Multistable processes and localisability

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Abstract

We use characteristic functions to construct α-multistable measures and integrals, where the measures behave locally like stable measures, but with the stability index $\alpha(x)$ varying with x. This enables us to construct α-multistable processes on $\mathbb{R}$, that is processes whose scaling limit at time t is an $\alpha(t)$-stable process. We present several examples of such multistable processes and examine their localisability.

1 Introduction

Stochastic processes where the local regularity varies with a parameter t (usually time) are useful both theoretically and for practical applications. A well-known example is multifractional Brownian motion, where the Hurst exponent h of fractional Brownian motion is replaced by a functional parameter $h(t)$, permitting the Hölder exponent to vary in a prescribed way, see [2, 3, 5, 12] and references therein. Close to time t, the process behaves like index-$h(t)$ fractional Brownian motion, but, nevertheless, this local form may vary considerably with time. This variability is suited to modelling phenomena where the volatility is time dependent, for example in financial or medical data.

However, there are situations where non-Gaussian processes may be more suitable. In particular, stable processes allow the possibility of divergent moments and discontinuities in sample paths. For example, linear fractional stable motion, see (4.5), involves both a self-similarity parameter which relates to the degree of long range dependence of the process and a stability parameter which reflects the heaviness of the tails of the marginal distributions. Such processes are used to model, for example, financial data, epileptic episodes in EEG, internet traffic, noise on telephone lines, signal processing and atomospheric noise, see [14] for many references, but again the nature of irregularity, including the stability level, may vary. Thus it is natural to set up α-multistable processes, where $\alpha : \mathbb{R} \to \mathbb{R}^+$, which behave locally like $\alpha(t)$-stable processes close to time t, in the sense that the local scaling limits are $\alpha(t)$-stable processes, but where the stability index $\alpha(t)$ varies with t.

For a fixed α_0 there are several ways of constructing α_0-stable processes, that is stochastic processes such that the finite-dimensional distributions of the process at any finite set of m times is an m-dimensional α_0-stable vector, see [17]. Similarly, a number of constructions for multistable processes have been given recently, generalizing the various constructions of stable processes. One approach is based on Poisson point process [8] and another is based on sums of random series [11]. Here we first use characteristic functions to construct multistable integrals and measures. We show that these multistable measures behave locally like stable measures and may be approximated by sums of many independent α_0-stable measures defined on short intervals but with differing constants α_0. We then define multistable processes in terms of multistable integrals and give sufficient conditions for such processes to be localisable or strongly localisable, that is to have a local scaling limit at time t that is an $\alpha(t)$-stable process. It turns out that these multistable processes differ significantly from those of [8, 11] in the nature of their finite-dimensional distributions. We give a range of examples of these processes.

2 Definition of $\alpha(x)$-multistable measure and integral

Throughout this paper, for given $0 < a \le b < \infty$, the function $\alpha : \mathbb{R} \to [a,b]$ will be a Lebesgue measurable function that will play the role of a varying stability index. We will work with various linear spaces of measurable functions on $\mathbb{R}$. For $0 < p < \infty$ let

$$\mathcal{F}_p := \{f : f \text{ is measurable with } \|f\|_p < \infty\} \text{ where } \|f\|_p = \left(\int |f(x)|^p dx\right)^{1/p};$$

thus $\|\cdot\|_p$ is a quasinorm (i.e. there is a weak triangle inequality $\|f+g\|_p \le k(\|f\|_p + \|g\|_p)$ for some $k > 0$) which becomes a norm if $1 \le p < \infty$. It is convenient to write

$$|f(x)|^{a,b} := \max\left\{|f(x)|^a, |f(x)|^b\right\},$$

and to define the space of the functions

$$\mathcal{F}_{a,b} := \mathcal{F}_a \cap \mathcal{F}_b = \{f : f \text{ is measurable with } \int |f(x)|^{a,b} dx < \infty\}.$$

We also define variable exponent Lebesgue spaces (special cases of Orlicz spaces, see for example [6]) by

$$\mathcal{F}_\alpha := \{f : f \text{ is measurable with } \|f\|_\alpha < \infty\} \text{ where } \|f\|_\alpha := \left\{\lambda > 0 : \int \left|\frac{f(x)}{\lambda}\right|^{\alpha(x)} dx = 1\right\},$$

where $\alpha : \mathbb{R} \to [a,b]$. Then $\|\cdot\|_\alpha$ is a quasinorm that reduces to $\|\cdot\|_p$ if $\alpha(x) \equiv p$ is constant, and is a norm if $1 \le a \le \alpha(x) \le b$ for all x.

Note that with $a \le \alpha(x) \le b$ we have $\mathcal{F}_\alpha \subseteq \mathcal{F}_{a,b}$ with

$$\|f\|_\alpha \le c_{a,b} \max\left\{\|f(x)\|_a, \|f(x)\|_b\right\},$$

where $c_{a,b}$ depends only on a and b.

We will define the multistable stochastic integral $I(f)$ of a function $f \in \mathcal{F}_\alpha$ by specifying the finite-dimensional distributions of I as a stochastic process on the space of functions $\mathcal{F}_{a,b}$ and then using the Kolmogorov Extension Theorem to show that the process is well-defined.

Given $f_1, f_2, ..., f_d \in \mathcal{F}_\alpha$, the following proposition shows that we can define a probability distribution on the vector $(I(f_1), I(f_2), ..., I(f_d)) \in \mathbb{R}^d$ by the characteristic function $\phi_{f_1,...f_d}$ given by (2.1). The essential point is that $\alpha(x)$ may vary with x.

Lemma 2.1. *Let $d \in \mathbb{N}$ and $f_1, f_2, ..., f_d \in \mathcal{F}_\alpha$, where $0 < a \leq \alpha(x) \leq 2$ for all $x \in \mathbb{R}$. Then*

$$\begin{aligned}\phi_{f_1,...f_d}(\theta_1, ..., \theta_d) &= \mathbb{E}\Big(\exp i \sum_{j=1}^{d} \theta_j I(f_j)\Big) \\ &= \exp\Big\{-\int \big|\sum_{j=1}^{d} \theta_j f_j(x)\big|^{\alpha(x)} dx\Big\} \qquad (2.1)\end{aligned}$$

for $(\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$, is the characteristic function of a probability distribution on the random vector $(I(f_1), I(f_2), ..., I(f_d))$.

Proof. First, assume that $\alpha(x)$ is given by the simple function

$$\alpha(x) = \sum_{k=1}^{m} \alpha_k \mathbf{1}_{A_k}(x), \qquad (2.2)$$

where $\alpha_k \in (0,2]$ are constants, and A_k are disjoint Lebesgue measurable sets with $\cup_{k=1}^{m} A_k = \mathbb{R}$.

For $(\theta_1, \ldots, \theta_d) \in \mathbb{R}^d$

$$\begin{aligned}\exp\Big\{-\int \big|\sum_{j=1}^{d} \theta_j f_j(x)\big|^{\alpha(x)} dx\Big\} &= \exp\Big\{-\sum_{k=1}^{m}\int \big|\sum_{j=1}^{d} \theta_j f_j(x)\mathbf{1}_{A_k}(x)\big|^{\alpha_k} dx\Big\} \\ &= \prod_{k=1}^{m} \exp\Big\{-\int \big|\sum_{j=1}^{d} \theta_j f_j(x)\mathbf{1}_{A_k}(x)\big|^{\alpha_k} dx\Big\}. \qquad (2.3)\end{aligned}$$

Now, $\exp\big\{-\int \big|\sum_{j=1}^{d} \theta_j f_j(x)\mathbf{1}_{A_k}(x)\big|^{\alpha_k} dx\big\}$ is the characteristic function of the α_k-stable symmetric random vector $(I(f_1\mathbf{1}_{A_k}), \ldots, I(f_d\mathbf{1}_{A_k}))$, see [17, Equation (3.2.2)]. Hence (2.3) is the product of the characteristic functions of m α_k-stable random vectors and so is the characteristic function of a d-dimensional random vector given by the independent sum of α_k-stable random vectors. Hence (2.1) is a valid characteristic function of a random vector $(I(f_1), \ldots, I(f_n))$ in the case when $\alpha(x)$ is a simple function (2.2).

Now let $0 < a \leq \alpha(x) \leq 2$ be measurable. Given $f_1, \ldots f_d \in \mathcal{F}_\alpha$ write $A = \{x : \sum_{j=1}^{d} |f_j(x)| \leq 1\}$. Take a sequence of simple functions $\alpha_p : \mathbb{R} \to (0,2]$, $(p = 1, 2\ldots.)$ such that $\alpha_p(x) \to \alpha(x)$ pointwise almost everywhere; we may assume that $\alpha_p(x) \geq \alpha(x)$ if $x \in A$ and $\alpha_p(x) \leq \alpha(x)$ if $x \notin A$, for all x and p. Then

$$\begin{aligned}\big|\sum_{j=1}^{d} \theta_j f_j(x)\big|^{\alpha_p(x)} &\leq \max_l \big\{\max\{|\theta_l|^a, |\theta_l|^2\}\big\}\Big(\sum_{j=1}^{d} |f_j(x)|\Big)^{\alpha_p(x)} \\ &\leq \max_l \big\{\max\{|\theta_l|^a, |\theta_l|^2\}\big\}\Big(\sum_{j=1}^{d} |f_j(x)|\Big)^{\alpha(x)},\end{aligned}$$

an expression that is integrable since $f_1,\dots f_d \in \mathcal{F}_\alpha$. By the dominated convergence theorem,

$$\int \Big|\sum_{j=1}^{d} \theta_j f_j(x)\Big|^{\alpha_p(x)} dx \to \int \Big|\sum_{j=1}^{d} \theta_j f_j(x)\Big|^{\alpha(x)} dx, \tag{2.4}$$

and so

$$\exp\Big\{\int \Big|\sum_{j=1}^{d} \theta_j f_j(x)\Big|^{\alpha_p(x)} dx\Big\} \to \exp\Big\{\int \Big|\sum_{j=1}^{d} \theta_j f_j(x)\Big|^{\alpha(x)} dx\Big\}, \tag{2.5}$$

as $p \to \infty$, for all $\theta_1,\dots,\theta_d \in \mathbb{R}$.

For $f_1,\dots,f_d \in \mathcal{F}_\alpha$,

$$\int \Big|\sum_{j=1}^{d} \theta_j f_j(x)\Big|^{\alpha(x)} dx \le \max_j \max\{|\theta_j|^a, |\theta_j|^2\} \sum_{j=1}^{d} |f_j(x)|^{\alpha(x)} \to 0$$

as $\max_j\{|\theta_j|\} \to 0$. Thus (2.1) is continuous at 0. Moreover from (2.3) $\exp\big\{-\int \big|\sum_{j=1}^{d} \theta_j f_j(x)\big|^{\alpha_p(x)} dx\big\}$ is a valid characteristic function of a d-dimensional random vector for all p. Applying Lévy's continuity theorem to (2.5), there is a probability distribution on the random vector $(I(f_1), I(f_2), ..., I(f_d))$, with characteristic function given by (2.1). □

As with α_0-stable integrals for constant α_0, see [17], Kolmogorov's extension theorem allows us to define $\alpha(x)$-stable integrals consistently on $\mathcal{F}_\alpha$. (Note that we use the variable 'x' here; whilst it might be thought of as a 'time' we reserve 't' for the time variable in processes defined as integrals with respect to $\alpha(x)$-stable measures in Section 4.)

Theorem 2.2. *Let $0 < a \le \alpha(x) \le 2$. There exists a stochastic process $\{I(f), f \in \mathcal{F}_\alpha\}$ with finite-dimensional distributions given by (2.1), that is with $\phi_{I(f_1),\dots,I(f_d)} = \phi_{f_1,\dots,f_d}$ for all $f_1,\dots,f_d \in \mathcal{F}_\alpha$.*

Proof. For $f_1,\dots,f_d \in \mathcal{F}_\alpha$ it follows from (2.1) that, for any permutation $(\pi(1), \pi(2), ..., \pi(d))$ of $(1,2,...,d)$, we have

$$\phi_{f_{\pi(1),\dots,\pi(d)}}(\theta_{\pi(1)}, ..., \theta_{\pi(d)}) = \phi_{f_1,\dots,f_d}(\theta_1, ..., \theta_d),$$

and also that, for any $n \le d$,

$$\phi_{f_1,\dots,f_n}(\theta_1, ..., \theta_n) = \phi_{f_1,\dots,f_n,\dots,f_d}(\theta_1, ..., \theta_n, 0, ..., 0).$$

Thus the probability distributions given by (2.1) satisfy the consistency conditions for Kolmogorov's Extension Theorem, so, applying this theorem to the space of functions $\mathcal{F}_\alpha$, there is a stochastic process on $\mathcal{F}_\alpha$ which we denote by $\{I(f), f \in \mathcal{F}_\alpha\}$, whose finite-dimensional distributions are given by the characteristic functions (2.1). □

We call $I(f)$ the α*-multistable integral* or $\alpha(x)$*-multistable integral* of f. By applying (2.1) with functions $(a_1 f_1 + a_2 f_2)$, f_1, f_2 and variables θ, $-a_1\theta$, $-a_2\theta$ it follows that the multistable integral is linear, that is if $f_1, f_2 \in \mathcal{F}_\alpha$ and $a_1, a_2 \in \mathbb{R}$, then

$$I(a_1 f_1 + a_2 f_2) = a_1 I(f_1) + a_2 I(f_2) \quad \text{a.s.} \tag{2.6}$$

Let L be Lebesgue measure on $\mathbb{R}$, let $\mathcal{E}$ be the Lebesgue measurable subsets of $\mathbb{R}$ and let $\mathcal{E}_0 = \{A \in \mathcal{E} : L(A) < \infty\}$ be the sets of finite Lebesgue measure. Let $\alpha : \mathbb{R} \to [a,b]$ be measurable where $0 < a \le b \le 2$. Analogously to [17, Section 3.3] for α_0-stable measures, we define the α-*multistable random measure* M_α by

$$M_\alpha(A) := I(\mathbf{1}_A) \tag{2.7}$$

for $A \in \mathcal{E}_0$, where $\mathbf{1}_A$ is the indicator function of the set A; thus $M_\alpha(A)$ is a random variable for each $A \in \mathcal{E}_0$.

It is natural to write

$$\int f(x)dM_\alpha(x) := I(f), \quad f \in \mathcal{F}_\alpha, \tag{2.8}$$

since there are many analogues to usual integration with respect to a measure. Clearly, linearity of this integral is a restatement of (2.6), and

$$\int \mathbf{1}_A(x)dM_\alpha(x) = M_\alpha(A).$$

With this notation the characteristic function (2.1) may be written as

$$\mathbb{E}\Big(\exp i\Big\{\sum_{j=1}^d \theta_j \int f_j(x)dM_\alpha(x)\Big\}\Big) = \exp\Big\{-\int \Big|\sum_{j=1}^d \theta_j f_j(x)\Big|^{\alpha(x)} dx\Big\} \tag{2.9}$$

for $f_j \in \mathcal{F}_\alpha$. For the random measures, taking $f_j = \mathbf{1}_{A_j}$ with $A_j \in \mathcal{E}_0$,

$$\mathbb{E}\Big(\exp i\Big\{\sum_{j=1}^d \theta_j M_\alpha(A_j)\Big\}\Big) = \exp\Big\{-\int \Big|\sum_{j=1}^d \theta_j \mathbf{1}_{A_j}(x)\Big|^{\alpha(x)} dx\Big\}. \tag{2.10}$$

We may estimate the moments of an α-multistable integral in terms of the norm $\|\cdot\|_\alpha$.

Proposition 2.3. *Let $0 < a \le \alpha(x) \le b \le 2$ and let $g \in \mathcal{F}_\alpha$. Then there is a number c_1 depending only on a and b such that for all $\lambda > 0$*

$$\mathbb{P}\Big(\Big|\int g(x)dM_\alpha(x)\Big| \ge \lambda\Big) \le c_1 \int \Big|\frac{g(x)}{\lambda}\Big|^{\alpha(x)} dx. \tag{2.11}$$

Moreover, if $0 < p < \inf_{x\in\mathbb{R}} \alpha(x)$ there is a number c_2 depending only on p and b such that

$$\mathbb{E}\Big(\Big|\int g(x)dM_\alpha(x)\Big|^p\Big) \le c_2 \|g\|_\alpha^p. \tag{2.12}$$

Proof. A simple calculation using distribution functions (see [4, p.47]) gives

$$\begin{aligned}
\mathbb{P}\Big(\Big|\int g(x)dM_\alpha(x)\Big| \ge \lambda\Big) &\le \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(1 - \mathbb{E}\Big(\exp\big(i\theta\int g(x)dM_\alpha(x)\big)\Big)\Big)d\theta \\
&= \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(1 - \exp\big(-\int |\theta g(x)|^{\alpha(x)}dx\big)\Big)d\theta \\
&\le \frac{\lambda}{2}\int_{-2/\lambda}^{2/\lambda}\Big(\int |\theta|^{\alpha(x)}|g(x)|^{\alpha(x)}dx\Big)d\theta \\
&\le c_1 \int \Big|\frac{g(x)}{\lambda}\Big|^{\alpha(x)} dx.
\end{aligned}$$

Assuming as we may that $c_1 \geq 1$ and writing $\lambda_0 = \|g\|_\alpha > 0$ for the number such that $\int \lambda_0^{-\alpha(x)} |g(x)|^{\alpha(x)} dx = 1$, we have

$$\begin{aligned}
\mathbb{E}\Big(\Big|\int g(x)dM_\alpha(x)\Big|^p\Big) &= p\int_0^\infty \lambda^{p-1}\mathbb{P}\Big(\Big|\int g(x)dM_\alpha(x)\Big| \geq \lambda\Big)d\lambda \\
&\leq c_1 p\int_0^\infty \lambda^{p-1}\min\Big\{1, \int \lambda^{-\alpha(x)}|g(x)|^{\alpha(x)}dx\Big\}d\lambda \\
&\leq c_1 p\int_0^{\lambda_0} \lambda^{p-1}d\lambda + c_1 p\int\int_{\lambda_0}^\infty \lambda^{p-1-\alpha(x)}|g(x)|^{\alpha(x)}d\lambda dx \\
&\leq c_3\lambda_0^p + c_3\lambda_0^p\int \lambda_0^{-\alpha(x)}|g(x)|^{\alpha(x)}dx \\
&= c_2\|g\|_\alpha^p.
\end{aligned}$$

□

Note that Ayache [1] has recently pointed out that the right hand side of (2.12) essentially characterizes the tail behaviour of the multistable integrals, from which it follows that $\|g\|_\alpha < \infty$ is also a necessary condition for $\int g(x)dM_\alpha(x)$ to have a finite pth moment.

Recall that a random measure M on $\mathbb{R}$ is *independently scattered* if $M(A_1), M(A_2), \ldots, M(A_d)$ are independent whenever $A_1, A_2, \ldots, A_d \in \mathcal{E}_0$ are pairwise disjoint, and is σ*-additive* if whenever $A_1, A_2, \ldots \in \mathcal{E}_0$ are disjoint and $\bigcup_{j=1}^\infty A_j \in \mathcal{E}_0$ then almost surely

$$M\Big(\bigcup_{j=1}^\infty A_j\Big) = \sum_{j=1}^\infty M(A_j),$$

taking an independent sum.

Theorem 2.4. *The α-multistable measure M_α is independently scattered and σ-additive.*

Proof. This is a slight variant of [17, Section 3.3]. Let $A_1, A_2, \ldots, A_k \in \mathcal{E}_0$ be pairwise disjoint. Then using (2.10)

$$\mathbb{E}\Big(\exp\Big\{i\sum_{j=1}^d \theta_j M_\alpha(A_j)\Big\}\Big) = \prod_{j=1}^d \exp\Big\{-\int \big|\theta_j \mathbf{1}_{A_j}(x)\big|^{\alpha(x)}dx\Big\} = \prod_{j=1}^d \mathbb{E}(\exp\{i\theta_j M_\alpha(A_j)\}).$$

so $M_\alpha(A_1), M_\alpha(A_2), \ldots, M_\alpha(A_d)$ are independent, and M_α is independently scattered.

If $A_1, A_2, \ldots, A_k \in \mathcal{E}_0$ is a finite collection of disjoint sets, using (2.7) and (2.6),

$$M_\alpha\Big(\bigcup_{j=1}^k A_j\Big) = I\big(\mathbf{1}_{\cup_{j=1}^k A_j}\big) = I\Big(\sum_{j=1}^k \mathbf{1}_{A_j}\Big) = \sum_{j=1}^k I(\mathbf{1}_{A_j}) = \sum_{j=1}^k M_\alpha(A_j).$$

For a countable family of disjoint sets $A_1, A_2, \ldots \in \mathcal{E}_0$ with $B \equiv \bigcup_{j=1}^\infty A_j \in \mathcal{E}_0$, so that $B = \bigcup_{j=1}^k A_j \cup \Big(\bigcup_{j=k+1}^\infty A_j\Big)$, it follows from above that

$$M_\alpha(B) = M_\alpha\Big(\bigcup_{j=1}^k A_j\Big) + M_\alpha\Big(\bigcup_{j=k+1}^\infty A_j\Big) = \sum_{j=1}^k M_\alpha(A_j) + M_\alpha\Big(\bigcup_{j=k+1}^\infty A_j\Big). \tag{2.13}$$

Since $\lim_{k\to\infty} \mathcal{L}(\bigcup_{j=k+1}^{\infty} A_j) = 0$ and $\alpha(x) \in [a,b]$, for each $\theta \in \mathbb{R}$

$$\lim_{k\to\infty} \mathbb{E}\Big(\exp i\big\{ \theta M_\alpha(\bigcup_{j=k+1}^{\infty} A_j)\big\}\Big) = \lim_{k\to\infty} \exp\Big\{ -\int |\theta \mathbf{1}_{\cup_{j=k+1}^{\infty} A_j}(x)|^{\alpha(x)} dx \Big\} = 1,$$

so $M_\alpha(\bigcup_{j=k+1}^{\infty} A_j) \stackrel{\mathrm{d}}{\to} 0$ as $k \to \infty$ by Lévy's Continuity Theorem.

By (2.13) we get $M_\alpha(B) - \sum_{j=1}^{k} M_\alpha(A_j) \stackrel{\mathrm{d}}{\to} 0$ and so $M_\alpha(B) - \sum_{j=1}^{k} M_\alpha(A_j) \stackrel{\mathrm{p}}{\to} 0$ as $k \to \infty$. Thus $\lim_{k\to\infty} \sum_{j=1}^{k} M_\alpha(A_j) \stackrel{p}{=} M_\alpha(B)$, and, since the summands $M_\alpha(A_j)$ are independent, this implies convergence almost surely, by a theorem of Kolmogorov, see [10, Theorem 6.1]. Thus M_α is σ-additive. □

Next we obtain conditions for convergence of a sequence of multistable measures with different multistable indexes.

Proposition 2.5. *Let $0 < a \le b \le 2$ and $\alpha_n, \alpha : \mathbb{R} \to [a,b]$ be Lebesgue measurable. Let M_{α_n}, M_α be the associated α_n-multistable and α-multistable measures characterised by (2.10). Suppose $\alpha_n(x) \to \alpha(x)$ for almost all $x \in \mathbb{R}$. Then $M_{\alpha_n} \stackrel{\mathrm{fdd}}{\to} M_\alpha$ as $n \to \infty$, that is for all $m \in \mathbb{N}$ and $A_1, A_2, \ldots, A_m \in \mathcal{E}_0$,*

$$(M_{\alpha_n}(A_1), M_{\alpha_n}(A_2), \ldots, M_{\alpha_n}(A_m)) \stackrel{\mathrm{d}}{\to} (M_\alpha(A_1), M_\alpha(A_2), \ldots, M_\alpha(A_m)).$$

Proof. Let A_1, A_2,...,$A_m \in \mathcal{E}_0$. Then for all n and all $x \in \mathbb{R}$

$$\Big|\sum_{j=1}^{m} \theta_j \mathbf{1}_{A_j}(x)\Big|^{\alpha_n(x)} \le c \mathbf{1}_A(x)$$

where $A = \bigcup_{j=1}^{m} A_j \in \mathcal{E}_0$ and $c = \max\big\{ \big(\sum_{j=1}^{m} |\theta_j|\big)^a, \big(\sum_{j=1}^{m} |\theta_j|\big)^b \big\}$. Since $\int \mathbf{1}_A(x)dx < \infty$, the dominated convergence theorem implies that

$$\lim_{n\to\infty} \exp\Big(-\int \Big| \sum_{j=1}^{m} \theta_j \mathbf{1}_{A_j}(x)\Big|^{\alpha_n(x)} dx\Big) = \exp\Big(-\int \Big| \sum_{j=1}^{m} \theta_j \mathbf{1}_{A_j}(x)\Big|^{\alpha(x)} dx\Big),$$

so from (2.10),

$$\mathbb{E}\Big(\exp i\big\{ \sum_{j=1}^{m} \theta_j M_{\alpha_n}(A_j)\big\}\Big) \to \mathbb{E}\Big(\exp i\big\{ \sum_{j=1}^{m} \theta_j M_\alpha(A_j)\big\}\Big)$$

as $n \to \infty$. By Lévy's continuity theorem $M_{\alpha_n} \stackrel{\mathrm{fdd}}{\to} M_\alpha$. □

To get a feel for α-mutistable measures, we show that, for a continuous α, the α-multistable measure M may be approximated by random measures that are the sum of many independent α_0-stable measures defined on short intervals but with differing constants α_0.

Assume that $\alpha : \mathbb{R} \to [a,b] \subset (0,2]$ is continuous and let M_α be the α-multistable measure on the sets $\mathcal{E}_0$. We now use the same procedure but with piecewise constant functions $\alpha_n : \mathbb{R} \to [a,b]$ to obtain approximating measures M_{α_n}.

For each n, let $\alpha_n : \mathbb{R} \to [a,b] \subset (0,2)$ be given by

$$\alpha_n(x) = \alpha(r2^{-n}) \text{ if } x \in [r2^{-n},(r+1)2^{-n}) \text{ for all } r \in \mathbb{Z}$$

and let M_{α_n} be the α_n-multistable measure obtained from α_n as above, so in particular M_{α_n} has finite-dimensional distributions given by the characteristic function

$$\mathbb{E}\Big(\exp i\Big\{\sum_{j=1}^{d}\theta_j M_{\alpha_n}(A_j)\Big\}\Big) = \exp\Big\{-\int\Big|\sum_{j=1}^{d}\theta_j \mathbf{1}_{A_j}(x)\Big|^{\alpha_n(x)}dx\Big\}.$$

It follows from Theorem 2.4 that each M_{α_n} is independently scattered and σ-additive.

Theorem 2.6. *Let $0 < a \le b \le 2$ and $\alpha : \mathbb{R} \to [a,b]$ be continuous. Let $M_{n,r}$ denote the restriction of $\alpha(r2^{-n})$-stable symmetric measure to the interval $[r2^{-n},(r+1)2^{-n}))$, that is*

$$M_{n,r}(A) := M_{\alpha(r2^{-n})}(A \cap [r2^{-n},(r+1)2^{-n})) = M_{\alpha_n}(A \cap [r2^{-n},(r+1)2^{-n})),$$

where $M_{\alpha(r2^{-n})}$ is $\alpha(r2^{-n})$-stable symmetric measure. Then M_{α_n} is a random measure given by the independent sum of random measures

$$M_{\alpha_n}(A) = \sum_{r\in\mathbb{Z}} M_{n,r}(A)$$

almost surely for $A \in \mathcal{E}_0$. Moreover $M_{\alpha_n} \stackrel{\text{fdd}}{\to} M_\alpha$ as $n \to \infty$.

Proof. Since M_{α_n} is independently scattered, we have that for each $A \in \mathcal{E}$

$$M_{\alpha_n}(A \cap [r2^{-n},(r+1)2^{-n})) = M_{n,r}(A)$$

are independent for distinct r.

Let $A \in \mathcal{E}_0$. Since M_{α_n} is σ-additive,

$$\begin{aligned} M_n(A) &= M_{\alpha_n}(A) \\ &= M_{\alpha_n}\Big(\bigcup_{r\in\mathbb{Z}} A \cap [r2^{-n},(r+1)2^{-n})\Big) \\ &= \sum_{r\in\mathbb{Z}} M_{\alpha_n}(A \cap [r2^{-n},(r+1)2^{-n})) \\ &= \sum_{r\in\mathbb{Z}} M_{n,r}(A) \end{aligned}$$

where the summands are independent.

For each n we have $\alpha_n(x) = \alpha(r2^{-n})$ for all $x \in [r2^{-n},(r+1)2^{-n})$. Since α is assumed continuous, we have $\lim_{n\to\infty}\alpha_n(x) = \alpha(x)$ for all x. Thus by Proposition 2.5, $M_{\alpha_n} \stackrel{\text{fdd}}{\to} M_\alpha$ as $n \to \infty$. □

One would expect an α-multistable measure to 'look like' an $\alpha(u)$-stable measure in a small interval around u. We now make this idea precise.

For $u \in \mathbb{R}$, $r > 0$, let $T_{u,r} : \mathbb{R} \to \mathbb{R}$ be the scaling map, $T_{u,r}(x) = (x-u)/r$. This induces a mapping $T^{\#}_{u,r}$ on random integrals and measures, given by

$$\int f(x) d(T^{\#}_{u,r} M_\alpha)(x) = \int f\Big(\frac{x-u}{r}\Big) dM_\alpha(x) \equiv I\Big(f\Big(\frac{.-u}{r}\Big)\Big).$$

In particular

$$(T^{\#}_{u,r} M_\alpha)(A) = M_\alpha(T^{-1}_{u,r}(A)) = I(\mathbf{1}_{T^{-1}_{u,r}(A)})$$

for $A \in \mathcal{E}_0$ by (2.7).

We show that scaling an α-multistable random measure about a point u yields $\alpha(u)$-stable measure $M_{\alpha(u)}$.

Theorem 2.7. *Let $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2]$ be continuous with*

$$|\alpha(x+r) - \alpha(x)| = o(1/\log r) \tag{2.14}$$

uniformly on bounded intervals and let $u \in \mathbb{R}$. Then for all functions $f_1, \ldots, f_d \in \mathcal{F}_{a,b}$ with compact support, the vectors

$$\Big(r^{-1/\alpha(u)} \int f_1(x) d(T^{\#}_{u,r} M_\alpha)(x), \ldots, r^{-1/\alpha(u)} \int f_d(x) d(T^{\#}_{u,r} M_\alpha)(x)\Big)$$
$$\xrightarrow{\mathrm{d}} \Big(\int f_1(x) dM_{\alpha(u)}(x), \ldots, \int f_d(x) dM_{\alpha(u)}(x)\Big) \tag{2.15}$$

as $r \to 0$. In particular,

$$r^{-1/\alpha(u)}\big((T^{\#}_{u,r} M_\alpha)(A_1), \ldots, (T^{\#}_{u,r} M_\alpha)(A_d)\big) \xrightarrow{\mathrm{d}} (M_{\alpha(u)}(A_1), \ldots, M_{\alpha(u)}(A_d)) \tag{2.16}$$

as $r \to 0$, for all bounded sets $A_1, \ldots, A_d \in \mathcal{E}_0$.

Proof. Let $f_1, f_2, ..., f_m \in \mathcal{F}_{a,b}$ be functions with compact support, say in $[-z_0, z_0]$. Let $\theta_j \in \mathbb{R}$, $j = 1, 2, ..., m$, and consider the characteristic functions.

$$\begin{aligned}
&\mathbb{E}\Big(\exp i \sum_{j=1}^m \theta_j r^{-1/\alpha(u)} \int f_j(x) d(T^{\#}_{u,r} M_\alpha)(x)\Big) \\
&= \mathbb{E}\Big(\exp i \sum_{j=1}^m \theta_j r^{-1/\alpha(u)} \int f_j\Big(\frac{x-u}{r}\Big) dM_\alpha(x)\Big) \\
&= \exp\Big(-\int \Big|\sum_{j=1}^m \theta_j r^{-1/\alpha(u)} f_j\Big(\frac{x-u}{r}\Big)\Big|^{\alpha(x)} dx\Big) \\
&= \exp\Big(-\int \Big|\sum_{j=1}^m \theta_j r^{-1/\alpha(u)} f_j(z)\Big|^{\alpha(rz+u)} r dz\Big) \\
&= \exp\Big(-\int \Big|\sum_{j=1}^m \theta_j f_j(z)\Big|^{\alpha(rz+u)} r^{1-\alpha(rz+u)/\alpha(u)} dz\Big),
\end{aligned} \tag{2.17}$$

on writing $(x-u)/r = z$.

From condition (2.14) it is easy to see that $\lim_{r\to 0} r^{1-\alpha(rz+u)/\alpha(u)} = 1$ uniformly for $z \in [-z_0, z_0]$, and also $\lim_{r\to 0}\alpha(rz+u) = \alpha(u)$ uniformly for all $z \in [-z_0, z_0]$ since α is continuous. Noting that there is a constant c such that for r sufficiently small,

$$|\sum_{j=1}^{m} \theta_j f_j(z)\big|^{\alpha(rz+u)} r^{1-\alpha(rz+u)/\alpha(u)} \leq c \sum_{j=1}^{m} |f_j(z)|^{a,b},$$

for $z \in [-z_0, z_0]$ and $f_j \in \mathcal{F}_{a,b}$, the dominated convergence theorem gives

$$\lim_{r\to 0} \exp\Big(-\int |\sum_{j=1}^{m} \theta_j f_j(z)\big|^{\alpha(rz+u)} r^{1-\alpha(rz+u)/\alpha(u)} dz\Big) = \exp\big(-\int |\sum_{j=1}^{m} \theta_j f_j(x)\big|^{\alpha(u)} dx\big),$$

so by (2.17)

$$\lim_{r\to 0} \mathbb{E}\Big(\exp i \sum_{j=1}^{m} \theta_j r^{-1/\alpha(u)} \int f_j(x) d(T^{\#}_{u,r} M_\alpha)(x)\Big) = \mathbb{E}\Big(\exp i \sum_{j=1}^{m} \theta_j \int f_j(x) dM_{\alpha(u)}(x)\Big).$$

Lévy's continuity theorem now implies (2.15) and (2.16). □

3 Multistable processes and localisability

In this section we introduce processes defined by multistable integrals, and in particular consider their local form, with the aim of constructing 'multistable' processes with a prescribed local form. Thus, given $\alpha : \mathbb{R} \to [a,b] \subset (0,2]$, we write

$$Y(t) := \int f(t,x) dM_\alpha(x), \tag{3.1}$$

for $t \in \mathbb{R}$ and $f \in \mathcal{F}_{a,b}$, where the integrals are with respect to an α-multistable measure M_α as in (2.8). By (2.9), for each $(t_1, t_2, \ldots, t_d) \in \mathbb{R}^d$, the characteristic function of the random vector $(Y(t_1), Y(t_2), \ldots, Y(t_d))$ is

$$\begin{aligned} \mathbb{E}\big(\exp i \sum_{j=1}^{d} \theta_j Y(t_j)\big) &= \mathbb{E}\big(\exp i \sum_{j=1}^{d} \int f(t_j, x) dM_\alpha(x)\big) \\ &= \exp\Big(-\int |\sum_{j=1}^{d} \theta_j f(t_j, x)\big|^{\alpha(x)} dx\Big) \qquad (3.2) \end{aligned}$$

for all $(\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d$.

We first give conditions for Y in (3.1) to have a continuous version.

Proposition 3.1. *Let $\alpha : \mathbb{R} \to [a,b] \subseteq (1,2]$ be measurable and suppose $f(t,\cdot) \in \mathcal{F}_\alpha$ for all $t \in \mathbb{R}$. Let Y be given by (3.1). Suppose that there exists $1/a < \eta < 1$, such that for each bounded interval J we can find $c > 0$ such that*

$$\|f(t,\cdot) - f(u,\cdot)\|_\alpha \leq c|t-u|^\eta \quad (t, u \in J). \tag{3.3}$$

Then Y has a continuous version satisfying an a.s. β-Hölder condition on each bounded interval for all $0 < \beta < (\eta a - 1)/a$. In particular, (3.3) holds if

$$\int \left|f(t,x) - f(u,x)\right|^{\alpha(x)} dx \leq c_1 |t-u|^{a\eta} \quad (t,u \in J), \tag{3.4}$$

a form that may be easier to check in practice.

Proof. Take p such that $1/\eta < p < a$. By Proposition 2.3

$$\mathbb{E}\big(|Y(t)-Y(u)|^p\big) = \mathbb{E}\left(\left|\int \big(f(t,x)-f(u,x)\big) dM_\alpha(x)\right|^p\right) \leq c_2 \|f(t,\cdot)-f(u,\cdot)\|_\alpha^p \leq c_2 c |t-u|^{\eta p}.$$

The conclusion follows from Kolmogorov's continuity theorem, see [16, Theorem 25.2]. □

Localisability has been considered by a number of authors, particularly in the context of Gaussian processes and fractional Brownian motion, see for example [3, 12] where it is termed *local asymptotic self-similarity*. Note that 'stable-like' processes were defined and studied in [13]. These processes are 'localizable' in a different sense, in that they are solutions of an $\alpha(x)$-fractional stochastic differential equation. In [13, Theorem 2.1] the local form of sample paths is considered, rather than the limiting process. Moreover, stable-like processes are Markov whereas, in general, the multistable processes considered here are not.

Here we say that a stochastic process Y is localisable at a point if it has a unique non-trivial scaling limit, formally $Y = \{Y(t) : t \in \mathbb{R}\}$ is *h-localisable* at u with *local form* or *tangent process* $Y'_u = \{Y'_u(t) : t \in \mathbb{R}\}$ if

$$\frac{Y(u+rt) - Y(u)}{r^h} \overset{\text{fdd}}{\to} Y'_u(t) \tag{3.5}$$

as $r \to 0$. If Y and Y'_u have versions in $C(\mathbb{R})$ (the space of continuous functions on $\mathbb{R}$) and convergence in (3.5) occurs in distribution with respect to the metric of uniform convergence on bounded intervals we say that Y is *strongly localisable*. For the simplest example, a self-similar process with stationary increments Y is localisable at all u with local form $Y'_u = Y$ and is strongly localisable if it has a version in $C(\mathbb{R})$. In general there are considerable restrictions on the possible local forms, see [7].

We call a stochastic process $\{Y(t), t \in \mathbb{R}\}$ *multistable* if for almost all u, Y is localisable at u with Y'_u an α-stable process for some $\alpha = \alpha(u)$, where $0 < \alpha(u) \leq 2$. Various constructions of multistable processes are given in [8, 9, 11].

For a stochastic process Y, it is natural to ask under what conditions Y is localisable. The following theorem, which is a multistable analogue of [9, Proposition 2.1], gives a sufficient condition.

Theorem 3.2. *Let*

$$Y(t) = \int f(t,x) dM_\alpha(x), \tag{3.6}$$

where M_α is an α-multistable measure for continuous $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2]$. Assume that $f(t,.) \in \mathcal{F}_{a,b}$ for all t and

$$\lim_{r\to 0} \int \left|\frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{h-1/\alpha(u+rz)}} - h(t,z)\right|^{a,b} dz = 0 \tag{3.7}$$

for a jointly measurable function $h(t,z)$ with $h(t,.) \in \mathcal{F}_{a,b}$ for all t. Then Y is h-localisable at u with local form

$$Y'_u = \left\{ \int h(t,z) dM_{\alpha(u)}(z) : t \in \mathbb{R} \right\} \tag{3.8}$$

where $M_{\alpha(u)}$ is $\alpha(u)$-stable measure.

If, in addition, there exists $\eta > 1/a$ such that for each bounded interval J we can find $c > 0$ such that

$$\left\| \frac{f(u+rt,\cdot) - f(u+rv,\cdot)}{r^h} \right\|_\alpha \leq c|t-v|^\eta \quad (t,v \in J) \tag{3.9}$$

for all sufficiently small $r > 0$, then Y is strongly localisable at u. Condition (3.9) is implied by

$$\int \left| \frac{f(u+rt,u+rz) - f(u+rv,u+rz)}{r^{h-1/\alpha(u+rz)}} \right|^{\alpha(u+rz)} dz \leq c_1 |t-v|^{a\eta} \quad (t,v \in J) \tag{3.10}$$

which can be more convenient to use in practice.

To prove Theorem 3.2, we need some convergence estimates.

Lemma 3.3. *Let $0 < a \leq b$. There is a constant c that depends only on a and b such that, for all measurable $\alpha : \mathbb{R} \to [a,b]$ and $g,k \in \mathcal{F}_{a,b}$,*

$$\begin{aligned} &\left| \int |g(x)|^{\alpha(x)} dx - \int |k(x)|^{\alpha(x)} dx \right| \\ \leq & c \left(\|g-k\|_a \|k\|_a^{\max\{0,a-1\}} + \|g-k\|_a^a + \|g-k\|_b \|k\|_b^{\max\{0,b-1\}} + \|g-k\|_b^b \right). \end{aligned} \tag{3.11}$$

Proof. If $0 < a \leq \alpha(x) \leq b \leq 1$ for all $x \in \mathbb{R}$, then

$$\left| \int |g(x)|^{\alpha(x)} dx - \int |k(x)|^{\alpha(x)} dx \right| \leq \int |g(x) - k(x)|^{\alpha(x)} dx \leq \|g-k\|_a^a + \|g-k\|_b^b.$$

On the other hand, if $1 \leq a \leq \alpha(x) \leq b$ for all $x \in \mathbb{R}$, then by the mean value theorem there exists $0 < \lambda(x) < 1$ such that

$$\begin{aligned} \left| |g(x)|^{\alpha(x)} - |k(x)|^{\alpha(x)} \right| &= \alpha(x) \Big| |g(x)| - |k(x)| \Big| \Big| |k(x)| + \lambda(x) \big(|g(x)| - |k(x)| \big) \Big|^{\alpha(x)-1} \\ &\leq b \Big| |g(x)| - |k(x)| \Big| \Big| |k(x)| + |g(x) - k(x)| \big) \Big|^{a-1} \\ &\quad + b \Big| |g(x)| - |k(x)| \Big| \Big| |k(x)| + |g(x) - k(x)| \big) \Big|^{b-1}. \end{aligned}$$

Integrating and using Hölder's inequality gives

$$\left| \int |g(x)|^{\alpha(x)} dx - \int |k(x)|^{\alpha(x)} dx \right| \leq c \left(\|g-k\|_a \Big\| |k| + |g-k| \Big\|_a^{a-1} + \|g-k\|_b \Big\| |k| + |g-k| \Big\|_b^{b-1} \right),$$

which gives (3.11) in this case.

In general, for $0 < a \leq \alpha(x) \leq b$, letting $A = \{x : a \leq \alpha(x) \leq 1\}$, inequality (3.11) holds for $g\mathbf{1}_A$ and $k\mathbf{1}_A$ and also for $g\mathbf{1}_{\mathbb{R}\setminus A}$ and $k\mathbf{1}_{\mathbb{R}\setminus A}$, and combining these cases we get (3.11) for g and k for an appropriate c. □

We require the following Corollary.

Corollary 3.4. *Let* $0 < a \leq b$ *and* $g : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *with* $g(r,.) \in \mathcal{F}_{a,b}$ *for all* $r > 0$*. Let* $k \in \mathcal{F}_{a,b}$ *and let* $\beta : \mathbb{R} \to [a,b]$ *be continuous at* 0*. If*

$$\lim_{r\to 0} \int |g(r,z) - k(z)|^{a,b} dz = 0, \tag{3.12}$$

then

$$\lim_{r\to 0} \int |g(r,z)|^{\beta(rz)} dz = \int |k(z)|^{\beta(0)} dz. \tag{3.13}$$

Proof. By (3.12) and Lemma 3.3

$$\lim_{r\to 0} \Big| \int |g(r,z)|^{\beta(rz)} dz - \int |k(z)|^{\beta(rz)} dz \Big| = 0.$$

Since $k \in \mathcal{F}_{a,b}$, the dominated convergence theorem gives

$$\lim_{r\to 0} \Big| \int |k(z)|^{\beta(rz)} dz - \int |k(z)|^{\beta(0)} dz \Big| = 0,$$

and (3.13) follows on combining these two limits. □

We can now complete the proof of Theorem 3.2.

Proof of Theorem 3.2 Fix $u \in \mathbb{R}$. We consider the characteristic function of the finite-dimensional distributions of $r^{-h}(Y(u+rt) - Y(u))$. Let $\theta_j \in \mathbb{R}$ and $t_j \in \mathbb{R}$ for $j = 1,2,...,m$. Then, using (3.6) and (2.9),

$$\begin{aligned}
&\mathbb{E}\Big(\exp i \sum_{j=1}^{m} \theta_j r^{-h} (Y(u+rt_j) - Y(u)) \Big) &(3.14)\\
&= \mathbb{E}\Big(\exp i \sum_{j=1}^{m} \theta_j r^{-h} \int (f(u+rt_j,x) - f(u,x)) dM_\alpha(x) \Big) \\
&= \exp\Big\{ -\int \Big| \sum_{j=1}^{m} \theta_j r^{-h} (f(u+rt_j,x) - f(u,x)) \Big|^{\alpha(x)} dx \Big\} \\
&= \exp\Big\{ -\int \Big| \sum_{j=1}^{m} \theta_j r^{-h+1/\alpha(rz+u)} (f(u+rt_j, rz+u) - f(u, rz+u)) \Big|^{\alpha(rz+u)} dz \Big\}, \\
& &(3.15)
\end{aligned}$$

after setting $x = rz + u$.

Defining

$$Z(t) = \int h(t,z) dM_{\alpha(u)}(z),$$

where $M_{\alpha(u)}$ is $\alpha(u)$-stable symmetric measure, its finite-dimensional distributions are given by the characteristic function

$$\mathbb{E}\Big(\exp i \sum_{j=1}^{m} \theta_j Z(t_j) \Big) = \exp\Big\{ -\int \Big| \sum_{j=1}^{m} \theta_j h(t_j,z) \Big|^{\alpha(u)} dz \Big\}. \tag{3.16}$$

We now use Corollary 3.4, taking

$$g(r,z) = \sum_{j=1}^{m} \theta_j \frac{f(u+rt_j, rz+u) - f(u, rz+u)}{r^{h-1/\alpha(rz+u)}},$$

$$k(z) = \sum_{j=1}^{m} \theta_j h(t_j, z),$$

and

$$\beta(x) = \alpha(u+x).$$

Then

$$\int |g(r,z) - k(z)|^{a,b} dz \to 0,$$

as $r \to 0$, using (3.7) and the quasi norm properties of $\|\cdot\|_a$ and $\|\cdot\|_b$. Thus by Corollary 3.4

$$\int \Big| \sum_{j=1}^{m} \theta_j r^{-h+1/\alpha(rz+u)} (f(u+rt_j, rz+u) - f(u, rz+u)) \Big|^{\alpha(rz+u)} dz \to \int \Big| \sum_{j=1}^{m} \theta_j h(t_j, z) \Big|^{\alpha(u)} dz,$$

as $r \to 0$.

Since the exponential function is continuous, (3.15), and hence (3.14), is convergent to (3.16) as $r \to 0$ for all $(\theta_1, \ldots, \theta_m)$. By Lévy's Continuity Theorem, $r^{-h}(Y(u+rt) - Y(u)) \overset{\text{fdd}}{\to} Z(t)$ as $r \to 0$, noting that (3.16) is a characteristic function. Thus Y is h-localisable with local form Y'_u given by (3.8).

Finally, if (3.9) holds then by Proposition 2.3, for $0 < p < a$,

$$\begin{aligned} \mathbb{E}(|Y_r(t) - Y_r(v)|^p) &= \mathbb{E}\Big(\Big| \int \frac{f(u+rt, x) - f(u-rv, x)}{r^h} dM_\alpha(x) \Big|^p \Big) \\ &\le c_2 \Big\| \frac{f(u+rt, \cdot) - f(u-rv, \cdot)}{r^h} \Big\|_\alpha^p \\ &\le c_3 |t-v|^\eta p. \end{aligned}$$

By choosing p such that $1/\eta < p < a$, Kolmogorov's continuity theorem, see [16, Theorem 25.2], implies that, for each $0 < \beta < (\eta a - 1)/a$ and each bounded interval J, the process Y_r satisfies an a.s. Hölder condition

$$|Y_r(t) - Y_r(v)| \le C_r |t-v|^\beta \quad (t, v \in J),$$

where the random constants behave uniformly in r, i.e, $\sup_{0<r\le r_0} \mathbb{P}(C_r \ge m) \to 0$ as $m \to \infty$. Thus for all $\varepsilon, \tau > 0$ there exists $\delta > 0$ such that

$$\limsup_{r\to 0} \mathbb{P}\Big(\sup_{|t-v|<\delta,\ t,v\in J} |Y_r(t) - Y_r(v)| > \tau \Big) < \varepsilon.$$

In other words, the Y_r are strongly stochastically equicontinuous on J which, along with convergence of the finite-dimensional distributions, implies that Y_r converges to Y' in distribution on the space of continuous functions with the metric of convergence on bounded intervals, see [4, Theorem 8.2] or[15, Theorem 10.2] . □

4 Examples

We give a number of examples to illustrate Theorem 3.2. Some of these are considered in [8, 9, 11] using alternative definitions of multistable processes.

It is convenient to make the convention that

$$\mathbf{1}_{[u,v]} = -\mathbf{1}_{[v,u]},$$

if $v < u$ in the following examples.

Example 4.1. *Weighted multistable Lévy motion.*

Let

$$Y(t) = \int w(x)\mathbf{1}_{[0,t]}(x)dM_\alpha(x),$$

where $\alpha : \mathbb{R} \to [a,2]$ *is continuous and* $a > 0$*, and* $w : \mathbb{R} \to \mathbb{R}$ *is continuous. Let* $u \in \mathbb{R}$ *be such that* $w(u) \neq 0$ *and suppose that as* $v \to u$*,*

$$|\alpha(u) - \alpha(v)| = o\big(1/\big|\log|u-v|\big|\big). \tag{4.1}$$

Then Y *is* $1/\alpha(u)$*-localisable at* u *with local form*

$$Y_u' = \Big\{ \int w(u)\mathbf{1}_{[0,t]}(z)dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \Big\} = w(u)L_{\alpha(u)},$$

where $M_{\alpha(u)}$ *is* $\alpha(u)$*-stable measure and* $L_{\alpha(u)}$ *is a* $\alpha(u)$*-stable Lévy motion*

Proof. Take $f(t,x) = w(x)\mathbf{1}_{[0,t]}(x)$ and $h(t,z) = w(u)\mathbf{1}_{[0,t]}(z)$. Condition (4.1) ensures that $r^{1/\alpha(u)-1/\alpha(u+rz)} \to 1$ as $r \to 0$ uniformly for $z \in [0,t]$ which is needed to ensure that (3.7) holds. Then Theorem 3.2 gives the conclusion. □

Next we consider multistable reverse Ornstein-Uhlenbeck motion. Notice that in the multistable case, we get a curious restriction on the range of α.

Example 4.2. *Multistable reverse Ornstein-Uhlenbeck motion.*

Let

$$Y(t) = \int_t^\infty \exp(-\lambda(x-t))dM_\alpha(x), \tag{4.2}$$

where α*:* $\mathbb{R} \to [a,b] \subseteq (1,2]$ *is continuous with* $1 < \sqrt{b} < a \leq b \leq 2$*. Let* $u \in \mathbb{R}$ *and suppose that as* $v \to u$*,*

$$|\alpha(u) - \alpha(v)| = o\big(1/\big|\log|u-v|\big|\big). \tag{4.3}$$

Then Y *is* $1/\alpha(u)$*-localisable at* u *with local form*

$$Y_u' = \Big\{ \int -\mathbf{1}_{(0,t)}(z)dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \Big\}, \tag{4.4}$$

where $M_{\alpha(u)}$ *is* $\alpha(u)$*-stable measure.*

Proof. We take $f(t,x) = \exp(-\lambda(x-t))\mathbf{1}_{[t,\infty)}(x)$ and $h(t,z) = -\mathbf{1}_{[0,t)}(z)$ in Theorem 3.2. After a little simplification,

$$
\begin{aligned}
&\int \Big| \frac{f(u+rt,u+rz) - f(u,u+rz)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} - h(t,z)\Big|^{a,b} dz \\
&= \int_{-|t|}^{|t|} \Big| \frac{-\exp(-\lambda rz)\mathbf{1}_{[0,t)}(z)}{r^{1/\alpha(u)-1/\alpha(u+rz)}} + \mathbf{1}_{[0,t)}(z)\Big|^{a,b} dz + \int_{|t|}^{\infty} \Big| \frac{\exp(-\lambda rz)(\exp(\lambda rt)-1)\mathbf{1}_{[t,\infty)}(z)}{r^{1/\alpha(u)-1/\alpha(u+rz)}}\Big|^{a,b} dz.
\end{aligned}
$$

The first integral converges to 0, noting that $r^{1/\alpha(u)-1/\alpha(u+rz)} \to 1$ as $r \to 0$, uniformly on $z \in [-t,t]$. The second integral is bounded by

$$
\begin{aligned}
&\int_{|t|}^{\infty} \big| r^{-1/a+1/b} \exp(-\lambda rz)(\exp(\lambda rt)-1)\big|^{a,b} dz \\
&\quad \leq \; r^{1-b/a} \int_{|t|}^{\infty} \big| \exp(-\lambda rz)(\exp(\lambda rt)-1)\big|^{a,b} dz \\
&\quad \leq \; c_1 r^{1-b/a} |\exp(\lambda rt)-1|^a \int_{|t|}^{\infty} \big|\exp(-\lambda arz)\big| dz \\
&\quad \leq \; c_2 r^{1-b/a} (\lambda r|t|)^a \exp(-\lambda ar|t|)(\lambda ra)^{-1} \\
&\quad \leq \; c_3 r^{a-b/a},
\end{aligned}
$$

for fixed t, where c_1, c_2 and c_3 are independent of $r < 1$. Since $a - b/a > 0$ the second integral converges to 0, so the conclusion follows from Theorem 3.2. □

The next example is linear fractional multistable motion. Recall from [17] that asymmetric linear fractional α_0-stable motion, $\alpha_0 \in (0,2]$, is given by

$$
L_{\alpha_0,h,b^+,b^-}(t) = \int_{-\infty}^{\infty} \rho_{\alpha_0,h}(b^+,b^-,t,x) dM_{\alpha_0}(x) \tag{4.5}
$$

where $t, b^+, b^- \in \mathbb{R}$, and

$$
\rho_{\alpha_0,h}(b^+,b^-,t,x) = b^+\big((t-x)_+^{h-1/\alpha_0} - (-x)_+^{h-1/\alpha_0}\big) + b^-\big((t-x)_-^{h-1/\alpha_0} - (-x)_-^{h-1/\alpha_0}\big),
$$

and M_{α_0} is α_0-stable random measure ($0 < \alpha < 2$). By convention, if $h - 1/\alpha_0 = 0$, we take

$$
\rho_{\alpha_0,h}(b^+,b^-,t,x) = (b^+ - b^-)\mathbf{1}_{[0,t]}(x)
$$

if $t \geq 0$, and

$$
\rho_{\alpha_0,h}(b^+,b^-,t,x) = -(b^+ - b^-)\mathbf{1}_{[t,0]}(x)
$$

if $t < 0$. Then (4.5) is an α_0-stable process.

For a multistable version, let $\alpha : \mathbb{R} \to [a,b] \subseteq (0,2)$ be continuous. We define *linear fractional α-multistable motion* by

$$
L_{\alpha,h,b^+,b^-}(t) = \int_{-\infty}^{\infty} \rho_{\alpha,h}(b^+,b^-,t,x) dM_{\alpha}(x) \tag{4.6}
$$

where $t \in \mathbb{R}$, $b^+, b^- \in \mathbb{R}$, and

$$\rho_{\alpha,h}(b^+, b^-, t, x) = b^+\big((t-x)_+^{h-1/\alpha(x)} - (-x)_+^{h-1/\alpha(x)}\big) + b^-\big((t-x)_-^{h-1/\alpha(x)} - (-x)_-^{h-1/\alpha(x)}\big),$$

where M_α is α-multistable random measure.

It may be checked directly that if $t \in \mathbb{R}$ and $1/a - 1/b < h < 1 + 1/b - 1/a$ then $\rho_{\alpha,h}(b^+, b^-, t, .) \in \mathcal{F}_{a,b}$ so that (4.6) is well-defined.

We show that linear fractional multistable motion has linear stable motion as its local form. We consider the case when $b^+ = 1$ and $b^- = 0$, the argument is similar for other b^+ and b^-.

Proposition 4.3. *Linear fractional multistable motion.*

Let

$$\begin{aligned} Y(t) &= \int (t-x)_+^{h-1/\alpha(x)} - (-x)_+^{h-1/\alpha(x)} dM_\alpha(x) \\ &= \int \rho_{\alpha,h}(1,0,t,x) dM_\alpha(x) \\ &= L_{\alpha,h,1,0}(t), \end{aligned}$$

where α*:* $\mathbb{R} \to [a,b] \subseteq (0,2)$ *is continuous. If*

$$1/a - 1/b < h < 1 + 1/b - 1/a, \tag{4.7}$$

then Y *is* h*-localisable at each* $u \in \mathbb{R}$ *with local form*

$$\begin{aligned} Y'_u(t) &= \left\{ \int \big((t-z)_+^{h-1/\alpha(u)} - (-z)_+^{h-1/\alpha(u)}\big) dM_{\alpha(u)}(z), \quad t \in \mathbb{R} \right\} \\ &= L_{\alpha(u),h,1,0}(t), \end{aligned}$$

where $M_{\alpha(u)}$ *is* $\alpha(u)$*-stable measure. Furthermore, if* $1/a < h < 1 + 1/b - 1/a$*, then* Y *has a continuous version and is strongly localisable at each* $u \in \mathbb{R}$*.*

Proof. We take $f(t,x) = (t-x)_+^{h-1/\alpha(x)} - (-x)_+^{h-1/\alpha(x)} \in \mathcal{F}_{a,b}$, given (4.7), and $h(t,z) = (t-z)_+^{h-1/\alpha(u)} - (-z)_+^{h-1/\alpha(u)}$ in Theorem 3.2. Then

$$\begin{aligned} &\int \left| \frac{f(u+rt, u+rz) - f(u, u+rz)}{r^{h-1/\alpha(u+rz)}} - h(t,z) \right|^{a,b} dz \\ &= \int \left| (t-z)_+^{h-1/\alpha(u+rz)} - (-z)_+^{h-1/\alpha(u+rz)} - (t-z)_+^{h-1/\alpha(u)} + (-z)_+^{h-1/\alpha(u)} \right|^{a,b} dz. \end{aligned}$$

This integral converges to 0 as $r \to 0$. This may be established by breaking the range of integration in the parts: $|z| < \delta, |z-t| < \delta, |z| > M$ and $A = \{z : \delta \le |z| \le M \text{ and } \delta \le |z-t|\}$. By choosing sufficiently small δ and large M, the integral over the first three parts can be made arbitrarily small, uniformly as $r \to 0$. The integrand converges to 0 pointwise on A and the bounded convergence theorem gives the integral over A convergent to 0. The conclusion follows from Theorem 3.2.

Finally, if $1/a < h < 1$ it is easily checked by a routine integral estimate (if $t > u$ splitting the resulting integral at u) that (3.4) holds so Y has a continuous version, and similarly that (3.10) if $1/a < \eta < h$, so Y is strongly localisable by Theorem 3.2. □

5 Further remarks

In the classical theory of stable distributions and processes, see [17], stable integrals may be defined in several ways. In particular, by first setting up the theory of stable random variables and random vectors, stable integrals may be defined as stochastic processes whose finite-dimensional distributions are jointly stable, with the properties of integrals following from the theory of stable distributions. Here, multistable integrals are set up in Lemma 2.1 in terms of the finite-dimensional distributions of integrals of a set of functions, rather than based on any notion of 'joint multistable distribution' of random vectors.

In this light, we may compare the definition of the multistable processes of Section 3 with the other definitions of multistable processes given in [8, 11] which depend on defining a random field in terms of a suitable function $f(t,v,x)$ and taking a diagonal section by setting $v=t$. It is shown in [11, Proposition 6.13] that the finite-dimensional distributions of the processes obtained by both the Poisson point representation [8] and the random series representation [11] are given by

$$\mathbb{E}\Big(\exp i\sum_{j=1}^{d}\theta_j Y(t_j)\Big) = \exp\Big(-2\int_0^\infty\int_0^\infty \sin^2\Big(\sum_{j=1}^{m}\theta_j b(t_j) C_{\alpha(t_j)}^{1/\alpha(t_j)} 2y^{-1/\alpha(t_j)} f(t_j,t_j,x)\Big)dydx\Big)$$

where $b(t)$ is an amplitude factor and $C_\alpha^{-1}=\int_0^\infty x^{-\alpha}\sin x dx$. Note that this form is intrinsically different from the finite-dimensional distributions (3.2) of our processes, in that they depend only on the values of α at the t_i whereas in (3.2) they depend on the values of α at all x; thus the multistable processes obtained here differ from the other constructions.

The multistable measures and integrals that we have considered in this paper are a generalization of stable symmetric processes. We could extend our definition to permit asymmetry by taking the characteristic function in Lemma 2.1 to be

$$\phi_{f_1,\dots f_d}(\theta_1,\dots,\theta_d) = \exp\Big\{-\int \Big|\sum_{j=1}^{d}\theta_j f_j(x)\Big|^{\alpha(x)}\Big(1-i\beta(x)\mathrm{sign}\Big(\sum_{j=1}^{d}\theta_j f_j(x)\Big)\tan\big(\tfrac{1}{2}\pi\alpha(x)\big)\Big)dx\Big\}$$

where $\beta:\mathbb{R}\to[-1,1]$ is a skewness function, with a logarithmic variant when $\alpha(x)=1$; compare [17, Equation (3.2.2)]. Provided $\alpha(x)\neq 1$ a similar argument shows that this process exists and similar properties should hold, albeit with more awkward algebra. However, if $\alpha(x)$ passes through the value 1 then convergence difficulties arise.

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