = \cdots = b^{i+1}$
a	b	$a^{i-1}b^2 = \cdots = b^{i+1}$
b	b	b^{i+1}
c	b	$c^{i-1}a^2 = c^{i-2}b^2a = c^{i-2}a^3 = \cdots = a^{i+1}$
a	c	$a^{i-1}c^2 = \cdots = c^{i+1}$
b	c	$b^{i-1}c^2 = c^{i+1}$
c	c	c^{i+1}

Table 5.6: The multiplication of x^i and y where $x,\ y \in \{a, b, c\}$ in the presentation $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \rangle$

So by the above table and Lemma 5.1.1, each element of our presentation is of the form a^i, b^i or c^i. Thus

$$\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \rangle =$$

$$\{a^i : i \in \mathbb{N}\} \cup \{b^i : i \in \mathbb{N}\} \cup \{c^i : i \in \mathbb{N}\} =$$

$$A \cup B \cup C.$$

Now we show that $A,\ B$ and C are pairwise disjoint. Let $w \in A$. Then we have

$$w = a^i = a^{i-2}a^2 = a^{i-2}cb = a^{i-3}c^2b,$$

or

$$w = a^i = a^{i-2}a^2 = a^{i-2}ba = a^{i-3}b^2a.$$

So we obtain a word that ends with aa, cb or ba. So we want to prove that if w is a word ending in aa, cb or ba and $w = v$ in S then v ends in aa, cb or ba.
First we start with w ends in aa. Let $w = w'aa$. Then if we apply a single relation to w', this does not change our claim. If we apply a relation to aa we get cb or ba which is the same of our claim. Now if we apply a relation in the interface of w' and aa,

we have $w'aa = w''aaa$ or $w'aa = w''baa = w''aaa$ or $w'aa = w''caa = w''b^2a$.
Now if w ends with cb. Let $w = w'cb$. Then if we apply a single relation to w', this does not change our claim. If we apply a relation to cb we get aa which is the same of our claim. Now if we apply a relation in the interface of w' and cb, we have $w'cb = w''acb = w''ccb$ or $w'cb = w''bcb = w''ccb$ or $w'cb = w''ccb$.
If w ends with ba. Let $w = w'ba$. Then if we apply a single relation to w' that does not change our claim. If we apply a relation to ba we get aa which is the same of our claim. Now if we apply a relation in the interface of w' and ba, we have $w'ba = w''aba = w''bba$ or $w'ba = w''bba$ or $w'bc = w''cba = w''bba$. Thus each word in A ends with aa, cb or ba. Similarly by the same argument, if $w \in B$ then

$$w = b^i = b^{i-2}b^2 = b^{i-2}ab = b^{i-3}a^2b,$$

or

$$w = b^i = b^{i-2}b^2 = b^{i-2}ca = b^{i-3}c^2a.$$

So we can notice that each word in B ends with $b^2,\ ca,\ ab$ and based on this $A \cap B = \emptyset$. Moreover, if $w \in C$ then

$$w = c^i = c^{i-2}c^2 = c^{i-2}ac = c^{i-3}b^2c,$$

or

$$w = c^i = c^{i-2}c^2 = c^{i-2}bc = c^{i-3}a^2c,$$

thus each word in C ends with c and then $B \cap C = \emptyset$ because there is no word in B which ends with c. Also $A \cap C = \emptyset$ because the end of each word in A differs from the end of each word in C. The last step we show that A, B and C are infinite. By considering the mapping φ from $\{a, b, c\}$ to $\mathbb{N}_0$ with $\varphi(a) = 1$, $\varphi(b) = 1$, $\varphi(c) = 1$, there exists a homomorphism $\psi : \{a, b, c\}^+ \to \mathbb{N}_0$ such that $\psi \upharpoonright_{\{a,b,c\}} = \varphi$ by Proposition 1.6.1. It is obvious that ψ preserves the all above relations and based on this we deduce that

$$\Re = \{ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2\} \subseteq \ker \psi.$$

The semigroup S' which is defined by the presentation (8)

$$\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2 \rangle,$$

is a disjoint union of three copies of the free monogenic semigroup because of the following:
We show that $x^i y = z^j$ where x, y, z are in $\{a, b, c\}$ in the following table:

x	y	$x^i y$
a	a	a^{i+1}
b	a	$b^{i-1}a^2 = \cdots = a^{i+1}$
c	a	$c^{i-1}a^2 = \cdots = a^{i+1}$
a	b	$a^{i-1}b^2 = \cdots = b^{i+1}$
b	b	b^{i+1}
c	b	$c^{i-1}a^2 = \cdots = a^{i+1}$
a	c	$a^{i-1}c^2 = \cdots = c^{i+1}$
b	c	$b^{i-1}c^2 = \cdots = c^{i+1}$
c	c	c^{i+1}

Table 5.7: The multiplication of x^i and y where $x,\ y \in \{a, b, c\}$ in the presentation $\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2 \rangle$

So by the above table and Lemma 5.1.1, each element of our presentation is of the form a^i, b^i or c^i. Thus

$$\langle a, b, c | ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2 \rangle =$$

$$\{a^i : i \in \mathbb{N}\} \cup \{b^i : i \in \mathbb{N}\} \cup \{c^i : i \in \mathbb{N}\} =$$

$$A \cup B \cup C.$$

Now we show that $A,\ B \ and \ C$ are pairwise disjoint. Let $w \in A$ and hence

$$w = a^i = a^{i-2}a^2 = a^{i-2}cb = a^{i-3}c^2b,$$

or

$$w = a^i = a^{i-2}a^2 = a^{i-2}ca = a^{i-3}c^2a,$$

or

$$w = a^i = a^{i-2}a^2 = a^{i-2}ba = a^{i-3}b^2a,$$

so we want to prove that if w is a word ending in $aa,\ cb,\ ca$ or ba and $w = v$ in S then v ends in $aa,\ cb,\ ca$ or ba. First we start with w ends in aa. Let $w = w'aa$. Then if we apply a single relation to w' that does not change our claim. If we apply a relation to aa we get cb, ca or ba which is the same of our claim. Now if we apply a relation in the interface of w' and aa, we have $w'aa = w''aaa$ or $w'aa = w''baa = w''a^2a$ or $w'aa = w''caa = w''a^2a$. If w ends with cb. Let $w = w'cb$ then if we apply a single relation to w' that does not change our claim. If we apply a relation to cb we get aa which is the same of our claim. Now if we apply a relation in the interface of w' and cb, we have $w'cb = w''acb = w''aab$ or $w'cb = w''bcb = w''ccb$ or $w'cb = w''ccb$. Similarly, if w ends with ca or ba. Thus each word in A ends with aa, cb, ca or ba. Analogously, if $w \in B$ then

$$w = b^i = b^{i-2}b^2 = b^{i-2}ab = b^{i-3}a^2b.$$

Hence each word in B ends with bb or ab by the same previous arguments. Thus $A \cap B = \emptyset$. Besides if $w \in C$ then

$$w = c^i = c^{i-2}c^2 = c^{i-1}bc = c^{i-2}a^2c$$

or

$$w = c^i = c^{i-2}c^2 = c^{i-2}ac = c^{i-3}a^2c.$$

Therefore each word in C ends with cc, ac or bc by the same previous argument. So no word ends with b and hence $B \cap C = \emptyset$. Now it is clear that $A \cap C = \emptyset$ and therefore A, B and C are a pairwise disjoint. The last step we show that A, B and C are infinite. Let $\varphi : \{a, b, c\} \to \mathbb{N}_0$ be a mapping with $\varphi(a) = 1$, $\varphi(b) = 1$ and $\varphi(c) = 1$. Then there exists a homomorphism $\psi : \{a, b, c\}^+ \to \mathbb{N}_0$ by Proposition 1.6.1, in which $\{a, b, c\}^+$ is the free semigroup on $\{a, b, c\}$ and ψ uniquely determined by the images of the generators a, b and c. It is obvious that ψ preserves all the above relations and based on this we deduce that

$$\Re = \{ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2\} \subseteq \ker \psi.$$

Hence, $\rho \subseteq \ker \psi$. Therefore, by Lemma 4.1.1 all elements of the semigroup $A \sqcup B \sqcup C$ are distinct as required.

The semigroup S' which is defined by the presentation (9)

$$\langle a, b, c | ab = b^2, ba = a^2, ac = b^i, ca = a^i, bc = a^i, cb = b^i \rangle,$$

is a disjoint union of three copies of the free monogenic semigroup because of the

following:
We show that $x^j y = z^k$ where x, y, z are in $\{a, b, c\}$ in the following table:

x	y	$x^j y$
a	a	a^{j+1}
b	a	$b^{j-1}a^2 = \cdots = a^{j+1}$
c	a	$c^{j-1}a^i = a^{j-2}a^i a^{i-1} = \cdots = a^{ij-j+1}$
a	b	$a^{j-1}b^2 = \cdots = b^{j+1}$
b	b	b^{j+1}
c	b	$c^{j-1}b^i = \cdots = b^{ij-j+1}$
a	c	$a^{j-1}b^i = \cdots = b^{i+j-1}$
b	c	$b^{j-1}a^i = \cdots = a^{i+j-1}$
c	c	c^{j+1}

Table 5.8: The multiplication of x^j and y where $x,\ y \in \{a, b, c\}$ in the presentation $\langle a, b, c | ab = b^2, ba = a^2, ac = b^i, ca = a^i, bc = a^i, cb = b^i \rangle$

So by the above table and Lemma 5.1.1, each element of our presentation is of the form a^i, b^i or c^i. Thus

$$\langle a, b, c | ab = b^2, ba = a^2, ac = b^i, ca = a^i, bc = a^i, cb = b^i \rangle =$$

$$\{a^i : i \in \mathbb{N}\} \cup \{b^i : i \in \mathbb{N}\} \cup \{c^i : i \in \mathbb{N}\} =$$

$$A \cup B \cup C.$$

Now we show that $A,\ B$ and C are pairwise disjoint. Let T be the semigroup with multiplication table:

	1	e	f	g
1	1	e	f	g
e	e	e	f	f
f	f	e	f	e
g	g	e	f	1

We define a mapping $\varphi : \{a, b, c\} \to T$ with $\varphi(a) = e$, $\varphi(b) = f$, $\varphi(c) = g$. Then there exists a homomorphism $\psi : \{a, b, c\}^+ \to T$. It is obvious that ψ preserves our relation as follows:

$$\psi(ab) = \psi(a)\psi(b) = ef = f = \psi(b^2),$$

$$\psi(ba) = \psi(b)\psi(a) = fe = e = \psi(a^2),$$
$$\psi(ac) = \psi(a)\psi(c) = eg = f = \psi(b^i),$$
$$\psi(ca) = \psi(c)\psi(a) = ge = e = \psi(a^i),$$
$$\psi(bc) = \psi(b)\psi(c) = fg = e = \psi(a^i),$$
$$\psi(cb) = \psi(c)\psi(b) = gf = f = \psi(b^i).$$

Therefore, ψ is a homomorphism from S into T and therefore $A \cap B = A \cap C = B \cap C = \emptyset$. So A, B and C are a pairwise disjoint. The last step is th show that A, B and C are infinite. Define a mapping $\varphi : \{a, b, c\} \to \mathbb{N}_0$ with $\varphi(a) = 1$, $\varphi(b) = 1$ and $\varphi(c) = i - 1$. Then there exists a homomorphism $\psi : \{a, b, c\}^+ \to \mathbb{N}_0$ in which $\{a, b, c\}^+$ is the free semigroup on $\{a, b, c\}$ and ψ uniquely determined by the images of the generators a, b and c by Proposition 1.6.1. It is obvious that ψ preserves the all above relations and based on this we deduce that

$$\Re = \{ab = b^2, ba = a^2, ac = b^i, ca = a^i, bc = a^i, cb = b^i\} \subseteq \ker \psi.$$

Hence, $\rho \subseteq \ker \psi$. Therefore, by Lemma 4.1.1 all elements of the semigroup $A \sqcup B \sqcup C$ are distinct as required. □

5.3 Semigroups of the same type

Theorem 5.3.1. *All semigroups of one of the following types*

$$(a, a, a, a, a, a), (a, a, a, a, b, b), (a, a, a, a, c, b), (a, a, c, a, c, a), (a, a, c, a, c, c), (b, a, b, a, a, b)$$

are non-isomorphic.

PROOF. We show now that all semigroups of the type (a, a, a, a, a, a) are non-isomorphic. For $k \geq 1$, let $S_{i,j,k} = A \sqcup B \sqcup C$ be a semigroup of the type (a, a, a, a, a, a) with $ab = a^i$, $ba = a^i$, $ac = a^j$, $ca = a^j$, $bc = a^k$, $cb = a^k$. Suppose that $S_{i,j,k} \cong S_{l,m,n}$ for some $i, j, k, l, m, n \in \mathbb{N}$ with $(i, j, k) \neq (l, m, n)$ that means there is an isomorphism $\varphi : S_{i,j,k} \to S_{l,m,n}$. Clearly, A is the only ideal which is isomorphic to $\mathbb{N}$ in both of $S_{i,j,k}$ and $S_{l,m,n}$, because the presentation says that any word which has a letter a will be in A and hence each ideal must consist of elements from just A or from A and B or C, or from A and B and C but each of the last two cases has at least two generators which implies that it is not isomorphic to $\mathbb{N}$. Consequently, $\varphi(a) = a^t$

where $t \in \mathbb{N}$ and hence $\varphi(a) = a$ by Lemma 4.2.4. Now if $\varphi(b) = b^t$, $\varphi(c) = c^u$ then $\varphi(b) = b$, $\varphi(c) = c$ by Lemma 4.2.4 with $\varphi(a) = a$ and hence we get an identity map with $\varphi(bc) \neq \varphi(b)\varphi(c)$. Similarly, if $\varphi(b) = c^t$ then $\varphi(c) = b^u$ and $t = u = 1$ by Lemma 4.2.4, with $\varphi(a) = a$. However, $\varphi(bc) = \varphi(a^i) = \Big(\varphi(a)\Big)^i = a^i$, $\varphi(b)\varphi(c) = cb = a^l$ and that implies $\varphi(bc) \neq \varphi(b)\varphi(c)$ by $(i \neq l)$.

We show that all semigroups of the type (a,a,a,a,b,b) are non-isomorphic. For $i,k,p \geq 1$, let $S_{i,k,p} = A \sqcup B \sqcup C$ be a semigroup of the type (a,a,a,a,b,b) with $ab = a^i$, $ba = a^i$, $ac = a^k$, $ca = a^k$, $bc = b^p$, $cb = b^p$. Suppose $S_{i,k,p} \cong S_{j,l,q}$ via an isomorphism $\varphi : S_{i,k,p} \to S_{j,l,q}$ with $(i,k,p) \neq (j,l,q)$. Clearly A is the only ideal which is isomorphic to $\mathbb{N}$ in both of $S_{i,k,p}$ and $S_{j,l,q}$ because the presentation says that any word which has a letter a will be in A and hence each ideal must consist of elements from just A or from A and B or C, or from A and B and C, but the last two possibilities have at least two generators which implies that it is not isomorphic to $\mathbb{N}$. Consequently, $\varphi(a) = a^t$ for some t in $\mathbb{N}$ and hence $t = 1$ by Lemma 4.2.4. Now, since B or subsemigroup of B is the only ideal, which is isomorphic to $\mathbb{N}$, in both of the two subsemigroups $B \sqcup C$ in $S_{i,k,p}$ and $S_{j,l,q}$. So $\varphi(b) = b^t$ for some $t \in \mathbb{N}$ and hence $\varphi(b) = b$ by Lemma 4.2.4. Thus $\varphi(c) = c^t$ and analogously $t = 1$ and then we obtain the identity mapping with $\varphi(a)\varphi(b) \neq \varphi(ab)$, a contradiction.

We show that all semigroups of the type (a,a,a,a,c,b) are non-isomorphic. For $k \geq 1$, let $S_k = A \sqcup B \sqcup C$ be a semigroup of the type (a,a,a,a,c,b) with $ab = a^i$, $ba = a^i$, $ac = a^i$, $ca = a^i$, $bc = c^2$, $cb = b^2$ and $S_i \cong S_j$ via an isomorphism $\varphi : S_i \to S_j$ with $i \neq j$. Clearly, A is the only ideal which is isomorphic to $\mathbb{N}$ in both of S_i and S_j because the presentation says that any word which has a letter a will be in A and hence each ideal must consist of elements from just A or from A and B or C, or from A and B and C, but the last two possibilities have at least two generators which implies that it is not isomorphic to $\mathbb{N}$. Consequently, $\varphi(a) = a^t$ for some t in $\mathbb{N}$ and hence $t = 1$ by Lemma 4.2.4. Now If $\varphi(b) = c^t$ and thus $\varphi(c) = b^u$ and $u = t = 1$ by Lemma 4.2.4. Hence, $\varphi(c) = b$ and $\varphi(b) = c$ but $\varphi(ca) = \varphi(a^i) = a^i$ and $\varphi(c)\varphi(a) = ba = a^j$. So $\varphi(ca) \neq \varphi(c)\varphi(a)$ by $i \neq j$. Therefore, $\varphi(b) = b$, $\varphi(c) = c$ by Lemma 4.2.4 which is the identity mapping with $\varphi(ac) \neq \varphi(a)\varphi(c)$ since $i \neq j$, a contradiction.

We show that all semigroups of the type (a,a,c,a,c,a) are non-isomorphic. For $i \geq 1$, let $S_i = A \sqcup B \sqcup C$ be a semigroup of the type (a,a,c,a,c,a) with $ab = a^i$,

$ba = a^i$, $ac = c^2$, $ca = a^2$, $bc = c^i$, $cb = a^i$. Suppose $S_i \cong S_j$ for some $i, j \in \mathbb{N}$ via the isomorphism $\varphi : S_i \to S_j$ with $i \neq j$. The only subsemigroup in both of S_i and S_j, which is a disjoint union of two left ideals free monogenic semigroups is $A \sqcup C$, or a subsemigroup of $A \sqcup C$. Thus $\varphi(b) = b$ by Lemma 4.2.4. Now if $\varphi(a) = c^t$, $\varphi(c) = a^u$ then $t = u = 1$ by Lemma 4.2.4. Thus $\varphi(ba) = \varphi(a^i) = c^i$ and $\varphi(b)\varphi(a) = bc = c^j$, a contradiction. Thus, $\varphi(a) = a^t$, $\varphi(c) = c^u$ and then $t = u = 1$ by Lemma 4.2.4. So we have the identity mapping with $\varphi(a)\varphi(b) \neq \varphi(ab)$ by $i \neq j$, a contradiction.

We show that all semigroups of the type (a, a, c, a, c, c) are non-isomorphic. For $k \geq 1$, let $S_k = A \sqcup B \sqcup C$ be a semigroup of the type (a, a, c, a, c, c) with $ab = a^i$, $ba = a^i$, $ac = c^2$, $ca = a^2$, $bc = c^i$, $cb = c^i$. Suppose that $S_i \cong S_j$ via an isomorphism $\varphi : S_i \to S_j$ with $i \neq j$. The only subsemigroup in both of S_i and S_j, which is a disjoint union of two left ideals free monogenic semigroups is $A \sqcup C$, or a subsemigroup of $A \sqcup C$. Thus, $\varphi(b) = b^t$ for some t in $\mathbb{N}$ and hence $t = 1$ by Lemma 4.2.4. Now If $\varphi(a) = c^t$ and thus $\varphi(c) = a^u$ and $u = t = 1$ by Lemma 4.2.4. Hence $\varphi(a) = c$ and $\varphi(c) = a$, but $\varphi(ab) = \varphi(a^i) = c^i$ and $\varphi(a)\varphi(b) = cb = c^j$. So $\varphi(ab) \neq \varphi(a)\varphi(b)$ by $i \neq j$. Therefore, $\varphi(a) = a$, $\varphi(b) = b$ by Lemma 4.2.4 which is the identity mapping with $\varphi(a)\varphi(b) \neq \varphi(ab)$ by $i \neq j$, a contradiction.

We show that all semigroups of the type (b, a, b, a, a, b) are non-isomorphic. For $i \geq 1$, let $S_i = A \sqcup B \sqcup C$ be the semigroup of the type (b, a, b, a, a, b) with $ab = b^2$, $ba = a^2$, $ac = b^i$, $ca = a^i$, $bc = a^i$, $cb = b^i$. Suppose $S_i \cong S_j$ for some $i, j \in \mathbb{N}$ which means there is an isomorphism $\varphi : S_i \to S_j$ with $i \neq j$. The only subsemigroup in both of S_i and S_j, which is a disjoint union of two left ideals free monogenic semigroups is $A \sqcup B$, or a subsemigroup of $A \sqcup B$. Hence, $\varphi(c) = c^t$ for some t in $\mathbb{N}$ and hence $t = 1$ by Lemma 4.2.4. Now if $\varphi(a) = b^t$, $\varphi(b) = a^u$ then $t = u = 1$ by Lemma 4.2.4. Thus, $\varphi(ac) = \varphi(b^i) = a^i$ and $\varphi(a)\varphi(c) = bc = a^j$, a contradiction. Thus $\varphi(a) = a^t$, $\varphi(b) = b^u$ and then $t = u = 1$ by Lemma 4.2.4. So we have the identity mapping with $\varphi(a)\varphi(c) \neq \varphi(ac)$ by $i \neq j$, a contradiction.

□

5.4 Comparison between disjoint unions of three copies of the free monogenic semigroup and three copies of the infinite cyclic group

In Chapter 3 we classified the semigroups which are disjoint unions of two or three copies of a group and this appears clearly in Theorems (3.2.1, 3.2.2, 3.3.1, 3.3.2). In Chapter 2 we proved that if we substituted the two copies of the infinite cyclic group by the free monogenic semigroup we obtain parallel results. In this section we take an infinite cyclic group as an example and we show the two theorems (3.3.1, 3.3.2) on an infinite cyclic group in details to see how we define homomorphisms between these three groups to get Clifford semigroups, Rees matrix semigroups and combination semigroups and then we compare these semigroups with the semigroups which are disjoint unions of three copies of the free monogenic semigroup. We will see two more types as disjoint unions of three copies of the free monogenic semigroup which do not exist as disjoint unions of three copies of the infinite cyclic group.

Consider the semigroup $S = S_\alpha \sqcup S_\beta \sqcup S_\gamma$ where $S_\alpha = \langle a, a^{-1} | aa^{-1} = a^{-1}a = 1_\alpha \rangle$, $S_\beta = \langle b, b^{-1} | bb^{-1} = b^{-1}b = 1_\beta \rangle$ and $S_\gamma = \langle c, c^{-1} | cc^{-1} = c^{-1} = 1_\gamma \rangle$ three copies of the infinite cyclic group. Then by Theorem 3.3.1 and Theorem 3.3.2 we have the following seven cases:

Case 1. S_γ is an ideal in S and $S_\alpha S_\beta \subseteq S_\gamma$, $S_\beta S_\alpha \subseteq S_\gamma$. Then we have three groups S_α , S_β and S_γ with

$$ac = ca = c^i \ , \ bc = cb = c^j \ , \ ab = ba = c^k,$$

where $k = i + j - 2$ and the reason for this is, S_γ is an ideal in S, and the associativity on $c(ac),\ c(bc), a(ba), c(ba)$. Analogously, with the remaining relations:

$$a^{-1}c^{-1} = c^{-1}a^{-1} = c^l \ , \ b^{-1}c^{-1} = c^{-1}b^{-1} = c^p \ , \ a^{-1}b^{-1} = b^{-1}a^{-1} = c^q,$$

$$a^{-1}c = ca^{-1} = c^r,\ b^{-1}c = cb^{-1} = c^s,\ a^{-1}b = ba^{-1} = c^t,$$

$$ac^{-1} = c^{-1}a = c^u,\ bc^{-1} = c^{-1}b = c^v,\ ab^{-1} = b^{-1}a = c^w.$$

Thus, there is a semilattice $Y = \{\alpha, \beta, \gamma\}$ where to each element $\alpha \in Y$ we assign a group S_α such that $\alpha\beta = \beta\alpha = \gamma$, $\alpha\gamma = \gamma\alpha = \gamma$, $\gamma\beta = \beta\gamma = \gamma$ if $S_\alpha S_\beta \subseteq S_\gamma$, $S_\beta S_\alpha \subseteq S_\gamma$, $S_\alpha S_\gamma \subseteq S_\gamma$, $S_\gamma S_\alpha \subseteq S_\gamma$ and $S_\beta S_\gamma \subseteq S_\gamma$, $S_\gamma S_\beta \subseteq S_\gamma$. There is also two pairs of elements $\alpha,\ \gamma$ in which we assign a map $\phi_{\alpha,\gamma}$ of S_α into S_γ with $\phi_{\alpha,\gamma}(a) = c^{i-1}$ and $\beta,\ \gamma$ in which we assign a map $\phi_{\beta,\gamma}$ of S_β into S_γ with $\phi_{\beta,\gamma}(b) = c^{j-1}$. Also there are three identity maps $\phi_{\alpha,\alpha},\ \phi_{\beta,\beta},\ \phi_{\gamma,\gamma}$ on S_α, S_β and S_γ respectively. Clearly, $\phi_{\alpha,\alpha}\phi_{\alpha,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \geq \alpha \geq \gamma$, $\phi_{\alpha,\gamma}\phi_{\gamma,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \geq \gamma$, $\phi_{\beta,\gamma}\phi_{\gamma,\gamma} = \phi_{\beta,\gamma}$ if $\beta \geq \gamma$ and finally $\phi_{\beta,\beta}\phi_{\beta,\gamma} = \phi_{\beta,\gamma}$ if $\beta \geq \gamma$. Now, notice that

$$\begin{aligned}
a^x c^y &= a^{x-1}c^i c^{y-1} \\
&= a^{x-2}c^i c^{i-1} c^{y-1} \\
&= a^{x-(x-1)}c^i c^{(i-1)(x-2)} c^{y-1} \\
&= c^i c^{i-1} c^{(i-1)(x-2)} c^{y-1} \\
&= c^{x(i-1)+y},
\end{aligned}$$

and

$$\phi_{\alpha,\gamma}(a^x)\phi_{\gamma,\gamma}(c^y) = c^{x(i-1)}c^y = c^{x(i-1)+y} = a^x c^y.$$

Also

$$\begin{aligned}
a^x b^y &= a^{x-1}c^k b^{y-1} \\
&= a^{x-2}c^i c^{k-1} b^{y-1} \\
&= c^{i+(i-1)(x-3)+k-1} b^{y-1} \\
&= c^{i+i-1+ix-3i-x+3+k-1} b^{y-1} \\
&= c^{-i+ix-x+k+1} b^{y-1} \\
&= c^{-i+ix-x+k} c^j b^{y-2} \\
&= c^{-i+ix-x+k} c^{j-1} c^j b^{y-3} \\
&= c^{-i+ix-x+k} c^{(j-1)(y-3)+2j-1} \\
&= c^{-i+ix-x+k} c^{jy-3j-y+3+2j-1} \\
&= c^{x(i-1)+y(j-1)+k-i-j+2} \\
&= c^{x(i-1)+y(j-1)} \qquad \text{(by } k = i+j-2\text{)},
\end{aligned}$$

besides

$$\phi_{\alpha,\gamma}(a^x)\phi_{\gamma,\gamma}(b^y) = c^{x(i-1)}c^{y(j-1)} = a^x b^y.$$

Analogously with the negative values of the powers i, j, k. The remaining relations follow similarly. Thus, S is a strong semilattice of groups and then S is a Clifford semigroup of type V.

Case2. S_γ is an ideal in S and S_β is an ideal in $S_\alpha \sqcup S_\beta$. Then we have three groups S_α , S_β and S_γ with

$$ac = ca = c^i \ , \ bc = cb = c^j \ , \ ab = ba = b^k,$$

and this because S_γ is an ideal in S and S_β is an ideal in $S_\alpha \sqcup S_\beta$ and the associativity on $c(ac), c(bc), b(ab)$. Similarly, with the remaining relations:

$$a^{-1}c^{-1} = c^{-1}a^{-1} = c^l \ , \ b^{-1}c^{-1} = c^{-1}b^{-1} = c^p \ , \ a^{-1}b^{-1} = b^{-1}a^{-1} = b^q,$$

$$a^{-1}c = ca^{-1} = c^r, \ b^{-1}c = cb^{-1} = c^s, \ a^{-1}b = ba^{-1} = b^t,$$

$$ac^{-1} = c^{-1}a = c^u, \ bc^{-1} = c^{-1}b = c^r, \ ab^{-1} = b^{-1}a = b^w.$$

Thus, there is a semilattice $Y = \{\alpha, \beta, \gamma\}$ where to each element $\alpha \in Y$ we assign a group S_α such that $\alpha\beta = \beta\alpha = \beta$, $\alpha\gamma = \gamma\alpha = \gamma$, $\gamma\beta = \beta\gamma = \gamma$ if $S_\alpha S_\beta \subseteq S_\beta$, $S_\beta S_\alpha \subseteq S_\beta$, $S_\alpha S_\gamma \subseteq S_\gamma$, $S_\gamma S_\alpha \subseteq S_\gamma$ and $S_\beta S_\gamma \subseteq S_\gamma$, $S_\gamma S_\beta \subseteq S_\gamma$. There is also three pairs of elements α, β in which we assign a map $\phi_{\alpha,\beta}$ of S_α into S_β with $\phi_{\alpha,\beta}(a) = b^{k-1}$ and β, γ in which we assign a map $\phi_{\beta,\gamma}$ of S_β into S_γ with $\phi_{\beta,\gamma}(b) = c^{j-1}$ and α, γ in which we assign a map $\phi_{\alpha,\gamma}$ of S_α into S_γ with $\phi_{\alpha,\gamma}(a) = c^{i-1}$. Also there are three identity maps $\phi_{\alpha,\alpha}$, $\phi_{\beta,\beta}$, $\phi_{\gamma,\gamma}$ on S_α, S_β and S_γ respectively. Clearly, $\phi_{\alpha,\alpha}\phi_{\alpha,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \geq \gamma$, $\phi_{\alpha,\gamma}\phi_{\gamma,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \geq \gamma$, $\phi_{\beta,\gamma}\phi_{\gamma,\gamma} = \phi_{\beta,\gamma}$ if $\beta \geq \gamma$, $\phi_{\beta,\beta}\phi_{\beta,\gamma} = \phi_{\beta,\gamma}$ if $\beta \geq \gamma$, $\phi_{\alpha,\beta}\phi_{\beta,\beta} = \phi_{\alpha,\beta}$ if $\alpha \geq \beta$,$\phi_{\alpha,\alpha}\phi_{\alpha,\beta} = \phi_{\alpha,\beta}$ if $\alpha \geq \beta$ and eventually $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ if $\alpha \geq \beta \geq \gamma$. Now, notice that

$$a^x b^y = a^{x-1} b^k b^{y-1} = b^{x(k-1)+y} \ , \ \phi_{\alpha,\beta}(a^x)\phi_{\beta,\beta}(b^y) = b^{x(k-1)+y}.$$

And

$$b^x c^y = b^{x-1} c^j c^{y-1} = c^{x(j-1)+y} \ , \ \phi_{\beta,\gamma}(b^x)\phi_{\gamma,\gamma}(c^y) = c^{x(j-1)+y}.$$

Besides

$$a^x c^y = a^{x-1} c^i c^y = c^{x(i-1)+y} \ , \ \phi_{\alpha,\gamma}(a^x)\phi_{\gamma,\gamma}(c^y) = c^{x(i-1)} c^y = c^{x(i-1)+y}.$$

Analogously, if i, j, k are negative. Similarly with the remaining relations. Thus, S is a strong semilattice of groups and then S is a Clifford semigroup of type I.

Case 3. S_γ is an ideal in S and S_α , S_β are left ideals in $S_\alpha \sqcup S_\beta$. Then we have a group S_γ and a Rees matrix semigroup $S_\alpha \sqcup S_\beta$ with

$$ac = ca = c^i \ , \ bc = cb = c^j \ , \ ab = b^k \ , \ ba = a^l,$$

and this because, as we have mentioned before, of the associativity on $c(ac)$, $c(bc)$. Similarly with the remaining relations:

$$a^{-1}c^{-1} = c^{-1}a^{-1} = c^p \ , \ b^{-1}c^{-1} = c^{-1}b^{-1} = c^q \ , \ a^{-1}b^{-1} = b^r \ , \ b^{-1}a^{-1} = a^s,$$

$$a^{-1}c = ca^{-1} = c^t, \ b^{-1}c = cb^{-1} = c^u, \ a^{-1}b = b^v \ , \ ba^{-1} = a^w,$$

$$ac^{-1} = c^{-1}a = c^e, \ bc^{-1} = c^{-1}b = c^f, \ ab^{-1} = b^g \ , \ b^{-1}a = a^h.$$

Thus, there is a semilattice $Y = \{\gamma, \delta\}$ where to each $\gamma \in Y$ we assign a group S_γ such that $S_\delta = S_\alpha \sqcup S_\beta$ where $\gamma\delta = \delta\gamma = \gamma$ if $S_\delta S_\gamma \subseteq S_\gamma$, $S_\gamma S_\delta \subseteq S_\gamma$. There is also a pair of elements δ, γ in which we assign a map $\phi_{\delta,\gamma}$ of S_δ into S_γ with $\phi_{\delta,\gamma}(a) = c^{i-1}$, and $\phi_{\delta,\gamma}(b) = c^{j-1}$. Furthermore, there are two identity maps $\phi_{\delta,\delta}$, $\phi_{\gamma,\gamma}$ on S_δ and S_γ respectively. Clearly, $\phi_{\delta,\delta}\phi_{\delta,\gamma} = \phi_{\delta,\gamma}$ if $\delta \geq \gamma$, $\phi_{\delta,\gamma}\phi_{\gamma,\gamma} = \phi_{\delta,\gamma}$ if $\delta \geq \gamma$. Now, notice that

$$a^x c^y = a^{x-1}c^i c^{y-1} = c^{x(i-1)+y},$$

$$\phi_{\delta,\gamma}(a^x)\phi_{\gamma,\gamma}(c^y) = c^{x(i-1)}c^y = a^x c^y.$$

Also

$$b^x c^y = b^{x-1}c^j c^{y-1} = c^{x(j-1)+y},$$

$$\phi_{\delta,\gamma}(b^x)\phi_{\gamma,\gamma}(c^y) = c^{x(j-1)}c^y = b^x c^y.$$

Analogously with the negative powers. Remaining relations follow similarly. Thus S is a combination semigroup of the type $\top$. Similarly, if S_α , S_β are right ideals in $S_\alpha \sqcup S_\beta$.

Case 4. $S_\alpha \sqcup S_\beta$ is an ideal in S and S_α, S_β are left ideals in $S_\alpha \sqcup S_\beta$. Then we have one group S_γ and a semigroup $S_\alpha \sqcup S_\beta$ with the relations

$$ac = ca = a^i \ , \ bc = cb = b^i \ , \ ab = b^2 \ , \ ba = a^2,$$

and this because of the associativity on $c(ac), c(bc), a(ba), a(bc)$. Similarly with the remaining relations:

$$a^{-1}c^{-1} = c^{-1}a^{-1} = a^p \ , \ b^{-1}c^{-1} = c^{-1}b^{-1} = b^q \ , \ a^{-1}b^{-1} = b^r \ , \ b^{-1}a^{-1} = a^{r'},$$

$$a^{-1}c = ca^{-1} = a^s, \ b^{-1}c = cb^{-1} = b^t, \ a^{-1}b = b^u \ , \ ba^{-1} = a^v,$$

$$ac^{-1} = c^{-1}a = a^w, \ bc^{-1} = c^{-1}b = b^e, \ ab^{-1} = b^f \ , \ b^{-1}a = a^h.$$

In addition, there exists a semilattice $Y = \{\gamma, \delta\}$ where to each $\gamma \in Y$ we assign a group S_γ such that $S_\delta = S_\alpha \sqcup S_\beta$ where $\gamma\delta = \delta\gamma = \delta$ if $S_\gamma S_\delta \subseteq S_\delta$ and to each pair of elements γ, δ we assign a map $\phi_{\gamma,\delta}$ of S_γ into S_δ with $\phi_{\gamma,\delta}(c) = a^{i-1}$, $\phi_{\gamma,\delta}(c^{-1}) = a^{p-1}$. Also there are two identity maps $\phi_{\gamma,\gamma}$ and $\phi_{\delta,\delta}$ on S_γ and S_δ respectively. Clearly, $\phi_{\gamma,\gamma}\phi_{\gamma,\delta} = \phi_{\gamma,\delta}$ if $\gamma \geq \delta$ and $\phi_{\gamma,\delta}\phi_{\delta,\delta} = \phi_{\gamma,\delta}$ if $\gamma \geq \delta$. Now notice that

$$c^x a^y = c^{x-1}a^i a^{y-1} = a^{x(i-1)+y} \ , \ \phi_{\gamma,\delta}(c^x)\phi_{\gamma,\delta}(a^y) = a^{x(i-1)+y}.$$

And

$$c^x b^y = c^{x-1}b^i b^{y-1} = b^{x(i-1)+y}, \phi_{\gamma,\delta}(c^x)\phi_{\delta,\delta}(b^y) = a^{x(i-1)}b^y = b^{x(i-1)+y}.$$

The remaining relations follow similarly and the argument works with the negative powers as well. So, we come to the conclusion that S is a strong semilattice of two semigroups one is a group and the other is a Rees matrix semigroup. Thus S is a combination between a Clifford semigroup and a Rees matrix semigroup of the type $\perp$. Similarly, if S_α , S_β are right ideals in $S_\alpha \sqcup S_\beta$.

Case 5. $S_\alpha \sqcup S_\beta$ is an ideal in S and S_α, S_β are left ideals in $S_\alpha \sqcup S_\beta$. Then we have one group S_γ and a semigroup $S_\alpha \sqcup S_\beta$ with the relations

$$ac = b^i \ , \ ca = a^i \ , \ bc = cb = b^i \ , \ ab = b^2 \ , \ ba = a^2,$$

and this because of the associativity on $a(ca)$, $b(cb)$, $a(ba)$, $a(bc)$. Similarly, with the relations

$$a^{-1}c^{-1} = b^q \ , \ c^{-1}a^{-1} = a^r \ , \ b^{-1}c^{-1} = c^{-1}b^{-1} = b^s \ , \ a^{-1}b^{-1} = b^t \ , \ b^{-1}a^{-1} = a^{t'},$$

$$a^{-1}c = b^u \ , \ ca^{-1} = a^v, \ b^{-1}c = cb^{-1} = b^w, \ a^{-1}b = b^e \ , \ ba^{-1} = a^f,$$

$$ac^{-1} = b^g \ , \ c^{-1}a = a^h, \ bc^{-1} = c^{-1}b = b^d, \ ab^{-1} = b^{d'} \ , \ b^{-1}a = a^{d''}.$$

In addition, there exists a semilattice $Y = \{\gamma, \delta\}$ where to each element $\gamma \in Y$ we assign a group S_γ such that $S_\delta = S_\alpha \sqcup S_\beta$ where $\gamma\delta = \delta\gamma = \delta$ if $S_\gamma S_\delta \subseteq S_\delta$ and to each pair of elements γ, δ we assign a map $\phi_{\gamma,\delta}$ of S_γ into S_δ with $\phi_{\gamma,\delta}(c) = b^{i-1}$. Also there are two identity maps $\phi_{\gamma,\gamma}$ and $\phi_{\delta,\delta}$ on S_γ and S_δ respectively. Clearly, $\phi_{\gamma,\gamma}\phi_{\gamma,\delta} = \phi_{\gamma,\delta}$ if $\gamma \geq \delta$ and $\phi_{\gamma,\delta}\phi_{\delta,\delta} = \phi_{\gamma,\delta}$ if $\gamma \geq \delta$. Now notice that

$$c^x a^y = c^{x-1}a^i a^{y-1} = a^{x(i-1)+y},$$

and

$$\begin{aligned}
\phi_{\gamma,\delta}(c^x)\phi_{\delta,\delta}(a^y) = b^{x(i-1)}a^y &= b^{xi-x-1}a^2a^{y-1} \\
&= b^{xi-x-2}a^3a^{y-1} \\
&= b^{xi-x-(xi-x-1)}a^{xi-x}a^{y-1} \\
&= a^{2+xi-x-1+y-1} \\
&= a^{x(i-1)+y}.
\end{aligned}$$

Also we have

$$c^x b^y = c^{x-1}b^i b^{y-1} = b^{x(i-1)+y} \ , \ \phi_{\gamma,\delta}(c^x)\phi_{\delta,\delta}(b^y) = b^{x(i-1)+y}.$$

And finally

$$
\begin{aligned}
a^x c^y &= a^{x-1} b^i c^{y-1} \\
&= a^{x-2} b^2 b^{i-1} c^{y-1} \\
&= a^{x-3} b^3 b^{i-1} c^{y-1} \\
&= a b^{x-1} b^{i-1} c^{y-1} \\
&= b^{2+x-1+i-2} c^{y-1} \\
&= b^{x+i-2} b^i c^{y-2} \\
&= b^{x+i-2} b^{(i-1)(y-3)} b^{i-1} b^i \\
&= b^{x+y(i-1)+i-i} \\
&= b^{x+y(i-1)},
\end{aligned}
$$

$$
\begin{aligned}
\phi_{\delta,\delta}(a^x)\phi_{\gamma,\delta}(c^y) &= a^x b^{y(i-1)} \\
&= a^{x-1} b^2 b^{yi-y-1} \\
&= a^{x-2} b^3 b^{yi-y-1} \\
&= a b^x b^{yi-y-1} \\
&= b^2 b^{x-1} b^{yi-y-1} \\
&= b^{x+y(i-1)}.
\end{aligned}
$$

Similarly with the remaining relations. Thus, S is a strong semilattice of a group and a Rees matrix semigroup. Thus S is a combination semigroup of the type $\perp$. Similarly, if S_α , S_β are right ideals in $S_\alpha \sqcup S_\beta$.

Case 6. $S_\alpha \sqcup S_\beta$ is an ideal in S and S_α, S_β are left ideals in $S_\alpha \sqcup S_\beta$. Then we have one group S_γ and a semigroup $S_\alpha \sqcup S_\beta$ with the relations

$$ac = b^i \ , \ ca = a^{i'} \ , \ bc = a^j \ , \ cb = b^{j'} \ , \ ab = b^k \ , \ ba = a^{k'},$$

$$a^{-1}c^{-1} = b^l \ , \ c^{-1}a^{-1} = a^{l'} \ , \ b^{-1}c^{-1} = a^p \ , \ c^{-1}b^{-1} = b^{p'} \ , \ a^{-1}b^{-1} = b^q \ , \ b^{-1}a^{-1} = a^{q'},$$

$$a^{-1}c = b^r \ , \ ca^{-1} = a^{r'}, \ b^{-1}c = a^s \ , \ cb^{-1} = b^{s'}, \ a^{-1}b = b^t \ , \ ba^{-1} = a^{t'},$$

$$ac^{-1} = b^u \ , \ c^{-1}a = a^{u'}, \ bc^{-1} = a^v \ , \ c^{-1}b = b^{v'}, \ ab^{-1} = b^w \ , \ b^{-1}a = a^{w'}.$$

In addition, there exists a semilattice $Y = \{\gamma, \delta\}$ where to each element $\gamma \in Y$ we assign a group S_γ such that $S_\delta = S_\alpha \sqcup S_\beta$ where $\gamma\delta = \delta\gamma = \delta$ if $S_\gamma S_\delta \subseteq S_\delta$. So S

is a semilattice of a group and a Rees matrix semigroup. Thus S is a combination semigroup of the type $\perp$. Similarly, if S_α and S_β are right ideal in S_δ.

Case 7. S_α, S_β and S_γ are left ideals in S. Then $S_\alpha \sqcup S_\beta \sqcup S_\gamma$ is a completely simple semigroup and that means S is isomorphic to a Rees matrix semigroup of a particular shape by Theorem 1.5.4. Now we want to construct all potential Rees matrix semigroups. In fact, we have three left ideals S_α, S_β and S_γ. So,

$$S_\alpha = L_1 = S1_\alpha,\ S_\beta = L_2 = S1_\beta,\ S_\gamma = L_3 = S1_\gamma$$

and

$$R_1 = 1_\alpha S = 1_\beta S = 1_\gamma S = S$$

Clearly, L_1, L_2, L_3 are $L-$classes and R_1 is $R-$class. Hence $I = \{1\}$, $\Lambda = \{1,2,3\}$ and since P is a $|\Lambda| \times |I|$ matrix then it consists of one column and three rows as $P = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Observe that $R_1 \cap L_1 = H_{11}$, $R_1 \cap L_2 = H_{12}$ and $R_1 \cap L_3 = H_{13}$. Therefore, $H_{11} = L_1 = S_\alpha$, $H_{12} = L_2 = S_\beta$, $H_{13} = L_3 = S_\gamma$ and then $S = \mathcal{M}\left[S_\alpha, \{1\}, \{1,2,3\}, P\right]$ or $S = \mathcal{M}\left[S_\beta, \{1\}, \{1,2,3\}, P\right]$ or $S = \mathcal{M}\left[S_\gamma, \{1\}, \{1,2,3\}, P\right]$ but simply we can notice that the three Rees matrix semigroups are isomorphic and that because all of them have the same sets I, Λ and the three semigroups S_α, S_β, S_γ are isomorphic. Analogously, if S_α, S_β and S_γ are right ideals in S.

Remark 5.4.1. Consider the previous cases. Clearly, in each of the above constructions we can substitute the free cyclic group by the free monogenic semigroup, to obtain unions of three copies of the free monogenic semigroup of these 7 types. In a different way, applying all the above constructions on a free monogenic semigroup, we get a collection of semigroups which are disjoint union of three copies of the free monogenic semigroup and we can show this directly by a comparison between these constructions and Theorem 5.2.1, as these cases 1,2,3,4,5,6,7 are, respectively, have the same constructions of the semigroups which defined by the presentations 1,2,3,5,4,9,7 in Theorem 5.2.1.

Remark 5.4.2. The semigroups which have arisen under these constructions in Remark 5.4.1 are not all semigroups which are disjoint unions of three copies of the free monogenic semigroup. There are two more types which are not strong semilattices of semigroups, Rees matrix semigroups nor combination semigroups and they are

1) The semigroup with presentation $\langle a, b, c | \ ab = b^2, ba = a^2, ac = c^2, ca = a^2, bc = c^2, cb = a^2 \rangle$ (Theorem 5.2.1)(8).

2) The semigroup with presentation $\langle a, b, c | \ ab = b^2, ba = a^2, ac = c^2, ca = b^2, bc = c^2, cb = a^2 \rangle$(Theorem 5.2.1)(7).

5.5 Disjoint union of an infinite cyclic group and a free monogenic semigroup

So far we looked at unions of a small number of copies of a group and copies of a semigroup and then we want to know what happen if we mixed them, for instance the semigroup which is a disjoint union of an infinite cyclic group and a free monogenic semigroup. Interestingly, we find in this section that the number of such semigroups decreases remarkably, because of not having the Rees matrix semigroups as they are not compatible with each other.

Theorem 5.5.1. *Let $S_\alpha = \langle a, a^{-1} | aa^{-1} = a^{-1}a = 1_\alpha \rangle$, $S_\beta = \langle c \rangle$. Suppose that S is a semigroup which is a disjoint union of S_α and S_β. Then one of the following must hold:*

i) $ac = ca = a^i$, $a^{-1}c = ca^{-1} = a^{i-2}$; *or*

ii) $ac = ca = ca^{-1} = a^{-1}c = c$.

PROOF. Now we have $ac,\ ca,\ a^{-1}c,\ ca^{-1} \ \in \{a^i : i \in \mathbb{Z}\} \cup \{c^i : i \in \mathbb{N}\}$. We distinguish the following cases.
Case 1. $\boldsymbol{ac = a^i}$. Then $a^{-1}(ac) = a^{-1}a^i = a^{i-1}$, $(a^{-1}a)c = 1_\alpha c$ and then $1_\alpha c = a^{i-1}$. Thus $a^{-1}1_\alpha c = a^{-1}a^{i-1} = a^{i-2}$. So we have so far $a^{-1}c = a^{i-2}$.
Subcase 1a. $ca = a^j$ that implies $a(ca) = a^{j+1}$, $(ac)a = a^{i+1}$. Thus $i = j$ and then $(ca)a^{-1} = a^i a^{-1} = a^{i-1} = c(aa^{-1}) = c1_\alpha$. Thus $c1_\alpha a^{-1} = a^{i-1}a^{-1} = a^{i-2}$. So, $ac = ca = a^i$, $a^{-1}c = ca^{-1} = a^{i-2}$.

Subcase 1b. $ca = c^j$ and we already have $a^{-1}c = a^{i-2}$. Then

$$\begin{aligned} a^{-1}(ca) &= a^{-1}c^j \\ &= a^{i-2}c^{j-1} \\ &= a^{i-3}a^i c^{j-2} \\ &= a^{i-3}a^{i-1}a^i c^{j-3} \\ &= a^{i-3}a^i a^{(i-1)(j-2)} \\ &= a^{ij-j-1}, \end{aligned}$$

and

$$(a^{-1}c)a = a^{i-2}a = a^{i-1}.$$

Thus, $j(i-1) = i$, it follows that $i = j = 2$ and then $a^{-1}c = 1_\alpha$. If $ca^{-1} = a^t$ then $a^{-1}(ca^{-1}) = a^{t-1}, (a^{-1}c)a^{-1} = a^{-1}$. Thus, $t = 0$ and then $ca^{-1} = 1_\alpha$ and we have $a^{-1}c = 1_\alpha$. Hence, $c(1_\alpha c) = ca = c^2$, $(c1_\alpha)c = ac = a^2$, a contradiction. And if $ca^{-1} = c^t$ then $a(ca^{-1}) = ac^t = a^2c^{t-1} = a^{t+1}$ and $(ac)a^{-1} = a^2a^{-1} = a$. Hence $t = 0$, a contradiction with $t \in \mathbb{N}$. Hence if $ac = a^i$ then $ac = ca = a^i, a^{-1}c = ca^{-1} = a^{i-2}$.

Case 2. $\boldsymbol{ac = c^i}$.

Subcase 2a. $ca = a^j$ and then $(ca)a^{-1} = a^j a^{-1} = a^{j-1}$, $c(aa^{-1}) = c1_\alpha$. Thus, $c1_\alpha = a^{j-1}$ and then $ca^{-1} = a^{j-2}$. Then

$$\begin{aligned} (ac)a^{-1} &= c^i a^{-1} \\ &= c^{i-1}a^{j-2} \\ &= c^{i-2}a^j a^{j-3} \\ &= c^{i-3}a^j a^{j-1}a^{j-3} \\ &= a^j a^{j-1}a^{(j-1)(i-3)}a^{j-3} \\ &= a^{ij-i-1}, \end{aligned}$$

and

$$a(ca^{-1}) = aa^{j-2} = a^{j-1}.$$

Thus, $ij - i - 1 = j - 1$ and then $i = j = 2$ and then $ca^{-1} = 1_\alpha$. If $a^{-1}c = a^t$ then $a^{-1}(ca^{-1}) = a^{-1}, (a^{-1}c)a^{-1} = a^{t-1}$. Thus, $t = 0$ and then $a^{-1}c = ca^{-1} = 1_\alpha$. Hence, $c(1_\alpha c) = ca = a^2$, $(c1_\alpha)c = ac = c^2$, a contradiction. And if $a^{-1}c = c^t$ then $a^{-1}(ca) = a^{-1}a^2 = a$ and $(a^{-1}c)a = c^t a = a^{t+1}$. Hence $t = 0$, a contradiction with

$t \in \mathbb{N}$.

Subcase 2b. $ca = c^j$. If $a^{-1}c = a^k$ then $1_\alpha c = a^{k+1}$ and then $ac = a^{k+2}$, a contradiction with the assumption. Now if $a^{-1}c = c^k$ then $i = j$ by the associativity on $c(ac)$. We also have that $1_\alpha c = c^{i+k-1}$ by the associativity on $(aa^{-1})c$ and thus $ac = c^{2i+k-2}$. Hence, $i = 2i + k - 2$, which implies $i = k = 1$ because $i, k \in \mathbb{N}$. Since $ca = c$, $c1_\alpha = ca^{-1}$ and then $ca = ca^{-1}a = c1_\alpha = c$ and then $ca^{-1} = c$. Therefore, $ac = ca = a^{-1}c = ca^{-1} = c$. □

Lemma 5.5.2. *Let S_α be an infinite cyclic group and S_β a free monogenic semigroup. For every semigroup S, which is a disjoint union of S_α and S_β, one of the following cases must hold.*

i) *S_α is an ideal in S; or*

ii) *S_β is an ideal in S.*

PROOF. Directly by Theorem 5.5.1.

Theorem 5.5.3. *Let $S_\alpha = \langle a, a^{-1} | aa^{-1} = a^{-1}a = 1_\alpha \rangle$ be an infinite cyclic group and let $S_\beta = \langle c \rangle$ be a free monogenic semigroup. Every semigroup S, which is a disjoint union of S_α, S_β, and S_α is an ideal in S, is a strong semilattice of semigroups.*

PROOF. In this case we have a semilattice $Y = \{\alpha, \beta\}$ where $\alpha\beta = \beta\alpha = \alpha$ when $S_\alpha S_\beta \subseteq S_\alpha$ and $S_\beta S_\alpha \subseteq S_\alpha$. We define a homomorphism $\psi_{\beta,\alpha}$ from the semigroup S_β to the group S_α by $\psi(c) = a^{i-1}$ such that $\psi_{\alpha,\alpha}$ and $\psi_{\beta,\beta}$ are the identity maps on S_α and S_β respectively. Also $\psi_{\beta,\alpha}\psi_{\alpha,\alpha} = \psi_{\beta,\alpha}$ if $\beta \geq \alpha$, $\psi_{\beta,\beta}\psi_{\beta,\alpha} = \psi_{\beta,\alpha}$ if $\beta \geq \alpha$. Now we have the following:

$$\begin{aligned} a^x c^y &= a^{x-1}a^i c^{y-1} \\ &= a^{x-1}a^{i-1}a^i c^{y-2} \\ &= a^{x-1}a^{(i-1)(y-1)}a^i \\ &= a^{x-1+iy-i-y+1+i} \\ &= a^{y(i-1)+x} \end{aligned}$$

and

$$\psi_{\alpha,\alpha}(a^x)\psi_{\beta,\alpha}(c^y) = a^x a^{y(i-1)} = a^{y(i-1)+x}.$$

Thus, $a^x c^y$ in S is equal to $\psi_{\alpha,\alpha}(a^x)\psi_{\beta,\alpha}(c^y)$ in the semilattice $\mathcal{S}[Y; \{S_\alpha, S_\beta\}; \psi_{\beta,\alpha}]$.

Analogously,

$$\begin{aligned}
(a^{-1})^x c^y &= (a^{-1})^{x-1} a^{i-2} c^{y-1} \\
&= (a^{-1})^{x-1} a^{i-3} a^i c^{y-2} \\
&= (a^{-1})^{x-1} a^{i-3} a^i a^{(i-1)(y-2)} \\
&= (a^{-1})^{x-1} a^{y(i-1)-1} \\
&= (a^{-1})^{x-1} a^{-1} a^{y(i-1)} \\
&= (a^{-1})^x a^{y(i-1)},
\end{aligned}$$

and

$$\psi_{\alpha,\alpha}((a^{-1})^x)\psi_{\beta,\alpha}(c^y) = (a^{-1})^x a^{y(i-1)}.$$

Thus, $(a^{-1})^x c^y$ in S is equal to $\psi_{\alpha,\alpha}((a^{-1})^x)\psi_{\beta,\alpha}(c^y)$ in the semilattice $\mathcal{S}[Y; \{S_\alpha, S_\beta\}; \psi_{\beta,\alpha}]$.

□

CHAPTER

SIX

CLASSIFICATION OF DISJOINT UNIONS OF TWO COPIES OF THE FREE SEMIGROUP OF RANK TWO

6.1 Introduction

In Chapters 3, 4, 5, we have described the semigroups which are disjoint unions of a small number of copies of a group or free monogenic semigroup, we have tried also to undertake such a classification with free semigroups of higher ranks, but at present it seems a very hard problem. So in this chapter we restrict our attention to a special case as follows.

Definition 6.1.1. A *balanced semigroup* is a semigroup defined by a presentation of the following form:

$$\begin{aligned}\langle a,b,c,d \mid \ & ac = w_{ac},\ ca = w_{ca},\ bc = w_{bc},\ cb = w_{cb},\ ad = w_{ad},\ da = w_{da},\\ & bd = w_{bd},\ db = w_{db}\ \rangle\end{aligned}$$

where $w_{x,y} \in \{a,b,c,d\}^+$, $|w_{xy}| = 2$.

The main topic of this chapter is to describe all possible balanced semigroups which are disjoint unions of two copies of the free semigroup of rank two, $S_\alpha = \langle a,b \mid \ \rangle$ and $S_\beta = \langle c,d \mid \ \rangle$, but the situation is still quite complicated. However, we remark that the problem becomes significantly easier if we ask about unions of free monoids, $S_\alpha = \langle a,b \mid \ \rangle$, $S_\beta = \langle c,d \mid \ \rangle$ and S_α is an ideal. Firstly, notice that $1_\alpha w_\beta = w_\beta 1_\alpha$ where w_β is an arbitrary word in S_β. Indeed, if $1_\alpha w_\beta = w_1$, $w_\beta 1_\alpha = w_2$ where $w_1, w_2 \in S_\alpha$, then $1_\alpha(w_\beta 1_\alpha) = 1_\alpha w_2 = w_2$ and $(1_\alpha w_\beta)1_\alpha = w_1 1_\alpha = w_1$. Thus $w_1 = w_2$ which implies that $1_\alpha w_\beta = w_\beta 1_\alpha$.

Now we have a homomorphism ψ from S_β to S_α with $\psi(w_\beta) = w_\beta 1_\alpha$ because

$$\psi(w_{\beta_1})\psi(w_{\beta_2}) = w_{\beta_1}1_\alpha w_{\beta_2}1_\alpha = w_{\beta_1}w_{\beta_2}1_\alpha 1_\alpha = w_{\beta_1}w_{\beta_2}1_\alpha = \psi(w_{\beta_1}w_{\beta_2}),$$

with two identity maps $\psi_{\alpha,\alpha}$ and $\psi_{\beta,\beta}$ on S_α and S_β respectively. Also $\psi_{\beta,\alpha}\psi_{\alpha,\alpha} = \psi_{\beta,\alpha}$ if $\beta \geq \alpha \geq \alpha$, $\psi_{\beta,\beta}\psi_{\beta,\alpha} = \psi_{\beta,\alpha}$ if $\beta \geq \beta \geq \alpha$. Now clearly

$$\psi_{\beta,\alpha}(w_\beta)\psi_{\alpha,\alpha}(w_\alpha) = w_\beta 1_\alpha w_\alpha = w_\beta w_\alpha.$$

Thus S is a semilattice $\mathcal{S}[Y;\{S_\alpha, S_\beta\};\psi_{\beta,\alpha}]$.

Main Thoerem. *Up to isomorphism, every balanced semigroup S is a disjoint union of two copies of the free semigroup of rank two $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ if and only if S is isomorphic to the semigroup which is defined by one of the following presentations:*

(1) $\langle p,q,r,s| \ \ pr = r^2,\ rp = r^2,\ qr = sr,\ rq = rs,\ ps = rs,\ sp = sr,\ qs = s^2,\ sq = s^2 \rangle$.

(2) $\langle p,q,r,s| \ \ pr = r^2,\ rp = r^2,\ qr = r^2,\ rq = r^2,\ ps = rs,\ sp = sr,\ qs = rs,\ sq = sr \rangle$.

(3) $\langle p,q,r,s| \ \ pr = r^2,\ rp = p^2,\ qr = sr,\ rq = pq,\ ps = rs,\ sp = qp,\ qs = s^2,\ sq = q^2 \rangle$.

(4) $\langle p,q,r,s| \ \ pr = s^2,\ rp = q^2,\ qr = rs,\ rq = qp,\ ps = sr,\ sp = pq,\ qs = r^2,\ sq = p^2 \rangle$.

(5) $\langle p,q,r,s| \ \ pr = p^2,\ rp = r^2,\ qr = qp,\ rq = rs,\ ps = pq,\ sp = sr,\ qs = q^2,\ sq = s^2 \rangle$.

(6) $\langle p,q,r,s| \ \ pr = q^2,\ rp = s^2,\ qr = pq,\ rq = sr,\ ps = qp,\ sp = rs,\ qs = p^2,\ sq = r^2 \rangle$.

This Chapter is devoted to proving this Theorem. Firstly, we classify all balanced semigroups which are disjoint unions of two copies of the free semigroup of rank two, and then we find - after some long proofs - that there are just six such balanced semigroups. In two of these semigroups S_α is an ideal in S. In two of these semigroups S_α, S_β are left ideals in S and in two of these semigroups S_α, S_β are right ideals in S.

Lemma 6.1.2. *Let $A = \{a, b, c, d\}$ be a set. Let $\Re$ be a set of relations on A^+. Let ρ be the smallest congruence on A^+ generated by $\Re$. Let $\varphi : A \to T$ be a mapping where T is a free semigroup of rank two and let $\psi : A^+ \to T$ be the unique homomorphism determined by φ. If $\rho \subseteq \ker \psi$ then $\langle a/\rho \rangle \cup \langle b/\rho \rangle$ is an infinite subsemigroup of A^+/ρ.*

PROOF. Since $\rho \subseteq \ker \psi$ we have that the mapping $\overline{\psi} : A^+/\rho \to T$ defined by $\overline{\psi}(w/\rho) = \psi(w)$ is a homomorphism by Theorem 1.3.2. Since a homomorphism maps elements of finite order to elements of finite order and there is no element of finite order in T, it follows that $\langle a/\rho \rangle \cup \langle b/\rho \rangle$ must have infinite order in A^+/ρ. □

Lemma 6.1.3. *Suppose that S' is a semigroup which is a disjoint union of two copies of the free semigroup of rank two $\langle a, b | \ \rangle$, $\langle c, d | \ \rangle$ and S' is defined by the presentation*

$$\langle a, b, c, d | \ xy = z^i \rangle$$

where $x, y, z \in \{a, b, c, d\}$, $i \in \mathbb{N}$. Then if S is a semigroup which is a disjoint union of two copies of the free monogenic semigroup and S is satisfied relations in the presentation then $S' \cong S$.

PROOF. Since S is satisfied relations in the presentation then S is a homomorphic image of S' by Proposition 1.6.2. If it is a proper homomorphic image then there is without loss of generality u_1 and $u_2 \in \langle a, b | \ \rangle$ or u and v in $\langle a, b | \ \rangle$ and $\langle c, d | \ \rangle$ respectively such that $u_1 = u_2$ or $u = v$ which contradicts with the fact that there is no element in the free semigroup of rank two is of finite order or contradicts with $\langle a, b | \ \rangle \cap \langle c, d | \ \rangle = \emptyset$. Thus $S' \cong S$. □

6.2 Six balanced semigroups

Theorem 6.2.1. *Let S' be a semigroup which is defined by the presentation*

$$\langle a, b, c, d \mid ac = c^2,\ ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = d^2,\ db = d^2 \rangle.$$

Let S be a semigroup which is a homomorphic image of S'. Then S is a disjoint union of two copies of the free semigroup of rank two, $\langle p, q \mid \ \rangle$ and $\langle r, s \mid \ \rangle$ if and only if $S \cong S'$.

PROOF. ($\Rightarrow$) Since S is a homomorphic image of S', $S \cong S'$ by Lemma 6.1.3.
($\Leftarrow$) Suppose that $S \cong S'$ and we want to show that S' is a disjoint union of two copies of the free semigroup of rank two. Let w be a word in S'. Then, if w does not have any letter from $\langle c, d | \ \rangle$ then w will be a word in $\langle a, b | \ \rangle$ and if w has at

least one letter from $\langle c, d | \ \rangle$ then w will be in $\langle c, d | \ \rangle$, which means that each word in the semigroup S' consists of a's and b's or c's and d's. Thus

$$\langle a, b, c, d \mid ac = c^2,\ ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = d^2,\ db = d^2 \ \rangle$$

$$\{w_i : w_i \text{ word in } a's \text{ and } b's\} \cup \{w_i : w_i \text{ word in } c's \text{ and } d's\} = S_\alpha \cup S_\beta.$$

Secondly, as a result of not having a relation which can be applied to a word from $\{a, b\}^+$, we have $S_\alpha \cap S_\beta = \emptyset$ and S_α is free.

Thirdly, we show that S_β is free as well. Let φ be a mapping from $\{a, b, c, d\}$ to the free semigroup T generated by f and g with $\varphi(a) = f,\ \varphi(b) = g,\ \varphi(c) = f,\ \varphi(d) = g$ and then there exists a homomorphism $\psi : \{a, b, c, d\}^+ \to T$ since $\{a, b, c, d\}^+$ is the free semigroup on $\{a, b, c, d\}$ and ψ uniquely determined by the images of the generators a, b, c, d by Proposition 1.6.1. Observe that

$$\psi(ac) = \psi(a)\psi(c) = ff = f^2 = \psi(c^2),$$

and similarly with $ca,\ bc,\ cb,\ ad,\ da,\ bd,\ db$, from which we deduce that

$$\Re = \{ac = ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = db = d^2\} \subseteq \ker\psi.$$

Hence, $\rho \subseteq \ker\psi$ where ρ is the smallest congruence generated by $\Re$. Therefore, by Lemma 6.1.2 all elements of semigroup $S_\alpha \sqcup S_\beta$ are distinct as required. □

Corollary 6.2.2. *The semigroup S which is a disjoint union of two copies of the free semigroup of rank two, $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ with*

$$ac = dc,\ ca = cd,\ bc = c^2,\ cb = c^2,\ ad = d^2,\ da = d^2,\ bd = cd,\ db = dc,$$

is isomorphic to the semigroup which is defined in Theorem 6.2.1.

PROOF. The semigroup S_1 which is defined in Theorem 6.2.1 has the relations

$$ac = c^2,\ ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = d^2,\ db = d^2,$$

and the semigroup S_2 which is defined in the Corollary 6.2.2 has the relations

$$ac = dc,\ ca = cd,\ bc = c^2,\ cb = c^2,\ ad = d^2,\ da = d^2,\ bd = cd,\ db = dc.$$

The proof is straightforward by just replacing c by d and d by c in S_1.Thus the semigroup S_1 is isomorphic to the semigroup S_2. $\square$

Theorem 6.2.3. *Let S' be a semigroup which is defined by the presentation*

$$\langle a,b,c,d \mid ac = ca = bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd,\ db = dc\rangle.$$

Let S be a semigroup which is a homomorphic image of S'. Then S is a disjoint union of two copies of the free semigroup of rank two, $\langle p,q \mid\ \rangle$ and $\langle r,s \mid\ \rangle$ if and only if $S \cong S'$.

PROOF. Similarly to the proof of Theorem 6.2.1 we can prove that

$$\langle a,b,c,d \mid ac = ca = bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd,\ db = dc\ \rangle$$

$$= \{w_i : w_i \text{ word in } a's \text{ and } b's\} \cup \{w_i : w_i \text{ word in } c's \text{ and } d's\} = S_\alpha \cup S_\beta$$

and $S_\alpha \cap S_\beta = \emptyset$ and S_α is free. The last step is the same as the last step in the same proof but we define the mapping from $\{a,b,c,d\}$ to the free semigroup T generated by f and g with $\varphi(a) = f,\ \varphi(b) = f,\ \varphi(c) = f,\ \varphi(d) = g$. $\square$

Corollary 6.2.4. *The semigroup S which is a disjoint union of two copies of the free semigroup of rank two, $S_\alpha = \langle a,b \mid\ \rangle$ and $S_\beta = \langle c,d \mid\ \rangle$ with*

$$ac = dc,\ ca = cd,\ bc = dc,\ cb = cd,\ ad = da = bd = db = d^2,$$

is isomorphic to the semigroup which is defined in Theorem 6.2.3.

PROOF. The semigroup S_1 which is defined in Theorem 6.2.3 has the relations

$$ac = ca = bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd,\ db = dc,$$

and the semigroup S_2 which is defined in the Corollary 6.2.4 has the relations

$$ac = dc,\ ca = cd,\ bc = dc,\ cb = cd,\ ad = da = bd = db = d^2.$$

The proof is just by replacing c by d and d by c in S_1. Thus the semigroup S_1 is isomorphic to the semigroup S_2. $\square$

Theorem 6.2.5. *Let S' be a semigroup which is defined by the presentation*

$$\langle a,b,c,d \mid ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2\ \rangle.$$

Let S be a semigroup which is a homomorphic image of S'. Then S is a disjoint union of two copies of the free semigroup of rank two, $\langle p, q \mid \ \rangle$ and $\langle r, s \mid \ \rangle$ if and only if $S \cong S'$.

PROOF. ($\Rightarrow$) Since S is a homomorphic image of S', $S \cong S'$ by Lemma 6.1.3.
($\Leftarrow$) Suppose that $S \cong S'$ and we want to show that S' is a disjoint union of two copies of the free semigroup of rank two. Firstly, we want to show that $x^i y^j = w$ where x, y are in $\{a, b, c, d\}$ and w is a word in S_α or S_β.

x	y	$x^i y^j$
a	a	a^{i+j}
b	a	$b^i a^j$
c	a	$c^i a^j = a^{i+j}$
d	a	$d^i a^j = d^{i-1} baa^{j-1} = d^{i-1} ba^j = \cdots = b^i a^j$
a	b	$a^i b^j$
b	b	b^{i+j}
c	b	$c^i b^j = c^{i-1} abb^{j-1} = \cdots = a^i b^j$
d	b	$d^i b^j = b^{i+j}$
a	c	$a^i c^j = c^{i+j}$
b	c	$b^i c^j = b^{i-1} dcc^{j-1} = \cdots = d^i c^j$
c	c	c^{i+j}
d	c	$d^i c^j$
a	d	$a^i d^j = a^{i-1} cdd^{j-1} = \cdots = c^i d^j$
b	d	$b^i d^j = d^{i+j}$
c	d	$c^i d^j$
d	d	d^{i+j}

Table 6.1: The multiplication of x^i and y^j where $x, y \in \{a, b, c, d\}$ in the presentation $\langle a, b, c, d \mid ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2 \rangle$

Claim: For every word $w \in \{a, b\}^+$, we have $wc = w' \in \{c, d\}^+$ and $wd = w'' \in \{c, d\}^+$.
Proof: We prove the claim by induction on $|w|$. So if $|w| = 1$ then it is obvious. Assume that the statement holds for every $k \leq n$. Thus if $|w| = k + 1$ and $w = ua$ then

$$uac = uc^2 = (uc)c = u'c \in \{c, d\}^+ \quad \text{and} \quad uad = ucd = (uc)d = u''d \in \{c, d\}^+,$$

this is when w ends with a and similarly when w ends with b. Analogously, when $w \in \{c, d\}^+$. So each word of our presentation is entirely in S_α or S_β as required.

Secondly, each word in S_α ends with a or b and each word in S_β ends with c or d. So once you have a word in S_α ends with a then all words which equal to this word will have the same end a by the presentation. Similarly if the word ends with b. Analogously if the word is in S_β and then $S_\alpha \cap S_\beta = \emptyset$.

Thirdly, analogous to the proof of the third part of Theorem 6.2.1, we define the mapping from $\{a, b, c, d\}$ to the free semigroup T generated by f and g with $\varphi(a) = f,\ \varphi(b) = g,\ \varphi(c) = f,\ \varphi(d) = g$ which shows that all elements of $S_\alpha \sqcup S_\beta$ are distinct. $\square$

Corollary 6.2.6. *The semigroup S which is a disjoint union of two copies of the free semigroup of rank two, $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ with*

$$ac = dc,\ ca = ba,\ bc = c^2,\ cb = b^2,\ ad = d^2,\ da = a^2,\ bd = cd,\ db = ab,$$

is isomorphic to the semigroup which is defined in Theorem 6.2.5.

PROOF. The semigroup S_1 which is defined in Theorem 6.2.5 has the relations

$$ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2,$$

and the semigroup S_2 which is defined in the Corollary 6.2.6 has the relations

$$ac = dc,\ ca = ba,\ bc = c^2,\ cb = b^2,\ ad = d^2,\ da = a^2,\ bd = cd,\ db = ab.$$

Now just by replacing c by d and d by c in S_1 we have that the semigroup S_1 is isomorphic to the semigroup S_2. $\square$

Theorem 6.2.7. *Let S' be a semigroup which is defined by the presentation*

$$\langle a, b, c, d \mid ac = d^2,\ ca = b^2,\ bc = cd,\ cb = ba,\ ad = dc,\ da = ab,\ bd = c^2,\ db = a^2 \rangle.$$

Let S be a semigroup which is a homomorphic image of S'. Then S is a disjoint union of two copies of the free semigroup of rank two, $\langle p, q \mid \ \rangle$ and $\langle r, s \mid \ \rangle$ if and only if $S \cong S'$.

PROOF. ($\Rightarrow$) Since S is a homomorphic image of S', $S \cong S'$ by Lemma 6.1.3.
($\Leftarrow$) Suppose that $S \cong S'$ and we want to show that S' is a disjoint union of two copies of the free semigroup of rank two. Firstly, we want to show that $x^i y^j = w$ where x, y are in $\{a, b, c, d\}$ and w is a word in S_α or S_β.

x	y	$x^i y^j$
a	a	a^{i+j}
b	a	$b^i a^j$
c	a	$c^i a^j = c^{i-1}b^2a^{j-1} = c^{i-2}baba^{j-1} = \cdots = ba^{i-1}ba^{j-1} = w \in S_\alpha$
d	a	$d^i a^j = d^{i-1}aba^{j-1} = d^{i-2}abba^{j-1} = \cdots = ab^i a^{j-1} = w \in S_\alpha$
a	b	$a^i b^j$
b	b	b^{i+j}
c	b	$c^i b^j = c^{i-1}bab^{j-1} = c^{i-2}baab^{j-1} = \cdots = ba^i b^{j-1} = w \in S_\alpha$
d	b	$d^i b^j = d^{i-1}a^2b^{j-1} = d^{i-2}abab^{j-1} = \cdots = ab^{i-1}ab^{j-1} = w \in S_\alpha$
a	c	$a^i c^j = a^{i-1}d^2c^{j-1} = a^{i-2}dcdc^{j-1} = \cdots = dc^{i-1}dc^{j-1} = w \in S_\beta$
b	c	$b^i c^j = b^{i-1}cdc^{j-1} = b^{i-2}cddc^{j-1} = \cdots = cd^i c^{j-1} = w \in S_\beta$
c	c	c^{i+j}
d	c	$d^i c^j$
a	d	$a^i d^j = a^{i-1}dcd^{j-1} = a^{i-2}dccd^{j-1} = \cdots = dc^i d^{j-1} = w \in S_\beta$
b	d	$b^i d^j = b^{i-1}c^2d^{j-1} = b^{i-2}cdcd^{j-1} = \cdots = cd^{i-1}cd^{j-1} = w \in S_\beta$
c	d	$c^i d^j$
d	d	d^{i+j}

Table 6.2: The multiplication of x^i and y^j where $x, y \in \{a, b, c, d\}$ in the presentation $\langle a, b, c, d \mid ac = d^2,\ ca = b^2,\ bc = cd,\ cb = ba,\ ad = dc,\ da = ab,\ bd = c^2,\ db = a^2 \rangle$

Claim: For every word $w \in \{a, b\}^+$, we have $wc = w' \in \{c, d\}^+$ and $wd = w'' \in \{c, d\}^+$.

Proof: We prove the claim by induction on $|w|$. So if $|w| = 1$ then it is obvious. Assume that the statement holds for every $k \leq n$. Thus if $|w| = k + 1$ and $w = ua$ then

$$uac = ud^2 = (ud)d = u'd \in \{c, d\}^+ \text{ and } uad = udc = (ud)c = u''c \in \{c, d\}^+,$$

this is when w ends with a and similarly when w ends with b. Analogously when $w \in \{c, d\}^+$. So each word of our presentation is entirely in S_α or S_β as required.

Secondly, each word in S_α ends with a or b and each word in S_β ends with c or d as we have explained this previously, and then $S_\alpha \cap S_\beta = \emptyset$.

Now we want to show that all elements of $S_\alpha \sqcup S_\beta$ are distinct. In this semigroup we could not follow the same technique of the previous theorems.

So we let $W = \{a,b\}^+ \cup \{c,d\}^+$, define mappings $\tau_a, \tau_b, \tau_c, \tau_d$ on W as follows:

$$\tau_a(w) = \begin{cases} aw & \text{if } w \in \{a,b\}^+; \\ d^2w' & \text{if } w = cw' \in \{c,d\}^+; \\ dcw' & \text{if } w = dw' \in \{c,d\}^+. \end{cases}, \quad \tau_b(w) = \begin{cases} bw & \text{if } w \in \{a,b\}^+; \\ cdw' & \text{if } w = cw' \in \{c,d\}^+; \\ c^2w' & \text{if } w = dw' \in \{c,d\}^+. \end{cases}$$

$$\tau_c(w) = \begin{cases} b^2w' & \text{if } w = aw' \in \{a,b\}^+; \\ baw' & \text{if } w = bw' \in \{a,b\}^+; \\ cw & \text{if } w \in \{c,d\}^+. \end{cases}, \quad \tau_d(w) = \begin{cases} abw' & \text{if } w = aw' \in \{a,b\}^+; \\ a^2w' & \text{if } w = bw' \in \{a,b\}^+; \\ dw & \text{if } w \in \{c,d\}^+. \end{cases}$$

We will abbreviate the above information by means of arrays:

$$\begin{pmatrix} aw \\ d^2w' \\ dcw' \end{pmatrix}, \begin{pmatrix} bw \\ cdw' \\ c^2w' \end{pmatrix}, \begin{pmatrix} b^2w' \\ baw' \\ cw \end{pmatrix}, \begin{pmatrix} abw' \\ a^2w' \\ dw \end{pmatrix}.$$

Now we show that all the compositions of all these mappings satisfy our relations in the presentation. So we have to check eight relations,
$\tau_a\tau_c = \tau_d\tau_d$, $\tau_c\tau_a = \tau_b\tau_b$, $\tau_b\tau_c = \tau_c\tau_d$, $\tau_c\tau_b = \tau_b\tau_a$, $\tau_a\tau_d = \tau_d\tau_c$, $\tau_d\tau_a = \tau_a\tau_b$, $\tau_b\tau_d = \tau_c\tau_c$, $\tau_d\tau_b = \tau_a\tau_a$. I explain the first relation in detail and the same with remaining relations.

$$\tau_a\tau_c(w) = \tau_a \begin{pmatrix} b^2w' \\ baw' \\ cw \end{pmatrix}.$$

Then by the definition of τ_a we replace each $w \in \{a,b\}^+$ by aw and each $w \in \{c,d\}^+$ starts with c by d^2w' or starts with d by dcw' as

$$\tau_a(b^2w') = ab^2w', \ \tau_a(baw') = abaw', \ \tau_a(cw) = d^2w,$$

and then we put these values in an array as $\begin{pmatrix} ab^2w' \\ abaw' \\ d^2w \end{pmatrix}$. Similarly

$$\tau_d\tau_d(w) = \tau_d \begin{pmatrix} abw' \\ a^2w' \\ dw \end{pmatrix} = \begin{pmatrix} ab^2w' \\ abaw' \\ d^2w \end{pmatrix}.$$

$$\tau_c\tau_a(w) = \tau_c \begin{pmatrix} aw \\ d^2w' \\ dcw' \end{pmatrix} = \begin{pmatrix} b^2w \\ cd^2w' \\ cdcw' \end{pmatrix} = \tau_b \begin{pmatrix} bw \\ cdw' \\ c^2w' \end{pmatrix} = \tau_b\tau_b(w).$$

$$\tau_b\tau_c(w) = \tau_b \begin{pmatrix} b^2w' \\ baw' \\ cw \end{pmatrix} = \begin{pmatrix} b^3w' \\ b^2aw' \\ cdw \end{pmatrix} = \tau_c \begin{pmatrix} abw' \\ a^2w' \\ dw \end{pmatrix} = \tau_c\tau_d(w).$$

$$\tau_c\tau_b(w) = \tau_c \begin{pmatrix} bw \\ cdw' \\ c^2w' \end{pmatrix} = \begin{pmatrix} baw \\ c^2dw' \\ c^3w' \end{pmatrix} = \tau_b \begin{pmatrix} aw \\ d^2w' \\ dcw' \end{pmatrix} = \tau_b\tau_a(w).$$

$$\tau_a\tau_d(w) = \tau_a \begin{pmatrix} abw' \\ a^2w' \\ dw \end{pmatrix} = \begin{pmatrix} a^2bw' \\ a^3w' \\ dcw \end{pmatrix} = \tau_d \begin{pmatrix} b^2w' \\ baw' \\ cw \end{pmatrix} = \tau_d\tau_c(w).$$

$$\tau_d\tau_a(w) = \tau_d \begin{pmatrix} aw \\ d^2w' \\ dcw' \end{pmatrix} = \begin{pmatrix} abw \\ d^3w' \\ d^2cw' \end{pmatrix} = \tau_a \begin{pmatrix} bw \\ cdw' \\ c^2w' \end{pmatrix} = \tau_a\tau_b(w).$$

$$\tau_b\tau_d(w) = \tau_b \begin{pmatrix} abw' \\ a^2w' \\ dw \end{pmatrix} = \begin{pmatrix} babw' \\ ba^2w' \\ c^2w \end{pmatrix} = \tau_c \begin{pmatrix} b^2w' \\ baw' \\ cw \end{pmatrix} = \tau_c\tau_c(w).$$

$$\tau_d\tau_b(w) = \tau_d \begin{pmatrix} bw \\ cdw' \\ c^2w' \end{pmatrix} = \begin{pmatrix} a^2w \\ dcdw' \\ dc^2w' \end{pmatrix} = \tau_a \begin{pmatrix} aw \\ d^2w' \\ dcw' \end{pmatrix} = \tau_a\tau_a(w).$$

Therefore we can define a homomorphism ϕ from S to the semigroup $\langle \tau_a, \tau_b, \tau_c, \tau_d \rangle$ as $\phi(a) = \tau_a$, $\phi(b) = \tau_b$, $\phi(c) = \tau_c$, $\phi(d) = \tau_d$ and hence if $u, v, w \in \{a, b\}^+$ then $\tau_u(w) = uw$ and $\tau_v(w) = vw$. Thus $\tau_u(w) \neq \tau_v(w)$ which implies $\tau_u \neq \tau_v$. Therefore $u \neq v$ for every $u, v \in \{a, b\}^+$. Similarly for every $u, v \in \{c, d\}^+$.

Corollary 6.2.8. *The semigroup S which is a disjoint union of two copies of the free semigroup of rank two, $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ with*

$$ac = cd,\ ca = ab,\ bc = d^2,\ cb = a^2,\ ad = c^2,\ da = b^2,\ bd = dc,\ db = ba,$$

is isomorphic to the semigroup which is defined in Theorem 6.2.7.

PROOF. The semigroup S_1 which is defined in Theorem 6.2.7 has the relations

$$ac = d^2,\ ca = b^2,\ bc = cd,\ cb = ba,\ ad = dc,\ da = ab,\ bd = c^2,\ db = a^2,$$

and the semigroup S_2 which is defined in the Corollary 6.2.8 has the relations

$$ac = cd,\ ca = ab,\ bc = d^2,\ cb = a^2,\ ad = c^2,\ da = b^2,\ bd = dc,\ db = ba.$$

Thus by replacing c by d and d by c in S_1 the semigroup S_1 is isomorphic to the semigroup S_2. □

Notice that in the two Theorems (6.2.5, 6.2.7) , S_α and S_β are left ideals. In the two following theorems S_α and S_β are right ideals.

Theorem 6.2.9. *Let S' be a semigroup which is defined by the presentation*

$$\langle a, b, c, d \mid ac = a^2,\ ca = c^2,\ bc = ba,\ cb = cd,\ ad = ab,\ da = dc,\ bd = b^2,\ db = d^2 \rangle.$$

Let S be a semigroup which is a homomorphic image of S'. Then S is a disjoint union of two copies of the free semigroup of rank two, $\langle p, q \mid \ \rangle$ and $\langle r, s \mid \ \rangle$ if and only if $S \cong S'$.

PROOF. The proof is similar to the proof of Theorem 6.2.5.

Corollary 6.2.10. *The semigroup S which is a disjoint union of two copies of the free semigroup of rank two, $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ with*

$$ac = ab,\ ca = cd,\ bc = b^2,\ cb = c^2,\ ad = a^2,\ da = d^2,\ bd = ba,\ db = dc,$$

is isomorphic to the semigroup which is defined in Theorem 6.2.9.

PROOF. Analogously to the proof of Corollary 6.2.6.

Theorem 6.2.11. *Let S' be a semigroup which is defined by the presentation*

$$\langle a, b, c, d \mid ac = b^2,\ ca = d^2,\ bc = ab,\ cb = dc,\ ad = ba,\ da = cd,\ bd = a^2,\ db = c^2 \rangle.$$

Let S be a semigroup which is a homomorphic image of S'. Then S is a disjoint union of two copies of the free semigroup of rank two, $\langle p, q \mid \ \rangle$ and $\langle r, s \mid \ \rangle$ if and only if $S \cong S'$.

PROOF. Similarly to the proof of Theorem 6.2.7.

Corollary 6.2.12. *The semigroup S which is a disjoint union of two copies of the free semigroup of rank two, $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ with*

$$ac = ba,\ ca = dc,\ bc = a^2,\ cb = d^2,\ ad = b^2,\ da = c^2,\ bd = ab,\ db = cd,$$

is isomorphic to the semigroup which is defined in Theorem 6.2.11.

PROOF. Analogously to the proof of Corollary 6.2.8.

6.3 Preliminary, technical results

Lemma 6.3.1. *Suppose that S is a balanced semigroup generated by $\{a, b, c, d\}$. If we have either of the following:*

i) $xy = y'y,\ x'y' = x'x$; or

ii) $xy = y'y,\ x'y' = x^2,\ xy' = x'x$;

for some $x, x' \in \{a, b\}$ and $y, y' \in \{c, d\}$ where $(x, y) \neq (x', y')$, then S is not a disjoint union of $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$.

PROOF. Notice that the only words equal to the word $x'y'y$ are $x'xy,\ x^2y,\ xy'y$. Neither of them belongs to $S_\alpha \sqcup S_\beta$. □

Now we introduce the basic theorem in this chapter which enables us to classify the balanced semigroups under consideration.

Theorem 6.3.2. *Suppose that $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ are two copies of the free semigroup of rank two. Suppose that S is a balanced semigroup which is a disjoint union of S_α and S_β. Then*

i) If $ac = c^2$ then $ca = a^2$ or c^2 and the two products $bc,\ ad$ belong to S_β.

ii) If $ac = dc$ then $ca = ba$ or cd and the two products $bc,\ ad$ belong to S_β.

iii) If $ac = d^2$ then $ca = b^2$ and the two products $bc,\ ad$ belong to S_β.

iv) If $ac = cd$ then $ca = ab$ and the two products $bc,\ ad$ belong to S_β.

PROOF. Before starting the proof notice that when we say S is not a disjoint union of S_α and S_β we mean this by using Lemma 6.3.1. We firstly divide (i) into three parts and prove each part separately.

Part 1: If $ac = c^2$ then $ca = a^2$ or c^2.

Suppose that $ac = c^2$. Then $ca \in \{ab, ba, b^2, a^2, c^2\}$ because $c(ac) = c^3$ and by associativity, $(ca)c$ must be c^3, which means that $ca \notin \{d^2, dc, cd\}$. So, $ca \in \{ab, ba, b^2, a^2, c^2\}$.

Case 1: $ca = ab$. Then $a(ca) = aab = a^2b$ and $(ac)a = c^2a = cab = ab^2$. Thus $a(ca) \neq (ac)a$ and then $ca \neq ab$.

Case 2: $ca = ba$. Then $(ca)c = bac = bc^2$ and then $bc \notin \{d^2, cd, dc\}$ by the fact that $c(ac) = c^3$. Thus $bc \in \{b^2, ab, ba, a^2, c^2\}$.

Subcase 2a. $bc = b^2$. Then $b^2c = b^3$ and then $bc^2 = b^3$, a contradiction.

Subcase 2b. $bc = ab$. Then $bc^2 = abc = a^2b$, a contradiction.

Subcase 2c. $bc = ba$. Then $bac = bc^2 = bac$. Hence S is not a disjoint union of S_α and S_β.

Subcase 2d. $bc = a^2$. Then $b(ca) = b^2a$ and $(bc)a = a^2a = a^3$, a contradiction.

Subcase 2e. $bc = c^2$. Then we have $b(ca) = b^2a$ and $(bc)a = c^2a = (cb)a$. To get that $b(ca) = (bc)a$, $cb \notin \{a^2, ab, ba\}$ and in addition $cb \neq c^2$ because if this happened we get $(cb)a = c^2a = cba = c^2a$ which implies that S is not a disjoint union of S_α and S_β, a contradiction. That means $cb \in \{b^2, d^2, cd, dc\}$ but this impossible because if $cb = b^2$ then $a(ca) = aba$ and $(ac)a = c^2a = cba = b^2a$. So, $a(ca) \neq (ac)a$. Similarly, if $cb \in \{d^2, cd, dc\}$ then $c(bc) = c^3$ and $(cb)c = d^2c, cdc, dc^2$ respectively, which implies that $c(bc) \neq (cb)c$.

Case 3: $ca = b^2$. Then $a(ca) = ab^2$ and $(ac)a = c^2a = cb^2 = (cb)b$. Therefore, $cb \notin \{a^2, b^2, ba, c^2, dc\}$. There remains three possible values for cb and they are $\{ab, d^2, cd\}$.

Subcase 3a. $cb = ab$. Then $a(cb) = a^2b$ and $(ac)b = c^2b = cab = b^3$ and then $a(cb) \neq (ac)b$.

Subcase 3b. $cb \in \{d^2, cd\}$. Then $c(ac) = c^3$ and $(ca)c = b^2c = b(bc)$. Hence, $bc \notin \{a^2, b^2, ab, ba, cd\}$ because $b(bc) \in \{ba^2, b^3, bab, b^2a, cd^2\}$ respectively. That implies $bc \in \{dc, d^2, c^2\}$ and then $c(bc) \in \{cdc, cd^2, c^3\}$ respectively but we know that $cb \in \{d^2, cd\}$ and then $(cb)c \in \{d^2c, cdc\}$ respectively. Therefore, $cb = cd$, $bc = dc$. Then $a(ca) = ab^2$ and $(ac)a = c^2a = cb^2 = cdb$ which implies that $db \notin \{a^2, ab, c^2, d^2, cd, dc\}$ because $cdb \in \{b^2a, b^3, c^3, cd^2, c^2d, cdc\}$ respectively, a contradiction with $a(ca) = ab^2$. In addition, if $db = b^2$ then $cdb = cb^2 = cdb$ which means that S is not a disjoint union of S_α and S_β, a contradiction. The last potential value for db is ba but even this, is impossible because $b(ca) = b^3$ and $(bc)a = dca = db^2 = bab$ it follows that $b(ca) \neq (bc)a$. Finally, $ca \notin \{b^2, ab, ba, d^2, dc, cd\}$. Therefore, $ca = a^2$ or $ca = c^2$.

Part 2: If $ac = c^2$ then $bc \in S_\beta$ in the two cases.
Case 1: $ac = c^2$, $ca = c^2$. Suppose that bc is a word in S_α and then we get a contradiction by the following:

$$bc = a^2 \Longrightarrow b(ca) = bc^2 = a^2c = ac^2 = c^3,\ (bc)a = a^3,$$

$$bc = b^2 \Longrightarrow b(ca) = bc^2 = b^2c = b^3,\ (bc)a = b^2a,$$

$$bc = ab \Longrightarrow b(ca) = bc^2 = abc = a^2b,\ (bc)a = aba,$$

$$bc = ba \Longrightarrow b(ca) = bc^2 = bac = bc^2 = bac,\ (bc)a = ba^2.$$

Thus $bc \in S_\beta$.
Case 2: $ac = c^2$, $ca = a^2$. Suppose that bc is a word in S_α then $b(ca) = ba^2$ and by the associativity on $(bc)a$, $bc = ba$ but $b(ac) = bc^2 = (ba)c$ and so S is not a disjoint union of S_α and S_β. It follows that $bc \notin S_\alpha$ and hence $bc \in S_\beta$.

Part 3: If $ac = c^2$ then $ad \in S_\beta$ in the two cases.
Case 1: $ac = c^2$, $ca = c^2$. Suppose that $ad \in S_\alpha$. We have already proved that $bc \in S_\beta$ and by symmetry, the word cb is in S_β as well. So that means $bc = w_{c,d}$, $cb = w'_{c,d}$ and then $c(bc) = cw_{c,d}$, $(cb)c = w'_{c,d}c$. Thus, $w_{c,d}$ ends with the letter c and $w'_{c,d}$ starts with the letter c and hence we have two possibilities here, $bc = cb = c^2$ or $bc = dc$, $cb = cd$. However, in the first possibility we have $(ca)d = c^2d$ and $c(ad) = c^3$ for every $ad \in \{a^2, b^2, ab, ba\}$, a contradiction. In the second possibility we have $(ca)d = c^2d$ and $c(ad) = c^3$ where $ad = a^2$, a contradiction. Hence, if $ad = ab$ then we have $adc = abc = adc$ which implies that S is not a disjoint union of S_α and S_β. Therefore, $ad \in \{b^2, ba\}$ and then $(ad)a \in \{b^2a, ba^2\}$ respectively. Thus, by the associativity on $a(da)$, we have $da \notin \{a^2, b^2, ab, ba, c^2, cd\}$, that means da must start with d. So $da \in \{d^2, dc\}$. At this stage we have so far the following relations:

$$ac = ca = c^2,\ bc = dc,\ cb = cd,\ ad \in \{ba, b^2\}, da \in \{d^2, dc\}.$$

Subcase 1a. $ad = ba, da = d^2$. Then $(ad)a = ba^2$, $a(da) = ad^2 = bad = b^2a$, a contradiction.
Subcase 1b. $ad = ba, da = dc$. Then $(ad)a = ba^2$, $a(da) = adc = bac = bc^2 = dc^2$, a contradiction. Thus, $ad \neq ba$.

Subcase 1c. $ad = b^2, da = d^2$. Then $(ad)a = b^2a, a(da) = ad^2 = b^2d$ and $bd = ba$ because if $bd \in \{a^2, b^2, ab, c^2, cd, d^2, dc\}$ then $(ad)a \neq a(da)$, but $(bd)a = ba^2, b(da) = bd^2 = bad = b^3$, a contradiction.
Subcase 1d. $ad = b^2, da = dc$. Then $(ad)b = b^3$ and by the associativity on $a(db)$, we have $db \notin S_\alpha \cup \{c^2, cd\}$ and hence $db \in \{d^2, dc\}$. If $db = d^2$ then by Theorem 6.3.2(i), $bd = b^2$ or d^2 but if $bd = b^2$ then $b^2c = bdc = b^2c$ then S is not a disjoint union of S_α and S_β and if $bd = d^2$ then $a(db) = ad^2 = b^2d = bd^2 = d^3$, $(ad)b = b^3$, a contradiction in the both cases. Therefore, the last potential possible is $db = dc$ and then we have $(ad)b = b^3, a(db) = adc = b^2c = bdc$, that means $bd \notin S_\beta$ and

$$bd = a^2 \Longrightarrow bdc = a^2c = ac^2 = c^3,$$

$$bd = b^2 \Longrightarrow bdc = b^2c = bdc = b^2c,$$

$$bd = ab \Longrightarrow bdc = abc = adc = b^2c = bdc,$$

$$bd = ba \Longrightarrow bdc = bac = bc^2 = dc^2,$$

a contradiction in each case. Thus the word $ad \in S_\beta$ as required.

Case 2: $ac = c^2, ca = a^2$. By the same technique, we start by assuming that $ad \in S_\alpha$, which means $ad \in \{a^2, ba, b^2, ab\}$ and then by Theorem 6.3.2(i) and symmetry, $bc \in S_\beta$ and $cb \in S_\alpha$.
Subcase 2a. $ad = a^2$, it follows that $da = a^2$ or d^2 by Theorem 6.3.2(i). Hence, $d(ac) = dc^2, (da)c \in \{c^3, d^2c\}$ respectively, a contradiction. Therefore, $ad \neq a^2$.
Subcase 2b. $ad = ba$. Then $(ca)d = a^2d = aba$, $c(ad) = cba$ and thus $cb = ab$ where $cb \in S_\alpha$. So,

$$bc = c^2 \Longrightarrow b(cb) = bab, \ (bc)b = c^2b = cab = a^2b,$$

$$bc = d^2 \Longrightarrow c(bc) = cd^2, \ (cb)c = abc = ad^2 = bad = b^2a,$$

$$bc = dc \Longrightarrow c(bc) = cdc, \ (cb)c = abc = adc = bac = bc^2 = dc^2,$$

a contradiction in each case and also we have

$$bc = cd \Longrightarrow b(cb) = bab, \ (bc)b = cdb \Longrightarrow db \notin S_\beta \Longrightarrow db \in S_\alpha,$$

but $(ad)b = bab$ and by $a(db)$ we must have $db \notin S_\alpha$, a contradiction. Thus $ad \neq ba$.

Subcase 2c. $ad = b^2$. Then $(ca)d = a^2d = ab^2,\ c(ad) = cb^2 = (cb)b \Longrightarrow cb = ab$.

$$bc = c^2 \Longrightarrow b(cb) = bab,\ (bc)b = c^2b = cab = a^2b,$$

a contradiction.

$$bc = cd \Longrightarrow b(cb) = bab,\ (bc)b = cdb \Longrightarrow db \in S_\alpha,$$

but $(ad)b = b^3$ and by $a(db)$ we must have $db \notin S_\alpha$, a contradiction.

Also if $bc = dc$ then $(ad)a = b^2a$ but by $a(da)$ we have $da \in S_\beta$ and since $d(ac) = dc^2$, and by $(da)c$, we have $da = dc$. In addition

$$(ad)a = b^2a,\ a(da) = adc = b^2c = bdc \Longrightarrow bd \in S_\alpha.$$

So

$$bd = a^2 \Longrightarrow bdc = a^2c = ac^2 = c^3 \Longrightarrow (ad)a \neq a(da),$$

$$bd = b^2 \Longrightarrow bdc = b^2c = bdc \Longrightarrow S \text{ is not a disjoint union of } S_\alpha \text{ and } S_\beta,$$

$$bd = ab \Longrightarrow bdc = abc = adc = b^2c = bdc \Longrightarrow S \text{ is not a disjoint union of } S_\alpha \text{ and } S_\beta,$$

$$bd = ba \Longrightarrow bdc = bac = bc^2 = dc^2 \Longrightarrow (ad)a \neq a(da),$$

a contradiction.

The last possibility is $bc = d^2$. Then

$$b(cb) = bab,\ (bc)b = d^2b = d(db) \Longrightarrow db \in S_\alpha,$$

but $(ad)b = b^3$ and by $a(db)$ we have $db \notin S_\alpha$, a contradiction. Therefore, $ad \neq b^2$.
Subcase 2d. $ad = ab$. Then

$$ad = ab \Longrightarrow (ad)b = ab^2, \text{ and by } a(db),\ db \in \{b^2\} \cup S_\beta.$$

Then

$$db = c^2 \Longrightarrow a(db) = ac^2 = c^3,$$

$$db = cd \Longrightarrow a(db) = acd = c^2d,$$

a contradiction. Now if $db = d^2$ then by Theorem 6.3.2(i), $bd \in \{b^2, d^2\}$ and then

$(db)c = d^2c$ and by $d(bc)$ we have $bc = dc$. Thus $(ad)b = ab^2$ and $a(db) = ad^2 = abd = ab^2$ or $ad^2 = abd = ad^2$ in which the latter means that S is not a disjoint union of S_α and S_β which implies that $bd = b^2$ and this contradicts with $b^2c = bdc = b^2c$. Therefore, $db \neq d^2$.

If $db = dc$ then we have $(db)c = dc^2$ and by $d(bc)$ we have $bc = c^2$ but $(ad)b = ab^2$ and $a(db) = adc = abc = ac^2 = c^3$, a contradiction. Hence, $bd \neq dc$.

If $db = b^2$ then by Theorem 6.3.2(i), $bd \in \{b^2, d^2\}$ but the possibility of $bd = d^2$ is rejected because $a(bd) = ad^2 = abd$ and then S is not a disjoint union of S_α and S_β. Thus $bd = b^2$. However, $d(bc) = dw_{c,d}$ and $(db)c = b^2c = bw_{c,d}$. Thus,

$$bc = c^2 \Longrightarrow b^2c = bc^2 = c^3,$$

$$bc = cd \Longrightarrow b^2c = bcd = cd^2,$$

$$bc = d^2 \Longrightarrow b^2c = bd^2 = b^2d = b^3,$$

$$bc = dc \Longrightarrow b^2c = bdc = b^2c,$$

a contradiction in each case. Therefore, $db \neq b^2$ and hence $ad \neq ab$. So $ad \in S_\beta$. ii, iii, iv are proved analogously. □

6.4 Classification of balanced semigroups

Corollary 6.4.1. *Suppose that S_α, S_β are two copies of the free semigroup of rank two. For every balanced semigroup S, which is a disjoint union of S_α and S_β, one of the following must hold:*

$i)$ S_β is an ideal in S.

$ii)$ S_α and S_β are left ideals.

$iii)$ S_α and S_β are right ideals.

PROOF. Directly by Theorem 6.3.2.

Theorem 6.4.2. *Up to isomorphism, every balanced semigroup S is a disjoint union of two copies of the free semigroup of rank two $S_\alpha = \langle a, b \mid \ \rangle$ and $S_\beta = \langle c, d \mid \ \rangle$ if and only if S is isomorphic to the semigroup which is defined by one of the following presentations:*

(1) $\langle p,q,r,s| \quad pr = r^2,\ rp = r^2,\ qr = sr,\ rq = rs,\ ps = rs,\ sp = sr,\ qs = s^2,\ sq = s^2 \rangle.$

(2) $\langle p,q,r,s| \quad pr = r^2,\ rp = r^2,\ qr = r^2,\ rq = r^2,\ ps = rs,\ sp = sr,\ qs = rs,\ sq = sr \rangle.$

(3) $\langle p,q,r,s| \quad pr = r^2,\ rp = p^2,\ qr = sr,\ rq = pq,\ ps = rs,\ sp = qp,\ qs = s^2,\ sq = q^2 \rangle.$

(4) $\langle p,q,r,s| \quad pr = s^2,\ rp = q^2,\ qr = rs,\ rq = qp,\ ps = sr,\ sp = pq,\ qs = r^2,\ sq = p^2 \rangle.$

(5) $\langle p,q,r,s| \quad pr = p^2,\ rp = r^2,\ qr = qp,\ rq = rs,\ ps = pq,\ sp = sr,\ qs = q^2,\ sq = s^2 \rangle.$

(6) $\langle p,q,r,s| \quad pr = q^2,\ rp = s^2,\ qr = pq,\ rq = sr,\ ps = qp,\ sp = rs,\ qs = p^2,\ sq = r^2 \rangle.$

PROOF. ($\Rightarrow$) Let S be a semigroup which is a disjoint union of S_α, S_β. Firstly, by Corollary 6.4.1 we start with the ideal case. So let S_β be an ideal in S. If $ac = c^2$ then $ca = c^2$ and then all the words bc, cb, ad, da, bd, db are in S_β by Theorem 6.3.2. Then $c(ad) = cw_{ad}$, $(ca)d = c^2d$ and that implies $w_{ad} = ad = cd$ by associativity. Hence $da = dc$ by Theorem 6.3.2. Also, $d(bc) = dw_{bc}$ and $(db)c = w_{db}c$, which means that the word w_{bc} ends with the letter c and the word w_{db} starts with the letter d. So there are two possibilities, $w_{bc} = c^2$ or dc and $w_{db} = d^2$ or dc and then if $w_{bc} = c^2$ that gives us $w_{db} = dc$ by associativity, and if $w_{bc} = dc$ that implies $w_{db} = d^2$ and after that we can continue to get cb and bd. So when $w_{bc} = c^2$ that means $cb = c^2$ and $bd = cd$ and when $w_{bc} = dc$ that means $cb = cd$ and $bd = d^2$ by Theorem 6.3.2. So we have two types of semigroups when S_β is an ideal. The first type is the semigroup S with relations

$$ac = ca = c^2,\ bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd,\ db = dc,$$

and this type of S is defined by the presentation

$$\langle a,b,c,d\ |ac = ca = c^2,\ bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd,\ db = dc\rangle$$

by Theorem 6.2.3 . The second type is the semigroup S with the relations

$$ac = ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = d^2,\ db = d^2,$$

and this type is defined by the presentation

$$\langle a,b,c,d\ | ac = ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = db = d^2 \rangle$$

by Theorem 6.2.1 . If $ac = dc$ then see Remark 6.4.3 below. This proves (1) and (2).

Secondly, the left ideal case. So let S_α and S_β be left ideals in S. If $ac = c^2$ then $ca = a^2$ and the words $bc,\ ad,\ bd$ are in S_β, $cb,\ da,\ db$ are in S_α by Theorem 6.3.2. Now by $b(ca) = ba^2$, $(bc)a = w_{bc}a$, w_{bc} must be equal to one of $cd,\ dc$, or d^2. But if $bc = cd$ then $cb = ba$ by Theorem 6.3.2, and this contradicts with the associativity on $c(bc)$. Similarly if $bc = d^2$ then $cb = a^2$ and then $c(bc) \neq (cb)c$. Thus bc must be equal to dc and hence $cb = ab$ by Theorem 6.3.2, and then $da = ba$ because $b(cb) = bab,\ (bc)b = dcb = dab$ and as a result of $da \in S_\alpha,\ da = ba$. So $ad = cd$ by Theorem 6.3.2. Finally, $d(bc) = d^2c,\ (db)c = w_{db}c$ and $db \in S_\alpha$. So $db \in \{ab, ba, a^2, b^2\}$ but if $db \in \{ab, ba, a^2\}$ then we have $d(bc) \neq (db)c$. Thus $db = b^2$ and that implies $bd = d^2$ by Theorem 6.3.2. Therefore we have the semigroup which is defined by the presentation

$$\langle a,b,c,d\ |\ ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2\ \rangle$$

by Theorem 6.2.5. This proves (3).

If $ac = cd$ then $ca = ab$ and the words $bc,\ ad,\ bd\ \in S_\beta,\ cb,\ da,\ db \in S_\alpha$ by Theorem 6.3.2. Now by $d(ac) = dcd,\ (da)c = w_{da}c,\ da \in \{ab, ba, b^2\}$. So we study each case separately. Starting with $da = ab$ and then $ad = dc$ by Theorem 6.3.2, but $a(da) = a^2b,\ (ad)a = dca = dab = ab^2$ and so $a(da) \neq (ad)a$. And if $da = ba$ then $ad = cd$ by Theorem 6.3.2, but $a(da) = aba,\ (ad)a = cda = cba$ and then $cb = ab$ and that implies $bc = dc$ by Theorem 6.3.2, but by $a(ca) = a^2b,\ (ac)a = cda = cba = aba$. Then $a(ca) \neq (ac)a$. Therefore, it just remains for us $da = b^2$ and then $ad = c^2$ by Theorem 6.3.2. So $c(ad) = c^3,\ (ca)d = abd$ and bd is a word in S_β. If $bd = c^2$ that implies $(ca)d = abd = ac^2 = cdc$ which contradicts with $c(ad)$. If $bd = d^2$ then $(ca)d = abd = ad^2 = c^2d$ which contradicts with $c(ad) = c^3$ as well. Similarly, if $bd = cd$ then $(ca)d = abd = acd = cd^2 \neq c(ad)$. Thus bd must be equal to dc and then $db = ba$ by Theorem 6.3.2, and by $d(bd) = d^2c,\ (db)d = bad = bc^2$ but bc is in S_β. So $bc = d^2$ and then $cb = a^2$ by Theorem 6.3.2. Hence we have the semigroup which is defined by the presentation

$$\langle a,b,c,d\ |\ ac = cd,\ ca = ab,\ bc = d^2,\ cb = a^2,\ ad = c^2,\ da = b^2,\ bd = dc,\ db = ba\ \rangle,$$

and this semigroup is isomorphic to the semigroup which is defined by the presentation

$$\langle a,b,c,d \mid ac = d^2,\ ca = b^2,\ bc = cd,\ cb = ba,\ ad = dc,\ da = ab,\ bd = c^2,\ db = a^2 \rangle$$

by Corollary 6.2.8. This proves (4). Similarly if S_α and S_β are right ideals in S, which proves (5), (6).

($\Leftarrow$) Let S' be a semigroup which is defined by one of the presentations in the theorem. Then S' is a disjoint union of two copies of the free semigroup of rank two by Theorems 6.2.1, 6.2.3, 6.2.5, 6.2.7, 6.2.9 and 6.2.11. Thus $S' \cong S$. □.

Remark 6.4.3. Analogously,

i) If $ac = dc$ in the ideal case then we can construct two semigroups one of which is defined by the presentation

$$\langle a,b,c,d \mid ac = dc,\ ca = cd,\ bc = dc,\ cb = cd,\ ad = d^2,\ da = d^2,\ bd = d^2,\ db = d^2 \rangle,$$

which is isomorphic to the semigroup which defined by the presentation

$$\langle a,b,c,d \mid ac = ca = bc = cb = c^2,\ ad = cd,\ da = dc,\ bd = cd,\ db = dc \rangle$$

by Corollary 6.2.4. The other semigroup is the semigroup which is defined by the presentation

$$\langle a,b,c,d \mid ac = dc,\ ca = cd,\ bc = cb = c^2,\ ad = da = d^2,\ bd = cd,\ db = dc \rangle,$$

which is isomorphic to the semigroup which is defined by the presentation

$$\langle a,b,c,d \mid ac = ca = c^2,\ bc = dc,\ cb = cd,\ ad = cd,\ da = dc,\ bd = db = d^2 \rangle$$

by Corollary 6.2.2.

ii) If $ac = dc$ in the left ideals case then we can construct the semigroup which defined by the presentation

$$\langle a,b,c,d \mid ac = dc,\ ca = ba,\ bc = c^2,\ cb = b^2,\ ad = d^2,\ da = a^2,\ bd = cd,\ db = ab \rangle,$$

which is isomorphic to the semigroup which defined by the presentation

$$\langle a,b,c,d \mid ac=c^2,\ ca=a^2,\ bc=dc,\ cb=ab,\ ad=cd,\ da=ba,\ bd=d^2,\ db=b^2\rangle$$

by Corollary 6.2.6.

iii) If $ac=d^2$ in the left ideals case then we can construct the semigroup which is defined by the presentation

$$\langle a,b,c,d \mid ac=d^2,\ ca=b^2,\ bc=cd,\ cb=ba,\ ad=dc,\ da=ab,\ bd=c^2,\ db=a^2\rangle,$$

which is isomorphic to the semigroup with the presentation

$$\langle a,b,c,d \mid ac=cd,\ ca=ab,\ bc=d^2,\ cb=a^2,\ ad=c^2,\ da=b^2,\ bd=dc,\ db=ba$$

by Corollary 6.2.8.

Remark 6.4.4. There is no balanced semigroup S which is a disjoint union of two copies of the free semigroup of rank two $S_\alpha=\langle a,b \mid \ \rangle$, $S_\beta=\langle c,d \mid \ \rangle$ if S_β is ideal in S and $ac=d^2$ or $ac=cd$ see Theorem 6.3.2.

The next step is to show that if we add the identity to the free semigroup of rank two and define it as in Theorem 6.2.5 we obtain a Rees matrix semigroup over a free monoid of rank two.

Theorem 6.4.5. *Suppose* $P=\begin{bmatrix} t \\ z \end{bmatrix}$ *is a* 2×1 *matrix over the free monoid* S_β *in two generators and* S *is a Rees matrix semigroup over* S_β *of type* 2×1 *(*$S=\mathcal{M}[S_\beta;\ \{1\},\ \{1,2\};\ P]$*). Then* S *is the union of two copies of the free monoid in two generators if and only if* $t=z=1$.

PROOF. ($\Rightarrow$) Suppose that $S=S_\alpha\sqcup S_\beta$ where $S_\alpha=\langle a,b \mid \ \rangle$, $S_\beta=\langle c,d \mid \ \rangle$ and we want to show that $t=z=1$. Suppose that $t=w_1^*$, $z=w_2^*$ where w_1^*, w_2^* are two words in S_β. Since $S=\mathcal{M}[S_\beta;\ \{1\},\ \{1,2\};\ P]$, there is an isomorphism ψ from S into $\mathcal{M}[S_\beta;\ \{1\},\ \{1,2\};\ P]$ with

$$\psi(a)=(1,w_1,1),\ \psi(b)=(1,w_2,1),\ \psi(c)=(1,w_1,2),\ \psi(d)=(1,w_2,2).$$

However, the element $(1,w_1w_2,1)$ which is in $\mathcal{M}[S_\beta;\ \{1\},\ \{1,2\};\ P]$ does not have an original w^* in S such that $\psi(w^*)=(1,w_1w_2,1)$ because one of the two words w_1^*, w_2^* or both of them will appear in the product of any two elements in $\mathcal{M}[S_\beta;\ \{1\},\ \{1,2\};\ P]$, for instance

$$\psi(a)\psi(b) = (1, w_1, 1)(1, w_2, 1) = (1, w_1 w_1^* w_2, 1).$$

Thus ψ is not surjective, a contradiction.

($\Leftarrow$) Suppose that $t = z = 1$ and we want to show that S is a disjoint union of S_α and S_β. Since the isomorphism ψ is defined as above, we obtain the relations $ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2$. So we have the semigroup S with the presentation
$\langle a,\ b,\ c,\ d\ |ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2\rangle$
which is $S_\alpha \sqcup S_\beta$ (Theorem 6.2.5). □

Theorem 6.4.6. *Suppose that $S_\alpha = \langle a, b \mid\ \rangle$ and $S_\beta = \langle c, d \mid\ \rangle$ are two copies of the free semigroup of rank two. Suppose that S is a balanced semigroup which is a disjoint union of S_α and S_β and is defined by the presentation*

$$\langle a, b, c, d \mid ac = c^2,\ ca = a^2,\ bc = dc,\ cb = ab,\ ad = cd,\ da = ba,\ bd = d^2,\ db = b^2\ \rangle.$$

Then S is a Rees matrix semigroup of the type 2×1.

PROOF. First we adjoin the identity to S_β and then we can choose $I = \{1\}$, $\Lambda = \{1,2\}$ and P to be the $|\Lambda| \times |I|$ matrix over S_β^1 as $P = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus,
$\bar{S} \cong \mathcal{M}[S_\beta^1, \{1\}, \{1,2\}, P]$ is a semigroup. Now, S is isomorphic to

$$\{(i, c^t, \lambda),\ (i, d^t, \lambda) : t \neq 0\} \leq \bar{S},$$

because there exists a mapping ψ from S into $\{(i, c^t, \lambda),\ (i, d^t, \lambda)\ : t \neq 0\}$ with

$$\psi(a) = (1, c, 1),\ \psi(b) = (1, d, 1),\ \psi(c) = (1, c, 2),\ \psi(d) = (1, d, 2).$$

Obviously, ψ is injective and surjective, so it remains to verify that ψ is a homomorphism as follows:

$$\begin{aligned}
\psi(a)\psi(c) &= (1,c,1)(1,c,2) = (1,c^2,2) = \psi(c^2) = \psi(ac), \\
\psi(c)\psi(a) &= (1,c,2)(1,c,1) = (1,c^2,1) = \psi(a^2) = \psi(ca), \\
\psi(b)\psi(c) &= (1,d,1)(1,c,2) = (1,dc,2) = \psi(dc) = \psi(bc), \\
\psi(c)\psi(b) &= (1,c,2)(1,d,1) = (1,cd,1) = \psi(ab) = \psi(cb), \\
\psi(a)\psi(d) &= (1,c,1)(1,d,2) = (1,cd,2) = \psi(cd) = \psi(ad), \\
\psi(d)\psi(a) &= (1,d,2)(1,c,1) = (1,dc,1) = \psi(ba) = \psi(da), \\
\psi(b)\psi(d) &= (1,d,1)(1,d,2) = (1,d^2,2) = \psi(d^2) = \psi(bd), \\
\psi(d)\psi(b) &= (1,d,2)(1,d,1) = (1,d^2,1) = \psi(b^2) = \psi(db).
\end{aligned}$$

Hence,

$$S \cong \{(i,c^t,\lambda),\ (i,d^t,\lambda) : t \neq 0\} \cong \{(i,a^t,\lambda),\ (i,b^t,\lambda) : t \neq 0\}.$$

Thus S is a Rees matrix semigroup of the type 2×1. □

6.5 Residual finiteness

Since in this chapter we consider balanced semigroups which are defined by a presentation where the relations preserve length, we introduce the following definition.

Definition 6.5.1. A presentation $\langle A \mid R \rangle$ is *balanced* if for every relation $u = v$ from R we have $|u| = |v|$.

Theorem 6.5.2. *Every semigroup S which is defined by a balanced presentation is residually finite.*

PROOF. Let $k \in \mathbb{N}$ and let the set T_k consists of all the words of length l where $l \geq k$. Thus T_k is an ideal in S of finite index. Now we define a relation ρ_k on S as follows:

$$x\rho_k y \iff \text{either } x = y \text{ or } x, y \in T_k.$$

It is clear that ρ_k is an equivalence relation on S. Furthermore, ρ_k is a congruence of finite index because T_k is an ideal of finite index. Now if we take arbitrary elements $x, y \in S$ and $(x,y) \notin \rho_k$ then we are done. If $(x,y) \in \rho_k$ we choose

$k = max(m, n) + 1$ where m, n are the length of x, y respectively and hence $(x, y) \notin \rho_k$. □

Corollary 6.5.3. *Every balanced semigroup is residually finite.*

PROOF. The proof follows immediately by Theorem 6.5.2. □

Part II

DISJOINT UNIONS OF ANY NUMBER OF COPIES OF THE FREE MONOGENIC SEMIGROUP

CHAPTER

SEVEN

RECTANGULAR BANDS OF FINITELY MANY COPIES OF THE FREE MONOGENIC SEMIGROUP

7.1 Introduction

A rectangular band semigroup is a band S which satisfies $xyx = x$ for every $x, y \in S$ and equivalently $xyz = xz$. For instance, given arbitrary non-empty sets I, J, we can define a semigroup operation on $I \times J$ by $(i, j)(k, l) = (i, l)$. This means that the $I \times J$ rectangular band S is a disjoint union of subsemigroups P_{ij} $(i \in I, j \in J)$ and $P_{ij}P_{kl} \subseteq P_{il}$. This can be visualized by means of the following diagram:

P_{11}	P_{12}	$\dots$	P_{1j}
P_{21}	P_{22}	$\dots$	P_{2j}
$\vdots$	$\vdots$	$\ddots$	$\vdots$
P_{i1}	P_{i2}	$\dots$	P_{ij}

Table 7.1: The multiplication of a rectangular band

Thus S is a special "nice" case of disjoint unions of semigroups. In this chapter we have a general theorem on finite presentability and a theorem on residual finiteness in the special case in which each block is a free monogenic semigroup.

7.2 Finite presentability

Theorem 7.2.1. *Every rectangular band of finitely many finitely presented semigroups is finitely presented.*

PROOF. Consider the rectangular band $S = \bigsqcup_{i=1}^{n} S_i$. We prove the theorem by induction on n. Now if $n = 1$ then it is clear that S is finitely presented. Suppose that the statement holds for every $k \leq n$. Thus S is a $p \times q$ rectangular band where $p \neq 1$ or $q \neq 1$. Since $S = \bigsqcup_{i=1}^{k+1} S_i$ then we have the following:

S_1	S_2	$\dots$	S_i
$\vdots$	$\vdots$	$\dots$	S_j
S_{j+1}	$\vdots$	$\dots$	S_h
S_{h+1}	$\dots$	$\dots$	S_{k+1}

Table 7.2: The rectangular band semigroup $\bigsqcup_{i=1}^{k+1} S_i$

and then we split S to the two following diagrams

S_1	S_2	$\dots$	S_i
$\vdots$	$\vdots$	$\dots$	S_j
S_{j+1}	$\dots$	$\dots$	S_h

S_{h+1}	$\dots$	$\dots$	S_{k+1}

Table 7.3: A split of the semigroup $\bigsqcup_{i=1}^{k+1} S_i$ to $\bigsqcup_{i=1}^{h} S_i$ and $\bigsqcup_{i=h+1}^{k+1} S_i$

Thus $S = A \sqcup B$ where $A = \bigsqcup_{i=1}^{h} S_i$ and $B = \bigsqcup_{i=h+1}^{k+1} S_i$. Notice that A, B are subsemigroups by the definition of the rectangular band and they are finitely presented by the inductive hypothesis. Furthermore, they are right ideals in S. Hence S is finitely presented by Theorem 2.3.1. Notice that if S is a $1 \times q$ rectangular band. Then we have the following:

S_1	S_2	$\dots$	$\dots$	S_k	S_{k+1}

Table 7.4: The one row rectangular band semigroup $\bigsqcup_{i=1}^{k+1} S_i$

and then we split S to the two following diagrams

S_1	S_2	$\dots$	$\dots$	S_k		S_{k+1}

Table 7.5: A split of the one row semigroup $\bigsqcup_{i=1}^{k+1} S_i$ to $\bigsqcup_{i=1}^{k} S_i$ and S_{k+1}

Thus $S = A \sqcup B$ where $A = \bigsqcup_{i=1}^{k} S_i$ and $B = S_{k+1}$. Notice that A, B are subsemigroups by the definition of the rectangular band and they are finitely presented by the inductive hypothesis. Furthermore, they are left ideals in S. Hence S is finitely presented by Theorem 2.3.1. Similarlly if S is a $p \times 1$ rectangular band. $\square$

The proof of this theorem actually gives us a presentation for S as in the following corollary.

Corollary 7.2.2. *Every semigroup S which is a rectangular band of finitely many finitely presented semigroups defined by presentations $\langle A_{ij} | R_{ij} \rangle$, has a presentation of the form:*

$$\Big\langle \bigsqcup_{(i,j)\in I\times J} A_{ij} \mid \bigsqcup_{(i,j)\in I\times J} R_{ij},\ a_{ij}a_{kl} = \beta(a_{ij}, a_{kl}) \Big\rangle$$

where $a_{ij} \in A_{ij}$, $a_{kl} \in A_{kl}$ and $\beta(a_{ij}, a_{kl})$ is a word in ${A_{il}}^{+}$.

PROOF. Follows immediately by Theorem 2.3.1 and the definition of the rectangular band. To clarify this, we prove it on the 2×2 rectangular band and this will do for all. Thus we have 4 finitely presented semigroups, A_{11}, A_{12}, A_{21}, A_{22}. By the definition of the rectangular band, we have two left ideals in S,

$$S_1 = A_{11} \sqcup A_{21} \text{ and } S_2 = A_{12} \sqcup A_{22}.$$

If u, v were two words in $\{A_{11}, A_{12}, A_{21}, A_{22}\}^+$ and $u = v$ holds in S then $u = v$ holds in S_1 or S_2. So let $u = v$ holds in S_1. Each word in S_1 ends with w_{11} or w_{21} where $w_{11},\ w_{21} \in \{A_{11}, A_{21}\}^+$ because S_1 and S_2 are left ideals in S. Now if u, v

don't have any letter from S_2 then $u = v$ holds in A_{11} or A_{21}. Suppose that $u = v$ holds in A_{11}. Then as each word in A_{11} starts with $w'_{11} \in A_{11}^+$ because A_{11} and A_{21} are right ideals in S_1, we eliminate all occurrences of letters of A_{21} from u, v by the relation $a_{11}a_{21} = \beta(a_{11}, a_{21})$ and then we obtain two words $\bar{u}, \bar{v} \in A_{11}^+$. Thus $\bar{u} = \bar{v}$ is a consequence of R_{11}. Hence $u = v$ is a consequence of the relations in the presentation. Analogously if $u = v$ holds in A_{21}. Now if u, v have letters from S_2 then we apply the relations

$$a_{12}a_{11} = \beta(a_{12}, a_{11}),\ a_{12}a_{21} = \beta(a_{12}, a_{21}),\ a_{22}a_{11} = \beta(a_{22}, a_{11}),\ a_{22}a_{21} = \beta(a_{22}, a_{21}),$$

to eliminate all occurrences of letters of S_2 from u, v. We get two words $\bar{u}$ and $\bar{v}$ with $\bar{u} = \bar{v}$ holds in S_1 with no letters from S_2 and then we repeat the same argument. Thus we have $u = \bar{u} = \bar{\bar{u}}$ and $v = \bar{v} = \bar{\bar{v}}$ and then $u = v$ is a consequence of the relations in the presentation. Similarly, if $u = v$ holds in S_2.

Corollary 7.2.3. *Every rectangular band of finitely many copies of the free monogenic semigroup is finitely presented.*

PROOF. Directly by Theorem 7.2.1.

7.3 Special example

Let P be the semigroup free product of two trivial semigroups as follows:

$$P = \langle h, t | h^2 = h,\ t^2 = t \rangle. \tag{7.1}$$

Every element of P is equal to a unique alternating product of the form $htht \dots$ or $thth \dots$. Let $S_{P_4} = P \setminus \{h, t\}$. Thus S_{P_4} is a subsemigroup and is a rectangular band of four copies of the free monogenic semigroup as follows:
$S_{P_4} = \{(th)^n : n \in \mathbb{N}\} \cup \{(th)^n t : n \in \mathbb{N}\} \cup \{(ht)^n h : n \in \mathbb{N}\} \cup \{(ht)^n : n \in \mathbb{N}\}$. Thus we can respectively notate this as $S_{P_4} = N_a \sqcup N_b \sqcup N_c \sqcup N_d$ with the multiplication table:

N_a	N_b
N_c	N_d

Table 7.6: The multiplication on the subsemigroup S_{P_4} of the semigroup free product of two trivial semigroups

where N_i is a free monogenic semigroup generated by i for every $i \in \{a, b, c, d\}$. Therefore S_{P_4} is defined by the presentation

$$\begin{aligned} \langle a, b, c, d \mid \ & ab = b^2,\ ac = a^2,\ ad = b, \\ & ba = a^2,\ bc = a^3,\ bd = b^2, \\ & ca = c^2,\ cb = d^3,\ cd = d^2, \\ & da = c,\ db = d^2,\ dc = c^2 \rangle. \end{aligned}$$

First notice that $ab = b^i$ by the definition of the rectangular band. Hence $i = 2$ by Theorem 4.2.1. Similarly,

$$ba = a^2,\ ac = a^2,\ ca = c^2,\ dc = c^2,\ cd = d^2,\ bd = b^2,\ db = d^2.$$

We also have

$$ad = (th)(ht) = th^2t = tht = b,$$

$$da = (ht)(th) = hth = c,$$

$$cb = (hth)(tht) = (ht)(ht)(ht) = (ht)^3 = d^3,$$

$$bc = (tht)(hth) = (th)^3 = a^3.$$

Therefore, S_{P_4} satisfies the relation in the presentation.

Now we want to prove that each relation in S_{P_4} is a consequence of the relations in the presentation and this follows from Corollary 7.2.2.

7.4 Residual finiteness

We started working on the residual finiteness for a rectangular band S of finitely many copies of the free monogenic semigroup by trying to find a homomorphism ϕ from S to a finite semigroup H such that for every $s \neq t \in S : \phi(s) \neq \phi(t)$. We found that in order to obtain such a homomorphism, H must be defined by the multiplication of either the Rees matrix semigroup or S_{P_4} (see Section 7.3).

Theorem 7.4.1. *Every semigroup S which is a rectangular band of four copies of the free monogenic semigroup is either a Rees matrix semigroup or S_{P_4}.*

Proof. We prove the theorem by taking four copies of N, they are N_a, N_b, N_c, N_d. So the multiplication is defined by the following table:

N_a	N_b
N_c	N_d

Table 7.7: Rectangular band of four copies of the free monogenic semigroup

Hence, by Theorem 4.2.1, we have the following

$$ab = b^2, ba = a^2, ac = a^2, ca = c^2, bd = b^2, db = d^2, cd = d^2, dc = c^2.$$

Since S is a rectangular band, $ad = b^i, da = c^j, bc = a^k, cb = d^l$ and by

$$(ad)a = b^i a = a^{i+1}, a(da) = ac^j = a^{j+1},$$

$i = j$. Analogously $k = l$. Furthermore,

$$a(dc) = ac^2 = a^3, (ad)c = b^i c = b^{i-1}a^k = a^{k+i-1}$$

and thus $k + i = 4$ which implies that $i = j = k = l = 2$ or $k = l = 1, i = j = 3$ or $k = l = 3, i = j = 1$. Thus we have the following three types of semigroups:

Type 1. A rectangular band S of four copies of the free monogenic semigroup with the relations $ab = b^2, ba = a^2, ac = a^2, ca = c^2, bd = b^2, db = d^2, cd = d^2, dc = c^2, ad = b^2, da = c^2, bc = a^2, cb = d^2$.

Now, as we have done before, we adjoin the identity N_a^1 and we have the 2×2 matrix P with identity entries. Thus we get the semigroup $\bar{S} = \mathcal{M}\left[N_a^1; I, \Lambda; P\right]$. It is clear that S is isomorphic to $\{(i, a^t, \lambda) : t \neq 0\} \leq \bar{S}$, because there exists a mapping ψ from S into $\{(i, a^t, \lambda) : t \neq 0\}$ with

$$\psi(a) = (1, a, 1),\ \psi(b) = (1, a, 2),\ \psi(c) = (2, a, 1),\ \psi(d) = (2, a, 2).$$

Obviously, ψ is injective and surjective, so it remains to verify that ψ is a homomorphism as follows:

$$\psi(a)\psi(b) = (1, a, 1)(1, a, 2) = (1, a^2, 2) = \psi(b^2) = \psi(ab),$$

$$\psi(b)\psi(a) = (1, a, 2)(1, a, 1) = (1, a^2, 1) = \psi(a^2) = \psi(ba),$$

$$\psi(a)\psi(c) = (1, a, 1)(2, a, 1) = (1, a^2, 1) = \psi(a^2) = \psi(ac),$$

$$\psi(c)\psi(a) = (2, a, 1)(1, a, 1) = (2, a^2, 1) = \psi(c^2) = \psi(ca),$$

$$\psi(b)\psi(d) = (1, a, 2)(2, a, 2) = (1, a^2, 2) = \psi(b^2) = \psi(bd),$$
$$\psi(d)\psi(b) = (2, a, 2)(1, a, 2) = (2, a^2, 2) = \psi(d^2) = \psi(db),$$

Hence,
$$S \cong \{(i, a^t, \lambda) : t \neq 0\}.$$

Thus S is a Rees matrix semigroup of the type 2×2.

Type 2. A rectangular band S of four copies of the free monogenic semigroup with the relations $ab = b^2, ba = a^2, ac = a^2, ca = c^2, bd = b^2, db = d^2, cd = d^2, dc = c^2, ad = b, da = c, bc = a^3, cb = d^3$.

So we can define an isomorphism ϕ from this semigroup to S_{P4} by $\phi(a) = th$, $\phi(b) = tht$, $\phi(c) = hth$, $\phi(d) = ht$.

Type 3. A rectangular band S of four copies of the free monogenic semigroup with the relations $ab = b^2, ba = a^2, ac = a^2, ca = c^2, bd = b^2, db = d^2, cd = d^2, dc = c^2, ad = b^3, da = c^3, bc = a, cb = d$.

This semigroup is isomorphic to the semigroup of Type 2. Therefore, S is a Rees matrix semigroup or S_{P_4}. □

Lemma 7.4.2. *Suppose S is a semigroup which is a rectangular band of four copies of the free monogenic semigroup. If $a^i b^j = c^k$ holds in S for some a, b, c in the generating set A, $i, j, k \in \mathbb{N}$ then $i, j \leq k$.*

PROOF. By Theorem 7.4.1, S is a Rees matrix semigroup or S_{P_4}. If S is a Rees matrix semigroup then $k = i + j$. If S is S_{P_4} then $k \in \{i + j, i + j - 1, i + j + 1\}$. Thus $i, j \leq k$. □

Theorem 7.4.3. *Every semigroup S which is a rectangular band of finitely many copies of the free monogenic semigroup is residually finite.*

PROOF. First we define a relation $\rho_k, k \in \mathbb{N}$ on S as follows:

$$(x, y) \in \rho_k \iff \textit{either } x = y \textit{ or } x = a^i, y = a^j \text{ for some } a \in A \text{ and } i, j \geq k.$$

It is clear that ρ_k is an equivalence relation on S. Moreover, ρ_k is a congruence by Lemma 7.4.2 and the fact that $N_a N_b \subseteq N_c$. Notice that $|N_a/\rho_k| = k$ and then there are nk equivalence classes where n is the number of the copies that we have. Therefore ρ_k is of finite index. Now take any two $a^i, a^j \in N_a$. Let

$$k = max(i, j) + 1.$$

Then $(a^i, a^j) \notin \rho_k$. Notice that if we take two elements a^i and b^j from different copies, then they are in separate classes because of the definition of the congruence. Hence ρ_k is a congruence on S of finite index which separates each two elements in S. $\square$

CHAPTER

EIGHT

FINITENESS CONDITIONS FOR DISJOINT UNIONS OF FINITELY MANY COPIES OF THE FREE MONOGENIC SEMIGROUP

8.1 Preliminaries: multiplication and arithmetic progressions

Let S be a semigroup which is a disjoint union of n copies of the free monogenic semigroup:

$$S = \bigsqcup_{a \in A} N_a,$$

where A is a finite set and $N_a = \langle a \rangle$ for $a \in A$. In this section we gather some background facts about S. The common feature is that they all elucidate a strong regularity with which elements of S multiply. We begin with two preliminary lemmas.

Lemma 8.1.1. *Let $a \in A$ and $q \in \mathbb{N}$ be fixed. There can be only finitely many elements $x \in S$ such that $a^p x = a^{p+q}$ for some $p \in \mathbb{N}$.*

PROOF. Suppose to the contrary that there are infinitely many such x. Two of these elements must belong to the same block N_c. Suppose these elements are c^r and c^s for $r \neq s$, and suppose we have $a^{p_1} c^r = a^{p_1+q}$ and $a^{p_2} c^s = a^{p_2+q}$. Note that these equalities imply $a^p c^r = a^{p+q}$ for all $p \geq p_1$, and $a^p c^s = a^{p+q}$ for all $p \geq p_2$. Let

$p = \max(p_1, p_2)$, and evaluate the element $a^p c^{rs}$ twice:

$$a^p c^{rs} = a^p (c^r)^s = a^p \underbrace{c^r \dots c^r}_{s} = a^{p+q} \underbrace{c^r \dots c^r}_{s-1} = \dots = a^{p+sq},$$

and, similarly,

$$a^p c^{rs} = a^p (c^s)^r = a^{p+rq}.$$

But from $r \neq s$ it follows that $a^{p+sq} \neq a^{p+rq}$, a contradiction. $\square$

Lemma 8.1.2. *If $a^p b^q = a^r$ holds in S for some $a, b \in A$ and $p, q, r \in \mathbb{N}$ then $p \leq r$.*

PROOF. Suppose to the contrary that $r = p - s < p$. Note that for every $t \geq p$ we have

$$a^t \cdot a^s b^q = a^{t+s} b^q = a^{t+s-p} a^p b^q = a^{t+s-p} a^r = a^{t+s-p+p-s} = a^t.$$

Hence, for every $u \geq 1$ we have

$$a^t (a^s b^q)^u a^s = a^t a^s = a^{t+s}.$$

By Lemma 8.1.1 we must have

$$(a^s b^q)^u a^s = (a^s b^q)^v a^s$$

for some distinct $u, v \in \mathbb{N}$. Post-multiplying by b^q we obtain

$$(a^s b^q)^{u+1} = (a^s b^q)^{v+1}.$$

This means that the element $a^s b^q \in S$ has finite order, a contradiction. $\square$

The next result shows that multiplication by $x \in S$ cannot 'reverse' the order of elements from the copies of N.

Lemma 8.1.3. *If $a, b \in A$ and $x \in S$ are such that*

$$a^p x = b^r, \; a^{p+q} x = b^s$$

for some $p, q, r, s \in \mathbb{N}$, then $r \leq s$.

PROOF. The assertion follows from

$$a^q b^r = a^q a^p x = a^{p+q} x = b^s,$$

and (the dual of) Lemma 8.1.2. □

The next lemma is absolutely pivotal for proofs of both finite presentability and residual finiteness.

Lemma 8.1.4. *If*

$$a^p x = b^r,\ a^{p+q} x = b^{r+s} \tag{8.1}$$

for some $a, b \in A$, $x \in S$, $p, q, r \in \mathbb{N}$, $s \in \mathbb{N}_0$, then

$$a^{p+qt} x = b^{r+st}$$

for all $t \in \mathbb{N}_0$.

PROOF. First note that from (9.2) we have

$$b^{r+s} = a^{p+q} x = a^q a^p x = a^q b^r. \tag{8.2}$$

We now prove the lemma by induction on t. For $t = 0$ we get the first relation in (9.2). Assume the statement holds for some t. Then, by induction and (9.3),

$$a^{p+q(t+1)} x = a^q a^{p+qt} x = a^q b^{r+st} = a^q b^r b^{st} = b^{r+s} b^{st} = b^{r+s(t+1)},$$

proving the lemma. □

Motivated by Lemma 8.1.4 we introduce the sets

$$T(a, x, b) = \{y \in N_a \ :\ yx \in N_b\}\ (a, b \in A,\ x \in S).$$

The following is immediate.

Lemma 8.1.5. *For any $a \in A$ and $x \in S$ we have*

$$N_a = \bigsqcup_{b \in A} T(a, x, b).$$

By Lemmas 8.1.3, 8.1.4, if a set $T(a, x, b)$ contains more than one element, then it contains an arithmetic progression, and hence is infinite. In fact, if $T(a, x, b)$ is infinite then it actually stabilizes into an arithmetic progression.

Lemma 8.1.6. *If $T = T(a, x, b)$ is infinite then there exist sets $F = F(a, x, b)$, $P = P(a, x, b)$ such that the following hold:*

(i) $T = F \sqcup P$;

(ii) $P = \{a^{p+qt} \ : \ t \in \mathbb{N}_0\}$ *for some* $p = p(a,x,b), q = q(a,x,b) \in \mathbb{N}$ *and* $a^{p-q} \notin T$;

(iii) $F \subseteq \{a, \ldots, a^{p-1}\}$ *is a finite set.*

PROOF. Let $q \in \mathbb{N}$ be the smallest number such that $a^p, a^{p+q} \in T$ for some $p \in \mathbb{N}$. Furthermore, let p be the smallest such; in particular $a^{p-q} \notin T$. Let $\{a^{p+qt} \ : \ t \in \mathbb{N}_0\}$. By Lemmas 8.1.3, 8.1.4 we have $P \subseteq T$, and by minimality of q we have $a^{p+tq+r} \notin T$ for any $t \in \mathbb{N}_0$ and any $r \in \{1, \ldots, q-1\}$. Hence $F = T \setminus P \subseteq \{a, \ldots, a^{p-1}\}$, and the lemma is proved. □

The next lemma discusses the values in the set $T(a,x,b) \cdot x$.

Lemma 8.1.7. *For* $T = T(a,x,b)$ *we either have* $|Tx| \leq 1$ *or else* $yx \neq zx$ *for all distinct* $y, z \in T$.

PROOF. Suppose that for some $p, q, r, s \in \mathbb{N}$ we have

$$a^p x = b^r, \ a^{p+q} x = b^{r+s}, \tag{8.3}$$

while for some $u, v, w \in \mathbb{N}$ we have

$$a^u x = b^w, \ a^{u+v} = b^w. \tag{8.4}$$

From (8.3), (8.4) and Lemma 8.1.4 we have:

$$a^{p+qt} x = b^{r+st} \quad (t \in \mathbb{N}), \tag{8.5}$$

$$a^{u+vt} x = b^w \qquad (t \in \mathbb{N}). \tag{8.6}$$

Let $t_1 \in \mathbb{N}$ be such that

$$r + st_1 > w, \tag{8.7}$$

and let $t_2 \in \mathbb{N}$ be such that

$$u + vt_2 > p + qt_1. \tag{8.8}$$

The inequalities (8.7), (8.8) and relations (8.5), (8.6) with $t = t_1$ and $t = t_2$ respectively contradict Lemma 8.1.3. □

The rest of this section will be devoted to proving that there are only finitely many distinct sets $T(a,x,b)$, a fact that will be crucial in Section 8.3. We accomplish this (in Lemma 8.1.13) by proving that there are only finitely many distinct numbers

$q(a,x,b)$ (Lemma 8.1.9), finitely many distinct numbers $p(a,x,b)$ (Lemma 8.1.10), and finitely many distinct sets $F(a,x,b)$ (Lemma 8.1.12). We begin, however, with an elementary observation, which must be well-known, but we prove it for completeness.

Lemma 8.1.8. *For every $n \in \mathbb{N}$ and every $r \in \mathbb{Q}^+$ the set*

$$\left\{(m_1,\dots,m_n) \in \mathbb{N}^n \ : \ \frac{1}{m_1} + \cdots + \frac{1}{m_n} = r\right\}$$

is finite.

PROOF. Prove the assertion by induction on n, the case $n = 1$ being obvious. Let $n > 1$, and assume the assertion is true for $n-1$. Consider an n-tuple $(m_1,\dots,m_n) \in \mathbb{N}^n$ such that

$$\frac{1}{m_1} + \cdots + \frac{1}{m_n} = r.$$

Without loss of generality assume $m_1 \geq \cdots \geq m_n$. Then we must have $1/m_n \geq r/n$, and so $m_n \leq n/r$. Thus there are only finitely many possible values for m_n. For each of them, the remaining $n-1$ numbers satisfy

$$\frac{1}{m_1} + \cdots + \frac{1}{m_{n-1}} = r - \frac{1}{m_n},$$

and by induction there are only finitely many such $(n-1)$-tuples. □

Lemma 8.1.9. *The set*

$$\{q(a,x,b) \ : \ a,b \in A,\ x \in S\}$$

is finite.

PROOF. Fix $a \in A$, $x \in S$, and notice that at least one of the sets $T(a,x,b)$ ($b \in A$) is infinite by Lemma 8.1.5. Let

$$m = \mathrm{lcm}\{q(a,x,b) \ : \ b \in A,\ |T(a,x,b)| = \infty\}.$$

Recall that all the sets $F(a,x,b)$ are finite, and let $r \in \mathbb{N}$ be such that

$$r > \max \bigcup_{b \in A} F(a,x,b).$$

Let $I = \{a^r, a^{r+1}, \dots, a^{r+m-1}\}$, an 'interval' of size m. From Lemma 8.1.6 (iii) we have $I \cap F(a,x,b) = \emptyset$ for all $b \in A$, so Lemma 8.1.5 implies that I is the disjoint union of sets $I \cap P(a,x,b)$ ($b \in A$). Since for every $b \in A$ with $P(a,x,b) \neq \emptyset$ we have $q(a,x,b) \mid m$, the set I contains precisely $m/q(a,x,b)$ elements from $P(a,x,b)$. It follows that

$$\sum_{b \in A} \frac{m}{q(a,x,b)} = m,$$

and hence

$$\sum_{b \in A} \frac{1}{q(a,x,b)} = 1.$$

The assertion now follows from Lemma 8.1.8. □

Lemma 8.1.10. *The set*

$$\{p(a,x,b) \ : \ a,b \in A, \ x \in S\}$$

is finite.

PROOF. Fix $a,b \in A$, $x \in S$, and for brevity write $p = p(a,x,b)$, $q = q(a,x,b)$. Recall that p has been chosen to be the smallest possible with respect to the condition that

$$p + qt \in T(a,x,b) \ (t \in \mathbb{N}_0). \tag{8.9}$$

Recalling Lemma 8.1.9, let

$$Q = \max\{q(c,y,d) \ : \ c,d \in A, \ y \in S\}.$$

Assume, aiming for contradiction, that

$$p > 2nQ.$$

Since $Q \geq q$ we have that $p - 2nq > 0$. Consider the n pairs

$$\{a^{p-(2t-1)q}, a^{p-2tq}\} \ (t = 1, \ldots, n).$$

By Lemmas 8.1.3, 8.1.4 and minimality of p we cannot have both members of one of these pairs belong to $T(a,x,b)$. Hence at least one member in each pairs belongs to some $T(a,x,c)$ with $c \neq b$. By the pigeonhole principle two of these must belong to the same $T(a,x,c)$, say

$$a^{p-uq}, a^{p-vq} \in T(a,x,c)$$

for some $1 \leq v < u \leq 2n$. Again by Lemmas 8.1.3, 8.1.4 we have that

$$a^{p-uq+(u-v)qt} \in T(a,x,c)$$

for all $t \in \mathbb{N}_0$. On the other hand for t sufficiently large (e.g. $t \geq u$) we have

$$p - uq + (u-v)qt \geq p,$$

so that

$$p - uq + (u-v)qt = p + wq$$

for some $w \in \mathbb{N}_0$, and so from (8.9) we have

$$a^{p-uq+(u-v)qt} \in T(a,x,b) \neq T(a,x,c),$$

a contradiction. We conclude that $p(a,x,b) \leq 2nQ$ for all $a, b \in A$, $x \in S$, where the right hand side does not depend on a, b or x. □

In order to prove our final ingredient, that there are only finitely many distinct sets $F(a,x,b)$, we require one more elementary fact.

Lemma 8.1.11. *Consider a finite collection of arithmetic progressions:*

$$R_i = \{p_i + tq_i \ : \ t \in \mathbb{N}_0\} \ (i = 1, \dots, n).$$

If there exists $p \in \mathbb{N}$ such that

$$[p, \infty) \subseteq \bigcup_{i=1}^{n} R_i,$$

then

$$[p', \infty) \subseteq \bigcup_{i=1}^{n} R_i,$$

where $p' = \max\{p_1, \dots, p_n\}$.

PROOF. If $p \leq p'$ there is nothing to prove. Otherwise the assertion follows from the fact that for every $m \geq p'$ and every $i = 1, \dots, n$ we have $m \in R_i$ if and only if $m + q_i \in R_i$. □

Lemma 8.1.12. *The set*

$$\{F(a,x,b) \ : \ a, b \in A, \ x \in S\}$$

is finite.

PROOF. Fix $a \in A$, $x \in S$. Finitely many arithmetic progressions $P(a,x,b)$ ($b \in A$, $|T(a,x,b)| = \infty$) eventually cover the block N_a by Lemmas 8.1.5, 8.1.6. Hence, by Lemma 8.1.11, they contain all elements a^t with

$$t \geq M = \max\{p(a,x,b) \ : \ b \in A\}.$$

Hence every $F(a,x,b)$ ($b \in A$) is contained in $\{a,\ldots,a^{M-1}\}$. Since the numbers $p(a,x,b)$ are uniformly bounded by Lemma 8.1.10 the assertion follows. □

Lemma 8.1.13. *The set*

$$\{T(a,x,b) \ : \ a,b \in A, \ x \in S\}$$

is finite.

PROOF. follows from Lemmas 8.1.6, 8.1.9, 8.1.10, 8.1.12. □

8.2 Finite presentability

Theorem 8.2.1. *Every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup is finitely presented.*

PROOF. Continue using notation from Section 8.1. Thus $S = \bigsqcup_{a\in A} N_a$, and $N_a = \langle a \rangle$. The set

$$W = \{a^k \ : \ a \in A, \ k \in \mathbb{N}\}$$

is a set of normal forms for S. Hence for any $a,b \in A$ and $k,l \in \mathbb{N}$ there exist unique $\alpha(a,k,b,l) \in A$ and $\kappa(a,k,b,l) \in \mathbb{N}$ such that

$$a^k b^l = [\alpha(a,k,b,l)]^{\kappa(a,k,b,l)}. \tag{8.10}$$

It is easy to see that generators A and relations (8.10) provide an (infinite) presentation for S; for instance, condition (P3) is clearly satisfied.
Now we claim that the (still infinite) presentation with generators A and relations

$$a^k b = [\alpha(a,k,b,1)]^{\kappa(a,k,b,1)}, \ (a,b \in A, \ k \in \mathbb{N}) \tag{8.11}$$

also defines S. Indeed, the above set of relations is contained in (8.10), and so S satisfies (9.1). We now show that a general relation from (8.10) is a consequence of (9.1). We do this by induction on l. For $l = 1$ we actually have a relation from (9.1), and there is nothing to prove. Assume the assertion holds for some l. Then we have

$$\begin{aligned} a^k b^{l+1} &\equiv a^k b^l b \\ &= [\alpha(a,k,b,l)]^{\kappa(a,k,b,l)} b && \text{(by induction)} \\ &= [\alpha(\alpha(a,k,b,l),\kappa(a,k,b,l),b,1)]^{\kappa(\alpha(a,k,b,l),\kappa(a,k,b,l),b,1)} \\ & && \text{(by (9.1))} \\ &\equiv [\alpha(a,k,b,l+1)]^{\kappa(a,k,b,l+1)}. && \text{(by uniqueness of normal forms)} \end{aligned}$$

Therefore, every relation (8.10) is a consequence of (9.1). Since (8.10) is a presentation for S, so is (9.1).

For any $a, b, c \in A$ consider the set $T(a,b,c)$. Note that for every $a^i \in T(a,b,c)$ there exists a unique $j \in \mathbb{N}$ such that $a^i b = c^j$. Let $R_{a,b,c}$ be the set of all these relations; clearly $|R_{a,b,c}| = |T(a,b,c)|$.

Next we claim that for any $a, b, c \in A$ there exists a finite set of relations $R^\circ_{a,b,c} \subseteq R_{a,b,c}$ such that all relations in $R_{a,b,c}$ are consequences of $R^\circ_{a,b,c}$. Indeed, if $T(a,b,c)$ is finite (i.e. $|T(a,b,c)| \leq 1$) the assertion is obvious. So suppose that $T(a,b,c)$ is infinite. By Lemma 8.1.6 we have

$$T(a,b,c) = F \cup P,$$

where $P = \{a^{p+tq} \ : \ t \in \mathbb{N}_0\}$ and $F \subseteq \{a, \ldots, a^{p-1}\}$. Now, if

$$a^p b = c^r, \ a^{p+q} b = c^{r+s}, \tag{8.12}$$

then by Lemma 8.1.4 we have

$$a^{p+tq} b = c^{r+ts} \ (t \in \mathbb{N}_0). \tag{8.13}$$

A closer inspection of the proof of Lemma 8.1.4 shows that in fact relations (8.13) are consequences of (8.12), in the technical sense above. On the other hand, relations (8.13) are precisely all the relations $a^i b = c^j$ with $a^i \in P$. There remain finitely many relations with $a^i \in F$, and the claim follows.

To complete the proof of the theorem, note that the set of defining relations (9.1) is the union $\bigcup_{a,b,c \in A} R_{a,b,c}$. Hence all these relations are consequences of $\bigcup_{a,b,c} R^\circ_{a,b,c}$, which is a finite set because A and all $R^\circ_{a,b,c}$ are finite. □

8.3 Residual finiteness

Let S be such a semigroup, and let all the notation be as in Section 8.1. Define a relation ρ on S as follows:

$$(x,y) \in \rho \Leftrightarrow (\forall z \in S^1)(\exists a \in A)(xz, yz \in N_a).$$

Intuitively, two elements (x,y) of S are ρ-related if every pair of translates by the same element of S^1 belongs to a single block. In particular, if $(x,y) \in \rho$ then x and y are powers of the same generator $a \in A$, i.e.

$$\rho \subseteq \bigcup_{a \in A} N_a \times N_a. \tag{8.14}$$

The following is obvious from the definition.

Lemma 8.3.1. *ρ is a right congruence.*

An alternative description of ρ is provided by:

$$(a^i, a^j) \in \rho \Leftrightarrow (\forall x \in S)(\exists b \in A)(a^i, a^j \in T(a,x,b)); \tag{8.15}$$

the proof is obvious. This description enables us to prove:

Lemma 8.3.2. *ρ has finite index.*

PROOF. From (8.15) it follows that the ρ-class of an element $a^i \in S$ is

$$a^i/\rho = \bigcap \{T(a,x,b) \ : \ x \in S,\ b \in A,\ a^i x \in N_b\}. \tag{8.16}$$

By Lemma 8.1.13 there are only finitely many distinct sets $T(a,x,b)$. Hence there are only finitely many intersections (8.16), and the assertion follows. □

For each $a \in A$, consider the restriction

$$\rho_a = \rho \restriction_{N_a}.$$

From Lemmas 8.3.1, 8.3.2 it follows that ρ_a is a right congruence of finite index on N_a. But N_a, being free monogenic, is commutative, and so ρ_a is actually a congruence. Furthermore, congruences on a free monogenic semigroup are well understood, and we have that

$$\rho_a = \{(a^i, a^j) \ : \ i = j \text{ or } (i, j \geq p_a \ \& \ i \equiv j \pmod{q_a})\},$$

for some $p_a, q_a \in \mathbb{N}$; see [How95, Section 1.2].
Motivated by this, for any pair $(x, y) \in \rho$ we define their *distance* as

$$d(x, y) = \frac{|i - j|}{q_a} \text{ if } x = a^i, \ y = a^j.$$

Lemma 8.3.3. *If $x, y, z \in S$ are such that $(x, y) \in \rho$ and $x \neq y$ then*

$$d(x, y) \mid d(xz, yz).$$

PROOF. Since ρ is a right congruence we have $(xz, yz) \in \rho$, and so $d(xz, yz)$ is defined. If $xz = yz$ there is nothing to prove, so suppose $xz \neq yz$. Write

$$x = a^r, \ y = a^s,$$

where $r, s \geq p_a$, $r \equiv s \pmod{q_a}$, $r \neq s$. Without loss of generality assume $s > r$ so that $s = r + tq_a$ for some $t \in \mathbb{N}$. Notice that $(a^r, a^{r+q_a}) \in \rho$; furthermore we must have $a^r z \neq a^{r+q_a} z$ by Lemma 8.1.7. Therefore

$$a^r z = b^u, \ a^{r+q_a} z = b^v, \tag{8.17}$$

for some $u, v \geq p_b$, $u \equiv v \pmod{q_b}$, $u < v$. Write $v = u + wq_b$, $w \in \mathbb{N}$. Equalities (8.17) become

$$a^r z = b^u, \ a^{r+q_a} z = b^{u+wq_b},$$

and Lemma 8.1.4 yields

$$a^{r+tq_a} z = b^{u+twq_b}.$$

Therefore

$$d(xz, yz) = d(a^r z, a^{r+tq_a} z) = d(b^u, b^{u+twq_b}) = tw = wd(a^r, a^{r+tq_a}) = wd(x, y),$$

as required. □

We are now ready to prove:

Theorem 8.3.4. *Every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup is residually finite.*

PROOF. Let S be such a semigroup, with all the foregoing notation remaining in force. Let $x, y \in S$ be two arbitrary distinct elements. By (RF3) it is sufficient to prove that x and y are separated by a right congruence of finite index. If $(x, y) \notin \rho$ then ρ is such a congruence by Lemmas 8.3.1, 8.3.2. So suppose $(x, y) \in \rho$, say with $x, y \in N_b$, and let

$$d(x, y) = d > 0.$$

Let σ be the right congruence on S generated by the set

$$G = \{(a^{p_a}, a^{p_a+2dq_a}) \ : \ a \in A\}.$$

Clearly σ is a refinement of ρ (i.e. $\sigma \subseteq \rho$). Notice that G contains one pair of distinct elements from each block N_a. Hence the restriction of σ to each N_a is a non-trivial congruence, and so has finite index. Therefore σ itself has finite index too.
We claim that $(x, y) \notin \sigma$. Suppose otherwise; this means that there is a sequence

$$x = u_1, u_2, \ldots, u_m = y$$

of elements of S, such that for each $i = 1, \ldots, m-1$ we can write

$$u_i = v_i z_i, \ u_{i+1} = w_i z_i,$$

for some $v_i, w_i \in S$, $z_i \in S^1$, satisfying $(v_i, w_i) \in G$ or $(w_i, v_i) \in G$. (This is a well-known general fact; see for example [How95, Section 8.1] .) Without loss of generality we may assume that all u_i are distinct. From $\sigma \subseteq \rho$ it follows that all u_i belong to the block N_b, say

$$u_i = b^{s_i} \ (i = 1, \ldots, m).$$

By definition of G and Lemma 8.3.3 we have that $2d \mid d(u_i, u_{i+1})$ for all $i = 1, \ldots, m-1$. This is equivalent to

$$s_i \equiv s_{i+1} \pmod{2q_b d} \ (i = 1, \ldots, m-1),$$

from which it follows that $s_1 \cong s_m \pmod{2q_b d}$, and hence $2d \mid d(x, y) = d$, a contradiction. □

8.4 Hopficity

We say that a semigroup S is *hopfian* if it is not isomorphic to any of its proper homomorphic images. Equivalently, any surjective endomorphism of S is injective and thus an automorphism. Also, we say that a semigroup S is *co-hopfian* if it is not isomorphic to any of its subsemigroups. Equivalently, any injective endomorphism of S is surjective and thus an automorphism. For further information see [VRed].

Proposition 8.4.1. *Let S be a semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup. Let ϕ be an endomorphism on S. Then*

i) $\phi(N_a) \subseteq N_b$ *where* $a, b \in A$.

ii) *If ϕ is surjective then*

1) $\phi(N_i) \subseteq N_j$, $\phi(N_k) \subseteq N_l \Longrightarrow j \neq l$.
2) $\phi(a^i) = b^j \Longrightarrow i = j$.

PROOF. (i) If $\phi(a) = b^j$ then $\phi(a^i) = b^{ij}$ for every $i \in \mathbb{N}$. Thus $\phi(N_a) \subseteq N_b$.
(ii) it is clear because if we assume the opposite in both of (1) and (2) we get a contradiction with the fact that ϕ is a surjective endomorphism . □

Theorem 8.4.2. *Every semigroup S which is a disjoint union of finitely many copies of the free monogenic semigroup is hopfian.*

PROOF. It is well-known that every finitely generated residually finite group or semigroup is hopfian by [Mal40] and as we know that S is finitely generated and residually finite by Theorem 8.3.4 and then S is hopfian. Additionally, we can prove it easily by Proposition 8.4.1. □

Remark 8.4.3. Notice that S is not co-hopfian because there is a subsemigroup T of S where $T \cong S$. For instance the semigroup of even numbers $2\mathbb{N}$ is a subsemigroup of $\mathbb{N}$ and $2\mathbb{N} \cong \mathbb{N}$.

8.5 Commutative semigroups

A semigroup S is rational if, for each $a, b \in S$, there exist integers m, n such that $a^m = b^n$. This semigroup has also been called power joined see [Lev68]. Archimedean semigroup is a semigroup with the property that for every $a, b \in S$ there is $n \in \mathbb{N}$

such that $a|b^n$, which means that each element in S divides a power of another element.

Theorem 8.5.1 ([MO70], Theorem 3.4)**.** *A finitely generated Archimedean semigroup without idempotents is a rational semigroup.*

Theorem 8.5.2. *A commutative semigroup S is a finite disjoint union of copies of the free monogenic semigroup if and only if it is a strong semilattice of copies of the free monogenic semigroup.*

PROOF. ($\Rightarrow$) Suppose that S is a disjoint union of n copies of the free monogenic semigroup. Since S is commutative , S is a semilattice of commutative Archimedean semigroups see ([Gri95], Theorem 4.2.2). Thus each component of the semilattice is a finitely generated semigroup without idempotents. Therefore, by Theorem 8.5.1, each component is a rational semigroup which implies that each component must have only one copy of the free monogenic semigroup. Hence the semilattice is of order n. So we have a semilattice $Y = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ where each element of Y we assign a semigroup N_{α_i} such that $\alpha_i\alpha_j = \alpha_j$ if $N_{\alpha_i}N_{\alpha_j} \subseteq N_{\alpha_j}$ (notice that $ab = b^k, a \in N_{\alpha_i}, b \in N_{\alpha_j}, k \in \mathbb{N}$) and to each pair of elements α_i, α_j we assign a map ϕ_{α_i,α_j} of N_{α_i} into N_{α_j} with $\phi_{\alpha_i,\alpha_j}(a) = b^{k-1}$. Also there is an identity map ϕ_{α_i,α_i} on N_{α_i} for every $i \in \{1, 2, \ldots, n\}$. Clearly, $\phi_{\alpha_i,\alpha_j}\phi_{\alpha_j,\alpha_j} = \phi_{\alpha_i,\alpha_j}$ if $\alpha_i \geq \alpha_j$ and $\phi_{\alpha_i,\alpha_i}\phi_{\alpha_i,\alpha_j} = \phi_{\alpha_i,\alpha_j}$ if $\alpha_i \geq \alpha_j$. Now, notice that

$$\begin{aligned}
a^x b^y &= a^{x-1} b^k b^{y-1} \\
&= a^{x-2} b^k b^{k-1} b^{y-1} \\
&= a^{x-(x-1)} b^k b^{(k-1)(x-2)} b^{y-1} \\
&= b^k b^{k-1} b^{(k-1)(x-2)} b^{y-1} \\
&= b^{kx-x+y}
\end{aligned}$$

and

$$\phi_{\alpha,\beta}(a^x)\phi_{\beta,\beta}(b^y) = b^{x(k-1)} b^y = b^{kx-x+y} = a^x b^y.$$

The above argument works for every $k > 1$ and a small modification is needed for the case $k = 1$ as we have mentioned in the proof of Theorem 4.3.1. This based on the same idea as near Rees matrix semigroup, but this time for strong semilattices of semigroups. More precisely, we adjoin the identity to N_{α_j} and let $\bar{S} = N_{\alpha_i} \sqcup N_{\alpha_j}^{1_{\alpha_j}}$. Next define $\phi_{\alpha_i,\alpha_j}(a) = 1_{\alpha_j}$ to form $\bar{S}$ into a strong semillatice of semigroups with multiplication $ab = ba = b$. Finally let $S = N_{\alpha_i} \sqcup N_{\alpha_j} \leq \bar{S}$. Therefore

$$S = \mathcal{S}\big[Y; \{N_{\alpha_i} : \alpha_i \in Y\}; \{\phi_{\alpha_i,\alpha_j} : \alpha_i, \alpha_j \in Y, \alpha_i \geq \alpha_j\}\big].$$

Thus S is a strong semilattice of n copies of the free monogenic semigroup.
($\Leftarrow$) Obvious by the definition of the strong semilattice of semigroups. $\square$

CHAPTER

NINE

DECIDABILITY FOR DISJOINT UNIONS OF FINITELY MANY COPIES OF THE FREE MONOGENIC SEMIGROUP

9.1 Introduction

In this chapter we prove that any semigroup S which is a disjoint union of finitely many copies of the free monogenic semigroup N has decidable word problem. Furthermore, our proof of finite presentability for S provides an explicit solution to the word problem as is shown in the second section of this chapter. In addition we look at the subsemigroup of S and itself is a finite disjoint union of subsemigroups of the free monogenic semigroups as well. Subsemigroups of the free monogenic semigroups, and in particular the so called numerical semigroups (subsemigroups of finite complement) have been subject to extensive investigation over the years; see [RGS09] for a comprehensive introduction. We take a complementary view-point, instead of looking at subsemigroups of N, we investigate semigroups which are 'composed' of finitely many subsemigroups of the free monogenic semigroup N. Now we introduce necessary well-known theorems about subsemigroups of the natural number semigroup $\mathbb{N}$. We will use these theorems to devise an algorithm to solve the subsemigroups membership problem in the third section of this chapter.

9.2 Word problem

A semigroup S generated by a finite set A, has a soluble word problem (with respect to A) if there exists an algorithm which, for any two words $u, v \in A^+$, decides whether the relation $u = v$ holds in S or not. For finitely generated semigroups it is easy to see that solubility of the word problem does not depend on the choice of (finite) generating set for S.

In order to prove that the semigroup S which is a disjoint union of finitely many copies of the free monogenic semigroup, has a soluble word problem, we need some additional information about the presentation in Section 8.2. In [AGR13] we proved that the semigroup S has the finite presentation

$$\left\langle A \mid a^k b = [\alpha(a,k,b,1)]^{\kappa(a,k,b,1)},\ (a,b \in A,\ k \in \{1,2,\ldots,j\} \subseteq \mathbb{N}) \right\rangle \tag{9.1}$$

Now we introduce the necessary Lemmas from Chapter 8 to add more informations to the presentation (9.1). In Chapter 8 we have introduced the set

$$T(a,x,b) = \{y \in N_a \ : \ yx \in N_b\}\ (a,b \in A,\ x \in S),$$

which played a significant role in the proofs. By Lemma 8.1.5 we have

$$N_a = \bigsqcup_{b \in A} T(a,x,b),$$

for any $a \in A$ and $x \in S$. Lemma 8.1.6 describes the set $T(a,x,b)$ as follows:
If $T = T(a,x,b)$ is infinite then there exist sets $F = F(a,x,b)$, $P = P(a,x,b)$ such that the following hold:

(i) $T = F \sqcup P$;

(ii) $P = \{a^{p+qt} \ : \ t \in \mathbb{N}_0\}$ for some $p = p(a,x,b), q = q(a,x,b) \in \mathbb{N}$ and $a^{p-q} \notin T$;

(iii) $F \subseteq \{a, \ldots, a^{p-1}\}$ is a finite set.

And then by Lemmas (8.1.9, 8.1.10, 8.1.12) we proved that The sets

$$\{q(a,x,b) \ : \ a,b \in A,\ x \in S\},\ \{p(a,x,b) \ : \ a,b \in A,\ x \in S\},\ \{F(a,x,b) \ : \ a,b \in A,\ x \in S\},$$

are finite. After this we proved that the set

$$\{T(a,x,b)\ :\ a,b\in A,\ x\in S\}$$

is finite by Lemma 8.1.13.

Now we specify the presentation (9.1) as follows. Since there are finitely many differences q in S by Lemma 8.1.9, take the the least common multiple (LCM) of all these differences D. Thus

$$R'_{a,b}=\bigcup\{a^ib=c_{\alpha(a,i,b)}^{\tau(a,i,b)}\ :i=1,\dots,r(a,b)\},$$

$$R_{a,b}=\bigcup\{a^kb=c_k^{\tau(a,k,b)},\ a^{k+D}b=c_k^{\tau(a,k+D,b)}\ :r(a,b)+1\leq k\leq r(a,b)+D\},\tag{9.2}$$

and then we get the required presentation as

$$R=\bigcup_{a,b\in A}(R'_{a,b}\cup R_{a,b}),$$

where $k=lD$, $l\geq 1$. Notice that from (9.2) we have

$$a^{k+D}b=a^Dc_k^{\tau(a,k,b)}=c_k^{\tau(a,k+D,b)}\tag{9.3}$$

So within $T(a,b,c)$ we have finitely many arithmetic progressions P_t by Lemma 8.1.10, where t is the remainder of division of $r(a,b)+q$ by D for every $q\in\{1,2,\cdots,D\}$ as follows:

$$P_0=\{a^{r(a,b)+1},a^{r(a,b)+1+D},a^{r(a,b)+1+2D},\dots\},$$

$$P_1=\{a^{r(a,b)+2},a^{r(a,b)+2+D},a^{r(a,b)+2+2D},\dots\},$$

$$\vdots$$

$$P_{D-1}=\{a^{r(a,b)+D},a^{r(a,b)+2D},a^{r(a,b)+3D},\dots\}.$$

Lemma 9.2.1. *In S, if we had $a^sb=c^j$ then we can determine j in a finite number of steps.*

PROOF. If the relation $a^sb=c^j$ belongs to R, we are done. Now, suppose that the given relation does not appear in R, that means $s>k$ where $k=lD$ for some l and then $s=hD+t$ where $0\leq t<D$ and thus $a^s\in P_t$. Notice that P_t starts with the two elements $a^{r(a,b)+(t+1)}$, $a^{r(a,b)+(t+1+D)}$ and by doing some calculations as follows:

First we know that

$$s = hD + t,$$

and

$$s - r(a,b) - t - 1 = hD + t - r(a,b) - t - 1 = fD$$

for some f. Thus,

$$hD = fD + r(a,b) + 1.$$

So,

$$s = r(a,b) + t + 1 + fD,$$

which means that a^s is in the f position. Hence,

$$\begin{aligned}
a^s b &= a^{fD+r(a,b)+t+1} b \\
&\equiv \underbrace{a^D a^D \cdots a^D}_{f} a^{r(a,b)+t+1} b \\
&= \underbrace{a^D a^D \cdots a^D}_{f} c_{r(a,b)+t+1}^{\tau\left(a,r(a,b)+t+1,b\right)} && \text{(by (9.2))} \\
&= \underbrace{a^D \cdots a^D}_{f-1} c_{r(a,b)+t+1}^{\tau\left(a,r(a,b)+t+1+D,b\right)} && \text{(by (9.3))} \\
&\vdots \\
&= c_{r(a,b)+t+1}^{\tau\left(a,r(a,b)+t+1+fD,b\right)}
\end{aligned}$$

Therefore, we can obtain j in finitely many steps. □

Theorem 9.2.2. *Every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup has soluble word problem.*

PROOF. Let $S = \bigsqcup_{a \in A} N_a$, and $N_a = \langle a \rangle$. Thus the **Algorithm** is as follows:
Input: $u, v \in A^+$ and $u = x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ and $v = y_1^{j_1} y_2^{j_2} \cdots y_n^{i_n}$, where $x_k, y_l \in A$ for every $1 \leq k \leq m$ and $1 \leq l \leq n$.

Output: $u = v$ or $u \neq v$.

Step 1. Take $D = LCM$ of all differences.

Step 2. Put $k = Dl$ for some $l \geq 1$.

Step 3. Specify the presentation (9.1) as follows:

$$R'_{a,b} = \bigcup \{a^i b = c_{\alpha(a,i,b)}^{\tau(a,i,b)} \ : i = 1, \ldots, r(a,b)\},$$

$$R_{a,b} = \bigcup \{a^k b = c_k^{\tau(a,k,b)},\ a^{k+D} b = c_k^{\tau(a,k+D,b)} \ : k = r(a,b)+1, \ldots, r(a,b)+D\},$$

and then the required presentation is

$$R = \bigcup_{a,b \in A} (R'_{a,b} \cup R_{a,b}).$$

Step 4. For every $T(a,b,c_k)$ in S arrange the arithmetic progressions as follows:

$$P_t = \{a^{r(a,b)+t+1}, a^{r(a,b)+t+1+D}, a^{r(a,b)+t+1+2D}, \ldots\},$$

where $t \in \{0, 1, \cdots, D-1\}$.

Step 5. Transfer u to its normal form as follows:

$$\begin{aligned} u &\equiv x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m} \\ &\equiv (x_1^{i_1} x_2) x_2^{i_2-1} \cdots x_m^{i_m} \\ &= x_{i_{12}}^{i_{12}} x_2^{i_2-1} \cdots x_m^{i_m} && \text{(by Lemma 9.2.1)} \\ &\equiv (x_{i_{12}}^{i_{12}} x_2) x_2^{i_2-2} \cdots x_m^{i_m}. \end{aligned}$$

So, by taking the first power $x_1^{i_1}$ with the next first element x_2 in i_2 time, we get rid of $x_2^{i_2}$ and using the same process with all $x_3^{i_3}, x_4^{i_4}, \cdots, x_m^{i_m}$, we ends with $x_I^{I_M}$ after $i_2 + i_3 + \cdots + i_m$ step. So we have $u = x_I^{I_M}$.

Step 6. Transfer v to its normal form $x_J^{J_N}$ analogously to step 5 .

Step 7. If $I = J$ and $I_M = J_N$ then $u = v$, otherwise $u \neq v$.

Therefore, S has soluble word problem. □

9.3 Subsemigroup membership problem

A finitely generated semigroup S has a soluble subsemigroup membership problem if there exists an algorithm which for any $x \in S$, decides whether $x \in T$ or not where T is a finitely generated subsemigroup of S.

Now we introduce necessary well-known theorems about subsemigroups of the natural number semigroup $\mathbb{N}$. We will use these theorems to devise an algorithm to solve the subsemigroups membership problem for the semigroup under consideration.

Corollary 9.3.1. *Every subsemigroup of* $\mathbb{N}$ *is finitely generated.*

PROOF. This corollary is well known and the proof is not hard as follows. Suppose that S is a subsemigroup of $\mathbb{N}$ and the greatest common divisor of S is 1. Thus the generating set for S is $S \cap \{1, 2, \ldots, 2k\}$ where $k \in \mathbb{N}$ such that for every $n \geq k \;:\; n \in S$ and this because if $m > 2k$ then $m = qk + f$. Thus $m = (q-1)k + k + f$ where $k + f \in S \cap \{1, 2, \ldots, 2k\}$. □

Fact: If S is a subsemigroup of $\mathbb{N}$ then the greatest common divisor (g.c.d) of S is the g.c.d of the generator set of S.

Theorem 9.3.2 ([Hig72], Theorem 1)**.** *Let S be a subsemigroup of* $\mathbb{N}$*, then*

i) *There is $s \in \mathbb{N}$ such that for $n \geq s$, $n \in S$, or*

ii) *There is $n \in \mathbb{N}$, $n > 1$ such that n is a factor of all $s \in S$.*

We prove this theorem as the proof itself leads us to Corollary 9.3.5.
PROOF. Assume that there exist $s_1, s_2, \ldots, s_m \in S$ such that the g.c.d of the collection $(s_1, s_2, \ldots, s_m)$ is 1. Let S' be the subsemigroup of $\mathbb{N}$ generated by $\{s_1, s_2, \ldots, s_m\}$, notice that $S' \subseteq S$. Let $s = 2s_1s_2 \ldots s_m$ and for $b > s$, since the g.c.d of $(s_1, s_2, \ldots, s_m)$ is 1, we may find integers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $\alpha_1 s_1 + \cdots + \alpha_m s_m = b$.
Hence there exist integers q_i and r_i such that $\alpha_i = q_i s_1 \ldots s_{i-1} s_{i+1} \ldots s_m + r_i$ where $0 < r_i \leq s_1 \ldots s_{i-1} s_{i+1} \ldots s_m$ $(i = 2, 3, \ldots, m)$. Now put

$$\beta_1 = \alpha_1 + (q_2 + \cdots + q_m) s_2 s_3 \ldots s_m, \; \beta_i = r_i, \; (i = 2, 3, \ldots, m).$$

Thus $b = \beta_1 s_1 + \beta_2 s_2 + \cdots + \beta_m s_m$. Note that $\beta_i > 0$ for $i = 2, 3 \ldots, m$. But since

$$\beta_2 s_2 + \cdots + \beta_m s_m = r_2 s_2 + \cdots + r_m s_m \leq 2 s_1 s_2 \ldots s_m < b,$$

clearly $\beta_1 > 0$. □

Thus there are two types of subsemigroups of $\mathbb{N}$. The first type contains all natural numbers greater than some fixed natural number, and will be called relatively prime subsemigroups of $\mathbb{N}$. The second type is a fixed integral multiple of a relatively prime subsemigroup.

Corollary 9.3.3. *Every subsemigroup of* $\mathbb{N}$ *has the form*

$$F \cup D_{\mathcal{N},d},$$

where F *is a finite set and* $D_{\mathcal{N},d} = \{da \,:\, a \geq \mathcal{N}\}$.

Definition 9.3.4. Suppose that the semigroup S is generated by $\{n_1, n_2, \cdots, n_k\}$. If there exist two elements $d, \mathcal{N} \in S$ and a set $F \subseteq S$ such that

$$F = S \cap \{1, 2, \cdots, \mathcal{N} - 1\};$$

$$S \cap \{\mathcal{N}, \mathcal{N} + 1, \cdots\} = \{dk \,:\, k \in \mathbb{N},\ dk \geq \mathcal{N}\},$$

then we say that S is defined by the *triple* $[d, \mathcal{N}, F]$.

Corollary 9.3.5. *Suppose that* S *is a subsemigroup of the natural number semigroup* $\mathbb{N}$. *Suppose that* S *is generated by* $n_1, n_2, \cdots, n_k$. *Then* S *is defined by the triple* $[d, \mathcal{N}, F]$ *where* d *is the greatest common divisor of* $\{n_1, n_2, \cdots, n_k\}$,

$$\mathcal{N} = 2dn_1 n_2 \cdots n_k,$$

and

$$F \subseteq \{1, 2, \cdots, \mathcal{N} - 1\}.$$

PROOF. Follows immediately from Theorem 9.3.2 and Corollary 9.3.1.

Corollary 9.3.6. *Suppose that* S *is a subsemigroup of the free monogenic semigroup* N. *Suppose that* S *is generated by* $a^{n_1}, a^{n_2}, \cdots, a^{n_k}$. *Then* S *is defined by the triple* $[d, \mathcal{N}, F]$ *where* d *is the greatest common divisor of* $\{a^{n_1}, a^{n_2}, \cdots, a^{n_k}\}$,

$$\mathcal{N} = a^2 d a^{n_1} a^{n_2} \cdots a^{n_k},$$

and

$$F \subseteq \{a, a^2, \cdots, a^{\mathcal{N}-1}\}.$$

PROOF. Directly by Corollary 9.3.5. □

After understanding how subsemigroups of N behaves we are ready to start designing the algorithm. Since

$$S = N_1 \sqcup N_2 \sqcup \cdots \sqcup N_n,$$

and T is a subsemigroup of S, then

$$T = T_1 \sqcup T_2 \sqcup \cdots \sqcup T_m,$$

where $T_i \leq N_i$ for every $i \in \{1, 2, \cdots, m\}$. Consequently, the generator set for T is

$$A_T = \bigcup_{i \in \{1,2,\cdots,m\}} A_{T_i},$$

where A_{T_i} is the generator set of T_i for every $i \in \{1, 2, \cdots, m\}$. Thus T is finitely generated ([ABF$^+$01], Proposition 3.1).

Lemma 9.3.7. *Suppose that the subsemigroup $U_j = \langle N_j \cap A_T \rangle$ is defined by the triple $[d_j, \mathcal{N}_j, F_j]$. Then there is an algorithm which takes arbitrary U_i, U_j and $b \in A_T$ and test whether*

$$U_i b \cap N_j \subseteq U_j$$

or not.

PROOF. Let $a_j^r \in U_i b \cap N_j$. Then

$$a_j^r \in U_j \iff a_j^r \in F_j \text{ or } a_j^r = a_j^{d_j h_j} \text{ for some } d_j h_j \geq d_j t_j \text{ where } d_j t_j = \mathcal{N}_j,$$

by Corollary 9.3.5. $\square$

Remark 9.3.8. We use the phrase " description $(U_1, \cdots, U_k)$" by which we mean "$U_1 \cup \cdots \cup U_k$."

Theorem 9.3.9. *Every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup has a soluble subsemigroup membership problem.*

PROOF. Let S be such a semigroup, with all the foregoing notation remaining in force. Then the **Algorithm** is as follows:
Input. $T = \langle A_T \rangle$ and $T \leq S$, $x \in S$.

Output. $x \in T$ or not.

Step 1. Define each $U_j = \langle N_j \cap A_T \rangle$ by the triple $\left[d_j, \mathcal{N}_j, F_j\right]$, for every $1 \leq j \leq m$.

Step 2. Check if

$$(U_1, U_2, \cdots, U_m) = T,$$

which means check whether

$$U_i x \subseteq \bigcup_{i=1}^{m} U_i \text{ for every } i \in \{1, 2, \cdots, m\} \text{ and for every } x \in A_T,$$

by Lemma 9.3.7. If yes then go to step 5. If there was $a_i^{r_i} x = a_j^{r_j}$ and $a_j^{r_j} \notin U_j$ then go to step 2.

Step 3. Add the missing element $a_j^{r_j}$ to U_j and then we have

$$U_j^{(+1)} = \langle A_{U_j} \cup a_j^{r_j} \rangle,$$

and then we have the new description

$$\left(U_1, U_2, \cdots, U_{j-1}, U_j^{(+1)}, U_{j+1}, \cdots, U_m\right), \tag{9.4}$$

Step 4. We start again with the new description (9.4) and we keep adding these missing elements with all $i \in \{1, 2, \cdots, m\}$.

Step 5. We reach to the final description

$$\left(U_1^{(+s_1)}, U_2^{(+s_2)}, \cdots, U_j^{(+s_j)}, \cdots, U_m^{(+s_m)}\right) = T.$$

Which means that $U_j^{(+s_j)} b \subseteq \bigcup_{i=1}^{m} U_i^{(+s_i)}$ for every $b \in A_T$ and for every $j \in \{1, 2, \cdots, m\}$ and that because as we explained before each U_j is defined by the triple $\left[d_j, \mathcal{N}_j, F_j\right]$. So if we add an element $a_j^{r_j}$ to U_j that means, by Corollary 9.3.3, we reduce the gaps in F_j and they are finite, or we reduce the difference d_j and we can do this just finitely often. Thus we add finitely many elements in each U_j, which implies that this process terminates. So now each $U_j^{(+s_j)}$ is defined by the triple

$$\left[d_j^{(+s_j)}, \mathcal{N}_{|j}^{(+s_j)}, F_j^{(+s_j)}\right].$$

Step 6. If we were given $x = a_h^{r_h} \in S$ and we want to see if $x \in T$ or not then we just take this element and see in $U_h^{(+s_h)}$ if

$$a_h^{r_h} \in F_h^{(+s_h)},$$

or

$$r_h = d_h^{(+s_h)} k \text{ for some } d_h^{(+s_h)} k \geq d_h^{(+s_h)} t \text{ where } d_h^{(+s_h)} t = \mathcal{N}_h^{(+s_h)},$$

then $x \in T$ otherwise $x \notin T$. □

Remark 9.3.10. Since residual finiteness is preserved under taking substructures, the subsemigroup of the semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup is residually finite by Theorem 8.3.4.

CHAPTER

TEN

CONCLUDING REMARKS AND CONJECTURES

Arguably, the free monogenic semigroup N is the most fundamental commutative semigroup. It is well-known that all finitely generated commutative semigroups are finitely presented and residually finite. Finite presentability was first proved by Rédei [Réd65]; see also [Gri95, Section 9]. Residual finiteness was proved by Malcev [Mal83]; see also [Car01] and [Lal71]. In this thesis we have shown that disjoint unions of copies of N (which, of course, need not be commutative) in this respect behave like commutative semigroups. It would be interesting to know if this generalises to unions of commutative semigroups.

Question 1. *Is it true that every semigroup which is a finite disjoint union of finitely generated commutative semigroups is necessarily: (a) finitely presented; (b) residually finite?*

By way of contrast, there is no reason to believe that our results would generalize to disjoint unions of copies of a free (non-commutative) semigroup of rank > 1. We proved in Chapter 6 that every balanced semigroup which is a disjoint union of two copies of the free semigroup of rank two is finitely presented and residually finite and this because we put a condition which is the product of any two generators preserves length. So what if we removed this condition, is it still the same result?

Question 2. *Does there exist a semigroup S which is a disjoint union of two copies of a free semigroup of rank two which is not: (a) finitely presented; (b) residually finite?*

We proved in Chapter 7 that every rectangular band of finitely presented semigroups is finitely presented (Theorem 7.2.1), and we proved the same theorem about residual finiteness but in the free monogenic semigroups case (Theorem 7.4.3). We therefore pose the following question.

Question 3. *Is every semigroup which is a rectangular band of residually finite semigroups residually finite?*

Finally, we investigated in Chapter 9 how the subsemigroups of disjoint unions of finitely many copies of the free monogenic semigroup behave and we show that such semigroups are finitely generated and residually finite by the facts that each subsemigroup of N is finitely generated, and residual finiteness is preserved under taking substructures.

Conjecture 4. *Every subsemigroup of every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup is finitely presented.*

We have thought about this conjecture and we tried to follow the same technique, which we believe in the rightness of it, in the proof of the Theorem 8.2.1 but we faced some complicated notations that prohibited us from continuing the proof as it follows:
We have $S = \bigsqcup_{a \in A} N_a$, $T \leq S$ and $T = \bigsqcup_{i \in \{1,\dots,m\}} N_i'$ where $N_i' \leq N_i$ for every $i \in \{1,\dots,m\}$. Also T is generated by the set A_T. Thus the set

$$W = \{(a^i)^m,\ (a^i)^m(a^j)^n\ :\ a^i, a^j \in A_T,\ m,n \in \mathbb{N}\},$$

is the set of normal forms for T. Hence for any $m, m', n, n' \in \mathbb{N}_0$, there exist unique $\alpha(a^i, m, a^j, m', b^k, n, b^l, n') \in A_T$ and $\kappa(a^i, m, a^j, m', b^k, n, b^l, n') \in \mathbb{N}$ such that

$$(a^i)^m(a^j)^{m'}(b^k)^n(b^l)^{n'} = [\alpha(a^i, m, a^j, m', b^k, n, b^l, n')]^{\kappa(a^i,m,a^j,m',b^k,n,b^l,n')}. \qquad (10.1)$$

So the generators A_T and relations (10.1) provide an infinite presentation for T. Now the difficulty appears when we want to prove that the relations (10.1) are consequences of the relations

$$(a^i)^m(a^j)(b^k)^n(b^l) = [\alpha(a^i, m, a^j, b^k, n, b^l)]^{\kappa(a^i,m,a^j,b^k,n,b^l)}, \qquad (10.2)$$

and as far as we are concerned, such a notation is not "nice" to complete the proof although it is doable.

Now we will list some open questions which are related to some finiteness conditions for the semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup.

Question 5. *Does every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup have a finite complete rewriting system?*

Question 6. *Is every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup automatic?*

Question 7. *Is every semigroup which is a disjoint union of finitely many copies of the free monogenic semigroup FA-presentable?*

Question 8. *Is every semigroup which is a disjoint union of finitely many copies of N unary FA-presentable?*

For the background of the last 4 questions see [BO93], [CRRT01, HT03], [CORT09], [CRT12], respectively.

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