EVERY GROUP IS A MAXIMAL SUBGROUP OF THE FREE IDEMPOTENT GENERATED SEMIGROUP OVER A BAND

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Dedicated to Stuart W. Margolis on the occasion of his 60th birthday

ABSTRACT. Given an arbitrary group G we construct a semigroup of idempotents (band) B_G with the property that the free idempotent generated semigroup over B_G has a maximal subgroup isomorphic to G. If G is finitely presented then B_G is finite. This answers several questions from recent papers in the area.

1. INTRODUCTION

Let S be a semigroup. The set $E = E(S)$ of all idempotents of S carries a structure of a partial algebra, called the *biordered set* of S, by retaining the products of the so-called *basic pairs*: these are pairs of idempotents $\{e, f\}$ such that $\{ef, fe\} \cap \{e, f\} \neq \emptyset$. It should be noted that if $ef \in \{e, f\}$ then fe is also an idempotent, possibly different from e, f and ef. Also, if S is an idempotent semigroup (i.e. a *band*) then its biordered set is in general different from S itself, since not every pair is necessarily basic. The term 'biordered set' comes from an alternative (but equivalent) approach, where one considers $E(S)$ as a relational structure equipped with two partial pre-orders; here we shall not pursue this approach, directing instead to [4, 5, 6, 11, 16] for further background.

The class of idempotent generated semigroups is of prime importance in semigroup theory, with a host of natural examples, such as the semigroups of singular (non-bijective) transformations of a finite set (Howie [12]) or singular $n \times n$ matrices over a field (Erdos [7]). It is not difficult to show that the category of all idempotent generated semigroups with a fixed biordered set E has an initial object $\mathsf{IG}(E)$, called the *free idempotent generated semigroup* over E (we shall also say 'over S' when $E = E(S)$). This semigroup is defined by the presentation

$$\mathsf{IG}(E) = \langle E \mid e \cdot f = ef \ (\{e, f\} \text{ is a basic pair}) \rangle.$$

Here $e \cdot f$ stands for a word of length 2 in the free semigroup E^+, while ef is the element of E to which the product equals in S. Unsurprisingly, $\mathsf{IG}(E)$ plays a crucial rule in understanding the structure of semigroups with a prescribed biordered set of idempotents.

For reasons that are intrinsic to basic structure theory of semigroups [11, 13], this in turn depends upon the knowledge of maximal subgroups of $\mathsf{IG}(E)$. It was conjectured for a long time that the maximal subgroups of $\mathsf{IG}(E)$ are always free; this conjecture was widely circulated back in the 1980s, and was explicitly recorded in [15]. The conjecture was proved in a number of particular cases, see e.g. [15, 17, 19]. In 2009, Brittenham, Margolis and Meakin [1] disproved the conjecture by means of

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an explicit 72-element semigroup S such that $\mathsf{IG}(E(S))$ has a maximal subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, the free abelian group of rank 2. This was followed by Gray and Ruškuc [9] who proved that *every* group arises as a maximal subgroup of $\mathsf{IG}(E(S))$ for a suitably chosen semigroup S; if the group in question is finitely presented then a finite S will suffice. Further ensuing work such as [10, 3, 8] investigates maximal subgroups of $\mathsf{IG}(S)$ for some specific natural semigroups S, and the first author [2] initiates the study of $\mathsf{IG}(B)$, where B is a band.

The aim of the present note is to prove the result announced in the title:

Theorem 1. *Let G be a group. Then there exists a band B_G such that $\mathsf{IG}(B_G)$ has a maximal subgroup isomorphic to G. Furthermore, if G is finitely presented, then B_G can be constructed to be finite.*

This single construction provides an alternative, simpler proof of *all* the main results of [9] (Theorems 1–4), resolves [9, Problem 1] which asks whether every finitely presented group is a maximal subgroup of $\mathsf{IG}(S)$ for some finite *regular* semigroup S, and solves [2, Problem 2] which calls for a characterisation of maximal semigroups of free idempotent generated semigroups over bands.

2. Presentation for maximal subgroups

A general presentation for maximal subgroups of $\mathsf{IG}(S)$ in terms of parameters that depend only on the structure of S has been exhibited in [9, Theorem 5]. Since we are interested here only in the case of bands, we utilise the particular form of this theorem, deduced in [2, Corollary 5].

First of all, recall [13, Theorem 4.4.1] that any band B decomposes into a semilattice of rectangular bands, which are the $\mathscr{D}$-classes of B. Thus a $\mathscr{D}$-class D of S can be viewed as an $I \times J$ 'table' of idempotents e_{ij} ($i \in I, j \in J$), where $\{R_i : i \in I\}$ and $\{L_j : j \in J\}$ are the $\mathscr{R}$- and $\mathscr{L}$-classes in D respectively. For $i, k \in I$ and $j, l \in J$ we refer to the tuple $(e_{ij}, e_{il}, e_{kj}, e_{kl})$ as the $(i, k; j, l)$ *square*.

Suppose now we have an element $f \in B$ belonging to a $\mathscr{D}$-class above D. From the basic theory of bands (see, for example, [13, Section 4.4]) we know that f induces idempotent mappings $\sigma : I \to I, i \mapsto \sigma(i)$, and $\tau : J \to J, j \mapsto (j)\tau$, such that for all $i \in I, j \in J$ we have

$$fe_{ij} = e_{\sigma(i),j},\ e_{ij}f = e_{i,(j)\tau}.$$

We say that the square $(i, k; j, l)$ is *singular* induced by f if one of the following holds:

(a) $\sigma(i) = i, \sigma(k) = k$ and $(j)\tau = (l)\tau \in \{j, l\}$; or
(b) $\sigma(i) = \sigma(k) \in \{i, k\}$ and $(j)\tau = j$, $(l)\tau = l$.

We talk of a *left-right* or *up-down* singular square depending on whether (a) or (b) applies.

With the above conventions the general presentation we need is as follows:

Proposition 2 ([9, 2]). *The maximal subgroup H of $\mathsf{IG}(B)$ containing $e_{11} \in D$ is presented by*

$$\langle f_{ij}\ (i \in I,\ j \in J) \mid \quad f_{i1} = f_{1j} = 1 \qquad (i \in I,\ j \in J), \tag{1}$$

$$f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} \qquad ((i, k; j, l) \text{ a singular square in } D)\rangle. \tag{2}$$

3. Construction of B_G

Let G be any group. Let us choose and fix a presentation $\langle A \mid R \rangle$ for G in which every relation has the form $ab = c$ for some $a, b, c \in A$. It is clear that G has such a presentation – for instance the Cayley table would do. What is less obvious, but nonetheless still true, is that if G is finitely presented then it has a *finite* presentation of this form. One way of seeing this is as follows: A relation $a_1 \dots a_k = b_1 \dots b_l$ can be replaced by two relations of the form $a_1 \dots a_k = c$, $b_1 \dots b_l = c$, at the expense of introducing a new generator c. Furthermore, the relation $a_1 \dots a_k = c$ can be replaced by $k-1$ relations $a_1 a_2 = d_2$, $d_2 a_3 = d_3$, $\dots$, $d_{k-2} a_{k-1} = d_{k-1}$, $d_{k-1} a_k = c$ of the desired form, with new generators $d_2, \dots, d_{k-1}$.

Define sets

$$A_0 = A \cup \{0\},\ A_0' = \{a' \,:\, a \in A_0\},\ I = A_0 \cup A_0',\ J = A_0 \cup \{\infty\},$$

where 0, ∞ and a' ($a \in A_0$) are symbols distinct from each other and those already in A. Consider the semigroup $\mathcal{T} = \mathcal{T}_I^{(l)} \times \mathcal{T}_J^{(r)}$, where $\mathcal{T}_I^{(l)}$ (respectively $\mathcal{T}_J^{(r)}$) is the semigroup of all mappings $I \to I$ (resp. $J \to J$) written on the left (resp. right). The semigroup $\mathcal{T}$ has a unique minimal ideal K consisting of all (σ, τ) with both σ and τ constant. This ideal is naturally isomorphic to the rectangular band $I \times J$, and we will identify the two. We will visualise K as in Figure 1.

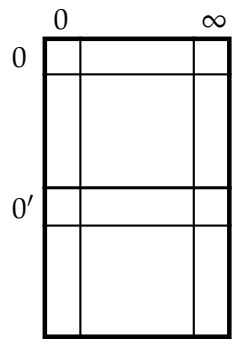

FIGURE 1. A visual representation of $K = I \times J$, highlighting the partition $I = A_0 \cup A_0'$, as well as the four distinguished rows and columns.

We now define a set $L \subseteq \mathcal{T}$. All elements $(\sigma, \tau) \in L$ will have

$$\sigma^2 = \sigma,\ \tau^2 = \tau,\ \ker(\sigma) = \{A_0, A_0'\},\ \mathrm{im}(\tau) = A_0. \tag{3}$$

Recall that $\ker(\sigma)$ is the equivalence on I defined by $(i, i') \in \ker(\sigma)$ if and only if $\sigma(i) = \sigma(i')$, and that it can be identified with the resulting partition of I into equivalence classes. Therefore, each (σ, τ) will be uniquely determined by $\mathrm{im}(\sigma)$ which must be a two-element set that is a cross-section of $\{A_0, A_0'\}$, and the value $(\infty)\tau \in A_0$. The elements of L come in four groups: $\boldsymbol{Z}$ – the initial pair; $\boldsymbol{G}$, $\overline{\boldsymbol{G}}$ – the elements arising from the generators A; $\boldsymbol{R}$ – the elements arising from the relations R:

Type	Notation	Indexing	$\mathrm{im}(\sigma)$	$(\infty)\tau$
$\boldsymbol{Z}$	(σ_0, τ_0)	–	$\{0, 0'\}$	0
$\boldsymbol{G}$	(σ_a, τ_a)	$a \in A$	$\{0, a'\}$	a
$\overline{\boldsymbol{G}}$	$(\overline{\sigma}_a, \overline{\tau}_a)$	$a \in A$	$\{a, a'\}$	0
$\boldsymbol{R}$	$(\sigma_{\boldsymbol{r}}, \tau_{\boldsymbol{r}})$	$\boldsymbol{r} = (ab, c) \in R$	$\{b, c'\}$	a

These elements can be visualised as shown in Figure 2.

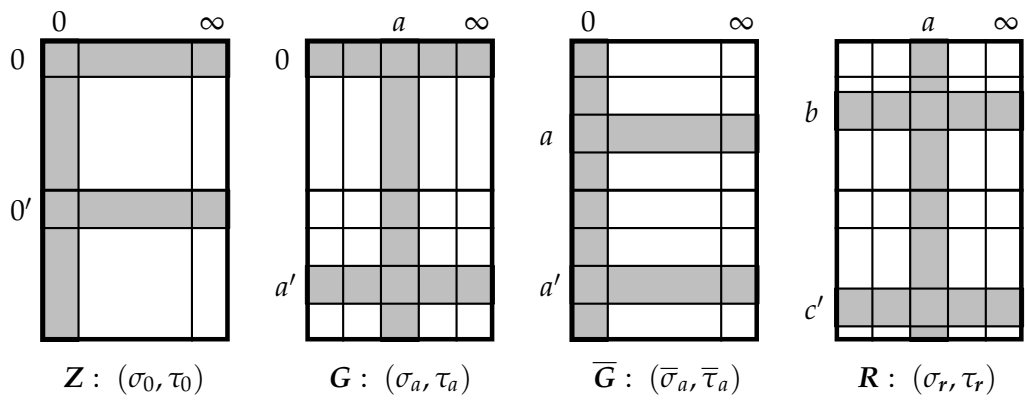

FIGURE 2. The elements (σ_0, τ_0), (σ_a, τ_a), $(\overline{\sigma}_a, \overline{\tau}_a)$ $(a \in A)$, $(\sigma_{\boldsymbol{r}}, \tau_{\boldsymbol{r}})$ $(\boldsymbol{r} = (ab, c) \in R)$ of L. For each (σ, τ) shaded are the two rows corresponding to $\mathrm{im}(\sigma)$ and one column corresponding to $(\infty)\tau$. They all have $\ker(\sigma) = \{A_0, A_0'\}$ and $\mathrm{im}(\tau) = A_0$.

Because $\ker(\sigma)$ and $\mathrm{im}(\tau)$ are the same for all $(\sigma, \tau) \in L$ it follows that L is a left zero semigroup (i.e. $xy = x$ for all $x, y \in L$). Furthermore, since K is an ideal in $\mathcal{T}$ (i.e. $xy, yx \in K$ for all $x \in K$, $y \in \mathcal{T}$), the set $B_G = K \cup L$ is a subsemigroup of $\mathcal{T}$. We remark that, strictly speaking, B_G depends not only on G, but crucially on the chosen presentation for G.

4. PROOF OF THEOREM 1

We will now use the presentation given in Proposition 2 to compute the maximal subgroup H of $\mathsf{IG}(B_G)$ containing the idempotent $e_0 = (0, 0) \in K$. Relations (1) in our context read

$$f_{0j} = f_{i0} = 1 \ (i \in I,\ j \in J). \tag{4}$$

The remaining relations (2) arise from the singular squares induced by the elements of L acting on K. Each up-down singular square is of one of the following forms:

$$(a_1, a_2; c_1, c_2),\ (a_1', a_2', c_1, c_2)\ (a_1, a_2, c_1, c_2 \in A_0).$$

The square $(a_1, a_2; c_1, c_2)$ yields the relation

$$f_{a_1,c_1}^{-1} f_{a_1,c_2} = f_{a_2,c_1}^{-1} f_{a_2,c_2}\ (a_1, a_2, c_1, c_2 \in A_0). \tag{5}$$

Putting $a_1 = c_1 = 0$, $a_2 = a$, $c_2 = c$ and using (4) yields

$$f_{a,c} = 1\ (a, c \in A_0); \tag{6}$$

clearly, all the remaining relations (5) are consequences of (6). Similarly, the squares (a_1', a_2', c_1, c_2) yield the relations

$$f_{a',c} = f_{0',c} \ (a, c \in A_0). \tag{7}$$

(Note that we do not necessarily have $f_{0',c} = 1$, and so cannot deduce $f_{a',c} = 1$.)

Turning to the left-right singular squares, each $(\sigma, \tau) \in L$ induces precisely one. Below we list respectively the squares introduced by (σ_0, τ_0) of type $\boldsymbol{Z}$, (σ_a, τ_a) of type $\boldsymbol{G}$, $(\overline{\sigma}_a, \overline{\tau}_a)$ of type $\overline{\boldsymbol{G}}$, and $(\sigma_{\boldsymbol{r}}, \tau_{\boldsymbol{r}})$ of type $\boldsymbol{R}$, together with the relations they yield:

$$(0, 0'; 0, \infty): \qquad f_{0,0}^{-1} f_{0,\infty} = f_{0',0}^{-1} f_{0',\infty} \tag{8}$$

$$(0, a'; a, \infty): \qquad f_{0,a}^{-1} f_{0,\infty} = f_{a',a}^{-1} f_{a',\infty} \qquad (a \in A) \tag{9}$$

$$(a, a'; 0, \infty): \qquad f_{a,0}^{-1} f_{a,\infty} = f_{a',0}^{-1} f_{a',\infty} \qquad (a \in A) \tag{10}$$

$$(b, c'; a, \infty): \qquad f_{b,a}^{-1} f_{b,\infty} = f_{c',a}^{-1} f_{c',\infty} \qquad (\boldsymbol{r} = (ab, c) \in R). \tag{11}$$

Using the relations (4), (6), (7), we can transform (8)–(11) into:

$$f_{0',\infty} = 1 \tag{12}$$

$$f_{a',\infty} = f_{0',a} \qquad (a \in A) \tag{13}$$

$$f_{a,\infty} = f_{a',\infty} = f_{0',a} \qquad (a \in A) \tag{14}$$

$$f_{0',b} = f_{0',a}^{-1} f_{0',c} \qquad (\boldsymbol{r} = (ab, c) \in R). \tag{15}$$

So, the group H is defined by the generators $f_{i,j}$ $(i \in I,\ j \in J)$ and relations (4), (6), (7), (12)–(15). The relations (4), (6), (7), (12)–(14) can be used simply to eliminate all the generators except $f_{0',a}$ $(a \in A)$. Replacing each symbol $f_{0',a}$ by the symbol a, the remaining relations (15) become

$$ab = c \ (\boldsymbol{r} = (ab, c) \in R).$$

In other words, we obtain the original presentation for G. This proves that $H \cong G$.

Finally note that if $\langle A \mid R \rangle$ is a finite presentation, the semigroup B_G is also finite, with

$$|B_G| = (2|A| + 2)(|A| + 2) + 1 + 2|A| + |R|,$$

and this completes the proof of our theorem.

5. An example, two remarks and an open problem

It may be instructive to follow in a specific example the sequence of Tietze transformations constituting the brunt of the above proof. Let us take $G = Q_8$, the quaternion group, with the well known Fibonacci $F(2, 3)$ presentation (see [14, Section 7.3]):

$$\langle a, b, c \mid ab = c,\ bc = a,\ ca = b \rangle.$$

The dimension of K in this case is 8×5, and Proposition 2 gives a presentation in terms of 40 generators. This is then simplified by a sequence of generator eliminations, using relations (1), up-down singular squares, and left-right singular squares induced by the elements of L of types $\boldsymbol{Z}$, $\boldsymbol{G}$, $\overline{\boldsymbol{G}}$. In the final step further singular squares are revealed, giving back the original presentation.

If we record the original generators in a natural 8×5 grid, this process may be encapsulated as shown in Figure 3.

$f_{0,0}$	$f_{0,a}$	$f_{0,b}$	$f_{0,c}$	$f_{0,\infty}$
$f_{a,0}$	$f_{a,a}$	$f_{a,b}$	$f_{a,c}$	$f_{a,\infty}$
$f_{b,0}$	$f_{b,a}$	$f_{b,b}$	$f_{b,c}$	$f_{b,\infty}$
$f_{c,0}$	$f_{c,a}$	$f_{c,b}$	$f_{c,c}$	$f_{c,\infty}$
$f_{0',0}$	$f_{0',a}$	$f_{0',b}$	$f_{0',c}$	$f_{0',\infty}$
$f_{a',0}$	$f_{a',a}$	$f_{a',b}$	$f_{a',c}$	$f_{a',\infty}$
$f_{b',0}$	$f_{b',a}$	$f_{b',b}$	$f_{b',c}$	$f_{b',\infty}$
$f_{c',0}$	$f_{c',a}$	$f_{c',b}$	$f_{c',c}$	$f_{c',\infty}$

$\xrightarrow{(1)}$

1	1	1	1	1
1	$f_{a,a}$	$f_{a,b}$	$f_{a,c}$	$f_{a,\infty}$
1	$f_{b,a}$	$f_{b,b}$	$f_{b,c}$	$f_{b,\infty}$
1	$f_{c,a}$	$f_{c,b}$	$f_{c,c}$	$f_{c,\infty}$
1	$f_{0',a}$	$f_{0',b}$	$f_{0',c}$	$f_{0',\infty}$
1	$f_{a',a}$	$f_{a',b}$	$f_{a',c}$	$f_{a',\infty}$
1	$f_{b',a}$	$f_{b',b}$	$f_{b',c}$	$f_{b',\infty}$
1	$f_{c',a}$	$f_{c',b}$	$f_{c',c}$	$f_{c',\infty}$

$\xrightarrow{\text{U/D}}$

1	1	1	1	1
1	1	1	1	$f_{a,\infty}$
1	1	1	1	$f_{b,\infty}$
1	1	1	1	$f_{c,\infty}$
1	a	b	c	$f_{0',\infty}$
1	a	b	c	$f_{a',\infty}$
1	a	b	c	$f_{b',\infty}$
1	a	b	c	$f_{c',\infty}$

$\xrightarrow{\text{L/R: } \boldsymbol{Z}}$

1	1	1	1	1
1	1	1	1	$f_{a,\infty}$
1	1	1	1	$f_{b,\infty}$
1	1	1	1	$f_{c,\infty}$
1	a	b	c	1
1	a	b	c	$f_{a',\infty}$
1	a	b	c	$f_{b',\infty}$
1	a	b	c	$f_{c',\infty}$

$\xrightarrow{\text{L/R: } \boldsymbol{G}}$

1	1	1	1	1
1	1	1	1	$f_{a,\infty}$
1	1	1	1	$f_{b,\infty}$
1	1	1	1	$f_{c,\infty}$
1	a	b	c	1
1	a	b	c	a
1	a	b	c	b
1	a	b	c	c

$\xrightarrow{\text{L/R: } \overline{\boldsymbol{G}}}$

1	1	1	1	1
1	1	1	1	a
1	1	1	1	b
1	1	1	1	c
1	a	b	c	1
1	a	b	c	a
1	a	b	c	b
1	a	b	c	c

$\xrightarrow{\text{L/R: } \boldsymbol{R}}$

1	1	1	1	1
1	1	1	1	a
1	1	1	1	b
1	1	1	1	c
1	a	b	c	1
1	a	b	c	a
1	a	b	c	b
1	a	b	c	c

FIGURE 3. The sequence of Tietze transformations constituting the proof of Theorem 1.

Remark 3. It is possible to describe completely the structure of the free idempotent generated semigroup $\mathsf{IG}(B_G)$. By known results (see e.g. [9, (IG1)–(IG4)]) $\mathsf{IG}(B_G)$ has precisely two regular $\mathscr{D}$-classes. The 'upper' one is a left zero semigroup $\overline{L}$ isomorphic to L (as all products in L are basic), while the 'lower' one $\overline{K}$, the completely simple minimal ideal, has a Rees matrix representation with structure group G and (normalised) sandwich matrix (a_{ji}^{-1}), where (a_{ij}) is the $|I| \times |J|$ table that is the end-product of Tietze transformations performed in the proof of Theorem 1 (in our example this is the last table in Figure 3). We claim that in fact $\mathsf{IG}(B_G) = \overline{L} \cup \overline{K}$. To confirm this, and see that the structure is completely determined, we need to show how to write products ef and fe with $e \in L$, $f \in K$ as products of idempotents from K in $\mathsf{IG}(B)$. For the product ef note that there exists $g \in K$ such that $f \mathscr{R} g$ and $ge = g$; both pairs $\{e, g\}$ and $\{f, g\}$ are critical and we have $ef = egf = hf$, where $h = eg \in K$. The product fe can be treated similarly.

Remark 4. Associated to the biorder E of idempotents of a *regular* semigroup S there is another free idempotent generated object $\mathsf{RIG}(E)$, the *free regular idempotent generated semigroup* on E. It is the largest *regular* semigroup with the biorder of idempotents E, and its presentation can be obtained by adding further relations to the defining presentation for $\mathsf{IG}(E)$. For definition and references we refer the

reader to [9]. In particular, from (IG1)–(IG4), (RIG1), (RIG2) in [9] it follows that $\mathsf{IG}(B_G) = \mathsf{RIG}(B_G)$.

One way of interpreting Remarks 3, 4 is to say that the word problem for $\mathsf{IG}(B_G)$ is decidable if and only if the word problem for G is decidable. It is the authors' belief that the next stage in the ongoing exploration of free idempotent generated semigroups is precisely an analysis of the word problem for $\mathsf{IG}(S)$. This at present seems a daunting task, even in the case where S is finite. Nonetheless, we propose the following problem which may just be within reach at this stage:

Question 1. Let B be a finite band such that all maximal subgroups of $\mathsf{IG}(B)$ have recursively soluble word problems. Is the word problem of $\mathsf{IG}(B)$ necessarily recursively soluble?

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