Sharp global nonlinear stability for a fluid overlying a highly porous material

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The stability of convection in a two-layer system in which a layer of fluid with a temperature dependent viscosity overlies and saturates a highly porous material is studied. Due to the difficulties associated with incorporating the nonlinear advection term in the Navier Stokes equations into a stability analysis, previous literature on fluid/porous thermal convection has modelled the fluid using the linear Stokes equations. This paper derives global stability for the full nonlinear system, by utilising a model proposed by Ladyzhenskaya. The nonlinear stability boundaries are shown to be sharp when compared with the linear instability thresholds.

Keywords: superposed porous-fluid convection; temperature dependent viscosity; energy method

1. Introduction

Thermal convection within a two-layer system constructed by a layer of fluid overlying a porous material saturated with the same fluid has numerous geophysical and industrial applications, such as the manufacturing of composite materials used in the aircraft and automobile industries, flow of water under the Earth's surface, flow of oil in underground reservoirs and growing of compound films in thermal chemical vapor deposition reactors. A detailed review is given by Nield & Bejan (2006), with current highly relevant literature including Chen & Chen (1988), Ewing (1998), Blest *et al.* (1999), Straughan (2002, 2008), Carr (2004), Chang (2004, 2005, 2006), Hirata *et al.* (2007), Hoppe *et al.* (2007), Mu & Xu (2007) and Hill & Straughan (2009).

Assessing the onset and type of convection is crucial in understanding and controlling these geophysical and industrial processes. This is achieved by analyzing both the linear instability and nonlinear stability thresholds of the governing model. Comparing these thresholds allows the assessment of the suitability of linear theory to predict the physics of the onset of convection. The derivation of sharp unconditional stability thresholds is particularly physically useful due to the lack of restrictions on the initial data (Straughan 2004).

Nonlinear energy stability analyses of thermal fluid/porous systems are not widespread in the current literature, with the only previous work being that of Payne & Straughan (1998) and Hill & Straughan (2009). In both these papers, due to the difficulties associated with incorporating the nonlinear $\mathbf{v}\cdot\nabla\mathbf{v}$ advection term in the Navier-Stokes equations into a stability analysis, the fluid is modelled using the linear Stokes equations.

This paper utilitses a model proposed by Ladyzhenskaya (Ladyzhenskaya 1967, 1968, 1969; Straughan 2002, 2004, 2008), which is used as an alternative to Navier-Stokes. This allows for the development of an unconditional nonlinear energy stability analysis for thermal convection with temperature dependent viscosity in a fluid/porous system, without the need to remove the nonlinear advection term $\mathbf{v} \cdot \nabla \mathbf{v}$. It is important to note that the viscosity of a liquid is usually strongly dependent on temperature (cf. Capone & Gentile 1994, 1995; Galiano 2000). Convection problems for which the viscosity or conductivity is a function of temperature has received much recent attention in the literature (see e.g. Payne & Straughan 2000; Shevtsova *et al.* 2001; Manga *et al.* 2001), making this work particularly timely.

The stability calculations required to construct the neutral curves involve determining eigenvalues and eigenfunctions, where the associated eigenvalue problems are not solvable analytically. The results are derived numerically using the Chebyshev tau - QZ method (Dongarra *et al.* 1996), which is a spectral method coupled with the QZ algorithm. All numerical results were checked by varying the number of polynomials to verify convergence. Standard indicial notation is employed throughout and $\mathbf{k} = (0, 0, 1)$.

2. Formation of the problem

Consider a fluid occupying the three-dimensional layer $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (0, d)\}$ and saturating an underlying homogeneous porous medium $\{(x, y) \in \mathbb{R}^2\} \times \{z \in (-d_m, 0)\}$. The interface between the saturated porous medium and the fluid is at $z = 0$.

We assume that the dynamic viscosity μ has a linear temperature dependence of the form

$$\mu(T) = \mu_0(1 - \gamma(T - T_L)),$$

for a constant $\gamma > 0$, where T, μ_0 and T_L are temperature and reference viscosity and temperature values, respectively. Although we only consider liquids which have a viscosity which decreases with increasing temperature, the analysis can be easily generalized to a more general viscosity-temperature relationship. The governing model for the fluid layer we select is

$$\begin{aligned} \rho_0\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) &= -\frac{\partial p}{\partial x_i} + 2\frac{\partial}{\partial x_j}\left[(\mu(T) + \mu_1|\mathbf{D}|)\, D_{ij}\right] \\ &\quad -g\rho_0 k_i(1 - \alpha(T - T_L)), \\ \frac{\partial v_i}{\partial x_i} &= 0, \\ \frac{\partial T}{\partial t} + v_j\frac{\partial T}{\partial x_j} &= \frac{\kappa_f}{(\rho_0 c_p)_f}\nabla^2 T, \end{aligned} \tag{2.1}$$

(Straughan 2002, 2004; Antontsev *et al.* 2001) where v_i, p, t and ρ_0 are velocity, pressure, time and reference density; and κ_f, g, c_p and α are the thermal conductivity, acceleration due to gravity, specific heat at a constant pressure and coefficient of thermal expansion. A variation of this model was suggested by Ladyzhensakaya (1967, 1968, 1969) as an alternative to the Navier-Stokes equations, and is a generalisation of a well known model in viscoelasticity (Antontsev *et al.* 2001). The parameter $\mu_1 > 0$ is a constant, $D_{ij} = (v_{i,j} + v_{j,i})/2$ and $|\mathbf{D}| = \sqrt{D_{ij}D_{ij}}$. The subscripts (or superscripts) f and m denote the fluid and porous layers respectively.

In the porous medium we assume a high porosity $\phi > 0.75$, such that the governing equations are given by

$$\begin{aligned}
\frac{\rho_0}{\phi}\left(\frac{\partial v_i^m}{\partial t} + \frac{1}{\phi}v_j^m\frac{\partial v_i^m}{\partial x_j}\right) &= -\frac{\partial p^m}{\partial x_i} + \frac{2}{\phi}\frac{\partial}{\partial x_j}\left[\left(\mu(T^m) + \mu_1|\mathbf{D}^m|\right)D_{ij}^m\right] \\
&\quad -\frac{\mu(T^m)}{K}v_i^m - g\rho_0 k_i\left(1 - \alpha\left(T^m - T_L\right)\right), \\
\frac{\partial v_i^m}{\partial x_i} &= 0, \qquad\qquad (2.2) \\
\frac{(\rho_0 c_p)^*}{(\rho_0 c_p)_f}\frac{\partial T^m}{\partial t} + v_j^m\frac{\partial T^m}{\partial x_j} &= \frac{\kappa^*}{(\rho_0 c_p)_f}\nabla^2 T^m,
\end{aligned}$$

where the variables v_i^m, p^m, T^m and K are the velocity, pressure, temperature and permeability, respectively. The starred quantities are defined in terms of the fluid and porous variables such that $S^* = \phi S_f + (1-\phi)S_m$, where $S^* = \kappa^*$ or $(\rho_0 c_p)^*$. A comprehensive discussion of the variances and various physical attributes of modelling transport through porous media is given in Alazmi & Vafai (2000).

The temperatures at the upper and lower boundaries are held fixed at T_U and T_L, respectively, with continuity of temperature, velocity and heat flux at the interface $z = 0$. The remaining boundary conditions at $z = 0$ are the continuity of normal stresses

$$-p + 2(\mu(T) + \mu_1|\mathbf{D}|)D_{33} = -p^m + \frac{2}{\phi}(\mu(T^m) + \mu_1|\mathbf{D}^m|)D_{33}^m, \qquad (2.3)$$

and tangential stresses

$$(\mu(T) + \mu_1|\mathbf{D}|)D_{\beta 3} = \frac{1}{\phi}(\mu(T^m) + \mu_1|\mathbf{D}^m|)D_{\beta 3}^m, \qquad (2.4)$$

for $\beta = 1, 2$. The derivation of appropriate boundary conditions at the fluid/porous interface is non-trivial, cf. Vafai & Thiyagaraja (1987), Alazmi & Vafai (2001), Vafai (2005).

Under these boundary conditions, the governing equations (2.1) $-$ (2.2) admit a steady state solution in which the velocity field is zero and

$$\begin{aligned}
\overline{T} &= T_L - \frac{\epsilon_T(T_L - T_U)}{\hat{d} + \epsilon_T} - \frac{(T_L - T_U)}{d_m(\hat{d} + \epsilon_T)}z, \qquad z \in (0, d), \\
\overline{T}^m &= T_L - \frac{\epsilon_T(T_L - T_U)}{\hat{d} + \epsilon_T} - \frac{\epsilon_T(T_L - T_U)}{d_m(\hat{d} + \epsilon_T)}z, \qquad z \in (-d_m, 0),
\end{aligned}$$

where $\epsilon_T = \tau_f/\tau_m$, $\tau_f = \kappa_f/(\rho_0 c_p)_f$, $\tau_m = \kappa^*/(\rho_0 c_p)_f$ and $\hat{d} = d/d_m$, with the overbar denoting the steady state. To study the stability of the steady state we introduce the perturbations $(u_i, \theta, \pi, u_i^m, \theta^m, \pi^m)$, where $d_{ij} = (u_{i,j} + u_{j,i})/2$, and non-dimensionalisize with the scalings

$$u_i = \frac{\mu_0}{\rho_0 d} u_i^*, \quad \pi = \frac{\mu_0^2}{\rho_0 d^2}\pi^*, \quad \theta = \theta^* \sqrt{\frac{\mu_0^3 (T_L - T_U)}{\rho_0^3 g\alpha d^3 \tau_f}}, \quad x_i = d x_i^*,$$
$$t = \frac{\rho_0 d^2}{\mu_0} t^*, \quad R = \sqrt{\frac{g\alpha\rho_0 d^3 (T_L - T_U)}{\mu_0 \tau_f}},$$

where $R_a = R^2$ is the fluid Rayleigh number. By replacing d and τ_f by d_m and τ_m, respectively, the porous layer scalings follow analogously, where $R_a^m = (R^m)^2$ is the porous Rayleigh number. This yields the non-dimensional perturbation equations

$$\begin{aligned}
\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} &= -\frac{\partial \pi}{\partial x_i} + k_i R\theta - \frac{2\Gamma Pr}{R}\frac{\partial}{\partial x_j}(\theta d_{ij}) + 2\omega \frac{\partial}{\partial x_j}(|\mathbf{d}| d_{ij}) \\
&\quad + 2\frac{\partial}{\partial x_j}(f_1 d_{ij}), \\
\frac{\partial u_i}{\partial x_i} &= 0, \qquad (2.5) \\
Pr\left(\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j}\right) &= R M_1 u_3 + \nabla^2 \theta,
\end{aligned}$$

in $\mathbb{R}^2 \times (0,1) \times (0,\infty)$ with $f_1 = 1 + \Gamma(M_2 + M_1 z)$, $d_{ij} = (u_{i,j} + u_{j,i})/2$ and

$$\begin{aligned}
\frac{1}{\phi}\frac{\partial u_i^m}{\partial t} + \frac{1}{\phi^2} u_j^m \frac{\partial u_i^m}{\partial x_j} &= -\frac{\partial \pi^m}{\partial x_i} + k_i R^m \theta^m - \frac{f_2}{\delta} u_i^m + \frac{2\hat{d}^2 \omega}{\phi}\frac{\partial}{\partial x_j}(|\mathbf{d}^m| d_{ij}^m) \\
&\quad + \frac{2}{\phi}\frac{\partial}{\partial x_j}(f_2 d_{ij}^m) + \frac{\Gamma Pr \epsilon_T}{R^m}\left(\frac{1}{\delta} u_i^m \theta^m - \frac{2}{\phi}\frac{\partial}{\partial x_j}(\theta^m d_{ij}^m)\right), \\
\frac{\partial u_i^m}{\partial x_i} &= 0, \qquad (2.6) \\
Pr\epsilon_T\left(G_m \frac{\partial \theta^m}{\partial t} + u_j^m \frac{\partial \theta^m}{\partial x_j}\right) &= R^m M_2 u_3^m + \nabla^2 \theta^m,
\end{aligned}$$

in $\mathbb{R}^2 \times (-1,0) \times (0,\infty)$, with $f_2 = 1 + \Gamma M_2(1+z)$, $d_{ij}^m = (u_{i,j}^m + u_{j,i}^m)/2$. The remaning parameters are the Prandtl number $Pr = \mu_0/(\kappa_f \rho_0)$, Darcy number $\delta = K/d_m^2$, $\omega = \mu_1/(\rho_0 d^2)$, $\Gamma = \gamma(T_L - T_U)$, $G_m = (\rho_0 c_p)^*/(\rho_0 c_p)_f$, $M_1 = \hat{d}/(\hat{d} + \epsilon_T)$ and $M_2 = \epsilon_T/(\hat{d} + \epsilon_T)$.

3. Linear Instability Analysis

To proceed with a linear analysis, the nonlinear terms from (2.5) and (2.6) are discarded. We assume normal modes of the form

$$u_i = u_i(z)\mathrm{e}^{\sigma t + i(a_1 x + a_2 y)}, \quad \pi = \pi(z)\mathrm{e}^{\sigma t + i(a_1 x + a_2 y)}, \quad \theta = \theta(z)\mathrm{e}^{\sigma t + i(a_1 x + a_2 y)},$$

with analogous definitions in the porous medium. Taking the double curls of $(2.5)_1$ and $(2.6)_1$ to remove the pressure terms, where the third component is chosen, leads to the linearised equations

$$
\begin{aligned}
f_1(D^2-a^2)^2w+2\Gamma M_1(D^2-a^2)Dw-a^2R\theta &= \sigma(D^2-a^2)w\\
(D^2-a^2)\theta+RM_1w &= Pr\sigma\theta\\
\frac{f_2}{\phi}(D^2-a_m^2-\frac{\phi}{\delta})(D^2-a_m^2)w^m &+ \frac{2\Gamma M_2}{\phi}(D^2-a_m^2-\frac{\phi}{2\delta})Dw^m\\
-a_m^2R^m\theta^m &= \frac{\sigma^m}{\phi}(D^2-a_m^2)w^m\\
(D^2-a_m^2)\theta^m+R^mM_2w^m &= Pr\sigma^m\epsilon_TG_m\theta^m
\end{aligned}
$$

where $D=\mathrm{d}/\mathrm{d}z$, $a^2=a_1^2+a_2^2$ and $a_m^2=(a_1^m)^2+(a_2^m)^2$. The boundary conditions for the twelfth order system at $z=1$ are

$$w=Dw=\theta=0,$$

and

$$w^m=Dw^m=\theta^m=0$$

at $z=-1$. On the interface $z=0$, we have

$$
\begin{aligned}
&w=\hat{d}w, \qquad Dw=\hat{d}^2Dw^m,\\
&\phi(D^2+a^2)w=\hat{d}^3(D^2+a_m^2)w^m, \qquad \theta=\sqrt{\epsilon_T\hat{d}^3}\theta^m,\\
&D\theta=\sqrt{\frac{\hat{d}^5}{\epsilon_T}}D\theta^m,
\end{aligned}
$$

and

$$
\begin{aligned}
f_1(D^2-3a^2)Dw+\Gamma M_1(D^2+a^2)w-\sigma Dw &= \frac{\hat{d}^4f_2}{\phi}(D^2-3a_m^2)w^m+\\
&\hat{d}^4\frac{\Gamma M_2}{\phi}(D^2+a_m^2)w^m-\frac{f_2\hat{d}^4}{\delta}Dw^m-\frac{\hat{d}^4\sigma_m}{\phi}Dw^m.
\end{aligned}
$$

The numerical results are presented in §5.

4. Nonlinear Stability Analysis

Let us define Ω_f and Ω_m to represent the period cells in the fluid and porous layers respectively, and introduce the notation of norm and inner product on the spaces $L^2(\Omega_f)$ and $L^2(\Omega_m)$, where

$$\|f\|_\alpha^2=\int_{\Omega_\alpha}f_if_i\,d\Omega_\alpha, \qquad (f,\,g)_\alpha=\int_{\Omega_\alpha}f_ig_i\,d\Omega_\alpha, \qquad \alpha=f,\,m.$$

To obtain global nonlinear stability bounds in the stability measure $L^2(\Omega_f)$ we multiply equations $(2.5)_1$ and $(2.5)_3$ by u_i and θ respectively, and integrate over the period cell. An analogous process is applied to $(2.6)_1$ and $(2.6)_3$. We may now define the functional $E(t)$ by

$$2E(t)=\|\mathbf{u}\|_f^2+\lambda_1 Pr\|\theta\|_f^2+\frac{\lambda_2}{\phi}\|\mathbf{u}^m\|_m^2+\lambda_3\epsilon_T G_m Pr\|\theta^m\|_m^2,$$

for coupling parameters λ_1, λ_2, $\lambda_3>0$, such that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= (u_i,\ [-u_j u_{i,\,j}-\pi_{,\,i}+k_i R\theta-\frac{2\Gamma Pr}{R}(\theta d_{ij})_{,\,j}+2\omega(|\mathbf{d}|d_{ij})_{,\,j}+2(f_1 d_{ij})_{,\,j}])_f\\ &+\lambda_1(\theta,\ [-Pru_i\theta_{,\,i}+RM_1 w+\nabla^2\theta])_f+\lambda_2(u_i^m,\ [-\frac{1}{\phi^2}u_j^m u_{i,\,j}^m-\pi_{,\,i}^m\\ &+k_i R^m\theta^m-\frac{f_2}{\delta}u_i^m-\frac{2\Gamma Pr\epsilon_T}{\phi R^m}(\theta^m d_{ij}^m)_{,\,j}+\frac{\Gamma Pr\epsilon_T}{\delta R^m}\theta^m u_i^m+\frac{2\hat{d}^2\omega}{\phi}(|\mathbf{d}^m|d_{ij}^m)_{,\,j}\\ &+\frac{2}{\phi}(f_2 d_{ij}^m)_{,\,j}])_m+\lambda_3(\theta^m,\ [-\epsilon_T Pru_i^m\theta_{,\,i}^m+R^m M_2 w^m+\nabla^2\theta^m])_m.\end{aligned}\qquad(4.1)$$

Utilising a similar approach to Hill & Straughan (2009), the first and third terms on the right hand side of (4.1) are integrated by parts, and the nondimensionalised versions of boundary condtions (2.3) and (2.4) are employed to yield

$$\begin{aligned}&(u_i,\ [-u_j u_{i,\,j}-\pi_{,\,i}-\frac{2\Gamma Pr}{R}(\theta d_{ij})_{,\,j}+2\omega(|\mathbf{d}|d_{ij})_{,\,j}+2(f_1 d_{ij})_{,\,j}])_f\\ &+\lambda_2(u_i^m,\ [-\frac{1}{\phi^2}u_j^m u_{i,\,j}^m-\pi_{,\,i}^m-\frac{2\Gamma Pr\epsilon_T}{\phi R^m}(\theta^m d_{ij}^m)_{,\,j}+\frac{2\hat{d}^2\omega}{\phi}(|\mathbf{d}^m|d_{ij}^m)_{,\,j}+\frac{2}{\phi}(d_{ij}^m)_{,\,j}])_m\\ &\quad=\frac{1}{2}\int_\Lambda\left(|\mathbf{u}|^2 w-\frac{\hat{d}^3}{\phi^2}|\mathbf{u}^m|^2 w^m\right)dS-2\omega\int_{\Omega_f}|\mathbf{d}|^3\,d\Omega_f-2\int_{\Omega_f}f_1|\mathbf{d}|^2\,d\Omega_f\\ &\quad+\frac{2\Gamma Pr}{R}\int_{\Omega_f}\theta|\mathbf{d}|^2\,d\Omega_f-\frac{2\hat{d}^5\omega}{\phi}\int_{\Omega_m}|\mathbf{d}^m|^3\,d\Omega_m-\frac{2\hat{d}^3}{\phi}\int_{\Omega_m}f_2|\mathbf{d}|^2\,d\Omega_m\\ &\quad+\frac{2\Gamma Pr\epsilon_T\hat{d}^3}{\phi R^m}\int_{\Omega_m}\theta^m|\mathbf{d}^m|^2\,d\Omega_m,\end{aligned}$$

where $\lambda_2=\hat{d}^3$, and Λ represents the fluid/porous interface at $z=0$. Similarly, by integrating by parts and utilising the non-dimensionalised boundary conditions

$$\begin{aligned}\lambda_1(\theta,\ [-Pru_i\theta_{,\,i}+\nabla^2\theta])_f &+ \lambda_3(\theta^m,\ [-Pr\epsilon_T u_i^m\theta_{,\,i}^m+\nabla^2\theta^m])_m\\ &= -\lambda(\|\nabla\theta\|_f^2+\hat{d}^4\|\nabla\theta^m\|_m^2)\end{aligned}$$

where $\lambda_1=\lambda$ and $\lambda_3=\lambda\hat{d}^4$.

Combining these definitions it follows that

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} \;=\;& \frac{1}{2}\int_{\Lambda}\left(|\mathbf{u}|^2 w - \frac{\hat{d}^3}{\phi^2}|\mathbf{u}^m|^2 w^m\right) dS + R < \theta,\, w > -2\omega\int_{\Omega_f}|\mathbf{d}|^3\, d\Omega_f \\
& -2\int_{\Omega_f} f_1|\mathbf{d}|^2\, d\Omega_f + \frac{2\Gamma Pr}{R}\int_{\Omega_f}\theta|\mathbf{d}|^2\, d\Omega_f - \lambda\|\nabla\theta\|^2 + \lambda R M_1 < w,\, \theta > \\
& +\hat{d}^3 R^m < \theta^m,\, u_i^m > -\frac{\hat{d}^3}{\delta}\int_{\Omega_m} f_2|\mathbf{u}^m|^2\, d\Omega_m + \frac{\Gamma Pr\epsilon_T\hat{d}^3}{\delta R^m}\int_{\Omega_m}\theta^m|\mathbf{u}^m|^2\, d\Omega_m \\
& -\frac{2\hat{d}^5\omega}{\phi}\int_{\Omega_m}|\mathbf{d}^m|^3\, d\Omega_m - \frac{2\hat{d}^3}{\phi}\int_{\Omega_m} f_2|\mathbf{d}^m|^2\, d\Omega_m \qquad (4.2)\\
& +\frac{2\Gamma Pr\epsilon_T\hat{d}^3}{\phi R^m}\int_{\Omega_m}\theta^m|\mathbf{d}^m|^2\, d\Omega_m + R^m\hat{d}^4 M_2\lambda < w^m,\, \theta^m > -\lambda\hat{d}^4\|\nabla\theta^m\|^2.
\end{aligned}
$$

To address the cubic nonlinearities in (4.2) we introduce the L^3 norm $\|\cdot\|_3$. Multiplying $(2.5)_3$ and $(2.6)_3$ by θ^2 and $(\theta^m)^2$, respectively, integrating over the period cell, and using Poincaré's inequality we find

$$
\begin{aligned}
\frac{\lambda_4 Pr}{3}\frac{d}{dt}\|\theta\|_3^3 \;+\;& \frac{\lambda_5 Pr\epsilon_T G_m}{3}\frac{d}{dt}\|\theta^m\|_3^3 \le \lambda_4 R M_1\int_{\Omega_f} w\theta^2(\mathrm{sgn}\ \theta)\, d\Omega_f \\
& +\lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}} R M_2\int_{\Omega_m} w^m(\theta^m)^2(\mathrm{sgn}\ \theta^m)\, d\Omega_m \\
& -\frac{8\pi^2\lambda_4}{9}\int_{\Omega_f}|\theta^3|\, d\Omega_f - \frac{8\pi^2\lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{9}\int_{\Omega_m}|\theta^m|^3\, d\Omega_m, \qquad (4.3)
\end{aligned}
$$

where $\lambda_5 = \lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}$ to ensure the removal of the boundary integrals (Hill & Straughan 2009). We now use Young's inequality on the cubic integral terms in both (4.2) and (4.3), such that

$$
\int_{\Omega} Q_1 Q_2^2\, d\Omega \le \frac{c^2}{3}\|Q_1\|_3^3 + \frac{1}{3c}\|Q_2\|_3^3
$$

for $c > 0$, where $Q_1 \neq Q_2$.

Letting

$$
E_1 = \frac{1}{2}E + \frac{\lambda_4 Pr}{3}\|\theta\|_3^3 + \frac{\lambda_5 Pr\epsilon_T G_m}{3}\|\theta^m\|_3^3
$$

and combining (4.2) and (4.3) we now have

$$
\begin{aligned}
\frac{\mathrm{d}E_1}{\mathrm{d}t} \le\ & R(1+\lambda M_1) < \theta,\, w >_f -2\int_{\Omega_f} f_1|\mathbf{d}|^2\, d\Omega_f - \lambda\|\nabla\theta\|_f^2 - \frac{\hat{d}^3}{\delta}\int_{\Omega_m} f_2|\mathbf{u}^m|^2\, d\Omega_m \\
- & \frac{2\hat{d}^3}{\phi}\int_{\Omega_m} f_2|\mathbf{d}^m|^2\, d\Omega_m - \lambda\hat{d}^4\|\nabla\theta^m\|_m^2 + R^m\hat{d}^3(1+\lambda M_2\hat{d}) < \theta^m,\, w^m >_m \\
+ & \frac{1}{2}\int_\Lambda \left(|\mathbf{u}|^2 w - \frac{\hat{d}^3}{\phi^2}|\mathbf{u}^m|^2 w^m\right) dS + \frac{RM_1\beta_1^2\lambda_4}{3}\int_{\Omega_f} |\mathbf{u}^3|\, d\Omega_f \\
+ & \left(\frac{R^m M_2\beta_2^2\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}\lambda_4}{3} + \frac{2\Gamma Pr\epsilon_T\hat{d}^3}{3\delta R^m\alpha_3^2}\right)\int_{\Omega_m} |\mathbf{u}^m|^3\, d\Omega_m \qquad (4.4) \\
- & \left(\frac{8\pi^2\lambda_4}{9} - \frac{2RM_1\lambda_4}{3\beta_1} - \frac{2\Gamma Pr\alpha_1}{3R}\right)\int_{\Omega_f} |\theta|^3\, d\Omega_f \\
- & \left(2\omega - \frac{4\Gamma Pr}{3\alpha_1^2 R}\right)\int_{\Omega_f} |\mathbf{d}|^3\, d\Omega_f - \left(\frac{2\omega\hat{d}^5}{\phi} - \frac{4\Gamma Pr\epsilon_T\hat{d}^3}{3\alpha_2^2 R^m\phi}\right)\int_{\Omega_m} |\mathbf{d}^m|^3\, d\Omega_m \\
- & \left(\frac{8\pi^2\lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{9} - \frac{2R^m M_2\lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{3\beta_2} - \frac{\Gamma Pr\epsilon_T\hat{d}^3}{3R^m}\left(\frac{2\alpha_2}{\phi} + \frac{\alpha_3}{\delta}\right)\right)\int_{\Omega_m} |\theta^m|^3\, d\Omega_m,
\end{aligned}
$$

where α_i, β_j are positive constants for $i = 1, 2, 3$; $j = 1, 2$ introduced by using Young's inequality on the cubic terms. Before we choose the coefficients to bind the cubic $\mathbf{d}$, θ, $\mathbf{d}^m$ and θ^m integrals, we must address the boundary integrals and cubic $\mathbf{u}$ and $\mathbf{u}^m$ terms in (4.4).

To achieve this we utilise the following Poincaré like inequalities:

$$\int_{\Omega_f} |\mathbf{u}|^3\, d\Omega_f \le c_1 \int_{\Omega_f} |\mathbf{d}|^3\, d\Omega_f \qquad (4.5)$$

and

$$\int_\Lambda |\mathbf{u}|^3\, dS + c_2\int_{\Omega_f} |\mathbf{u}|^3\, d\Omega_f \le c_3\int_{\Omega_f} |\mathbf{d}|^3\, d\Omega_f, \qquad (4.6)$$

where c_1, c_2, $c_3 > 0$. Similar inequalities follow in the porous case, for constants c_1^m, c_2^m, c_3^m. A proof of these inequalities and the definitions of the constants are given in Appendix A. Applying (4.5) and (4.6) to (4.4) yields

$$
\begin{aligned}
\frac{\mathrm{d}E_1}{\mathrm{d}t} \quad \leq \quad & R(1+\lambda M_1) < \theta,\, w >_f -2\int_{\Omega_f} f_1|\mathbf{d}|^2\, d\Omega_f - \lambda\|\nabla\theta\|_f^2 - \frac{\hat{d}^3}{\delta}\int_{\Omega_m} f_2|\mathbf{u}^m|^2\, d\Omega_m \\
& -\frac{2\hat{d}^3}{\phi}\int_{\Omega_m} f_2|\mathbf{d}^m|^2\, d\Omega_m - \lambda\hat{d}^4\|\nabla\theta^m\|_m^2 + R^m\hat{d}^3(1+\lambda M_2\hat{d}) < \theta^m,\, w^m >_m \\
& -\left(\frac{8\pi^2\lambda_4}{9} - \frac{2RM_1\lambda_4}{3\beta_1} - \frac{2\Gamma Pr\alpha_1}{3R}\right)\int_{\Omega_f}|\theta|^3\, d\Omega_f \\
& -\left(2\omega - \frac{4\Gamma Pr}{3\alpha_1^2 R} - \frac{RM_1\beta_1^2\lambda_4 c_1}{3} - \frac{c_3 - c_1c_2}{2}\right)\int_{\Omega_f}|\mathbf{d}|^3\, d\Omega_f \qquad (4.7) \\
& -\left(\frac{2\omega\hat{d}^5}{\phi} - \frac{2\Gamma Pr\epsilon_T\hat{d}^3}{3R^m}\left(\frac{2}{\phi\alpha_2^2} + \frac{c_1^m}{\delta\alpha_3^2}\right) - \frac{R^m M_2\beta_2^2\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}\lambda_4 c_1^m}{3}\right. \\
& \qquad \left. - \frac{\hat{d}^3(c_3^m - c_1^m c_2^m)}{2\phi^2}\right)\int_{\Omega_m}|\mathbf{d}^m|^3\, d\Omega_m \\
& -\left(\frac{8\pi^2\lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{9} - \frac{2R^m M_2\lambda_4\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{3\beta_2} - \frac{\Gamma Pr\epsilon_T\hat{d}^3}{3R^m}\left(\frac{2\alpha_2}{\phi} + \frac{\alpha_3}{\delta}\right)\right)\int_{\Omega_m}|\theta^m|^3\, d\Omega_m.
\end{aligned}
$$

Now put $\lambda_4 = \lambda_4' + k\varepsilon$, and let

$$
\begin{aligned}
& \frac{8\pi^2\lambda_4'}{9} - \frac{2RM_1\lambda_4'}{3\beta_1} - \frac{2\Gamma Pr\alpha_1}{3R} = 0, \\
& \frac{8\pi^2\lambda_4'\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{9} - \frac{2R^m M_2\lambda_4'\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{3\beta_2} - \frac{\Gamma Pr\epsilon_T\hat{d}^3}{3R^m}\left(\frac{2\alpha_2}{\phi} + \frac{\alpha_3}{\delta}\right) = 0,
\end{aligned}
$$

which is satisfied for

$$
\beta_2 = \beta_1\left(\frac{\epsilon_T^3}{\hat{d}^5}\right)^{\frac{1}{2}}, \quad \frac{2\alpha_2}{\phi} + \frac{\alpha_3}{\delta} = 2\alpha_1\hat{d}, \quad \lambda_4' = \frac{3\Gamma Pr\alpha_1\beta_1}{4\pi^2 R\beta_1 - 3R^2 M_1}.
$$

We now minimize

$$
\frac{4\Gamma Pr}{3\alpha_1^2 R} + \frac{\Gamma Pr\alpha_1 M_1\beta_1^2 c_1}{4\pi^2\beta_1 - 3RM_1}
$$

with respect to α_1 and β_1 to yield

$$
\alpha_1 = \frac{8\pi^2 4^{\frac{1}{3}}}{9RM_1 c_1^{\frac{1}{3}}}, \quad \beta_1 = \frac{9RM_1}{8\pi^2},
$$

and choose α_3 to minimize

$$
\frac{2\Gamma Pr\epsilon_T\hat{d}^3}{3R^m}\left(\frac{2}{\phi\alpha_2^2} + \frac{c_1^m}{\delta\alpha_3^2}\right).
$$

From (4.7), choosing $k = 27/(8\pi^2)$ we can now deduce

$$\begin{aligned}\frac{\mathrm{d}E_1}{\mathrm{d}t} \leq\ & R(1+\lambda M_1) < \theta,\, w >_f -2\int_{\Omega_f} f_1|\mathbf{d}|^2\, d\Omega_f - \lambda\|\nabla\theta\|_f^2 - \frac{\hat{d}^3}{\delta}\int_{\Omega_m} f_2|\mathbf{u}^m|^2\, d\Omega_m \\ & -\frac{2\hat{d}^3}{\phi}\int_{\Omega_m} f_2|\mathbf{d}^m|^2\, d\Omega_m - \lambda\hat{d}^4\|\nabla\theta^m\|_m^2 + R^m\hat{d}^3(1+\lambda M_2\hat{d}) < \theta^m,\, w^m >_m \\ & -\varepsilon\int_{\Omega_f}|\theta|^3\, d\Omega_f - \hat{\omega}\int_{\Omega_f}|\mathbf{d}|^3\, d\Omega_f - \hat{\omega}^m\int_{\Omega_m}|\mathbf{d}^m|^3\, d\Omega_m - \varepsilon\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}\int_{\Omega_m}|\theta^m|^3\, d\Omega_m,\end{aligned}$$

where we require

$$\begin{aligned}\hat{\omega} &= 2\omega - \frac{81\Gamma Pr 4^{\frac{1}{3}} R M_1^2 c_1^{\frac{2}{3}}}{64\pi^4} - \frac{c_3 - c_1c_2}{2} - \varepsilon\frac{729R^3M_1^3c_1}{512\pi^6} > 0 \\ \hat{\omega}^m &= \frac{2\omega\hat{d}^5}{\phi} - \frac{2\Gamma Pr\epsilon_T\hat{d}^3}{3R^m}\left(\frac{2}{\phi\alpha_2^2} + \frac{c_1^m}{\delta\alpha_3^2}\right) - \frac{27\Gamma Pr 4^{\frac{1}{3}} R^m M_2^2 c_1^m \epsilon_T^{\frac{5}{2}}\hat{d}^{\frac{3}{2}}}{32\pi^4 c_1^{\frac{1}{3}}} \\ & \quad -\frac{\hat{d}^3(c_3^m - c_1^m c_2^m)}{2\phi^2} - \varepsilon\frac{729(R^m)^3M_2^3c_1^m\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}}{512\pi^6} > 0.\end{aligned} \tag{4.8}$$

Defining

$$\begin{aligned}\mathcal{I} &= R(1+\lambda M_1) < \theta,\, w >_f + R^m\hat{d}^3(1+\lambda M_2\hat{d}) < \theta^m,\, w^m >_m, \\ \mathcal{D} &= 2\int_{\Omega_f} f_1|\mathbf{d}|^2\, d\Omega_f + \lambda\|\nabla\theta\|_f^2 + \frac{\hat{d}^3}{\delta}\int_{\Omega_m} f_2|\mathbf{u}^m|^2\, d\Omega_m \\ & \quad + \frac{2\hat{d}^3}{\phi}\int_{\Omega_m} f_2|\mathbf{d}^m|^2\, d\Omega_m + \lambda\hat{d}^4\|\nabla\theta^m\|_m^2,\end{aligned}$$

it follows that

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} \leq -\mathcal{D}\left(\frac{R_E - 1}{R_E}\right) - \varepsilon\int_{\Omega_f}|\theta|^3\, d\Omega_f - \varepsilon\epsilon_T^{\frac{1}{2}}\hat{d}^{\frac{11}{2}}\int_{\Omega_m}|\theta^m|^3\, d\Omega_m,$$

where

$$\frac{1}{R_E} = \max_{\mathcal{H}}\left(\frac{\mathcal{I}}{\mathcal{D}}\right) < 1. \tag{4.9}$$

Utilising Poincaré like inequalities (cf. Payne & Straughan 1998) allows us to deduce that

$$\frac{\mathrm{d}E_1}{\mathrm{d}t} \leq -mE_1$$

where $m > 0$. Integrating, we have $E_1(t) \leq E_1(0)\mathrm{e}^{-mt} \to 0$ as $t \to \infty$, where convergence is at least exponential, so we have established unconditional nonlinear stability provided (4.8) and (4.9) hold.

The corresponding Euler Lagrange equations which arise at the sharpest threshold $R_E = 1$ are

$$\begin{gathered}2f_1\nabla^2 u_i + 2f_1' w_{,\,i} + 2f_1' u_{i,\,3} + k_i R(M_1\lambda + 1)\theta = L_{,\,i} \\ 2\lambda\nabla^2\theta + R(M_1\lambda + 1)w = 0\end{gathered} \tag{4.10}$$

in the fluid layer, and

$$2\delta\hat{d}^3(f_2\nabla^2u_i^m + f_2'w_{,i}^m + \phi f_2'u_{i,3}^m) - 2\phi\hat{d}^3 f_2 u_i^m + k_i\phi\delta R^m\hat{d}^3(M_2\lambda\hat{d}+1)\theta^m = \phi\delta L_{,i}^m$$
$$2\lambda\hat{d}\nabla^2\theta^m + R^m(M_2\lambda\hat{d}+1)w^m = 0 \tag{4.11}$$

in the porous layer, where L and L^m are Lagrange multipliers.

By taking the double *curl* of equations $(4.10)_1$ and $(4.11)_1$ and adopting normal mode representations, the twelfth-order eigenvalue problem (4.10) – (4.11) can be utilised to locate the critical nonlinear Rayleigh number Ra_E, which is given by

$$Ra_E = \max_{\lambda}\min_{a^2} R^2(a^2,\lambda).$$

Numerical results for the nonlinear energy approach are presented in §5.

5. Results and conclusions

The first key result that can be derived is for the case $\Gamma = 0$, which corresponds to the viscosity of the fluid being constant with respect to temperature. Under this condition the equations for linear instability and nonlinear stability are identical to those of Hill & Straughan (2009), for which excellent agreement was shown between the two. It is important to note, though, that in this paper the nonlinear advection term $\mathbf{v}\cdot\nabla\mathbf{v}$ is included in the analysis, whereas the analysis of Hill & Straughan (2009) is limited to the nonlinear Stokes problem.

For $\Gamma \neq 0$ we now solve the eigenvalue problem (4.10) $-$ (4.11) by means of a D^2 Chebyshev tau method. The details are similar to those given by Dongarra *et al.* (1996). The parameters, unless stated otherwise, are fixed at $\delta = 5\times10^{-6}$, $G_m = 10$, $Pr = 6$ and $\epsilon_T = 0.7$. The porous material is assumed to be that of a Foametal (which is used extensively in industrial applications such as heat exchangers, chemical reactors and fluid filters), with physical values of permeability and porosity of $8.19\times10^{-8}m^2$ and 0.79, respectively (cf. Straughan 2002, Goyeau *et al.* 2003).

Figure 1 shows the neutral curves for a variation of Γ values, where the linear instability and nonlinear stability thresholds are represented by solid and dashed lines respectively.

It is clear that an increase in Γ causes the system to become more stable. Assuming that the temperature at the boundaries remains fixed, this corresponds to the strength of the linear dependence of viscosity on temperature increasing. As the viscosity decreases with an increase in temperature, this physcially makes sense. An interesting result is that the bimodal nature of the neutral curve is unaffected by the change in Γ.

Since the linear instability and nonlinear stability results clearly show excellent agreement, we can conclude that the linear theory accurately encapsulates the physics of the onset of convection.

Appendix A. A proof of inequalities (4.5) and (4.6)

Let Ω_f represent the fluid period cell, and $u_i \in C^1$ be a solenoiodal function satisfying the boundary condition $u_i = 0$ on Λ_0, where the boundary of Ω_f is given by

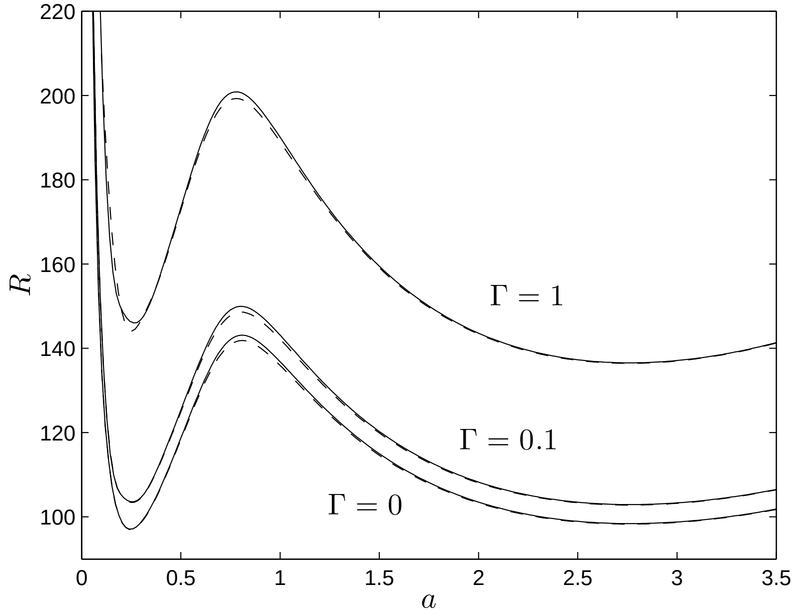

Figure 1. Visual representation of linear instability (solid lines) and nonlinear stability (dashed lines) thresholds, with critical thermal Rayleigh number R plotted against wavenumber a, for $\Gamma = 0$, 0.1 and 1. The remaining parameters are $\hat{d} = 0.116$, $Pr = 6$, and $\phi = 0.79$.

$\partial\Omega_f = \Lambda + \Lambda_0$. In the same vein as §4, Λ represents the fluid/porous interface at $z = 0$. For a C^1 function f_i to be chosen at our discretion, we observe that

$$\int_\Lambda f_i n_i |\mathbf{u}|^3 \, dS = \int_{\Omega_f} f_{i,\,i} |\mathbf{u}|^3 \, d\Omega_f + 3 \int_{\Omega_f} f_i |\mathbf{u}| u_j u_{j,\,i} \, d\Omega_f$$

and

$$3 \int_\Lambda f_i n_j |\mathbf{u}| u_j u_i \, dS = 3 \int_{\Omega_f} f_i |\mathbf{u}|_{,\,j} u_j u_i \, d\Omega_f$$
$$+ 3 \int_{\Omega_f} f_i |\mathbf{u}| u_j u_{i,\,j} \, d\Omega_f + 3 \int_{\Omega_f} f_{i,\,j} |\mathbf{u}| u_j u_i \, d\Omega_f.$$

By letting

$$\mathbf{f} = (-p_1 x, \, -p_1 y, \, -p_1 z - p_2),$$

where p_1 and p_2 are constants to be chosen at our discretion, it follows that

$$p_2 \int_\Lambda |\mathbf{u}|^3 \, dS - 3 \int_\Lambda f_i u_i |\mathbf{u}| w \, dS + 6 p_1 \int_{\Omega_f} |\mathbf{u}|^3 \, d\Omega_f$$
$$= 3 \int_{\Omega_f} f_i |\mathbf{u}| u_j (u_{i,\,j} + u_{j,\,i}) \, d\Omega_f + 3 \int_{\Omega_f} f_i |\mathbf{u}|_{,\,j} u_j u_i \, d\Omega_f.$$

The arithmetic-geometric mean inequality leads us to

$$\int_\Lambda x_\beta u_\beta w|\mathbf{u}|\,dS \leq \frac{d_1}{2}\left(\frac{1}{\alpha}\int_\Lambda w^2|\mathbf{u}|\,dS + \alpha\int_\Lambda u_\beta^2|\mathbf{u}|\,dS\right)$$

where $\alpha > 0$ is a constant, and $d_1 = \max_{i=1,2}(1, |x_i|)$. Employing this inequality yields

$$\left(4p_2 - \frac{3p_1 d_1}{2\alpha}\right)\int_\Lambda w^2|\mathbf{u}|\,dS + \left(p_2 - \frac{3p_1 d_1\alpha}{2}\right)\int_\Lambda u_\beta^2|\mathbf{u}|\,dS + 6p_1\int_{\Omega_f}|\mathbf{u}|^3\,d\Omega_f$$
$$\leq 3\int_{\Omega_f} f_i|\mathbf{u}|u_j(u_{i,j} + u_{j,i})\,d\Omega_f + 3\int_{\Omega_f} f_i|\mathbf{u}|_{,j}u_j u_i\,d\Omega_f. \qquad \text{(A 1)}$$

Letting $\alpha = 1/2$ and $p_2 = 3p_1 d_1/4$ to remove the boundary integrals, we now have

$$6p_1\int_{\Omega_f}|\mathbf{u}|^3\,d\Omega_f \leq \frac{63p_1 d_1}{4}\int_{\Omega_f}|\mathbf{u}|^2|\mathbf{d}|\,d\Omega_f.$$

By using the Cauchy-Schwartz inequality on the right hand side, we find

$$\int_{\Omega_f}|\mathbf{u}|^3\,d\Omega_f \leq \left(\frac{21d_1}{8}\right)^3\int_{\Omega_f}|\mathbf{d}|^3\,d\Omega_f,$$

which is inequality (4.5).

To derive inequality (4.6) we return to (A 1). By letting

$$p_2 = \frac{p_1 d_1}{2\alpha}(1-\alpha^2),$$

and using (4.5) it follows that

$$\int_\Lambda |\mathbf{u}|^3\,d\Lambda + \frac{12\alpha}{d_1(1-4\alpha^2)}\int_{\Omega_f}|\mathbf{u}|^3\,d\Omega_f \leq \frac{9(21d_1)^2(2\alpha-\alpha^2+1)}{64(1-4\alpha^2)}\int_{\Omega_f}|\mathbf{d}|^3\,d\Omega_f$$

as required, where α is a constant to be chosen at our discretion.

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