GENERATING TRANSFORMATION SEMIGROUPS USING ENDOMORPHISMS OF PREORDERS, GRAPHS, AND TOLERANCES

J. D. MITCHELL, M. MORAYNE, Y. PÉRESSE, AND M. QUICK

ABSTRACT. Let Ω^Ω be the semigroup of all mappings of a countably infinite set Ω. If U and V are subsemigroups of Ω^Ω, then we write $U \approx V$ if there exists a finite subset F of Ω^Ω such that the subsemigroup generated by U and F equals that generated by V and F. The relative rank of U in Ω^Ω is the least cardinality of a subset A of Ω^Ω such that the union of U and A generates Ω^Ω. In this paper we study the notions of relative rank and the equivalence $\approx$ for semigroups of endomorphisms of binary relations on Ω.

The semigroups of endomorphisms of preorders, bipartite graphs, and tolerances on Ω are shown to lie in two equivalence classes under $\approx$. Moreover such semigroups have relative rank 0, 1, 2, or $\mathfrak{d}$ in Ω^Ω where $\mathfrak{d}$ is the minimum cardinality of a dominating family for $\mathbb{N}^\mathbb{N}$. We give examples of preorders, bipartite graphs, and tolerances on Ω where the relative ranks of their endomorphism semigroups in Ω^Ω are 0, 1, 2, and $\mathfrak{d}$.

We show that the endomorphism semigroups of graphs, in general, fall into at least four classes under $\approx$ and that there exist graphs where the relative rank of the endomorphism semigroup is $2^{\aleph_0}$.

1. INTRODUCTION

1.1. **Background and Preliminaries.** Bergman and Shelah [2] introduced the following preorder (i.e. reflexive and transitive binary relation) on the subsets of the symmetric group $\mathrm{Sym}(\Omega)$ on a countably infinite set Ω. If G and H are subsets of $\mathrm{Sym}(\Omega)$, then $G \preccurlyeq H$ if there exists a finite subset F of $\mathrm{Sym}(\Omega)$ such that G is contained in the subgroup generated by $H \cup F$. Galvin [6] proved that every countable set of permutations on Ω is contained in a 2-generated subgroup of $\mathrm{Sym}(\Omega)$. Hence if there exists a countable subset F such that G is contained in the subgroup generated by $H \cup F$, then $G \preccurlyeq H$. The preorder $\preccurlyeq$ gives rise to an equivalence relation $\approx$ on the subsets of $\mathrm{Sym}(\Omega)$ defined by $G \approx H$ whenever $G \preccurlyeq H$ and $H \preccurlyeq G$. In [2] it was shown that the subgroups of $\mathrm{Sym}(\Omega)$ that are closed in the topology of pointwise convergence fall into four classes with respect to $\approx$. Furthermore, the partial order on these four equivalence classes induced by $\preccurlyeq$ is a total order.

The situation for the semigroup Ω^Ω of all mappings from Ω to Ω (the semigroup theoretic analogue of $\mathrm{Sym}(\Omega)$) is somewhat different. Of course, it is straightforward to give a definition of $\preccurlyeq$ for Ω^Ω: if U, V are subsets of Ω^Ω, then $U \preccurlyeq V$ if there exists a finite subset F of Ω^Ω such that U is contained in the subsemigroup generated by $V \cup F$. Throughout the remainder of the paper we will denote the subsemigroup generated by a subset U of Ω^Ω by $\langle\, U \,\rangle$. Analogous to the theorem

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of Galvin mentioned above, a classical theorem of Sierpiński [12] states that every countable set of mappings on Ω is contained in a 2-generated subsemigroup of Ω^Ω. Hence if $U, V \subseteq \Omega^\Omega$ such that $U \subseteq \langle V, F \rangle$ for some countable $F \subseteq \Omega^\Omega$, then $U \preccurlyeq V$.

Mesyan [11] proved an analogue of Bergman and Shelah's theorem for a restricted collection of closed (again in the topology of pointwise convergence) subsemigroups of Ω^Ω. Namely for subsemigroups U with the following properties (throughout this article we will write functions on the right of their arguments and compose them from left to right):

- if $\Sigma \subseteq \Omega$ is finite, then $U \approx \{ f \in U : \sigma f = \sigma \text{ for all } \sigma \in \Sigma \}$; and
- the set of functions in U that are injective on a cofinite subset of Ω are dense in U.

Letting $\Omega = \{\alpha_1, \alpha_2, \ldots\}$ and $\mathbb{N} = \{1, 2, \ldots\}$, Mesyan showed that such subsemigroups must be equivalent under $\approx$ to one of the following semigroups:

(i) the trivial semigroup $\{1_\Omega\}$;
(ii) $S_{1,\alpha} = \{ f \in \Omega^\Omega : \alpha f \in \{\alpha_1, \alpha\} \text{ for all } \alpha \in \Omega \}$;
(iii) $S_2 = \{ f \in \Omega^\Omega : \{\alpha_{2n-1} f, \alpha_{2n} f\} \subseteq \{\alpha_{2n-1}, \alpha_{2n}\} \text{ for all } n \in \mathbb{N}\}$;
(iv) $S_\leq = \{ f \in \Omega^\Omega : \alpha_n f \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \text{ for all } n \in \mathbb{N} \}$;
(v) the full transformation semigroup Ω^Ω.

It was also shown that if $\mathfrak{F} = \{ f \in \Omega^\Omega : |\Omega f| < \aleph_0 \}$, then

$$\{1_\Omega\} \prec \mathfrak{F} \prec S_{1,\alpha} \prec S_2 \prec S_\leq \prec \Omega^\Omega$$

where $\prec$ denotes $\preccurlyeq$ but not $\approx$. Mesyan also proved that $\preccurlyeq$ contains an infinite chain and at least two incomparable elements. However, there is no complete characterisation of the closed subsemigroups of Ω^Ω with respect to $\preccurlyeq$. It is not even known how many equivalence classes there are on subsets of Ω^Ω under $\approx$.

In this paper rather than considering all closed subsemigroups of Ω^Ω we will consider subsemigroups arising as the endomorphism semigroups of preorders, graphs and tolerances (reflexive and symmetric binary relations). In the main theorems of this paper, we will prove that if S is the endomorphism semigroup of a preorder, bipartite graph, or tolerance on Ω, then either $S \approx \Omega^\Omega$ or $S \approx S_\leq$. Whether $S \approx \Omega^\Omega$ or $S \approx S_\leq$ depends on certain simple structural properties of the underlying relation; further details can be found in Section 1.3.

The notion of $\approx$ among subsets of Ω^Ω is related to that of relative rank. The *relative rank* of a subset U of Ω^Ω is defined to be the least cardinality of a set A such that $\langle U, A \rangle = \Omega^\Omega$ and is denoted by $\operatorname{rank}(\Omega^\Omega : U)$. Relative ranks of subsets of Ω^Ω have been previously studied, for example, see [4], [5], or [9].

Using Sierpiński's Theorem [12] it is straightforward to prove that $\operatorname{rank}(\Omega^\Omega : U)$ is $0, 1, 2$ or uncountable for any $U \subseteq \Omega^\Omega$. Moreover, it follows immediately from the definitions that $\operatorname{rank}(\Omega^\Omega : U) = 0, 1, 2$ if and only if $U \approx \Omega^\Omega$. On the other hand, if $U, V \leq \Omega^\Omega$ with $U \preccurlyeq V$ and $\operatorname{rank}(\Omega^\Omega : U) > \aleph_0$, then $\operatorname{rank}(\Omega^\Omega : U) \geq \operatorname{rank}(\Omega^\Omega : V)$.

Assuming the Continuum Hypothesis holds the relative rank of any U in Ω^Ω is $0, 1, 2$, or $2^{\aleph_0}$. However, if the Continuum Hypothesis is not assumed, then it is natural to ask what values $\operatorname{rank}(\Omega^\Omega : U)$ can have when it is uncountable.

We will prove that if U and V are semigroups of endomorphisms of a preorder, bipartite graph, or tolerance, where $\operatorname{rank}(\Omega^\Omega : U), \operatorname{rank}(\Omega^\Omega : V) > \aleph_0$, then $\operatorname{rank}(\Omega^\Omega : U) = \operatorname{rank}(\Omega^\Omega : V)$. We require the following well-known notion to define the cardinal equalling any such relative ranks.

If Ω is well-ordered by $\leq$, then a function $f \in \Omega^\Omega$ is said to *dominate* $g \in \Omega^\Omega$ if $\alpha f \geq \alpha g$ for all $\alpha \in \Omega$. The study of the notion of dominance and related ideas gave rise to the following cardinal number, introduced by van Douwen. A *dominating family for* Ω^Ω is a subset $\mathcal{F}$ of Ω^Ω such that for all $f \in \Omega^\Omega$ there exists $g \in \mathcal{F}$ where g dominates f. Of course, whether a subset is a dominating family for Ω^Ω depends on the well-ordering of Ω, but the least cardinality of a dominating family does not depend on the well-ordering. Thus we can define (without ambiguity) the cardinal $\mathfrak{d}$ to be the least cardinality of a dominating family for Ω^Ω. The following relations are not hard to obtain: $\aleph_1 \leq \mathfrak{d} \leq 2^{\aleph_0}$. If the Continuum Hypothesis holds, then $\mathfrak{d} = 2^{\aleph_0}$. However, without the Continuum Hypothesis, it is consistent with the usual axioms of set theory (ZFC) that $\mathfrak{d} = \aleph_1 < 2^{\aleph_0} = \aleph_2$ or $\aleph_1 < \mathfrak{d} = 2^{\aleph_0} = \aleph_2$, see [1].

1.2. **Definitions and notation.** As usual a *binary relation* R on a set Ω is just a subset of $\Omega \times \Omega$. Let Ω and Λ be sets, and R and S be binary relations on Ω and Λ, respectively. Then a *homomorphism* from (Ω, R) to (Λ, S) is a function $f : \Omega \longrightarrow \Lambda$ such that $(\alpha f, \beta f) \in S$ for all $(\alpha, \beta) \in R$. A homomorphism is an *isomorphism* if it is bijective and its inverse is also a homomorphism. An *endomorphism* is a homomorphism from (Ω, R) to (Ω, R). An *automorphism* is an isomorphism from (Ω, R) to (Ω, R). We denote the semigroup of endomorphisms on (Ω, R) under composition of mappings by $\operatorname{End}(\Omega, R)$. Let $R \subseteq \Omega \times \Omega$ and let $\Lambda \subseteq \Omega$. We define the *subrelation* of R *induced by* Λ to be $R \cap \Lambda \times \Lambda$.

A *walk* from $\alpha \in \Omega$ to $\beta \in \Omega$ in (Ω, R) is a sequence of elements of Ω

$$\alpha = \gamma_0, \gamma_1, \gamma_2, \dots, \gamma_n = \beta$$

such that $(\gamma_i, \gamma_{i+1}) \in R$ or $(\gamma_{i+1}, \gamma_i) \in R$ for all i. We will say that such a walk has length n. Two points are *connected* if there exists a walk from one to the other. Being connected is an equivalence relation on Ω and the equivalence classes are called the *components* of (Ω, R). We will say that (Ω, R) is *connected* if it only has one component. If R is a binary relation on Ω, then a *path* in (Ω, R) is a walk in which all points are distinct.

The *degree* of an element $\alpha \in \Omega$ is the size of the set $\{\ \beta \in \Omega\ :\ (\alpha, \beta) \in R \text{ or } (\beta, \alpha) \in R\ \}$. We say that (Ω, R) is *locally finite* if all the elements of Ω have finite degree.

A *preorder* is a reflexive and transitive binary relation. A *partial order* is a preorder that is also anti-symmetric. A set with a partial order is called a *partially ordered set* or *poset*. A *graph* $G = (\Omega, E)$ is a set Ω together with a binary relation E that is symmetric and irreflexive. If G is a graph, then for the sake of consistency with the literature, we will call the elements of Ω the *vertices* of G, the elements of E the *edges* of G, and a subrelation induced by a set will be referred to as the *subgraph* induced by that set. Two vertices $\alpha, \beta \in \Omega$ are *adjacent* if $(\alpha, \beta) \in E$. A graph G is *bipartite* if its vertices can be partitioned into two sets where adjacent vertices lie in distinct sets. A binary relation is called a *tolerance* if it is reflexive and symmetric.

In what follows we will always, unless stated otherwise, assume that Ω is the countably infinite set $\{\alpha_1, \alpha_2, \dots\}$ and we will always assume that $\mathbb{N} = \{1, 2, \dots\}$.

1.3. **Overview.** Let R be a preorder, bipartite graph, or tolerance. Then the main theorems of this paper can be summarized as follows:

- if R has finitely many components and is locally finite, then $\mathrm{End}(\Omega, R) \approx S_{\leq}$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, R)) = \mathfrak{d}$;
- if R has infinitely many components or is not locally finite, then $\mathrm{End}(\Omega, R) \approx \Omega^\Omega$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, R)) \in \{0, 1, 2\}$;

see Theorems 2.3, 3.1, 4.4, 4.5, and 5.1.

The picture is more complicated for arbitrary non-bipartite graphs. In particular, there exist examples of graphs G where:

- G has infinitely many components, $\mathrm{End}(G) \approx \{1_\Omega\}$ or $\mathrm{End}(G) \approx S_2$, and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 2^{\aleph_0}$;
- G has infinitely many components, $\mathrm{End}(G) \approx S_{\leq}$, and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = \mathfrak{d}$;
- G is connected and locally finite, $\mathrm{End}(G) \approx \{1_\Omega\}$, and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 2^{\aleph_0}$;
- G is connected and not locally finite, $\mathrm{End}(G) \approx S_{\leq}$, and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = \mathfrak{d}$;

see Examples 6.1, 6.2, and 6.3.

The following weaker version of the theorems regarding bipartite graphs hold for an arbitrary graph G:

- if G has finitely many components and is locally finite, then $\mathrm{End}(G) \preccurlyeq S_{\leq}$;
- if all the components of G are finite, then one of the following holds: $\mathrm{End}(G) \approx \{1_\Omega\}$, $S_{1,\alpha} \preccurlyeq \mathrm{End}(G) \preccurlyeq S_{\leq}$, or $\mathrm{End}(G) \approx \Omega^\Omega$;

see Theorems 2.4 and 4.3.

2. Uncountable ranks and binary relations

The following theorem connects the notions of relative rank, domination, and the preorder $\preccurlyeq$. We require the following notion for a subset F of Ω^Ω. We say that F is an *almost disjoint family* if for all $f, g \in F$ there are only finitely many $\alpha \in \Omega$ such that $\alpha f = \alpha g$. It is reasonably straightforward to show that there exists an almost disjoint family F in Ω^Ω with $|F| = 2^{\aleph_0}$; see, for example, [10, Theorem 1.3].

Theorem 2.1. *Let U be a subset of Ω^Ω. If $U \approx S_{\leq}$, then* $\mathrm{rank}(\Omega^\Omega : U) = \mathfrak{d}$.

On the other hand, if $U \preccurlyeq S_2$, then $\mathrm{rank}(\Omega^\Omega : U) = 2^{\aleph_0}$.

Proof. For a proof of the fact that $\mathrm{rank}(\Omega^\Omega : S_{\leq}) = \mathfrak{d}$ see [5, Lemma 3.5].

We will show that $\mathrm{rank}(\Omega^\Omega : S_2) = 2^{\aleph_0}$. Let A be a subset of Ω^Ω such that $\langle\, S_2, A \,\rangle = \Omega^\Omega$. Seeking a contradiction assume that $|A| < 2^{\aleph_0}$. Let $(a_1, a_2, \dots, a_m)$ be an m-tuple of elements of A. Then define

$$B_{(a_1, a_2, \dots, a_m)} = \{\, s_0 a_1 s_1 a_2 s_2 \dots a_m s_m \;:\; s_0, s_1, \dots, s_m \in S_2 \,\}.$$

The semigroup Ω^Ω can be given as the union of the sets $B_{(a_1,a_2,\ldots,a_m)}$ over all finite tuples of elements of A.

Let $F \subseteq \Omega^\Omega$ be a family of almost disjoint functions of size $2^{\aleph_0}$. If $B_{(a_1,a_2,\ldots,a_m)} \cap F$ were finite for all $(a_1, a_2, \ldots, a_m)$, then $|F| \leq \min\{\aleph_0, |A|\}$. But $|F| = 2^{\aleph_0}$ and so there exists a tuple $(b_1, b_2, \ldots, b_n)$ of elements from A such that $B_{(b_1,b_2,\ldots,b_n)} \cap F$ is infinite.

Define

$$C_\alpha = \{ \alpha h \;:\; h \in B_{(b_1,b_2,\ldots,b_n)} \}.$$

Then $|C_\alpha| \leq 2^{n+1}$ for all $\alpha \in \Omega$ by the definition of S_2. Let $N = 2^{n+1}$ and $f_1, f_2, \ldots, f_{N+1}$ be distinct elements of $B_{(b_1,b_2,\ldots,b_n)} \cap F$. Then, since F is a family of almost disjoint functions, there exists $\beta \in \Omega$ such that $\beta f_1, \beta f_2, \ldots, \beta f_{N+1}$ are distinct. But $|C_\beta| \leq N$, a contradiction. $\square$

It is straightforward to classify those binary relations whose endomorphism semigroups equal Ω^Ω. The proof follows immediately from the definitions and is omitted.

Lemma 2.2. *Let Ω be an infinite set and let R be a binary relation on Ω. Then the relative rank of $\mathrm{End}(\Omega, R)$ in Ω^Ω is 0 if and only if R is one of $\emptyset$, $\Omega\times\Omega$, or $\Delta_\Omega = \{(\alpha, \alpha) : \alpha \in \Omega\}$.*

In light of Lemma 2.2 we will assume throughout that R is a non-empty proper subset of $\Omega \times \Omega$ not equal to $\Delta_\Omega = \{ (\alpha, \alpha) \;:\; \alpha \in \Omega \}$.

Theorem 2.3. *Let R be a reflexive binary relation on Ω such that (Ω, R) has infinitely many components. Then $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, R)) \leq 1$ and so $\mathrm{End}(\Omega, R) \approx \Omega^\Omega$.*

Proof. Recall that $\Omega = \{\alpha_1, \alpha_2, \ldots\}$. Let the components of (Ω, R) be $L_1, L_2, \ldots$ and let $\gamma_i \in L_i$ be fixed for all i. Define $g \in \Omega^\Omega$ by $\alpha_i g = \gamma_i$.

Let $f \in \Omega^\Omega$ be arbitrary. Let $\widehat{f} \in \Omega^\Omega$ map all points in L_i to $\alpha_i f$ for $i = 1, 2, \ldots$. Since R is reflexive, $\widehat{f} \in \mathrm{End}(\Omega, R)$. Then for all $\alpha_i \in \Omega$ we have $\alpha_i g\widehat{f} = \gamma_i\widehat{f} = \alpha_i f$. Thus $f \in \langle\, \mathrm{End}(\Omega, R), g \,\rangle$. Since f was chosen arbitrarily we conclude that $\Omega^\Omega = \langle\, \mathrm{End}(\Omega, R), g \,\rangle$ and hence $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, R)) \leq 1$. $\square$

Theorem 2.4. *Let R be a binary relation on Ω such that (Ω, R) has finitely many components and is locally finite. Then $\mathrm{End}(\Omega, R) \preccurlyeq S_{\leq}$ and hence $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, R)) \geq \mathfrak{d}$.*

We require the following result to prove Theorem 2.4. Let $d : \Omega \times \Omega \longrightarrow \mathbb{R}$ be a metric on Ω. A function $f \in \Omega^\Omega$ is *Lipschitz* if there exists a constant $C \in \mathbb{N}$ such that $d(\alpha f, \beta f) \leq Cd(\alpha, \beta)$ for all $\alpha, \beta \in \Omega$. We may also say that f is *Lipschitz with constant C*. Denote the semigroup of all Lipschitz functions on Ω by $\mathfrak{L}_\Omega$.

Proposition 2.5. [5, Theorem 3.1] *Let Ω be a countably infinite set and let d be a metric on Ω that is unbounded on every infinite subset of Ω. Then $\mathfrak{L}_\Omega \preccurlyeq S_{\leq}$ and $\mathrm{rank}(\Omega^\Omega : \mathfrak{L}_\Omega) \geq \mathfrak{d}$.*

Proof of Theorem 2.4. Let $L_1, L_2, \ldots, L_n$ be the components of R. To show that $\mathrm{End}(\Omega, R) \preccurlyeq S_{\leq}$ we define a metric on Ω and prove that $\mathrm{End}(\Omega, R) \subseteq \mathfrak{L}_\Omega$.

Let $d_{L_i} : L_i \times L_i \longrightarrow \mathbb{N}\cup\{0\}$ be defined so that $d_{L_i}(\alpha, \beta)$ is the minimal length of a walk from α to β. It is straightforward to verify that d_{L_i} is a metric on L_i for all

i. We will now extend the metrics d_{L_i} to a metric d on the entire set Ω. Let $\gamma_i \in L_i$ be fixed. Then define d by

$$d(\alpha,\beta) = \begin{cases} d_{L_i}(\alpha,\beta) & \text{if } \alpha,\beta \in L_i \\ d_{L_i}(\alpha,\gamma_i) + d_{L_j}(\gamma_j,\beta) + 1 & \text{if } \alpha \in L_i \text{ and } \beta \in L_j \text{ where } i \neq j. \end{cases}$$

It can easily be seen that d is indeed a metric on Ω and that it is unbounded above on every infinite subset.

We will now show that all functions in $\mathrm{End}(\Omega, R)$ are Lipschitz with respect to d. Let $f \in \mathrm{End}(\Omega, R)$ be arbitrary and let $M = \max\{\ d(\gamma_i, \gamma_j f) \ :\ 1 \leq i, j \leq n\ \}$. If α and β are in the same component L_i, then $\alpha f, \beta f \in L_j$ for some j and since $f \in \mathrm{End}(\Omega, R)$ we have that

$$d(\alpha f, \beta f) = d_{L_j}(\alpha f, \beta f) \leq d_{L_i}(\alpha, \beta) = d(\alpha, \beta).$$

Next, if $\alpha \in L_i$, $\beta \in L_j$ with $i \neq j$, and $\alpha f \in L_k$, $\beta f \in L_l$, then

$$\begin{aligned} d(\alpha f, \beta f) &\leq d(\alpha f, \gamma_i f) + d(\gamma_i f, \gamma_k) + d(\gamma_k, \gamma_l) + d(\gamma_l, \gamma_j f) + d(\gamma_j f, \beta f) \\ &\leq d_{L_k}(\alpha f, \gamma_i f) + M + 1 + M + d_{L_l}(\gamma_j f, \beta f) \\ &\leq d_{L_i}(\alpha, \gamma_i) + M + 1 + M + d_{L_j}(\gamma_j, \beta) \\ &= d(\alpha, \beta) + 2M \\ &\leq d(\alpha, \beta) + 2M d(\alpha, \beta) = (2M+1) d(\alpha, \beta). \end{aligned}$$

Thus f is Lipschitz with constant $2M + 1$. Therefore it follows from Theorem 2.5 that $\mathrm{End}(\Omega, R) \preccurlyeq S_{\leq}$. □

3. PREORDERS

In this section we completely classify the endomorphisms of preorders $\sqsubseteq$ on Ω with respect to $\preccurlyeq$. Since preorders are reflexive, the case where $(\Omega, \sqsubseteq)$ has infinitely many components follows directly from Theorem 2.3. That is, if $\sqsubseteq$ is a preorder on Ω such that $(\Omega, \sqsubseteq)$ has infinitely many components, then $\mathrm{End}(\Omega, \sqsubseteq) \approx \Omega^\Omega$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) \leq 1$.

The case where $\sqsubseteq$ is a partial order was considered in [9]. It was shown that the endomorphisms of a poset $(\Omega, \sqsubseteq)$ have finite relative rank in Ω^Ω precisely when $(\Omega, \sqsubseteq)$ is locally finite or $(\Omega, \sqsubseteq)$ has infinitely many components. Here we will show that this classification extends to preorders and show that the only infinite value that can arise for $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq))$ is $\mathfrak{d}$.

Theorem 3.1. *Let $\sqsubseteq$ be a preorder on Ω such that $(\Omega, \sqsubseteq)$ has finitely many components.*

(i) *If $(\Omega, \sqsubseteq)$ is locally finite, then $\mathrm{End}(\Omega, \sqsubseteq) \approx S_{\leq}$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) = \mathfrak{d}$.*
(ii) *If $(\Omega, \sqsubseteq)$ is not locally finite, then $\mathrm{End}(\Omega, \sqsubseteq) \approx \Omega^\Omega$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) \leq 2$.*

It is natural to ask if the bound given in Theorem 3.1(ii) is the best possible. The answer is yes: two examples of connected posets with $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) = 1$ and 2, respectively, were given in [9].

To prove Theorem 3.1 we require the following four lemmas.

Lemma 3.2. *Let R be a binary relation on Ω, let $g \in \operatorname{End}(\Omega, R)$ be any endomorphism with infinite image, let R' be the subrelation of R induced by $\operatorname{im}(g)$, and let S be any relation on Ω such that $(\operatorname{im}(g), R')$ is isomorphic to (Ω, S). Then $\operatorname{End}(\Omega, R) \succcurlyeq \operatorname{End}(\Omega, S)$.*

Proof. Let $\Psi : (\operatorname{im}(g), R') \longrightarrow (\Omega, S)$ be an isomorphism. Then $g\Psi \in \Omega^\Omega$ is a surjective homomorphism from (Ω, R) to (Ω, S).

Let $\overline{g} \in \Omega^\Omega$ be any function such that $\alpha\overline{g} \in \alpha(g\Psi)^{-1} = \{ \beta \in \Omega \ : \ \beta g \Psi = \alpha \}$ for all $\alpha \in \Omega$. Then $\overline{g}g\Psi = 1_\Omega$ where 1_Ω denotes the identity map on Ω. Likewise, if $\Psi^* \in \Omega^\Omega$ is an extension of Ψ, then $\Psi^{-1}\Psi^* = 1_\Omega$.

Let $f \in \operatorname{End}(\Omega, S)$ be arbitrary. Then $g\Psi f \Psi^{-1} \in \operatorname{End}(\Omega, R)$. Thus

$$f = \overline{g}g\Psi f \Psi^{-1}\Psi^* \in \langle \operatorname{End}(\Omega, R), \overline{g}, \Psi^* \rangle.$$

Since f was arbitrary, $\operatorname{End}(\Omega, S) \subseteq \langle \operatorname{End}(\Omega, R), \overline{g}, \Psi^* \rangle$. □

Lemma 3.3. *Let $\Omega = \{\alpha_1, \alpha_2, \ldots\}$ and let*

(i) $R = \{ (\alpha_i, \alpha_{i+1}), (\alpha_{i+1}, \alpha_i) \ : \ i \in \mathbb{N} \}$;
(ii) $S = \{ (\alpha_{2i-1}, \alpha_{2i}), (\alpha_{2i+1}, \alpha_{2i}) \ : \ i \in \mathbb{N} \}$.

Then (Ω, R) is a graph with $\operatorname{End}(\Omega, R) \succcurlyeq S_\le$, and (Ω, S) is a poset with $\operatorname{End}(\Omega, S) \succcurlyeq S_\le$.

Proof. It suffices to show that $\operatorname{End}(\Omega, R) \cap \operatorname{End}(\Omega, S) \succcurlyeq S_\le$. Let $g \in \Omega^\Omega$ be defined by $\alpha_n g = \alpha_{n(n-1)+1}$ for all $n \in \mathbb{N}$ and let $h \in \Omega^\Omega$ be any function such that $(\alpha_{2n-1})h = \alpha_n$ for every $n \in \mathbb{N}$.

Let $f \in S_\le$ be arbitrary. We will define a function $\widehat{f} \in \operatorname{End}(\Omega, R) \cap \operatorname{End}(\Omega, S)$ in two steps so that f can be written as a product of $\widehat{f}$, g, and h. The first step is to let $\widehat{f}$ be defined on the elements of the form $\alpha_{n(n-1)+1}$ by

$$(\alpha_{n(n-1)+1})\widehat{f} = \alpha_{2k-1}$$

whenever $\alpha_n f = \alpha_k$.

The second step is to define $\widehat{f}$ on all the elements α_m with indices in the range $n(n-1) + 2$ to $n(n + 1)$. If $\alpha_n f = \alpha_k$ and $\alpha_{n+1} f = \alpha_l$, then $k \le n$ and $l \le n + 1$ since $f \in S_\le$. It follows that the length of the path on (Ω, R) from α_{2k-1} to α_{2l-1} is an even number not greater than $2n$. Hence there exists a walk

$$\beta_0 = \alpha_{2k-1}, \beta_1, \ldots, \beta_{2n} = \alpha_{2l-1}$$

of length $2n$. The definition of $\widehat{f}$ is completed by setting

$$(\alpha_{n(n-1)+1+i})\widehat{f} = \beta_i$$

for all $i \in \{1, 2, \ldots, 2n - 1\}$. By construction, $\widehat{f}$ is an endomorphism of (Ω, R).

We will now show that $\widehat{f}$ is also an element of $\operatorname{End}(\Omega, S)$. By construction,

$$\{\alpha_1, \alpha_3, \alpha_5, \ldots\}\widehat{f} \subseteq \{\alpha_1, \alpha_3, \alpha_5, \ldots\} \text{ and } \{\alpha_2, \alpha_4, \alpha_6, \ldots\}\widehat{f} \subseteq \{\alpha_2, \alpha_4, \alpha_6, \ldots\}.$$

Let $\alpha, \beta \in \Omega$ with $(\alpha, \beta) \in S$. Then $\alpha = \alpha_{2i-1}$ and $\beta = \alpha_{2i}$ or α_{2i-2} for some $i \in \mathbb{N}$. Since α and β are adjacent in (Ω, R) their images $\alpha\widehat{f}$ and $\beta\widehat{f}$ are also adjacent in (Ω, R). Thus either $(\alpha\widehat{f}, \beta\widehat{f}) \in S$ or $(\beta\widehat{f}, \alpha\widehat{f}) \in S$. In fact, $(\alpha\widehat{f}, \beta\widehat{f}) \in S$ since $\alpha\widehat{f} = \alpha_{2i-1}\widehat{f} \in \{\alpha_1, \alpha_3, \alpha_5, \ldots\}$. So, $\widehat{f} \in \operatorname{End}(\Omega, R) \cap \operatorname{End}(\Omega, S)$, as required.

To conclude the proof, let $\alpha_i \in \Omega$ be arbitrary and let $\alpha_j = \alpha_i f$. Then

$$\alpha_i g\widehat{f}h = (\alpha_{i(i-1)+1})\widehat{f}h = (\alpha_{2j-1})h = \alpha_j = \alpha_i f.$$

Thus $S_{\leq} \subseteq \langle\, \mathrm{End}(\Omega, R) \cap \mathrm{End}(\Omega, S), g, h \,\rangle$ and so $\mathrm{End}(\Omega, R) \cap \mathrm{End}(\Omega, S) \succcurlyeq S_{\leq}$. □

Lemma 3.4 (König's Lemma). *Let G be an infinite connected locally finite graph. Then there exists an infinite path in G, that is, a sequence of distinct vertices $\beta_1, \beta_2, \ldots$ such that β_i and β_{i+1} are adjacent for all i.*

For a proof see [3, Lemma 19.2.1].

The following lemma is an analogue of König's Lemma for arbitrary binary relations. It is also slightly stronger, in so far as when it is applied to graphs the subgraph induced by $\beta_1, \beta_2, \ldots$ from Lemma 3.4 is isomorphic to the graph defined in Lemma 3.3(i).

Lemma 3.5. *Let Ω be countably infinite and let $R \subseteq \Omega \times \Omega$ be such that (Ω, R) is connected and locally finite. Then there exists a sequence $\gamma_1, \gamma_2, \ldots$ of distinct elements of Ω such that, for $i \neq j$, $\gamma_i R \gamma_j$ or $\gamma_j R \gamma_i$ if and only if i and j are consecutive integers.*

Proof. Let E be the symmetric closure of $R \setminus \Delta_\Omega$. Then $G = (\Omega, E)$ is a graph. Hence by Lemma 3.4 there exist a infinite path $\beta_1, \beta_2, \ldots$ in G. But β_i is adjacent to β_{i+1} in G if and only if (β_i, β_{i+1}) or $(\beta_{i+1}, \beta_i) \in R$.

Let $\gamma_1 = \beta_1$. Assume that γ_{i-1} has been defined for some $i > 1$. Then define

$$n_i = \max\{\, n \in \mathbb{N} \,:\, (\gamma_{i-1}, \beta_n) \text{ or } (\beta_n, \gamma_{i-1}) \in R \,\}$$

and set $\gamma_i = \beta_{n_i}$. The number n_i exists since (Ω, R) is locally finite. The sequence $\gamma_1, \gamma_2, \ldots$ obtained in this way has the required property. □

Proof of Theorem 3.1. **(i).** As $(\Omega, \sqsubseteq)$ is locally finite, it follows immediately from Theorem 2.4 that $\mathrm{End}(\Omega, \sqsubseteq) \preccurlyeq S_{\leq}$.

To prove that $\mathrm{End}(\Omega, \sqsubseteq) \succcurlyeq S_{\leq}$, we show that there exists $g \in \mathrm{End}(\Omega, \sqsubseteq)$ such that the preorder induced by the image of g is isomorphic to that given in Lemma 3.3(ii). This allows us to apply Lemma 3.2 to conclude the proof.

Since $(\Omega, \sqsubseteq)$ has finitely many components there is at least one infinite component. By Lemma 3.5 that component contains a sequence of distinct elements $\gamma_1, \gamma_2, \ldots$ such that $\gamma_i \sqsubseteq \gamma_j$ or $\gamma_j \sqsubseteq \gamma_i$ if and only if i and j are consecutive integers.

Let γ_n be arbitrary. If $\gamma_n \sqsubseteq \gamma_{n+1}$, then $\gamma_{n+1} \sqsupseteq \gamma_{n+2}$ as otherwise $\gamma_n \sqsubseteq \gamma_{n+2}$ by transitivity of $\sqsubseteq$, a contradiction. Likewise, if $\gamma_n \sqsupseteq \gamma_{n+1}$, then $\gamma_{n+1} \sqsubseteq \gamma_{n+2}$. Assume without loss of generality that $\gamma_1 \sqsubseteq \gamma_2$. We conclude that the subposet induced by $\{\gamma_1, \gamma_2, \ldots\}$ is isomorphic to that defined in Lemma 3.3(ii).

Next, we specify $g \in \mathrm{End}(\Omega, \sqsubseteq)$ with image equal to $\{\gamma_1, \gamma_2, \ldots\}$ by defining it on the components of $(\Omega, \sqsubseteq)$. Let K be any component of $(\Omega, \sqsubseteq)$. Then since $\sqsubseteq$ is transitive and $(\Omega, \sqsubseteq)$ is locally finite, it follows that there exists $\beta_1 \in K$ such that for all $\beta \in K$ with $\beta \sqsubseteq \beta_1$ we have that $\beta \sqsupseteq \beta_1$. Note that, in some sense, β_1 is a minimal element of K.

Let $L_1 = \{\, \beta \in K \,:\, \beta \sqsubseteq \beta_1 \,\}$ and define $L_2, L_3, \ldots$ recursively as follows:

$$L_{2i} = \{\, \beta \in K \,:\, \text{there exists } \delta \in L_{2i-1} \text{ with } \beta \sqsupseteq \delta \,\} \setminus (L_1 \cup \cdots \cup L_{2i-1})$$

and

$$L_{2i+1} = \{\ \beta \in K\ :\ \text{there exists } \delta \in L_{2i} \text{ with } \beta \sqsubseteq \delta\ \} \setminus (L_1 \cup \cdots \cup L_{2i}).$$

Of course, since $(\Omega, \sqsubseteq)$ is locally finite, L_i is finite for all $i \in \mathbb{N}$. As K is connected, every element in K lies in some L_i. Also, if K is infinite, then L_i is non-empty for all i.

So, if $g_K : K \longrightarrow \Omega$ is defined so that $\alpha g_K = \gamma_i$ for all $\alpha \in L_i$, then by construction g_K is a homomorphism from $(K, \sqsubseteq)$ to the preorder induced by $\{\gamma_1, \gamma_2, \ldots\}$. Let $g : \Omega \longrightarrow \{\gamma_1, \gamma_2, \ldots\}$ be the union of the functions g_K over all the components K of $(\Omega, \sqsubseteq)$. Then $g \in \mathrm{End}(\Omega, \sqsubseteq)$ and, as $(\Omega, \sqsubseteq)$ has at least one infinite component, g is surjective.

If R is the preorder induced by $\gamma_1, \gamma_2, \ldots$, then, by Lemma 3.2, $\mathrm{End}(\Omega, \sqsubseteq) \succcurlyeq \mathrm{End}(\Omega, R)$. Moreover, by Lemma 3.3, it follows that $\mathrm{End}(\Omega, R) \succcurlyeq S_{\leq}$ and the proof of this case is concluded.

(ii). Recall that in this case we assume that $(\Omega, \sqsubseteq)$ is not locally finite. If $\alpha, \beta \in \Omega$ such that $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$, then we will write $\alpha \equiv \beta$. If all the equivalence classes of $\equiv$ are finite, then there are infinitely many such classes and they can be given as $E_1, E_2, \ldots$. Let $\beta_n \in E_n$ be fixed for every $n \in \mathbb{N}$ and let $g \in \Omega^\Omega$ be defined by $\alpha g = \beta_n$ for all $\alpha \in E_n$ and for all n. It is straightforward to verify that $g \in \mathrm{End}(\Omega, \sqsubseteq)$. Furthermore the preorder induced by the image of g is a partial order which is not locally finite. In [9] it was shown that the set of endomorphisms of a non-locally finite poset is always equivalent under $\approx$ to Ω^Ω. Thus by Lemma 3.2 we have that $\mathrm{End}(\Omega, \sqsubseteq) \approx \Omega^\Omega$ and so, as mentioned in the introduction, it follows by Sierpiński [12] that $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) \leq 2$.

Next, we assume that there exists an infinite equivalence class E of $\equiv$. Let $k : \Omega \longrightarrow E$ be any bijection and let $k^* \in \Omega^\Omega$ be any extension of k^{-1}. Let $f \in \Omega^\Omega$ be arbitrary and define $\widehat{f} \in \Omega^\Omega$ by

$$\alpha \widehat{f} = \begin{cases} \alpha k^{-1} f k & \text{if } \alpha \in E \\ \alpha & \text{if } \alpha \in \Omega \setminus E. \end{cases}$$

Then $\widehat{f} \in \mathrm{End}(\Omega, \sqsubseteq)$ since f fixes $\Omega \setminus E$ pointwise and maps elements of E to elements of E. Furthermore, if $\alpha \in \Omega$, then

$$\alpha k \widehat{f} k^* = \alpha k (k^{-1} f k) k^* = \alpha f.$$

Thus $\Omega^\Omega = \langle\ \mathrm{End}(\Omega, \sqsubseteq), k, k^*\ \rangle$ and so $\mathrm{End}(\Omega, \sqsubseteq) \approx \Omega^\Omega$. In fact, $k \in \mathrm{End}(\Omega, \sqsubseteq)$ and so $\Omega^\Omega = \langle\ \mathrm{End}(\Omega, \sqsubseteq), k^*\ \rangle$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) = 1$. □

4. Graphs

In this section we consider semigroups of endomorphisms of graphs. These semigroups fall into more equivalence classes under $\approx$ than endomorphisms of preorders and we do not achieve a full classification in this case.

Lemma 4.1. *If G contains a subgraph isomorphic to the complete graph K_Ω on Ω, then* $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 1$.

Proof. Let H denote the subgraph of G isomorphic to K_Ω, let $H_1, H_2, \ldots$ be infinite sets partitioning the vertices of H, and let $g \in \Omega^\Omega$ be a function that maps all elements of H_i to α_i for $i = 1, 2, \ldots$. Note that $g \notin \mathrm{End}(G)$.

Pick an arbitrary $f \in \Omega^\Omega$. Let $\widehat{f}$ be an injection such that $\alpha_i \widehat{f} \in H_j$ whenever $\alpha_i f = \alpha_j$. Since $\mathrm{im}(\widehat{f}) \subseteq H$ all image points are adjacent and so $\widehat{f} \in \mathrm{End}(G)$. Now $\alpha_i \widehat{f} g = \alpha_j = \alpha_i f$ for all $\alpha_i \in \Omega$. Hence $\Omega^\Omega = \langle\, \mathrm{End}(G), g \,\rangle$. □

Let G be a graph and define $K(G)$ to be the set of components. If $L, M \in K(G)$, then we will write $L \ll M$ whenever there exists a homomorphism from L to M. Denote by $L^{\ll}$ the set $\{\, M \in K(G) \,:\, L \ll M \,\}$.

Theorem 4.2. *Let G be a graph such that for infinitely many components L of G the set $L^{\ll}$ is infinite. Then* $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) \leq 2$.

Proof. Let $L_1, L_2, \ldots$ be the components of G with $L_i^{\ll}$ infinite for all $i \in \mathbb{N}$.

First, let

$$\{A_{(i,1)}, A_{(i,2)}, \ldots\} \subseteq L_i^{\ll}$$

such that $\{A_{(i,1)}, A_{(i,2)}, \ldots\} \cap \{A_{(j,1)}, A_{(j,2)}, \ldots\} = \emptyset$ for $i \neq j$.

Let $\Omega = \{\alpha_1, \alpha_2, \ldots\}$, let $g \in \Omega^\Omega$ be any function with $\alpha_i g \in L_i$, let $h \in \Omega^\Omega$ be any function such that $\alpha h = \alpha_j$ for all $\alpha \in A_{(i,j)}$, and let $f \in \Omega^\Omega$ be arbitrary. Since $A_{(i,k)} \in L_i^{\ll}$ for all i, k, there exists a homomorphism from L_i to $A_{(i,k)}$. A function that is a homomorphism on all the components of G is an endomorphism of G. So there exists $\widehat{f} \in \mathrm{End}(G)$ such that $L_i \widehat{f} \subseteq A_{(i,k)}$ whenever $\alpha_k = \alpha_i f$. Let $\alpha_i \in \Omega$ be arbitrary and let $\alpha_k = \alpha_i f$. Then $\alpha_i g \in L_i$ and so $(\alpha_i g)\widehat{f} \in A_{(i,k)}$. Hence $\alpha_i g \widehat{f} h = \alpha_k = \alpha_i f$. So $f = g \widehat{f} h$ and $\Omega^\Omega = \langle\, \mathrm{End}(G), g, h \,\rangle$. □

In Theorem 2.3, we prove that endomorphisms of reflexive relations with infinitely many components have relative rank at most 1 in Ω^Ω. However for graphs the analogous statement is not true. Examples of graphs G and H satisfying the hypothesis of Theorem 4.2 where $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 1$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(H)) = 2$ can be found in Example 6.4 and Proposition 7.8, respectively.

We use a result from Mesyan [11] to show that the converse of Theorem 4.2 holds in the case that all the components of G are finite.

Theorem 4.3. *Let G be a countably infinite graph such that every component of G is finite. Then the following are equivalent:*

(i) *$L^{\ll}$ is finite for all but finitely many components L of G;*
(ii) $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) > 2$*;*
(iii) $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) \geq \mathfrak{d}$*;*
(iv) $S_{1,\alpha} \preccurlyeq \mathrm{End}(G) \preccurlyeq S_{\leq}$ *or* $\mathrm{End}(G) \approx \{1_\Omega\}$.

Proof. By Theorem 2.1 it follows that (iv) implies (iii). Also (iii) implies (ii) immediately. Theorem 4.2 tells us that (ii) implies (i).

It remains to show that (i) implies (iv). Under this assumption, the set $\{\, \alpha f \,:\, f \in \mathrm{End}(G) \,\}$ is finite for all but finitely many $\alpha \in \Omega$, since an endomorphism must map components into components. Let ρ be the preorder on Ω defined by $(\alpha, \beta) \in \rho$ if $\beta = \alpha f$ for some $f \in \mathrm{End}(G)$ and let $E(\rho) = \{\, f \in \Omega^\Omega \,:\, (\alpha, \alpha f) \in \rho \text{ for all } \alpha \in \Omega \,\}$. Then $\{\, \beta \in \Omega \,:\, (\alpha, \beta) \in \rho \,\}$ is finite for all but finitely many $\alpha \in \Omega$.

It was shown in [11, Section 7] that $E(\rho) \preccurlyeq S_{\leq}$ for such a preorder ρ. It follows from the definition of $E(\rho)$ that $\mathrm{End}(G) \subseteq E(\rho)$ and thus $\mathrm{End}(G) \preccurlyeq S_{\leq}$.

It remains to prove that either $\mathrm{End}(G) \succcurlyeq S_{1,\alpha}$ or $\mathrm{End}(G) \approx \{1_G\}$. There are two possibilities. Suppose that, for all but finitely many components L, the only homomorphism from L into G is the identity map. It follows that $\mathrm{End}(G)$ is countable since all the components of G are finite. Thus $\mathrm{End}(G) \approx \{1_G\}$ as the equivalence class of $\{1_G\}$, consists of all countable subsets of Ω^Ω.

On the other hand, suppose there exist infinitely many components $L_1, L_2, \ldots$ of G and non-identity homomorphisms $g_i : L_i \longrightarrow G$ for all $i \in \mathbb{N}$. We will define an infinite subset $\{\delta_1, \delta_2, \ldots\}$ of the union of $L_1, L_2, \ldots$ such that

(a) if δ_i and δ_j are in the same component, then $i = j$;
(b) if $\delta_i \in L_j$, then $\delta_i g_j \notin \{\delta_1, \delta_2, \ldots\}$ for all $i \in \mathbb{N}$.

Since g_i is not the identity on L_i, for all $i \in \mathbb{N}$ there exists $\gamma_i \in L_i$ such that $\gamma_i g_i \neq \gamma_i$. There are two cases to consider. If there exists $j \in \mathbb{N}$ such that

$$A = \{\, \gamma_i \ :\ \gamma_i g_i = \gamma_j \,\}$$

is infinite, then A satisfies conditions (a) and (b) above.

Otherwise, we define $\{\delta_1, \delta_2, \ldots\}$ recursively as follows. Let $\delta_1 = \gamma_1$. Assume that $\delta_1, \delta_2, \ldots, \delta_{n-1} \in \{\gamma_1, \gamma_2, \ldots\}$ have already been defined and set

$$B_n = \{\, \gamma_i \ :\ \gamma_i g_i \in \{\delta_1, \delta_2, \ldots, \delta_{n-1}\} \,\}.$$

Since by assumption $\{\, \gamma_i \ :\ \gamma_i g_i = \delta_j \,\}$ is finite for all $j \in \{1, \ldots, n-1\}$, B_n is finite. Hence we may choose δ_n to be any element of

$$\{\gamma_1, \gamma_2, \ldots\} \setminus (B_n \cup \{\delta_1, \delta_2, \ldots, \delta_{n-1}\}).$$

It follows, by construction, that $\{\delta_1, \delta_2, \ldots\}$ satisfies (a) and (b).

Let $h : \Omega \longrightarrow \{\delta_1, \delta_2, \ldots\}$ be the map defined by $\alpha_i h = \delta_i$ and let $k \in \Omega^\Omega$ be defined by

$$\alpha k = \begin{cases} \alpha_i & \text{if } \alpha = \delta_i \text{ for some } i \\ \alpha_1 & \text{if } \alpha \notin \{\delta_1, \delta_2, \ldots\}. \end{cases}$$

Let $f \in S_{1,\alpha}$ be arbitrary. Then define $\widehat{f} \in \Omega^\Omega$ as follows. Let $\alpha \in \Omega$ and let L_j be the component of G containing α. If $\delta_i \in L_j$ for some $i \in \mathbb{N}$ and $\alpha_i f = \alpha_1$, then we define

$$\alpha\widehat{f} = \alpha g_j.$$

Otherwise define $\alpha\widehat{f} = \alpha$. Since $\widehat{f}$ is a homomorphism on each component, $\widehat{f} \in \mathrm{End}(G)$.

Let $\alpha_i \in \Omega$ be arbitrary. Then either $\alpha_i f = \alpha_1$ or $i > 1$ and $\alpha_i f = \alpha_i$. In the former case, if $\delta_i \in L_j$, then

$$\alpha_i h\widehat{f}k = \delta_i\widehat{f}k = \delta_i g_j k = \alpha_1 = \alpha_i f$$

as $\delta_i g_j \notin \{\delta_1, \delta_2, \ldots\}$.

In the latter case,

$$\alpha_i h\widehat{f}k = \delta_i\widehat{f}k = \delta_i k = \alpha_i = \alpha_i f.$$

Thus $S_{1,\alpha} \subseteq \langle\, \mathrm{End}(G), h, k \,\rangle$ and the proof is complete. $\square$

In Example 6.3 we give an instance of a graph G with infinitely many components, all of which are finite, and where $\mathrm{End}(G) \approx S_{\leq}$. In Example 6.2 we show that there exists a graph with infinitely many components and where $S_{1,\alpha} \prec \mathrm{End}(G) \approx S_2 \prec S_{\leq}$. It is not known if there exists a graph G such that $S_{1,\alpha} \approx \mathrm{End}(G)$.

If G is a graph with finitely many components and G is locally finite, then it follows immediately from Theorem 2.4 that $\mathrm{End}(G) \preccurlyeq S_{\leq}$ and $\mathrm{rank}(\Omega^{\Omega} : \mathrm{End}(G)) \geq \mathfrak{d}$. The converse of this statement does not hold and Example 6.1 is a counterexample. This contrasts with the analogous situation for preorders described in Theorem 3.1. In Lemma 3.3 and Example 6.2 we give examples of graphs G and H with finitely many components and where $\mathrm{End}(G) \approx S_{\leq}$ and $\mathrm{End}(H) \approx \{1_{\Omega}\} \prec S_{\leq}$.

Note that in the proofs of Theorems 4.2 and 4.3 neither symmetry nor irreflexivity is used and that these theorems generalise to arbitrary binary relations with infinitely many components. We chose not to phrase these results in the most general way since the only other kinds of relations considered in this paper are preorders and tolerances for which the much stronger Theorem 2.3 holds.

We have not succeeded in proving any general theorem relating to graphs with finitely many components that are not locally finite. However, we will show that there exist such graphs where the relative rank of their endomorphisms in Ω^{Ω} is any of $1, 2, \mathfrak{d}$, or $2^{\aleph_0}$. Moreover, if we restrict our attention to the class of bipartite graphs, then we again obtain a complete classification.

Theorem 4.4. *Let G be a graph with infinitely many bipartite components. Then* $\mathrm{rank}(\Omega^{\Omega} : \mathrm{End}(G)) = 1$ *and so* $\mathrm{End}(G) \approx \Omega^{\Omega}$.

Proof. There are two cases to consider.

Case 1: *there exist infinitely many singleton components $\{\beta_1\}, \{\beta_2\}, \ldots$ in G.* Let $g \in \Omega^{\Omega}$ be defined by $\alpha_i g = \beta_i$ for all $i \in \mathbb{N}$. If $f \in \Omega^{\Omega}$ is arbitrary, then define $\widehat{f}$ by $\beta_i \widehat{f} = \alpha_i f$ for all i and $\alpha \widehat{f} = \alpha$ for all $\alpha \neq \beta_i$ for any i. Then $\widehat{f} \in \mathrm{End}(G)$ and $\alpha_i g \widehat{f} = \beta_i \widehat{f} = \alpha_i f$. Hence $\langle\, \mathrm{End}(G), g \,\rangle = \Omega^{\Omega}$.

Case 2: *there exist infinitely many bipartite components $L_1, L_2, \ldots$ in G with at least two vertices.* Let $\gamma_n \in L_n$ be fixed for all $n \in \mathbb{N}$ and let

$$I = \{\, i \in \mathbb{N} \,:\, \alpha_i \notin L_j \text{ for all } j \in \mathbb{N} \,\}.$$

Then, by definition, $\gamma_m \neq \alpha_n$ for all $m \in \mathbb{N}$ and for all $n \in I$. Also $\mathbb{N} \setminus I$ is infinite as clearly there are infinitely vertices α_i in $L_1 \cup L_2 \cup \cdots$. It follows that there exists an injective $g \in \Omega^{\Omega}$ such that $\gamma_i g = \alpha_i$ for all $i \in I$ and where $(\Omega \setminus \{\gamma_i : i \in I\})g \subseteq \{\gamma_i : i \in \mathbb{N} \setminus I\}$. Hence g^2 is an injection and $\mathrm{im}(g^2) \subseteq \{\gamma_i : i \in \mathbb{N} \setminus I\}$.

Let L_i and L_j be arbitrary and let $\alpha \in L_i$ and $\beta \in L_j$. Since L_i and L_j are bipartite and contain at least two vertices, there exists a homomorphism $\phi_{\alpha,\beta} : L_i \longrightarrow L_j$ such that $\alpha\phi_{\alpha,\beta} = \beta$.

Let $f \in \Omega^{\Omega}$ be arbitrary. We require two endomorphisms $\widehat{f_1}$ and $\widehat{f_2}$ of G that together with g will generate f.

We define $\widehat{f_1}$ on an arbitrary component L as follows. Either there exist $i \in I$, $j \in \mathbb{N}$, and $\alpha \in \Omega$ such that $\alpha f = \alpha_i$, $L = L_j$, and $\alpha g^2 = \gamma_j$, or not. If i, j, and α exist, then define

$$\beta\widehat{f_1} = \beta\phi_{\gamma_j,\gamma_i}$$

for all $\beta \in L$. Otherwise, we define $\beta\widehat{f_1} = \beta$ for all $\beta \in L$. In particular, if $\alpha f = \alpha_i$ for some $i \notin I$, then $\widehat{f_1}$ fixes αg^2. Since $\widehat{f_1}$ is a homomorphism on every component of G, it is an element of $\mathrm{End}(G)$.

We define $\widehat{f_2}$ on an arbitrary component L of G as follows. As above, either there exist $i \in \mathbb{N} \setminus I$, $j \in \mathbb{N}$, and $\alpha \in \Omega$ such that $\alpha f = \alpha_i$, $L = L_j$, and $\alpha g^3 = \gamma_j$, or not. If i, j, and α exist, then, since $i \notin I$, there exists $k \in \mathbb{N}$ such that $\alpha_i \in L_k$. It follows that ϕ_{γ_j,α_i} is well-defined and so we define

$$\beta\widehat{f_2} = \beta\phi_{\gamma_j,\alpha_i}$$

for all $\beta \in L$. Otherwise, we define $\beta\widehat{f_2} = \beta$ for all $\beta \in L$. In particular, if $i \in I$, then, from the definition of I, $\alpha_i \notin L_j$ for all $j \in \mathbb{N}$ and so $\widehat{f_2}$ fixes α_i. Again since $\widehat{f_2}$ is a homomorphism on all the components of G, it follows that $\widehat{f_2} \in \mathrm{End}(G)$.

We will now show that $g^2\widehat{f_1}g\widehat{f_2} = f$. Let $\alpha \in \Omega$ be arbitrary. Then $\alpha f = \alpha_i$ for some $i \in \mathbb{N}$. If $i \in I$ and $\alpha g^2 = \gamma_j$ for some j, then

$$\alpha g^2\widehat{f_1}g\widehat{f_2} = \gamma_j\widehat{f_1}g\widehat{f_2} = \gamma_j\phi_{\gamma_j,\gamma_i}g\widehat{f_2} = \gamma_i g\widehat{f_2} = \alpha_i\widehat{f_2} = \alpha_i = \alpha f.$$

If $i \notin I$ and $\alpha g^3 = \gamma_k$ for some k, then

$$(\alpha g^2)\widehat{f_1}g\widehat{f_2} = \alpha g^3\widehat{f_2} = \gamma_k\widehat{f_2} = \gamma_k\phi_{\gamma_k,\alpha_i} = \alpha_i = \alpha f.$$

Thus $\Omega^\Omega = \langle\, \mathrm{End}(G), g\,\rangle$ and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 1$. $\square$

Theorem 4.5. *Let G be a bipartite graph with finitely many components. Then either:*

(i) *G is locally finite, $\mathrm{End}(G) \approx S_{\leq}$, and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = \mathfrak{d}$; or*
(ii) *G is not locally finite, $\mathrm{End}(G) \approx \Omega^\Omega$, and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) \leq 2$.*

Before we prove Theorem 4.5 we require the following lemma.

Lemma 4.6. *Let G be the graph with edges (α_1, α_i) for all $i > 1$ (see Figure 1 for a diagram). Then $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 1$.*

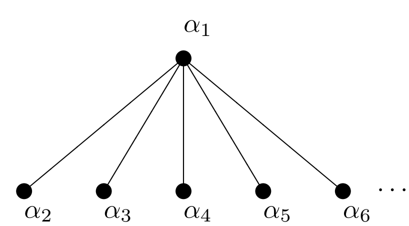

FIGURE 1. The graph from Lemma 4.6

Proof. Note that if $f : \Omega \longrightarrow \Omega$ such that $\alpha_1 f = \alpha_1$ and $\alpha_i f \neq \alpha_1$ for all $i > 1$, then $f \in \mathrm{End}(G)$. Let $g, h \in \mathrm{End}(G)$ be defined by

$$\alpha_i g = \begin{cases} \alpha_i & i = 1 \\ \alpha_{i+1} & i > 1 \end{cases} \qquad \alpha_i h = \begin{cases} \alpha_i & 1 \leq i \leq 2 \\ \alpha_{i-1} & i > 2. \end{cases}$$

Let $t \in \Omega^\Omega$ be a transposition with $\alpha_1 t = \alpha_2$ and vice versa. Then $\alpha_i g t = \alpha_{i+1}$ and $\alpha_{i+1} t h = \alpha_i$ for all $i \in \mathbb{N}$.

Let f be an arbitrary element of Ω^Ω. Define the function $\widehat{f}$ by $\alpha_1\widehat{f} = \alpha_1$ and $\alpha_{i+1}\widehat{f} = \alpha_{k+1}$ whenever $\alpha_i f = \alpha_k$. Then $\widehat{f} \in \mathrm{End}(G)$ by our earlier remark. Furthermore, for an arbitrary vertex $\alpha_i \in \Omega$ with $\alpha_i f = \alpha_k$ we have that

$$\alpha_i gt\widehat{f}th = \alpha_{i+1}\widehat{f}th = \alpha_{k+1}th = \alpha_k = \alpha_i f$$

and so $\langle\, \mathrm{End}(G), t \,\rangle = \Omega^\Omega$. □

Proof of Theorem 4.5. Let G be a bipartite graph with finitely many components $L_1, L_2, \ldots, L_n$.

(i). If G is locally finite, then by Theorem 2.4 we have that $\mathrm{End}(G) \preccurlyeq S_{\leq}$. We will show that $\mathrm{End}(G) \succcurlyeq S_{\leq}$. By Lemma 3.5, there exists a sequence $\gamma_1, \gamma_2, \ldots$ of vertices that induce a subgraph H of G isomorphic to the graph defined in Lemma 3.3(i).

Let $\delta_i \in L_i$ be fixed. For $m = 0, 1, 2, \ldots$ define

$$L_i^{m+1} = \{\, \alpha \in L_i \ : \ \text{the shortest path from } \alpha \text{ to } \delta_i \text{ has length } m \,\}.$$

Let $g \in \Omega^\Omega$ map every point in $L_1^m \cup L_2^m \cup \ldots \cup L_n^m$ to γ_m. Since G is locally finite and at least one L_i is infinite, it follows that g is surjective. If $(\alpha, \beta) \in E$, then, since G is bipartite, $\alpha \in L_j^m$ and $\beta \in L_j^{m+1}$ or $\beta \in L_j^m$ and $\alpha \in L_j^{m+1}$ for some j and m. Hence $(\alpha g, \beta g) = (\gamma_m, \gamma_{m+1}) \in E$ or $(\alpha g, \beta g) = (\gamma_{m+1}, \gamma_m) \in E$. Thus $g \in \mathrm{End}(G)$. So, by Lemma 3.2 and Lemma 3.3 it follows that $\mathrm{End}(G) \succcurlyeq S_{\leq}$.

(ii). Since G is bipartite we may partition Ω into sets R and B such that the edges of G only join vertices in R to vertices in B. Since G is not locally finite it has a vertex of infinite degree. Without loss of generality we assume that $\alpha_1 \in R$ and that α_1 has infinite degree.

Let g be any function such that $\alpha g = \alpha_1$ for all $\alpha \in R$ and

$$Bg \subseteq \{\beta \in \Omega : (\alpha_1, \beta) \in E\} \subseteq B$$

with $|Bg| = \aleph_0$. Then g is an endomorphism of G and the image of g induces a graph isomorphic to that defined in Lemma 4.6. So, by Lemmas 3.2 and 4.6 it follows that $\mathrm{End}(G) \approx \Omega^\Omega$. □

Lemma 4.6 provides an example of a graph G satisfying the hypothesis of Theorem 4.5(ii) and where $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 1$. In Section 6 we give an example of such a bipartite graph H with $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(H)) = 2$.

5. TOLERANCES

Let f be a homomorphism of a graph G with vertices Ω and edges E. Then f cannot map adjacent vertices to the same vertex, since $(\alpha, \alpha) \notin E$ for all $\alpha \in \Omega$. It might be argued that the definition of a homomorphism of a graph could be modified to allow $\alpha f = \beta f$ for $(\alpha, \beta) \in E$. This would be equivalent to considering the endomorphisms of $(\Omega, E \cup \Delta_\Omega)$ where $\Delta_\Omega = \{\, (\alpha, \alpha) \ : \ \alpha \in \Omega \,\}$, that is, the endomorphisms of a tolerance on Ω.

We completely classify the semigroups of endomorphisms of tolerances R on Ω according to $\preccurlyeq$. If (Ω, R) has infinitely many components, then it follows from Theorem 2.3 that $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R)) = 1$.

Theorem 5.1. *Let R be a tolerance on Ω such that (Ω, R) has finitely many components. Then either:*

(i) *(Ω, R) is locally finite,* $\operatorname{End}(\Omega, R) \approx S_{\leq}$*, and* $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R)) = \mathfrak{d}$*; or*
(ii) *(Ω, R) is not locally finite,* $\operatorname{End}(\Omega, R) \approx \Omega^\Omega$*, and* $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R)) \leq 2$.

Proof. Recall that R is a symmetric and reflexive relation, and let $L_1, L_2, \ldots, L_n$ be the components of (Ω, R).

(i). By Theorem 2.4, it follows that $\operatorname{End}(\Omega, R) \preccurlyeq S_{\leq}$. We must prove that $\operatorname{End}(\Omega, R) \succcurlyeq S_{\leq}$. Then, by Lemma 3.5, there exists $\Gamma = \{\gamma_1, \gamma_2, \ldots\}$ such that, for $i \neq j$, $(\gamma_i, \gamma_j) \in R$ if and only if $\{i, j\} = \{k, k+1\}$ for some $k \in \mathbb{N}$.

Let L_i^m be the sets and $g \in \Omega^\Omega$ be the function defined in the proof of Theorem 4.5(i). If $(\alpha, \beta) \in R$, then either $\alpha, \beta \in L_j^m$ or $\alpha \in L_j^m$ and $\beta \in L_j^{m+1}$ for some j and m. In the first case, $(\alpha g, \beta g) = (\gamma_m, \gamma_m) \in R$ and in the second case $(\alpha g, \beta g) = (\gamma_m, \gamma_{m+1}) \in R$. Hence $g \in \operatorname{End}(\Omega, R)$.

Let R' be the subrelation of R induced by Γ. Then by Lemma 3.2 we have that $\operatorname{End}(\Omega, R) \succcurlyeq \operatorname{End}(\Omega, S)$ where (Ω, S) is isomorphic to (Γ, R'). Now, $(\Omega, S \setminus \Delta_\Omega)$ is a graph isomorphic to that defined in Lemma 3.3(i). Thus, by Lemma 3.3, $\operatorname{End}(\Omega, S \setminus \Delta_\Omega) \succcurlyeq S_{\leq}$. As $\operatorname{End}(\Omega, S) \supseteq \operatorname{End}(\Omega, S \setminus \Delta_\Omega)$, it follows that $\operatorname{End}(\Omega, R) \succcurlyeq \operatorname{End}(\Omega, S) \succcurlyeq \operatorname{End}(\Omega, S \setminus \Delta_\Omega) \succcurlyeq S_{\leq}$.

(ii). There exists an element of Ω with infinite degree. Assume without loss of generality that α_1 has infinite degree, that is, $A = \{\beta \in \Omega : (\alpha_1, \beta) \in R\}$ is infinite. It is a straightforward consequence of Ramsey's Theorem [3, Theorem 10.6.1], applied to $(\Omega, R \setminus \Delta_\Omega)$, that the subrelation induced by A contains an infinite subset B such that $(B \times B) \cap R = B \times B$ or Δ_B.

Note that $(\Omega, R \setminus \Delta_\Omega)$ is a graph and $\operatorname{End}(\Omega, R \setminus \Delta_\Omega) \subseteq \operatorname{End}(\Omega, R)$. If $(B \times B) \cap R = B \times B$, then, by Lemma 4.1, $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R \setminus \Delta_\Omega)) = 1$ and so $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R)) = 1$.

If $(B \times B) \cap R = \Delta_B$, then define $g \in \Omega^\Omega$ by $\alpha g = \alpha$ for all $\alpha \in B$ and define $\alpha g = \alpha_1$ for all $\alpha \in \Omega \setminus B$. Since R is reflexive and $(\alpha_1, \beta) \in R$ for all $\beta \in B$, it follows that $g \in \operatorname{End}(\Omega, R)$. Therefore by an argument analogous to that in the previous paragraph, by Lemmas 3.2 and 4.6, $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R)) \leq 2$. ☐

If $G = (\Omega, E)$ is the graph in Lemma 4.6, then $(\Omega, E \cup \Delta_\Omega)$ is a tolerance where $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, E \cup \Delta_\Omega)) = 1$. In Section 8 we construct a tolerance with $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(\Omega, R)) = 2$.

It is natural to ask whether Theorems 3.1 and 5.1 generalise to endomorphisms of reflexive binary relations without the respective assumptions of transitivity and symmetry. The answer is no. In Example 6.5 we construct an example of a reflexive binary relation R such that (Ω, R) is not locally finite but where $\operatorname{End}(\Omega, R) \not\approx \Omega^\Omega$. In Example 6.6, we give an example of a reflexive binary relation R such that (Ω, R) is locally finite but where $\operatorname{End}(\Omega, R) \not\approx S_{\leq}$.

6. Examples I

The following example shows that, in general, the converse of Theorem 2.4 is not true.

Example 6.1. Let G denote the graph with edges (α_1, α_i) and (α_i, α_{i+1}) for all $i \in \mathbb{N}$ (for a diagram see Figure 2). Then G is not locally finite. However, we will show that $\operatorname{End}(G) \preccurlyeq S_{\leq}$ and thus $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(G)) \geq \mathfrak{d}$.

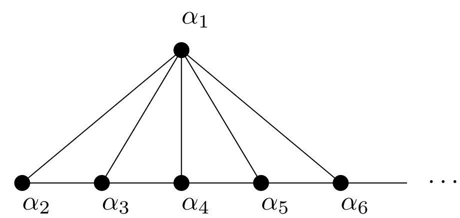

FIGURE 2. The graph from Example 6.1

Let $F = \{\, f \in \operatorname{End}(G) \ : \ \alpha_1 f = \alpha_1 \,\}$ and $U = \operatorname{End}(G) \setminus F$. If H is the graph obtained from G by deleting all the edges incident to α_1, then $F \subseteq \operatorname{End}(H)$. But $\operatorname{End}(H) \approx S_{\leq}$ by Theorem 4.5 and so $F \preccurlyeq S_{\leq}$. In Section 1.1 we defined $\mathfrak{F} = \{\, f \in \Omega^\Omega \ : \ |\Omega f| < \aleph_0 \,\}$ and we noted that Mesyan [11] proved that $\mathfrak{F} \prec S_{\leq}$. Since $U \preccurlyeq \mathfrak{F}$, this implies that $U \prec S_{\leq}$. It follows that $\operatorname{End}(G) = U \cup F \preccurlyeq S_{\leq}$.

In fact, an argument analogous to that used in the proof of Lemma 3.3 shows that $\operatorname{End}(G) \approx S_{\leq}$ and so $\operatorname{rank}(\Omega^\Omega : \operatorname{End}(G)) = \mathfrak{d}$.

Example 6.2. A graph G is called *rigid* if $\operatorname{End}(G) = \{1_\Omega\}$. It follows from [8, Theorem 3] that there exists a locally finite countably infinite rigid graph H with infinitely many components.

We will construct a graph G from the components of H such that $\operatorname{End}(G) \approx S_2$. Let $L_1, L_2, \ldots$ be distinct components of H. Then define G to have components $M_1, M_2, \ldots$ and $N_1, N_2, \ldots$ such that $M_i \neq N_j$ and M_i, N_i, and L_i are isomorphic for all $i, j \in \mathbb{N}$. The only homomorphisms between components of G are the isomorphisms $h_i : M_i \longrightarrow N_i$. Thus for all $\alpha \in \Omega$ the set $\{\, \alpha f \ : \ f \in \operatorname{End}(G) \,\}$ has two elements: α and αh_i if $\alpha \in M_i$ for some i or α and αh_i^{-1} if $\alpha \in N_i$ for some i. If Ω is enumerated in such a way that $\{\alpha_{2i-1}, \alpha_{2i}\} = \{\, \alpha_{2i} f \ : \ f \in \operatorname{End}(G) \,\}$ for all $i \in \mathbb{N}$, then clearly $\operatorname{End}(G) \leq S_2$ and, in particular, $\operatorname{End}(G) \preccurlyeq S_2$.

To show that $\operatorname{End}(G) \succcurlyeq S_2 = \big\{ f \in \Omega^\Omega : \{\alpha_{2i-1} f, \alpha_{2i} f\} \subseteq \{\alpha_{2i-1}, \alpha_{2i}\}$ for all $i \in \mathbb{N} \big\}$, let $m_1 \in M_1, m_2 \in M_2, \ldots$ be fixed, let $g : \Omega \longrightarrow \{\, m_i, m_i h_i \, : \, i \in \mathbb{N} \,\}$ be defined by $\alpha_{2i-1} g = m_i$ and $\alpha_{2i} g = m_i h_i \in N_i$, and let h be any mapping extending g^{-1} to an element of Ω^Ω. If $f \in S_2$ is arbitrary, then there exists $\widehat{f} \in \operatorname{End}(G)$ such that $m_i \widehat{f} = (\alpha_{2i-1} f) g \in \{m_i, m_i h_i\}$ and $m_i h_i \widehat{f} = (\alpha_{2i} f) g \in \{m_i, m_i h_i\}$. Hence $\alpha_{2i-1} g \widehat{f} h = m_i \widehat{f} h = \alpha_{2i-1} f g h = \alpha_{2i-1} f$ and, likewise, $\alpha_{2i} g \widehat{f} h = \alpha_{2i} f$. Therefore $\operatorname{End}(G) \succcurlyeq S_2$, as required.

Let $\operatorname{Aut}(G)$ denote the group of automorphisms from a graph G to G. A *cycle* of length n is a graph G with vertices $\beta_1, \beta_2, \ldots, \beta_n$ and with edges (β_1, β_n) and (β_i, β_{i+1}) for $1 \leq i \leq n-1$.

Example 6.3. Let G be a graph with components $O_1, O_3, O_5, \ldots$ where O_{2i+1} is an odd cycle of length $2i+1$ for all $i \in \mathbb{N}$.

We will show that $\mathrm{End}(G) \approx \mathrm{Aut}(G) \approx S_{\leq}$ and so $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = \mathrm{rank}(\Omega^\Omega : \mathrm{Aut}(G)) = \mathfrak{d}$. It is well-known (and not difficult to verify) that the image of any element in O_{2i+1} under an endomorphism of G lies in O_{2j+1} with $j \leq i$; for a proof see [7, Corollary 1.4]. In other words, $|O_{2i+1}^{\ll}| \leq i$ for all $i \in \mathbb{N}$. It follows, by Theorem 4.3, that $\mathrm{End}(G) \preccurlyeq S_{\leq}$.

Let $\omega(i,1), \omega(i,2), \ldots, \omega(i, 2i+1)$ be the vertices of O_{2i+1}. Then define $g, h \in \Omega^\Omega$ by $\alpha_i g = \omega(i,1)$ and $(\omega(i,j))h = \alpha_j$ for all $i, j \in \mathbb{N}$.

Let $f \in S_{\leq}$ be arbitrary and let $t : \mathbb{N} \longrightarrow \mathbb{N}$ be the map such that $\alpha_i f = \alpha_{it}$ for all $i \in \mathbb{N}$. Note that $it \leq i < 2i+1$ for all i and so the vertex $\omega(i, it)$ exists for all i. Now, for all $i \in \mathbb{N}$ there exists an automorphism of O_{2i+1} mapping $\omega(i,1)$ to $\omega(i, it)$. Let $\widehat{f} \in \Omega^\Omega$ be the union of these automorphisms. By definition, $\widehat{f} \in \mathrm{Aut}(G)$ and

$$\alpha_i g \widehat{f} h = (\omega(i,1))\widehat{f} h = (\omega(i, it))h = \alpha_{it} = \alpha_i f.$$

Thus $S_{\leq} \subseteq \langle \mathrm{Aut}(G), g, h \rangle$ and our claim follows.

Example 6.4. An n-*clique* of a graph G is a subgraph of G isomorphic to the complete graph K_n with n vertices. Let G be a graph with only finite components and let G have arbitrarily large n-cliques. We will show that $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 1$.

Let $L_1, L_2, \ldots$ be the components of G. Then there exist infinitely many disjoint sets $\mathfrak{L}_0, \mathfrak{L}_1, \mathfrak{L}_2, \ldots$ of components such that for all $k \in \mathbb{N} \cup \{0\}$, the set $\mathfrak{L}_k$ contains a component with an n-clique for all $n \in \mathbb{N}$.

Let $M_1, M_2, \ldots$ be distinct elements of $\mathfrak{L}_0$ where M_i contains a clique of size at least $|L_i|$ for all i. Then define g to be any injective endomorphism so that $L_i g$ is contained in M_i for all i. Let $h \in \Omega^\Omega$ be any function which, for $j \geq 1$, maps every vertex lying in a component belonging to $\mathfrak{L}_j$ to α_j and which maps the vertex $\alpha_i g$ (belonging to one of the components in $\mathfrak{L}_0$) into one of the components in $\mathfrak{L}_i$.

Let $f \in \Omega^\Omega$ be arbitrary. Then let $\widehat{f}$ be any endomorphism of G such that: if $\alpha_j = \alpha_i f$, then $L\widehat{f}$ equals the set of vertices of an $|L|$-clique in some component in $\mathfrak{L}_j$ for all $L \in \mathfrak{L}_i$ and $\alpha \widehat{f} = \alpha$ for all α belonging to a component in $\mathfrak{L}_0$. Note that since $\Omega = \{\alpha_1, \alpha_2, \ldots\}$, i and j in the preceding definition are strictly greater than 0.

If $\alpha_i \in \Omega$ is arbitrary, then $\alpha_i gh$ lies in a component in $\mathfrak{L}_i$, $i > 0$. Thus $(\alpha_i gh)\widehat{f}$ lies in a component in $\mathfrak{L}_j$ where $\alpha_j = \alpha_i f$ and $j > 0$. So $(\alpha_i gh\widehat{f})h = \alpha_j = \alpha_i f$. Hence $f = gh\widehat{f}h$ and $\Omega^\Omega = \langle \mathrm{End}(G), h \rangle$.

The purpose of the next two examples is to show that Theorems 3.1 and 5.1 do not generalise to arbitrary reflexive binary relations.

Example 6.5. We construct a relation R on Ω such that (Ω, R) is connected, not locally finite, and $\mathrm{End}(\Omega, R) \preccurlyeq S_{\leq}$. Let $G = (\Omega, E)$ be a connected, locally finite graph, let $B = \{ (\beta_0, \gamma) : \gamma \in \Omega \}$ for a fixed $\beta_0 \in \Omega$, and let $R = E \cup B \cup \Delta_\Omega$. The relation R was constructed so that it is reflexive and (Ω, R) is not locally finite.

Let $\alpha, \beta \in \Omega$ such that α, β are adjacent in G and let $f \in \mathrm{End}(\Omega, R)$. Then $(\alpha f, \beta f) \in R$ and $(\beta f, \alpha f) \in R$. Hence $\alpha f = \beta f$ or αf and βf are adjacent in G. We conclude that $\mathrm{End}(\Omega, R) \subseteq \mathrm{End}(\Omega, E \cup \Delta_\Omega) \approx S_{\leq}$ by Theorem 5.1(i) and so $\mathrm{End}(\Omega, R) \preccurlyeq S_{\leq}$.

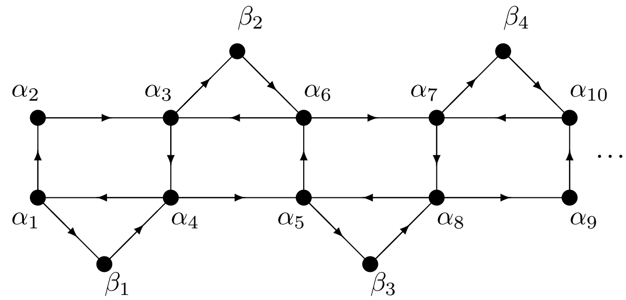

FIGURE 3. The binary relation from Example 6.6. The relations (α, α) for all $\alpha \in \Omega$ are not shown.

Example 6.6. Let $\Omega = \{\alpha_1, \alpha_2, \ldots\} \cup \{\beta_1, \beta_2, \ldots\}$ and define the following relation R on Ω. Let $(\alpha, \alpha) \in R$ for all $\alpha \in \Omega$ and let

$$(\alpha_i, \alpha_{i+1}), (\alpha_{2i+2}, \alpha_{2i-1}), (\alpha_{2i-1}, \beta_i), (\beta_i, \alpha_{2i+2}) \in R$$

for all $i \in \mathbb{N}$. A diagram of (Ω, R) can be found in Figure 3. The relation R is reflexive and (Ω, R) is connected and locally finite. We will prove that $\operatorname{End}(\Omega, R) \preccurlyeq \mathfrak{F} \prec S_{\leq}$.

Let $f \in \operatorname{End}(\Omega, R)$, let $A_i = \{\alpha_{2i-1}, \alpha_{2i}, \alpha_{2i+1}, \alpha_{2i+2}\}$, and let $B_i = \{\alpha_{2i-1}, \alpha_{2i+2}, \beta_i\}$ for all $i \in \mathbb{N}$. We start by proving that for all $i \in \mathbb{N}$ one of the following holds: $A_i f$ is a singleton, $A_i f = A_j$, or $A_i f = B_j$ for some $j \in \mathbb{N}$. We will also show that if $A_i f = A_j$, then

(1) $\quad \beta_i f = \beta_j$ and $(\alpha_{2i-1} f, \alpha_{2i} f, \alpha_{2i+1} f, \alpha_{2i+2} f) = (\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2})$.

Since f is a homomorphism, $A_i f = \{\gamma_1, \ldots, \gamma_k\}$ where $1 \leq k \leq 4$ and for all $1 \leq j \leq k-1$ we have that $(\gamma_k, \gamma_1), (\gamma_j, \gamma_{j+1}) \in R$. The only subsets of Ω that satisfy this condition are singletons, A_j, or B_j for some $j \in \mathbb{N}$. Thus $A_i f$ is either a singleton, $A_i f = A_j$, or $A_i f = B_j$ for some $j \in \mathbb{N}$.

In the case that, $A_i f = A_j$, since f is an endomorphism, we have that

$$(\alpha_{2i+2} f, \alpha_{2i-1} f), (\alpha_{2i-1} f, \beta_i f), (\beta_i f, \alpha_{2i+2} f) \in R.$$

The only $\gamma, \delta \in A_j$ with $(\gamma, \delta) \in R$ such that there exists $\lambda \in \Omega$ with $(\delta, \lambda), (\lambda, \gamma) \in R$ are α_{2j-1} and α_{2j+2}. It follows that $\beta_i f = \beta_j$ and $(\alpha_{2i-1} f, \alpha_{2i} f, \alpha_{2i+1} f, \alpha_{2i+2} f) = (\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2})$.

We will now prove that there are only countably many elements of $\operatorname{End}(\Omega, R)$ with infinite image. Note that the only element of Ω not in any B_j is α_2. There are 3 cases to consider.

Case 1: $A_1 f = A_j$ **for some** $j \in \mathbb{N}$. In this case, from (1), $\beta_1 f = \beta_j$ and $(\alpha_1 f, \alpha_2 f, \alpha_3 f, \alpha_4 f) = (\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2})$. Since $\alpha_3 f$ and $\alpha_4 f$ are distinct, $A_2 f$ is not a singleton. Also if $\alpha_3 f \in B_i$ and $\alpha_4 f \in B_k$, then $i \neq k$ and so $A_2 f \neq B_i$ for all $i \in \mathbb{N}$. Hence $A_2 f = A_k$ for some $k \in \mathbb{N}$. It follows from (1) that $\alpha_3 f = \alpha_{2(j+1)-1}$ and $\alpha_4 f = \alpha_{2(j+1)}$. Thus $A_2 f = A_{j+1}$ and so again, from (1), $\alpha_5 f = \alpha_{2(j+1)+1}$, $\alpha_6 f = \alpha_{2(j+1)+2}$ and $\beta_2 f = \beta_{j+1}$.

Repeating this process it follows that $\alpha_i f = \alpha_{2(j-1)+i}$ and $\beta_i f = \beta_{(j-1)+i}$ for all $i \in \mathbb{N}$. In particular, there are only countably many endomorphisms f with $A_1 f = A_j$ for some $j \in \mathbb{N}$.

Case 2: $A_1 f \subseteq B_j$ for some $j \in \mathbb{N}$. In this case,

$$\alpha_3 f, \alpha_4 f \in B_j = \{\alpha_{2j-1}, \alpha_{2j+2}, \beta_j\}.$$

Since $(\alpha_3 f, \alpha_4 f) \neq (\alpha_{2k-1}, \alpha_{2k})$ for all $k \in \mathbb{N}$, it follows by (1) that $A_2 f \neq A_k$ for all $k \in \mathbb{N}$. Thus either $A_2 f = B_j$ or $A_2 f$ is a single element of B_j and in either case $A_2 f \subseteq B_j$.

Repeating this argument, we conclude that $\omega f \in B_j$ for all $\omega \in \Omega$ and f has finite image.

Case 3: $A_1 f = \{\alpha_2\}$. In particular, $\alpha_3 f = \alpha_4 f$ and so $|A_2 f| < 4$. Thus by (1) $A_2 f \neq A_k$ for all $k \in \mathbb{N}$. Furthermore, $\alpha_2 \notin B_k$ for all $k \in \mathbb{N}$ and so $A_2 f \neq B_k$ for all $k \in \mathbb{N}$. Thus $A_2 f = \{\alpha_2\}$. Repeating this argument it follows that $\mathrm{im}(f) = \{\alpha_2\}$.

Since there are only countably many endomorphisms of (Ω, R) with infinite image we conclude that $\mathrm{End}(\Omega, R) \preccurlyeq \mathfrak{F} \prec S_{\leq}$. Note that, on the other hand, it is possible to show that $|\mathrm{End}(\Omega, R)| = 2^{\aleph_0}$ and so $\mathrm{End}(\Omega, R) \succ \{1_\Omega\}$.

7. Examples II – graphs with rank 2

In this section we construct two examples of graphs G, one connected and one with infinitely many components, such that $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 2$.

Lemma 7.1. *Let U be a subsemigroup of Ω^Ω such that $f \in U$ is injective if and only if f is surjective. Then $\mathrm{rank}(\Omega^\Omega : U) \geq 2$.*

Proof. Let $g \in \Omega^\Omega$ be arbitrary. Seeking a contradiction assume that $\langle\, U, g \,\rangle = \Omega^\Omega$. Let $h \in \Omega^\Omega$ be injective but not surjective and let $k \in \Omega^\Omega$ be surjective but not injective. Then there exist $h_1, h_2, \ldots, h_m, k_1, k_2, \ldots, k_n \in U \cup \{g\}$ such that $h = h_1 h_2 \cdots h_m$ and $k = k_1 k_2 \cdots k_n$.

Let

$$M = \min\{\, i \ :\ h_1 h_2 \cdots h_i \text{ is not surjective} \,\}$$

and

$$N = \max\{\, i \ :\ k_i k_{i+1} \cdots k_n \text{ is not injective} \,\}.$$

Then h_M is injective, as h is injective, and so $h_M = g$. On the other hand, k_N is surjective, as k is surjective, and so $k_N = g$. But then g is injective and not injective, a contradiction. □

An example of a connected but not locally finite poset $(\Omega, \sqsubseteq)$ where the only injective or surjective endomorphism is the identity is given in [9, Section 6]. It follows from Theorem 3.1 and Lemma 7.1 that $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(\Omega, \sqsubseteq)) = 2$. We will use this poset to define a bipartite graph with the same property. The poset $(\Omega, \sqsubseteq)$ is described as follows.

Let $A = \{\, a_i \ :\ i \in \mathbb{N} \,\}$ be a countably infinite set. Let $\mathcal{E}$ denote the set of all finite subsets E of A such that $|E| \geq 2$ and where $a_n \in E$ implies that $|E| \leq n + 1$. Thus any set in $\mathcal{E}$ containing a_1 has cardinality 2, any set in $\mathcal{E}$ containing a_2 has cardinality 2 or 3, any set in $\mathcal{E}$ containing a_3 has cardinality 2, 3 or 4, etc. We enumerate the elements of $\mathcal{E}$ as $A_1, A_2, \ldots$ Now, we assign in a one-to-one way a new element b_E, not in A, to every E in $\mathcal{E}$. Let $B = \{\, b_E \ :\ E \in \mathcal{E} \,\}$. Also, let $C = \{c_0, c_1, c_2, \ldots\}$ be any set disjoint from $A \cup B$.

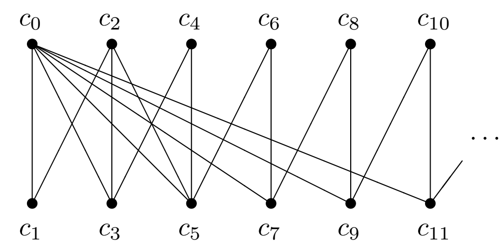

FIGURE 4. The poset $\sqsubseteq$ restricted to C

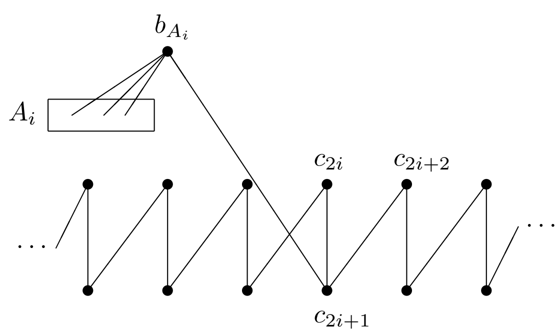

FIGURE 5. A portion of the poset $(\Omega, \sqsubseteq)$.

We define the partial order $\sqsubseteq$ on the elements of $\Omega = A \cup B \cup C$ by: $a \sqsubseteq b_E$ for all $a \in E$; $c_{2i+1} \sqsubseteq c_0$ for all $i \geq 0$; $x \sqsubseteq c_2$ for all $x \in \{c_1, c_3, c_5\}$; $c_{2i-1} \sqsubseteq c_{2i}$, $c_{2i+1} \sqsubseteq c_{2i}$ for all $i \geq 2$; and $c_{2i+1} \sqsubseteq b_{A_i}$ for all $i \geq 0$. See Figures 4 and 5 for two diagrams of portions of $(\Omega, \sqsubseteq)$.

Theorem 7.2. *Let $\sqsubseteq$ be the partial order defined above and let $f \in \mathrm{End}(\Omega, \sqsubseteq)$ be injective or surjective. Then f is the identity mapping on Ω.*

For a proof see [9, Theorem 6.7].

We construct a graph $G = (\Omega, E)$ from the poset $(\Omega, \sqsubseteq)$ by letting

$$(\alpha, \beta), (\beta, \alpha) \in E \text{ whenever } \alpha \neq \beta \text{ and } \alpha \sqsubseteq \beta.$$

Let $P = A \cup \{\, c_{2i+1} \;:\; i \in \mathbb{N} \cup \{0\} \,\}$ and $Q = B \cup \{\, c_{2i} \;:\; i \in \mathbb{N} \cup \{0\} \,\}$. Note that if $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$ and $\alpha \sqsubseteq \beta$, then $\alpha \in P$ and $\beta \in Q$. Note that every edge in G connects a vertex in P to one in Q and so G is bipartite.

Lemma 7.3. *Let $f \in \mathrm{End}(G)$. If there exists $\alpha \in P$ such that $\alpha f \in P$, then $f \in \mathrm{End}(\Omega, \sqsubseteq)$. Likewise, if there exists $\alpha \in Q$ such that $\alpha f \in Q$, then $f \in \mathrm{End}(\Omega, \sqsubseteq)$.*

Proof. We will prove the lemma in the case where $\alpha, \alpha f \in P$. The proof of the other case is identical. Let $\beta \in P$. Since G is connected there exists a path from α to β. Furthermore, this path has even length since $\alpha, \beta \in P$ and G is bipartite. Thus there is a walk of even length from αf to βf. It follows that $\beta f \in P$ since $\alpha f \in P$. On the other hand, if $\beta \in Q$, then any path from α to β has odd length and so there is a walk of odd length from $\alpha f \in P$ to βf. Thus $\beta f \in Q$. It follows that $Pf \subseteq P$ and $Qf \subseteq Q$. Now let $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$ and $\alpha \sqsubseteq \beta$. Then $(\alpha f, \beta f) \in E$ and $\alpha f \in P, \beta f \in Q$. Thus $\alpha f \sqsubseteq \beta f$ and hence $f \in \mathrm{End}(\Omega, \sqsubseteq)$. $\square$

Using Lemma 7.3 we prove that the graph obtained from $(\Omega, \sqsubseteq)$ has no non-identity injective or surjective endomorphisms. To do so, we will make use of the following notion.

If R is a binary relation on Ω, $\alpha \in \Omega$ and $n \in \mathbb{N}$, then let

$$B(\alpha, n) = \{\, \beta \in \Omega \;:\; \text{there exists a path of length at most } n \text{ from } \alpha \text{ to } \beta \,\}.$$

The proof of the following lemma is straightforward and omitted.

Lemma 7.4. *Let $R \subseteq \Omega \times \Omega$ and let $\alpha \in \Omega$. If $B(\alpha, n) = \Omega$ for some $n \in \mathbb{N}$ and $f \in \mathrm{End}(\Omega, R)$ is surjective, then $B(\alpha f, n) = \Omega$.*

Theorem 7.5. *Let G be the graph defined above and let $f \in \mathrm{End}(\Omega, G)$ be injective or surjective. Then f is the identity mapping on Ω.*

Proof. Let $g \in \mathrm{End}(G)$ be injective. Note that all vertices of $A \subseteq P$ have infinite degree but c_0 is the only vertex of Q with infinite degree. Since injective endomorphisms map vertices of infinite degree to vertices of infinite degree, it follows that $ag \in Q$ for at most one $a \in A$. In particular, there exists $a \in A$ such that $af \in P$ and so, by Lemma 7.3, $g \in \mathrm{End}(\Omega, \sqsubseteq)$. By Theorem 7.2 this implies that g is the identity on Ω.

Let $h \in \mathrm{End}(G)$ be surjective. We will show that $c_0 h = c_0$. From the definition of G we have that $B(c_0, 1) = \{c_0\} \cup \{\, c_{2i+1} : i \in \mathbb{N} \cup \{0\} \,\}$ and thus $B(c_0, 2) = B \cup C$ and $B(c_0, 3) = \Omega$. We will prove that $B(\alpha, 3) \neq \Omega$ for all $\alpha \neq c_0$.

If $a_i \in A$, then $B(a_i, 3) \cap \{\, c_{2k+1} : k \in \mathbb{N} \cup \{0\} \,\} = \{\, c_{2j+1} : a_i \in A_j \,\} \neq \{\, c_{2k+1} : k \in \mathbb{N} \cup \{0\} \,\}$. If $b_E \in B$, then $B(b_E, 3) \cap B = \{\, b_F \in B : E \cap F \neq \emptyset \,\} \neq B$. If $i \geq 0$, then $B(c_{2i+1}, 3) \cap A = \{\, a_j \in A : a_j \in A_i \,\} \neq A$. Finally, if $i \geq 1$, then $B(c_{2i}, 3) \cap A$ is finite.

Thus c_0 is the unique vertex α of G such that $B(\alpha, 3) = \Omega$. It follows by Lemma 7.4 that $c_0 h = c_0$. Thus $h \in \mathrm{End}(\Omega, \sqsubseteq)$ by Lemma 7.3 and hence h is the identity on Ω by Theorem 7.2. □

Corollary 7.6. *Let G be the graph obtained from $(\Omega, \sqsubseteq)$. Then $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 2$.*

Proof. Since G is bipartite and not locally finite, by Theorem 4.5(ii), $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) \leq 2$. On the other hand, G has no non-identity injective or surjective endomorphisms by Theorem 7.5. Thus $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) \geq 2$ by Lemma 7.1. □

The following example shows that there are graphs G with infinitely many components and $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 2$. We require the following notion. A graph is a *core* if every endomorphism is an automorphism. If G is a graph where every component is a core and no two components are isomorphic, then the preorder $\ll$ defined in Section 4 is a partial order on the set of components of G.

Theorem 7.7. [7, Theorem 3.3] *Let P be a countable poset. Then there exists a graph G where every component is a finite core and the set of components of G under $\ll$ is isomorphic to P.*

Example 7.8. Let G be a graph with finite components the distinct cores $L_1, L_2, \ldots$ and $M_1, M_2, \ldots$ such that there exists a homomorphism from $L_i \longrightarrow M_j$ for all i, j and there are no further homomorphisms between components. Such a graph exists by Theorem 7.7.

Now $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) \leq 2$ by Theorem 4.3. Furthermore, every injective endomorphism of G must fix each component setwise. It follows that every injective endomorphism is surjective. Likewise all surjective endomorphisms are also injective. So, using Lemma 7.1, we conclude that $\mathrm{rank}(\Omega^\Omega : \mathrm{End}(G)) = 2$.

8. Examples III – A tolerance with rank 2

Let $\Omega = A \cup B$ where A and B are the sets defined in Section 7 and let $\sqsubseteq$ be the partial order defined in Section 7 restricted to $A \cup B$.

Lemma 8.1. *Let $\sqsubseteq$ be the partial order defined above and let $f \in \mathrm{End}(\Omega, \sqsubseteq)$ be surjective. Then f is the identity mapping on Ω.*

For a proof see [9, Lemma 6.5].

We define a tolerance R based on $\sqsubseteq$ by letting $(\alpha, \beta), (\beta, \alpha) \in R$ whenever $\alpha = \beta$ or $\alpha \sqsubseteq \beta$.

The following lemma is routine and the proof omitted.

Lemma 8.2. *If $f \in \mathrm{End}(\Omega, R)$ such that $Af \subseteq A$, then $f \in \mathrm{End}(\Omega, \sqsubseteq)$.*

Next, we prove that (Ω, R) has no non-identity surjective endomorphisms.

Lemma 8.3. *Let R be the tolerance defined above and let $f \in \mathrm{End}(\Omega, R)$ be surjective. Then f is the identity mapping on Ω.*

Proof. Let $a_i \in A$. For any $a_j \in A$ there exists $b_E \in B$ such that $a_i, a_j \in E$. Hence $B(a_i, 2) \supseteq A$ and so $B(a_i, 3) = \Omega$. On the other hand, if $b_E \in B$ is arbitrary, then $B(b_E, 3) \cap B = B(b_E, 2) \cap B = \{ b_F \in B \ : \ E \cap F \neq \emptyset \} \neq B$ by construction. Thus $B(b_E, 3) \neq \Omega$. Let $f \in \mathrm{End}(\Omega, R)$ be surjective. It follows by Lemma 7.4 that $Af \subseteq A$. Hence $f \in \mathrm{End}(\Omega, \sqsubseteq)$ by Lemma 8.2 and thus f is the identity on Ω by Lemma 8.1. $\square$

Although (Ω, R) has no non-identity surjective endomorphisms, it does have injective endomorphisms that are not surjective. So, in order to apply Lemma 7.1, we will define a new tolerance R^* on a set Σ based on (Ω, R) such that $f \in \mathrm{End}(\Sigma, R^*)$ is injective if and only if f is surjective.

Let $\{ c(i,j) \ : \ i, j \in \mathbb{N} \}$ be a set of new points with no elements in A and B, let B be as above, let $a_i^* = \{c(i,1), c(i,2), \ldots, c(i, i+2)\}$, let $C = a_1^* \cup a_2^* \cup \cdots$ and let $\Sigma = B \cup C$. Then define R^* to be the symmetric and reflexive closure of the set containing:

(i) $(c(i,j), c(i,j+1))$ for all $j \in \{1, \ldots, i+1\}$ and $(c(i,i+2), c(i,1))$ for all i;
(ii) (b_E, c) for all $c \in a_i^*$ and for all i such that $a_i \in E$.

Note that a_i^* is a cycle of length $i+2$ for all i.

Theorem 8.4. *Let (Σ, R^*) be the tolerance defined above. Then $f \in \mathrm{End}(\Sigma, R^*)$ is injective if and only if f is surjective.*

Proof. Let $c(i,j) \in C$ and $b_E \in B$. Then, by a similar argument to the one in the proof of Lemma 8.3, $B(c(i,j), 3) = \Sigma$ and $B(b_E, 3) \neq \Sigma$. Let $f \in \mathrm{End}(\Sigma, R^*)$ be surjective. It follows, by Lemma 7.4, that $cf \in C$ for all $c \in C$. Furthermore, since

f is a homomorphism, for any $i \in \mathbb{N}$ we have that $a_i^* f \subseteq a_j^*$ for some $j \in \mathbb{N}$. We may thus define $\widehat{f} \in \Omega^\Omega$ (recall that $\Omega = A \cup B$) by

$$\alpha\widehat{f} = \begin{cases} a_j & \text{if } \alpha = a_i \text{ and } a_i^* f \subseteq a_j^* \\ a_j & \text{if } \alpha \in B \text{ and } \alpha f \in a_j^* \\ \alpha f & \text{if } \alpha \in B \text{ and } \alpha f \in B. \end{cases}$$

Then $\widehat{f}$ is surjective since f is surjective. Moreover, if $a_i \in E$, then $(a_i, b_E) \in R$ and so $(c(i,j), b_E) \in R^*$ for all j. Hence $(c(i,j)f, b_E f) \in R^*$ for all j. If $a_i^* f \subseteq a_j^*$ and $b_E f \in C$, then $b_E f \in a_j^*$ and so $(a_i\widehat{f}, b_E\widehat{f}) = (a_j, a_j) \in R$. Otherwise, $b_E f = b_F \in B$ for some $F \in \mathcal{E}$ and so $a_j \in F$. Hence $(a_i\widehat{f}, b_E\widehat{f}) = (a_j, b_F) \in R$. Therefore $\widehat{f} \in \mathrm{End}(\Omega, R)$ and it follows that $\widehat{f}$ is the identity by Lemma 8.3. Therefore $bf = b$ for all $b \in B$ and the components a_j^* are fixed setwise by f. Since every a_j^* is finite and f is surjective, it follows that $f \in \mathrm{Aut}(\Sigma, R^*)$.

Let $f \in \mathrm{End}(\Sigma, R^*)$ be injective. Since every element in C has infinite degree and every element in B has finite degree, it follows that $Cf \subseteq C$. Hence for all $i \in \mathbb{N}$ we have that $a_i^* f \subseteq a_j^*$ for some $j \in \mathbb{N}$. But since f is injective and $|a_i^* f| = |a_i^*| = i + 2$ it follows that $j \geq i$. On the other hand, there does not exist an injective homomorphism from the cycle a_i^* to any cycle a_j^* where $j > i$. Hence $i = j$ and so $f \in \mathrm{Aut}(\Sigma, R^*)$. □

Corollary 8.5. *Let* (Σ, R^*) *be the tolerance defined above. Then* $\mathrm{rank}(\Sigma^\Sigma : \mathrm{End}(\Sigma, R^*)) = 2$.

Proof. By Theorem 5.1, $\mathrm{rank}(\Sigma^\Sigma : \mathrm{End}(\Sigma, R^*)) \leq 2$. By Theorem 8.4 and Lemma 7.1, $\mathrm{rank}(\Sigma^\Sigma : \mathrm{End}(\Sigma, R^*)) \geq 2$. □

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M. Morayne: Institute of Mathematics and Computer Science, Wrocław University of Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland.

e-mail: michal.morayne@pwr.wroc.pl

J. D. Mitchell, Y. Péresse, and M. R. Quick: Mathematics Institute, University of St Andrews, North Haugh, St Andrews, Fife, KY16 9SS, Scotland.
emails: jdm3@st-and.ac.uk, yhp1@st-and.ac.uk, mq3@st-and.ac.uk