ENDS OF SEMIGROUPS

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Ends of Semigroups

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Abstract

The aim of this thesis is to understand the algebraic structure of a semigroup by studying the geometric properties of its Cayley graph. We define the notion of the partial order of ends of the Cayley graph of a semigroup. We prove that the structure of the ends of a semigroup is invariant under change of finite generating set and at the same time is inherited by subsemigroups and extensions of finite Rees index. We prove an analogue of Hopfs Theorem, stating that a group has 1, 2 or infinitely many ends, for left cancellative semigroups and that the cardinality of the set of ends is invariant in subsemigroups and extension of finite Green index in left cancellative semigroups. We classify all semigroups with one end and make use of this classification to prove various finiteness properties for semigroups with one end.

We also consider the ends of digraphs with certain algebraic properties. We prove that two quasi-isometric digraphs have isomorphic end sets. We also prove that vertex transitive digraphs have 1, 2 or infinitely many ends and construct a topology that reflects the properties of the ends of a digraph.

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Chapter 1

Introduction and Background

The ends of groups have been studied in depth over the last century and the results have had an impact on many other areas of group theory. The notion of ends is a geometric property and has over the years given rise to an algebraic characterisation of groups in terms of their ends. In this thesis I use a generalisation of the notion of ends of a graph to the ends of a digraph to study the ends of semigroups with the aim of getting a similar characterisation. This chapter is devoted to introducing the notion of the ends of a graph, some of the main theorems relating to the ends of a group and to introducing the generalised notion of the ends of a digraph. We delay introducing most of the well-known or fundamental concepts and theorems relating to other relevant areas until Chapter 2 to give the reader a clearer insight into the material.

1.1 Ends of Graphs and Groups

The notion of the ends of a graph was introduced by Freudenthal in [13] which is linked with his 1931 thesis [12]. The ends of a graph can be thought of as the appearance of the graph at infinity. There is no mention of graphs in [13] but rather countable, discrete spaces with adjacency of points. The definition I give of the ends of a graph is that of Halin from [19] which is equivalent to Freundenthal's in [13] in the case of locally finite graphs. To introduce the concept of the ends of a graph we shall begin by defining some basic graph theoretical concepts. We first introduce digraphs and then graphs as a special kind of digraph.

A *digraph* Γ is a pair $(V\Gamma, E\Gamma)$ consisting of a set of elements $V\Gamma$ referred to as *vertices* and a subset $E\Gamma \subseteq V\Gamma \times V\Gamma$ referred to as *edges*. For brevity we write $v \to w$ if $(v, w) \in E\Gamma$ and which graph Γ we are referring to is clear from the context. A *graph* is a digraph Γ where $(w, v) \in E\Gamma$ for all $(v, w) \in E\Gamma$. A *loop* is an edge of the form $(v, v) \in E\Gamma$, we allow graphs to have loops.

Let Γ be a digraph. A *walk* is a sequence of vertices $(v_0, v_1, \ldots, v_n)$ such that $(v_i, v_{i+1}) \in E\Gamma$ for all $0 \leq i \leq n-1$. We say this walk is of *length* n. We also use the notation $v_0 \to v_1 \to \ldots \to v_n$ to denote a walk. We also call an infinite sequence $(v_0, v_1, \ldots)$ such that $(v_i, v_{i+1}) \in E\Gamma$ for all $i \in \mathbb{N}_0$ a *walk* and say it is of infinite length. A *path* is a walk containing no repeated vertices. For the sake of brevity we will refer to a path with initial vertex in a set A and final vertex in a set B as a path from A to B. A *cycle* is a walk where the first and last vertices are the same but all others are distinct.

Let Γ be a graph. A *ray* is an infinite sequence $(v_0, v_1, \ldots)$ of pairwise distinct vertices such that $(v_i, v_{i+1}) \in E\Gamma$ for all $i \in \mathbb{N}_0$. We say two rays $\mathbf{x}$ and $\mathbf{y}$ are *equivalent* if there exist infinitely many pairwise disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. Of course, since Γ is a graph, if there are infinitely many pairwise disjoint paths from $\mathbf{x}$ to $\mathbf{y}$ then there are infinitely many pairwise disjoint paths from $\mathbf{y}$ to $\mathbf{x}$, namely the paths from $\mathbf{x}$ to $\mathbf{y}$ with the order of the vertices reversed. The property of rays being equivalent forms, as the name would suggest, an equivalence relation. We will demonstrate the details of this later in Proposition 1.9 when we define the ends of a digraph. The equivalence classes on the set of rays of a graph Γ are called the *ends* of Γ and denoted by $\Omega(\Gamma)$.

In 1944 Hopf [22] applied the definition of the ends of a graph to the Cayley graph of a group with the aim of better understanding the geometric properties of groups. Let G be a group and let A be an inverse-closed generating set for G. The *Cayley graph of G with respect to A*, which we denote by $\Gamma(G, A)$, is the graph with vertices G and edge set $E\Gamma = \{(g, ga) : g \in G, a \in A\}$ for each $g \in G$ and each $a \in A$. Hopf defined the ends of the group G with respect to the generating set A to be $\Omega(\Gamma(G, A))$. The first important result relating to the ends of a group is the following theorem. This was first mentioned in [22] but not explicitly proved. However, it can be recovered through Theorem 1.3, which is a result of Möller in [32, Proposition 1].

Theorem 1.1. *[22] Let G be a group and let A and B be finite generating sets for G. Then $|\Omega(\Gamma(G, A))| = |\Omega(\Gamma(G, B))|$.*

This theorem allows one to talk about the number of ends of a finitely generated group rather than just the number of ends with respect to some generating set. If G is a finitely generated group we use $\Omega(G)$ to denote $\Omega(\Gamma(G, A))$ where A is some finite generating set.

Subgroups of finite index are in a sense close to being nearly the whole group. They often share the same properties as the group. These properties have been studied in papers such as [34] and [35] and include notions such as being finitely generated, finitely presented, residually finite and having solvable word problem. These notions are defined later in Chapter 5. Hopf considered how the number of ends of a subgroup of finite index related to the number of ends of the group and proved the following result:

Theorem 1.2. *[22, Satz IV] Let G be a finitely generated group and let H be a subgroup of finite index. Then $|\Omega(G)| = |\Omega(H)|$.*

This theorem shows that subgroups of finite index behave the same way as a group when considering their number of ends. This is further justification that the number of ends reflects an algebraic property and is not just a geometric property.

A theorem which incorporates both of these results came in [32] and is staged in purely graph theoretical terms. It relates the number of ends of two graphs that are quasi-isometric. To understand the definition of a quasi-isometry I shall introduce some relevant concepts from the theory of metric spaces.

Firstly, we equip a connected graph with a notion of distance and consider it as a metric space. The *graph metric* of the graph Γ, which we denote by d_Γ, defines $d_\Gamma(v, w)$ to be the shortest distance of a path from v to w.

We say two metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there exists a function $f : X \to Y$ and there exist $\lambda, \epsilon, \mu \in \mathbb{R}^+$ such that for all $x, y \in X$

$$\frac{1}{\lambda} d_X(x, y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \epsilon$$

and for all $x' \in Y$ there exists $x \in X$ such that

$$d_Y(f(x), x') \leq \mu.$$

Quasi-isometries describe spaces which globally look the same but can look different locally. I mean this in the sense that if two metric spaces were quasi-isometric they might look very different in a small area around points but if one was to look at them both from far away they would look very similar. An example of two metric spaces which are quasi-isometric are $\mathbb{Z}$ and $\mathbb{R}$ equipped with the usual Euclidean metric. An example of a quasi-isometry between them is $f : \mathbb{Z} \to \mathbb{R}$ defined by $f(x) = x$. We can choose $\lambda = 1$, $\epsilon = 0$ and $\mu = 0.5$ as every real number can be expressed in the form $x + r$ where $r \in [-0.5, 0.5)$ and $x \in \mathbb{Z}$.

We say two connected graphs Γ_1 and Γ_2 are *quasi-isometric* if Γ_1 equipped with its graph metric is quasi-isometric to Γ_2 equipped with its graph metric.

Let Γ be a graph. A vertex $v \in V\Gamma$ is called *locally finite* if the set $\{u \in V\Gamma : (v, u) \in E\Gamma\}$ is finite. A graph is called *locally finite* if every vertex is locally finite.

Theorem 1.3. *[32, Proposition 1] Let* Γ_1 *and* Γ_2 *be locally finite connected graphs. If* Γ_1 *is quasi-isometric to* Γ_2 *then* $|\Omega(\Gamma_1)| = |\Omega(\Gamma_2)|$.

We say two finitely generated groups G_1 and G_2 are *quasi-isometric* if there exists a finite generating set A for G_1 and a finite generating set B for G_2 such that $\Gamma(G_1, A)$ is quasi-isometric to $\Gamma(G_2, B)$. In [10] de la Harpe says

> A property $(\mathcal{P})$ of finitely generated groups is *geometric* if, for a pair (Γ_1, Γ_2) of finitely generated groups which are quasi-isometric,

Γ_1 has property $(\mathcal{P})$ if and only if Γ_2 has property $(\mathcal{P})$.

Examples of geometric properties include finite presentability and having solvable word problem, see [10, Section IV.B] for details and more information on quasi-isometries. An algebraic characterisation of groups in terms of their ends, which we shall shortly discuss, was proved before Theorem 1.3. This does not diminish the usefulness of Theorem 1.3 when considering groups but reinforces how the geometric property of ends reflects algebraic structure.

In terms of getting an algebraic characterisation for ends the most important result from Hopf's paper restricts the number of ends it is possible for a group to have.

Theorem 1.4. *[22, Satz I] Let G be an infinite, finitely generated group. Then G has one, two or $2^{\aleph_0}$ ends.*

Theorem 1.4 may seem to be fairly inconsequential, it merely notes that an infinite finitely generated group has one of three prescribed numbers of ends. However, it is Theorem 1.4 that lead to an algebraic characterisation of groups possessing a given number of ends. The first step toward this was the following theorem:

Theorem 1.5. *[22, Satz 5] A group G has two ends if and only if G contains an infinite cyclic group as a subgroup of finite index.*

The next major result in the area of ends of groups came in 1972 with Stallings' celebrated classification of groups with more than one end. This result was combinatorial in flavour but with links to 3-dimensional manifold theory and Bass–Serre theory. There have been many different versions of this proof. A very succinct graph-theoretic version is provided by Krön in [26]. This version is far more accessible than the original proof.

To understand Stallings' result we introduce some group-theoretic constructions. The first is that of a free product with amalgamation. Let A, B and C be groups, let $\phi_A : C \to A$ and $\phi_B : C \to B$ be monomorphisms, let $\langle X_1 \,|\, R_1 \rangle$ be a presentation for A and let $\langle X_2 \,|\, R_2 \rangle$ be a presentation for B. Then the *free product of A and B amalgamated at C* is the group with presentation $\langle X_1 \cup X_2 \,|\, R_1, R_2, \phi_A(c) = \phi_B(c)\, (c \in C) \rangle$.

The second definition is due to Higman, Neumann and Neumann (hence the name). Let $G = \langle X \,|\, R \rangle$ be a group, let H be a subgroup of G, let $\phi : H \to G$ be a monomorphism and let t be an element not in G. The *HNN-extension* of G by ϕ and t, which we denote by $G *_H t$, is the group $\langle X, t \mid R, t^{-1}ht = \phi(h)\ (h \in H) \rangle$.

Both of these constructions can also be defined via universality properties, which means they are independent of the choice of presentation of their foundation groups. We include the universality properties here to demonstrate the similarities between these constructions but do not prove the existence or uniqueness, for details see [36, Chapter 6.4].

For the free product with amalgamation we let A, B and C be groups and let $\phi_A : C \to A$ and $\phi_B : C \to B$ be monomorphisms. Then the free product of A and B amalgamated at C is a group $A *_C B$ such that for any group G and homomorphisms $\sigma_A : A \to G$ and $\sigma_B : B \to G$ such that $\sigma_A(\phi_A(c)) = \sigma_B(\phi_B(c))$ for all $c \in C$ there exists a unique homomorphism $\sigma : A *_C B \to G$.

For the HNN-extension, let G be a group, let $H, K \leq G$ and let $\theta : H \to K$ be an isomorphism. The HNN-extension of G is a group $G *_H t$ such that for any group X and homomorphism $\sigma : G \to X$ such that $\sigma(H)$ is conjugate to $\sigma(K)$ there exists a unique homomorphism $\phi : G *_H t \to X$.

Theorem 1.6. *[40, 1.B.6] Let G be a finitely generated group. Then G has more than one end if and only if G can be written as a non-trivial free product with amalgamation $B *_C D$ where C is finite, or G can be written as a non-trivial HNN-extension $B *_C x$ where C is finite.*

For this result Stalling was awarded the Cole prize in 1970 and Theorem 1.6 has had repercussions in many areas such as in [11], [28], and [33].

1.2 Ends of Digraphs

The notion of the ends of a digraph was introduced by Zuther in [43]. The ends of a digraph are a generalisation of the ends of a graph and are built up as equivalence classes of “rays” in a similar fashion.

Let Γ be a digraph. We define an *out-ray* to be an infinite sequence $(v_0, v_1, \ldots)$ such that $(v_i, v_{i+1}) \in E\Gamma$ for all $i \in \mathbb{N}_0$ and $v_i \neq v_j$ for $i \neq j$. We use the notation $v_0 \to v_1 \to \ldots$ to denote an out-ray. We define an *in-ray* to be an infinite sequence $(v_0, v_1, \ldots)$ such that $(v_{i+1}, v_i) \in E\Gamma$ for all $i \in \mathbb{N}_0$ and $v_i \neq v_j$ for $i \neq j$. We use the notation $v_0 \leftarrow v_1 \leftarrow \ldots$ to denote an in-ray. We collectively refer to in-rays and out-rays as *rays*.

Let $\preccurlyeq$ be a binary relation on a set X. A *preorder* is a reflexive and transitive binary relation. We define a pre-order, $\preccurlyeq$, on the rays of a digraph Γ. Let $\mathbf{x}$ and $\mathbf{y}$ be rays. We say $\mathbf{x}$ is *greater* than $\mathbf{y}$, written $\mathbf{x} \succcurlyeq \mathbf{y}$, if there exist infinitely many pairwise disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. We say two rays $\mathbf{x}$ and $\mathbf{y}$ are *equivalent* if $\mathbf{x} \succcurlyeq \mathbf{y}$ and $\mathbf{x} \preccurlyeq \mathbf{y}$. In [43] it is shown that this is an equivalence relation but we include a proof for demonstration's sake. However, first we will give some equivalent definitions for when a ray $\mathbf{x}$ is greater than a ray $\mathbf{y}$.

Lemma 1.7. *Let Γ be a digraph and let $\mathbf{x}$ and $\mathbf{y}$ be rays. Then $\mathbf{x} \not\succcurlyeq \mathbf{y}$ if and only if there exists a finite set $F \subseteq V\Gamma$ such that all paths from $\mathbf{x}$ to $\mathbf{y}$*

contain an element of F.

Proof. If $\mathbf{x} \not\succcurlyeq \mathbf{y}$ then there do not exist infinitely many disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. If for any finite set of paths $\pi_1, \pi_2, \ldots, \pi_n$ from $\mathbf{x}$ to $\mathbf{y}$ there existed another path π from $\mathbf{x}$ to $\mathbf{y}$ such that π was disjoint from every π_i then by induction there would be infinitely many disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. This means there exists a finite set of paths $\pi_1, \pi_2, \ldots, \pi_n$ such that any other path from $\mathbf{x}$ to $\mathbf{y}$ intersects π_i for some $1 \leq i \leq n$. If we set F to be the union of the vertices in the paths $\pi_1, \pi_2, \ldots, \pi_n$ then it follows that any path from $\mathbf{x}$ to $\mathbf{y}$ passes through F.

Assume that there exists a finite set $F \subseteq V\Gamma$ such that all paths from $\mathbf{x}$ to $\mathbf{y}$ pass through F. Then by the pigeonhole principle any collection of at least $|F| + 1$ paths contains two paths that pass through the same vertex of F, and hence these paths are not disjoint. □

Corollary 1.8. *Let Γ be a digraph and let $\mathbf{x}$ and $\mathbf{y}$ be rays. Then $\mathbf{x} \succcurlyeq \mathbf{y}$ if and only if for any finite set of vertices F there exists a path from $\mathbf{x}$ to $\mathbf{y}$ that does not pass through F.*

We now prove that $\preccurlyeq$ is a pre-order.

Proposition 1.9. *[43, Proposition 2.2] Let Γ be a digraph. The relation $\preccurlyeq$ forms a pre-order on the set of rays of Γ.*

Proof. The relation $\preccurlyeq$ is clearly reflexive as there is always a path of length 0 from any vertex to itself.

It remains to show that $\preccurlyeq$ is transitive. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be rays and assume $\mathbf{x} \succcurlyeq \mathbf{y}$ and $\mathbf{y} \succcurlyeq \mathbf{z}$. Let F be a finite set and let $\mathbf{y} = (y_0, y_1, \ldots)$.

Since $\mathbf{y}$ is a ray there exists an $N \in \mathbb{N}$ such that $y_i \notin F$ for all $i \geq N$.

Assume $\mathbf{y}$ is an out-ray. As $\mathbf{x} \succcurlyeq \mathbf{y}$ there exists a path π from $\mathbf{x}$ to $\mathbf{y}$ that does not pass through $\{y_0, y_1, \ldots, y_N\} \cup F$ and so has final vertex y_m for some $m \geq N$. As $\mathbf{y} \succcurlyeq \mathbf{z}$ there exists a path σ from $\mathbf{y}$ to $\mathbf{z}$ that does not pass through $\{y_0, y_1, \ldots y_m\} \cup F$ and has initial vertex y_{m+k} for some $k \in \mathbb{N}$. By concatenating the paths π, $y_m \rightarrow y_{m+1} \rightarrow \ldots \rightarrow y_{m+k}$ and σ we get a path from $\mathbf{x}$ to $\mathbf{y}$ that does not pass through F. Hence, for any finite set F there exists a path from $\mathbf{x}$ to $\mathbf{z}$ that does not pass through F. By Corollary 1.8 it follows that $\mathbf{x} \succcurlyeq \mathbf{z}$.

Assume $\mathbf{y}$ is an in-ray. As $\mathbf{y} \succcurlyeq \mathbf{z}$ there exists a path π from $\mathbf{y}$ to $\mathbf{z}$ that does not pass through $\{y_0, y_1, \ldots, y_N\} \cup F$ and so has initial vertex y_m for some $m \geq N$. As $\mathbf{x} \succcurlyeq \mathbf{y}$ there exists a path σ from $\mathbf{x}$ to $\mathbf{y}$ that does not pass through $\{y_0, y_1, \ldots y_m\} \cup F$ and has terminal vertex y_{m+k} for some $k \in \mathbb{N}$. By concatenating the paths σ, $y_{m+k} \rightarrow y_{m+k-1} \rightarrow \ldots \rightarrow y_m$ and π we get a path from $\mathbf{x}$ to $\mathbf{y}$ that does not pass through F. Hence, for any finite set F there exists a path from $\mathbf{x}$ to $\mathbf{z}$ that does not pass through F. By Corollary 1.8 it follows that $\mathbf{x} \succcurlyeq \mathbf{z}$. □

Let Γ be a digraph. We have shown that the rays of Γ under $\preccurlyeq$ form a pre-order. Recall that a ray $\mathbf{x}$ was said to be equivalent to a ray $\mathbf{y}$ if $\mathbf{x} \succcurlyeq \mathbf{y}$ and $\mathbf{x} \preccurlyeq \mathbf{y}$. The *end poset* of Γ, which we denote by $\Omega(\Gamma)$, is the partially ordered set induced by the pre-order $\preccurlyeq$ on equivalence classes of rays. When not making use of the order structure of $\Omega(\Gamma)$ we will sometimes refer to $\Omega(\Gamma)$ as the ends of Γ.

1.3 Results

In this thesis I consider the ends of the Cayley graph of a semigroup. The aim is to relate the algebraic structure of a semigroup to the structure of its ends.

The definition of the Cayley graph of a semigroup is analogous to that of the Cayley graph of a group. Although the Cayley graph of a group is defined to be a digraph it can be thought of as a graph since for every edge (g, ga) there is an edge $(ga, gaa^{-1}) = (ga, g)$. However, when considering semigroups this is not the case. Moreover, when considering a semigroup the digraph defined by the edges (g, ga) is not necessarily isomorphic to the digraph defined by the edges (g, ag). This is in contrast to the situation with groups.

Let S be a semigroup and let A be a generating set for S. The *right*

Cayley graph of S with respect to A is the digraph with vertices S and edges (s, sa) for $s \in S$ and $a \in A$, which is denoted by $\Gamma_r(S, A)$. The *left Cayley graph* of S with respect to A is the digraph with vertices S and edges (s, as) for $s \in S$ and $a \in A$, which is denoted by $\Gamma_l(S, A)$. In a group G with generating set A we have a graph isomorphism between $\Gamma_r(G, A)$ and $\Gamma_l(G, A)$, namely $g \mapsto g^{-1}$. This means when considering groups one does not need to state whether one is working in the right or left Cayley graph. However, for semigroups these two graphs may be very different, see for instance Example 5.59. We define the *left ends* of S with respect to A as the ends of $\Gamma_l(S, A)$ and the *right ends* of S with respect to A as the ends of $\Gamma_r(S, A)$.

Let S be a semigroup with operation $\cdot$. The *dual* semigroup of S, denoted S^*, is the semigroup with set S and multiplication $*$ defined by $s * t = t \cdot s$ for all $s, t \in S$. One may see that the left Cayley graph of S is equal to the right Cayley graph of the dual semigroup S^*. It follows that any results that are obtained when considering the right Cayley graph of a semigroup S also hold in the left Cayley graph of the dual semigroup S^*. Henceforth, we will only work in the right Cayley graph of semigroups unless stated otherwise.

In Chapter 3 we prove the foundational results about the ends of a semigroup. These results are similar in flavour to those proved in [22]. The first main theorem in this thesis, which is the basis on which the other main results rely, is, roughly speaking, that the ends of the right Cayley graph of a semigroup do not depend on the generating set used to define that graph.

Theorem 1.10. *Let S be a semigroup and let A and B be finite generating sets. Then $\Omega(\Gamma_r(S, A))$ is isomorphic as a poset to $\Omega(\Gamma_r(S, B))$.*

Bearing this theorem in mind we define the *ends of a finitely generated semigroup S* to be $\Omega(S) = \Omega(\Gamma_r(S, A))$ where A is any finite generating set for S.

The next two theorems are semigroup analogues of Theorem 1.2. Recall Theorem 1.2 states that a subgroup of finite index has the same number of ends as the parent group.

Let S be a semigroup and let T be a subsemigroup of S. We say T is of finite *Rees index* in S if $|S \setminus T| < \infty$. We say the *index* of T in S is $|S \setminus T| + 1$. Rees index may seem to be an artificial way of defining index when compared with the notion in group theory, however, it encapsulates many basic semigroup constructions. These include both adjoining an identity element and adjoining a zero element.

Theorem 1.11. *Let S be a finitely generated semigroup and let T be a subsemigroup of finite Rees index. Then $\Omega(S)$ is isomorphic as a poset to $\Omega(T)$.*

Another possible definition of index is Green index, which will be defined later. The analogue of Theorem 1.11 does not hold in full generality when finite Rees index is replaced by finite Green index. However the analogue does hold for a special class of semigroups, namely left-cancellative semigroups. This will be discussed in Section 3.3.

Theorem 1.12. *Let S be finitely generated left-cancellative semigroup and let T be a subsemigroup of finite Green index. Then $|\Omega(S)| = |\Omega(T)|$.*

Examples are provided of semigroups with any finite number of left ends or right ends. This is not conducive to an algebraic classification other than by looking at semigroups with a given number of ends. However, an analogue of Hopf's Theorem (Theorem 1.4) is given for left-cancellative semigroups. This reinforces the impression given by Theorem 1.12 that maybe it is possible to get an algebraic classification of left-cancellative semigroups from their ends.

Theorem 1.13. *An infinite, finitely generated, left-cancellative semigroup has one, two or infinitely many ends.*

Unlike groups it is possible for a left-cancellative semigroup to have $\aleph_0$ ends. This is further explored in Section 5.4.

We finish Chapter 3 by comparing our results with the existing notion of the ends of a semigroup given by Jackson and Kilibarda in [24]. We will give further details of this definition and the results contained in [24] in Section

3.5 so as to not overload the reader with too much notation. We also include analogues of the finite index theorems under their definition.

In Chapter 4 we consider the behaviour of ends under the action of semigroups. We consider certain classes of digraphs and actions in the hope of generalising Theorem 1.13.

A digraph Γ is said to be *out-locally finite* if the set $\{u \in V\Gamma : (v, u) \in E\Gamma\}$ is finite for all $v \in V\Gamma$.

Let X be a set. A *semi-metric* is a function $d : X \times X \to \mathbb{R}^\infty$ such that the following hold for all $x, y, z \in X$

1. $d(x, y) \geq 0$,

2. $d(x, y) = 0$ if and only if $x = y$,

3. $d(x, y) + d(y, z) \geq d(x, z)$.

We generalise the graph metric for digraphs by defining the *digraph semi-metric*. We define the digraph semi-metric for the digraph Γ as d_Γ where $d_\Gamma(u, v)$ is the length of the shortest path from u to v for $u, v \in V\Gamma$. When Γ is a connected graph this coincides with the graph metric.

The following definition is due to Gray and Kambites [16]. We say two semi-metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there exists a function $f : X \to Y$ and there exist $\lambda, \epsilon, \mu \in \mathbb{R}^+$ such that for all $x, y \in X$

$$\frac{1}{\lambda}d_X(x,y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x,y) + \epsilon$$

and for all $x' \in Y$ there exists $x \in X$ such that

$$d_Y(f(x), x'), d_Y(x', f(x)) \leq \mu.$$

We say two digraphs Γ_1 and Γ_2 are *quasi-isometric* if Γ_1 equipped with its digraph semi-metric is quasi-isometric to Γ_2 equipped with its digraph semi-metric.

We prove an analogue of Theorem 1.3, namely:

Theorem 1.14. *If Γ_1, Γ_2 are two out-locally finite quasi-isometric digraphs then $\Omega(\Gamma_1) = \Omega(\Gamma_2)$.*

A digraph Γ is *connected* if there is an undirected path between any two vertices of Γ. A digraph Γ is *vertex transitive* if for any $x, y \in V\Gamma$ there exists an automorphism g of Γ such that $g(x) = y$.

We use this theorem to describe the end structure of vertex transitive digraphs.

Theorem 1.15. *Infinite, out-locally finite, connected, vertex transitive digraphs have one, two or infinitely many ends.*

We end this section by constructing a topology on a digraph together with its ends. The end topology on a graph is instrumental in the higher level work involving ends, such as in [26]. The topology on digraphs is by its nature not as well behaved as that on graphs but carries many of the same properties and characteristics.

Chapter 5 contains a characterisation of all semigroups with one end.

Theorem 1.16. *Let S be a finitely generated semigroup with one end. Then S is the disjoint union of a semigroup T, where tS is infinite for all $t \in T$, and a possibly empty ideal I such that $|iS| < \infty$ for all $i \in S$, and one of the following holds:*

(A) T has a right group $R = G \times E$ as a subsemigroup of finite Rees index, where E is a finite right-zero semigroup and G is a finitely generated group with one end; or

(B) T has a presentation of the form

$$\begin{aligned} \langle a, u_1, u_2, \ldots, u_n | \quad u_i \cdot u_j &= a^{\alpha(i,j)} u_{\beta(i,j)}, \\ u_i \cdot a &= a^{f(i)} u_{g(i)}, \\ a^{\pi(i,j)} u_i &= a^{\pi(j,i)} u_j \rangle, \end{aligned}$$

where

$$\alpha : \{1,2,\ldots,n\} \times \{1,2,\ldots,n\} \to \mathbb{N}_0,$$

$$\beta : \{1,2,\ldots,n\} \times \{1,2,\ldots,n\} \to \{1,2,\ldots,n\},$$

$$f : \{1,2,\ldots,n\} \to \mathbb{N}_0,$$

$$g : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\},$$

$$\pi : \{1,2,\ldots,n\} \times \{1,2,\ldots,n\} \to \mathbb{N}.$$

This theorem is closer to Theorem 1.5, in the approach and strength of result, than Stallings' classification of groups. We prove some interesting properties of semigroups with one end using the classification. We also prove that a cancellative semigroup which cannot be embedded in a group has $2^{\aleph_0}$ ends.

Chapter 6 details some open questions and directions for further research that lead on from the results in this thesis.

Chapter 2

Preliminary Definitions

This chapter consists of definitions, notation and useful theorems which will be used throughout the rest of the thesis. Much of the content of this chapter is of a basic nature but is given for the sake of completeness. We use the notation $\mathbb{N}$ to denote the set of natural numbers $\{1, 2, 3, \ldots\}$. We will use the notation $\mathbb{N}_0$ to denote the set $\mathbb{N} \cup \{0\}$. The notation $\mathbb{R}^+$ will be used to denote the set of positive real numbers.

Let X be a set and let $R \subseteq X \times X$. We call R a *binary relation* over X. We now introduce various properties that it is possible for a binary relation to have.

We say R is:

- *reflexive* if $(x, x) \in R$ for all $x \in X$;

- *symmetric* if for all $(x, y) \in R$ we have $(y, x) \in R$;
- *anti-symmetric* if $(x, y) \in R$ and $(y, x) \in R$ implies that $x = y$;
- *transitive* if for all $(x, y), (y, z) \in R$ we have $(x, z) \in R$;
- *total* if $(x, y) \in R$ or $(y, x) \in R$ for all $x, y \in R$;
- an *equivalence relation* if R is reflexive, symmetric and transitive;
- a *preorder* if R is reflexive and transitive;
- a *partial order* if R is reflexive, anti-symmetric and transitive;
- a *total order* if R is reflexive, anti-symmetric, total and transitive.

The next two definitions apply only to sets equipped with a partial order, later referred to as *posets*. Let X be a set and let R be a partial order. We say a subset A of X is an *anti-chain* if $(a, a') \notin R$ for any $a, a' \in A$ with $a \neq a'$. We say $x \in X$ is *minimal* if for all $(y, x) \in R$ we have $y = x$. We say a partially ordered set X satisfies the *minimal condition* if every non-empty subset of X contains a minimal element.

Let R, S be binary relations over a set X. The *composition* of R and S, denoted by $R \circ S$, is the binary relation consisting of pairs (x, y) where $(x, z) \in R$ and $(z, y) \in S$ for some $z \in X$. Let R be a binary relation over the set X and let S be a binary relation over the set Y. We define the binary relation $R \cup S$ to be the binary relation over the set $X \cup Y$ with elements $(x, y) \in R \cup S$ if and only if $(x, y) \in R$ or $(x, y) \in S$.

2.1 Semigroups

A *semigroup* is a set S with a binary operation $\cdot$ satisfying the property that for all $x, y, z \in S$ we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. We call a semigroup M a *monoid* if there exists an element $1 \in M$ such that $m \cdot 1 = 1 \cdot m = m$ for all $m \in M$. We call a monoid G a *group* if for all $g \in G$ there exists $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = 1$. From now onwards we will omit the operation $\cdot$, unless we want to emphasise a product, and will simply juxtapose elements to indicate multiplication.

An important notion in any algebraic structure is that of substructures. Let S be a semigroup and let T be a subset of S. We say T is a *subsemigroup* of S if $xy \in T$ for all $x, y \in T$. We say T is a *right ideal* if $ts \in T$ for all $t \in T$ and all $s \in S$. We call T a *left ideal* if $st \in T$ for all $t \in T$ and $s \in S$. We say T is an *ideal* if T is both a right and left ideal. A *principal right ideal* is a right ideal I such that $I = sS$ for some $s \in S$. Similarly, a *principal left ideal* is a left ideal I such that $I = Ss$ for some $s \in S$. If S is a monoid with identity 1 we call a subsemigroup T a *submonoid* if $1 \in T$. If S is a group we call a subsemigroup T a *subgroup* if for all $t \in T$ the element $t^{-1} \in T$ where

t^{-1} is the inverse of t in S.

Let S be a semigroup and let $A \subseteq S$. We define $\langle A \rangle$ to be the least subsemigroup of S with respect to containment that contains the subset A. If S is a monoid we define $Mon\langle A \rangle$ to be the least submonoid of S with respect to containment that contains the subset A and $\langle A \rangle$ to be the least subsemigroup of S with respect to containment that contains the subset A. If S is a group we define $Gp\langle A \rangle$ to be the least subgroup of S with respect to containment that contains the subset A, $Mon\langle A \rangle$ to be the least submonoid of S with respect to containment that contains the subset A and $\langle A \rangle$ to be the least subsemigroup of S with respect to containment that contains the subset A.

A *semigroup generating set* for a semigroup S is a subset $A \subseteq S$ such that $\langle A \rangle = S$. A *monoid generating set* for a monoid M is a subset $A \subseteq M$ such that $Mon\langle A \rangle = M$. A *group generating set* for a group G is a subset $A \subseteq G$ such that $Gp\langle A \rangle$. We will normally just refer to these as generating sets and the type will be clear from the context. If S is a semigroup and A is a generating set then we can see that for every element $s \in S$ there exists $a_1, a_2, \ldots, a_n \in A$ such that $s = a_1 \cdot a_2 \cdot \ldots \cdot a_n$.

For a semigroup S, a generating set A and an element $s \in S$ we define $|s|_A = \min\{n \in \mathbb{N}_0 : a_1 \cdot a_2 \cdot \ldots \cdot a_n = s, a_i \in A\}$.

Let S be a semigroup. Then we denote by S^1 the semigroup obtained from

S by adjoining an identity, that is a new element 1 such that $1 \cdot s = s \cdot 1 = s$ for all $s \in S \cup \{1\}$. We denote by S^0 the semigroup obtained from S by adjoining a zero, that is a new element 0 such that $0 \cdot s = s \cdot 0 = 0$ for all $s \in S \cup \{0\}$.

2.2 Green's Relations

Green's relations were introduced in 1951 by Green, see [18]. Green's relations are binary relations whose structure in some sense reflects the algebraic structure of the semigroup.

Let S be a semigroup and let $x, y \in S$. We say x is *$\mathcal{R}$-related* to y, denoted $x\mathcal{R}y$, if there exist $s, t \in S^1$ such that $xs = y$ and $yt = x$. Similarly, we say x is *$\mathcal{L}$-related* to y, denoted $x\mathcal{L}y$, if there exist $s, t \in S^1$ such that $sx = y$ and $ty = x$. We say x is *$\mathcal{H}$-related* to y, denoted $x\mathcal{H}y$, if $x\mathcal{R}y$ and $x\mathcal{L}y$. We say x is *$\mathcal{D}$-related* to y, denoted $x\mathcal{D}y$ if there exists $z \in S$ such that $x\mathcal{R}z$ and $z\mathcal{L}y$.

Equivalently we may say x is $\mathcal{D}$-related to y if there exists $z \in S$ such that $x\mathcal{L}z$ and $z\mathcal{R}y$. We may also phrase $\mathcal{R}$ in terms of principal right ideals. Namely, in a semigroup S we have $x\mathcal{R}y$ if and only if $xS^1 = yS^1$.

The $\mathcal{R}$-relation gives an equivalence relation on elements of S. We denote the $\mathcal{R}$-class of S containing an element s by R_s. We use the notation L_s,

H_s and D_s similarly. There is a natural partial order on the $\mathcal{R}$-classes of S. We define a preorder on S by $x \leq_{\mathcal{R}} y$ if there exists $z \in S^1$ such that $yz = x$. Here we shall demonstrate this preorder and the poset arising from the preorder.

Let S be a semigroup. To demonstrate that $\leq_{\mathcal{R}}$ is reflexive we note that for all $x \in S$ we have $1 \in S^1$ such that $x \cdot 1 = x$. For transitivity we let $x, y, z \in S$ such that there exist $s, t \in S^1$ with $ys = x$ and $zt = y$. Then clearly we have $zts = ys = x$ so the binary relation $\leq_{\mathcal{R}}$ is transitive. For any preorder one can construct a poset by identifying all equivalent elements, in the case of $\leq_{\mathcal{R}}$ these are the $\mathcal{R}$-classes of the semigroup.

We say a semigroup is $\mathcal{R}$-simple if it only has one $\mathcal{R}$-class. We define $\mathcal{L}$-simple and $\mathcal{D}$-simple similarly.

A related notion which makes use of subsemigroups is due to Wallace in [41]. Let S be a semigroup, let T be a subsemigroup of S and let $x, y \in S$. We say x is $\mathcal{R}^T$*-related* to y, denoted $x\mathcal{R}^T y$ if there exists $s, t \in T^1$ such that $xs = y$ and $yt = x$. We may define $\mathcal{L}^T$, $\mathcal{H}^T$ and $\mathcal{D}^T$ analogously. As with Green's $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$ and $\mathcal{D}$ relations $\mathcal{R}^T$, $\mathcal{L}^T$, $\mathcal{H}^T$ and $\mathcal{D}^T$ are all equivalence relations. The relation $\mathcal{R}^T$ has the property that the equivalence class of an element $s \in S$, denoted by R_s^T, is either contained in T or is contained in $S \setminus T$. The same property also holds for $\mathcal{L}^T$, $\mathcal{H}^T$ and $\mathcal{D}^T$.

Lemma 2.1. *Let S be a semigroup, let T be a subsemigroup of S and let U be a subsemigroup of T. Then $\mathcal{R}^U \subseteq \mathcal{R}^T$.*

Proof. If $(s,t) \in \mathcal{R}^U$ then there exist $u, v \in U$ such that $su = t$ and $tv = s$. However, $U \leq T$ so $(s,t) \in \mathcal{R}^T$. □

Let S be a semigroup and let $s \in S$. The *right translation of S by s* is the map $\rho_s : S \to S$ defined by $x \mapsto xs$. The *left translation of S by s* is the map $\rho_s : S \to S$ defined by $x \mapsto sx$. Let X, Z be sets, let Y be a subset of X and let $f : X \to Z$. The *restriction of f to Y*, denoted by $f|_Y$, is the function $f|_Y : Y \to Z$ given by $f|_Y(y) = f(y)$.

Lemma 2.2 (Green's Lemma)**.** *Let a, b be $\mathcal{R}$-related elements in a semigroup S and let $s, t \in S^1$ be such that $as = b$ and $bt = a$. Then the right translations $\rho_s|_{L_a}$ and $\rho_t|_{L_b}$ are mutually $\mathcal{R}$-class preserving bijections from L_a onto L_b and from L_b onto L_a respectively.*

2.3 Indices and Congruences

There are various notions of how large a substructure is inside of an algebraic object. We introduce some of these notions as they apply to semigroups.

In groups the notion of how large a substructure is inside a group is called the *group index*. Let G be a group and let H be a subgroup of G. We say two elements, x, y, of G lie in the same *coset* of H in G if $xy^{-1} \in H$. The *index* of H in G is defined to be the number of distinct cosets of H in G. Or equivalently, we say H is of index n if n is the least cardinal such that there exists $g_1, g_2, \ldots, g_n$ such that for any $g \in G$ there exists i such that $gH = g_iH$. We say H is of *finite index* if n is finite. Many properties are shared between groups and subgroups of finite index. In particular if G is a finitely generated group and H is a subgroup of finite index then H is finitely generated.

In semigroups there are many notions of index, for example Rees index, Green index and syntactic index. Here we shall introduce just two and provide theorems relating to finite generation and behaviour of subsemigroups of groups.

Let S be a semigroup and T be a subsemigroup of S. We say T is of finite *Rees index* in S if $|S \setminus T| < \infty$. We say the *Rees index* of T in S is $|S \setminus T| + 1$. The notion of Rees index has been used to study how finiteness properties, such as finite presentability and solvable word problem, pass to subsemigroups in papers such as [7] and [20].

The following theorem is attributed to Jura in [25], however, a more constructive proof can be found in [7, Corollary 3.2].

Theorem 2.3. *[25] Let S be a semigroup and T be a subsemigroup of S of* *finite Rees index. Then S is finitely generated if and only if T is finitely generated.*

Lemma 2.4. *Let S be an infinite group and let T be a subsemigroup of S of finite Rees index. Then $T = S$.*

Proof. Let $x \in S \setminus T$. As S is infinite and $S \setminus T$ is finite it follows that T is infinite. As S is a group the set xT must be of the same cardinality as T and hence is infinite. This means $xT \cap T$ is also infinite as $S \setminus T$ is finite. As $S \setminus T$ is finite there are only finitely many $t \in T$ such that $xt \in T$ and $t^{-1} \in S \setminus T$. It follows there exists $t \in T$ such that $xt \in T$ and $t^{-1} \in T$. Now $x = xtt^{-1}$ but $xt \in T$ and $t^{-1} \in T$ and therefore $x \in T$, a contradiction. □

Another definition of index is that of *Green index* introduced in [17]. Let S be a semigroup and let T be a subsemigroup of S. We say T is of finite *Green index* in S if $S \setminus T$ has finitely many $\mathcal{H}^T$-classes. We say the *Green index* of T in S is the number of $\mathcal{H}^T$-classes in $S \setminus T$ plus one. The proof that this is well-defined is covered in [17].

Theorem 2.5. *[6, Corollary 9.2] Let S be a semigroup and T be a subsemigroup of S of finite Green index. Then S is finitely generated if and only if T is finitely generated.*

Theorem 2.6. *[17, Corollary 34] Let S be a group and let T be a subsemigroup of S of finite Green index. Then T is a subgroup of S of finite group* *index.*

The notions of Rees index and Green index are quite different. If a subsemigroup T of a semigroup S is of finite Rees index then T is also of finite Green index, however, the values of their respective index may be different. In contrast if a subsemigroup has finite Green index then it may not have finite Rees index.

A related notion to index is that of a *quotient*. Let S be a semigroup and let ρ be an equivalence relation on S. We say ρ is a *right-congruence* if for all $(x, y) \in \rho$ and for all $s \in S$ we have $(xs, ys) \in \rho$. A left-congruence is defined similarly. We call an equivalence relation ρ on a semigroup S a *congruence* if it is both a left- and a right-congruence. We say a congruence is of *finite index* if it has finitely many equivalence classes.

Let S be a semigroup and let ρ be a congruence. Let ${}^{s}/_{\rho}$ be the equivalence class of s in ρ. The *quotient semigroup* of S with respect to ρ is the semigroup with elements ${}^{S}/_{\rho}$ and multiplication ${}^{s}/_{\rho} \cdot {}^{t}/_{\rho} = {}^{st}/_{\rho}$. For details, such as the fact that this product is well-defined, see [9, Section 1.5]

A special example of this is the Rees quotient. Let S be a semigroup and let I be an ideal. The *Rees quotient*, denoted ${}^{S}/_{I}$, is the quotient of S by the congruence $I \times I \cup \{(s, s) : s \in S\}$.

2.4 Rewriting Systems

A useful tool in semigroup theory is the notion of a string rewriting system. String rewriting systems can sometimes be used to obtain a unique representation of each element of a semigroup given by a presentation. For greater detail and proofs of lemmas see [4, Chapter 1].

Let A be a finite set of symbols. We define the following sets

- $A^n = \{a_1 a_2 \cdots a_n : a_i \in A\}$;
- $A^* = \bigcup_{n \geq 0} A^n$;
- $A^+ = \bigcup_{n \geq 1} A^n$.

Let A be a finite set and let $\rightarrow$ be a binary relation on A^*. Let $\rightarrow^{-1}$ be the reverse of $\rightarrow$ and let $\circ$ be composition of relations.

- $\rightarrow^0$ is the identity relation.
- $\rightarrow^n = \rightarrow \circ \rightarrow^{n-1}$ for $n > 0$.
- $\rightarrow^* = \cup_{n \geq 0} \rightarrow^n$ and $\rightarrow^+ = \cup_{n > 0} \rightarrow^n$.
- $\leftrightarrow = \rightarrow \cup \rightarrow^{-1}$.
- $\leftrightarrow^0$ is the identity relation.
- $\leftrightarrow^n = \leftrightarrow \circ \leftrightarrow^{n-1}$ for $n > 0$.

- $\leftrightarrow^* = \cup_{n \geq 0} \leftrightarrow^n$ and $\leftrightarrow^+ = \cup_{n>0} \leftrightarrow^n$.

The relation $\to^*$ is the reflexive and transitive closure of $\to$ and the relation $\leftrightarrow^*$ is the smallest equivalence relation on A^* containing $\to$.

A *string rewriting system* is a pair (A, R), where A is a finite alphabet and R is a set of pairs (l, r), known as *rewriting rules*, drawn from $A^* \times A^*$. The *single reduction relation* is defined as follows: $u \to_R v$ (where $u, v \in A^*$) if there exists a rewriting rule $(l, r) \in R$ and words $x, y \in A^*$ such that $u = xly$ and $v = xry$. That is, $u \to_R v$ if one can obtain v from u by substituting the word r for a subword l of u, where (l, r) is a rewriting rule. The *reduction relation* $\to_R^*$ is the reflexive and transitive closure of $\to_R$.

Let (A, R) be a string rewriting system. We say $w \in A^*$ is *irreducible* if there is no $v \in A^*$ such that $w \to_R v$. We denote the set of all irreducible elements of A^* by $IRR(A, R)$. If $x, y \in A^*$ and $x \to_R^* y$ and y is irreducible then y is a *normal form* for x.

Let $S = (A, R)$ be a string rewriting system.

- We say S is *confluent* if for all $w, u, v \in A^*$ such that $w \to_R^* u$ and $w \to_R^* v$ there exists $x \in A^*$ such that $u \to_R^* x$ and $v \to_R^* x$.

- We say S is *locally confluent* if for all $w, u, v \in A^*$ such that $w \to_R u$ and $w \to_R v$ there exists $x \in A^*$ such that $u \to_R^* x$ and $v \to_R^* x$.

- We say S has the *Church-Rosser property* if for all $u, v \in A^*$ if $u \leftrightarrow_R^* v$ then there exists $w \in A^*$ such that $u \rightarrow_R^* w$ and $v \rightarrow_R^* w$.

Lemma 2.7. *Let $S = (A, R)$ be a string rewriting system. Then S is Church-Rosser if and only if S is confluent.*

Corollary 2.8. *Let (A, R) be a confluent string rewriting system. Then each $w \in A^*$ has at most one normal form.*

Let (A, R) be a string rewriting system. The relation $\rightarrow_R$ is *noetherian* if there is no infinite sequence $x_0, x_1, \ldots$ such that $x_i \rightarrow_R x_{i+1}$ for every $i \in \mathbb{N}_0$.

Lemma 2.9. *Let (A, R) be a string rewriting system. If $\rightarrow_R$ is noetherian then every $w \in A^*$ has a normal form.*

If $S = (A, R)$ is a noetherian and confluent string rewriting system then we say S is *complete*.

Theorem 2.10. *If (A, R) is a complete string rewriting system then every $w \in A^*$ has a unique normal form.*

Theorem 2.11. *Let $S = (A, R)$ be a noetherian string rewriting system. Then S is confluent if and only if S is locally confluent.*

Let (A, R) be a string rewriting system. The *Thue congruence* generated by R is the relation $\leftrightarrow_R^*$. The set of equivalence classes of the Thue congruence of a string rewriting system forms a monoid under composition of representatives.

Theorem 2.12. *The monoid with presentation $\langle A|R\rangle$ is isomorphic to the quotient semigroup of $(A \cup \{1\})^*$ with respect to the Thue congruence $\leftrightarrow_R^*$.*

2.5 Regular and Simple Semigroups

Regularity is an important property in semigroup theory. Regular semigroups are similar in some ways to groups. Let e be an element of a semigroup S. We say e is *idempotent* if $e^2 = e$.

The following lemma is well-known but a proof is provided nonetheless.

Lemma 2.13. *Let S be a semigroup and let $s \in S$. If there exists $i, j \in \mathbb{N}$ with $i \neq j$ such that $s^i = s^j$ then S contains an idempotent.*

Proof. Without loss of generality there exists $i < j$ with $s^i = s^j$. If $j = 2i$ then $(s^i)^2 = s^{2i} = s^j = s^i$ and s^i is an idempotent. If $j > 2i$ then $(s^{j-i})^2 = s^{2j-2i} = s^j s^{j-2i} = s^i s^{j-2i} = s^{j-i}$. If $i < j < 2i$ then we show there exists $k \in \mathbb{N}$ such that $s^k = s^i$ and $k \geq 2i$. As $i < j < 2i$ there exists $1 \leq n < i$ such that $j = i + n$, we prove by induction that for all $m \in \mathbb{N}$ we have $s^{j+mn} = s^i$. For $m = 1$ we get that $s^{j+n} = s^{i+n} = s^j = s^i$. Assuming that for all $m < p$ that $s^{j+mn} = s^i$ it follows that $s^{j+pn} = s^{j+n}s^{(p-1)n} = s^{i+n}s^{(p-1)n} = s^{j+(p-1)n} = s^i$. Thus for a suitably large m we have $j + mn \geq 2i$. □

Corollary 2.14. *Any finite semigroup contains an idempotent.*

Proof. Let S be a finite semigroup and let $|S| = n$. Let $s \in S$ and consider $s, s^2, s^3, \ldots, s^{n+1}$. There are $n+1$ of these so $s^i = s^j$ for some i, j. □

Let S be a semigroup and let $e, f \in S$ be idempotents. We say $e \leq f$ if $ef = fe = e$. This gives a partial order on the idempotents of S. An idempotent e is called *primitive* if e is minimal in the partial order on idempotents.

Let S be a semigroup and let $s \in S$. We say s is *regular* if there exists $t \in S$ such that $sts = s$. A semigroup S is called *regular* if s is regular for all $s \in S$.

A special kind of regular semigroup is a *right zero semigroup*. This is a semigroup S in which $st = t$ for all $s, t \in S$.

The proof of the following theorems can be found in [23, Chapter 2].

A $\mathcal{D}$-class is regular if all elements in the $\mathcal{D}$-class are regular.

Theorem 2.15. *[9, Theorem 2.11]*

1. *If a $\mathcal{D}$-class D of a semigroup S contains a regular element, then every element is regular.*

2. *If D is regular, then every $\mathcal{L}$-class and every $\mathcal{R}$-class contained in D contains an idempotent.*

Let S be a semigroup and let $s \in S$. We say t is an *inverse* of s if $sts = s$ and $tst = t$.

Theorem 2.16. *Let s be an element of a regular $\mathcal{D}$-class D in a semigroup S.*

1. *If t is an inverse of s then $t \in D$ and the two $\mathcal{H}$-classes $R_s \cap L_t$ and $L_s \cap R_t$ contain, respectively, the idempotents st and ts.*

2. *If $t \in D$ is such that $R_s \cap L_t$ and $L_s \cap R_t$ contain idempotents e, f respectively, then H_t contains an inverse s^* of s such that $ss^* = e$ and $s^*s = f$.*

3. *No $\mathcal{H}$-class contains more than one inverse of s.*

Simple semigroups play a similar role in semigroups as simple groups do in group theory. This is in the sense that they often arise arise as the "blocks" from which other semigroups are constructed. The two notions, however, are not the same but are similar in nature. A semigroup is called *simple* if it has no proper ideals.

Lemma 2.17. *A semigroup is simple if and only if $SxS = S$ for all $x \in S$.*

A semigroup is *completely simple* if it is simple and contains a primitive idempotent.

An important construction that relates to simple semigroups is that of a Rees matrix semigroup. Let G be a group, let I and Λ be sets and let P be a $\Lambda \times I$ matrix with elements from G. The *Rees matrix semigroup* associated with G, I, Λ and P, denoted by $\mathcal{M}[G; I, \Lambda; P]$, is the set $I \times G \times \Lambda$ with multiplication $(i, g, \lambda) \cdot (j, h, \mu) = (i, gp_{\lambda,j}h, \mu)$. Here $p_{\lambda i}$ is the (λ, i) entry in the matrix P.

Theorem 2.18 (Rees)**.** *A semigroup is completely simple if and only if it is isomorphic to a Rees matrix semigroup* $\mathcal{M}[G; I, \Lambda; P]$*.*

A semigroup S is called *completely regular* if there exists a unary operation $s \mapsto s^{-1}$ such that $(s^{-1})^{-1} = s$, $ss^{-1}s = s$ and $ss^{-1} = s^{-1}s$.

Lemma 2.19. *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is completely regular;*
2. *every element of S lies in a subgroup of S;*
3. *every $\mathcal{H}$-class in S is a group.*

Recall a poset X satisfies the minimality condition if every non-empty subset of X contains a minimal element.

Theorem 2.20. *Let S be a simple semigroup. The following are equivalent:*

1. S is completely simple;

2. S is completely regular;

3. the $\mathcal{R}$-class and the $\mathcal{L}$-class posets both satisfy the minimal condition;

4. S contains at least one minimal left ideal and one minimal right ideal.

2.6 Properties of Left Cancellative Semigroups

Some of these properties can be derived from work in [39] but full proofs are included for completeness.

Let S be a semigroup. We say S is *left-cancellative* if for all $x, y, z \in S$ such that $xy = xz$ then we must have $y = z$. A semigroup is *right cancellative* if for all $x, y, z \in S$ such that $yx = zx$ then $y = z$. If S is both left and right cancellative then we shall call it *cancellative.*

Lemma 2.21. *Let S be a left-cancellative semigroup. Then all idempotents of S are $\mathcal{R}$-related.*

Proof. Let $e, f \in S$ be idempotents. As e is idempotent we have $e^2 = e$ and hence $e^2 s = es$ for all $s \in S$. Then by left-cancellativity we have $es = s$ for all $s \in S$. This means that $ef = f$ and $fe = e$, as required. □

Corollary 2.22. *Let S be a left-cancellative semigroup. All regular elements are $\mathcal{R}$-related.*

Proof. Let $x \in S$ be a regular element. This means there exists $y \in S$ such that $xyx = x$. The element xy is an idempotent and $xy \cdot x = x$ so $x\mathcal{R}xy$. Hence, every regular element is $\mathcal{R}$-related to an idempotent and as all idempotents are $\mathcal{R}$-related, all regular elements are $\mathcal{R}$-related. $\square$

Lemma 2.23. *Let S be a left-cancellative semigroup. If S has more than one $\mathcal{R}$-class then there exists $x \in S$ such that $(x^i, x^j) \notin \mathcal{R}$ for all $i \neq j$.*

Proof. If $x^2\mathcal{R}x$ for all $x \in S$ then there exists $s \in S^1$ such that $x^2s = x$. This means $x^2sx = x^2$ and then left cancellativity means $xsx = x$, so x is regular. If $x^2\mathcal{R}x$ for all $x \in S$ then all elements of S are regular and by Corollary 2.22 are all $\mathcal{R}$-related.

It follows there exists $x \in S$ such that $(x, x^2) \notin \mathcal{R}$. If $(x, x^2) \notin \mathcal{R}$ for some $x \in S$ then if $x^i\mathcal{R}x^j$ for some $i < j$ there exists $s \in S^1$ such that $x^js = x^i$. This means $x^jsx^{j-i-1} = x^ix^{j-i-1} = x^{j-1}$ and hence by left cancellativity $x^2sx^{j-i-1} = x$, a contradiction. Thus $(x^i, x^j) \notin \mathcal{R}$ for all $i \neq j$. $\square$

Corollary 2.24. *A left cancellative semigroup has one or infinitely many $\mathcal{R}$-classes.*

Lemma 2.25. *Let S be a left-cancellative semigroup. If $x \in S$ has a non-trivial $\mathcal{R}$-class the $\mathcal{R}$-class of x is of the form xU where U is the set of regular elements.*

Proof. Let R be a non-trivial $\mathcal{R}$-class and let $x \in R$. As R is non-trivial there exists $y \in R$ such that $y \neq x$. This means there exist $s, t \in S$ such that $xs = y$ and $yt = x$. We have that $xst = x$ and it follows that $xsts = yts = xs$ and hence by left-cancellativity $sts = s$. Hence, S contains regular elements and as y was arbitrary $R \subseteq xU$. Clearly, if $s\mathcal{R}t$ then $xs\mathcal{R}xt$ and by Corollary 2.22 elements of U are all $\mathcal{R}$-related so $xU \subseteq R$, as required. □

Corollary 2.26. *Let S be a left-cancellative semigroup then the set of regular elements is an $\mathcal{R}$-simple regular subsemigroup of S.*

Proof. If S has no regular elements the statement trivially holds. If S has only one regular element, say x, then there exists $y \in S$ such that $xyx = x$. But then xy is a regular element so $xy = x$ and hence $x^2 = xyx = x$ so x is an idempotent so the statement holds. If S has at least two regular elements then the set U of regular elements is a non-trivial $\mathcal{R}$-class. This means $U = uU$ for all $u \in U$ so U is an $\mathcal{R}$-simple regular subsemigroup. □

Lemma 2.27. *Let S be a left-cancellative semigroup and let U be the subsemigroup of regular elements. Then $S \setminus U$ is an ideal.*

Proof. Let $s, t \in S$ such that $st \in U$. For all idempotents $e \in S$ and for all $x \in S$ we have $e^2x = ex$ and hence $ex = x$. It follows from Corollary 2.22 that if $st \in U$ there exists $t' \in S$ such that $st' = e$. As $es = s$ it follows that $s\mathcal{R}e$ and hence $s \in U$. Also if $st \in U$ then st is regular so there exists $x \in S$

such that $stxst = st$ but then by left-cancellativity $txst = t$ so t is regular and hence $t \in U$. Thus if $st \in U$ we have $s, t \in U$ so $S \setminus U$ is an ideal. $\square$

Lemma 2.28. *Let S be a left-cancellative semigroup. If S has infinitely many $\mathcal{R}$-classes and an infinite $\mathcal{R}$-class then it has infinitely many infinite $\mathcal{R}$-classes.*

Proof. If S has infinitely many $\mathcal{R}$-classes then by Lemma 2.23 there exists $x \in S$ such that $(x^i, x^j) \notin \mathcal{R}$ for all $i \neq j$. As S has an infinite $\mathcal{R}$-class the set U of regular elements is infinite by Lemma 2.25. As U is infinite it is certainly non-empty and hence contains an idempotent e. Now consider the sets x^iU. If $x^iU \cap x^jU \neq \emptyset$ for some $i < j$ then there exists $u \in U$ such that $x^ie = x^ju$. This means that $e = x^{j-i}u$, but then as $e^2 = e$ we have that $e^2x = ex$. By left-cancellativity $ex = x$ so $x\mathcal{R}e$ and hence $x \in U$. However, U is a subsemigroup of S so $x^2 \in U$, a contradiction. Therefore, all x^iU are distinct infinite $\mathcal{R}$-classes. $\square$

Here we provide an example which exhibits many of the properties we have discussed in this section.

Example 2.29. Let $G = \langle A \mid R \rangle$ be an infinite finitely presented group. We construct a semigroup $(S, \cdot)$ by introducing a new element x and defining $S = \langle A, x \mid R, 1_G \cdot x = x \rangle$ where 1_G is the identity element of G. These relations form a complete rewriting system with the following unique normal forms. As $1_G \cdot v = v$ for any $v \in G \cup \{x\}$ we have unique normal forms for S given by alternating products from $\{x^i : i \in \mathbb{N}\}$ and $G \setminus 1$, possibly ending in 1_G. Now $u1_G \cdot v = u \cdot v$ for all $u, v \in S$, however, this does not contradict the cancellativity of S as $1_G \cdot v = v$ for all $v \in S$.

We now consider products with normal forms $u \cdot v$. If $u = u'x$ for some $u' \in S^1$ then $u \cdot v = uv$ for all $v \in S$. Similarly if $v = xv'$ for some $v' \in S^1$ we have $u \cdot v = uv$. Now we consider the cases when $u = u'g$ and $v = g'v'$ where $u', v' \in S$ and $g, g' \in G$. Either $v = g'$ or $v = g'xv''$. If v' is trivial then $u \cdot v = u'(gg')$. If $v = g'xv''$ and $g' \neq g^{-1}$ then $u \cdot v = u'(gg')xv''$. Finally if $v = g^{-1}xv''$ then $u \cdot v = u'xv''$. By comparing these multiplication forms one can see that S is left cancellative. However, it is not right cancellative as $x1_G \cdot g = x \cdot g$ but $x \neq x1_G$.

We now describe the $\mathcal{R}$-classes of S. If $u \in S$ is of the form $u'x$ then for any $v \in S$ there can be no $w \in S$ such that $uvw = u$. This means each element of S of the form $u'x$ lies in a trivial $\mathcal{R}$-class. If $u \in S$ is of the form $u'g$, however, then for any $g' \in G$ we have $u'g \cdot g^{-1}g' = u'g'$. These $u'g'$ are all distinct so each element of the form $u'g$ lies in an infinite $\mathcal{R}$-class. Now if we multiply by anything other than an element of G we add an x. As previously seen these block us from forming $\mathcal{R}$-classes. This means the non-trivial $\mathcal{R}$-classes are of the form $u'G$.

From the multiplication we saw $x^i \mathcal{R} x^j$ only holds if $i = j$ and furthermore all x^i, x^j are distinct for $i \neq j$. We then get infinitely many infinite $\mathcal{R}$-classes of the form x^iG.

A *right group* is a semigroup isomorphic to the direct product of a group and a right-zero semigroup.

Lemma 2.30. *[23, Question 2.6.6] A semigroup S is a right group if and only if it is $\mathcal{R}$-simple and left-cancellative.*

Lemma 2.31. *A Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ is left cancellative if and only if $|I|$=1.*

Proof. Assume $|I| \geq 2$ and let i_1, i_2 be distinct elements of I. Let $i \in I$ and let $\lambda \in \Lambda$. Then

$$(i, 1_G, \lambda)(i_1, p^{-1}_{\lambda,i_1}, \lambda) = (i, 1_G, \lambda) = (i, 1_G, \lambda)(i_2, p^{-1}_{\lambda,i_2}, \lambda)$$

but $i_1 \neq i_2$ so $\mathcal{M}[G; I, \Lambda; P]$ is not left-cancellative.

Assume $I = \{i\}$. Let $(i, g_1, \lambda_1), (i, g_2, \lambda_2), (i, h, \mu) \in \mathcal{M}[G; I, \Lambda; P]$. If $(i, h, \mu)(i, g_1, \lambda_1) = (i, h, \mu)(i, g_2, \lambda_2)$ then we have the equality $(i, hp_{\mu,i}g_1, \lambda_1) = (i, hp_{\mu,i}g_2, \lambda_2)$. It follows that $\lambda_1 = \lambda_2$ and $hp_{\mu,i}g_1 = hp_{\mu,i}g_2$ and hence $g_1 = g_2$. □

Lemma 2.32. *Let S be a cancellative semigroup without identity. Then S^1 is a cancellative semigroup.*

Proof. With the aim of getting a contradiction, assume that S^1 is not left cancellative, the proof for S^1 not being right cancellative follows by symmetry. As S^1 is not left cancellative there exist $x, s, t \in S^1$ such that $xs = xt$ for some $s \neq t$. Clearly $x \neq 1$, so $x \in S$. If $s, t \in S$ then the equality $xs = xt$ is true in S, however S is cancellative, a contradiction. We assume that $x, t \in S$ and $s = 1$, it follows $x = xt$. Now for any $y \in S$ we have that $xy = xty$. These elements all lie in S so by left cancellativity of S we have $y = ty$ for all $y \in S$. This means t is a left identity for all elements in S. Also $x^2 = xtx$ holds in S and right cancellativity of S implies $x = xt$ and similarly we deduce that t also a right identity, a contradiction. □

Lemma 2.33. *Let S be a left cancellative monoid. If $s\mathcal{R}1$ then $s\mathcal{H}1$.*

Proof. If $s\mathcal{R}1$ then there exists $t \in S$ such that $st = 1$. It follows that $sts = s$ and then left cancellativity ensures that $ts = 1$. □

Lemma 2.34. *Let S be a cancellative monoid. If S is not a group then any generating set A contains an element a such that $\langle a \rangle \cong (\mathbb{N}, +)$.*

Proof. Let A be a generating set and let $a \in A$. If there exists $i > j \in \mathbb{N}$ such that $a^i = a^j$ then cancellativity ensures that $a^{i-j} = 1$ and it follows that $a\mathcal{H}1$. If $a\mathcal{H}1$ for all $a \in A$ then S is a group, this means there must exist $a \in A$ such that $a^i \neq a^j$ for all $i \neq j \in \mathbb{N}$. □

Corollary 2.35. *Let S be a cancellative semigroup which is not a monoid then S has an element a such that $\langle a \rangle \cong (\mathbb{N}, +)$.*

Lemma 2.36. *Let S be a semigroup. Let $s, t \in S$. Let $sR_t = \{sr : r \in R_t\}$. Then $sR_t \subset R_{st}$*

Proof. If $x\mathcal{R}y$ then $sx\mathcal{R}sy$. It follows $sR_t \subseteq R_{st}$. □

Lemma 2.37. *Let S be a left-cancellative semigroup. Let $t \in S$ and let R_{s_1}, R_{s_2} be $\mathcal{R}$-classes. If $tR_{s_1}, tR_{s_2} \subset R_{s_3}$ then $R_{s_1} = R_{s_2}$.*

Proof. As $ts_1\mathcal{R}ts_2$ there exist $u_1, u_2 \in S$ such that $ts_1u_1 = ts_2$ and $ts_2u_2 = ts_1$. But by cancellativity $s_1u_1 = s_2$ and $s_2u_2 = s_1$ so $R_{s_1} = R_{s_2}$. □

Lemma 2.38. *Let S be a left-cancellative semigroup. For any $s, t \in S$, $|sR_t| = |R_t|$.*

Proof. The set sR_t has at most as many elements as R_t. Let $t_1, t_2 \in R_t$. If $st_1 = st_2$ then by left-cancellativity $t_1 = t_2$. □

2.7 Actions

Let X be a set and let S be a semigroup. We say S *acts* on X if it comes equipped with a mapping $f : X \times S \to X$ denoted by $f(x, s) = x^s$ such that $(x^s)^t = x^{st}$ for all $x \in X$ and all $s \in S$. Furthermore, if S is a monoid with identity 1 we require $x^1 = x$.

The *orbit* of an element $x \in X$ under S is the set $x^S = \{x^s : s \in S\}$. We say S acts *transitively* if $x^S = X$ for all $x \in X$. We say S acts *weakly transitively* if there exists $x \in X$ such that $x^S = X$.

2.8 Graph Theory

The most basic definitions for graphs and digraphs were included in the introduction material. In this section we introduce a few other concepts which play a role.

The *underlying undirected graph*, denoted Γ^U, of a digraph $\Gamma = (V\Gamma, E\Gamma)$ is the graph $(V\Gamma, E\Gamma \cup \{(w, v) : (v, w) \in E\Gamma\})$. An *undirected walk* in a digraph Γ is a sequence $(v_0, v_1, \ldots, v_n)$ such that $v_i \to v_{i+1}$ or $v_i \leftarrow v_{i+1}$. A digraph is called *connected* if there exists an undirected path from u to v for all $u, v \in V\Gamma$. A digraph Γ is called *strongly connected* if there exists a (directed) path from u to v for all $u, v \in V\Gamma$.

Let Γ be a digraph. A *subdigraph* Δ of Γ is a digraph where $V\Delta \subseteq V\Gamma$ and $E\Delta \subseteq E\Gamma$. An *induced subdigraph* Δ of Γ is a digraph with $V\Delta \subseteq V\Gamma$ and $E\Delta = \{(u, v) \in E\Gamma : u, v \in V\Delta\}$.

Let Γ_1 and Γ_2 be digraphs. The *direct product of* Γ_1 *and* Γ_2, denoted $\Gamma_1 \times \Gamma_2$, is the digraph with vertices $V\Gamma_1 \times V\Gamma_2$ and where there is an edge $((x_1, x_2), (y_1, y_2))$ in $\Gamma_1 \times \Gamma_2$ if and only if $(x_1, y_1) \in E\Gamma_1$ and $(x_2, y_2) \in E\Gamma_2$.

We now introduce some special classes of mappings on digraphs that preserve structure. Let Γ_1 and Γ_2 be digraphs. A function $f : V\Gamma_1 \to V\Gamma_2$ is called a *homomorphism* if for all $(u, v) \in E\Gamma_1$ we have $(f(u), f(v)) \in E\Gamma_2$. A homomorphism f is called *injective* if f is injective on the set of vertices. A homomorphism f is called *strong* if $u, v \in V\Gamma_1$ satisfy $(f(u), f(v)) \in E\Gamma_2$ then $(u, v) \in E\Gamma_1$. We call a homomorphism f of Γ an *isomorphism* if f is a bijection and f is a strong homomorphism.

A homomorphism $f : V\Gamma \to V\Gamma$ is called an *endomorphism* of Γ. An endomorphism f of Γ is called a *monomorphism* if f is injective on the set of vertices. An endomorphism f of Γ is called a *strong endomorphism* if $u, v \in V\Gamma$ satisfy $(f(u), f(v)) \in E\Gamma$ then $(u, v) \in E\Gamma$. We call an endomorphism f of Γ an *automorphism* if f is a bijection on $V\Gamma$ and f is a strong endomorphism.

The set of endomorphisms of a digraph Γ forms a semigroup under composition, which we denote by $\mathrm{End}(\Gamma)$. Similarly, the set of automorphisms forms a group which we denote $\mathrm{Aut}(\Gamma)$.

We may also apply the above definitions to posets. Let X be a set and let $\mathcal{P} = (X, \leq)$ be a poset. A function $f : X \to X$ is called an *endomorphism* of $\mathcal{P}$ if for all $x, y \in X$ such that $x \leq y$ we have $f(x) \leq f(v)$. An endomorphism f of $\mathcal{P}$ is called a *monomorphism* if f is injective on X. An endomorphism f of $\mathcal{P}$ is called a *strong endomorphism* if for $x, y \in X$ we have that $f(x) \leq f(y)$

implies that $x \leq y$. We call an endomorphism f of $\mathcal{P}$ an *automorphism* if f is a bijection on X and f is a strong endomorphism.

An important notion in geometry is that of distance. We rigorously define this to work in spaces other than $\mathbb{R}^n$. Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}$ such that the following hold for all $x, y, z \in X$

1. $d(x, y) \geq 0$,

2. $d(x, y) = 0$ if and only if $x = y$,

3. $d(x, y) + d(y, z) \geq d(x, z)$,

4. $d(x, y) = d(y, x)$.

We generalise the graph metric for digraphs by defining the *digraph semi-metric*. We define the digraph semi-metric for the digraph Γ as d_Γ where $d_\Gamma(u, v)$ is the length of the shortest path for u to v for all $u, v \in V\Gamma$. When Γ is a connected graph this coincides with the graph metric.

Just as every digraph Γ has an associated underlying graph Γ^U, every digraph semi-metric d_Γ has an associated metric d_Γ^U. To build this we firstly define $\bar{d} : V\Gamma \times V\Gamma \to \mathbb{N}_0$ as the minimum of $d_\Gamma(x, y)$ and $d_\Gamma(y, x)$. Then $d_\Gamma^U(x, y)$ is defined to be the minimum of $\bar{d}(x, x_1) + \bar{d}(x_1, x_2) + \ldots + \bar{d}(x_n, y)$ over all sequences $(x_1, x_2 \ldots, x_n)$. This works in the case of semi-metric spaces with integer distances but may not necessarily form a metric space otherwise.

Let (X, d) be a semi-metric space, let $x \in X$ and let $n \in \mathbb{N}_0$. The *out-ball* of size n centred around x is the set $\overrightarrow{B}_n(x) = \{y \in X : d(x, y) \leq n\}$. The *ball* of size n centred around x is the set $B_n(x) = \{y \in X : d^U(x, y) \leq n\}$.

We say a digraph Γ is *out-locally finite* if $\overrightarrow{B}_1(v)$ is finite for all $v \in V\Gamma$. A digraph Γ is *locally finite* if $B_1(v)$ is finite for all $v \in V\Gamma$.

2.9 Ends of Digraphs

Ends of digraphs were proposed by Zuther in [43]. This section contains some technical lemmas which apply to ends of digraphs and are used throughout the rest of the thesis.

Let Γ be a digraph. An *anti-walk* is a sequence $(v_0, v_1, \ldots, v_n)$ over $V\Gamma$ such that $(v_{i+1}, v_i) \in E\Gamma$ for all $0 \leq i \leq n-1$. We also call an infinite sequence $(v_0, v_1, \ldots)$ such that $(v_{i+1}, v_i) \in E\Gamma$ for all $i \in \mathbb{N}_0$ an anti-walk.

Lemma 2.39. *Let Γ be a digraph and let $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be a walk. Then there exists a path with initial vertex w_1, terminal vertex w_n and such that all vertices of this path lie in $\mathbf{w}$.*

Proof. If $w_1 = w_n$ then the path of length zero from w_1 to w_n is a path satisfying the conditions of the lemma.

Otherwise, set $a(1) = 1$ and inductively define $a(i) = \max\{j \in \mathbb{N} : w_j = w_{a(i-1)}\} + 1$. The inductive steps to define $a(i)$s will eventually halt after $m \leq n$ steps because there are at most n different vertices in the walk $\mathbf{w}$. It follows that $a(m) = w_n$ as this is the only vertex of $\mathbf{w}$ with no successor. We then construct the sequence $\pi = (w_{a(1)}, w_{a(2)}, \ldots, w_{a(m)})$. By definition we have that the initial vertex of π is w_1 and the terminal vertex is w_n. The sequence π is a walk as $w_{a(i)} \rightarrow w_{a(i+1)}$ by the construction of $a(i+1)$. The walk π is in fact a path as $w_{a(i)} \neq w_{a(j)}$ for $i \neq j$. □

We now employ a similar technique to prove an infinite analogue of Lemma 2.39.

Lemma 2.40. *Let Γ be a digraph on Ω and let $\mathbf{a} = (\alpha_0, \alpha_1, \ldots)$ be an infinite walk (or anti-walk) in Γ such that every vertex of $\mathbf{a}$ occurs only finitely many times. Then $\mathbf{a}$ contains an out-ray $\mathbf{r}$ (or in-ray, respectively) such that $\mathbf{r}$ has infinitely many disjoint paths to every infinite subset of $\{\alpha_0, \alpha_1, \ldots\}$ and there are infinitely many disjoint paths from this set to $\mathbf{r}$. All such possible $\mathbf{r}$ are equivalent.*

Proof. We prove that $\mathbf{a}$ contains an out-ray in the case that $\mathbf{a}$ is a walk; an analogous argument proves that $\mathbf{a}$ contains an in-ray in the case that $\mathbf{a}$ is an anti-walk.

Let $a(0) = 1$ and for every $i \geq 1$ define $a(i) = \max\{j \in \mathbb{N} : \alpha_j =$

$\alpha_{a(i-1)}\} + 1$, i.e. $\alpha_{a(i)-1}$ is the last appearance of $\alpha_{a(i-1)}$ in $\mathbf{a}$. We will show that

$$\mathbf{r} = (\alpha_{a(0)}, \alpha_{a(1)}, \ldots)$$

is the required out-ray. Since $(\alpha_i, \alpha_{i+1}) \in E\Gamma$ for all i we have $(\alpha_{a(i-1)}, \alpha_{a(i)}) = (\alpha_{a(i)-1}, \alpha_{a(i)}) \in \Gamma$. Hence $\mathbf{r}$ is an infinite walk where $\alpha_{a(i)} \neq \alpha_{a(j)}$ for all $i, j \in \mathbb{N}$ such that $i \neq j$ and so $\mathbf{r}$ is an out-ray.

Let Σ be any infinite subset of $\{\alpha_0, \alpha_1, \ldots\}$. If infinitely many elements in Σ are vertices of $\mathbf{r}$, then we have infinitely many disjoint paths of length 0 from $\mathbf{r}$ to Σ and vice versa. If only finitely many elements of Σ belong to $\mathbf{r}$, then if we can construct infinitely many disjoint paths from $\mathbf{r}$ to $\Sigma \setminus \{\alpha_{a(0)}, \alpha_{a(1)}, \ldots\}$ we have infinitely many disjoint paths to Σ and vice versa. Hence may assume without loss of generality that Σ contains no elements in $\{\alpha_{a(0)}, \alpha_{a(1)}, \ldots\}$.

We define infinitely many disjoint paths from Σ to $\mathbf{r}$ by induction. Let $b(0) \in \mathbb{N}$ be any number such that $\alpha_{b(0)} \in \Sigma$. Then there exists $k(0) \in \mathbb{N}$ such that $a(k(0)) < b(0) < a(k(0)+1)$ and

$$\mathbf{b}_0 := (\alpha_{a(k(0))}, \alpha_{a(k(0))+1}, \ldots, \alpha_{b(0)}, \ldots, \alpha_{a(k(0)+1)})$$

is a walk from $\alpha_{a(k(0))}$ in $\mathbf{r}$ to $\alpha_{a(k(0))+1}$ in $\mathbf{r}$ via $a_{b(0)} \in \Sigma$. Lemma 2.39 states every finite walk contains a path, we conclude that there is a path contained in $\mathbf{b}_0$ from a vertex of $\mathbf{r}$ to $a_{b(0)} \in \Sigma$ and a path back from $a_{b(0)}$ to a vertex of $\mathbf{r}$.

Suppose that we have defined $b(0), \ldots, b(i-1), k(0), \ldots, k(i-1) \in \mathbb{N}$ and finite walks $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{i-1}$ for some $i \geq 1$. Choose $k(i), b(i) \in \mathbb{N}$ so that $b(i) \geq a(k(i))$, α_j does not equal any vertex in any of $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{i-1}$ for all $j \geq a(k(i))$, and $\alpha_{b(i)} \in \Sigma$. Then we define

$$\mathbf{b}_i = (\alpha_{a(k(i))}, \alpha_{a(k(i))+1}, \ldots, \alpha_{b(i)}, \ldots, \alpha_{a(k(i)+1)}).$$

By construction, if $i \neq j$, then $\boldsymbol{\beta}_i$ & $\boldsymbol{\beta}_j$ are disjoint and so we have infinitely many disjoint paths (contained in the $\mathbf{b}_i$) from $\mathbf{r}$ to Σ and back, as required. $\square$

For two sequences $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_m)$ we define their *concatenation*, denoted by $(x_0, x_1, \ldots, x_n)^\frown(y_0, y_1, \ldots, y_m)$, to be the sequence $(x_0, x_1, \ldots, x_n, y_0, y_1, \ldots, y_m)$.

Lemma 2.41. *Let Γ be an out-locally finite digraph on a set X and let $\mathbf{w}_0, \mathbf{w}_1, \ldots$ be finite walks of bounded length in Γ with distinct final vertices. Then every vertex in the sequence $\mathbf{w}_0^\frown\mathbf{w}_1^\frown\cdots$ occurs only finitely many times.*

Proof. Let $K \in \mathbb{N}$ be a bound on the lengths of $\mathbf{w}_0, \mathbf{w}_1, \ldots$. If a vertex v occurs in infinitely many of $\mathbf{w}_0, \mathbf{w}_1, \ldots$, then the set B of vertices that can be reached from v by a path of length at most K contains the final vertex of $\mathbf{w}_i$ for infinitely many $i \in \mathbb{N}$. But the final vertices of the $\mathbf{w}_i$ are distinct and so B is infinite, contradicting the assumption that Γ is out-locally finite. $\square$

Lemma 2.42. *Let Γ be an out-locally finite digraph, let $\Sigma \subseteq V\Gamma$ be infinite and let $v_0 \in V\Gamma$ such that there is a path from v_0 to every $v \in \Sigma$. Then there exists an out-ray $\mathbf{r}$ in Γ with initial vertex v_0 such that there exist infinitely many disjoint paths from $\mathbf{r}$ to Σ.*

Proof. We construct $\mathbf{r}$ recursively. Start by setting $\Sigma_0 := \Sigma$ and let P_0 be a set containing precisely one path q_w from v_0 to w for all $w \in \Sigma_0$. Then, since v_0 has finite out-degree and there is a path in P_0 from v_0 to every $w \in \Sigma_0$, there exists a vertex u_0 such that $(v_0, u_0) \in E\Gamma$ and there is a path $q_w \in P_0$ from v_0 via u_0 to every w in an infinite subset $\Sigma_1 \subseteq \Sigma_0$.

Let $w_1 \in \Sigma_1$ be fixed and also fix a path

$$p_1 = (x_1 = v_0, x_2 = u_0, x_3, \ldots, x_{n-1}, x_n = w_1).$$

Let $P_1 = \{q_w \in P_0 : w \in \Sigma_1\}$. If $w \in \Sigma_1$ is arbitrary and $q_w \in P_1$, then there exists $i(w) \in \mathbb{N}$ such that $x_{i(w)}$ is the last vertex belonging to both the paths p_1 and q_w. The number $i(w)$ exists since, in particular, both paths go through u_0. By the pigeonhole principle, there exists $m \in \mathbb{N}$ such that $2 \leq m \leq n$ and $\Sigma_2 = \{w \in \Sigma_1 : i(w) = m\}$ is infinite. Set $v_1 = x_m$. Since $m \geq 2$, $v_1 \neq v_0$ and, by construction, there is a path from v_1 to every element w of the infinite set Σ_2 (consisting of the vertices between v_1 and w in $q_w \in P_1$) such that the only vertex in p_1 and this path is v_1. Set P_2 to the set of paths from v_1 to $w \in \Sigma_2$ from the previous sentence.

We may repeat the above process *ad infinitum* to obtain for all $i > 0$: $w_i \in \Sigma_i$ and a path $p_i \in P_i$ from v_{i-1} to w_i, an v_i in p_i, an infinite $\Sigma_{i+1} \subseteq \Sigma_i$ and an infinite set P_{i+1} of paths from v_i to every element of Σ_{i+1} such that the only vertex in p_i and any path in P_{i+1} is v_i.

Hence there is a walk $\mathbf{r}$ containing $\{v_i : i \in \mathbb{N}\}$ consisting of the vertices on the paths p_{i+1} between v_i and v_{i+1}. In fact, by construction, the only vertex on both p_i and p_{i+1} is v_{i+1}, and so the walk $\mathbf{r}$ is a ray. Moreover, there are infinitely many paths from $\mathbf{r}$ to Σ consisting of the remaining vertices on p_{i+1} between v_{i+1} and w_{i+1}. Again by construction the only vertex on both p_i and p_{i+1} is v_{i+1} and so the paths from $v_{i+1} \in \mathbf{r}$ to $w_{i+1} \in \Sigma$ are disjoint for all i. □

Corollary 2.43. *Let S be a semigroup, let A be a finite generating set for S and let R be an infinite $\mathcal{R}$-class of S. There exists a ray $\mathbf{r}$ in $\Gamma_r(S, A)$ such that the vertices of $\mathbf{r}$ are elements of R.*

Proof. Let $\Gamma = \Gamma_r(S, A)$. As A is finite Γ is out-locally finite. Let $v_0 \in R$. There exist paths fromv_0 to infinitely many elements of R as the elements all lie in the same $\mathcal{R}$-class. Let $v \in R$. The vertices of any path from v_0 to v must also lie in R as there is a path from v to v_0. By Lemma 2.42 there exists a ray $\mathbf{r}$ with infinitely many disjoint paths to R. It follows the vertices of $\mathbf{r}$ must be elements of R. □

Lemma 2.44. *[43, Lemma 2.8] A ray either passes through infinitely many strongly connected components or is equivalent to a ray that lies in one strongly connected component (but not both).*

Lemma 2.45. *Let Γ_1 and Γ_2 be infinite, locally finite, connected graphs. Then $\Gamma_1 \times \Gamma_2$ has one end.*

Proof. Let $\mathbf{r} = ((x_1, y_1), (x_2, y_2), \ldots)$ be a ray in $\Gamma_1 \times \Gamma_2$. Fix $y \in V\Gamma_2$. We show that $\mathbf{r}$ is equivalent to a ray in $\Gamma_1 \times \{y\}$ and then that all rays in $\Gamma_1 \times \{y\}$ are equivalent.

As $\mathbf{r}$ is a ray either the set $\{x_1, x_2, \ldots\}$ is infinite or the set $\{y_1, y_2, \ldots\}$ is infinite. If $\{x_1, x_2, \ldots\}$ is infinite then for each y_i we fix a path π_i from y_i to y in Γ_2. We can do this as Γ_2 is connected. We then have infinitely many disjoint paths from $((x_1, y_1), (x_2, y_2), \ldots)$ to $((x_1, y), (x_2, y), \ldots)$ namely by taking the paths $\{x_i\} \times \pi_i$. Then by using Lemma 2.42 with basepoint (x_1, y) we get a ray in $V\Gamma_1 \times \{y\}$ that is equivalent to $\mathbf{r}$.

If $\{y_1, y_2, \ldots\}$ is infinite, then we fix some infinite set of vertices in $V\Gamma_1$, call these $\{z_1, z_2, \ldots\}$. Then we have infinitely many disjoint paths from $\mathbf{r}$ to the set of vertices $\{(z_1, y_1), (z_2, y_2), \ldots\}$. From $\{(z_1, y_1), (z_2, y_2), \ldots\}$ we have infinitely many disjoint paths to the set $\{(z_1, y), (z_2, y), \ldots\}$ and then applying Lemma 2.42 we get that $\mathbf{r}$ is equivalent to a ray in $V\Gamma_1 \times \{y\}$.

Let $\mathbf{r}_1 = (x_0, x_1, x_2, \ldots)$ and $\mathbf{r}_2 = (z_0, z_1, z_2, \ldots)$ be rays in Γ_1. If $\mathbf{r}_1$ is not equivalent to $\mathbf{r}_2$ then as Γ_1 is a connected graph we may assume $x_0 = z_0$ and they have no other points in common. Fix a ray $(y_1, y_2 \ldots)$ in Γ_2 with initial vertex $y_1 = y$. Then for each $i \in \mathbb{N}$ we have a path from (x_i, y) to (x_i, y_i), a path from (x_i, y_i) to (z_i, y_i) (namely $(x_i, y_i) \to (x_{i-1}, y_i) \to \ldots \to (z_i, y_i)$) and a path from (z_i, y_i) to (z_i, y). We call the concatenation of these paths π_i. The paths π_i are disjoint by construction and hence $\mathbf{r}_1 \times \{y\}$ is equivalent to $\mathbf{r}_2 \times \{y\}$. □

Chapter 3

Ends of Semigroups

3.1 Independence of Generating Set

Let S be a semigroup and let A be a generating set for S. We define the *right ends of S with respect to A* to be the ends of the right Cayley graph of S with respect to A. Recall we denote the right Cayley graph of S with respect to A by $\Gamma_r(S, A)$ and we denote the ends of a digraph Γ by $\Omega(\Gamma)$. We define the left ends of S dually. The first property we desire for this definition to have any hope of recovering information about a semigroup is that the poset of ends is invariant under change of finite generating set.

Let G be an infinite group, let A, B be any finite generating sets for G, and let C be an infinite generating set for G. Although the numbers of ends of $\Gamma(G, A)$ equals the number of ends of $\Gamma(G, B)$, it is not necessarily the case that $\Gamma(G, C)$ has the same number of ends. An example of this is if we take the group $\mathbb{Z} = Gp\langle a \mid \rangle$. Then $\{a\}$ is a finite generating set and $\mathbb{Z}$ is an infinite generating set. The Cayley graph $\Gamma(\mathbb{Z}, \{a\})$ has two ends. However, the Cayley graph $\Gamma(\mathbb{Z}, \mathbb{Z})$ contains the ray $(1, a, a^{-1}, a^2, a^{-2}, \ldots)$ and this ray contains every element of $\mathbb{Z}$ and hence will be equivalent to any other ray in $\Gamma(\mathbb{Z}, \mathbb{Z})$.

From this point onwards we will only concern ourselves with finitely generated semigroups.

Theorem 3.1. *Let S be a finitely generated semigroup and let A and B be finite generating sets for S. Then $\Omega(\Gamma_r(S, A))$ is isomorphic to $\Omega(\Gamma_r(S, B))$ as a poset.*

Proof. If $X \subseteq S$, then, for the sake of brevity, we denote $\Gamma_r(S, X)$ by Γ_X. It suffices to prove that $\Omega(\Gamma_A) \cong \Omega(\Gamma_{A\cup\{s\}})$ for all $s \in S$ since it follows that $\Omega(\Gamma_A) \cong \Omega(\Gamma_{A\cup B}) \cong \Omega(\Gamma_B)$. Let $s \in S$. We can express s as some product $a_1 a_2 \cdots a_n$ where $a_i \in A$. Note that Γ_A is a subdigraph of $\Gamma_{A\cup\{s\}}$ with the same vertices but, possibly, fewer edges. Hence every in-ray or out-ray in Γ_A is an in-ray or out-ray in $\Gamma_{A\cup\{s\}}$, respectively.

Firstly we will show that every out-ray in $\Gamma_{A\cup\{s\}}$ is equivalent to an out-ray with edges labelled by elements of A; the proof for in-rays is analogous. Let $\mathbf{r} = x_0 \rightarrow^{c_1} x_1 \rightarrow^{c_2} \ldots$ be an out-ray in $\Gamma_{A\cup\{s\}}$. If $c_i \neq s$ for all i, then there is nothing to prove. If $c_i = s$ then by replacing every edge

$$x_{i-1} \to^{s} x_i$$

by the path

$$x_{i-1} \to^{a_1} x_{i-1}a_1 \to^{a_2} \ldots \to^{a_k} x_{i-1}a_1 \cdots a_n = x_{i-1}s = x_i$$

we obtain a walk $\mathbf{w}$ with edges labelled by elements of A. If when replacing a path from x_{i-1} to x_i the walk now goes through some x_j where $j < i$ then there is an element $t \in S$ satisfying $|t|_A \leq n$ such that $x_i t = x_j$. Since $\overrightarrow{B}_n(x_i)$ is finite each vertex in $\mathbf{w}$ appears finitely many times and by Lemma 2.40 there is an infinite subray of $\mathbf{w}$ equivalent to $\mathbf{r}$.

From now on all rays in $\Gamma_{A\cup\{s\}}$ will have edges labelled by elements of A. We now show that rays $\mathbf{x}$ and $\mathbf{y}$ labelled by elements of A satisfy $\mathbf{x} \succcurlyeq \mathbf{y}$ in $\Gamma_{A\cup\{s\}}$ if and only if $\mathbf{x} \succcurlyeq \mathbf{y}$ in Γ_A. If $\mathbf{x} \succcurlyeq \mathbf{y}$ in Γ_A then clearly $\mathbf{x} \succcurlyeq \mathbf{y}$ in $\Gamma_{A\cup\{s\}}$ as Γ_A is contained in $\Gamma_{A\cup\{s\}}$. Assume, with the aim of reaching a contradiction, that there exist rays $\mathbf{x}, \mathbf{y}$ in $\Gamma_{A\cup\{s\}}$ such that $\mathbf{x} \not\succcurlyeq \mathbf{y}$ in Γ_A but $\mathbf{x} \succcurlyeq \mathbf{y}$ in $\Gamma_{A\cup\{s\}}$. Let $\Pi = \{\pi_i\}_{i\in\mathbb{N}}$ be a set of disjoint paths from $\mathbf{x}$ to $\mathbf{y}$ in $\Gamma_{A\cup\{s\}}$ where $\pi_k = (x_{i_k} = c_{k,0} \to^{d_{k,1}} c_{k,1} \to^{d_{k,2}} \ldots \to^{d_{k,n_k}} c_{k,n_k} = y_{j_k})$.

Any occurrences of an edge labelled by s in π_k can be replaced by $a_1 \cdots a_n$ in the same way as for rays. By Lemma 2.39 for any finite walk there exists a path with same initial and terminal vertices as the walk and only containing vertices and edges from walk. Hence, for each π_k we have a path σ_k with labels over A from x_{i_k} to y_{j_k}. As $\mathbf{x} \not\succeq \mathbf{y}$ in Γ_A there exists a finite set $F \subseteq T$ such that all paths from $\mathbf{x}$ to $\mathbf{y}$ pass through F. This means in particular all σ_k must pass through F. The construction of σ_k means that for any $v \in \sigma_k$ the out-ball $\overrightarrow{B}_n(v)$ contains an element of π_k. However, then the finite set $\overrightarrow{B}_n(F)$ must contain an element of every π_k, a contradiction. □

Dual arguments show that the end poset of the left Cayley graphs $\Gamma_l(S, A)$ and $\Gamma_l(S, B)$ are also isomorphic.

If S is a finitely generated semigroup and A is any finite generating set for S, then we will denote $\Omega(\Gamma_r(S, A))$ by $\Omega_r(S)$, which is well-defined by Theorem 3.1. Similarly, we denote $\Omega(\Gamma_l(S, A))$ by $\Omega_l(S)$.

3.2 Examples

For any finitely generated group G and generating set A for G we have that $|\Omega_r(G)| = |\Omega_l(G)|$. This is because $\Gamma_r(G, A)$ is isomorphic to $\Gamma_l(G, A)$. However, it is not true in general that $|\Omega_r(S)| = |\Omega_l(S)|$ for a finitely generated semigroup S.

Theorem 3.2. *Let S be the Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ where $|I| = n$, $|\Lambda| = m$, G is an infinite finitely generated group and P is a $m \times n$ matrix with entries in G. Then the right ends of S form an anti-chain of size $n \cdot |\Omega(G)|$ and the left ends of S form an anti-chain of size $m \cdot |\Omega(G)|$.*

Proof. Let X be a finite semigroup generating set for G containing 1_G and let

$$A = \{(i, p_{\mu,j}^{-1}x, \lambda) | x \in X,\ \lambda, \mu \in \Lambda,\ i, j \in I\}.$$

The set A is a finite generating set for S. To get the element (i, g, λ) one takes $x_1, x_2, \ldots x_n \in X$ such that $x_1 \cdot x_2 \cdots x_n = p_{\lambda,i}g$ then the product $(i, p_{i,\lambda}^{-1}x_1, \lambda) \cdot (i, p_{i,\lambda}^{-1}x_2, \lambda) \cdots (i, p_{i,\lambda}^{-1}x_n, \lambda)$ equals (i, g, λ).

Let Γ_i be the induced subdigraph of $\Gamma_r(S, A)$ with vertex set $\{i\} \times G \times \Lambda$ and let $\Gamma_{i,\lambda}$ be the subdigraph of Γ_i with vertex set $\{i\} \times G \times \{\lambda\}$ and edges with labels $(i, p_{\lambda,i}^{-1}x, \lambda)$ where $x \in X$. As $(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda,j}h, \mu)$ note that $\Gamma_r(S, A)$ is the disjoint union of the Γ_i. This means that $\Omega(\Gamma_r(S, A))$ is n incomparable copies of $\Omega(\Gamma_i)$. As in $\Omega(G)$ all ends are incomparable it suffices to show that $\Omega(\Gamma_i)$ is isomorphic to $\Omega(G)$ for all $i \in I$.

We first note that for a fixed $\lambda \in \Lambda$, the graph $\Gamma_{i,\lambda}$ is isomorphic to $\Gamma_r(G, X)$. We now prove that any out-ray in Γ_i is equivalent to an out-ray in $\Gamma_{i,\lambda}$, the proof for in-rays is analogous. Let $\mathbf{r} = ((i, g_0, \lambda_{j0}), (i, g_1, \lambda_{j_1}), \ldots)$ be an out-ray in Γ_i. We then define a corresponding sequence or vertices $\mathbf{r}' = ((i, g_0, \lambda), (i, g_1, \lambda), \ldots)$ in $\Gamma_{i,\lambda}$. We show that there is an infinite walk $\mathbf{w}$ in $\Gamma_{i,\lambda}$ containing $\mathbf{r}'$ in which every vertex appears finitely many times.

We construct $\mathbf{w}$ by concatenating the shortest paths in $\Gamma_{i,\lambda}$ between each (i, g_k, λ) and (i, g_{k+1}, λ), these shortest paths exist because $\Gamma_{i,\lambda}$ is isomorphic to $\Gamma_r(G, X)$. Next we show that there is a global bound on the lengths of these shortest paths. As $(i, g_k, \lambda_{j_k}) \to (i, g_{k+1}, \lambda_{j_{k+1}})$ there exists $(i, p_{\mu,j}^{-1}x, \nu) \in A$ such that $(i, g_k, \lambda_{j_k}) \cdot (i, p_{\mu,j}^{-1}x, \nu) = (i, g_{k+1}, \lambda_{j_{k+1}})$. It follows $\nu = \lambda_{j_{k+1}}$ and $g_{k+1} = g_k p_{\lambda_{j_k}, i} p_{\mu,j}^{-1} x$. This means the shortest path in $\Gamma_{i,\lambda}$ between any consecutive elements of $\mathbf{r}'$ is of length less than $K = \max\{|p_{j,\mu} p_{k,\nu}^{-1}|_X : j, k \in I, \mu, \nu \in \Lambda\} + 2$.

As $\mathbf{r}$ is a ray it follows there are at most $|\Lambda|$ repetitions of vertices in $\mathbf{r}'$. Every vertex of $\mathbf{w}$ has a path of length less than K to a vertex of $\mathbf{r}'$ and as $\Gamma_{i,\lambda}$ is out-locally finite this means that if some vertex v appears infinitely often in $\mathbf{w}$ then infinitely many elements of $\mathbf{r}'$ lie in an out-ball of size K around v. But each vertex in $\mathbf{r}'$ appears at most $|\Lambda|$ times so any infinite set of elements of $\mathbf{r}'$ contains infinitely many vertices, a contradiction. By Lemma 2.40, $\mathbf{w}$ contains a ray $\mathbf{s}$ with infinitely many disjoint paths from $\mathbf{s}$ to and from $\mathbf{r}'$. Hence, there are infinitely many paths from $\mathbf{s}$ to and $\mathbf{r}$ and vice versa.

This means any ray in Γ_i is equivalent to a ray in $\Gamma_{i,\lambda}$, to complete the proof we must now verify that if we have rays $\mathbf{r_1}$ and $\mathbf{r_2}$ in $\Gamma_{i,\lambda}$ such that $r_1 \not\approx r_2$ then $r_1 \not\approx r_2$ in Γ_i. Let $\mathbf{r_1}$ and $\mathbf{r_2}$ be incomparable rays in $\Gamma_{i,\lambda}$. As the rays are incomparable in $\Gamma_{i,\lambda}$ there exists a finite set $F = \{(i, f_1, \lambda), \ldots, (i, f_m, \lambda)\}$ such that all paths from $\mathbf{r_1}$ to $\mathbf{r_2}$ in $\Gamma_{i,\lambda}$ pass through F. For any edge $(i, g, \mu) \to^{(j, p_{\nu,k}^{-1}x, \xi)} (i, g p_{\mu,j} p_{\nu,k}^{-1} x, \xi)$ we have a word $w = x_1 x_2 \ldots x_p$ over X of minimal length such that $w =_G p_{\mu,j}p_{\nu,k}^{-1}x$ and a corresponding path

$$(i,g,\mu) \to^{(i,p_{\mu,i}^{-1},\lambda)} (i,g,\lambda) \to^{(i,p_{\lambda,i}^{-1}x_1,\lambda)} (i,gx_1,\lambda) \to \dots$$

$$\to (i,gx_1x_2\cdots x_p,\lambda) \to^{(i,p_{\lambda,i}^{-1},\xi)} (i,gp_{\mu,j}p_{\nu,k}^{-1}x,\xi).$$

This means that any path in Γ_i has a corresponding walk in $\Gamma_{i,\lambda}$ such that any point on the walk has a path of length less than $K+2$ to a vertex on the path in Γ_i. This means any path π from $\mathbf{r_1}$ to $\mathbf{r_2}$ in Γ_i has a corresponding walk in $\Gamma_{i,\lambda}$ and this must pass through F and hence π must pass through $\overrightarrow{B}_{K+2}(F)$. As Γ_i is out-locally finite this is a finite set so $\mathbf{r_1} \not\approx \mathbf{r_2}$. □

Corollary 3.3. *If $S = G \times E$ is a right group then the right ends of S are isomorphic as a poset to $\Omega(G)$.*

It follows that for any $m,n \in \mathbb{N}$ there exists a semigroup S such that $\Omega_l(S) = n$ and $\Omega_r(S) = m$.

The following example shows that unlike in the groups case it is possible for a semigroup to have $\aleph_0$ ends.

Example 3.4. The semigroup $S = \mathbb{N}_0 \times \mathbb{N}_0$ under componentwise addition has $\aleph_0$ ends. We let A be the generating set $\{(0,1),(1,0)\}$ for S and let $\Gamma = \Gamma_r(S,A)$. We show any ray in Γ is either equivalent to the ray $(0,0) \to (1,0) \to (1,1) \to (2,1) \to (2,2)\dots$ or is equivalent to one of the rays $(i,0) \to$

$(i,1) \to (i,2), \ldots$, or $(0,i) \to (1,i) \to (2,i) \to \ldots$ for some $i \in \mathbb{N}_0$. Firstly, we show that the rays given all lie in different ends of Γ. If $i < j$ then there are no paths from (j,k) to (i,l) for any $k,l \in \mathbb{N}_0$ and hence $((i,0),(i,1),\ldots)$ is inequivalent to $((j,0),(j,1),\ldots)$ for $i \neq j$. Similarly, $((0,i),(1,i),\ldots)$ is inequivalent to $((0,j),(1,j),\ldots)$ for $i \neq j$. Also, for each $i \in \mathbb{N}_0$ there are no paths from (j,k) to (i,l) or (l,i) for any $l \in N_0$ if $j,k > i$ so $(0,0) \to (1,0) \to (1,1) \to (2,1) \to (2,2) \ldots$ is inequivalent to any $((i,0),(i,1),\ldots)$ or any $((0,i),(1,i),\ldots)$.

Because there are no edges $(i,j) \to (k,l)$ for $k < i$ or $l < j$, for any fixed vertex (i,j) there are only finitely many vertices with a path to (i,j). This means there are no in-rays in Γ. Any ray either contains finitely many elements in the first component of its vertices, finitely many elements in the second component of its vertices or infinitely many distinct elements in both components. In the first case again since there are no edges $(i,j) \to (k,l)$ for $k < i$ or $l < j$ it follows that eventually the elements are of the form (i,j) for some fixed i. Hence, all but finitely many of the elements of the ray are in $(i,0) \to (i,1) \to (i,2), \ldots$, so the rays are equivalent. By symmetry, if the ray has finitely many elements in the second component of its vertices then it will be equivalent to $(0,i) \to (1,i) \to (2,i) \to \ldots$ for some $i \in \mathbb{N}_0$.

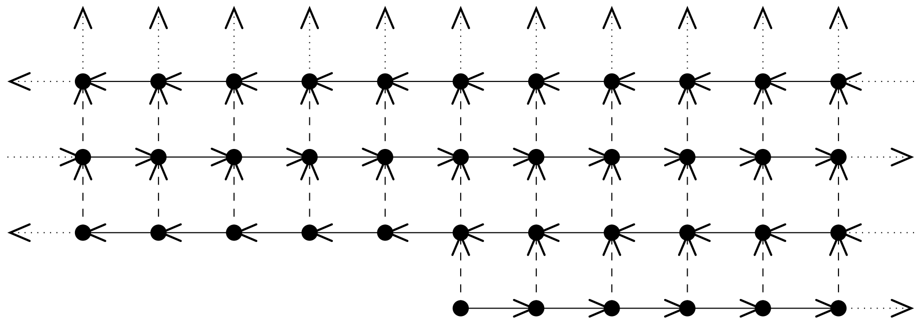
For any element (i,j) where $i < j$ there is a path from (i,i) to (i,j) to (j,j), similarly, if $i > j$ there is a path from (j,j) to (i,j) to (i,i). If a ray $\mathbf{r} = ((x_1, y_1), (x_2, y_2), \ldots)$ has infinitely many distinct elements in both components then there exists an injective function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $x_{f(i)} \geq y_{f(i)}$ for all $i \in \mathbb{N}_0$ or $x_{f(i)} \leq y_{f(i)}$ for all $i \in \mathbb{N}_0$. Furthermore, we can choose f such that $x_{f(i+1)} \geq y_{f(i)}$ if $x_{f(i)} \geq y_{f(i)}$ or $y_{f(i+1)} \geq x_{f(i)}$ if $x_{f(i)} \leq y_{f(i)}$. This ensures that the paths π_i from $(x_{f(i)}, x_{f(i)})$ to $(x_{f(i)}, y_{f(i)})$ to $(y_{f(i)}, y_{f(i)})$ are distinct for each $i \in \mathbb{N}_0$. Hence, $\mathbf{r}$ is equivalent to $(0,0) \to (1,0) \to (1,1) \to (2,1) \to (2,2) \ldots$ □

The next example demonstrates the existence of a semigroup whose Cayley graph contains an in-ray that is not equivalent to any out-ray. The example also shows that it is possible to have in-rays in the Cayley graph of a semigroup with trivial $\mathcal{R}$-classes.

Example 3.5. Let M be the monoid given by the presentation $Mon\langle a, b \mid aba = b\rangle$. It is easy to check $b^2a = ab^2$ is a consequence of the relation $aba = b$. The rules $aba \to b$ and $b^2a \to ab^2$ form a complete rewriting system defining a monoid isomorphic to M. It follows that elements of M have the form $b^\epsilon a^i b^j$, where $\epsilon \in \{0, 1\}$ and $i, j \in \mathbb{N}_0$. We have that $a^i b \cdot a = a^{i-1}b$ for all $i \geq 1$ and hence have the in-ray $b \leftarrow ab \leftarrow a^2b \leftarrow \ldots$. See Figure 3.1 for a portion of the Cayley graph. □

Let S and T be semigroups and let 0 be an element in neither S nor T. The *zero union* of S and T, denoted $S \cup_0 T$, is defined to be the semigroup

Figure 3.1: A portion of the right Cayley graph of $\langle a, b \mid aba = b \rangle$ from Example 3.5, edges labelled by a are represented by filled lines and b by dashed lines.



on the set $S \cup T \cup \{0\}$ with multiplication defined as $x \cdot y = xy$ if $x, y \in S$ or $x, y \in T$ and $x \cdot y = 0$ otherwise.

Lemma 3.6. *Let S and T be finitely generated semigroups. Then $\Omega_r(S \cup_0 T) = \Omega_r(S) \cup \Omega_r(T)$.*

Proof. Let A and B be finite generating sets for S and T respectively. It is easy to see the set $A \cup B$ is a finite generating set for $S \cup_0 T$. Any rays in $\Gamma_r(S, A)$ or $\Gamma_r(T, B)$ will also be rays in $\Gamma_r(S \cup_0 T, A \cup B)$ and if $\mathbf{x} \not\succcurlyeq \mathbf{y}$ in S or T then $\mathbf{x} \not\succcurlyeq \mathbf{y}$ in $S \cup_0 T$.

It remains to show that there are no other rays in $\Gamma_r(S \cup_0 T, A \cup B)$ except for those contained in S or T and that all rays in S are incomparable to rays in T. Let $x \in S$, if $x \to y$ in $\Gamma_r(S \cup_0 T, A \cup B)$ then either $y = xa$ for some $a \in A$ or $y = 0$. However, the only element u satisfying $0 \to u$ is $u = 0$ so any path $x \to y \to u$ containing 0 must have $u = 0$ and both x, y in either S or T. Hence, the only rays containing 0 are in-rays whose initial vertex is 0 and whose tails are in-rays in either S or T. It follows from this that there are no paths from S to T or vice versa in $\Gamma_r(S \cup_0 T, A \cup B)$ so all rays in S and T are incomparable. $\square$

One might ask which posets may arise as the end poset of a semigroup. This question remains unresolved. However, by combining the following lemmas we are able to partially answer the question.

Recall a semigroup is *decomposable* if $S^2 = S$.

Lemma 3.7. *Any finite partially ordered set that can be realised as the $\mathcal{R}$-class poset of a finite decomposable semigroup arises as the poset of right ends of some semigroup.*

Proof. Let S be a finite semigroup and let P be the $\mathcal{R}$-class poset of S. Then $\mathbb{Z} \times \mathbb{Z} \times S$ is finitely generated as $\mathbb{Z} \times \mathbb{Z}$ and S are finitely generated and decomposable. Let X be a finite generating set for $\mathbb{Z} \times \mathbb{Z} \times S$. The $\mathcal{R}$-class poset of $\mathbb{Z} \times \mathbb{Z} \times S$ is isomorphic to the $\mathcal{R}$-class poset of S as $(i, j, s)\mathcal{R}(k, l, t)$ if and only if $s\mathcal{R}t$. By Lemma 2.44, rays either pass through infinitely many strongly connected components or are equivalent to a ray contained in a strongly connected component. Strongly connected components in $\Gamma_r(\mathbb{Z} \times \mathbb{Z} \times S, X)$ correspond to the $\mathcal{R}$-classes of $\mathbb{Z} \times \mathbb{Z} \times S$. There are only finitely many $\mathcal{R}$-classes so all rays are equivalent to some ray contained in an $\mathcal{R}$-class. All rays inside an $\mathcal{R}$-class are equivalent as any ray in an $\mathcal{R}$-class will be equivalent to a ray in $\mathbb{Z} \times \mathbb{Z} \times \{r\}$ for some $r \in S$. The partial order of the end poset is inherited from the $\mathcal{R}$-class poset of S. $\square$

3.3 Indices

One important theorem in the theory of ends of groups is the following:

Theorem 3.8. *[22, Satz IV] Let G be a finitely generated group and let H be a subgroup of finite index. Then $|\Omega(G)| = |\Omega(H)|$.*

In this section we examine the the number of ends of subsemigroups of finite index for both Rees and Green index.

Let S be a semigroup and let T be a subsemigroup of S. Recall, we say T is of finite *Rees index* in S if $|S \setminus T| < \infty$.

Lemma 3.9. *Let S be a finitely generated semigroup, let T be a subsemigroup of finite Rees index and let A be a finite generating set for T. Let $u \in S \setminus T$ and let $c_1, \ldots, c_n \in A \cup (S \setminus T)$. If $|u \cdot c_1 \cdot \ldots \cdot c_n|_{A \cup (S \setminus T)} = n + 1$ and $n + 1 > |S \setminus T|$ then $u \cdot c_1 \cdot \ldots \cdot c_i \in T$ for some $i \leq n$.*

Proof. Let $U = S \setminus T$ and let $w = u \cdot c_1 \cdot \ldots \cdot c_n$. As the product $u \cdot c_1 \cdot \ldots \cdot c_n$ is of shortest possible length as a representative for w no two prefixes can be equal. Therefore, by the Pigeonhole Principle there exists $1 \leq i \leq |U|$ such that $uc_1 \cdot \ldots \cdot c_j \notin S \setminus T$. Hence, represents $uc_1 \cdot \ldots \cdot c_j$ is an element of T. □

Theorem 3.10. *Let S be a finitely generated semigroup and let T be a subsemigroup of finite Rees index. Then $\Omega(S) \cong \Omega(T)$.*

Proof. Let $U = S \setminus T$ and let A be a finite generating set for T. Then $A \cup U$ is a finite generating set for S. Firstly we will show that every out-ray in $\Gamma_r(S, A \cup U)$ is equivalent to an out-ray with vertices in T and edges labelled by elements of A, the proof for in-rays is analogous. Let $\mathbf{r} = x_0 \to^{c_1} x_1 \to^{c_2} \ldots$ be an out-ray in $\Gamma_r(S, A \cup U)$. As U is finite only finitely many elements of $\mathbf{r}$ can lie in U so there exists an x_i such thatl $x_j \in T$ for all $j \geq i$ giving an out-ray all of whose vertices lie in T.

Without loss of generality we will now assume all vertices of a ray are elements of T. We will now demonstrate that all rays are equivalent to a ray with elements in T with edges labelled by elements from A. Let $C := \{us_1s_2 \ldots s_i \in T \mid u \in U, s_j \in A \cup U, 1 \leq i \leq |U| + 1\}$. As $|C| < \infty$ it follows $K = \max\{|c|_A : c \in C\}$ exists. If $c_i \notin U$ for all i then there is nothing to prove so assume $c_i \in U$. Now by Lemma 3.9 one of $c_i, c_ic_{i+1}, \ldots, c_i \cdots c_{i+|U|+1}$ must lie in C. This means if $c_i = u$ we can replace the path $c_i \cdots c_j$ between x_{i-1} and x_j, for $j \leq i + |U| + 1$, by some path labelled by A of length less than K. By performing all such replacements of subpaths of $\mathbf{r}$ we get a walk

$\mathbf{w}$. The walk $\mathbf{w}$ is not necessarily a ray because of the new paths added. The walk $\mathbf{w}$ will satisfy the conditions of Lemma 2.40 as $\overrightarrow{B}_K(x_i)$ is finite for all $i \in \mathbb{N}$. This means there is a ray with vertices in T and edges labelled by elements of A which is equivalent to $\mathbf{r}$.

Without loss of generality we will assume all rays in S will have vertices from T and edges labelled by elements of A. If $\mathbf{x} \preceq \mathbf{y}$ in $\Gamma_r(T, A)$ then $\mathbf{x} \preceq \mathbf{y}$ in $\Gamma_r(S, A \cup U)$. Assume for the sake of obtaining a contradiction that there exist rays $\mathbf{x}, \mathbf{y}$ in $\Gamma_r(S, A \cup U)$ such that $\mathbf{x} \not\preceq \mathbf{y}$ in $\Gamma_r(T, A)$ but $\mathbf{x} \preceq \mathbf{y}$ in $\Gamma_r(S, A \cup U)$. Let $\Pi = \{\pi_i\}_{i \in \mathbb{N}}$ be a set of disjoint paths from $\mathbf{x}$ to $\mathbf{y}$ in $\Gamma_r(S, A \cup U)$ where $\pi_k = (x_{i_k} = c_{k,0} \to^{d_{k,1}} c_{k,1} \to^{d_{k,2}} \ldots \to^{d_{k,n_k}} c_{k,n_k} = y_{j_k})$. We can assume that no π_i passes through U as U is a finite set and only finitely many pairwise disjoint paths can pass through U. We may also assume that $j_1 > |U| + 1$ and $j_{k+1} - j_k > |U| + 1$ by inductively choosing paths satisfying this condition.

Let $\mathbf{y} = y_0 \to^{b_1} y_1 \to^{b_2} \ldots$, the proof for $\mathbf{y}$ being an in-ray is analogous. Extend π_k to $\pi'_k = x_{i_k} \to^{d_{k,1}} c_{k,1} \to^{d_{k,2}} \ldots \to^{d_{k,n_k}} y_{j_k} \to^{b_{j_k+1}} y_{j_k+1} \to^{b_{j_k+2}} \ldots \to^{b_{j_k+|U|+1}} y_{j_k+|U|+1}$, through the choice of π_k the π'_k are disjoint. By applying Lemma 3.9 any occurrences of $u \in U$ in π'_k can be replaced by a word over A of length less than K. Hence for each π'_k we have a path σ_k with vertices in T and labels over A from x_{i_k} to $y_{j_k+|U|+1}$. As $\mathbf{x} \not\preceq \mathbf{y}$ in $\Gamma_r(T, A)$ there exists a finite set $F \subseteq T$ such that all paths from $\mathbf{x}$ to $\mathbf{y}$ pass through F. This means in particular that all σ_k must pass through F. Let $A^K = \{a_1 a_2 \cdots a_n | a_i \in A, 1 \leq n \leq K\}$ then the construction of σ_k means that for any $v \in \sigma_k$ the set vA^K contains an element of π'_k. However, then the finite set FA^K must contain an element of every π'_k, a contradiction. $\square$

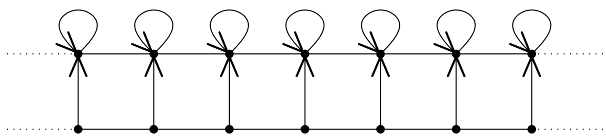
It should be noted that the proofs of Theorem 3.1 and Theorem 1.11 are rather similar. In fact they can be proved as a corollary of a more general theorem.

Let S be a semigroup and let T be a subsemigroup of S. Recall, we say T is of finite *Green Index* in S if $S \setminus T$ has finitely many $\mathcal{H}^T$- classes.

The following example demonstrates that in general a subsemigroup of finite Green index may have a different number of ends to the original semigroup.

Example 3.11. Let $\{0, 1\}$ be the semigroup with the usual multiplication (of real numbers) and let $\mathbb{Z}$ be the group of integers under addition. Consider the semigroup $S = \mathbb{Z} \times \{0, 1\}$. Then $T = \mathbb{Z} \times \{1\}$ is a subsemigroup. The semigroup S is commutative so $\mathcal{R}^T$ coincides with $\mathcal{L}^T$. It follows that $\mathcal{H}^T = \mathcal{R}^T$. Let $i, j \in \mathbb{Z}$ then $(i, 0) \cdot (-i + j, 1) = (j, 0)$ and so $(i, 0)\mathcal{H}^T(j, 0)$. It follows that $(S \setminus T)$ has only one $\mathcal{H}^T$ class and so $\mathbb{Z} \times \{1\}$ has finite Green index in $\mathbb{Z} \times \{0, 1\}$. As $\mathbb{Z} \times \{1\}$ is isomorphic to $\mathbb{Z}$ we readily see that T has two ends. However, $\mathbb{Z} \times \{0, 1\}$ has 4 ends corresponding to the rays

Figure 3.2: A portion of the right Cayley graph of the semigroup $\mathbb{Z} \times \{0, 1\}$ from Example 3.11.



$(0, 1) \to (1, 1) \to \ldots$, $(0, 1) \to (-1, 1) \to \ldots$, $(0, 0) \to (1, 0) \to \ldots$ and $(0, 0) \to (-1, 0) \to \ldots$.

For a diagram of a portion of the right Cayley graph of $\mathbb{Z} \times \{0, 1\}$ see Figure 3.2.

Recall that in Section 2.3 of the preliminaries we defined the index of a subsemigroup T in a semigroup S to be the number of $\mathcal{H}^T$-classes in $S \setminus T$ plus 1. As $\mathcal{R}^T$-classes are unions of $\mathcal{H}^T$-classes it follows that if there are finitely many $\mathcal{H}^T$-classes in $S \setminus T$ then there are finitely many $\mathcal{R}^T$-classes contained in $S \setminus T$. Hence, if T has finite Green index in S then $S \setminus T$ has finitely many $\mathcal{R}^T$-classes.

Theorem 3.12. *[6, Corollary 9.2] Let S be a semigroup and let T be a subsemigroup of finite Green index. Then S is finitely generated if and only if T is finitely generated.*

Recall from Theorem 2.6 that a subgroup H is of finite Green index in a group G if and only if H is of finite index in G. It is possible that the values of the indices may differ. Obviously if a semigroup is of finite Rees index then it is also of finite Green index.

Let S be a semigroup and let $U \leq T \leq S$. Recall that if T is not of finite Green index in S then U is not of finite Green index in S.

Lemma 3.13. *Let G be an infinite group and let E be a right zero semigroup. If T is a subsemigroup of finite Green index in $G \times E$ then $T = H \times E$, where H is a subgroup of finite index in G.*

Proof. Let $S = G \times E$. One can see S has only one $\mathcal{R}$-class, therefore the $\mathcal{H}^S$-classes of S are the $\mathcal{L}^S$-classes of S. As $(g, e) \cdot (h, f) = (gh, f)$ we see that $\mathcal{L}$-classes of S are of the form $G \times \{f\}$ for each $f \in E$.

Let F be a non-empty proper subset of E. The subsemigroup $G \times F$ forms a left ideal as $(g, e) \cdot (h, f) = (gh, f)$. It follows that $G \times F$ has infinite Green index in $G \times E$. Hence if $T \subseteq G \times F$ then $F = E$.

For each $e \in E$ we let H_e be those elements $h \in G$ such that $(h, e) \in T$. We now show each H_e contains 1_G. Let $e \in E$. One can see H_e is a subsemigroup of G as in particular $(g, f)(h, e) = (gh, e)$ so $H_f H_e \subseteq H_e$. If $H_e = G$ then $1_G \in H_e$. By Lemma 2.4 we know that a subsemigroup of finite Rees index in G is equal to G so we may assume $G \setminus H_e$ is infinite. As E is a right zero semigroup each $\mathcal{L}^S$-class, and hence each $\mathcal{H}^S$-class, must consist of elements of the form (g, e) for some fixed $e \in E$. Then as $\mathcal{H}^T$-classes are contained in $\mathcal{H}^S$-classes and as $G \backslash H_e$ is infinite we must have at least one non-trivial $\mathcal{H}^T$-class that contains distinct elements $(g, e), (g', e)$ with $g, g' \notin H_e$. As these elements are $\mathcal{H}^T$-related they are $\mathcal{R}^T$-related and hence there exist $(h, f), (h', f) \in T$ such that $(g, e)(h, f) = (g', e)$ and $(g', e)(h', f') = (g, e)$. This means $f = f' = e$ and furthermore that $ghh' = g$. Hence, $hh' = 1_G$ is an element of H_e. This means $H_e \subseteq H_f$ for all $e, f \in E$ so $H_e = H_f$ for all $e, f \in E$. We call this semigroup H. This means T is of the form $H \times E$ for some subsemigroup H of finite Green index in G.

As $T = H \times E$ has finite Green index in $G \times E$ it must follow that H has finite Green index in G. By Theorem 2.6 H is a subgroup of G with finite group index. □

Theorem 3.14. *Let S be finitely generated left-cancellative semigroup and let T be a subsemigroup of finite Green index. Then $|\Omega(S)| = |\Omega(T)|$.*

Proof. By Lemma 2.30 if S is right simple, then $S \cong G \times E$ for some finitely generated group G and some finite right zero semigroup E . Since T has finite Green index in S, it follows from Lemma 3.13 that $T \cong H \times E$ where H is a subgroup of finite group index in G. In other words, T is a right group and so $|\Omega(T)| = |\Omega(H)|$ by Corollary 3.3. By Theorem 1.2 $|\Omega(H)| = |\Omega(G)|$. It follows $|\Omega(T)| = |\Omega(H)| = |\Omega(G)| = |\Omega(S)|$.

Let U be the right group of regular elements in S. Since S is finitely generated, it follows from Theorem 3.12 that T is finitely generated. Let A and B be finite generating sets for S and T, respectively, such that $B \subseteq A$. Since $S \setminus U$ is an ideal, U is also finitely generated. Hence, as T is also left cancellative, the right group of regular elements V of T is finitely generated. It follows by Lemma 2.25 that $\mathcal{R}^T = \mathcal{R}^V$, and so V has finite Green index in U.

Suppose that S has more than one $\mathcal{R}$-class. Then, by Corollary 2.24, S has infinitely many $\mathcal{R}$-classes. If S has no infinite $\mathcal{R}$-classes, then since $\mathcal{R}^T$-classes are contained in $\mathcal{R}^S$-classes, it follows that T has finite Rees index in S and so by Theorem 1.11, the theorem follows. We consider the case that S has infinitely many infinite $\mathcal{R}$-classes. As U is a right group, by Corollary 3.3 U has either has 1, 2, or $2^{\aleph_0}$ ends.

If U has $2^{\aleph_0}$ ends, then, since $S \setminus U$ is an ideal, S has $2^{\aleph_0}$ ends. Since V has finite Green index in U and U is a right group, V has $2^{\aleph_0}$ ends and so T has $2^{\aleph_0}$ ends also.

Suppose that U has 1 or 2 ends. Then S and T have at least $\aleph_0$ ends, since every pair of infinite $\mathcal{R}$-classes contain a pair of inequivalent rays. Let $\Sigma(S)$ be the set of ends of S containing a ray that has non-empty intersection with infinitely many $\mathcal{R}^S$-classes. By Lemma 2.44, if ω is an end of $\Gamma_r(S, A)$, then every ray in ω is contained in a strongly connected component or intersects infinitely many strongly connected components (but not both). Since connected components of $\Gamma_r(S, A)$ are precisely $\mathcal{R}^S$-classes, it follows that $|\Omega(S)| = \max\{\aleph_0, |\Sigma(S)|\}$ and $|\Omega(T)| = \max\{\aleph_0, |\Sigma(T)|\}$. We conclude the proof by showing that $|\Sigma(S)| = |\Sigma(T)|$.

Let $\mathbf{r}$ be a ray in $\Gamma_r(S, A)$ that has non-empty intersection with infinitely many $\mathcal{R}^S$-classes. Since every $\mathcal{R}^S$-class is a union of $\mathcal{R}^T$-classes, $\mathbf{r}$ has non-empty intersection with infinitely many $\mathcal{R}^T$-classes. Since there are only finitely many $\mathcal{R}^T$-classes in $S \setminus T$, we may assume without loss of generality that the vertices in $\mathbf{r}$ are in T. Let n be the number of $\mathcal{R}^T$-classes in $S \setminus T$ and let $(xc_1, xc_1c_2, \ldots, xc_1 \cdots c_m)$ be a subpath of $\mathbf{r}$ that has non-empty intersection with $n+1$ $\mathcal{R}^T$-classes. By left cancellativity, the path $(c_1, c_1c_2, \ldots, c_1 \cdots c_m)$ has non-empty intersection with at least $n+1$ $\mathcal{R}^T$-classes also. It follows that there exists i such that $c_1 \cdots c_i \in T$. Hence $c_1 \cdots c_i$ is a product $b_1b_2 \cdots b_j$ of elements in the generating set B for T. Recursively replacing every such path $(xc_1, xc_1c_2, \ldots, xc_1 \cdots c_i)$ by the corresponding walk $(xb_1, xb_1b_2, \ldots, xb_1 \cdots b_j)$ we obtain a walk $\mathbf{w} = (w_0, w_1, \ldots)$ in $\Gamma_r(T, B)$ that has non-empty intersection with infinitely many $\mathcal{R}^T$-classes contained in T. If $i < j$ and $w_i \mathcal{R}^T w_j$, then $w_i \mathcal{R}^T w_{i+1} \mathcal{R}^T \cdots \mathcal{R}^T w_j$. But $\mathbf{w}$ has non-empty intersection with infinitely many $\mathcal{R}^T$-classes and so every vertex of $\mathbf{w}$ occurs only finitely many times. Hence, by Lemma 2.40, $\mathbf{w}$ is equivalent to a ray in with vertices in T and edges labelled by elements of B.

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be rays with vertices in T and edges labelled by elements of B such that $\mathbf{r}_1$ and $\mathbf{r}_2$ have non-empty intersection with infinitely many $\mathcal{R}^S$-classes. If $\mathbf{r}_1$ is equivalent to $\mathbf{r}_2$ in $\Gamma_r(T, B)$, then clearly $\mathbf{r}_1$ is equivalent to $\mathbf{r}_2$ in $\Gamma_r(S, A)$. Suppose that $\mathbf{r}_1$ is equivalent to $\mathbf{r}_2$ in $\Gamma_r(S, A)$. In this case, there are infinitely many disjoint paths in $\Gamma_r(S, A)$ from $\mathbf{r}_1$ to $\mathbf{r}_2$ and vice versa. We may assume that these paths all pass through at least $n+1$ $\mathcal{R}^T$-classes and the last n $\mathcal{R}^T$ classes lie in T, where n is the number of $\mathcal{R}^T$-classes in $S \setminus T$. This can be done by taking any collection of paths and extending them along the ray $\mathbf{r}_2$. By repeatedly replacing subpaths as we did in the previous paragraph there exist infinitely many paths from $\mathbf{r}_1$ to $\mathbf{r}_2$ labelled by elements of B. If infinitely many of these paths are disjoint and the same holds true with paths from $\mathbf{r}_2$ to $\mathbf{r}_1$, then the proof is complete. Otherwise infinitely many of these paths have non-empty intersection with a finite subset of S, and so infinitely many paths contain some fixed element $s \in S$. But then there exists a path from s to an element in $\mathbf{r}_2$ and a path from that vertex to an element in $\mathbf{r}_1$, and so infinitely many elements in $\mathbf{r}_1$ are $\mathcal{R}^S$-related, a contradiction. We have shown that for all rays $\mathbf{r}_1$ and $\mathbf{r}_2$ in $\Gamma_r(T, B)$ such that $\mathbf{r}_1$ and $\mathbf{r}_2$ have non-empty intersection with infinitely many $\mathcal{R}^S$-classes, $\mathbf{r}_1$ is equivalent to $\mathbf{r}_2$ in $\Gamma_r(T, B)$ if and only if they are equivalent in $\Gamma_r(S, A)$. Therefore $|\Sigma(S)| = |\Sigma(T)|$, as required. $\square$

The following example demonstrates that although the size of the poset of ends is the same, the poset of ends may be different for a subsemigroup of finite index.

Example 3.15. Let $\mathbb{Z}$ be the group of integers under addition and let $\mathbb{N}_0$ be the monoid of natural numbers under addition. Let $S = \mathbb{Z} \times \mathbb{Z} \times \mathbb{N}_0$ and let $T = \mathbb{Z} \times \mathbb{Z} \times (\mathbb{N}_0 \setminus \{1\})$. One can verify that S is a finitely generated cancellative monoid as it is the direct product of a group and a cancellative monoid. The subsemigroup T is of finite Green index as $(i, j, 1) \cdot (k - i, l - j, 0) = (k, l, 1)$ for any $i, j, k, l \in \mathbb{Z}$. The ends of S correspond to each $\mathbb{Z} \times \mathbb{Z} \times \{i\}$ and to the ray that passes through infinitely many $\mathcal{R}$-classes, we denote these ends by ω_i and ω_∞ respectively. One can see $\omega_i \succcurlyeq \omega_j$ if $i < j$ and also $\omega_i \succcurlyeq \omega_\infty$ for all $i \in \mathbb{N}_0$. In $\mathbb{Z} \times \mathbb{Z} \times \mathbb{N}_0 \setminus \{1\}$ the ends correspond to $\mathbb{Z} \times \mathbb{Z} \times \{i\}$ for $i \in \mathbb{N}_0 \setminus \{1\}$ and to the ray that passes through infinitely many $\mathcal{R}$-classes. However, there are no paths from any $\mathbb{Z} \times \mathbb{Z} \times \{i\}$ to $\mathbb{Z} \times \mathbb{Z} \times \{i+1\}$ or from $\mathbb{Z} \times \mathbb{Z} \times \{i+1\}$ to $\mathbb{Z} \times \mathbb{Z} \times \{i\}$. This means that in particular the ends in $\mathbb{Z} \times \mathbb{Z} \times \{2\}$ and $\mathbb{Z} \times \mathbb{Z} \times \{i\}$ are incomparable. For a portion of the Cayley graph $\Gamma_r(S, \{(a, 1, 1), (1, b, 1), (1, 1, c)\})$, where a, b generate a copy of $\mathbb{Z}$ and c generates $\mathbb{N}_0$, see Figure 3.3. For the end posets see Figure 3.4.

Figure 3.3: A portion of the Cayley graph of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{N}_0$ from Example 3.15 where the generator $(a, 1, 1)$ is represented by full lines, $(1, b, 1)$ by dashed dotted lines and $(1, 1, c)$ by dashed lines.

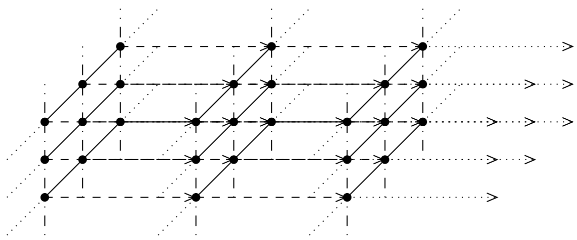
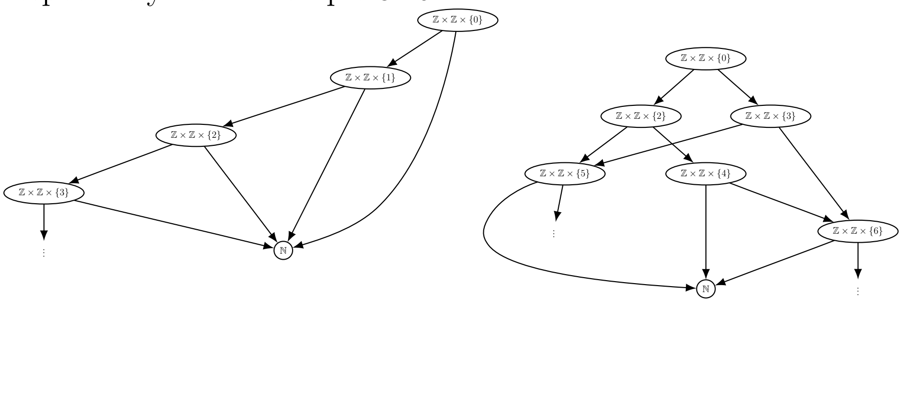


Figure 3.4: Portions of the poset of ends of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{N}_0$ and $\mathbb{Z} \times \mathbb{Z} \times (\mathbb{N} \setminus \{1\})$ respectively from Example 3.15.



3.4 1,2 or Infinitely Many Ends

An important step in the process of relating algebraic properties of a group to the properties of its ends was Hopf's result describing the possible number of ends of a group.

Theorem 3.16. *[22, Satz 1] Let G be an infinite, finitely generated group. Then G has 1, 2 or $2^{\aleph_0}$ ends.*

As we have seen from Theorem 3.2 it is possible to have any number of left or right ends in a semigroup. It is reasonable to consider whether there is a 'nice' class of semigroups where an analogue of this theorem holds. If a one, two, infinity type theorem existed for a smaller class of semigroups it could lead to a classification of the smaller class via the number of ends. The first class of group-like semigroups one may consider is inverse semigroups. A semigroup S is *inverse* if for every $s \in S$ there exists a unique $t \in S$ such that $sts = s$ and $tst = t$. However, the zero union of groups is an inverse semigroup so by taking the zero union of n copies of $\mathbb{Z} \times \mathbb{Z}$ and invoking Lemma 3.6 we get an inverse semigroup with n ends. As in Theorem 1.12 we see that left-cancellative semigroups preserve enough geometric properties to get an analogous result.

Theorem 3.17. *An infinite, finitely generated, left-cancellative semigroup has 1, 2 or infinitely many ends.*

Proof. Let S be a left cancellative semigroup and let A be a finite generating set for S. By Corollary 2.24 we know S has one or infinitely many $\mathcal{R}$-classes.

If S only has one $\mathcal{R}$-class then by Lemma 2.30 S is isomorphic to a right group $G \times E$ where E is a finite right-zero semigroup and G is a finitely generated group. By Corollary 3.3 $|\Omega(S)| = |\Omega(G)|$. Hence, by Theorem 1.4 the semigroup S has 1, 2 or $2^{\aleph_0}$ ends.

If S has only one $\mathcal{R}$-class the Theorem was shown to be true so without loss of generality we may assume S has infinitely many $\mathcal{R}$-classes. If S has an infinite $\mathcal{R}$-class then, by Lemma 2.28, the semigroup S has infinitely many infinite $\mathcal{R}$-classes. By Lemma 2.42 we may construct an out-ray in each infinite $\mathcal{R}$-class. Let R_1 and R_2 be infinite $\mathcal{R}$-classes and let $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ be out-rays with vertices in R_1 and R_2 respectively. If $\mathbf{x}$ is equivalent to $\mathbf{y}$ there exists a path from $x_0 \in R_1$ to some $y_i \in R_2$. There also exists a path from y_i to some $x_j \in R_1$. But $x_j \mathcal{R} x_0$ so there exists a path from y_i to x_0 and a path from x_0 to y_i. Hence $y_i \mathcal{R} x_0$ and $R_1 = R_2$. This means if S has infinitely many infinite $\mathcal{R}$-classes then S has infinitely many ends.

Henceforth we may assume without loss of generality that S has infinitely many $\mathcal{R}$-classes all of which are finite. If there exists $s \in S$ such that $s \in xS$ for infinitely many $x \in S$ then by Lemma 2.42 we may construct an out-ray $\mathbf{r} = (r_0, r_1, \ldots)$ where each r_i has a path to s. If s had a path to any r_i

then $r_i \mathcal{R} r_j$ for all $j \geq i$, as all $\mathcal{R}$-classes were assumed to be finite no such path exists. For an arbitrary $x \in S$ consider the sequence $(xr_0, xr_1, \ldots)$, if $xr_n = xr_m$ then by left-cancellativity $r_n = r_m$ and as $\mathbf{r}$ is a ray $m = n$. This means for any $x \in S$ the sequence $(xr_0, xr_1, \ldots)$ is an out-ray. We now consider the out-rays $(s^n r_0, s^n r_1, \ldots)$ for $n \in \mathbb{N}_0$. If there is a path from $(s^n r_0, s^n r_1, \ldots)$ to $(s^m r_0, s^m r_1, \ldots)$ for $m < n$ then by left-cancellativity there is a path from $(s^{n-m} r_0, s^{n-m} r_1, \ldots)$ to $\mathbf{r}$. But then there is a path from s to r_i via $s^{n-m} r_j$, a contradiction. This means there can be no paths from $(s^n r_0, s^n r_1, \ldots)$ to $(s^m r_0, s^m r_1, \ldots)$ for $m < n$ and hence all these rays are inequivalent and S has infinitely many ends.

Assume without loss of generality that for all $s \in S$ there exist only finitely many $x \in S$ such that $s \in xS$ and assume S has n ends where $2 \leq n < \infty$. Notice that if for all $s \in S$ there exist only finitely many $x \in S$ such that $s \in xS$ then S contains no in-rays. Let $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ be out-rays in distinct ends. One of these rays, say $\mathbf{r_1}$, must be minimal under $\preccurlyeq$. This means there exists a finite subset $F \subseteq S$ such that all paths from $\mathbf{r}_1$ to any other $\mathbf{r}_i$ pass through F. Let $F' = \{s \in S : \exists f \in F, f \in sS\}$, then by assumption F' is finite and there exists $r \in \mathbf{r_1}$ such that r has no path to F. Now consider the rays $r\mathbf{r}_1$ and $r\mathbf{r_2}$. Neither ray can pass through F as there are no paths from r to F. They cannot be equivalent as if there exist infinitely many disjoint paths between them then, by left cancellativity, there would exist infinitely many disjoint paths between $\mathbf{r}_1$ and $\mathbf{r}_2$. This means one ray must have infinitely many disjoint paths to some $\mathbf{r}_i$ for $i \neq 1$. As $\mathbf{r}_1$ was chosen to be minimal all such paths must pass through F but there are no paths from r to F, a contradiction. □

Note that, unlike in Theorem 1.4, we only prove that a left-cancellative semigroup has infinitely many ends rather than $2^{\aleph_0}$ ends. There are left-cancellative semigroups with $\aleph_0$ ends, see for instance Example 3.4. We consider the possible cardinalities that may arise as the cardinality of the poset of ends of a semigroup in Section 5.4.

3.5 Undirected Ends

In [24] Jackson and Kilibarda introduced various alternative notions for the ends of a semigroup. It was proved that all these notions coincided for a fixed generating set of a semigroup. We will refer to the number of ends as defined by Jackson and Kilibarda as the undirected ends. It is proved in [24] that the number of undirected ends is invariant under change of finite generating set. They also construct a class of semigroups such that for any prescribed $n, m \in \mathbb{N} \cup \{\infty\}$ there exists a semigroup S with n right undirected ends and m left undirected ends and proved that all subsemigroups of a free semigroup have either 1 or ∞ undirected ends. Although interesting in its own right one may argue that by removing the direction in the Cayley graph one makes it harder to regain algebraic structure. For instance, the ideal structure of a semigroup would no longer be visible in the Cayley graph.

Here we prove analogous results to Theorem 1.11 and Theorem 1.12 with Jackson and Kilibarda's notion of ends. We only use the notion of undirected ends in this section and will make no other reference to it for the rest of this thesis. Let S be a semigroup and let A be a generating set. We define the *right undirected ends* of S with respect to A, denoted by $\mathcal{E}_r(S, A)$, to be the maximum number of infinite components of the induced subdigraph $\Gamma_r(S, A) \setminus F$, where F is any finite subset of S, or ∞ if no such maximum exists. The number of left undirected ends is defined analogously via the left Cayley graph.

Similarly to the definition of ends used throughout the rest of the thesis we only consider the ends of the right Cayley graph, as similar results for the left Cayley graph follow from considering the dual semigroup.

Let Γ be a digraph and let $F \subseteq V\Gamma$. Recall that $\overrightarrow{\delta}(F) = \{v \in V\Gamma : \exists u \in F,\ u \to v\}$.

Theorem 3.18. *Let S be a finitely generated semigroup and let T be a subsemigroup of finite Rees index. Then $\mathcal{E}(S) = \mathcal{E}(T)$.*

Proof. Let A be a finite generating set for T. Clearly $A \cup (S \setminus T)$ is a finite generating set for S. Let $\Gamma_T = \Gamma_r(T, A)$ and let $\Gamma_S = \Gamma_r(S, A \cup S \setminus T) \setminus (S \setminus T)$. We note that Γ_S has the same number of undirected ends as $\Gamma_r(S, A \cup S \setminus T)$ because for any finite set F the set $F \cup (S \setminus T)$ is also finite so $\Gamma_S \setminus F$ is isomorphic to $\Gamma_r(S, A \cup S \setminus T) \setminus (F \cup S \setminus T)$.

Now, Γ_S has the same vertices as Γ_T but possibly more edges. Let $F \subset T$ be a finite set. We may assume that A is contained in F as this can only increase the number of infinite components of both Γ_S and Γ_T when F is removed. Then for each $v \in \overrightarrow{\delta}(F)$ we define $C_T(v, F)$ to be all those vertices of $\Gamma_T \setminus F$ that can be reached from v by an undirected path that does not pass through F. Similarly let $C_S(v, F)$ to be all those vertices of $\Gamma_S \setminus F$ that can be reached from v by an undirected path that does not pass through F. Now any vertex of $\Gamma_T \setminus F$ lies in at least one $C_T(v, F)$ as every vertex of T can be reached by a path from A. There are only finitely many $v \in \overrightarrow{\delta}(F)$ so there are only finitely many components of $\Gamma_T \setminus F$. The same argument shows that for any finite set F there are only finitely many components of $\Gamma_S \setminus F$. By adding edges to Γ_T we can join components. However, as there are only finitely many components we cannot join finite components to make a new infinite component for Γ_S. This means $\mathcal{E}(\Gamma_S) \leq \mathcal{E}(\Gamma_T)$.

We now show that given an infinite set of elements $u_1, u_2, \ldots \in S$ and an infinite set of elements $v_1, v_2, \ldots \in S$ such that $v_i x_i = u_i$ for some $x_i \in S \setminus T$ there exists a constant $K \in \mathbb{N}$ such that there is an undirected path of length

K from v_i to u_i for all $i \in \mathbb{N}$ in Γ_T. As for each u_i there exists v_i such that $v_i x_i = u_i$ for some $x_i \in S \setminus T$ there exists a $v_i' \in S$ of minimal length over A such that $v_i' x_i' = u_i$ for some $x_i' \in S \setminus T$. Only finitely many of these v_i' can be the same as $|v(S \setminus T)| \leq |S \setminus T|$. We only consider an infinite subset of the u_i whose associated v_i' are all distinct. Now as there are infinitely many v_i' and A is finite there must be infinitely many such that $|v_i'| > 1$. Hence, we may write $v_i' = w_i a_i$ for some $a_i \in A \cup (S \setminus T)$. Now $w_i a_i x_i' = v_i' x_i' = u_i$. If $a_i x_i' \in S \setminus T$ this would contradict the minimality of v_i'. This means $a_i x_i' \in T$ and we set $K = \max\{|ax|_A : a \in A,\ x \in S \setminus T,\ ax \in T\} + 1$. Then we have a path $v_i' \leftarrow w_i \rightarrow^{a_i x_i'} u_i$ of length less than or equal to K.

This property relates to the undirected ends of S and T as if we were to assume that $\mathcal{E}(\Gamma_S) < \mathcal{E}(\Gamma_T)$ then there would exist two components C_1 and C_2 of $\Gamma_T \setminus F$ contained in one component of $\Gamma_S \setminus F$. This would mean that there are infinitely many edges labelled by $x_i \in S \setminus T$ joining elements of C_1 and C_2. Without loss of generality we would have infinitely many edges from C_1 to C_2. By replacing these edges by paths in T of globally bounded length, as in the previous paragraph, we would get infinitely many disjoint paths from C_1 to C_2 in Γ_T, a contradiction. $\square$

As in our accepted notion of ends, subsemigroups of finite Green index need not have the same number of undirected ends as the original semigroup.

Here we prove that the number of ends is the same in a subsemigroup of finite Green index if the parent semigroup is cancellative. We make use of Möller's result, Theorem 1.3, that the number of ends is a quasi-invariant of locally finite graphs. It is clear that if a finitely generated semigroup is right cancellative then the degree of any vertex in the right Cayley graph is finite. To see this fix some finite generating set A. The out degree of any vertex is bounded by $|A|$ and the in-degree of any vertex is bounded by $|A|$ as if $xa = ya$ then $x = y$.

Lemma 3.19. *Let S be a cancellative monoid and let T be a subsemigroup of finite Green index in S with $1_S \in T$. Let A be a finite generating set for T and let C be a finite set of representatives for the $\mathcal{H}^T$-classes of $S \setminus T$. Let $t_1c_1, t_2c_2 \in S\setminus T$ be in the same $\mathcal{H}^T$ class as c_1 and c_2 respectively. Then there exists a constant $K_1 \in \mathbb{N}$ such that for any edge $t_1c_1 \to^z t_2c_2$ ($z \in A \cup C$) for $t_1c_1, t_2c_2 \in S \setminus T$ there exists a path between t_1 and t_2 with vertices in T and edges labelled by elements of A and of length $\leq K_1$.*

Proof. Let $N_1 = \max_{c\in C, a\in A}\{|t|_A : t \in T, \exists c' \in C$ such that $tc' = ca\}$, $N_2 = \max_{c\in C}\{|t|_A : t \in T, \exists c' \in C$ such that $tc' = c\}$ and $K_1 = N_1 + N_2$. As both C and A are finite there are only finitely many elements ca. By cancellativity for each $c, c' \in C$ and $a \in A$ there can only be one element $t \in T$ such that $tc' = ca$ otherwise $tc' = t'c'$. It follows there are at most $|C|$ elements of

S such that $tc' = tc$. Thus N_1 exists. Also note that N_2 exists for similar reasons.

As $t_1c_1z = t_2c_2$ notice that $c_1z \notin T$ as $t_2c_2 \notin T$. This means c_1z lies in the same $\mathcal{H}^T$-class as some $c_3 \in C$ and so we can express c_1z as $t'c_3$ for some $t' \in T$. Now as t_2c_2 is $\mathcal{L}^T$-related to c_2 we know t_2 has an inverse $t_2^{-1} \in T$ and it follows that $t_2^{-1}t_1t'c_3 = c_2$. Therefore by taking $t'_1 = t'$ and $t'_2 = t_2^{-1}t_1t'$ we get $t_1t'_1 = t_2t'_2$ giving a path $t_1 \rightarrow^{t'_1} t_1t'_1 \leftarrow^{t'_2} t_2$ of length $\leq K_1$ contained in T. $\square$

Lemma 3.20. *Let S be a cancellative monoid and let T be a subsemigroup of finite Green index in S with $1_S \in T$. Let A be a finite generating set for T and let C be a finite set of representatives for the $\mathcal{H}^T$-classes of $S \setminus T$. Then there exists $K_2 \in \mathbb{N}$ such that for any adjacent vertices $x \in T$ and $tc \in S \setminus T$, $c \in C$ there exists a path between x and t contained in T of length $\leq K_2$.*

Proof. Let $M_1 = \max_{c \in C, z \in C \cup A}\{|cz|_A : cz \in T\}$, $M_2 = \max_{c \in C, z \in A \cup C}\{|t|_A : t \in T, \exists c' \in C \text{ such that } tc' = cz\}$, $K_2 = \max\{M_1, M_2 + 1, N_2\}$ and $N_2 = \max_{c \in C}\{|t|_A : t \in T, \exists c' \in C \text{ such that } tc' = c\}$. These are all constants by similar arguments to those contained in the previous lemma.

There are two cases:

(1) If $tcz = x$ for some $z \in A \cup C$ then if $cz \in T$ there is a path $t \rightarrow^{cz} x$ of length $\leq M_1$ which is contained in T. Otherwise $cz \notin T$ so $cz = t'c'$ and

we have a path $t \to^{t'} tt' \to^{c'} x$ of length $\leq M_2 + 1$ contained in T.

(2) If $xc' = tc$ for some $c' \in C$ then t has an inverse $t' \in T$ and $t'xc' = t'tc = c$ so $t'x$ is a path contained in T from t to x and is of length $\leq N_2$. □

We use the notation $\mathcal{U}(S)$ to denote the largest subgroup (with respect to containment) of a cancellative semigroup S. The largest subgroup is well defined as there can only be one identity element in a cancellative semigroup.

Theorem 3.21. *Let S be a cancellative monoid, let T be a subsemigroup of finite Green index in S with $1_S \in T$, let A be a finite generating set for T and let C be a finite set of representatives for the $\mathcal{H}^T$-classes of $S \setminus T$. Then $\Gamma_r^U(S, A \cup C)$ is quasi-isometric to $\Gamma_r^U(T, A)$.*

Proof. We take $f : \Gamma_r(T, A) \to \Gamma_r(S, A \cup C)$ to be $f(t) = t$, we use d_T and d_S for the metrics in these two graphs. Every element of $S \setminus T$ is adjacent to something in T in $\Gamma_r(S, A \cup C)$ as elements of $S \setminus T$ can be expressed in the form tc. It is also obvious that for all $x, y \in T$ we have $d_T(x, y) \geq d_S(f(x), f(y))$ as $\Gamma_r(T, A)$ is a subdigraph of $\Gamma_r(S, A \cup C)$.

We claim that either S is a group and T is a subgroup or that for all $c \in C$ we have $cT \cap T \neq \emptyset$.

Proof of claim: Let $c \in C$ and assume that $cT \cap T = \emptyset$. Let $t \in T$ be arbitrary. Then $ct^i \in S \setminus T$ for all $i \in \mathbb{N}$. As there are finitely many $\mathcal{H}^T$-classes there must exist $i, j \in \mathbb{N}$ such that $ct^i \mathcal{H}^T ct^{i+j}$. Then using the cancellativity of

S we deduce that $t^j \in \mathcal{U}(T)$. As t was arbitrary we see all elements are invertible and T is a subgroup of S. It is shown in [6] that if T is a subgroup of finite Green index in a cancellative semigroup S then S is a group. If T is a subgroup of finite index in a group S it is a well known fact that this inequality holds, for instance see [10, Chapter IV.B]. Therefore for the rest of this we will assume that T is not a group.

We now show that there exists $\lambda \in \mathbb{N}$ such that $d_S(x, y) \geq \frac{1}{\lambda} d_T(x, y)$ for all $x, y \in T$. Let $c \in C$, if $cT \cap T \neq \emptyset$ then fix $t_c \in T$ such that $ct_c \in T$. Let $K_3 = \max_{c \in C}\{|t_c|_A + |ct_c|_A : cT \cap T \neq \emptyset\}$ and $\lambda = (K_1 + 2K_2)K_3$, where K_1 is the constant from Lemma 3.19 and K_2 is the constant from Lemma 3.20.

Let $x, y \in T$ and π be a path of minimal length from x to y in $\Gamma_r(S, A \cup C)$. We firstly show that we can find a path π' in $\Gamma_r(S, A \cup C)$ whose vertices lie in T, but possibly has edges labelled by elements from c, such that $|\pi'| \leq (K_1 + 2K_2)|\pi|$. We then show that we can replace those edges labelled by elements of c by an undirected path of globally bounded length labelled by elements of A.

We construct such a path π' as follows. The path π is either contained in T already or has some subpath $x', t_1c_1, \ldots, t_nc_n, y'$ where $x', y' \in T$ but $t_ic_i \in S \setminus T$. Now we can apply Lemma 3.19 and Lemma 3.20 to this subpath to get a walk whose vertices are all in T and the walk is of length less than $K_1 \cdot n + 2K_2$. We repeat this process on any subpaths of π of this form thus getting a walk of length $\leq (K_1 + 2K_2)|\pi|$. All walks contain a subpath of shorter or equal length gained by removing any loops, this is the required path.

As we assumed T is not a group we know that for all $c \in C$ there exists $t_c \in T$ such that $ct_c \in T$. Now as noted above the path π', although contained in T, may contain edges labelled by elements of C, we now show how to replace these. Let $c \in C$ and $t_1, t_2 \in T$ be such that $t_1 \to^c t_2$. We replace the edge c by the path $t_1 \to^{ct_c} t_1 \cdot ct_c \leftarrow^{t_c} t_2$. This path is labelled by elements from A, has vertices contained in T and is of length less than or equal to K_3. We can now apply this to the path π' getting a path from x to y with the required properties. It follows that $K_3 d_S(x, y) \leq d_T(x, y)$. □

Corollary 3.22. *Let S be a finitely generated cancellative semigroup and let T be a subsemigroup of finite Green index. Then $\mathcal{E}(S) = \mathcal{E}(T)$.*

One may note that in Theorem 1.12 it was proved that the number of ends passes to subsemigroups of finite Green index in left-cancellative semigroups, whereas here we only proved this for cancellative semigroups. This is because it is not the case that the Cayley graph of a left-cancellative semigroup is locally-finite.

Example 3.23. Let $S = \langle a, b | ab = b \rangle$. The rule $ab \to b$ gives a complete rewriting system and gives unique normal forms $b^i a^j$ for $i, j \in \mathbb{N}_0$. The product $b^i a^j \cdot b^k a^l$ is equal to $b^{i+k} a^l$ if $k > 0$ and $b^i a^{j+l}$ if $k = 0$. We then consider equalities of the form $b^i a^j \cdot b^k a^l = b^i a^j \cdot b^m a^n$.

This splits into three cases.

Case 1: If $k, m > 0$ then $b^{i+k} a^l = b^{i+m} a^n$ and hence $k = m$ and $l = n$.

Case 2: If $k > 0$ and $m = 0$ then $b^{i+k} a^l = b^i a^{j+n}$ and hence $k = 0$, a contradiction. When $m > 0$ and $k = 0$ follows by symmetry.

Case 3: If $k, m = 0$ then $b^i a^{j+l} = b^i a^{j+n}$ and hence $l = n$ and by assumption $m = k$.

This means that S is left-cancellative. However, $a^i b = b$ for all $i \in \mathbb{N}_0$ so $\Gamma_r(S, \{a, b\})$ is not locally-finite.

However, it is still open as to whether the number of undirected ends of a left-cancellative semigroup is preserved by finite Green index subsemigroups.

Chapter 4

Actions on Digraphs

The previous chapter has focused on generalising ends of graphs to ends of digraphs and generalising groups to semigroups. This chapter also concerns generalisations of ends but focuses on semigroups acting on graphs and on groups acting on digraphs. Particularly, it looks at how ends behave under graph homomorphisms and the end structure of vertex transitive digraphs.

Recall Theorem 1.4, stating that an infinite finitely generated group has 1, 2 or $2^{\aleph_0}$ ends. A generalised version of Hopf's theorem is attributed to Abel and Hopf. A graph Γ is said to be *almost vertex transitive* if the group of automorphisms of Γ has finitely many orbits on Γ.

Theorem 4.1. *(Abel and Hopf) Let* Γ *be an infinite, locally finite, connected, almost vertex transitive graph. Then* Γ *has 1, 2, or* $2^{\aleph_0}$ *many ends.*

4.1 Semigroups Acting on Ends

The proof of Theorem 4.1 makes use of the fact that an automorphism of the graph Γ gives rise to an automorphism of the poset of ends of Γ. This section explores the behaviour of ends under endomorphisms in the hope of an analogue for digraphs with certain properties such as weak transitivity. This first lemma demonstrates that an automorphism of a digraph induces an automorphism on the ends.

Lemma 4.2. *Let Γ be a digraph and let G be a group acting on Γ by automorphisms. Then G acts on the poset $\Omega(\Gamma)$ by automorphisms.*

Proof. Let $g \in G$, let $\omega \in \Omega(\Gamma)$ and let $\mathbf{r} \in \omega$. We show that G acts by automorphisms on the pre-order on rays and hence acts by automorphisms on the poset of ends. As $g \in \mathrm{Aut}(\Gamma)$ it induces a bijection on the vertices and edges of Γ and hence g gives a bijection on rays. We define the action of G on Σ, the set of rays, by $(\mathbf{r}, g) \to \mathbf{r}^g$. If $\mathbf{x} \succcurlyeq \mathbf{y}$ then as g preserves paths we have $\mathbf{x}^g \succcurlyeq \mathbf{y}^g$. This means that each $g \in G$ gives an endomorphism of $(\Sigma, \succcurlyeq)$. By symmetry, if $\mathbf{x}^{g^{-1}} \succcurlyeq \mathbf{y}^{g^{-1}}$ then $\mathbf{x} \succcurlyeq \mathbf{y}$ hence each g gives a strong endomorphism of $(\Sigma, \succcurlyeq)$. Therefore G acts on $(\Sigma, \succcurlyeq)$ by automorphisms and hence G acts on $\Omega(\Gamma)$ by automorphisms. $\square$

The following example shows that endomorphisms do not preserve end structure in such a straightforward way. In fact it is possible that under an endomorphism an infinite graph can be mapped to a finite graph.

Example 4.3. The graph consisting of vertices labelled by positive integers and edges $n \sim n+1$ has an endomorphism whose image is isomorphic to the induced subgraph $\{0,1\}$, namely $f(2n-1)=1, f(2n)=2$.

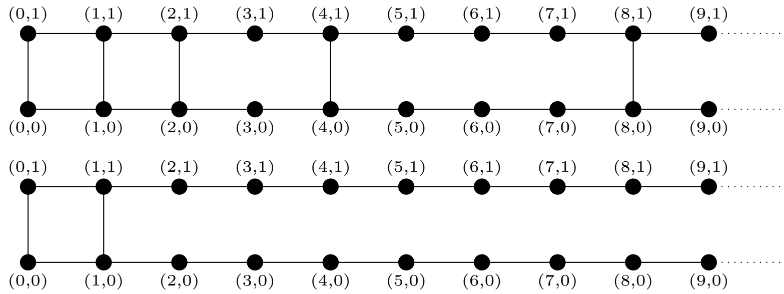
This next example shows that even if an endomorphism is surjective it does not necessarily preserve end structure.

Example 4.4. Let Γ be the graph with vertex set $V\Gamma = \mathbb{N}\times\{0,1\}$ and edge set $E\Gamma = \{\{(i,j),(i+1,j)\},\{(0,0),(0,1)\},\{(2^i,0),(2^i,1)\}|i \in \mathbb{N}_0, j \in \{0,1\}\}$. See Figure 4.1. Every positive natural number can be expressed in the form $n = 2^j + m$ where $0 \le m < 2^j$ and $j \in \mathbb{N}_0$. We define a map f as follows:

$$f(n,j) = \begin{cases} (0,j) & \text{if } n = 0 \\ (0,j) & \text{if } n = 2^{2i} \\ (m,j) & \text{if } n = 2^i + m,\ i \le 2 \\ (m,j) & \text{if } n = 2^i + m,\ m \le 2^{i-1} \\ (2^i - m, j) & \text{if } n = 2^i + m,\ m > 2^{i-1} \end{cases}$$

The map gives an endomorphism of Γ whose image has vertices and edges $\{\{(i,j),(i+1,j)\},\{(2,0),(2,1)\}|j \in \{0,1\}\}$. The graph Γ has one end as any ray is equivalent to either $\mathbb{N}\times\{1\}$ or $\mathbb{N}\times 0$ and these two rays are also equivalent because of the edges of the form $(2^{2i},0) \sim (2^{2i},1)$. However, $f(\Gamma)$

Figure 4.1: A portion of the graph in Example 4.4 and its image under the endomorphism



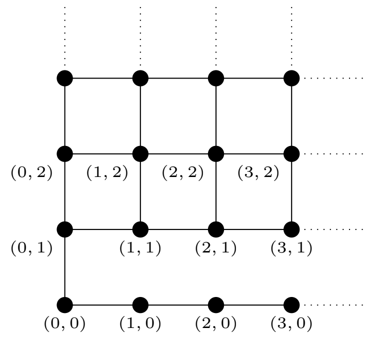
has two ends as there are no paths between the rays $\mathbb{N} \times \{1\}$ and $\mathbb{N} \times 0$ other than $(2,0) \sim (2,1)$. □

The previous two examples demonstrate that the number of ends of a digraph can both increase and decrease under endomorphisms. However, if an endomorphism is injective then it does preserve the structure of the ends of a digraph.

Lemma 4.5. *Let Γ be a digraph and let S be a semigroup acting on Γ by monomorphisms. Then S acts on $\Omega(\Gamma)$ by order-preserving homomorphisms.*

Proof. Let $s \in S$, let $\omega \in \Omega(\Gamma)$ and let $\mathbf{r} \in \omega$. We show that S acts by order-preserving homomorphisms on the pre-order on rays and hence acts by order-preserving homomorphisms on the poset of ends. As s is an injective homomorphism the image of a ray is a ray. If $\mathbf{x} \succcurlyeq \mathbf{y}$ then s preserves the infinitely many disjoint paths between $\mathbf{x}^s$ and $\mathbf{y}^s$ hence $\mathbf{x}^s \succcurlyeq \mathbf{y}^s$. □

Figure 4.2: A portion of the graph from Example 4.6



Example 4.6. Let Γ be the graph with vertices $\mathbb{N}_0 \times \mathbb{N}_0$ and edges $\{\{(i,j),(i+1,j)\} : i,j \in \mathbb{N}_0\}$, $\{(0,0),(0,1)\}$ and $\{\{(i,j),(i,j+1)\} : i \in \mathbb{N}_0, j \in \mathbb{N}\}$, see Figure 4.2. This graph has two ends, one corresponding to the ray $(0,0) \sim (1,0) \sim (2,0) \sim \ldots$ and another corresponding to the subgraph $\mathbb{N} \times \mathbb{N}$. The graph Γ has a monomorphism f whose image is the subgraph Γ' with vertices $\mathbb{N}\times\mathbb{N}$ and edges $\{\{(i,j),(i+1,j)\} : i,j \in \mathbb{N}\}$ and $\{\{(i,j),(i,j+1)\} : i,j \in \mathbb{N}\}$. The graph $f(\Gamma)$ has one end. This example shows that the number of ends can be decreased under the action of monomorphisms.

In the hope of a generalisation of Theorem 1.13 stating that a left-cancellative semigroup has one, two or infinitely many ends one might try to encode the properties of a left-cancellative semigroup acting by left multiplication on its right Cayley graph into the action of endomorphisms on a graph. Let S be a semigroup and let A be a finite generating set for S. The first property to note is that for each $s \in S$, the mapping $\rho_s : S \to S$ given by

$x \mapsto sx$ is an endomorphism of $\Gamma_r(S, A)$. In a left-cancellative semigroup left multiplication is in fact a strong endomorphism as if $sa \neq t$ then $xsa \neq xt$. Left multiplication is also injective on vertices because if $xs = xt$ then $s = t$. A digraph Γ is *almost weakly vertex transitive* if there exists a finite set $F \subseteq V\Gamma$ such that for all $v \in V\Gamma$ there exists $f \in F$ and $s \in \mathrm{End}(\Gamma)$ such that $f^s = v$. All Cayley graphs of finitely generated semigroups are almost weakly transitive as if A is a finite generating set for S we have $S = S^1A$. It follows that all Cayley graphs of left-cancellative semigroups are almost weakly transitive under the action of strong monomorphisms. The following example gives a graph which is weakly transitive under the action of strong monomorphisms, but has n ends for any prescribed $n \in \mathbb{N}$.

Example 4.7. Construct a graph Γ by taking the Cayley graph of the free monoid over the alphabet $A = \{1, 2, \ldots, n\}$. Then additionally add vertices labelled by pairs $\{(w, i) | w \in A^+, i \in \mathbb{N}\}$ and edges $\{((w, i), wj1^i) | 2 \leq j \leq n\}$ and $((w, i), w1n^i)$ for each $w \in A^+$ and $i \in \mathbb{N}$.

To see that this graph is weakly transitive under strong monomorphisms we demonstrate that we can map the vertex ϵ to any other vertex by using a strong monomorphism. We map ϵ to $v \in A^*$ by mapping $w \mapsto vw$ and $(w, i) \mapsto (vw, i)$.

To map ϵ to (v, i) is slightly more involved. We map $jw \mapsto vj1^iw$ for

$2 \leq j \leq n$ and $1w \mapsto v1n^i w$ where $w \in A^*$. We also map $(jw, k) \mapsto (vj1^i w, k)$ for $2 \leq j \leq n$ and $(1w, k) \mapsto (v1n^i, k)$.

By inspection one can see that these maps are injective. As the vertices labelled by A^+ are always mapped to an isomorphic subtree of the form vA^+ and the vertices of the form (w, k) are mapped to the corresponding (w', k) we see that these mappings are strong endomorphisms.

To argue for a fixed $n \in \mathbb{N}$ that this graph has n ends we consider the induced subgraphs with vertices jw and (jw, k) for a fixed $1 \leq j \leq n$ and for $w \in A^*$ and $k \in \mathbb{N}$. We call these induced subgraphs C_j. We firstly note that there are no paths between each C_j apart from through ϵ. There is certainly no path only involving vertices from A^* and (jw, k) is only adjacent to vertices of the form $jwl1^k$ for $2 \leq l \leq n$ and $jw1n^k$ and these lie in jA^*. It follows that Γ has at least n ends.

We claim that there is path between any $u, v \in jA^*$ with $|u| = |v|$ that does not pass through any words of A^* of length shorter than $|u| - 1$. Fix $1 \leq j \leq n$. We prove this claim inductively. Consider the vertices jkl for $1 \leq k \leq n$, $1 \leq l \leq n$, there is a path from jkn to jkl for any $1 \leq l \leq n$ via jk. The vertex $(j, 1)$ is adjacent to both $j1n$ and $jk1$ for any $2 \leq k \leq n$. It follows we have paths from any ju to jv for $|u| = |v| = 2$ that does not pass through any word of A^* of length less than 2.

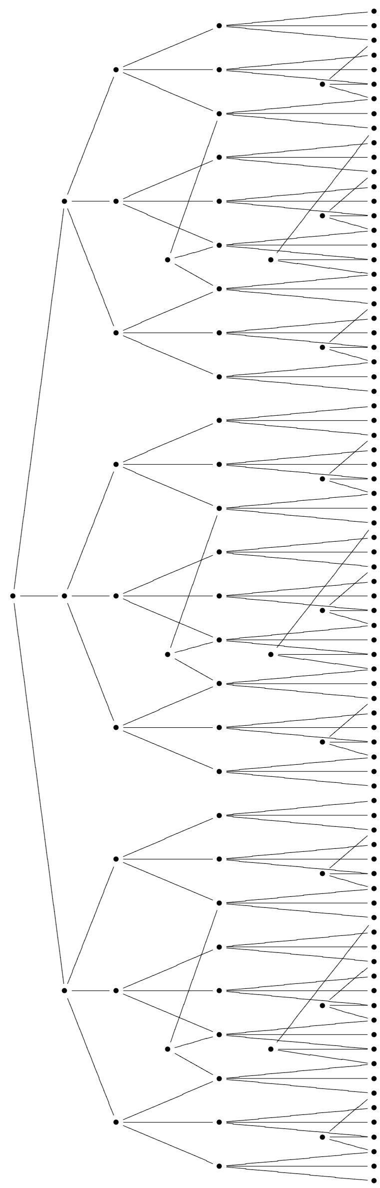
Assume we can pass from any ju to jv for $|u| = |v| = m$ via a path with vertices of size at least m. Let jku and jlv be vertices with $|u| = |v| = m+1$ and $k, l \in \{1, \ldots, n\}$. By assumption there exists a path from jku to $jk1^m$ and a path from jku to jkn^m via vertices of size at least $m + 1$. Similarly, there exists a path from jlv to $jl1^m$ and a path from jku to jkn^m via vertices of size at least $m + 1$. To get between $j1n^i$ and $jk1^i$ we pass through the vertex (j, i) which is not in the set $B_{m+1}(\epsilon)$. It follows that for any two rays in C_j and any finite set we can find a path between the rays which does not pass through the finite set. Hence, Γ has n ends. Figure 4.3 gives part of the graph Γ for $n = 3$.

The graph defined in the previous example gives n ends for an alphabet of size n. Hence, locally finite connected graphs which are weakly transitive under the action of strong monomorphisms do not satisfy a 1, 2, ∞ theorem. It is natural to ask whether this is true under some stronger conditions. One such condition would be to consider digraphs which are strongly transitive. Another is to consider groups rather than cancellative semigroups, Section 4.3 deals with this case.

4.2 Quasi-Isometries

Quasi-isometries are an important tool in the theory of metric spaces, and therefore also in graph theory. Quasi-isometries are maps between metric

Figure 4.3: A portion of the graph from Example 4.7 of a graph that is weakly transitive under the action of strong monomorphisms with n ends. Here$n = 3$.



spaces which divide metric spaces into equivalence classes of spaces which are globally similar. Various group-theoretic properties are quasi-isometric invariant, meaning that if a group G has property P and the Cayley graph of G is quasi-isometric to the Cayley graph of a group H then H has property P. These properties include such things as finite presentability, hyperbolicity and degree of growth. Also if a group G is finitely presented and has solvable word problem and G is quasi-isometric to H then H has solvable word problem. For more information consult [10, Chapter IV.B]. It was shown in Möller [32, Proposition 1] that if two locally finite connected graphs are quasi-isometric then they have the same number of ends. This next section contains a generalisation of this theorem to out-locally finite digraphs.

Recall we say two semi-metric spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there exists a function $f : X \to Y$ and there exist $\lambda, \epsilon, \mu \in \mathbb{R}^+$ such that for all $x, y \in X$

$$\frac{1}{\lambda} d_X(x, y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \epsilon$$

and for all $x' \in Y$ there exists $x \in X$ such that

$$d_Y(f(x), x'), d_Y(x', f(x)) \leq \mu.$$

We say two digraphs Γ_1 and Γ_2 are quasi-isometric if Γ_1 equipped with its digraph semi-metric is quasi-isometric to Γ_2 equipped with its digraph semi-metric.

We now show that this definition of quasi-isometry preserves the end poset of digraphs. We shall first need the following theorem.

Theorem 4.8 (Dilworth)**.** *Any infinite poset contains an infinite chain or an infinite anti-chain.*

Corollary 4.9. *Let Γ be an infinite digraph and let C be an infinite set of vertices. Then there exists an infinite sequence $\{c_1, c_2, \ldots\}$ of C such that either $d(c_i, c_j) < \infty$ for all $i < j$ or $d(c_i, c_j) < \infty$ for all $i > j$ or $d(c_i, c_j) = \infty$ for all $i < j$.*

Proof. If infinitely many elements of C lie an infinite strongly connected component then these vertices satisfy $d(c_i, c_j) < \infty$ for all $i < j$. If only finitely many elements of C lie in any given strongly connected component we define a preorder on $V\Gamma$ by $u \succcurlyeq v$ if $d(u, v) < \infty$, we then make a partial order by quotienting by equivalence under $\succcurlyeq$. This gives an infinite poset as the equivalence class of each vertex is finite because only finitely many elements of C lie in any given strongly connected components. By Theorem 4.8 the poset given by $\succcurlyeq$ contains an infinite chain or an infinite anti-chain. If it contains an infinite anti-chain then there exists an infinite set of vertices such that $d(c_i, c_j) = \infty$ for all $i \neq j$. If it contains an infinite chain then it contains an infinite set of vertices such that $d(c_i, c_j) < \infty$ for all $i < j$

or $d(c_i, c_j) < \infty$ for all $i > j$ depending on whether the chain is infinite ascending or infinite descending. □

Lemma 4.10. *Let Γ_1 and Γ_2 be out-locally finite digraphs and let f be a quasi-isometry from Γ_1 to Γ_2. If $C \subseteq V\Gamma_1$ is an infinite set then $f(C)$ is infinite.*

Proof. By Corollary 4.9 there exists an infinite sequence $\{c_1, c_2, \ldots\}$ of C such that either $d(c_i, c_j) < \infty$ for all $i < j$ or $d(c_i, c_j) < \infty$ for all $i > j$ or $d(c_i, c_j) = \infty$ for all $i < j$. Let $\epsilon, \lambda, \mu \in \mathbb{R}^+$ be the constants from the quasi-isometry f.

Assume $d(c_i, c_j) = \infty$ for all $i < j$. Let $i \leq j$. If $f(c_i) = f(c_j)$ then $d(f(c_i), f(c_j)) = 0$. It follows $d(c_i, c_j) \leq \lambda \cdot \epsilon$. Hence, $i = j$. It follows that $f(C)$ must be infinite.

Assume $d(c_i, c_j) < \infty$ for all $i < j$. It follows that $d(f(c_i), f(c_j))$ is finite so there exists a path from $f(c_i)$ to $f(c_j)$. If $|f(C)|$ is finite then there exists $v \in f(C)$ such that there exists an $n \in \mathbb{N}$ with $d(v, u) \leq n$ for all $u \in f(C)$. However, as Γ_1 is out-locally finite we have the property that for all $n \in \mathbb{N}$ there exists $K_n \in \mathbb{N}$ such that $d(c_{K_n}, c_0) > n$. Then $d(f(c_{K_{\lambda(n+\epsilon)}}), f(c_0)) \geq \frac{1}{\lambda} d(c_{K_{\lambda(n+\epsilon)}}, c_0) - \epsilon > n$, a contradiction. By symmetry we also see that if $d(c_i, c_j) < \infty$ for all $i > j$ then $f(C)$ is infinite. □

Lemma 4.11. *Let Γ_1, Γ_2 be out-locally finite digraphs and let $f : \Gamma_1 \to \Gamma_2$ be* *a quasi-isometry. The image of an in-ray (out-ray) under f is contained in an infinite anti-walk (walk) in which each vertex appears only finitely many times.*

Proof. It follows from Lemma 4.10 that the image of an in-ray $\mathbf{r} = r_0 \leftarrow r_1 \leftarrow \ldots$ is infinite. We make $f(\mathbf{r})$ into an infinite anti-walk $\mathbf{w}$ in Γ_2 by adjoining the shortest paths from each $f(r_{i+1})$ to $f(r_i)$. As $d(r_{i+1}, r_i) = 1$ it follows that $d(f(r_{i+1}), f(r_i)) \leq \lambda + \epsilon$ for all $i \in \mathbb{N}_0$. Hence, for all $v \in \mathbf{w}$ the out-ball $\overrightarrow{B}_{\lambda+\epsilon}(v)$ contains an element of $f(\mathbf{r})$. If a vertex $v \in \mathbf{w}$ appears infinitely often then the finite set $\overrightarrow{B}_{\lambda+\epsilon}(v)$ contains infinitely many $f(r_i)$, however, by the above argument the image of any infinite subset of $\mathbf{r}$ must be infinite, a contradiction. The argument for out-rays follows from considering a set C with the property $d(c_i, c_j) < \infty$ for $i < j$. □

Theorem 4.12. *If Γ_1, Γ_2 are two out-locally finite quasi-isometric digraphs then $\Omega(\Gamma_1) \cong \Omega(\Gamma_2)$.*

Proof. Let $f : \Gamma_1 \to \Gamma_2$ be a quasi-isometry with constants $\lambda, \epsilon, \mu \in \mathbb{R}^+$. By Lemma 4.11 the image of a ray under f is contained in an infinite walk $\mathbf{w}$ with each vertex in $\mathbf{w}$ appearing finitely often, then by Lemma 2.40 there exists a ray $\mathbf{r}' \in \Gamma_2$ with infinitely many disjoint paths from $\mathbf{r}'$ to $\mathbf{w}$ and vice versa. It follows that all possible choices for $\mathbf{r}'$ in Lemma 2.40 are equivalent, hence, for each ray $\mathbf{r}$ in Γ_1 there exists a unique (up to equivalence) ray $\mathbf{r}'$ in

Γ_2.

Let $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ be out-rays in Γ_1 and let $\mathbf{x}'$ and $\mathbf{y}'$ be the corresponding out-rays in Γ_2. To show $\Omega(\Gamma_1)$ is isomorphic to $\Omega(\Gamma_2)$ we prove $\mathbf{x} \succcurlyeq \mathbf{y}$ in Γ_1 if and only if $\mathbf{x}' \succcurlyeq \mathbf{y}'$, the proof for when $\mathbf{x}$ and $\mathbf{y}$ are not both out-rays is similar.

Firstly, assume that $\mathbf{x} \succcurlyeq \mathbf{y}$ but $\mathbf{x}' \not\succcurlyeq \mathbf{y}'$. Let $\Pi = \{\pi_k\}_{k \in \mathbb{N}}$ be a set of disjoint paths $\pi_k : x_{i_k} = c_{k,1} \to c_{k,2} \to \ldots \to c_{k,n_k} = y_{j_k}$. Construct a walk in Γ_2 from $f(x_{i_k})$ to $f(y_{j_k})$ by adjoining the shortest paths between each $f(c_{k,i})$ and $f(c_{k,i+1})$ and then let σ_k be the subpath of this walk from $f(x_{i_k})$ to $f(y_{j_k})$. We construct a walk w_1 from $\mathbf{x}'$ to $\mathbf{y}'$ by adjoining a path from $\mathbf{x}'$ to $f(x_{i_1})$ to σ_1 and then to a path from $f(y_{j_1})$ to $\mathbf{y}'$. Then we iteratively construct walks w_n by choosing a σ_{k_n} whose endpoints do not lie in any previous w_i and such that the path from $\mathbf{x}'$ to $f(x_{i_{k_n}})$ and the paths from $f(y_{j_{k_n}})$ do not pass through any previous w_i. This can be done as the image of the endpoints of all the π_k under f is infinite by Lemma 4.11 so there are infinitely many σ_k satisfying this condition. By Lemma 2.40 we know we can pick infinitely many disjoint paths from $\mathbf{x}'$ to the set of the initial points of these σ_k so we can pick infinitely many disjoint paths that avoid any prescribed finite set. The same follows for paths from the terminal vertices of the σ_k to $\mathbf{y}'$ and hence we have infinitely many choices for w_k and fix one which we call σ_{k_n}.

As $\mathbf{x}' \not\succcurlyeq \mathbf{y}'$ all w_k must pass through some finite set F. The w_k were constructed so the initial segments from $\mathbf{x}'$ to σ_{k_n} and terminal segments from σ_{k_n} to $\mathbf{y}'$ did not pass through any previous w_i, this means all the σ_{k_n} must pass through F. However, the σ_k were constructed in such a way that for any vertex v in σ_k there exists $f(c_{k,i_k})$ with $d(v, f(c_{k,i_k})) < \lambda + \epsilon$. This means the finite set $\overrightarrow{B}_{\lambda+\epsilon}(F)$ contains infinitely many $f(c_{k,i_k})$. It is a consequence of Corollary 4.9 and Lemma 4.10 that the image of these infinitely many c_{k,i_k} under f must be infinite, a contradiction.

Assume $\mathbf{x} \not\approx \mathbf{y}$, then there exists a finite set F such that all paths from $\mathbf{x}$ to $\mathbf{y}$ pass through F. Let $\mathbf{x}'$ and $\mathbf{y}'$ be the corresponding rays in Γ_2. We consider the set of all paths from $\mathbf{x}'$ to $\mathbf{y}'$. For each path σ from $\mathbf{x}'$ to $\mathbf{y}'$ we extend σ to a path from $f(\mathbf{x})$ to $f(\mathbf{y})$ by picking infinite disjoint sets of paths from $f(\mathbf{x})$ to $\mathbf{x}'$ and from $\mathbf{y}'$ to $f(\mathbf{y})$. Let $\pi = (f(x_i) = c_0, c_1, \ldots, c_n = f(y_j))$ be one such path from $f(\mathbf{x})$ to $f(\mathbf{y})$. For each c_i there exists $c_i' \in V\Gamma_1$ such that $d(f(c_i'), c_i), d(c_i, f(c_i')) \leq \mu$. This gives a sequence in $V\Gamma_1$ such that $d(f(c_i'), f(c_{i+1}')) \leq 2\mu + 1$ so $d(c_i', c_{i+1}') \leq \lambda(2\mu + 1 + \epsilon)$. We make this sequence of c_i' in Γ_1 into a walk in Γ_1 by adjoining shortest paths between consecutive c_i'. Now one vertex of this path must be in F so at least one of the c_i' must be in the finite set $F' = \overrightarrow{B}_{\lambda(2\mu+1+\epsilon)}(F)$. At least one $f(c_i')$ must be in $f(F')$ and then $c_i \in \overrightarrow{B}_\mu(f(F'))$ which is a finite set. This means every path we have constructed from $f(\mathbf{x})$ to $f(\mathbf{y})$ goes through this finite set. It may be that the vertex of the path that lies in $\overrightarrow{B}_\mu(f(F'))$ was not in the original path from $\mathbf{x}'$ to $\mathbf{y}'$, however, as these paths from $f(\mathbf{x})$ to $\mathbf{x}'$ and from $\mathbf{y}'$ to $f(\mathbf{y})$ were made up from an infinite set of disjoint paths only finitely many can pass through any finite set. This means $\mathbf{x}' \not\succcurlyeq \mathbf{y}'$.

We now show that for any ray $\mathbf{z}$ in Γ_2 there exists a ray $\mathbf{z}'$ in Γ_1 such that $\mathbf{z}$ has infinitely many disjoint paths to $f(\mathbf{z}')$ and vice versa. This will give an equivalence between the poset of ends of each of the digraphs. Let $\mathbf{z} = (z_0, z_1, \ldots)$ be an out-ray in Γ_2, the proof for in-rays is analogous. For each $z_i \in \mathbf{z}$ there exists $z_i' \in \Gamma_1$ such that $d(z_i, f(z_i')), d(f(z_i'), z_i) \leq \mu$. If $z_i' = z_j'$ then $d(z_i', z_i), d(z_i', z_j) \leq \mu$. As $\overrightarrow{B}_\mu(v)$ is finite for any $v \in V\Gamma_2$ it follows that only finitely many z_i have the same z_i'. We make an infinite walk $\mathbf{w}$ by adjoining the shortest paths from z_i' to z_{i+1}', as $d(z_i, z_{i+1}) = 1$ it follows that $d(f(z_i'), f(z_{i+1}')) \leq 2\mu + 1$ and hence $d(z_i', z_{i+1}') \leq \lambda(2\mu + 1 + \epsilon)$. If a vertex v appears infinitely many times in this walk then infinitely many z_i' lie in $\overrightarrow{B}_{\lambda(2\mu+1+\epsilon)}(v)$. However, this ball is finite and any infinite collection of z_i' contains infinitely many vertices. By Lemma 2.40 there exists a ray $\mathbf{z}'$ with infinitely many disjoint paths to $\mathbf{w}$ and vice versa. Now it follows from the proof of $\mathbf{x} \succcurlyeq \mathbf{y}$ implies $\mathbf{x}' \succcurlyeq \mathbf{y}'$ that there are infinitely many disjoint paths from $(\mathbf{z}')'$ to $f(\mathbf{w})$ and vice versa. This means that $(\mathbf{z}')'$ is equivalent to $\mathbf{z}$. □

Proposition 4.13. *[16, Proposition 4] Let A and B be finite generating sets* *for a semigroup S. Then $\Gamma_r(S, A)$ is quasi-isometric to $\Gamma_r(S, B)$.*

Corollary 4.14. *Let S be a semigroup and A and B finite generating sets for S then $\Omega(\Gamma_r(S, A)) \cong \Omega(\Gamma_r(S, B))$.*

Note that, unlike in the case of groups, quasi-isometric invariance is not sufficient to understand the end structure of subsemigroups of finite index. For example, consider the $FG(a, b)$. If we adjoin a zero element then $d(w, 0) = 1$ for all $w \in FG(a, b)$. However, in $\Gamma(FG(a, b), \{a, b\})$ there can be no vertex with such a property. The group $FG(a, b)$ is of finite Rees index (and hence finite Green index) in $FG(a, b) \cup \{0\}$.

4.3 Almost Vertex Transitive Digraphs

In this section we examine an analogue of Abel and Hopf's theorem (Theorem 4.1) for digraphs.

In general it is not true that locally finite, connected, almost vertex transitive digraphs have one, two or infinitely many ends. In fact it is possible to construct such a digraph with any finite number of ends including 0. These are demonstrated in the next two examples.

We refer to the graph with vertices $\mathbb{Z} \times \mathbb{Z}$ and edges $(i, j) \sim (i+1, j), (i, j) \sim (i, j+1)$ for $i, j \in \mathbb{Z}$ as the *infinite grid.*

Example 4.15. Let Γ be the direct product of the infinite grid with a directed line of length n. Then Γ is almost transitive as the infinite grid is vertex transitive and there are only finitely many copies of the infinite grid. Any ray must eventually lie in only one copy of the infinite grid so Γ has n ends and $\Omega(\Gamma) = \omega_1 > \omega_2 > \omega_3 > \ldots > \omega_n$.

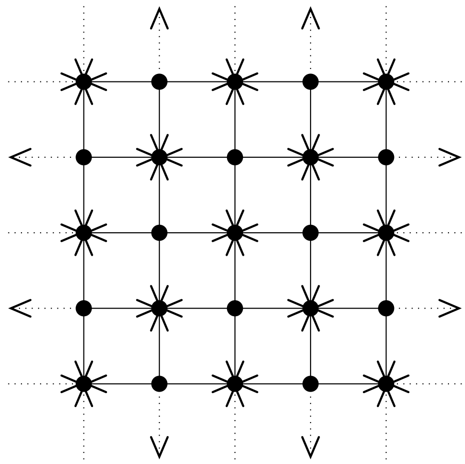
Example 4.16. Let Γ be the digraph with vertices $\mathbb{Z} \times \mathbb{Z}$ and edges

$$\begin{aligned}
E\Gamma &= \{(2i, 2j), (2i+1, 2j) : i, j \in \mathbb{Z}\} \\
&\cup \{(2i, 2j), (2i-1, 2j) : i, j \in \mathbb{Z}\} \\
&\cup \{(2i, 2j), (2i, 2j+1) : i, j \in \mathbb{Z}\} \\
&\cup \{(2i, 2j), (2i, 2j-1) : i, j \in \mathbb{Z}\} \\
&\cup \{(2i+1, 2j+1), (2i+2, 2j+1) : i, j \in \mathbb{Z}\} \\
&\cup \{(2i+1, 2j+1), (2i, 2j+1) : i, j \in \mathbb{Z}\} \\
&\cup \{((2i+1, 2j+1), (2i, 2j+2)) : i, j \in \mathbb{Z}\} \\
&\cup \{(2i+1, 2j+1), (2i+1, 2j) : i, j \in \mathbb{Z}\}.
\end{aligned}$$

See Figure 4.4 for a portion of the digraph. The map $f : V\Gamma \to V\Gamma$ defined by $f(a, b) = (a + 2, b)$ and the map $g : V\Gamma \to V\Gamma$ defined by $f(a, b) = (a, b = 2)$ are automorphisms. This digraph is almost vertex transitive as $\{(0, 0), (1, 0)\}$ can be mapped to any other vertex of Γ using f and g. However, the longest path is of length 1 so Γ has no ends.

Lemma 4.17. *A finite vertex transitive connected digraph is strongly connected.*

Figure 4.4: A portion of an almost vertex transitive digraph with no ends from Example 4.16.



Proof. Let Γ be a finite vertex transitive digraph with vertices $v_1, v_2, \ldots, v_n$. It is sufficient to show that if $u \to v$ then there exists a walk from v to u. Let $u \to v$. As Γ is vertex transitive there exists $g \in \mathrm{Aut}(\Gamma)$ such that $u^g = v$. Now consider the vertices $u, u^g, u^{g^2}, \ldots, u^{g^n}$. There are $n+1$ of these so there exists $i < j$ such that $u^{g^i} = u^{g^j}$. It follows that $u^{g^{j-i}} = u$ and that $v = u^g \to u^{g^2} \to \ldots \to u^{g^{j-i}} = u$ is a walk from v to u. □

Lemma 4.18. *A connected vertex transitive digraph has one or infinitely many strongly connected components.*

Proof. Let Γ be a connected vertex transitive digraph. Assume Γ has finitely many strongly connected components say C_1, C_2, $\ldots$, C_n. Then as Γ is vertex transitive the connected components have the same size and any automorphism permutes them. As $\mathrm{Aut}(\Gamma)$ acts transitively on Γ it follows that $\mathrm{Aut}(\Gamma)$ acts transitively on connected components. Let Γ' be a finite digraph with vertices $\{1, 2, \ldots, n\}$ and an edge (i, j) if there exist a vertex $x \in C_i$ and a vertex $y \in C_j$ such that $(x, y) \in E\Gamma$. As $\mathrm{Aut}(\Gamma)$ acts transitively on connected components it also acts transitively on the vertices of Γ'. By Lemma 4.17 finite vertex transitive digraphs are strongly connected so $n = 1$ and Γ has only one strongly connected component. □

Theorem 4.19. *An out-locally finite, strongly connected, almost vertex transitive digraph is quasi-isometric to its underlying undirected graph.*

Proof. Let Γ be an out-locally finite strongly connected almost vertex transitive digraph with semi-metric d and automorphism group G. Fix representatives $B = \{v_1, v_2, \ldots, v_n\}$ from each orbit, note there are only finitely many because Γ is almost vertex transitive. For each $1 \leq i \leq n$, let $\lambda_i = \max\{d(w, v_i) : (v_i, w) \in E\}$. The distance $d(w, v_i)$ is finite because Γ is strongly connected and the maximum exists because Γ is out-locally finite. Let $\lambda = \max\{\lambda_i : 1 \leq i \leq n\}$.

Let Γ^U be the underlying undirected graph of Γ and let $\bar{d}$ be the graph metric on Γ^U. We claim that the identity map $id : V\Gamma^u \to V\Gamma$ is a quasi-isometry. As we are using the identity map we have that for any $v \in V\Gamma$ the distance $d(v, id(v)) = d(id(v), v) = 0$. Also because any path in Γ is also a path in Γ^u it follows that $\bar{d}(x,y) \leq d(x,y)$.

Let $x, y \in V\Gamma$ be such that $x \to y$.There exists $v_i \in B$ and $g \in G$ such that $v_i^g = x$. There also exists $w \in V\Gamma$ such that $v_i \to w$ and $w^g = y$. It follows that

$$d(y,x) = d(w^g, v_i^g) = d(w, v_i) \leq \lambda.$$

From the above inequality it follows that for any undirected path there is a directed path gained by replacing every edge $u \leftarrow v$ by a directed path of length at most λ from u to v. Hence, for any $x, y \in V\Gamma$ we have $d(x,y) \leq \lambda \bar{d}(x,y)$. Thus $\frac{1}{\lambda} d(x,y) \leq \bar{d}(x,y) \leq \lambda d(x,y)$, as required. □

Lemma 4.20. *A connected vertex transitive digraph with more than one vertex and no non-trivial directed cycles has at least two ends.*

Proof. Let Γ be a connected vertex transitive digraph with no non-trivial directed cycles. As Γ is connected and has at least two vertices there exist vertices $u, v \in V\Gamma$ with $u \to v$. It follows from vertex transitivity that there exists an element $g \in \mathrm{Aut}(G)$ such that $u^g = v$. If $u^{g^i} = u^{g^j}$ for some $i, j \in \mathbb{Z}$ with $i < j$ then $u^{g^{j-i}} = u$. As $u \to u^g$ we have that $u^{g^k} \to u^{g^{k+1}}$ for all $k \in \mathbb{Z}$ and hence, there is a directed cycle $u \to u^g \to u^{g^2} \to \ldots \to u^{g^{j-i}} = u$. It follows that $u^{g^i} \neq u^{g^j}$ for all distinct i, j. This means we have a doubly infinite path $\ldots \to u^{g^{-1}} \to u \to u^g \to u^{g^2} \to \ldots$.

If there was a directed path from u^{g^i} to u^{g^j} for some $i > j$ in $\mathbb{Z}$ then there would be a directed cycle. In particular the out-ray $u \rightarrow u^g \rightarrow u^{g^2} \rightarrow \ldots$ cannot be equivalent to the in-ray $u \leftarrow u^{g^{-1}} \leftarrow u^{g^{-2}} \leftarrow \ldots$ so Γ has at least two ends. $\square$

As a consequence of the proof of Lemma 4.20 if a digraph Γ has no non-trivial directed cycles then out-rays cannot be equivalent to in-rays. Thus we are able to describe ends as being in- or out- as they can only contain rays of the same type.

Lemma 4.21. *Let* Γ *be a digraph in which the number of ends containing an in-ray (resp. out-ray) is finite and let* $\mathbf{x}$ *be an in-ray (out-ray). Then there exists a finite set* F *such that for any in-ray (out-ray)* $\mathbf{r} = (x_0, x_1, \ldots)$ *with the property* $\mathbf{r} \not\succeq \mathbf{x}$ $(\mathbf{x} \not\succeq \mathbf{r})$ *there exists an* $N \in \mathbb{N}$ *such that all paths from* x_n *to* $\mathbf{a}$ *(from* $\mathbf{d}$ *to* x_n*) pass through* F *for all* $n \geq N$*.*

Proof. Let $\omega_1, \omega_2, \ldots, \omega_n$ be the ends of Γ which contain an in-ray and let $\mathbf{r_i}$ be an in-ray in ω_i. Without loss of generality we will assume that $\mathbf{a} = \mathbf{r_1}$. For each $\mathbf{r_i}$ such that $\mathbf{r_1} \not\preceq \mathbf{r_i}$ there exists a finite set F_i such that all paths from $\mathbf{r_i}$ to $\mathbf{r_1}$ pass through F_i. We let $F = \bigcup_{i \in I} F_i$. Now any in-ray $\mathbf{r} = (x_0, x_1, \ldots)$ such that $\mathbf{r_1} \not\preceq \mathbf{r}$ must be equivalent to some $\mathbf{r_i}$. There exists a minimal $N \in \mathbb{N}$ such that for all $n > N$ there exists a path from $\mathbf{r_i}$ to x_n that does not pass through F. Any path from x_n to $\mathbf{r_1}$ can be extended to a walk from $\mathbf{r_i}$ to x_n to $\mathbf{r_1}$. As all paths from $\mathbf{r_i}$ to $\mathbf{r_1}$ pass through F_i, all walks must also pass through F_i. However, when we extended the path from x_n to $\mathbf{r_i}$ it was done in such a way that the path from $\mathbf{r_i}$ to x_n did not pass through F and, hence, the path from x_n to $\mathbf{r_i}$ must pass through F. $\square$

We say two rays $\mathbf{x}$ and $\mathbf{y}$ are of the same *type* if both $\mathbf{x}$ and $\mathbf{y}$ are in-rays or both $\mathbf{x}$ and $\mathbf{y}$ are out-rays.

Lemma 4.22. *Let Γ be a connected, vertex transitive digraph. If Γ contains two inequivalent rays of the same type then there exists two inequivalent rays of the same type with the same initial vertex.*

Proof. Let $\mathbf{x} = (x_0, x_1, \ldots)$ and $\mathbf{y} = (y_0, y_1, \ldots)$ be inequivalent rays of the same type. As Γ is connected there exists an undirected path $x_0 = z_0 \sim z_1 \sim \ldots \sim z_n = y_0$. As Γ is vertex transitive there exists $g_i \in \mathrm{Aut}(\Gamma)$ such that $x_0^{g_i} = z_i$. If $\mathbf{x}^{g_n}$ is inequivalent to $\mathbf{y}$ then the we have inequivalent rays of the same type with the same initial vertex. Otherwise there exists a maximal $0 \leq i < n$ such that $\mathbf{x}^{g_i}$ is inequivalent to $\mathbf{y}$ but $\mathbf{x}^{g_{i+1}}$ is equivalent to $\mathbf{y}$. As $x_0^{g_i} = z_i \sim z_{i+1} = x_0^{g_{i+1}}$ one of these rays can be extended so they start at the same vertex. $\square$

Lemma 4.23. *Let Γ be a vertex transitive digraph with no non-trivial cycles. If there exists a vertex $v \in V\Gamma$ and an out-ray (resp. out-ray) $\mathbf{r}$ such that there exist paths from infinitely many vertices of $\mathbf{r}$ to v (from v to infinitely many vertices of $\mathbf{r}$) then Γ has infinitely many ends.*

Proof. Assume Γ has finitely many ends. Let $\mathbf{r} = r_0 \to r_1 \to \ldots$ be a ray. Let $g \in \text{Aut}(\Gamma)$ be such that $r_0^g = v$. As Γ has finitely many ends it follows that $\mathbf{r}^{g^i} \equiv \mathbf{r}^{g^j}$ for some $i < j$. As g is an automorphism it follows that $\mathbf{r}^{g^k} \equiv \mathbf{r}$ for $k = j - i$. This means there is a path from $r_0^{g^k}$ to r_i for some $i \in \mathbb{N}$. However, there are paths from infinitely many r_j to r_0^g and so for any $j \in \mathbb{N}$ there exists a path from r_j to some $r_{j'}$ which has a path to r_0^g. Hence there is a path from all r_j to r_0^g. There are also paths from $r_0^{g^j}$ to $r_0^{g^{j+1}}$. This path gives rise to a cycle in contradiction to the fact that Γ does not contain any directed cycles. □

Lemma 4.24. *Let Γ be an out-locally finite, connected, vertex transitive digraph with no non-trivial directed cycles. If there exist two ends of the same type then $|\Omega(\Gamma)| = \infty$.*

Proof. Let $\mathbf{x} = (x_0, x_1, x_2, \ldots)$ and $\mathbf{y} = (y_0, y_1, y_2, \ldots)$ be two inequivalent rays of the same type and let $g_i \in \text{Aut}(\Gamma)$ such that $x_0^{g_i} = y_i$. Assume that Γ has finitely many ends. As Γ has finitely many ends we may assume that $\mathbf{y}$ is maximal in its type. By this we mean if a ray $\mathbf{r}$ satisfies $\mathbf{r} \succcurlyeq \mathbf{y}$ then if $\mathbf{r}$ is of the same type as $\mathbf{y}$ we have $\mathbf{r} \equiv \mathbf{y}$. By Lemma 4.22 we can assume $x_0 = y_0$. We claim that if $\mathbf{x}$ and $\mathbf{y}$ are in-rays then for infinitely many $i \in \mathbb{N}$ there exists an in-rays $\mathbf{r_i}$ whose initial vertex is y_i and such that $\mathbf{r_i} \not\succcurlyeq \mathbf{y}$.

Let $g \in \text{Aut}(\Gamma)$ then one of the following holds $\mathbf{x}^g \succ \mathbf{y}$, $\mathbf{x}^g \equiv \mathbf{y}$ or $\mathbf{x}^g \not\succcurlyeq \mathbf{y}$.

It follows that for some infinite subset I of $\mathbb{N}$ we have one of the following $\mathbf{x}^{g_i} \succ \mathbf{y}$, $\mathbf{x}^{g_i} \equiv \mathbf{y}$ or $\mathbf{x}^{g_i} \not\succeq \mathbf{y}$ for all $i \in I$.

As $\mathbf{y}$ was chosen to be maximal of its type and $\mathbf{x}^{g_i}$ is of the same type as $\mathbf{y}$ we cannot have $\mathbf{x}^{g_i} \succ \mathbf{y}$.

If $\mathbf{y} \equiv \mathbf{x}^{g_i}$ then $\mathbf{y}^{g_i} \not\equiv \mathbf{y}$, as $\mathbf{x}$ is not equivalent to $\mathbf{y}$. It follows either $\mathbf{y}^{g_i} \succ \mathbf{y}$ or $\mathbf{y}^{g_i} \not\succeq \mathbf{y}$. Again as $\mathbf{y}$ is maximal of its type we cannot have $\mathbf{y}^{g_i} \succ \mathbf{y}$. Hence, $\mathbf{y}^{g_i} \not\succeq \mathbf{y}$ for all $i \in I$. These $\mathbf{y}^{g_i}$ are the $\mathbf{r_i}$ from the claim.

By Lemma 4.21, and the assumption that Γ has finitely many ends, there exists a finite set F such that for any in-ray $\mathbf{r}$ such that $\mathbf{r} \not\succeq \mathbf{y}$ all paths, starting after a certain point in $\mathbf{r}$, from $\mathbf{r}$ to $\mathbf{y}$ intersect F. As the subpaths of each $\mathbf{r_i}$ ending at y_i can be viewed as paths from $\mathbf{r_i}$ to $\mathbf{y}$ it follows each $\mathbf{r_i}$ must pass through F. Then by Corollary 4.23 there exists some vertex $f \in F$ that has paths to infinitely many vertices of the in-ray $\mathbf{y}$ and, hence, Γ has infinitely many ends.

By symmetry if $\mathbf{x}$ and $\mathbf{y}$ are out-rays then there exists out-rays $\mathbf{r_i}$ whose initial vertex is y_i and such that $\mathbf{y} \not\succeq \mathbf{r_i}$. Again we may use Lemma 4.21 and Lemma 4.23 to show that Γ has infinitely many ends. □

Theorem 4.25. *Infinite, out-locally finite, connected, vertex transitive digraphs have* 1, 2 *or infinitely many ends.*

Proof. Let Γ be an infinite, out-locally finite, connected vertex transitive digraph. Then five situations can occur

1. Γ has finitely many strongly connected components.

2. Γ has infinitely many finite strongly connected components.

3. Γ has infinitely many infinite strongly connected components.

4. Γ has no directed cycles.

By Lemma 4.18 if Γ has finitely many strongly connected components it is strongly connected. It follows from Theorem 4.19 that an infinite strongly connected almost vertex transitive (and thus a vertex transitive) digraph is quasi-isometric to an infinite connected vertex transitive graph. Then by Theorem 4.12 the digraph Γ has the same number of ends as an infinite connected vertex transitive graph. Theorem 4.1 states that an infinite connected vertex transitive graph must have 1, 2, or $2^{\aleph_0}$ ends. Hence, Γ has 1, 2, or $2^{\aleph_0}$ ends in Case (1).

In a vertex transitive digraph all connected component have the same size. If Γ has infinitely many finite strongly connected components then Γ is quasi-isometric to the digraph gained by identifying each strongly connected component to a vertex. It follows from Theorem 4.12 that the possible number of ends of a digraph in Case (2) is the same as the possible number of ends of a digraph in Case (4). By Lemma 4.24 if Γ has no directed cycles it has two or infinitely many ends.

We are left with Case (3). If Γ has infinitely many infinite strongly connected components then by vertex transitivity each pair of the strongly connected components are isomorphic. Each strongly connected component is also vertex transitive as strongly connected components are preserved under automorphisms. Thus by Theorem 4.19 each strongly connected component has 1, 2, or $2^{\aleph_0}$ ends. Because Γ has countably many strongly connected components it follows Γ has $\aleph_0$ or $2^{\aleph_0}$ ends. $\square$

Here we give examples of vertex transitive digraphs with 'interesting' end sets.

Example 4.26. Let Γ be the digraph with vertices $\mathbb{Z} \times \mathbb{Z}$ and edges $(i, j) \to (i + 1, j)$ and $(i, j) \to (i, j + 1)$. Let f be the function $f : V\Gamma \to V\Gamma$ such that $f(i, j) = (i + 1, j)$ and let g be the function $g : V\Gamma \to V\Gamma$ such that $g(i, j) = (i, j + 1)$. Both f and g are automorphisms of Γ and by using the automorphisms f, g, f^{-1}, g^{-1} we can see Γ is vertex transitive. We see Γ has $\aleph_0$ ends corresponding to rays of the form $\mathbb{N} \times \{i\}$, $(-\mathbb{N}) \times \{i\}$, $\{i\} \times \mathbb{N}$, $\{i\} \times (-\mathbb{N})$, and the out-ray with vertices including each (i, i) for $i \in \mathbb{N}$ and the in-ray with vertices including (i, i) for $i \in -\mathbb{N}$. The digraph Γ reflects the fact that a digraph can have $\aleph_0$ ends whereas it is not possible for a vertex

Figure 4.5: A portion of the digraph from Example 4.26.

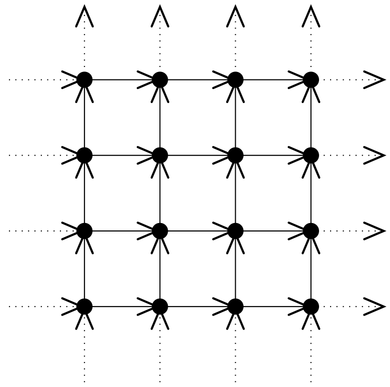
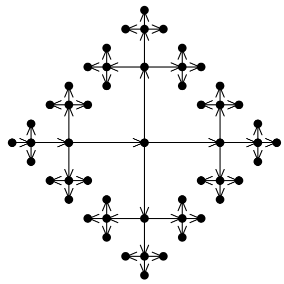


Figure 4.6: A portion of the digraph in Example 4.27.



transitive graph to have $\aleph_0$ ends. See Figure 4.5 for a portion of the digraph.

Example 4.27. Let Γ be the digraph with vertices labelled by reduced words in $\{a, a^{-1}, b, b^{-1}\}^*$ and edges $w \to wa$, $w \to wa^{-1}$ and $w \to wb$ if w is not of the form $w'b^{-1}$ and $w \to w'$ if w is of the form $w'b^{-1}$. The digraph Γ contains $2^{\aleph_0}$ ends but only one in-ray, $\ldots \to b^{-2} \to b^{-1} \to \epsilon$. See Figure 4.6 for a portion of the digraph.

Theorem 4.25 complements a result due to Gray and Kambites in [15]. The following definitions are needed to understand the result.

Definition 4.28. *Let* (X, d) *be a semi-metric space. A* path *of length* $n \in \mathbb{R}$ *from* x *to* y *is a map* $p : [0, n] \to X$ *such that* $p(0) = x$*,* $p(n) = y$ *and* $d(p(a), p(b)) \leq b?a$ *for all* $0abn$*.*

Definition 4.29. *If* $d(x, y) < \infty$ *then a* geodesic *from* x *to* y *is a path of length* $d(x, y)$ *from* x *to* y*. The semimetric space* X *is called* geodesic *if for all* $x, y \in X$ *with* $d(x, y) < \infty$ *there exists at least one geodesic from* x *to* y*.*

A digraph equipped with the usual digraph semimetric can be made into a geodesic semimetric space by glueing a copy of the unit interval onto each edge.

Definition 4.30. *Let* (X, d) *be a semi-metric space, let* $x \in X$ *and let* $r \geq 0$*. The* strong ball *of radius* r *around* x *is the set*

$$\overleftrightarrow{B}(x, r) = \{y \in X \ : \ d(x, y), d(y, x) \leq r\}.$$

Definition 4.31. *The action of a semigroup* S *on a semi-metric space* X *is called* cobounded *if there exists a strong ball* B *of finite radius such that* $B^S = X$*.*

Definition 4.32. *The action of a semigroup* S *on a semi-metric space* X *is called* outward proper *if for every out-ball* B *of finite radius the set* $\{s \in S : d(B, B^s) = 0\}$ *is finite.*

Definition 4.33. *Let (X, d) be a semi-metric space. An element $x_0 \in X$ is called a* basepoint *if $d(x_0, y) < \infty$ for all $y \in X$.*

Definition 4.34. *The action of a semigroup S on a semi-metric space X is called* idealistic at basepoint x_0 *if for all $s, t \in S$ such that $d(x_0^s, x_0^t) < \infty$ we have $tS \subseteq sS$. We say a semigroup acts* idealistically *if the action is idealistic at some basepoint x_0.*

Theorem 4.35. *[15, Theorem 4.1] Let M be a monoid acting idealistically, outward properly and coboundedly by isometric embeddings on a geodesic semi-metric space X. Then M is finitely generated and the left Cayley graph of Γ is quasi-isometric to X.*

A locally finite, vertex transitive digraph with a basepoint satisfies these conditions. It is easy to see that a vertex transitive digraph with a basepoint will necessarily be strongly connected. Thus it is possible to get Theorem 4.19 as a corollary of this theorem. However, not all vertex transitive digraphs have basepoints which is why examples like Examples 4.26 and 4.27 have end sets that are so different from the set of ends of the group acting on them.

Theorem 4.35 is also heavily linked with the considerations in Section 4.1. There we tried to describe how ends behaved under endomorphisms. In an attempt to have a 1,2, ∞ theorem for digraphs we tried to encode the properties of a cancellative semigroup into the endomorphisms. Using

Theorem 4.35 and Theorem 1.13 we have the following result.

Corollary 4.36. *Let* Γ *be a locally finite digraph with basepoint* x_0*. If there exists a semigroup of strong monomorphisms that acts idealistically, outward properly and coboundedly on* Γ *then* Γ *has* 1*,* 2 *or infinitely many ends.*

Proof. Let S be the set of monomorphisms. We may assume that for each $s, t \in S$ there exists $v \in V\Gamma$ such that $v^s \neq v^t$. Otherwise, these monomorphisms are equivalent. Let $s, t, x \in S$. If $v^{sx} = v^{tx}$ then $v^s = v^t$ as x is injective. Hence S forms a right-cancellative monoid. By Theorem 4.35 S is finitely generated and Γ is quasi-isometric to the left Cayley of S. As S is a right-cancellative monoid the left Cayley graph of S has 1, 2 or infinitely many ends by Theorem 1.13. □

The semigroup S acting on the graph Γ in Example 4.7 is cancellative and its action is cobounded and outward proper. The graph Γ also has a basepoint (because it is a graph), however, Γ does not satisfy the idealistic action property and thus can have n ends whereas S is cancellative and hence has 1, 2, or infinitely many ends.

4.4 End Topology

An important part of the theory of ends of graphs is the end topology of a graph. The end topology can be used to give a more geometric proof of Stallings' Theorem as well as showing that a locally finite graph has finitely many, $\aleph_0$ or $2^{\aleph_0}$ ends. In this section we give a topology for digraphs that generalises the end topology for graphs and shares similar properties. First we introduce some related notions and prove some technical lemmas.

Definition 4.37. *Let* Γ *be a digraph and let* C *be a set of vertices. The* out-boundary *of* C*, denoted* $\overrightarrow{\delta}(C)$*, is the the set* $\{v \in V\Gamma \setminus C : \exists u \in C,\, u \to v\}$.

Definition 4.38. *Let* Γ *be a digraph. We say a set* $C \subseteq V\Gamma$ *is a* cut *if* $|\overrightarrow{\delta}(C)| < \infty$.

Lemma 4.39. *Let* Γ *be a digraph, let* C *be a cut and let* $\mathbf{r}$ *be a ray. Then either only finitely many vertices of* $\mathbf{r}$ *lie in* C *or only finitely many vertices of* $\mathbf{r}$ *lie in* $V\Gamma \setminus C$ *but not both.*

Proof. We assume $\mathbf{r} = (r_0, r_1, \ldots)$ is an out-ray, the proof for when $\mathbf{r}$ is an in-ray is analogous. Assume there are infinitely many vertices of $\mathbf{r}$ in C. As $\overrightarrow{\delta}(C)$ is finite and each r_i is distinct there can be only finitely many r_i such that $r_i \in C$ and $r_{i+1} \notin C$. If all r_i are in C then the statement holds, so assume there exists i such that $r_i \notin C$. Let N be the largest integer such that $r_N \in C$ and $r_{N+1} \notin C$. As infinitely many vertices of $\mathbf{r}$ lie in C there exists N' such that $N' > N$ and $r_{N'} \in C$. But then all r_n must lie in C for $n \geq N'$. $\square$

It follows from Lemma 4.39 that we can say a ray is *contained* in a cut if it has infinitely many vertices in the cut.

Lemma 4.40. *Let Γ be a digraph, let C be a cut and let $\mathbf{x}$ and $\mathbf{y}$ be rays. If $\mathbf{x}$ is contained in C and $\mathbf{x} \succcurlyeq \mathbf{y}$ then $\mathbf{y}$ is contained in C.*

Proof. As $\mathbf{x} \succcurlyeq \mathbf{y}$ there exist infinitely many disjoint paths from $\mathbf{x}$ to $\mathbf{y}$. We may assume these paths all have initial vertices in C as all but finitely many vertices of $\mathbf{x}$ lie in C. Only finitely many of these paths can pass through $\overrightarrow{\delta}(C)$ as $\overrightarrow{\delta}(C)$ is finite and the paths are disjoint. It follows that infinitely many vertices of $\mathbf{y}$ lie in C. $\square$

It follows from Lemma 4.40 that if $\mathbf{x}$ is a ray contained in the cut C and if $\mathbf{y}$ is equivalent to $\mathbf{x}$ then $\mathbf{y}$ is contained in C. Hence, if C is a cut we can define $\Omega(C)$ to be all those ends containing a ray with infinitely many vertices in C.

Definition 4.41. *Let Γ be a digraph. The* end topology *of Γ is on the set of vertices and ends of Γ. The topology has basis $\mathcal{B} = \{C \cup \Omega(C) : C \text{ is a cut}\}$.*

Lemma 4.42. *Let Γ be a digraph and let C, D be cuts. Then $C \cap D$ and $C \cup D$ are cuts.*

Proof. For $C \cup D$ we note that $\overrightarrow{\delta}(C \cup D) \subseteq \overrightarrow{\delta}(C) \cup \overrightarrow{\delta}(D)$ so $|\overrightarrow{\delta}(C \cup D)| \leq |\overrightarrow{\delta}(C)| + |\overrightarrow{\delta}(D)|$. For $C \cap D$ we let $v \in \overrightarrow{\delta}(C \cap D)$. If $v \notin C$ then $v \in \overrightarrow{\delta}(C)$. If $v \in C$ then $v \notin D$ and $v \in \overrightarrow{\delta}(D)$. Hence $\overrightarrow{\delta}(C \cap D) \subseteq \overrightarrow{\delta}(C) \cup \overrightarrow{\delta}(D)$. $\square$

Definition 4.43. *A topological space X is T_0 if for any points $x, y \in X$ there exists an open set U such that $x \in U$ and $y \notin U$ or $x \notin U$ and $y \in U$.*

Definition 4.44. *A topological space X is T_1 if for any points $x, y \in X$ there exist open sets U, V such that $x \in U$ and $y \notin U$ and $y \in V$ and $x \notin V$.*

Definition 4.45. *A topological space X is* compact *if every open cover of X contains a finite subcover.*

Definition 4.46. *Let X be a topological space. A sequence $(x_0, x_1, \ldots)$* converges *to a limit x if for every open neighbourhood U of x there exists $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.*

Definition 4.47. *A topological space X is* sequentially compact *if every sequence has a convergent subsequence.*

Definition 4.48. *A topological space is* countably compact *if every countable cover has a finite subcover.*

Lemma 4.49. *[42, 17G.2] A sequentially compact topological space is countably compact.*

Definition 4.50. *A topological space X is* Lindelhöf *if every cover of X has a countable subcover.*

Lemma 4.51. *A sequentially compact, Lindelhöf space is compact.*

Proof. As the space is sequentially compact it follows from Lemma 4.49 that the space is countably compact. As the space is Lindelhöf every open cover has a countable subcover. However, as the space is countably compact any countable cover has a finite subcover. Thus every open cover for the space has a finite subcover. □

Definition 4.52. *A digraph Γ is* finitely based *if there exist finitely many elements $\{v_1, v_2, \ldots, v_n\}$ such that for any vertex v there exists a path from some v_i to v.*

Notation 4.53. *Let Γ be a digraph, let $F \subseteq V\Gamma$ and let $v \in V\Gamma$. We denote by $C(v, V\Gamma \setminus F)$ the set of all vertices that can be reached from v by a path not passing through F.*

Lemma 4.54. *Let Γ be a digraph, let $F \subseteq V\Gamma$ and let $v \in V\Gamma \setminus F$. Then $\overrightarrow{\delta}(C(v, V\Gamma \setminus F))$ is contained in F.*

Proof. From the sake of brevity let $C := C(v, V\Gamma \setminus F)$. Let $w \in \overrightarrow{\delta}(C)$ and let $u \in C$ such that $u \to w$. As there exists a path $v = v_0 \to v_1 \to \ldots \to v_n = u$ such that $v_i \in C$ for all i we have the path $v = v_1 \to \ldots \to u \to w$. As $w \notin C$ this path must pass through F, the vertices $v_0, v_1, \ldots, v_n$ all lie in C, and hence not F, thus $w \in F$. □

Theorem 4.55. *Let Γ be an out-locally finite and finitely based digraph. Then the end topology of Γ is compact, T_0 and has a countable basis.*

Proof. As Γ is out-locally finite and finitely based it follows that Γ is at most countable. There are only countably many finite subsets of a countable set, hence, there are only countably many choices for the out-boundary of a cut. Many cuts may of course have the same out-boundary. Let $\mathcal{B} = \{C(v, V\Gamma \setminus F) \ : \ v \in V\Gamma,\ F \subseteq V\Gamma,\ |F| < \infty\}$. As the number of finite sets is countable and the number of vertices is countable, $\mathcal{B}$ is countable. Let C be any cut and let $F = \overrightarrow{\delta}(C)$. We claim that C is equal to the union of $C(v, V\Gamma \setminus F)$ for each $v \in C$. Certainly C is contained in this union as each $C(v, V\Gamma \setminus F)$ contains v. Assume with the aim of reaching a contradiction that there exists $w \in C(v, V\Gamma \setminus F)$ that is not in C for some $v \in C$. Then $w \neq v$ so there exists a path $v \to v_1 \to \ldots v_n \to w$ that contains no elements of F, however, $w \notin C$ so the path must go through $\overrightarrow{\delta}(C) = F$, a contradiction. As any cut can be expressed as the union of $C(v, V\Gamma \setminus F)$, of which there are countably many, the space has a countable basis.

Let $x, y \in V\Gamma \cup \Omega(\Gamma)$ be distinct elements. If $x \in V\Gamma$ then $C(x, V\Gamma \setminus \overrightarrow{\delta}(x)) = \{x\}$. Hence, we have an open set containing x but not y. Similarly. if y is a vertex then $C(y, V\Gamma \setminus \overrightarrow{\delta}(y)) = \{y\}$ is an open set containing y but not x. If $x, y \in \Omega(\Gamma)$ then without loss of generality we may assume $x \not\preceq y$. Let $\mathbf{x} = (x_0, x_1, \ldots)$ be a ray in x and let $\mathbf{y} = (y_0, y_1, \ldots)$ be a ray in y. As $x \not\preceq y$ there exists a finite set F such that all paths from $\mathbf{x}$ to $\mathbf{y}$ pass through F. Let $C = \bigcup C(x_i, V\Gamma \setminus F)$. The union of cuts is an open set.The ray $\mathbf{x}$ is contained in C so $x \in \Omega(C)$. As all paths from $\mathbf{x}$ to $\mathbf{y}$ pass through F no y_i are in C. It follows $y \notin \Omega(C)$. Thus the end topology of Γ is T_0.

As the end topology has a countable basis it is Lindelhöf. By Lemma 4.51 it suffices to show that the end topology of Γ is sequentially compact.

Let $S = (x_0, x_1, \ldots)$ be a sequence over $V\Gamma \cup \Omega(\Gamma)$. Inductively we construct vertices v_i and cuts C_i such that $v_i \to v_{i+1}$, $C_{i+1} \subseteq C_i$ and each $C_i \cup \Omega(C_i)$ contains infinitely many elements of S.

Let V_0 be a finite set of base points for Γ. Consider the sets $C(v, V\Gamma)$ for each $v \in V_0$. The out-boundary of each $C(v, V\Gamma)$ is empty so in particular they are cuts. By the pigeon hole principle there exists $v_0 \in V_0$ such that $C(v_0, V\Gamma) \cup \Omega(C(v_0, V\Gamma))$ contains infinitely many elements of S, we denote $C(v_0, V\Gamma)$ by C_0. We construct v_{i+1} and C_{i+1} by considering the sets $C(v, V\Gamma \setminus \overrightarrow{B}_i(V_0))$ for each $v \in \overrightarrow{\delta}(v_i) \setminus \overrightarrow{B}_i(V_0)$. As $\overrightarrow{\delta}(C(v, V\Gamma \setminus \overrightarrow{B}_i(V_0))) \subseteq$

$\overrightarrow{B}_i(V_0)$ and Γ is out-locally finite we know that each $C(v, V\Gamma \setminus \overrightarrow{B}_i(V_0))$ is a cut. As each $v \in \overrightarrow{\delta}(v_i) \setminus \overrightarrow{B}_i(V_0)$ is also an element of C_i it follows that each $C(v, V\Gamma \setminus \overrightarrow{B}_i(V_0)) \subseteq C_i$. The union of these cuts contains $C_i \setminus \overrightarrow{B}_i(V_0)$. By the pigeon hole principle one of these cuts together with its ends must contain infinitely many elements of S, say $C(v_{i+1}, V\Gamma \setminus \overrightarrow{B}_i(V_0))$ and denote this set by C_{i+1}.

As $v_i \to v_{i+1}$ and each $v_i \in \overrightarrow{B}_i(V_0) \setminus \overrightarrow{B}_{i-1}(V_0)$ the sequence $\mathbf{r} = (v_0, v_1, \ldots)$ is a ray. Let ω be the end containing $\mathbf{r}$. We now demonstrate that the end ω is the limit of some subsequence of S. To do this we need to show that any open neighbourhood containing ω must contain infinitely many elements of S. It is sufficient to consider cuts D such that $\omega \in \Omega(D)$ as any open neighbourhood of ω is the union of cuts in the basis of the end topology. Let D be a cut containing ω. By Lemma 4.39 all but finitely many elements of $\mathbf{r}$ lie in D so there exists $N_1 \in \mathbb{N}$ such that $v_n \in D$ for all $n \geq N_1$. Also as D is a cut it has finite out-boundary and there exists $N_2 \in \mathbb{N}$ such that $\overrightarrow{\delta}(D) \subseteq \overrightarrow{B}_{N_2-1}(V_0)$. We let $N = \max\{N_1, N_2\}$. We now show $C_N \subseteq D$. Certainly v_N is in D and if $u \in C_N = C(v_n, V\Gamma \setminus \overrightarrow{B}_{N-1}(V_0))$ then there exists a path from v_N to u that does not pass through $\overrightarrow{B}_{N-1}(V_0)$. If no vertex on this path lies in $\overrightarrow{B}_{N-1}(V_0)$ then certainly no vertex of this path lies in $\overrightarrow{\delta}(D)$ so each vertex on this path lies in D, in particular $u \in D$. This means $C_N \subseteq D$ and then $C_N \cup \Omega(C_N) \subseteq D \cup \Omega(D)$. As $C_N \cup \Omega(C_N)$ contains infinitely many elements of S so does D. $\square$

Lemma 4.56. *Let Γ be an out-locally finite, finitely based digraph Γ. Then the end topology of Γ is T_1 if and only if all ends of Γ are incomparable.*

Proof. Let $x, y \in V\Gamma \cup \Omega(\Gamma)$. Let V_0 be a finite set of vertices such that every vertex can be reached by a path from V_0. If x is a vertex then the set $\{x\}$ is a cut and contains no ends. Hence, if x, y are both vertices then the open sets $\{x\}$ and $\{y\}$ have the property that $x \in \{x\}$, $y \notin \{x\}$ and $y \in \{y\}$, $x \notin \{y\}$. If x is a vertex and y is an end then there exists $N \in \mathbb{N}$ such that $x \in \overrightarrow{B}_{V_0}(N)$ and there is a ray $\mathbf{r} = (r_0, r_1, \ldots) \in y$ with no vertices in $\overrightarrow{B}_{V_0}(N)$. Then $C(r_0, V\Gamma \setminus \overrightarrow{B}_{V_0}(N))$ is a cut containing y and is disjoint from $\overrightarrow{B}_{V_0}(N)$. These facts hold in any out-locally finite finitely based digraph.

However, as shown in Lemma 4.40, if x, y are ends and $x \succcurlyeq y$ then any cut containing x must also contain y and hence any open set containing x must also contain y. If $x \not\succcurlyeq y$ then for two rays $\mathbf{r}_1 \in x$ and $\mathbf{r}_2 \in y$ there exists a finite set F such that all paths from $\mathbf{r}_1$ to $\mathbf{r_2}$ pass through F. There exists $N \in \mathbb{N}$ such that $\overrightarrow{B}_N(V_0)$ contains F then by considering $C(v, V\Gamma \setminus \overrightarrow{B}_N(V_0))$ for each $v \in \overrightarrow{\delta}(\overrightarrow{B}_N(V_0))$ we get a cut containing $\mathbf{r}_1$ but not $\mathbf{r}_2$. $\square$

A space X is said to be *first-countable* if for each point $x \in X$ there exists a sequence $U_1, U_2, \ldots$ of open neighbourhoods of x such that for any open neighbourhood V of x there exists an integer i with U_i contained in V. It is easy to see that if a space has a countable basis then it is first countable.

Theorem 4.57. *[29] Let X be a first countable compact T_1 space. If $|X|$ is uncountable then $|X|$ is at least $2^{\aleph_0}$.*

Corollary 4.58. *A finitely generated semigroup whose ends are all incomparable has finitely many, $\aleph_0$ or $2^{\aleph_0}$ ends.*

Proof. By Lemma 4.56 if all ends in a semigroup are incomparable then the end topology of S is T_1. Then by Theorem 4.55 and Theorem 4.57 the semigroup has finitely many, $\aleph_0$ or $2^{\aleph_0}$ ends. □

Chapter 5

Classification

The classification of groups in terms of their ends started in 1943 with Hopf's theorem that classifies groups with two ends.

Theorem 5.1. *[22, Satz V] Let G be a finitely generated group. Then G has two ends if and only if G has an infinite cyclic group as a subgroup of finite index.*

The next leap forward in the classification of groups in terms of their ends was by Stallings. He used a mixture of topology and actions on trees to prove the following result:

Theorem 5.2. *[40, 1.B.6] Let G be a finitely generated group. Then G has more than one end if and only if G can be written as a non-trivial free product with amalgamation $B *_C D$ where C is finite, or G can be written as* *a non-trivial HNN-extension $B *_C x$ where C is finite.*

In this section we give a classification of the semigroups with one end. This does not complete the classification of all groups with one end, as part of the classification of semigroups with one end is given in terms of groups with one end.

Theorem 5.3. *Let S be a finitely generated semigroup with one end. Then S is the disjoint union of a semigroup T where tS is infinite for all $t \in T$ and a possibly empty ideal I such that $|iS| < \infty$ for all $i \in S$ and one of the following holds:*

(A) T has a right group $R = G \times E$ as a subsemigroup of finite Rees index where E is a finite right-zero semigroup and G is a finitely generated group with one end; or

(B) T has a presentation of the form

$$\begin{aligned} \langle a, u_1, u_2, \ldots, u_n | \quad u_i \cdot u_j &= a^{\alpha(i,j)} u_{\beta(i,j)}, \\ u_i \cdot a &= a^{f(i)} u_{g(i)}, \\ a^{\pi(i,j)} u_i &= a^{\pi(j,i)} u_j \rangle, \end{aligned}$$

where

$$\alpha : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{N}_0,$$

$$\beta : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$

$$f : \{1, 2, \ldots, n\} \to \mathbb{N}_0,$$

$$g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$$

$$\pi : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{N}.$$

The first part of the proof is that T and S have the same number of ends. The proof is then split into two main parts. The first part concerns semigroups with one end which contain an infinite $\mathcal{R}$-class. These semigroups are shown to have an $\mathcal{R}$-simple subsemigroup of finite Rees index. These $\mathcal{R}$-simple semigroups are then shown to be regular. By combining this regularity with the fact that they have one end, we are able to deduce that they are in fact right groups.

The second part concerns semigroups with one end that have no infinite $\mathcal{R}$-class. These semigroups are shown to satisfy a certain density property. Combinatorial and lexicographical arguments are then used to obtain the presentation.

The following are some technical lemmas which apply to all semigroups with one end.

Lemma 5.4. *Let S be a finitely generated semigroup with one end. For all $x \in S$ one of the sets $S \setminus xS$ or xS is finite.*

Proof. Let $x \in S$ and let A be a finite generating set. If xS is finite then as S is infinite we have that $S \setminus xS$ is infinite. Assume that both xS and $S \setminus xS$ are infinite. As xS is infinite it contains a ray $\mathbf{x}$ by Lemma 2.42. If $S \setminus xS$ is infinite then by Lemma 2.42 there is a ray $\mathbf{y}$ with initial vertex $a \in A$ and with infinitely many disjoint paths to some infinite subset of $S \setminus xS$. If there is a path from $y \in S$ to $s \in S \setminus xS$ then $yS \cap (S \setminus xS) \neq \emptyset$. It follows that $y \notin xS$. As each vertex of $\mathbf{y}$ has a path to an element of $S \setminus xS$ this means each vertex of $\mathbf{y}$ lies in $S \setminus xS$. There can be no paths from $\mathbf{x}$ to $\mathbf{y}$, as $\mathbf{x} \subseteq xS$ and $\mathbf{y} \subseteq S \setminus xS$, in contradiction to the assumption that S has one end. □

Lemma 5.5. *Let S be a finitely generated semigroup. If S has one end then S is the disjoint union of a subsemigroup T such that tS is infinite for all $t \in T$ and a possibly empty ideal I such that iS is finite for all $i \in I$.*

Proof. Assume S has one right end. Let $T = \{s \in S : |sS| = \infty\}$ and $I = \{s \in S : |sS| < \infty\}$. We will show that T is a finitely generated subsemigroup of S and I is an ideal.

Let $s, t \in S$ and assume stS is infinite then clearly we have that $stS \subseteq sS$ so sS is infinite. We also have that tS is infinite otherwise it would not be possible for $s(tS)$ to be infinite. This shows that if $st \in T$ then both $s, t \in T$, hence, I is an ideal.

Let $s, t \in T$. As tS is infinite by Lemma 5.4 it follows $S \setminus tS$ must be finite. Now $S = (S \setminus tS) \cup tS$ and so $sS = s(S \setminus tS) \cup s(tS)$. As sS is infinite and $S \setminus tS$ is finite it follows $s(tS) = (st)S$ is infinite. Hence, $st \in T$ and so T is a subsemigroup of S. $\square$

Example 5.6. Note that it is not true for a general semigroup that the set of elements $s \in S$ such that $|sS| = \infty$ forms a subsemigroup of S. For example, let $S = \langle a| \ \rangle \cup_0 \langle b| \ \rangle$ be the zero-union of two copies of the free monogenic semigroup. Then aS and bS both are infinite, but abS is finite.

Lemma 5.7. *Let S be a semigroup which can be expressed as the disjoint union of a subsemigroup T and a possibly empty ideal I such that iS is finite for all $i \in I$. If S is finitely generated then there exist a finite set B such that B generates T and a finite set $D \subseteq I$ such that $DS \subseteq D$ and every element of I can be expressed in the form $T^1 D$.*

Proof. Let A be a finite generating set for S. As S is the disjoint union of a subsemigroup and an ideal $A = B \cup C$ where $B \subseteq T$ and $C \subseteq I$. Consider a product $s = a_1 a_2 \cdots a_n$ where $a_i \in A$. If $a_i \in I$ for some i then $s \in I$. This means that all products of generators that lie in T consist of elements from B and hence B generates T.

As the set C is finite and iS is finite for all $i \in I$ it follows that the set $D = \{cs : c \in C, s \in S^1\}$ is finite. Clearly, we have $D \subset I$ as I is an ideal. We also have that $DS \subseteq D$ from the definition of D. Again we consider the product $s = a_1 a_2 \cdots a_n$. If $s \in I$ we must have some $a_i \in C$ as T is a subsemigroup. By assuming a_i is the first occurrence of C in the product $s = a_1 a_2 \cdots a_i \cdots a_n$ and noting that $d = a_i a_{i+1} \cdots a_n \in D$ we get $s = a_1 a_2 \cdots a_{i-1} d$. □

Lemma 5.8. *Let S be a finitely generated semigroup which is the disjoint union of a subsemigroup T and a possibly empty ideal I such that iS is finite for all $i \in I$. Then $\Omega_r(S)$ is isomorphic as a poset to $\Omega_r(T)$.*

Proof. We claim that there exists $N \in \mathbb{N}_0$ such that $|iS| < N$ for all $i \in I$. As S is finitely generated by Lemma 5.7 there exists a finite set B such that B generates T and a finite set $D \subseteq I$ such that $DS \subseteq D$ and every element of I can be expressed in the form $T^1 D$. Let $i \in I$ and let $|D| = N$. Any element of iS can be expressed in the form xds for some $s \in S$, $x \in T^1$ and $d \in D$. However, as $ds \in D$ we have $|iS| \leq |xD| \leq |D| = N$.

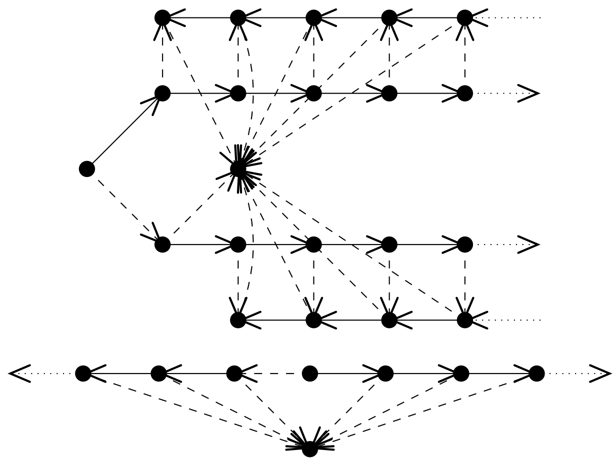
Let $\mathbf{x} = (x_0, x_1, \ldots)$ be a ray in $\Gamma_r(S, A)$. If $\mathbf{x}$ is an out-ray then $x_i \notin I$ as $x_i S$ is infinite. If $\mathbf{x}$ is an in-ray then $x_n \notin I$ for $n > N$ as $|x_n S| > N$. Thus any ray in $\Gamma_r(S, A)$ is equivalent to a ray with vertices in T. We now show that all but finitely many of the the labels of edges in $\mathbf{x}$ must also lie in T. It will follow that all rays in $\Gamma_r(S, A)$ are equivalent to a ray with vertices in T and edges labelled by elements of B. Let $\mathbf{x} = x_0 \rightarrow^{a_1} x_1 \rightarrow^{a_2} \ldots$. If $a_n \in I$ then $x_n \in I$. Now $x_m \in x_n S$ for all $m > n$, however, $|x_n S| \leq N$ in contradiction to the fact $\mathbf{x}$ is a ray. If $\mathbf{x} = x_0 \leftarrow^{a_0} x_1 \leftarrow^{a_1} \ldots$ is an in-ray then $x_n \notin I$ for all $n > N$. It follows $a_n \notin I$ for all $n > N$ otherwise $x_n \in I$.

Let $\mathbf{x}$ and $\mathbf{y}$ be rays with vertices in T and edges labelled by elements of B. If there exists a path in $\Gamma_r(S, A)$ from $\mathbf{x}$ to $\mathbf{y}$ then as I is an ideal the vertices of this path must all lie in T. Similarly the labels on the edges of this path must lie in B otherwise a vertex of this path would lie in I. This means $\Omega_r(S)$ is isomorphic as a poset to $\Omega_r(T)$. □

In a semigroup S the set of elements I such that $|iS| < \infty$ for all $i \in I$ always forms an ideal. One might wonder what can happen to the ends if you form the Rees quotient with respect to this ideal. This example shows that the number of ends can change if $S \setminus I$ is not a subsemigroup. Note that it only affects in-rays.

Example 5.9. Let S be the monoid given by the presentation $Mon\langle a, b, 0 | a0 = 0a = b0 = 0b = 0^2 = 0, aba = b, b^2 = 0\rangle$. The rules $aba \to b$ and $b^2 \to 0$ form a complete rewriting system and we get unique normal forms for elements $0, b^{\epsilon_1} a^i b^{\epsilon_2}$ where $i \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 \in \{0, 1\}$. The Cayley graph of S has four ends corresponding to the rays

Figure 5.1: A portion of the Cayley graph of $Mon\langle a, b | aba = b, b^2 = 0\rangle$ with edges labelled by b as dashed lines and edges labelled by a as full lines and the Rees quotient $Mon\langle a, b | ab = 0, b^2 = 0\rangle$.



$$a \to a^2 \to a^3 \to \dots,$$

$$ba \to ba^2 \to ba^3 \to \dots,$$

$$ab \leftarrow a^2b \leftarrow a^3b \leftarrow \dots,$$

$$bab \leftarrow ba^2b \leftarrow ba^3b \leftarrow \dots.$$

A portion of the Cayley graph is shown in Figure 5.9. The Rees quotient by the ideal $I = \{s \in S : |sS| < \infty\} = \{0, a^ib, ba^ib\}$ is isomorphic to $Mon\langle a, b | ab = 0, b^2 = 0\rangle$. This has normal forms a^i, ba^i for $i \in \mathbb{N}_0$. However, the quotient has two ends corresponding to $a \to a^2 \to \dots$ and $b \to ba \to ba^2 \to \dots$.

Lemma 5.10. *Let S be a finitely generated semigroup. If S is the disjoint* *union of a subsemigroup T such that tS is infinite for all $t \in T$ and a possibly empty ideal I such that iS is finite for all $i \in I$ then tT is infinite for all $t \in T$.*

Proof. Let $t \in T$. Assume with the aim of reaching a contradiction that tT is finite. By Lemma 5.7 there exists a finite set B that generates T and a finite set D such that $DS \subseteq D$ and every element of I can be expressed in the form T^1D. It follows that $tI = tT^1D$. If tT is finite then tT^1 is finite. Then as D is finite tT^1D is finite. However, $tS = tT \cup tI$ so tS is finite, a contradiction. □

5.1 Semigroups with an Infinite $\mathcal{R}$-class

The classification of semigroups with one end is split into two separate cases: those with an infinite $\mathcal{R}$-class and those with no infinite $\mathcal{R}$-class. This section deals with those semigroups with one end and an infinite $\mathcal{R}$-class.

Lemma 5.11. *Let S be a finitely generated semigroup with an infinite $\mathcal{R}$-class. If S has one right end then S has precisely one infinite $\mathcal{R}$-class.*

Proof. Let A be a finite generating set for S and let $\Gamma = \Gamma_r(S, A)$. Let R_s and R_t be infinite $\mathcal{R}$-classes and let $s \in R_s$ and $t \in R_t$. By Corollary 2.43 there exists an out-ray $\mathbf{x} = (x_0, x_1, \ldots)$ with initial vertex s contained in R_s

and an out-ray $\mathbf{y} = (y_0, y_1, \ldots)$ with initial vertex t contained in R_t in Γ. If there is no path from $\mathbf{x}$ to $\mathbf{y}$ then S has at least two ends. Assume there is a path from x_i to y_j for some $i, j \in \mathbb{N}_0$. There must also be a path from y_k to x_l for some $k \geq j$ otherwise $\mathbf{x}$ would not be equivalent to $\mathbf{y}$ ensuring S has at least two ends. Hence, there exists a path in Γ from x_i to y_j and a path from y_j to y_k to x_l to x_i. The final path exists because $x_l \mathcal{R} x_i$. This means that $x_i \mathcal{R} y_j$ and hence $R_s = R_t$. □

Lemma 5.12. *Let T be a finitely generated semigroup such that the set sT is infinite for all $s \in T$. If T has one end and an infinite $\mathcal{R}$-class then T has finitely many $\mathcal{R}$-classes.*

Proof. Let A be a finite generating set for T and let $\Gamma = \Gamma_r(T, A)$. We may assume T is a monoid because by Theorem 3.10 if T has one end then T^1 will have one end and if T^1 has finitely many $\mathcal{R}$-classes then T has finitely many $\mathcal{R}$-classes. By Lemma 5.11 it follows that T has precisely one infinite $\mathcal{R}$-class which we shall denote by R.

By Corollary 2.43 there exists a ray $\mathbf{r}$ in R. Let $\mathbf{x}$ be any ray in $\Gamma_r(T, A)$. We claim all but finitely many elements of $\mathbf{x}$ must lie in R. We demonstrate this for an out-ray, the proof for an in-ray is analogous. Assume $\mathbf{x} = x_0 \to x_1 \to \ldots$. Then there exists a path from $r \in \mathbf{r}$ to some x_n. Then for all $m \geq n$ there exists a path from x_m to $\mathbf{r}$ and hence to r. It follows $x_m \in R$

for all $m \geq n$. Assume $T \setminus R$ is infinite then by Lemma 2.42 there exists an out-ray $\mathbf{x}$ with initial vertex 1 and with infinitely many disjoint paths to an infinite subset of $T \setminus R$ and vice versa. However, all but finitely many elements of $\mathbf{x}$ must lie in R and hence the infinite subset of $S \setminus R$ must be contained in R, a contradiction. $\square$

Recall that a semigroup is called $\mathcal{R}$-simple if it contains precisely one $\mathcal{R}$-class.

Lemma 5.13. *If T is a semigroup with finitely many $\mathcal{R}$-classes, precisely one of which is infinite, then T has an $\mathcal{R}$-simple semigroup as a subsemigroup of finite Rees index.*

Proof. Without loss of generality one may assume that T is a monoid as adjoining an identity will preserve the properties of T as stated in the lemma. Let R be the infinite $\mathcal{R}$-class of T and let $U = T \setminus R$. As T has finitely many $\mathcal{R}$-classes and all $\mathcal{R}$-classes apart from R are finite it follows that $U = S \setminus R$ is finite.

We now prove that R is a subsemigroup of T. Suppose for the sake of contradiction there exist $s, t \in R$ such that $st \notin R$. Therefore for all $x \in T$ we have $stx \neq s$ otherwise $st\mathcal{R}s$. This means that $stS \cap R = \emptyset$. Now as T is a monoid $R \subseteq tS$ so $sR \cap R = \emptyset$ and $|sR| < \infty$. However, it then follows from the fact that sU is finite and that sR is finite that $R \subseteq sT = sR \cup sU$

is finite. This is a contradiction, and hence R is a subsemigroup".

Now it remains to prove that R is $\mathcal{R}^R$-simple. The first property we prove is that for all $r \in R$ and all $s \in T$ if rsT is infinite then $rs \in R$. As $T \setminus R$ is finite there exists $t \in T$ such that $rst \in R$. In fact we can choose t such that $rst = r$. This means $rs\mathcal{R}r$ as required.

Fix $r \in R$ and enumerate all other elements of R as $r_1, r_2, \ldots$. We know that there exists $x_i \in T$ such that $rx_i = r_i$. If for all $r \in R$ the corresponding x_i are elements of R then R is $\mathcal{R}^R$-simple. Assume otherwise. We know the x_i are distinct otherwise $r_i = rx_i = rx_j = r_j$. It follows that as $T \setminus R$ is finite, only finitely many $x_i \in T \setminus R$, say $x_1, x_2, \ldots, x_n$. Let $y_i \in T$ such that $r_{n+1}y_i = r_i$ for $1 \leq i \leq n$. Now $rx_{n+1}y_i = r_{n+1}y_i = r_i$. As $r_i \in R$ we have r_iT is infinite. This means $rx_{n+1}y_iT$ is infinite and hence $x_{n+1}y_iT$ is infinite. However, $x_{n+1} \in R$ so by the previous property we know that $x_{n+1}y_i \in R$ and hence R is $\mathcal{R}$-simple. □

It is not true in general that an $\mathcal{R}$-class which forms a subsemigroup is $\mathcal{R}$-simple.

Example 5.14. The *bicyclic monoid* $B = Mon\langle b, c | bc = 1\rangle$ has unique normal forms $c^i b^j$. The $\mathcal{R}$-class of 1, here denoted by R_1, is all elements of the form b^i for $i \in \mathbb{N}_0$. This forms a semigroup which is isomorphic to the free monoid on one generator. However, the $\mathcal{R}$-classes of R_1 as a semigroup are trivial.

The following is a result of Byleen, see [5, Proposition 2.5].

Theorem 5.15. *If R is a finitely generated $\mathcal{D}$-simple semigroup then R is regular.*

Proof. To prove R is regular it is sufficient, by Lemma 2.15, to show that R contains a regular element. We will do so by finding an idempotent.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite generating set for R. Assume R has a non-trivial $\mathcal{R}$-class, then by Green's Lemma 2.2 all $\mathcal{R}$-classes are non-trivial. Let $a \in A$ and let $b \in R$ such that $b \neq a$ and $b\mathcal{R}a$. Then there exist $s, t \in R$ such that $as = b$ and $bt = a$. It follows that $ast = a$. For each $a_i \in A$ let $x_i \in R$ such that $a_i x_i = a_i$. If $x_i \in Ra_i$ then there exists $s \in R$ such that $sa_i = x_i$ then $sa_i x_i = sa_i$ which implies $x_i^2 = x_i$. Let $1 \leq i, j \leq n$. If $x_i \in Ra_j$ then $a_i \in Ra_j$. If $s \in Rt$ and $t \in Ru$ then $s \in Ru$. Together this implies that we have an $m \leq n$ such that $a_{i_1} \in Ra_{i_2}$, $a_{i_1}, a_{i_2} \in Ra_{i_3}$, $\ldots a_{i_1}, a_{i_2}, \ldots a_{i_m} \in Ra_{i_k}$ for some $k \leq m$. This means $x_{i_m} \in Ra_{i_k} \subseteq Ra_{i_m}$ and hence R has an idempotent.

If R has trivial $\mathcal{R}$-classes then R must be $\mathcal{L}$-simple. Then by symmetry there exists $x_i \in R$ such that $x_i a_i = a_i$ for each $a_i \in A$. By the same argument as above we get $x_i \in a_i R$ for some i. It follows that x_i is an idempotent. □

The next theorem demonstrates that if a semigroup is $\mathcal{R}$-simple and finitely generated then it has quite restricted structure

Theorem 5.16. *If R is a finitely generated $\mathcal{R}$-simple semigroup it is isomorphic to a right group $G \times E$ where G is a finitely generated group and E is a finite right-zero semigroup.*

Proof. It is well known that each completely simple semigroups is isomorphic to some Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$, for a proof see [23, Theorem 3.3.1]. The first step is to show that R is completely simple. An $\mathcal{R}$-simple semigroup is simple as it has no proper right ideals and hence no proper ideals.

By Theorem 2.20 we know a simple semigroup is completely simple if and only if it is completely regular. As R is $\mathcal{R}$-simple it is $\mathcal{D}$-simple, so by Theorem 5.15 we know R is regular. By Lemma 2.19 it is sufficient to show every $\mathcal{H}$-class contains an idempotent. As R only has one $\mathcal{R}$-class this means every $\mathcal{L}$-class is an $\mathcal{H}$-class. By Theorem 2.15 every $\mathcal{L}$-class in a regular $\mathcal{D}$-class contains an idempotent. Therefore, R is completely regular.

This means R is completely simple and is isomorphic to some Rees matrix semigroup $\mathcal{M}[H; I, \Lambda; P]$ where I, Λ are index sets and H is a group. It is easy to see that R is finitely generated if and only if $|I|, |\Lambda| < \infty$ and H is finitely generated. If $|I| \geq 2$ then R has at least two $\mathcal{R}$-classes so $I = \{i\}$. By Lemma 2.31 R is left-cancellative. If R is left-cancellative and is $\mathcal{R}$-simple then by Lemma 2.30 R is a right group $G \times E$ where G is a finitely generated group and E is a finite right zero semigroup. $\square$

One might ask if it is possible to generalise Theorem 5.16 to $\mathcal{D}$-simple semigroups.

Example 5.17. Let $B = Mon\langle b, c | bc = 1\rangle$ be the bicyclic monoid. B is $\mathcal{D}$-simple as $c^i b^j \mathcal{R} c^i$ and $c^i \mathcal{L} 1$. However, the $\mathcal{H}$ classes are trivial but not all elements are idempotent, for instance b. Hence by Lemma 2.19 and Theorem 2.20, B is not completely simple.

Theorem 5.18. *A finitely generated semigroup S with an infinite $\mathcal{R}$-class has one end if and only if S is a finitely generated semigroup which is the disjoint union of a semigroup T satisfying $tS = \infty$ for all $t \in T$ and an ideal I such that the set iS is finite for all $i \in I$ and where T has a subsemigroup of finite Rees index isomorphic to a right group $G \times E$ where G is a finitely generated group with one end.*

Proof. ($\Rightarrow$) If S has an infinite $\mathcal{R}$-class and one end then by Lemma 5.11 it has precisely one infinite $\mathcal{R}$-class. By Lemma 5.5 S is a disjoint union of a semigroup T satisfying $|tS| = \infty$ for all $t \in T$ and a possibly empty ideal I satisfying the condition $|iS| < \infty$ for all $i \in I$. By Lemma 5.7 the semigroup T is finitely generated and by Lemma 5.8 T also has one end. The subsemigroup T must contain an infinite $\mathcal{R}^T$-class as any $\mathcal{R}^S$-class containing some $i \in I$ must be finite as iS is finite and as I is an ideal $xi \in I$ for all

$x \in S$ and all $i \in I$. By Lemma 5.10 the subsemigroup T must satisfy tT is infinite for all $t \in T$. By Lemma 5.12 T must have finitely many $\mathcal{R}^T$-classes. Then it follows from Lemma 5.13 that the infinite $\mathcal{R}^T$-class R is an $\mathcal{R}$-simple subsemigroup of finite Rees index in T. As Rees index preserves finite generation R is a finitely generated $\mathcal{R}^R$-simple semigroup. By Theorem 5.16 R is isomorphic to a right group $G \times E$ where G is a finitely generated group and E is a finite right-zero semigroup. This is a particular kind of Rees matrix semigroup where $|I| = 1$ and P is the matrix only containing identity elements. By Theorem 3.2 a Rees matrix semigroup of this form will have one end if and only if G has one end.

($\Leftarrow$) A right group $R = G \times E$ where G is a finitely generated semigroup with one end is a Rees matrix semigroup $\mathcal{M}[G; \{i\}, \{\lambda_1, \lambda_2, \ldots, \lambda_n\}; P]$ where G is a finitely generated group with one end will have one end. By Theorem 3.2 R has one end. If R is subsemigroup of finite Rees index in a semigroup T then by Theorem 1.11 T and R have the same end structure. Hence, T has one end. If S is a finitely generated semigroup which is the union of T and an ideal I such that iS is infinite for all $i \in I$ then by Theorem 5.8 the end poset of S is isomorphic to the end poset of T. Therefore, S has one end. $\square$

5.2 Semigroups with only Finite $\mathcal{R}$-classes

This section deals with semigroups with one end and no infinite $\mathcal{R}$-class.

Lemma 5.19. *Let Γ be an out-locally finite digraph. If there exists a ray $\mathbf{r} = (r_0, r_1, \ldots)$ and $N, M \in \mathbb{N}_0$ such that for all but finitely many $v \in V\Gamma$ there exists an $i \in \mathbb{N}_0$ with $d_\Gamma(r_i, v), d_\Gamma(v, r_{i+M}) \leq N$ then S has one end.*

Proof. Let $\mathbf{x} = (x_0, x_1, \ldots)$ be a ray. Let U be the finite set of vertices such that either $d_\Gamma(r_i, u) \geq N$ or $d_\Gamma(u, r_{i+M}) \geq N$ for all $i \in \mathbb{N}_0$. It follows there exists $P \in \mathbb{N}_0$ such that $x_p \notin U$ for all $p \geq P$. Without loss of generality we will assume that no vertex of $\mathbf{x}$ lies in U. Let σ_i be a path of length less than N from r_{s_i} to x_i and let π_i be a path of length less than N from x_i to r_{s_i+M} for each $i \in \mathbb{N}_0$. Let $I \subseteq \mathbb{N}_0$ be an infinite set. Then the set $S = \{s_i : i \in I\}$ must be infinite because Γ is locally finite and hence for any finite set F the set $\overrightarrow{B}_N(F)$ must be finite and $\overrightarrow{B}_N(S)$ contains each x_i. If S is infinite then the set $\{s_i + M : i \in I\}$ is infinite.

Assume infinitely many π_i pass through some finite set F. This means infinitely many π_i pass through some vertex f. Then $\overrightarrow{B}_N(f)$ contains $\{r_{s_i+M} : i \in I\}$ for some infinite set I. However, $\overrightarrow{B}_N(f)$ is finite, contradicting that infinitely many π_i intersect, thus $\mathbf{x} \succcurlyeq \mathbf{r}$.

Let I be an infinite subset of $\mathbb{N}$ and assume σ_i passes through some finite set F for all $i \in I$. It follows that there exists an infinite $J \subseteq I$ such that σ_i

passes through some vertex f for all $i \in J$. But then we have $x_i \in \overrightarrow{B}_N(f)$ for all $i \in J$, contradicting the fact $\overrightarrow{B}_N(f)$ is finite. Thus $\mathbf{r} \succcurlyeq \mathbf{x}$ and all rays are equivalent to $\mathbf{r}$. □

Theorem 5.20. *Let T be a finitely generated semigroup with no infinite $\mathcal{R}$-class such that $|sT| = \infty$ for all $s \in T$. Let A be a finite generating set. If T has one end then for any ray $\mathbf{r} = r_0 \to r_1 \to \ldots$ there exist $N, M \in \mathbb{N}$ such that for all but finitely many $x \in T$ there exists $i \in \mathbb{N}$ with $d_A(r_i, x), d_A(x, r_{i+M}) \leq N$.*

Proof. Fix a ray $\mathbf{r} = r_0 \to^{a_1} r_1 \to^{a_2} \ldots$. We define $\mathbf{s_i} = a_i \to a_i a_{i+1} \to \ldots$, these must be rays as if $a_i \cdots a_j = a_i \cdots a_k$ then $r_{i-1} a_i \cdots a_j = r_{i-1} a_i \cdots a_k$ and hence $r_j = r_k$.

We show that all but finitely many $x \in T$ lie in a unique set $r_i T \setminus r_{i+1} T$. As sT is infinite for all $s \in T$ we know that $r_0 T$ is infinite. By Lemma 5.4 it follows that $T \setminus r_0 T$ is finite. Thus all but finitely many elements of T can be expressed in the form $r_0 s$ for some $s \in T$. If $x \in r_n T$ for all $n \in \mathbb{N}$ then, as xT is infinite, there exists $s \in T$ such that $xs = r_i$ for some i. However, as $x \in r_n T$ for all $n \geq i$, this gives an infinite $\mathcal{R}$-class. This means for all but finitely many $x \in T$ there exists $i \in \mathbb{N}$ such that $x \in r_i T$ but $x \notin r_{i+1} T$. Every element of $r_i T \setminus r_{i+1} T$ can be expressed by an element of $r_i (T \setminus a_{i+1} T)$ and hence any $x \in r_i T \setminus r_{i+1} T$ can be written as $r_i u$ for some $u \in T \setminus a_{i+1} T$. Now

$T \setminus aT$ is finite for all $a \in A$ so by setting $N_3 = \min\{|u|_A : u \in T \setminus aT, a \in A\}$ we get $d_A(r_i, x) \leq N_3$.

We now show that there exists $x \in T$ such that for any $a \in A$ and for any element of T that can be expressed in the form $r_i u$ for some $u \in T \setminus aT$ there exists a path of globally bounded length from $r_i u$ to $r_i x$. As sT is infinite for each $s \in T$ there is a ray with initial vertex s for every $s \in T$ by Lemma 2.42. Because T only has one end there is a path from each $s \in T$ to the ray $\mathbf{r}$. In particular, for each $a \in A$ and for each $u \in T \setminus aT$ there exists a $t \in T$ such that $ut \in \mathbf{r}$. Let $M_1 = \min\{i \in \mathbb{N} : r_i \in uT, a \in A, u \in T \setminus aT\}$ and let $N_1 = \min\{|t|_A : ut = r_{M_1}, a \in A, u \in T \setminus aT\}$. This means there is a path from $r_i u$ to $r_i r_{M_1}$ of length N_1.

We now claim there is a path of bounded length from each $r_i r_{M_1}$ to an element of $\mathbf{r}$. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a map defined as $\sigma(i) = \min\{j \in \mathbb{N} : a_i \cdots a_{i+j} \in r_{M_1} T\}$. The set $T \setminus r_{M_1} T$ is finite and all $a_i \cdots a_{i+j}$ are distinct. It follows that some $a_i \cdots a_{i+j} \notin T \setminus r_{M_1} T$ so σ is well-defined. Let $M_2 = |T \setminus r_{M_1} T| + 1$ then one of $a_i, a_i a_{i+1}, \ldots a_i a_{i+1} \cdots a_{i+M_2}$ must lie in $r_{M_1} T$ and hence $\sigma(i) \leq M_2$ for all $i \in \mathbb{N}$. Let $P_i = \min\{|t|_A : r_{M_1} t = a_i \cdots a_{i+M_2}\}$ and let $N_2 = \max\{P_i : i \in \mathbb{N}\}$, this exists as A^{M_2} is finite. This means if $x \in r_i T \setminus r_{i+1} T$ then there exists an element s of length less than N_1 such that $xs = r_i r_{M_1}$ and an element t of length less than N_2 such that $r_{M_1} t = a_{i+1} \cdots a_{i+1+M_2}$. Hence, $xst = r_i x_{M_1} t = r_i a_i \cdots a_{i+1+M_2} = r_{i+1+M_2}$.

By taking $M = M_2 + 1$ and $N = \max\{N_1 + N_2, N_3\}$ we have $N, M \in \mathbb{N}$ such that $d_A(r_i, x), d_A(x, r_{i+M}) \leq N$. $\square$

Definition 5.21. *Let S be a semigroup and A be a generating set. The* growth of S with respect to generating set A *is the function $f : \mathbb{N} \to \mathbb{N}$ where $f(n) = |\{s \in S : |s|_A \leq n\}|$.*

The growth type of a semigroup (for example linear, polynomial of degree d, exponential) can be shown to be invariant under different finite generating sets. The details of this are almost exactly the same as in the group case, see [31].

Corollary 5.22. *Let T be a finitely generated semigroup with no infinite $\mathcal{R}$-class and such that $|sT| = \infty$ for all $s \in T$. If T has one end then T has linear growth.*

Proof. Let A be a finite generating set for T. Theorem 5.20 states that for any ray $\mathbf{r}$ in $\Gamma_r(T, A)$ we have all but finitely many elements of T lie in $\overrightarrow{B}_N(\mathbf{r})$. This means the growth of T with respect to A is linear. $\square$

Theorem 5.23. *[21, Theorem 4.2] If S is a finitely generated semigroup with linear growth then there exist finitely many elements $a_i, b_i, c_i \in S^1$ such that every element of S is represented by a word of the form $a_i b_i^n c_i$ for some i and some $n \in \mathbb{N}_0$.*

Theorem 5.24. *Let T be a finitely generated semigroup with the property that $|xT|$ is infinite for all $x \in T$ and with no infinite $\mathcal{R}$-classes and let A be a finite generating set. If T has one end then*

1. *there exists $s \in T$ of infinite order,*
2. *there exist $N, M \in \mathbb{N}$ such that for all but finitely many $x \in T$ there exists $i \in \mathbb{N}$ with $d_A(s^i, x), d_A(x, s^{i+M}) \leq N$.*

Proof. Corollary 5.22 states that T has linear growth. As T has one right end it must be infinite and, hence, by Theorem 5.23 it follows T must have an element of infinite order, say s.

Let $s = a_1 a_2 \cdots a_n$ and let $\mathbf{w} = 1 \to a_1 \to a_1 a_2 \to \ldots \to a_1 a_2 \cdots a_n = s \to s a_1 \to \ldots$ be an infinite walk.

If $s^i a_1 \cdots a_j = s^k a_1 \cdots a_j$ then $s^i a_1 \cdots a_j a_{j+1} \cdots a_n = s^k a_1 \cdots a_j a_{j+1} \cdots a_n$ and hence $s^{i+1} = s^{k+1}$. However, s is of infinite order so in any set of $n+1$ or more elements of $\mathbf{w}$ only at most n can be equal to any given element $s \in T$. Also note that for each $1 \leq i \leq n$ there exists

$$\max\{k : s^m a_1 \cdots a_i = s^{m+k} a_1 \cdots a_j,\ m \in \mathbb{N}_0,\ 1 \leq j \leq n\}$$

as if for $m_1 \leq m_2$ we have $s^{m_1} a_1 \cdots a_i = s^{m_1+k_1} a_1 \cdots a_j$ and $s^{m_2} a_1 \cdots a_i =$

$s^{m_2+k_2}a_1\cdots a_j$ then

$$\begin{aligned} s^{m_2+k_2}a_1\cdots a_j &= s^{m_2}a_1\cdots a_i \\ &= s^{m_2-m_1}s^{m_1}a_1\cdots a_i \\ &= s^{m_2-m_1}s^{m_1+k_1}a_1\cdots a_j \\ &= s^{m_2+k_1}a_1\cdots a_j \end{aligned}$$

and hence $k_1 = k_2$. This means that there exists $N_1 \in \mathbb{N}$ such that if $s^p a_1\cdots a_i = s^q a_1\cdots a_j$ for some i, j then $|q-p| \leq N_1$. Hence by the Lemma 2.40 there exists a ray $\mathbf{r}$ with paths from each element of $\mathbf{r}$ to an element of $\langle s\rangle$ and vice versa. These paths are of length less than $\mathbb{N}_1$ as loops in the walk are of size at most N_1. Theorem 5.20 states that for the ray $\mathbf{r} = (r_0, r_1, \ldots)$ there exist $N_2, M \in \mathbb{N}$ such that for all but finitely many $x \in T$ there exists $i \in \mathbb{N}$ with $d_A(r_i, x), d_A(x, r_{i+M}) \leq N_2$. Hence, for all but finitely elements of T there exists $i \in \mathbb{N}$ with $d_A(s^i, x), d_A(x, s^{i+M}) \leq N_2 + N_1$. □

Corollary 5.25. *Let T be a finitely generated semigroup with no infinite $\mathcal{R}$-class and such that $|sT| = \infty$ for all $s \in T$. If T has one right end then T has normal forms a^*u where a is of infinite order and $u \in T \setminus aT$.*

Proof. In Theorem 5.20 it was shown that for any finite generating set and ray $\mathbf{r} = r_0 \to^{a_1}\to r_1 \to^{a_2} \ldots$ all but finitely many elements of T can be expressed in the form $r_i u$ where $u \in T \setminus a_{i+1}T$. In Theorem 5.24 it was shown that T has an element of infinite order, say a. Combining these facts tells us that if A is a generating set containing a then $a \to a^2 \to a^3 \dots$ is a ray and every element can be expressed in the form $a^i u$ where $u \in T \setminus aT$. $\square$

However, note that these normal forms are not necessarily unique.

Theorem 5.26. *Let T be a finitely generated semigroup with the property that $|xT|$ is infinite for all $x \in T$ and no infinite $\mathcal{R}$-classes. If T has one right end then T has a presentation of the form*

$$\begin{aligned} \langle a, u_1, u_2, \dots, u_n | \quad u_i \cdot u_j &= a^{\alpha(i,j)} u_{\beta(i,j)}, \\ u_i \cdot a &= a^{f(i)} u_{g(i)}, \\ a^{\pi(i,j)} u_i &= a^{\pi(j,i)} u_j \rangle, \end{aligned}$$

where

$\alpha : \{1, 2, \dots, n\} \times \{1, 2, \dots, n\} \to \mathbb{N}_0$,

$\beta : \{1, 2, \dots, n\} \times \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$,

$f : \{1, 2, \dots, n\} \to \mathbb{N}_0$,

$g : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$,

$\pi : \{1, 2, \dots, n\} \times \{1, 2, \dots, n\} \to \mathbb{N}$

and where u_0 denotes a (possibly external) identity element.

Proof. By Corollary 5.25 there exists an element of infinite order $a \in T$ such that $\{a\} \cup (T \setminus aT)$ is a finite generating set for T and every element can be expressed in the form $a^i u$ for some $u \in T \setminus aT$ and some $i \in N_0$.

Note this means that for each $u_i \in T \setminus aT$ the element $u_i \cdot a$ can be expressed in the form $a^{f(i)}u_{g(j)}$ and the element $u_i \cdot u_j$ can be expressed as $a^{\alpha(i,j)}u_{\beta(i,j)}$. We have introduced a new element u_0 which may not be in the semigroup purely to signify cases such as $u_iu_j = a^p$. However, these forms are not necessarily unique.

If $a^iu = a^ju$ for some $i < j$ and $u \in T \setminus aT$ then $a^iu = a^ju = a^{j-i}a^iu = a^{2j-i}u$ and hence $a^{k(j-i)+i}u = a^iu$ for all $k \in \mathbb{N}$ which contradicts the fact that each element lies in a unique $a^iT \setminus a^{i+1}T$ as shown in Theorem 5.20. This means for each a^iu there are at most n words of the form a^ju' such that $a^iu = a^ju'$. For each $1 \leq i, j \leq n$ with $i \neq j$ either $a^cu_i \neq a^du_j$ for any $c, d \in \mathbb{N}$ or there exists $c, d \in \mathbb{N}$ such that $a^cu_i = a^du_j$. If there exist such c, d we let $\pi(i,j) = \min\{c \in \mathbb{N} : \exists d \in \mathbb{N}, a^cu_i = a^du_j\}$. We let $\sigma(i,j) = \min\{d \in \mathbb{N} : a^du_j = a^{\pi(i,j)}u_i\}$. We now show $\sigma(i,j) = \pi(j,i)$. Clearly, we have that $\sigma(i,j) \geq \pi(j,i)$ as $\pi(j,i)$ was chosen to be the smallest integer such that $a^{\pi(j,i)}u_j = a^cu_i$ for any $c \in \mathbb{N}$ and $\sigma(i,j)$ is the smallest integer such that $a^{\sigma(i,j)}u_j = a^{\pi(i,j)}u_i$. If $\pi(i,j)$ exists then we have the equality

$$\begin{aligned} a^{\pi(i,j)}u_i &= a^{\sigma(i,j)}u_j \\ &= a^{\sigma(i,j)-\pi(j,i)}a^{\pi(j,i)}u_j \\ &= a^{\sigma(i,j)-\pi(j,i)+\sigma(j,i)}u_i. \end{aligned}$$

However, each element of T can be expressed in the form a^iu for some unique i and, hence, $\pi(i,j) = \sigma(i,j) - \pi(j,i) + \sigma(j,i)$. As $\sigma(i,j) \geq \pi(j,i)$ for any

i, j it follows $\sigma(i, j) = \pi(j, i)$.

We now demonstrate that any other equality involving products of elements of $\{a\} \cup (T \setminus aT)$ must be a consequence of the rules we have just defined. By using the rules $u_i \cdot u_j = a^{\alpha(i,j)} u_{\beta(i,j)}$ and $u_i \cdot a = a^{f(i)} u_{g(i)}$ we may assume any product has the form $a^c u_i$ for some $c \in \mathbb{N}_0$ and some $u_i \in \{u_0, u_1, \ldots, u_n\}$. Let $c, d \in \mathbb{N}$ such that $a^c u_i = a^d u_j$. It follows that $c \geq \pi(i, j)$. If $c = \pi(i, j)$ then $d = \pi(j, i)$, otherwise we get a contradiction to the fact $a^d u_j$ lies in a unique set $a^i T \setminus a^{i+1} T$. If $c > \pi(i, j)$ then $a^d u_j = a^c u_i = a^{c-\pi(i,j)} a^{\pi(i,j)} u_i = a^{c-\pi(i,j)+\pi(j,i)} u_j$. To avoid a contradiction we must have that $d = c - \pi(i, j) + \pi(j, i)$. But then $a^d u_j = a^{c-\pi(i,j)} a^{\pi(j,i)} u_j = a^{c-\pi(i,j)} a^{\pi(i,j)} u_i = a^c u_i$ and hence $a^c u_i = a^d u_j$ is a consequence of $a^{\pi(i,j)} u_i = a^{\pi(j,i)} u_j$. □

Theorem 5.27. *Let S be a finitely generated semigroup with one end and no infinite $\mathcal{R}$-class. The S has a presentation of the form*

$$\begin{aligned}
\langle a, u_1, u_2, \ldots, u_n, i_1, i_2, \ldots i_m | \quad u_i \cdot u_j &= a^{\alpha(i,j)} u_{\beta(i,j)}, \\
u_i \cdot a &= a^{f(i)} u_{g(i)}, \\
i_j \cdot i_k &= i_{\gamma(j,k)}, \\
i_j \cdot a &= i_{\delta(j)}, \\
i_j \cdot u_k &= i_{\epsilon(j,k)}, \\
a^{\pi(b,c)} u_b &= a^{\pi(c,b)} u_c, \\
a^{\mu((b,j),(c,k))} u_b i_j &= a^{\nu((b,j),(c,k))} u_c i_k \rangle,
\end{aligned}$$

where

$\alpha : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{N}_0,$

$\beta : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$

$f : \{1, 2, \ldots, n\} \to \mathbb{N}_0,$

$g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\},$

$\gamma : \{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\},$

$\delta : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m\},$

$\epsilon : \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\},$

$\pi : \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \to \mathbb{N}$

$\mu, \nu : (\{0, 1, \ldots, n\} \times \{1, 2, \ldots, m\}) \times (\{0, 1, \ldots, m\} \times \{1, 2, \ldots, n\}) \to \mathbb{N}$
and where u_0 *denotes an (possibly external) identity element.*

Proof. Lemma 5.5 says that S is the union of a subsemigroup T with one end such that $|tS| = \infty$ for all $t \in T$ and an ideal I such that $|iS| < \infty$ for all

$i \in I$. Lemma 5.7 states that there exist a finite set B such that B generates T and a finite set $D \subseteq I$ such that $DS \subseteq D$ and every element of I can be expressed in the form $T^1 D$. Lemma 5.10 states that tT is infinite for all $t \in T$. By Theorem 5.26 we have that T has a presentation $\langle a, u_1, u_2, \ldots, u_n | u_i \cdot u_j = a^{\alpha(i,j)} u_{\beta(i,j)}, u_i \cdot a = a^{f(i)} u_{g(i)}, a^{\pi(i,j)} u_i = a^{\pi(j,i)} u_j, \rangle$. As S is the union of a semigroup and an ideal a presentation for S will contain a presentation for T and information on how the ideal I interacts with T.

Let $D = \{i_1, i_2, \ldots, i_n\}$. As D is a right ideal in S it follows that $i_j \cdot i_k = i_{\gamma(j,k)}$, $i_j \cdot a = i_{\delta(j)}$ and $i_j \cdot u_k = i_{\epsilon(j,k)}$.

By Lemma 5.7 every element of I can be expressed in the form $T^1 D$. By Lemma 5.25 every element of T can be expressed in the form $a^i u$ where $i \in \mathbb{N}_0$ and $u \in S \setminus aS$. This means that every element of I can be expressed in the form $a^p u_b i_j$ for $p \in \mathbb{N}_0$, $1 \leq b \leq n$ and $1 \leq j \leq l$.

Recall, that we use the notation u_0 for a possibly external identity element of T. For each pair $u_b, u_c \in \{u_0, u_1, \ldots, u_n\}$ and each pair $i_j, i_k \in \{i_1, i_2, \ldots, i_m\}$ with $(b, j) \neq (c, k)$ either there exist $p, q \in \mathbb{N}$ such that $a^p u_b i_j = a^q u_c i_k$ or not. If there exist such $p, q \in \mathbb{N}$ we define

$$\mu((b, j), (c, k)) = \min\{p \in \mathbb{N} : \exists q \in \mathbb{N} \text{ such that } a^p u_b i_j = a^q u_c i_k\}.$$

We then define

$$\nu((b, j), (c, k)) = \min\{q \in \mathbb{N} : a^{\mu(j,k)} u_b i_j = a^q u_c i_k\}.$$

If $(b, j) = (c, k)$ and there exist $p < q$ such that $a^p u_b i_j = a^q u_b i_j$ then we define

$$\mu((b, j), (b, j)) = \min\{p \in \mathbb{N} : \exists q > p \text{ such that } a^p u_b i_j = a^q u_b i_j\}.$$

We then define

$$\nu((b, j), (b, j)) = \min\{q \in \mathbb{N} : q > p, a^{\mu(j,k)} u_b i_j = a^q u_c i_k\}.$$

We now show that any other equality in S is a consequence of the rules we have just defined. In Theorem 5.26 we showed that the rules defining T were complete so here we focus on equalities arising in the ideal I. We have shown that any element of the ideal can be expressed in the form $a^p u_b i_j$ for $p \in \mathbb{N}_0$, $0 \leq b \leq n$ and $1 \leq j \leq m$.

Assume that $a^p u_b i_j = a^q u_c i_k$. We now show this equality is a consequence of $a^{\mu((b,j),(c,k))} u_b i_j = a^{\nu((b,j),(c,k))} u_c i_k$ and $a^{\mu((c,k),(c,k))} u_c i_k = a^{\nu((c,k),(c,k))} u_c i_k$ (if $\mu((c, k), (c, k))$ exists). We have that $p \geq \mu((b, j), (c, k))$ from the definition of $\mu((b, j), (c, k))$, hence, $a^q u_c i_k = a^p u_b i_j = a^{p-\mu((b,j),(c,k))} a^{\mu((b,j),(c,k))} u_b i_j$. This means that $a^q u_c i_k = a^{p+\nu((b,j),(c,k))-\mu((b,j),(c,k))} u_c i_k$.

If $q = p + \nu((b, j), (c, k)) - \mu((b, j), (c, k))$ then $a^p u_b i_j = a^q u_c i_k$ is a consequence of $a^{p(j,k)} i_j = a^{q(j,k)} i_k$.

Otherwise, if $q \neq p + \nu((b, j), (c, k)) - \mu((b, j), (c, k))$ then $\mu((c, k), (c, k))$ exists. It follows that

$$a^{\mu((c,k),(c,k))} u_c i_k = a^{\nu((c,k),(c,k))} u_c i_k = a^{\nu((c,k),(c,k))-\mu((c,k),(c,k))} a^{\mu((c,k),(c,k))} u_c i_k$$

and hence

$$a^{\mu((c,k),(c,k))}u_ci_k = a^{z(\nu((c,k),(c,k))-\mu((c,k),(c,k)))}a^{\mu((c,k),(c,k))}u_ci_k$$

for all $z \in \mathbb{N}$. This means for any $p > \mu((c,k),(c,k))$ the element $a^pu_ci_k$ is equal to $a^{\mu((c,k),(c,k))+r}u_ci_k$ for some $0 \leq r < \nu((c,k),(c,k)) - \mu((c,k),(c,k))$.

We deduce from the rule $a^{\mu((c,k),(c,k))}u_ci_k = a^{\nu((c,k),(c,k))}u_ci_k$ that $a^pu_bi_j = a^{p-\mu((b,j),(c,k))}a^{\nu((c,k),(c,k))}u_ci_k = a^{\mu((c,k),(c,k))+r}u_ci_k$ and $a^qu_ci_k = a^{\mu((c,k),(c,k))+r'}$. Without loss of generality we assume that $r \leq r'$. The arguments for when $r' \leq r$ follow by symmetry. It follows from $\mu((c,k),(c,k)) + r' \leq \nu((c,k),(c,k))$ that

$$\begin{aligned} a^{\nu((c,k),(c,k))-(r'-r)}u_ci_k &= a^{\nu((c,k),(c,k))-\mu((c,k),(c,k))-r'}a^{\mu((c,k),(c,k))+r}u_ci_k \\ &= a^{\nu((c,k),(c,k))-\mu((c,k),(c,k))-r'}a^{\mu((c,k),(c,k))+r'}u_ci_k \\ &= a^{\nu((c,k),(c,k))}u_ci_k \\ &= a^{\mu((c,k),(c,k))}u_ci_k. \end{aligned}$$

As $\nu((c,k),(c,k))$ was minimal it follows that $\nu((c,k),(c,k)) - (r' - r) = \nu((c,k),(c,k))$ or $\nu((c,k),(c,k)) - (r'-r) = \mu((c,k),(c,k))$. As $r' < \nu((c,k),(c,k)) - \mu((c,k),(c,k))$ we know that $\nu((c,k),(c,k)) - (r'-r) > \mu((c,k)(c,k))$ and hence $\nu((c,k),(c,k)) - (r'-r) = \nu((c,k),(c,k))$. This means that $r' = r$ and that $a^qu_ci_k = a^{p+\nu((b,j),(c,k))-\mu((b,j),(c,k))}u_ci_k$ is a consequence of $a^{\nu((c,k),(c,k))}u_ci_k = a^{\mu((c,k),(c,k))}u_ci_k$. Then we have that $a^pu_bi_j =$

$a^q u_c i_k$ is a consequence of $a^{\mu((b,j),(c,k))} u_b i_j = a^{\nu((b,j),(c,k))} u_c i_k$ and then it follows $a^{\mu((c,k),(c,k))} u_c i_k = a^{\nu((c,k),(c,k))} u_c i_k$, as required. □

Corollary 5.28. *Let S be a finitely generated semigroup with one end and no infinite $\mathcal{R}$-class and let u_0 be a possibly external identity element. If $A = \{a, u_1, \ldots, u_n, i_1, \ldots i_m\}$ is a generating set for S (as in Theorem 5.27) then there exists a finite set $X \subseteq A^+$ and $N \in \mathbb{N}$ such that S has unique normal forms $X \cup a^* a^N (Y + Z)$ for some $Y \subseteq \{u_0, u_1, \ldots, u_n\}$ and $Z \subseteq \{u_0 i_1, u_0 i_2, \ldots, u_0 i_m, \ldots, u_n i_1, \ldots, u_n i_m\}$.*

Proof. Theorem 5.27 states that every element of S can be written in the form $a^p u_j$ or $a^p u_j i_k$. Let $J = \{u_0 i_1, u_0 i_2, \ldots, u_0 i_m, \ldots u_n i_1, u_n i_2, \ldots, u_n i_m\}$. Let F be those $u_b i_j \in J$ such that $\mu((b,j),(b,j))$ exists. There are only finitely many elements of S of the form $a^p u_b i_j$ for $u_b i_j \in F$. These elements are $u_b i_j, a u_b i_j, \ldots, a^{\nu((b,j),(b,j))-1} u_b i_j$.

Let $N \geq \max\{\mu((b,j),(c,k)), \pi(j,k)\}$. We define a partial order $\preceq$ on $\{u_1, u_2, \ldots, u_n\}$ and $J \setminus F$ respectively as follows. We say $u_j \preceq u_k$ if $j \leq k$ and $\pi(k,j)$ exists, similarly $u_b i_j \preceq u_c i_k$ if $b \leq c$, $j \leq k$ and $\mu((b,j),(c,k))$ exists. Let Y and Z be minimal elements of $\{u_1, u_2, \ldots, u_m\}$ and $J \setminus F$ respectively. In particular if $u_b i_j \preceq u_c i_k$ then in $a^N S$ we have $a^p u_c i_k = a^{p+(\nu((b,j),(c,k))-\mu((b,j),(c,k)))} i_j$ so $a^N S = a^N a^* (Y + Z)$. The minimality of elements of Z ensures that $a^p u_b i_j \neq a^q u_c i_k$ for $(b,j) \neq (c,k)$ and the minimality of elements of Y ensures that $a^p u_j \neq a^q u_k$ for $j \neq k$. Hence every element of $a^N S$ is expressed uniquely. There are only finitely many elements in $S \setminus a^N S$ as S has one end and there are only finitely many elements of the form $a^p u_b i_j$ where $\mu((b,j),(b,j))$ exists so we fix a representative in A^* for each of these elements and call this set X. Thus we get that $X \cup a^* a^N (Y + Z)$ gives a unique representative for each element of S. $\square$

5.3 Properties

In this section we consider interesting properties that a semigroup with one end may have.

5.3.1 The Ideal Effect

As mentioned throughout this chapter when taking a semigroup and adding a suitably well-behaved ideal we may preserve the number of ends, whilst dramatically changing the algebraic properties of the semigroup. In this subsection we will try to give an insight into why the ideals have such an effect on the algebraic structure.

The reader should firstly recall Lemma 5.7 and Lemma 5.8. Lemma 5.7 gives a way of constructing a new semigroup S from a semigroup T and a finite semigroup D such that S will have the same number of ends as T. This is the process that will be used in Example 5.35 and Example 5.39. To form this construction we define the products $d \cdot t$ so that they lie in D and also define equalities within T^1D. The semigroup S is then defined to be $T \cup T^1D$. We may think of the products $d \cdot t$ as an action of T on the set of elements of D, this is because for the associativity of S we must have $(d \cdot s) \cdot t = d \cdot (st)$. The equalities that are defined in T^1D give rise to a binary relation on T. Namely for eachx $d \in D$ we define $\rho_d \subseteq T \times T$ as $(t_1, t_2) \in \rho_d$ if and only if $t_1d = t_2d$. Each ρ_d is clearly an equivalence relation. These relations are also left-congruences as if $t_1d = t_2d$ we have $st_1d = st_2d$ for all $s \in S$. Another property that holds is if there exists $s \in S$ such that $d_1s = d_2$ then $\rho_{d_1} \subseteq \rho_{d_2}$.

Lemma 5.29. *Let T be a finitely generated semigroup and let $\phi : T \to T$ be a homomorphism. Then there exists a finitely generated semigroup S which is the disjoint union of the semigroup T and an ideal I such that iS is finite for all $i \in I$ and $\phi(t_1) = \phi(t_2)$ if and only if there exists $d \in I$ with $t_1d = t_2d$.*

Proof. Let $D = \{d\}$ be the trivial group. We want $DT \subseteq D$ so we set $d \cdot t = d$ for all $t \in T$. This leaves a trivial action of T on D so we can focus on defining relations. We now define the equalities $t_1d = t_2d$ for each $t_1, t_2 \in T$ such that $\phi(t_1) = \phi(t_2)$. Then $S = T \cup T^1D$ is finitely generated and T^1D is an ideal. □

The ideal $I = T^1d$ is a semigroup and the elements look like the quotient

$T \big/ \phi(T)$, however, the elements of I are left-zeroes. This may not appear to greatly influence the algebraic structure of the semigroup but when looking at such things as finite presentability the only way to construct these elements is by using the semigroup T and the homomorphism ϕ. One may be able to better understand the properties arising from adjoining ideals of this type by considering the left-congruences ρ_d induced on T.

5.3.2 Finite Presentability

The first property we consider is finite presentability. Recall a semigroup S is finitely presentable if there exists a presentation $\langle A|R\rangle$ for S in which A and R are finite.

Theorem 5.30. *[27, Chapter 2.2 Proposition 2.1] Given a semigroup S with presentation $\langle A|R\rangle$ then S is finitely presentable if and only if there exists a presentation $\langle A'|R'\rangle$ for S where $A' \subseteq A$, $R' \subseteq R$ and A' and R' are finite.*

There are examples of both finitely generated semigroups and finitely generated groups which are not finitely presentable. One such example is the following.

Example 5.31. Let S be the semigroup $\langle a, b | ab^i a = aba \, (i \in \mathbb{N})\rangle$ then there exists no finite presentation for S. This follows from the fact that $ab^n a = aba$ cannot be deduced from any combinations of $ab^i a = aba$ for any $i < n$.

An example of a non-finitely presented group is $\mathbb{Z}\wr\mathbb{Z}$ which can be given by the presentation $Gp\langle a, t \mid [t^{-i}at^{i}, t^{-j}at^{j}] = 1\,(i, j \in \mathbb{Z})\rangle$. We use the notation $[a, b]$ to denote the product $aba^{-1}b^{-1}$ in a group.

For semigroups with an infinite $\mathcal{R}$-class we need some theorems from the literature.

Theorem 5.32. *[37, Theorem 1.3] Let S be a semigroup and let T be a subsemigroup of finite Rees index. Then S is finitely presented if and only if T is finitely presented.*

Theorem 5.33. *[1, Theorem 5.1] A Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ is finitely presented if and only if G is finitely presented and I, Λ are finite.*

Theorem 5.34. *(A) A semigroup with one end and no infinite $\mathcal{R}$-class is finitely presented.*

(B) Let S be a semigroup with one end, an infinite $\mathcal{R}$-class and a right group $G \times E$ as a subgroup of finite Rees index. Then S is finitely presented if and only if G is finitely presented and E is finite.

Proof. (A) A presentation for a semigroup with one end and no infinite $\mathcal{R}$-class is given in Theorem 5.27 and this presentation is finite.

(B) By Theorem 5.33 a right group $G \times E$ is finitely presented if and only if G is finitely presented and E is finite. It follows from Theorem 5.32 that

S is finitely presented if and only if $G \times E$ is finitely presented. Hence, S is finitely presented if and only if G is finitely presented and E is finite. $\square$

Notice that Theorem 5.34 does not cover a semigroup with one end, with an infinite $\mathcal{R}$-class and infinitely many elements in the ideal of elements such that iS is finite. Some of these semigroups may be finitely presented, however, some are not.

Example 5.35. Let G be the group $\mathbb{Z} \times Gp\langle a, b \mid \rangle$. By Lemma 2.45 G has one end. We may present G as $Gp\langle a, b, c \mid ac = ca, bc = cb\rangle$. We construct a semigroup S by adding another element d such that $d^2 = da = db = dc = d$ and $g^{-1}[b^{-i}ab^i, b^{-j}ab^j]gd = d$ for all $i, j \in \mathbb{Z}$ and all $g \in G$. This semigroup will have one end as we have adjoined an ideal I such that iS is finite for all $i \in I$. However, any presentation for S would need to contain a presentation for $\mathbb{Z} \wr \mathbb{Z}$ so S is not finitely presented. $\square$

The previous example further demonstrates that although the addition of an ideal I to a semigroup S such that iS is finite for all $i \in I$ makes no difference when considering the right ends of a semigroup, it can influence the algebraic structure quite significantly.

5.3.3 Word Problem

Let S be a semigroup, let A be a finite set and let $\psi : A^* \to S$ be a surjective homomorphism. For $u, v \in A^*$ we write $u =_S v$ if $\psi(u) = \psi(v)$. We say S has *solvable word problem* if there exists an algorithm that can decide for any two words $u, v \in A^+$ whether $u =_S v$.

Theorem 5.36. *[37, Theorem 5.1] Let S be a finitely generated semigroup and let T be a subsemigroup of finite Rees index. Then S has solvable word problem if and only if T has solvable word problem.*

Theorem 5.37. *[1, Theorem 5.4] A Rees matrix semigroup $S = [G; I, \Lambda; P]$ has solvable word problem if and only if G has solvable word problem.*

Theorem 5.38. *(A) If S is a finitely generated semigroup with one end and has no infinite $\mathcal{R}$-class then S has solvable word problem.*

(B) Let S be a finitely generated semigroup with one end, an infinite $\mathcal{R}$-class and a right group $G \times E$ as a subsemigroup of finite Rees index. Then S has solvable word problem if and only if G has solvable word problem.

Proof. (A) Let $A = \{a, u_1, u_2, \ldots, u_n, i_1, i_2, \ldots, i_m\}$ be a generating set for S as in Theorem 5.27. The normal forms for all but finitely many elements of S are given in Corollary 5.28. Thus if there exists an algorithm to decide if a word lies in this finite set X the semigroup S has solvable word problem.

From Corollary 5.25 and Lemma 5.7 we have elements of I are of the form $a^p u_b i_k$. We retain the notation u_0 for a possibly external identity element.

Let $s = a_1 a_2 \dots a_n$ be a word in A^+. We now show that we can decide if $s \in X$ or not. Recall from Corollary 5.25 elements of X come in two flavours, those that lie in $S \setminus a^N S$ and those of the form $a^p u_b i_j$ where $\mu((b, j), (b, j))$ exists. We can rewrite s to be in the form $a^p u_b$ or $a^p u_b i_j$ using Corollary 5.25. If $p \geq N$ then s is certainly in $a^N S$.

If $p < N$ and $s = a^p u_b$ then we apply rules of the form $a^{\pi(b,c)} u_b = a^{\pi(c,b)} u_c$ where possible to get different representatives of the form $a^{p_i} u$ for different elements u. We need only apply at most n rules as at most n elements of the form $a^p u$ can be equal. Otherwise $a^p u = a^q u$ for some $p < q$ which cannot happen. If there is a $p_i \geq N$ then s is in $a^N S$ otherwise $s \in X$.

If $p < N$ and $s = a^p u_b i_j$ then we first check if $\mu((b, j), (b, j))$ exists, if this is the case then $s \in X$. If not then we apply rules of the form $a^{\mu((b,j),(c,k))} u_b i_j = a^{\nu((b,j),(c,k))} u_c i_k$ where possible to get various elements of the form $a^{p_i} u_i$. If we apply these rules more than $(n+1)m$ times and get at least $(n+1)m + 1$ different forms then there will exist $0 \leq c \leq n$, $1 \leq k \leq m$

and $q_1 \leq q_2$ such that $a^{q_1}u_ci_k = a^{q_2}u_ci_k$. It follows that

$$\begin{aligned} a^p u_b i_j &= a^{q_1} u_c i_k \\ &= a^{q_2} u_c i_k \\ &= a^{q_2 - \nu((b,j),(c,k)) + \mu((b,j),(c,k))} u_b i_j \end{aligned}$$

and hence as $q_2 > q_1$ the rule $\mu((b,j),(b,j))$ exists. It follows the rules only need to be applied at most $(n+1)m$ times. If any $p_i \geq N$ then $s \in a^N S$ otherwise $s \in X$.

We are now able to decide if a word $s \in A^+$ is in X and if not we can apply rewrite rules from Corollary 5.28 to get it in unique normal form. Let $s \in A^*$ represent an element of X.

If s is of the form $a^p u_b$ then there are only at most n words of the form $a^q u_c$ such that $a^p u_b = a^q u_c$. We can list these using the rules of the form $a^{\pi(b,c)} u_b = a^{\pi(c,b)} u_c$.

If s is of the form $a^p u_b i_j$ where $\mu((b,j),(b,j))$ does not exist then there are only at most $(n+1)m$ words of the form $a^q u_c i_k$ such that $a^p u_b i_j = a^q u_c i_k$. We can list these using the rules of the form $a^{\mu((b,j),(c,k))} u_b i_j = a^{\nu((b,j),(c,k))} u_c i_k$.

If s is of the form $a^p u_b i_j$ where $\mu((b,j),(b,j))$ does exist then either $p < \mu((b,j),(b,j))$, in which case we apply the argument above, or $p \geq \mu((b,j),(b,j))$. If $p \geq \mu((b,j),(b,j))$ then we apply the rule $\mu((b,j),(b,j))$ to get s in the form $a^{\mu((b,j),(b,j))+r} u_b i_j$ where $0 \leq r < \nu((b,j),(b,j)) - \mu((b,j),(b,j))$. There can only be at most $(n+1)m$ words in this form.

We can list these using the rules of the form $a^{\mu((b,j),(c,k))}u_b i_j = a^{\nu((b,j),(c,k))}u_c i_k$ for $(b,j) \neq (c,k)$.

By inspection one can see that this is solvable in linear time.

(B) By Theorem 5.36 S has solvable word problem if and only if $G \times E$ has solvable word problem and from Theorem 5.37 $G \times E$ has solvable word problem if and only if G has solvable word problem. Thus S has solvable word problem if and only if G has solvable word problem. $\square$

As in the case for finite presentability we can use an ideal to make a semigroup with one end and an infinite $\mathcal{R}$-class with unsolvable word problem from a group G even if G has solvable word problem.

A subset S of $\mathbb{Z}^n$ is said to be *Diophantine* if there exists a polynomial $P(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots y_m)$ such that $(s_1, \ldots, s_n) \in S$ if and only if $P(s_1, s_2, \ldots, s_n, y_1, y_2, \ldots y_m)$ has an integer root.

A subset S of $\mathbb{Z}^n$ is said to be *recursive* if both S and $\mathbb{Z}^n \setminus S$ are Diophantine.

Example 5.39. Let Σ be a Diophantine non-recursive set of positive integers. The group $H = Gp\langle a, b, c, d \mid a^{-i}ba^i = c^{-i}dc^i \, (i \in \Sigma)\rangle$ has unsolvable word problem, see [27, Theorem 7.7]. The group $G = Gp\langle a, b, c, d \mid \rangle \times \mathbb{Z}$ has one end by Lemma 2.45 and has solvable word problem. However, the semigroup constructed by taking G and adding an element z with the property that

$z^2 = z$, $z \cdot g = z$ for all $g \in G$ and $g^{-1}a^{-i}ba^{i}gd = g^{-1}c^{-i}dc^{i}gd$ for all $i \in \Sigma$ and all $g \in G$ gives a semigroup with one end and an unsolvable word problem.

5.3.4 Automaticity

Definition 5.40. *Let S be a semigroup. A* rational structure *for S is a pair (A, L) such that A is a finite set, L is a regular subset of A^* and there exists a homomorphism $\psi : A^* \to S$ such that $\psi|_L$ is surjective.*

To define the notion of an automatic semigroup we need to introduce a 'padding' function.

Let A be a finite set. For two words $u, v \in A^*$ we inductively define the padded string $(u, v)\delta$ over $(A \cup \{\$\})^* \times (A \cup \{\$\})^*$ as follows. For any $a, b \in A$ we let $(a, b)\delta = (a, b)$. For any $a, b \in A$ and $u, v \in A^*$, we let $(au, bv)\delta = (a, b)(u, v)\delta$, $(au, \epsilon)\delta = (a, \$)(u, \epsilon)\delta$ and $(\epsilon, bv)\delta = (\$, b)(\epsilon, v)\delta$.

Definition 5.41. *A semigroup S is* automatic *if there exists a rational structure (A, L) for S such that the sets*

$$\{(u, v)\delta \in ((A \cup \{\$\}) \times (A \cup \{\$\}))^* : \psi(u) = \psi(v)\}$$

and

$$\{(u, v)\delta \in (A \cup \{\$\})^* \times (A \cup \{\$\})^* : \psi(ua) = \psi(v)\}$$

are regular for all $a \in A$.

A related notion is that of an asynchronously automatic semigroup.

Definition 5.42. *A semigroup S is* asynchronously automatic *if there exists a rational structure (A, L) for S such that the sets*

$$\{(u, v) \in L \times L : \psi(u) = \psi(v)\}$$

and

$$\{(u, v) \in L \times L : \psi(ua) = \psi(v)\}$$

are regular for all $a \in A$.

Theorem 5.43. *[2, Theorem A.6.1] An automatic semigroup is asynchronously automatic.*

Theorem 5.44. *[2, Theorem B.4.1] The direct product of two automatic groups is again automatic.*

Theorem 5.45. *[8, Theorem 1.1] Let S be a Rees matrix semigroup given by $\mathcal{M}[G; I, \Lambda; P]$ where I and Λ are finite. Then S is automatic if and only if G is automatic.*

Theorem 5.46. *[20, Theorem 1.1] Let S be a semigroup and let T be a subsemigroup of finite Rees index. Then S is automatic if and only if T is automatic.*

Theorem 5.47. *(A) If S is a finitely generated semigroup with one end and no infinite $\mathcal{R}$-class then S is asynchronously automatic.*

(B) Let S be a finitely generated semigroup with one end, an infinite $\mathcal{R}$-class and a right group $G \times E$ as a subsemigroup of finite Rees index. Then S is automatic if and only if G is automatic and E is finite.

Proof. (A). Suppose S is a finitely generated semigroup with one end and no infinite $\mathcal{R}$-class. Let $A = \{a, u_1, \ldots, u_n, i_1, \ldots, i_m\}$ be a generating set for S as in Theorem 5.27. By Corollary 5.28 there exists a finite set $X \subseteq A^+$ and $N \in \mathbb{N}$ such that S has unique normal forms $X \cup a^* a^N (Y + Z)$ for some $Y \subseteq \{u_0, u_1, \ldots, u_n\}$ and $Z \subseteq \{u_0 i_1, u_0 i_2, \ldots, u_0 i_m, \ldots, u_n i_1, \ldots, u_n i_m\}$. Let $L = X \cup a^* a^N (Y + Z)$. By inspection L is regular. Let $\psi : L \to S$ be a bijection.

As L is a regular language with unique representatives for elements of S the set $\{(u, v) \in L \times L : \psi(u) = \psi(v)\} = \{(u, u) \in L \times L\}$ is regular. Let $x \in A$. We now consider the sets $L_x = \{(u, v) \in L \times L : \psi(ux) = \psi(v)\}$. We may disregard the finitely many elements not in $a^N a^* (Y \cup Z)$.

As the finite union of regular languages is regular we consider the subsets of L that end with some $u_b \in Y$ or $u_b i_j \in Z$. Let $s \in Y \cup Z$. Either $a^N s \cdot x = a^q t$ for some $t \in Y \cup Z$ and $q \geq N$ or $a^N sx = a^q u_c i_k$ where $\mu((c, k), (c, k))$ exists. If $a^N sx = a^q t$ then $a^p sx = a^{p-N+q} t$ so we have L_x as the finite union over $(a, a)^* (a^N s, a^{q(s,x)} t)$ for $s \in Y \cup Z$. Otherwise $a^N sx = a^q u_c i_k$ where $\mu((c, k), (c, k))$ exists. It follows that $\mu((c, k), (c, k)) \leq q \leq$

$\nu((c,k),(c,k)) - 1$. We divide cases by powers of a mod P where $P = \nu((c,k),(c,k)) - \mu((c,k),(c,k))$ and express these pairs in L_x as

$$
\begin{aligned}
&(a^p, \epsilon)^*(a^N s, a^{q(s,x)} i_j),\\
&(a^p, \epsilon)^*(a^{N+1} s, a^{q(s,x)+1} i_j),\\
&\vdots\\
&(a^p, \epsilon)^*(a^{N+p-1} s, a^{q(s,x)+P-1} i_j).
\end{aligned}
$$

(B). A right group $G \times E$ is a special kind of Rees matrix semigroup so by Theorem 5.45 $G \times E$ is automatic if and only if G is automatic and E is finite. By Theorem 5.46 it follows S is automatic if and only if G is automatic and E is finite. □

Example 5.48. Let $H = Gp\langle A \mid R\rangle$ be a group which is not asynchronously automatic. Let $G = \mathbf{Z} \times Gp\langle A \mid \rangle$. By Theorem 5.44 G is automatic and by Lemma 2.45 G has one end. Let $\psi : (A \cup A^{-1})^* \to H$ be a surjective homomorphism. By extending the domain of ψ from $(A \cup A^{-1})^*$ to $(A \cup A^{-1})^* \times \mathbf{Z}$ we get a homomorphism from G to $\mathbf{Z} \times H$. By Lemma 5.29 we construct a semigroup S with one end of the form $G \cup (\mathbb{Z} \times H) \cdot d$ where $(\mathbb{Z} \times H) \cdot d$ is an ideal. The semigroup S is not asynchronously automatic as H was not asynchronously automatic.

The question as to whether semigroups with one end and no infinite $\mathcal{R}$-class are automatic is left open.

5.3.5 Residual Finiteness

Residual finiteness is a widely studied finiteness property in both semigroups and groups. This subsection introduces various notions involved when considering residual finiteness and describes semigroups with one end which are residually finite.

Definition 5.49. *A semigroup S is* residually finite *if for any two non-equal elements $s, t \in S$ there exists a homomorphism $\phi : S \to T$ for some finite T such that $\phi(s) \neq \phi(t)$.*

Lemma 5.50. *A semigroup S is residually finite if and only if for all $x, y \in S$ there exists a congruence ρ of finite index such that ${}^{x}/_{\rho} \neq {}^{y}/_{\rho}$.*

Proof. Let S be a semigroup and let $x, y \in S$.

If S is residually finite then there exists a homomorphism $\phi : S \to T$ for some finite T such that $\phi(x) \neq \phi(y)$. We define a relation ρ on S by $(s, t) \in \rho$ if and only if $\phi(s) = \phi(t)$. The relation ρ is clearly an equivalence relation. It is also a congruence because if $\phi(s) = \phi(t)$ then for any $a, b \in S$ we have $\phi(a)\phi(s)\phi(b) = \phi(a)\phi(t)\phi(b)$ and hence $\phi(asb) = \phi(atb)$. The relation has finite index because T is finite.

Let ρ be a congruence of finite index such that ${}^{x}/_{\rho} \neq {}^{y}/_{\rho}$. Let $T = \{t_1, t_2, \ldots, t_n\}$ where the t_i denote the equivalence classes of ρ. We define a map $\phi : S \to T$ by $\phi(s) = t_i$ if $(s, t_i) \in \rho$. If $\phi(s) = \phi(u)$ and $\phi(t) = \phi(v)$ then

$(s,u) \in \rho$ and $(t,v) \in \rho$. It follows $(st,ut) \in \rho$ and $(ut,uv) \in \rho$. This means $(st,uv) \in \rho$ and hence $\phi(st) = \phi(uv)$ so $\phi(s)\phi(t) = \phi(st)$ is well defined and ϕ is a homomorphism onto a finite set such that $\phi(x) \neq \phi(y)$. □

Lemma 5.51. *If for every $s \in S$ there exists an ideal I of finite Rees index such that $s \notin I$ then S is residually finite.*

Proof. Let $s,t \in S$ and let I be an ideal of finite Rees index not containing s. The Rees quotient ${}^{S}/_{I}$ gives a homomorphism ϕ in which the images of s and t are not equal. This is because either $t \notin I$ in which case by definition they are unequal or $t \in I$ in which case $\phi(t) = 0$ and $\phi(s) \neq 0$ and hence are unequal. □

Lemma 5.52. *Let S be a semigroup and let ρ be a right congruence of finite index. Then the largest congruence contained in ρ is also of finite index.*

Proof. Let $C_1, C_2, \ldots, C_n$ be the ρ-classes of S. We know for each $s \in S$ that if $(x,y) \in \rho$ then $(xs,ys) \in \rho$. This means if $x,y \in C_i$ then $xs, ys \in C_j$ for some j. We define a map $\psi : S \to \{1,2,\ldots,n\}^n$ by $\psi(s) = (i_1, i_2, \ldots, i_n)$ if $C_j s \in C_{i_j}$. We define a binary relation $\hat{\rho} = \{(x,y) \in \rho : \psi(x) = \psi(y)\}$. It is easy to check that $\hat{\rho}$ is an equivalence relation and it has finitely many classes as the set $\{1,2,\ldots,n\}^n$ is finite. Let $s \in S$ and $(x,y) \in \hat{\rho}$. For every $1 \leq j \leq n$ we know $C_j x$ and $C_j y$ are both contained in the same C_{i_j}, it follows that $C_j(xs)$ and $C_j ys$ are both contained in $C_{i_j}s$ which is contained within some C_k. This means $(xs, ys) \in \hat{\rho}$. Let $C_j s$ be contained in C_{i_j}. As $C_{i_j} x$ and $C_{i_j} y$ are contained in $C_{i_{i_j}}$ it follows that $C_j sx$ and $C_j sy$ are contained in $C_{i_{i_j}}$ and, hence, $\hat{\rho}$ is a left congruence. We have constructed a congruence contained in ρ that is of finite index so the largest congruence contained in ρ must also be of finite index. □

Lemma 5.53. *If for any $s \in S$ there exists a right ideal I of finite Rees index such that $x \notin I$ then S is residually finite.*

Proof. A right ideal gives a right congruence ρ and and then by Lemma 5.52 taking the largest congruence contained in ρ gives a congruence of finite index. □

Theorem 5.54. *[38, Corollary 4.6] Let T be a subgroup of finite Rees index in a semigroup S. Then S is residually finite if and only if T is residually finite.*

Theorem 5.55. *[14, Corollary 2] The semigroup $\mathcal{M}[G; I, \Lambda; P]$, for which at least one of I, Λ is finite, is residually finite if and only if G is residually finite.*

Theorem 5.56. *(A) Let S be a finitely generated semigroup with no infinite $\mathcal{R}$-class and one right end. Then S is residually finite.*

(B) Let S be a finitely generated semigroup with one end, an infinite $\mathcal{R}$-class

and a right group $G \times E$ as a subsemigroup of finite Rees index. Then S is residually finite if and only if G is residually finite.

Proof. (A). If S has no infinite $\mathcal{R}$-class and one end then by Corollary 5.28 we have a generating set $A = \{a, u_1, \ldots, u_n, i_1, \ldots i_m\}$ for S such that there exists a finite set $X \subseteq A^+$ and $N \in \mathbb{N}$ such that S has unique normal forms $X \cup a^* a^N (Y + Z)$ for some $Y \subseteq \{u_0, u_1, \ldots, u_n\}$ and $Z \subseteq \{u_0 i_1, u_0 i_2, \ldots, u_0 i_m, \ldots, u_n i_1, \ldots, u_n i_m\}$.

We prove the semigroup $a^N S$ is residually finite, it follows from Theorem 5.54 that S is residually finite. Let $s, t \in X$.

If $s = a^p u_b$ then s does not lie in the right ideal $a^{p+1}S$. The set $S \setminus a^p S$ is finite by Lemma 5.4. Hence, by Lemma 5.53 we see if $s \notin X$ then there exists a homomorphism $\phi : S \to F$ for some finite semigroup F such that $\phi(s) \neq \phi(t)$. Similarly, if $s = a^p u_b i_j$ and $\mu((b,j),(b,j))$ does not exist then s does not lie in the right ideal $a^{p+1}S$ and there exists a homomorphism $\phi : S \to F$ for some finite semigroup F such that $\phi(s) \neq \phi(t)$.

It remains to consider the case when $s = a^p u_b i_j$ and $t = a^q u_c i_k$ and where both $\mu((b,j),(b,j))$ and $\mu((c,k),(c,k))$ exist. As both $\mu((b,j),(b,j))$ and $\mu((c,k),(c,k))$ exist we may assume that $\mu((b,j),(b,j)) \leq p < \nu((b,j),(b,j))$ and $\mu((c,k),(c,k)) \leq q < \nu((c,k),(c,k))$. Let $M = \Pi_{(b,j)}(\nu((b,j),(b,j)) -$

$\mu((b,j),(b,j)))$. We define a binary relation ρ on $a^N S$ as follows:

$$(a^p, a^q), (a^p u_b, a^q u_c), (a^p u_b i_j, a^q u_c i_k) \in \rho \Leftrightarrow j = k \text{ and } p \equiv q \mod M.$$

The relation ρ is clearly an equivalence relation. It has finitely many classes as the sets $\{u_1, \ldots, u_n\}$ and $\{i_1, \ldots, i_m\}$ are finite and M is finite. The relation ρ is also a right congruence as if $(s,t) \in \rho$ and $s \neq t$ then without loss of generality we have that $a^{kM} s = t$ for some $k \in \mathbb{N}$. Then if $u \in a^N S$ we know that $s \cdot u \in a^N S$ and hence can be written in normal form, but then we know $a^{kM}(s \cdot u)$ is also in normal form so $(su, tu) \in \rho$. Note that if $s \neq t$ then $(s,t) \notin \rho$ so by Lemma 5.53 $a^N S$, and hence S, is residually finite.

(B). By Theorem 5.54 S is residually finite if $G \times E$ is residually finite. Then applying Theorem 5.55 we have that a Rees matrix semigroup $\mathcal{M}[G; \{i\}, \{1, \ldots, n\}; P]$ is residually finite if and only if G is residually finite. It follows S is residually finite if and only if G is residually finite. □

Let $m, n \in \mathbb{N}$. The *Baumslag-Solitar group*, denoted by $BS(m,n)$, is the group with presentation $Gp\langle a, b \mid b^{-1} a^m b = a^n \rangle$. This class of groups was introduced by Baumslag and Solitar in [3] to provide an example of a group that could be finitely presented with one relation but was not Hopfian. A group G is *Hopfian* if every surjective homomorphism $\psi : G \to G$ is an automorphism of G.

Theorem 5.57. *[30, Theorem C] Let $m, n \in \mathbb{N}$. The group $BS(m,n) = Gp\langle a, b \mid b^{-1}a^m b = a^n\rangle$ is residually finite if and only if $m = 1$,$n = 1$ or $m = n$.*

The following examples demonstrates that a semigroup with one end and an infinite $\mathcal{R}$-class need not be residually finite.

Example 5.58. Let $m, n \in \mathbb{N}$ be such that $m > 1$, $n > 1$ or $m \neq n$. Let G be the group $Gp\langle c \mid \rangle \times Gp\langle a, b \mid \rangle$ and let $\psi : \{a, b, a^{-1}, b^{-1}\}^* \to BS(m,n)$ be the canonical surjective homomorphism from the presentation. Let $\hat{\psi}((c^i, u)) = (c^i, \psi(u))$ for $u \in \{a, b, a^{-1}, b^{-1}\}^*$. Then by Lemma 5.29 there exists a semigroup S which is the union of G and an ideal with elements of the form $(Gp\langle c \mid \rangle \times BS(m,n))d$. Then S is not residually finite as $BS(m,n)$ is not residually finite.

5.3.6 Comparing Left and Right Ends

As mentioned in Chapter 3 the left Cayley graph of a group is isomorphic to the right Cayley graph of the group. One might pose the question as to when this is true for semigroups. At the start of this thesis we noted that it sufficed to study right Cayley graphs of semigroups as any properties that arose in the right Cayley graph of a semigroup S would also arise in the left Cayley graph of the dual of S. This gives an easy result that if a semigroup is isomorphic to it's dual it will have the same poset of left and right ends. However, Theorem 3.2 shows that in general the posets of left or right ends can be very different. A reasonable question might be whether semigroups with one right end also have one left end. The following example shows that this is not the case even for left cancellative semigroups.

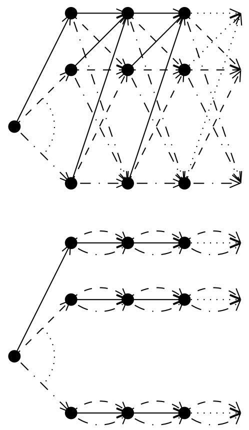
Example 5.59. The semigroup $S = Mon\langle a_1, a_2, \ldots, a_n | a_i a_j = a_j^2 (i \neq j)\rangle$ has normal forms a_i^j for $1 \leq i \leq n$ and $j \in \mathbb{N}_0$. In the right Cayley graph each ray of the form $a_i \rightarrow a_i^2 \rightarrow a_i^3 \rightarrow \ldots$ is equivalent to the ray $a_1 \rightarrow a_1^2 \rightarrow a_1^3 \rightarrow \ldots$ as $a_i^j a_1 = a_1^{j+1}$ and $a_1^j a_i = a_i^{j+1}$. By the pigeonhole principle any other ray must contain infinitely many vertices of the form a_i^j for some fixed i and hence $|\Omega_r(S)| = 1$. However, in the left Cayley graph sa_i is of the form a_i^j for all $s \in S$ hence S has n ends each corresponding to a ray of the form $a_i \rightarrow a_i^2 \rightarrow a_i^3 \rightarrow \ldots$. For a portion of both the right and left Cayley graphs see Figure 5.59.

If we restrict the question to cancellative semigroups we get a positive answer when the semigroup has one end.

Lemma 5.60. *Let S be a cancellative semigroup. Then S has one right end if and only if S has one left end.*

Proof. We prove that if S has one right end then S has one left end and the other direction follows by symmetry. The result follows because if S is

Figure 5.2: A portion of the right and left Cayley graphs of $Mon\langle a_1, a_2, \ldots, a_n | a_i a_j = a_j^2 \rangle$ from Example 5.59.



cancellative then the dual semigroup S^* is also cancellative and the right Cayley graph of S^* is isomorphic to the left Cayley graph of S. As S is cancellative and has one right end we know that $|sS| = \infty$ for all $s \in S$.

We show if S has an infinite $\mathcal{R}$-class then S is a group. If S has an infinite $\mathcal{R}$-class then Lemma 5.11 states that S has precisely one infinite $\mathcal{R}$-class. As S is cancellative it is certainly left-cancellative and by Lemma 2.28 the semigroup S must have precisely one $\mathcal{R}$-class. By Theorem 5.16 finitely generated semigroups with one $\mathcal{R}$-class are right groups $G \times E$ where G is a finitely generated group and E is a finite right-zero semigroup. These semigroups are only cancellative when $|E| = 1$ and thus S is a group. The left Cayley graph of a group is isomorphic to a right Cayley graph when the generating set is symmetric, hence, the statement holds.

If S has no infinite R-class then by Theorem 5.26 S has a presentation of the form $\langle a, u_1, u_2, \ldots, u_n | u_i \cdot u_j = a^{\alpha(i,j)} u_{\beta(i,j)}, u_i \cdot a = a^{f(i)} u_{g(i)}, a^{\pi(b,c)} u_b = a^{\pi(c,b)} u_c \rangle$. We consider the relations of the form $a^{\pi(b,c)} u_b = a^{\pi(c,b)} u_c$. If $\pi(b,c) < \pi(c,b)$ then by left-cancellativity we have $u_b = a^{\pi(c,b)-\pi(b,c)} u_c$ which contradicts the definition of the u_i. Similarly we cannot have $\pi(b,c) > \pi(c,b)$. Finally if $\pi(b,c) = \pi(c,b)$ then by left-cancellativity $u_b = u_c$, so we have an unnecessary generator. It follows that we have no relations of the form $a^{\pi(b,c)} u_b = a^{\pi(c,b)} u_c$.

Let $A = \{a, u_1, \ldots, u_n\}$. We will show that in $\Gamma_l(S, A)$ there exist $N, M \in \mathbb{N}_0$ such that for any element $x \in S$ there exists $i \in \mathbb{N}_0$ such that $d_A(a^i, x), d_A(x, a^{i+M}) \leq N$. We can express any element of S in the form $a^i u_j$, we do not worry about the uniqueness of this representative in this proof. We can then express $a^i u_j$ in the form $a^p u_q a^r$ where $r \in \mathbb{N}_0$ and $0 \leq p \leq N_1$ where $N_1 = \max\{f(i) : 0 \leq i \leq n\}$ using the identity $u_i \cdot a = a^{f(i)} u_{g(i)}$. This ensures that $d_A(a^i, x) \leq N_1 + 1$ for all $x \in S$.

Assume $\beta(i,k) = \beta(j,k)$ and assume without loss of generality that $\alpha(i,k) \leq \alpha(j,k)$. It follows that $a^{\alpha(j,k)-\alpha(i,k)} u_i u_k = a^{\alpha(j,k)-\alpha(i,k)} a^{\alpha(i,k)} u_{\beta(i,k)} = a^{\alpha(j,k)} u_{\beta(i,k)} = u_j u_k$. Then by right cancellativity $a^{\alpha(j,k)-\alpha(i,k)} u_i = u_j$. As $u_j \notin aS$ we have $\alpha(i,k) = \alpha(j,k)$ and hence $u_i = u_j$. It follows that for a fixed u_k all $\beta(1,k), \beta(2,k), \ldots, \beta(n,k)$ are distinct and therefore for each $1 \leq k \leq n$ there exists $1 \leq i \leq n$ such that $u_i u_k = a^{\alpha(i,k)} u_k$. It follows from right cancellativity that $u_i = a^{\alpha(i,k)}$. But $u_i \notin aS$ so u_i is an identity element. There also exists $1 \leq j \leq n$ such that $u_j u_k = a^{\alpha(j,k)} u_i = a^{\alpha(j,k)}$. For each $1 \leq k \leq n$ we define $\tau(k)$ to be such that $u_{\tau(k)} u_k = a^{\alpha(\tau(k),k)}$.

In a similar fashion we now show that g is a permutation. Assume $g(i) = g(j)$ and assume without loss of generality $f(i) \leq f(j)$. It follows $a^{f(j)-f(i)} u_i a = a^{f(j)} u_{g(i)} = u_j a$. Then by right cancellativity $a^{f(j)-f(i)} u_i = u_j$. As $u_j \notin aS$ we have $f(i) = f(j)$ and $u_i = u_j$, and hence g is an injection.

For any $a^p u_b$ we have

$$
\begin{aligned}
u_{g^{-p}(\tau(b))} a^p u_b &= a^{f(g^{-p}(\tau(b)))} u_{g^{p-1}(\tau(b))} a^{p-1} u_b \\
&\vdots \\
&= a^{\sum f(g^{-i}(\tau(b)))} u_{\tau(b)} u_b \\
&= a^{\sum f(g^{-i}(\tau(b))) + \alpha(\tau(b),b)}.
\end{aligned}
$$

We set $M_1 = N_1 \max\{f(i)\}$ and $M_2 = \max\{\alpha(\tau(b), b) \in \mathbb{N}_0 : u_{\tau(b)} u_b = a^{\alpha(\tau(b),b)}\}$. Then $d_A(a^p u_b a^r, a^{r+M_1+M_2}) \leq 1$. Then by Lemma 5.19 it follows that $\Gamma_l(S, A)$ has one end. □

5.4 Cardinality Questions

As mentioned in Section 4.4, it is not known what cardinalities $\Omega(S)$ can have, even for restricted types of semigroups such as those which are left cancellative. We prove that $\Omega(S)$ has cardinality $2^{\aleph_0}$ for a particular type of cancellative semigroup. The following is known as Ore's Theorem.

Theorem 5.61. *[9, Theorem 1.23] Let S be a cancellative semigroup. If $sS \cap tS \neq \emptyset$ for all $s, t \in S$ then S can be embedded in a group.*

Theorem 5.62. *A cancellative semigroup which cannot be embedded in a group has $2^{\aleph_0}$ ends.*

Proof. Let S be a cancellative semigroup that cannot be embedded in a group. As S is not group-embeddable there exist $s, t \in S$ such that $sS \cap tS = \emptyset$. Let $\cdot$ denote the operation in S and let $\psi : \{s, t\}^* \to S$ be the homomorphism defined by $\psi(u_1 u_2 \dots u_n) = u_1 \cdot u_2 \cdot \dots \cdot u_n$. We now show that ψ is an injection. Let $u = u_1 u_2 \dots u_n, v = v_1 v_2 \dots v_m \in \{s, t\}^*$ and assume $\psi(u) = \psi(v)$, without loss of generality we assume the length of u is less than or equal to the length of v.

If u is a prefix of v then $v = uv'$. It follows that $\psi(u)x = \psi(uv')x = \psi(u)\psi(v')x$ for all $x \in S$ but S is cancellative so $x = \psi(v')x$ for all $x \in S$. Hence, $\psi(v')$ is a left identity for all elements of S. The first letter of v' is (without loss of generality) s and hence $tx = \psi(v')tx \in sS$ for all $x \in S$, a contradiction.

If u is not a prefix of v then there exists a position $i \leq n$ such that $u_j = v_j$ for all $j < i$ but $u_i \neq v_i$. As $u_j = v_j$ for all $j < i$ and $\psi(u) = \psi(v)$ it follows by left-cancellativity that $\psi(u_i \dots u_n) = \psi(v_i \dots v_m)$ and $u_i \neq v_i$, however,

$sS \cap tS = \emptyset$, a contradiction.

We now show that for $u, v \in \{s,t\}^*$ we have $\psi(v) \in \psi(u)S$ if and only if u is a prefix of v. Clearly if u is a prefix of v then $\psi(v) \in \psi(u)S$. With the aim of getting a contradiction assume that $u = u_1u_2 \ldots u_n$ is not a prefix of $v = v_1v_2 \ldots v_m$ but $\psi(v) \in \psi(u)S$. This means there exists $x \in S$ such that $\psi(u)x = \psi(v)$. As u is not a prefix of v there exists $1 \leq i \leq n$ such that $u_j = v_j$ for all $j < i$ but $u_i \neq v_j$. But then by left-cancellativity $\psi(u_i \ldots u_n)x = \psi(v_i \ldots v_m)$. Then as $\{u_i, v_i\} = \{s,t\}$ it follows that $sS \cap tS \neq \emptyset$.

As ψ is an injection there is a copy T of the free semigroup on two generators as a subsemigroup of S. It follows that for any $u, v \in T$ there only exists $s \in S$ such that $u \cdot s = v$ if uw is equal to v as a word over $\{s,t\}$. It follows that any two rays $s \to s_1 \to s_2 \to$ and $s \to t_1 \to t_2$ are either equal or inequivalent. Thus S has at least $2^{\aleph_0}$ ends. This is also the maximum possible number of ends so $|\Omega(S)| = 2^{\aleph_0}$. $\square$

Chapter 6

Further Work and Open Questions

In this final chapter we will reiterate open questions mentioned in the thesis and possible further directions for research in this area.

The main aim when considering ends of semigroups is, of course, to obtain a result analogous to Stallings' classification of groups. Some questions which may provide an avenue to approach this problem are given here.

The first question pertains to the possible cardinality of the end poset of a semigroup. For this we obviously do not assume the Continuum Hypothesis. It follows there are possibly cardinals between $\aleph_0$ and $2^{\aleph_0}$.

Question 6.1. *Let Γ be an out locally finite digraph. If $|\Omega(\Gamma)|$ is uncountable* *then is the cardinality of the poset of ends of* Γ *necessarily* $2^{\aleph_0}$*?*

If this is not the case then we could ask the more specific question:

Question 6.2. *Let* S *be a finitely generated semigroup. If the cardinality of the poset of ends of* S *is uncountable then is the cardinality of the poset of ends necessarily* $2^{\aleph_0}$*?*

This is true when S is a group and when all ends of S are incomparable, see Section 4.4. One may be able to answer this by considering the end topology of S along with properties arising from the semigroup. If this is not the case then one could ask which cardinalities is it possible to get.

Question 6.3. *Given a cardinal* $\aleph_0 < \kappa < 2^{\aleph_0}$ *does there exist a semigroup* S *such that* $|\Omega(S)| = \kappa$*?*

In Section 5.4 it was demonstrated that a cancellative semigroup which cannot be embedded in a group has $2^{\aleph_0}$ ends. This may be a step in understanding the possible cardinalities one may obtain from a cancellative semigroup. This would be a different approach to the previous question as it may not involve using the end topology, just the algebraic structure.

Question 6.4. *Let* S *be a cancellative semigroup. If the cardinality of the poset of ends of* S *is uncountable then is the cardinality of the poset of ends necessarily* $2^{\aleph_0}$*?*

A reasonable approach to this may be by considering the number of ends of the group G in which the semigroup S embeds. By this we mean the group SS^{-1} which is constructed in the proof of Ore's Theorem, see for instance [9, Theorem 1.23].

Conjecture 6.5. *If S is a cancellative semigroup and has one end then it is either a group with one end or embeds in a group with two ends.*

If S is a semigroup with one end and an infinite $\mathcal{R}$-class then as S is cancellative it has only one $\mathcal{R}$-class and is therefore a group. It would remain to consider semigroups with no infinite $\mathcal{R}$-class. One might approach this by considering the presentation given by semigroups with one end and no infinite $\mathcal{R}$-classes. In further generality we may consider those semigroups which embed in a group with more than one end. These could be considered to be those cancellative semigroups with "tree-like" structure.

Question 6.6. *Let S be a cancellative semigroup. If S embeds in the group $G = SS^{-1}$ and G has $2^{\aleph_0}$ ends then does S have $2^{\aleph_0}$ ends?*

Another approach to classifying semigroups in terms of their end posets could be by classifying those semigroups with a given end poset. Firstly, one must consider which posets can arise as the end poset of a semigroup.

Question 6.7. *Does there exist a finite poset P such that there is no finitely generated semigroup S with $\Omega(S) \cong P$?*

If such a poset did exist then one may consider if this poset could arise as a subposet of a semigroup.

Question 6.8. *Given a poset P which does not arise as the end poset of any semigroup does there exist a semigroup S such that P is isomorphic to an induced subposet of $\Omega(S)$?*

Following on from this one may ask if the possible end posets of all semigroups are in some sense "large" in the set of the posets. Let P be a poset. We say the set of all posets in which P is *forbidden* is the set of all posets that do not have P as an induced subposet. We consider a subset $\mathcal{P}$ of the set of all posets to be large if $\mathcal{P}$ can be described by a finite set of forbidden posets.

Question 6.9. *Can the set of all attainable posets of ends of semigroups be given by a finite set of forbidden posets?*

The previous three questions may also be considered in terms of which posets can arise as the $\mathcal{R}$-class poset of a semigroup.

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