POINCARÉ AGAINST FOUNDATIONALISTS OLD AND NEW

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Poincaré Against Foundationalists Old and New

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Abstract: The early 20th century witnessed concerted research in foundationalism in mathematics. Those pursuing a basis for mathematics included Hilbert, Russell, Zermelo, Frege, and Dedekind. They found a vocal opponent in Poincaré, whose attacks were numerous, vituperative, and often indiscriminant. One of the objections was the *petitio* argument that claimed a circularity in foundationalist arguments. Any derivation of mathematical axioms from a supposedly simpler system would employ induction, one of the very axioms purportedly derived.

Historically, these attacks became somewhat moot as both Frege and Hilbert had their programs devastated—Frege's by Russell's paradox and Hilbert's by Gödel's incompleteness result. However, the publication of *Frege's Conception of Numbers as Objects* [63] by Crispin Wright began the neo-logicist program of reviving Frege's project while avoiding Russell's paradox. The neo-logicist holds that Frege's theorem—the derivation of mathematical axioms from Hume's Principle(HP) and second-order logic—combined with the transparency of logic and the analyticity of HP guarantees knowledge of numbers. Moreover, the neo-logicist conception of language and reality as inextricably intertwined guarantees the objective existence of numbers. In this context, whether or not a revived version of the *petitio* objection can be made against the revived logicist project.

The current project investigates Poincaré's philosophy of arithmetic—his psychologism, conception of intuition, and understanding of induction, and then evaluates the effectiveness of his *petitio* objection against three foundationalist groups: Hilbert's early and late programs, the logicists, and the neo-logicists.

I, Michael Thomas Sands, Jr., hereby certify that this thesis, which is approximately 39,616 words in length, has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.

Signature of Candidate :

Date :

I was admitted as a research student and as a candidate for the degree of M.Phil in June 2007, the higher study, for which this is a record, was carried out between 2007 and 2009.

Signature of Candidate :

Date :

I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of M.Phil in the University of St Andrews and that the candidate is qualified to submit this portfolio in application for that degree.

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Chapter 1

Introduction

The early 20th century witnessed a huge rise in interest in foundationalism in mathematics. Those pursuing a solid foundation for mathematics included Hilbert, Russell, Zermelo, Frege, and Dedekind. They found a vocal opponent in Poincaré, whose attacks were numerous, vituperative, and often indiscriminant. One of the objections was the *petitio* argument that claimed a circularity in the foundationalist argument. Any derivation of mathematical axioms from a supposedly simpler system would employ induction, one of the very axioms purportedly derived.

Historically, these attacks became somewhat moot as both Frege and Hilbert had their programs devestated—Frege's by Russell's paradox and Hilbert's by Gödel's incompleteness result. However, the publication of *Frege's Conception of Numbers as Objects* [63] by Crispin Wright began the neo-logicist program of reviving Frege's project while avoiding Russell's paradox. The neo-logicist holds that Frege's theorem—the derivation of mathematical axioms from Hume's Principle(HP) and second-order logic—combined with the transparency of logic and the analyticity of HP guarantees knowledge of numbers. Moreover, the neo-logicist conception of language and reality as inextricably intertwined guarantees the objective existence of numbers. In this context, whether or not a revived version of the *petitio* objection can be made against the revived logicist project.

The current piece has an ultimate goal of demonstrating that a revived Poincaréan objection can be posed to the neo-logicist and that the neo-logicist does not currently offer a solution. Along the way, I hope to achieve a number of historical and interpretive goals.

Chapter 2 has two separate interpretive goals. The first is to explicate Poincaré's usage of 'intuition' and demonstrate both its origins in Poincaré's thoroughly psychologistic background and its effect on his conception of induction. Poincaré's intuition will be separated into three distinct threads—mathematical ability, temporal awareness, and psychological necessity. Secondly, the chapter aims to differentiate Poincaré's views on the philosophy of mathematics from those of Kant, whose positions Poincaré largely adopts as his own. The chapter focuses on Poincaré's philosophy of arithmetic independently of his philosophy of geometry or science.

Chapter 3 divides Hilbert's work into the early and late periods. For the former, I will demonstrate exactly how Poincaré's *petitio* objection is effective; I will give Hilbert's proof of the consistency of the Peano axioms and highlight where this proof relies on induction. The second half of the chapter looks to impose an intepretive structure on Hilbert's papers of the 1920s. This structure will identify four contrasts fundamental to Hilbert's program and help evaluate a reformulated Poincaréan objection. The chapter also whether either of two other reconstructions, Michael Detlefesen's or Marcus Giaquinto's, fare any better (they do not).

Chapter 4 recapitulates the argument given by Warren Goldfarb [23] that Poincaré's objection was not effective against the original logicists because of their unique conception of logic. The goal of this recapitulation is twofold. First, it demonstrates why a similar defense for the neo-logicists is not possible. Second, it shows how the *petitio* could fail, in contrast to its success against Hilbert's program.

Finally, Chapter 5 spells out the neo-logicist program in detail and lodges two revised Poincaréan objections. The chapter also compares these objections to pre-existing ones so as to throw those into a Poincaréan light. Ultimately, it does not argue that these objections are knock down arguments, but rather that they require an explanation from the neo-logicist that does not currently exist.

Chapter 2

Poincaré

The current chapter has two distinct and sometimes competing goals. The first is historical; it seeks to give an accurate account of Poincaré's views on the mathematical notions of intuition and induction. Moreover, it looks to contrast these views with Kant's philosophy of arithmetic, which Poincaré is often labeled as following. One of the main problems with an attempt to accurately represent Poincaré's position is that his writings on arithmetic are scant and not always self-coherent. The second goal is therefore to impose an interpretive structure on Poincaré's writing so as to make his position a coherent whole. Both goals can be achieved by proposing a structure strongly supported by textual evidence.

To this end, the chapter will first identify four distinct aspects of Poincaré's psychologism. Second, it will identify three notions of mathematical intuition in Poincaré's work and trace their roots back to elements of his psychologism. Third, it will argue that these three underlying conceptions directly inform Poincaré's views on mathematical induction, which lies not only at the heart of mathematical science for Poincaré, but also as the basis for scientific progress. This conclusion will demonstrate that Poincaré's *petitio* objection stems immediately from his conception of induction. Finally, it will briefly present Kantian notions of intuition and syntheticity in order to argue that they differ significantly from Poincaré's conception of those concepts, thereby demonstrating in what way Poincaré's philosophy of arithmetic differs from Kant's.

Secondary literature directly pertaining Poincaré's philosophy of arithmetic is sparse, with most commentators focusing on Poincaré's broader philosophy of science (see Zahar [66] or Ben-Menahem [1] recent takes) or else his philosophy of geometry (Torretti [62]). Dedicated manuscripts primarily about his philosophy of arithmetic appear from Folina [17], de Lorenzo [10], and Mooij [39], with only the first in English. Much of the scarcity can probably be attributed to the limited quantity of Poincaré's writings directly on the subject. Instead, much of his philosophy of arithmetic, intuition, and induction must be gleaned from his objections to Hilbert and the Logicists. Some of the blame may be due to the fact that while Poincaré's views on geometry differed significantly from Kant, his views on arithmetic are seen to ape his predecessor. While true generally, I will discuss two aspects of Poincaré's philosophy of arithmetic that significantly differ from Kant—intuition and syntheticity.

One goal not taken up by this chapter is an for the correctness of Poincaré's views. There is an inherent plausibility in the formulation of these principles, and this chapter will attempt to present them in just such a favorable light. However, I will not provide detailed arguments for why Poincaré is ultimately correct in his views. Ultimately, the dialectical role of this chapter is to clarify Poincaré's philosophy of arithmetic, to expose differences between Poincaré and Kant, and to lay bare the assumptions brought to bear in Poincaréan objections to the logicists and neo-logicists. [1]

[1]Page citations of Poincaré are from [45], [46], and [44] where appropriate. Where possible, I have given the paper citations from the original paper as republished in Ewald [14].

2.1 Poincaré's Psychologism

The lens through which one must view Poincaré is thoroughly psychologistic; his psychologism is not limited to his belief that logic is psychological because the laws of logic are the laws of thought, nor is it limited to his belief that logics is psychological because it is ultimately justified on psychological grounds. Rather, psychology plays a fundamental role in all aspects of Poincaré's philosophy. This role can be categorized by four principles.

1. Regardless of field, psychological principles are the ultimate bedrock upon which justifications rest.

2. Arguments from psychological grounds are permissible and convincing.

The relationship between human psychology and the sciences is bilateral:

1. Advances in science can lead to greater understanding of the human psyche.

2. Our psychological experience of these subjects must accord with new theories and can weigh in favor of a particular theory over another.

2.1.1 Science Advancing Psychology

> [Mathematical science] reflecting upon itself is reflecting upon the human mind which has created it; the more so because, of all its creations, mathematics is the one for which it has borrowed the least from the outside [45, p. 36]

By examining the products of the human mind—by reflecting upon the structure the mind places on the world—one can come to know more about the structures of human psychology, Poincaré argues. These structures cannot help but reflect the structure of the mind that created them (Compare to the neo-logicist view of language

(§5.1.2). These structures are simply the scientific theories created to explain phenomena. Among these, mathematics holds a privileged place because it is informed the least by empirical data—it springs forth from the mind most "purely". Therefore, it bears most strongly the psychological structures of the human mind, and its study is best suited to reveal deep-seated psychological rules. Not only is mathematics most likely to be fruitful on examination, but also most likely to reveal the fundamental principles of human psychology.

Poincaré writes,

> The genesis of the mathematical discovery is a problem which must inspire the psychologist with the keenest interest. For this is the process in which the human mind seems to borrow least from the exterior world, in which it acts, or appears to act, only by itself and on itself, so that by studying the process of geometric thought we may hope to arrive at what is most essential in the human mind [45, p.46].

2.1.2 Psychology Advancing Science

> This method [of the logicists] is evidently contrary to all healthy psychology. It is certainly not in this manner that the human mind proceeded to construct mathematics" [45, pp.144-5].

Poincaré here objects to the logicist project by arguing that the construction of number from logical principles is not in accordance with the psychological history of the construction of mathematics. Given Poincaré's view that mathematics and arithmetic are structures of the human mind, any theory that correctly explains their foundations must be analogous to the personal psychological experience of learning mathematics(that personal experience is itself analogous to the original psychological construction of mathematics). Thus, reflection on psychological experiences at learning mathematics can and should, according to Poincaré, act as a guide to mathematical theory.

Poincaré goes further and argues that not only can rational cognizing on psychological experiences bring about scientific advances, but that *psychological experiences themselves* play an integral role in scientific examination, and thus in the science's epistemic justification. This position is put forth in an extended discussion of the process by which he discovered a particular theorem about Fuschian groups (Cf. [45, p.54]) Poincaré argues that aesthetic sense of the mathematician recognizes the inherent elegance of the correct proof and thus functions as a reliable tool. That the mathematician possesses an ability to discern correct proofs by their aesthetic value reinforces the view that mathematics is a pure product of the human mind. The mathematician's appreciation can be seen as an unconscious recognition of the accordance of the mathematical proof with pre-existent psychological structures. That psychological principles and experiences should be followed to scientific discovery is a consequence of Poincaré's view that science is a psychological structure.

2.1.3 Psychological Principles as Bedrock

> We cannot believe that two quantities which are equal to a third are not equal to one another, and we are thus led to suppose that A is different from B, and B from C [45, pp.22-3].

Poincaré holds that the only ultimate justification of bedrock principles are psychological in nature. Consider the Principle of Non-Contradiction. Poincaré's justification of it above is not logical, but rather based on our *psychological inability* to conceive that it is false. It is a product of psychological necessity. The *unavoidable belief* in the principle is what justifies it, not a sense of logical validity. Thus, if even the Law of Non-Contradiction has a psychological epistemic basis, than these psychological rules are bedrock.

It should be noted that these psychologically justified bedrock propositions inform Poincaré's definition of the synthetic *a priori*:

> ... the term employed by Kant to designate the judgments that can neither be demonstrated analytically, nor reduced to identity, nor established experimentally [46, p.146].

Something that cannot be reduced to identity, established experimentally, nor demonstrated analytically must be something that is self-evident upon reflection, for these options exhaust all other potential avenues of justification. This notion of self-evidentness, for Poincaré, is nothing more than psychological inevitability.[2] For Poincaré, psychological necessity is sufficient to constitute epistemic justification.

2.1.4 Permissibility of Psychological Arguments

> How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds, if its evidence is founded on principles that are common to all men, and that none but a madman would attempt to deny, how does it happen that there are so many people who are entirely impervious to it [45, p.47]?

In arguing mathematics being solely logical, Poincaré employs a purely psychological argument, consisting wholly of premises about human understanding and human psychology, yet making a metaphysical conclusion. This particular argument takes the following form:

P1 If mathematics were described wholly by the rules of logic, then everyone would be able to understand it;

P2 Some people do not understand mathematics;

Concl. Therefore, mathematics is not described wholly by the rules of logic.

P2 is true and **Concl.** follows from **P1** and **P2**. However, Poincaré needs to justify **P1**, and here is where Poincaré employs psychological principles as bedrock justification. Accordingly, his argument is outlined thus:

[2] Compare this notion of the synthetic *priori* for Kant (§2.4).

P1 If mathematics were described wholly by the rules of logic, then everyone would be able to understand it.

- **a)** The rules of logic are self-evident upon reflection.
- **b)** The rules of logic are universally valid, across space and time.
- **c)** If the rules of logic are universal, then they are shared by everyone.
- **d)** The rules of logic are shared by everyone. [From b) and c)]
- **e)** If the rules of logic are shared by everyone and they are self-evident upon reflection, then everyone has access to them, upon reflection.

  Subconcl. Everyone has access to the rules of logic, upon reflection. [from a), d), and e)]

P2 Some people do not understand mathematics, even when it is explained to them.

Concl. Therefore, mathematics is not described wholly by the rules of logic.

Note that **a)** is a psychological claim. That the rules of logic are self-evident upon reflection is a claim about what the human mind must believe; it claims that the rules of logic are forced upon human understanding by the very nature of human psychology. This rule of human psychology is the basis for **P1**, and consequently the entire argument. Thus, Poincaré's admission of psychological argument can be seen as another aspect of his admission of psychological principles as bedrock.

2.2 Intuition

Poincaré bases much of his views on the foundations of mathematics on intuition. However, he presents no clear coherent conception of intuition. There is no evolving notion that comes clear. Nor exists a single thread from which one can draw a single latent principle. Rather, there appear three separate, yet inextricable aspects of intuition in Poincaré's usage that he himself often conflates. These three are mathematical ability, grasp of the succession of moments, and immediate psychological acquaintance.

The first we have seen the foundation of in Poincaré's psychologism. There, he claim that part of a mathematician's ability derived from an unconscious aesthetic appreciation of mathematics. This appreciation is something that can be honed, but only in the presence of a pre-existing talent. Some people have it and are able to undertake mathematical research, while others do not. Poincaré describes it in the following passage:

> We can understand that this feeling, this intuition of mathematical order, which enables us to guess hidden harmonies and relations, cannot belong to every one. Some have neither this delicate feeling that is difficult to define, nor a power of memory and attention above the common, and so they are absolutely incapable of understanding even the first steps of higher mathematics. This applies to the majority of people. Other have the feeling only in a slight degree, but they are gifted with an uncommon memory and a great capacity for attention. They learn the details one after the other by heart, they can understand mathematics and sometimes apply them, but they are not in a condition to create. Lastly, others possess the special intuition I have spoken of more of less highly developed, and they can not only understand mathematics, even though their memory is in no way extraordinary, but they can become creators, and seek to make discovery with more or less chance of success, according as their intuition is more or less developed [45, p. 50].

Intuition as mathematical ability allows one to foresee the direction of proofs and perceive the correct path. As such, it demonstrates that Poincaré adheres to a Kantian position of mathematics as synthetic. A mathematical proof is not a series of deductive steps resulting in a theorem. Instead, every step in the proof requires an intuitive leap made possible by the mathematician's ability.

The second aspect of Poincaré's intuition is an immediate, instinctual grasp of temporal progression. More precisely, an understanding of mathematical notions can only be achieved against the backdrop of a succession of instants of time. Mathematical reasoning is done in a temporal context. This progression of time is not a continuum, but instead instants following after instants, cascading one after another constantly. It is an instinctual grasp of moments in time being well-ordered, but discontinuous.

The succession of instants to which we have immediate, instinctual access is analogous to the rationals embedded in the real line. Rational numbers exist prior to the irrationals, and hence real numbers. The fact that one learns about the rational numbers before the reals guarantees this fact for Poincaré (Cf. §2.1.2). In fact, the definition of real numbers—and hence human understanding for Poincaré—is built out of and in reference to the rationals. The two main definitions that in mind here are Cauchy sequences and Dedekind cuts. They are both comprised of sets of rational numbers and postulate the existence of a new number from these sets. That these definitions make reference to rationals reflects for Poincaré the structure of our psychology. Our understanding of the rationals is prior to our understanding of irrationals.

Similarly, our grasp of the succession of instants is psychologically (and therefore epistemically) prior to our grasp of a notion of a continuity of time—our pre-theoretic notion of "moment succession" is the foundation out of which our theoretical notion of the continuity of time springs. There are many additional parallels to be drawn between the two relationships.

First, just as there are infinitely many rationals, there are infinitely many instants in time. Moreover, just there are for any given moment in time, infinitely many moments of time clustered around that moment, so too the case with the rationals. Moreover, between any two rationals, there can be found another rational. Similarly, between any two moments of time, there can be found an additional moment. Finally, like the rationals, Poincaré's notion of moments of time is that they are *totally disconnected*. That is, there are no non-trivial connected subsets. Each moment in time, like each rational, is disconnected in some way from every other moment in time.

The third and final aspect of Poincaré's use of 'intuition' is perhaps the closest to traditional uses of the word, both within and outwith philosophy. Poincaré claims that one can know a proposition by intuition meaning that one has an immediate grasp, unquestioning readiness to accept, or the proposition is self-evident.[3] For example, recall that Poincaré insists that the Principle of Non-Contradiction can rely upon no further epistemic justification (§2.1.3). Our intuitive grasp (read:psychological certainty) of particular principles outstrips any potential objections to them. This strand of intuition is simply a repackaging of Poincaré's prior commitment to the acceptability of psychological facts about human nature(e.g. an inability to disbelieve a proposition) as epistemic justification for the truth of a proposition. To say that something is known intuitively on this usage is simply to claim that the psychologically inevitability of that assertion guarantees its truth. Thus, the "intuition" that underlies "intuitive" propositions is nothing more than psychological force.

These three notions do not interact in clearly delineated ways within Poincaré's texts. None is more fundamental than the other two. Instead they are mutually basic and alternatively reinforcing. There are instances of Poincaré using the notion of temporal continuity to explain the self-evidentness of a proposition; of Poincaré showing that strong mathematical abilities rely on strong temporal intuitions; and of Poincaré under-girding our understanding of the continuity of time via self-evidentness. Drawing out these three separate threads is not aimed at performing a reduction on Poincaré's arguments. Rather, it is aimed at explicating Poincaré's unique notions of intuition and identifying how these three conceptions coalesce in Poincaré's concep-

[3] Self-evidentness in particular among these construals is controversial insofar as one might be able to give an account of self-evidentness that is logical, rather than psychological in nature. For Poincaré, however, it would be undoubted that to say that something is self evident would be to make a psychological claim.

tion and justification of mathematical induction. A secondary goal of so explaining Poincaré's notions is to witness their character as distinct from Kantian notions of arithmetic, to be taken up in §{Kant.

2.2.1 Mathematical Ability

We begin with Poincaré's regard of intuition as mathematical ability. He starts *Science and Hypothesis* by claiming, "The very possibility of mathematical science seems an insoluble contradiction"[43, p. 1]. This contradiction springs forth from the belief that although mathematics might seem purely deductive, a purely deductive mathematical proof would generate no new truths. It would, in essence, simply be a retelling of known facts in a slightly different way, something akin to showing that 'All bachelors are unmarried' "proves" that 'If you are a bachelor, then you are unmarried. Poincaré's notion of deduction here is restrictive. He holds that deductive reasoning is complete obedience to a set of rules. Deduction can only be done in a formalized system, and because of these rules, deductive systems have all their theorems predetermined. New truths, are then those things that are not predetermined by the rules. Poincaré seems to have as a model of deduction a complete, sound system like first-order logic. Deductions within the system don't serve to introduce new truths because the truth of those theorems was pre-ordained by the formation rules of the formal system.

Were mathematics to be purely deductive, Poincaré argues, all mathematical progress would be reducible to mere tautology. Moreover, were mathematics to be complete–were mathematics to be merely the result of applications of deductive rules of logic—there could be a mind capable, with one glance, to immediately apprehend all of mathematics.[4] That such a mind is unimaginable is proof enough for

[4]Recall that the completeness of first-order logic had not been proven by this point. Poincaré's position can be construed as a poorly formed argument that mathematics is not part of first-order

Poincaré that there must be something beyond mere deductive prowess in mathematical proofs.[5] This something extra makes real mathematical progress possible, according to Poincaré. He makes the connection between creative ability and mathematical reasoning: "...it must be granted that mathematical reasoning has of itself a kind of creative virtue, and is therefore to be distinguished from the syllogism...[modes of reasoning that] retain the analytical character, *ipso facto*, lose their power"[43, p. 3]. The power being lost is the power to generate new knowledge and progress beyond the known truths. I will return to explore this seeming contradiction in depth later in §2.3 and argue that the something extra is inductive reasoning. For now, I leave this notion to the side to examine thoroughly Poincaré's detailed investigation of intuition as mathematical ability.

Poincaré's discussion of intuition as mathematical ability takes place under the guise of an examination of mathematical education and the psychology behind someone coming to learn mathematics. That Poincaré develops this notion via an exploration of pedagogical principles is emblematic of psychological bases playing a foundational role in his metaphysical arguments (§2.1.4).

However, we must attempt to clarify the type of mathematical ability Poincaré has in mind. He argues that "The principle aim of mathematical education is to develop certain faculties of the mind, and among these intuition is not the least precious"[45, p. 128]. This intuition, which is the aim of mathematical education, stands in contrast to logic:

> Logic teaches us that on such and such a road we are sure of not meeting an obstacle; it does not tell us which is the road that leads to the desired end. For this it is necessary to see the end from afar, and the faculty which teaches us to see is intuition. Without it, the geometrician would

logic, and whatever system mathematics is a part of, it is not complete. Of course, Poincaré does not seem to recognize the difference between first-order and second-order logic in his rants against the neo-logicist, so this interpretation may be too charitable.

[5]Note also that this argument is fundamentally psychologistic: the fact that a thing is inconceivable to us is taken as a premise in the argument.

be like a writer well up in grammar but destitute of ideas [45, p.130].

Logic appears to Poincaré to be without content—it is merely a set of rules that functions regardless of the subject of its inquiry. It is purely mechanical and non-creative. However, mathematics has substantial content and is fundamentally creative. Someone who is quite capable in logic might very well determine that a proof is correct, but for Poincaré, this logician would fail to recognize what the proof is *actually about*. This recognition of the substance of mathematics via the creative spark is the intuition of which Poincaré speaks.

An important feature of this description is a comprehension of unity. It is vital that mathematical ability is somehow able to perceive or apprehend the totality of a mathematical argument, rather than simply verifying each individual step. This verification process is precisely how Poincaré characterizes logic and deductive reasoning, which checks that one can reason safely from the premises

$$p$$

$$p \supset q$$

to the conclusion

$$q$$

Mathematical ability is marked by a comprehension of the whole proof and the collection of those individual thoughts and steps into a coherent whole. Those truly gifted with this ability (intuition) will be able to foresee a proof's steps and the general direction without actually verifying each and every one of the deductive steps. They will be able to sketch a proof and know it is correct without verification of the formal steps. This ability echoes Poincaré's notion of the principle of induction—moving from particular inferences to a globally true inference (See §2.3.1).

Intuitive mathematical ability as unifying particulars into a general principle appears again in the same chapter:

> It is the same in mathematics. When the logician has resolved each demonstration into a host of elementary operations, all of them correct, he will not yet be in possession of the whole reality; that indefinable something that constitutes the unity of the demonstration will still escape him completely.
>
> What good is it to admire the mason's work in the edifices erected by great architects, if we cannot understand the general plan of the master? Now pure logic cannot give us this view of the whole; it is to intuition we must look for it[45, p. 126].

Again, Poincaré draws significant contrast between the specifics of logic and the wider, grander vision of intuition. Poincaré attributes an aesthetic quality to mathematical ability—a psychological experience helping to improve mathematical discovery as in §2.1.2. Someone who possesses such intuition is likened to one with an appreciation of the general plan of a master architect.

Poincaré later develops a description of how this aesthetic intuition combines subconscious thoughts having been activated by conscious work into solutions hitherto unknown. The subconscious performs a mental re-arrangement on these agitated ideas to re-combine them into (true) conclusions unattainable via purely conscious thought. Poincaré then guarantees the accuracy of this subconscious work by a second appeal to aesthetic appreciation:

> More commonly the privileged unconscious phenomena, those that are capable of becoming conscious, are those which directly or indirectly, most deeply affect our sensibility.
>
> It may appear surprising that sensibility should be introduced in connexion with mathematical demonstrations, which, it would seem, can only interest the intellect. But not if we bear in mind the *feeling of mathematical beauty*, of the harmony of numbers and forms and of geometric elegance. It is a real aesthetic feeling that all true mathematicians recognize, and this is true sensibility.[45, pp.58-9, my emphasis]

Poincaré's notion of intuition as mathematical ability extends beyond intellectual skill and encompasses an emotion or sensibility. Recognition of the harmony and simplicity of correct mathematical arguments enables mathematical creativity. Aesthetic appreciation of mathematical beauty is that which distinguishes the mathematician from the logician, according to Poincaré. Because that intuition is not deductive, mathematics is not purely deductive and the paradox that Poincaré sets before himself at the beginning of the chapter is resolved.

Two features of Poincaré's discussion of mathematical ability are particularly important.

First, his argumentation is purely psychological, involving examples from his own personal experience as well as from the general experience of mathematics instruction. Psychological principles provide the foundation for arguments in favor of metaphysical conclusions. That aesthetic appreciation plays an important role in the basic functioning of this ability reinforces Poincaré's psychological basis.

Second, intuitive mathematical ability, is the recognition of unifying general principles and a perception of their truth from observation of individual cases via an aesthetic appreciation for the beauty and harmony of mathematics. Logic serves only to verify the accuracy mathematician's creative proof. It checks each individual step from the starting assumptions to the end conclusion. Mathematical genius is the comprehension of the general structure of mathematics beyond the individual moves permitted by the rules of logic; it takes specific permitted deductive steps and marshals them into a proof of the general structure of mathematics. This particular facet of intuition is manifested clearly in Poincaré's view on mathematical induction. It becomes the ability of the mathematician to move from the particular syllogisms of

$$P(1)$$
$$P(1) \supset P(2)$$
$$P(2)$$
$$P(2) \supset P(3)$$
$$P(3)$$
$$\vdots$$

to the general statement of

$$P(n)$$

for all n. We will return Poincaré's conception of induction in detail later in §2.3, once the two additional notions of intuition have been discussed.

2.2.2 Temporal Progression

Poincaré's conception of intuition as mathematical ability is relatively explicit in his text; his notion of intuition as an immediate grasp of time as a succession of instants is more subtle. I argue that Poincaré viewed an understanding of a sufficiently dense ordering of the passage of moments of time as a necessary presupposition to any mathematical reasoning. Mathematics for Poincaré is not atemporal manipulations of eternally unchanging quantities. Rather, it is dynamic and persists through the passage of time. Any mathematical reasoning is done against the backdrop of the succession of moments. Immediate access to intuitions of temporal succession underlies Poincaré's philosophy of mathematics. Poincaré's writing on two mathematical concepts—geometrical space and infinity—will serve to illuminate his conception of this aspect of intuition.

Time and Geometric Space

Poincaré discusses at length how it is that we come to conceptualize geometric space [49] and [46, pp.51-88]. In fact, an entire part of his collection *Science and Hypothesis* is dedicated to the examination of the theory of space and geometry. In order to understand the basic theory of space as it relates to geometry, Poincaré examines the initial way in which humans learn about geometry. In order to understand geometry's epistemic status (whether it is *a priori* or contingent) Poincaré examines how humans learn about geometry in the first place. This psychological learning process necessarily accords for Poincaré with the epistemic and metaphysical justification of geometry. Psychological justification is bedrock.

For this reason, Poincaré calls the following a paradox,

> Beings whose minds were made as ours, and with sense like ours, but without any preliminary education, might receive from a suitably-chosen external world impressions which would lead them to construct a geometry other than that of Euclid, and to localise the phenomena of this external world in a non-Euclidean space, or even in space of four dimensions.[46, p.51]

This constitutes a paradox for Poincaré because geometry (being a merely mathematical subject) is not empirical in any way because it is not verified by experiment. One does not go about measuring triangles in the world to verify the Pythagorean Theorem. It is thus a product purely of our minds and of our psychology. But if it is mathematical—and thus completely reflective of the internal psychological structures of our mind—then a being with identical psychological structures could only generate an identical geometry. The internal psychological structures pre-determine the mathematical structures created by that mind. Yet, Poincaré claims that external factors could result in the development of non-Euclidean geometry as the primary way in which the world is apprehended. Beings with the same psychology cannot conceive of a different geometry as primary (in the way that we might think Euclidean geometry is primary) unless geometry itself is empirical or somehow contingent. Hence, the paradox.

To resolve this problem, Poincaré claims that although the axioms of Euclidean geometry are "mere" conventions in the sense that they are only one of many potential sets of syntactic rules for geometry, they are impressed upon us by our external physical surroundings. An analogy can be drawn between the rules of geometry and units of measurement. Just as there are many different sets of rules for geometry (hyperbolic, Euclidean, parabolic, etc.), there are many different units of measurement that accurately describe the world. However, different situations impress the use of different units upon us. For instance, we cannot measure distances between cities in nanometers or microchip specifications in light-years. The impossibility is not a logical one, for surely we *could*, in some sense, measure the distance between stars in millimeters. Rather, the impossibility is a psychological and pragmatic one. We can't measure star distances in millimeters because our minds could not hold that many numbers in our head and we fail to have any useful perspective.

Similarly, it is not a logical impossibility that we would generate a non-Euclidean geometry as the standard geometry used in ordinary situations. Poincaré even goes so far as to show a conversion from non-Euclidean to Euclidean geometry.However, it is a psychological impossibility, given our external experiences. The psychological requirement arises because of the external environment. Pragmatic reasons force humans to use Euclidean Geometry while different reasons force those strange beings to construct non-Euclidean geometry even though they share our psychology. The paradox is resolved by the fact that geometries, Euclidean and non-Euclidean alike, are merely different tools to help understand the external world. Different geometries are appropriate in different situations, based on external impressions.

A key step in this argument, however, is to show how the phenomena impressed upon us actually bring about an impression of geometrical space. Poincaré provides a short summary of his explanation:

> *None of our sensations, if isolated, could have brought us to the concept of space; we are brought to it solely by studying the laws by which those sensations succeed one another.*[46, p.58, original emphasis]

Poincaré argues that there is no way static physical impingements on a retina could bring about a conception of space.[6] His thesis is that what enables the construction of a correct account of geometric space is a succession of sensations. These sensations need not be continuous, but they must *sufficiently dense* so as to construct a conception of space. Moreover, they must be *properly ordered*, or else the rules of geometric space will fail to be comprehended.

To witness the plausibility of this view, let us isolate vision as the only sense and consider a subject who is blind from birth. Moreover, our subject lacks the intuitive conception of time; she cannot perceive the passage of time whatsoever. Suppose we were to suddenly give her sight. If we hold her visual field completely static, she will be unable to form even the most basic rules of perspective or distance. For instance, if in her visual field one man is standing near while another is hundreds of feet away, she has no reason not to conclude that the two men are the same distance away, but the former can hold the latter in the palm of his hand. Thus, Poincaré concludes, there must be a succession of sensations rather than just the sensation itself to create the conception of geometric space.

Now, however, let us move the subject around the room, but only let her open her eyes for half a second every minute. Recall, also, that she does not have any conception of the passage of time, so all these sensations are not ordered, nor is

[6]Poincaré gives an argument similar to the following in [49]

she aware of how much time passes between opening her eyes. Again, she will be incapable of rendering a coherent conception of space. Objects will appear to blink in and out of existence or jump from one place to the next without occupying the space in between. The impressions must be sufficiently dense in order for her to form a correct conception.

Finally, suppose that we give her a sufficiently dense series of sense impressions, but that we do not put them in any order. That is, a lifetime of sense impressions appear randomly on her eyes. Certainly, no matter how dense this collection of sense impressions is, the subject will be unable to form a meaningful mental conception of geometric space. Instead, the sense impressions would merely represent a nonsense jumble with no inherent structure.

Thus, concludes Poincaré, the construction of the mental concept of geometric space require sense perceptions that are sufficiently dense and properly ordered. While the sense perceptions themselves come from biological organs (eyes, ears, nose, mouth, etc.), understanding their density and order presupposes some notion of the passage of time. This is precisely what Poincaré has in mind for the intuitive grasp of temporal succession. One's understanding of time must be immediate, sufficiently dense, and properly ordered. For Poincaré, failing to have such immediate access results in an inability to represent geometric space, and more generally, do mathematics at all. The presupposition of the understanding of the passage of time is, for Poincaré, an aspect of human intuition.

Time and Infinity

The instinctual grasp of temporal progression also underlies Poincaré's conception of infinity. His notion of infinity as an uncompleted potential stands in stark contrast to the Cantorians' (so-called by Poincaré himself) view of infinities as actual, existing, completed quantities. He attacks, "*There is no actual infinity* The Cantorians forgot this, and so fell into contradiction... Like the Cantorians, the logisticians have forgotten the fact, and they have met with the same difficulties"[48, p.195, original emphasis]. The debate arises because while speaking of *all* the natural numbers is acceptable to Cantorians or to the logicists, such talk was not permitted by Poincaré. For him, infinity was merely a way of expressing the possibility of going on forever. Infinity was a quantity that *could* surpass all other quantities, rather than one that *did* surpass all others. He writes,

> The notion of infinity had long since been introduced into mathematics, but this infinity was what philosophers call a *becoming*. Mathematical infinity was only a quantity susceptible of growing beyond all limit; it was a variable quantity of which it could not be said that it had passed, but only that it would pass, all limits[45, p.143, original emphasis].

Poincaré's conception of infinite implicitly assumes the passage of time in mathematics. This assumption is present in his definition of infinity as a 'becoming', because it must then be changing (or at least have to possibility to change). The original French, '*devenir*' has the same connotations of a process occurring through time. However, in order to change (or have the possibility of change), a thing must exist within a framework of time. In order for change to occur, the object in question must exist in at least two distinct times.

In contrast to constant mathematical concepts like 0 that are eternal and unchanging for Poincaré, infinity is not a static quantity. Infinity is a process, a merely potential quantity. Any talk of infinity as a whole, completed quantity is inadmissible for Poincaré. Infinity is not something that remains constant, but rather it is a dynamic, changing thing that is constantly in flux. We might say that it is constantly growing. The infinity now, as you read the first word of the sentence, is smaller than the infinity now as you read the last word. Both of these "infinities" will be smaller than what infinity will be if you re-read this paragraph. Poincaré points to this process of ever-increasing size in the last sentence of the quotation. If it could not be said that it "*had passed*" in the past, but only that it "*would* pass" in the future, then the implication is clear that Poincaré's notion of infinity is firmly within a temporal framework. Not only does infinity exist in such a framework, but all of mathematical reasoning exists within a temporal framework. To grasp mathematics, Poincaré holds that one must possess a pre-existing understanding of the temporal framework. The centrality of this position helps explain why mathematical induction is so vital for Poincaré in the sciences.

2.2.3 Psychological Necessity

The third and final aspect of Poincaré's intuition is to indicate knowing a proposition via an immediate acceptance of it or its self-evidentness. Moreover, this access to propositions has epistemic power for Poincaré; having an intuitive grasp of something counts as an epistemic basis, though not absolutely so, and ultimately is the only possible justification. This particular usage is most similar to the non-technical, pre-theoretical, or popular usage. Phrases like 'women's intuition' or 'mother's intuition' echo this meaning, as do the ethicist's questions about our 'moral intuitions' about immolating babies. The mother can point to no ostensible reason that she knows that her child is lying. We cannot say exactly how we know that burning infants alive is so obviously morally wrong. It simply *is*, and there is no way to answer *how* we know, yet it is beyond doubt that we do in fact know.

Poincaré's employs this immediacy criterion of intuition in his discussion of distance and the relativity of space. In it, he argues that we have no direct access to distance itself, but merely to relative distances. So, for example, I know that I am six feet tall only by dint of the fact that the measuring tape that I use is itself six feet tall. If one night, while I sleep, my height was radically changed to that of six miles, surely I would notice this change. However, were similar changes to be made to every item in existence, including, but not limited to, my bed, the measuring tape, door frames, etc., then there is no way for me to determine the change in my size. The point here for current purposes is to recognize Poincaré's denial of immediate access to distance and his use of 'intuition' to describe such access.

> We have so little the intuition of distance in itself that, in a single night, as we have said, a distance could become a thousand times greater without our being able to perceive it, if all other distances had undergone the same alteration [45, p.99].

'Intuition of' can most easily be replaced here by the phrase 'direct access to'. Though this usage is not identical to the immediacy notion that characterized the last instance, direct access and self-evidentness are intimately related.

Poincaré continues this usage when examining the epistemic status of mathematical axioms. He writes,

> Every conclusion presumes premises. These premises are either self-evident and need no demonstration, or can be established only if based on other propositions; and, as we cannot go back in this way to infinity, every deductive science and geometry in particular, must rest upon a certain number of indemonstrable axioms [46, p.35].

This quotation presumes that psychological self-evidentness possesses justificatory power. Moreover, Poincaré claims that all conclusions have a justification grounded ultimately in psychological inevitability, because the justification claim leads back to undeniable propositions. So, not only is self-evidentness a justification, in the end it is the ultimate epistemic justification for any proposition to Poincaré.

The indemonstrability of the axioms is replicated in Poincaré's response to Russell's theory of types in *Science and Method*. This time, however, the indemonstrability is attributed to intuition, rather than self-evidentness:

> He [Russell] similarly introduces principles which he declares to be undemonstrable. But these undemonstrable principles are appeals to intuition, *a priori* synthetic judgments [42, p.162].

Thus, intuitive grasp of the axioms, for Poincaré, is simply a recognition of their self-evidentness. A third notion is added in this quotation: synthetic *a priori.* Recall that for Poincaré, the synthetic *a priori* is "the term employed by Kant to designate the judgments that can neither be demonstrated analytically, nor reduced to identity, nor established experimentally."(p. 12

When looking at justification within a mathematical system, Poincaré writes,

> No doubt we may refer back to axioms which are at the source of all these reasonings. If it is felt that they cannot be reduced to the principle of contradiction, if we decline to see in them any more than experimental facts which have no part or lot in mathematical necessity, there is still one resource left to us: we may class them among a priori synthetic views [43, p.3].

The ultimate justification of these mathematical axioms cannot reside within the system (analytically), because they form the basis of the system itself. Empirical justification is not possible, as no experiment could demonstrate a mathematical axiom sufficiently. The judgment that the axiom is true must be synthetic *a priori*, or in other words, they must be psychologically necessary and self-evident.

However, the epistemic justification offered by intuition is limited; one cannot claim an immediate grasp of every mathematical proposition. Poincaré writes,

> [The mathematicians prior to the rigorization of the calculus in the 19th century] trusted to intuition, but intuition cannot give us exactness, nor even certainty, and this has been recognized more and more. It teaches us, for instance, that every curve has a tangent—that is to say, that every continuous function has a derivative—and that is untrue. As certainty was required, it has been necessary to give less and less place to intuition[45, p.123].

At first glance, it may seem that he is speaking about mathematical ability, as in our first interpretation of 'intuition' (p. 17). While this notion is central to Poincaré's thought, the last sentence points to a slightly different meaning, because he would most certainly disagree that "mathematical ability has less and less place" in proofs. Instead, what must have a less important place is the initial appearance of things that seem obvious. The rigorization of the calculus demonstrated that what we immediately accept as obviously correct in mathematics need not, in fact, be correct.

This quotation imposes a limitation intuition's justificatory power. Poincaré denies that all mathematical propositions are justified by intuition alone, in essence denying Kant's position on the matter (Cf. §2.4.2). Only the most basic principles are supported by an intuitive grasp of their truth. Moreover, due to their fundamental nature, the only support these principles *could* have is intuition. The farther one gets from these basic principles, the farther one proceeds from the situations in which an immediate grasp of truth is reliable. In more complex areas of math, logic and deductive procedures must guarantee the truth of proofs, rather than what is immediately thought to be true.

Intuition, under this construal of Poincaré, provides the epistemic basis for the axioms of scientific inquiry. It serves as the only justification for these base axioms, but this justification cannot be extended to the higher, more complicated propositions save through the proper application of logical rigor. Again, this position is fundamentally different from Kant's position, wherein intuition is a mode of representation that can only be directed towards individual objects (§2.4.2).

2.3 Induction

A clear statement of the principle of mathematical induction (PMI) must first be given, before an examination of Poincaré's conception. For some property of the numbers P, if

- P holds for the base case of 0, and
- If P holds of n, then P holds of $n+1$, for any n

then P holds for all natural numbers. We can rewrite this principle, again in modern form, in second-order logic as

$$\forall \mathbf{P} \left(\mathbf{P}(\mathbf{0}) \wedge \forall \mathbf{n} \left(\mathbf{P}(\mathbf{n}) \rightarrow \mathbf{P}(\mathbf{n}+\mathbf{1}) \right) \rightarrow \forall \mathbf{n} \left(\mathbf{P}(\mathbf{n}) \right) \right).$$

Poincaré elucidates his conception of induction by defining addition recursively and then deriving associativity and commutativity from that definition alone. In so doing, the reader is meant to witness the centrality of induction to mathematical reasoning. He assumes that the operation $x + 1$ is already defined and goes onto define addition of $x + a$ via the equation

$$x + a = [x + (a - 1)] + 1 \tag{I}$$

Note that this definition is recursive and thus employs mathematical induction. Poincaré remarks that through this process,

> ... we can define successively and "by recurrence" the operations $x+2$, $x+3$, etc. This definition deserves a moment's attention; it is of a particular nature which distinguishes it even at this stage from the purely logical definition; the equality (I), in fact, contains an infinite number of distinct definitions, each having only one meaning when we know the meaning of its predecessor.[43, pp.6-7]

There are three implicit notions of induction at work in the quotation. First, induction is shorthand for an infinite number of definitions. It is merely a convenient way of writing the following:

$$\begin{aligned} x + 2 =& (x + 1) + 1 \\ x + 3 =& (x + 2) + 1 \\ x + 4 =& (x + 3) + 1 \\ &\vdots \end{aligned}$$

Inductive definition prevents the mathematician from actually having (for epistemic certainty and complete formality) to go on indefinitely.

Second, by being shorthand for an infinite number of definitions, inductive definition, for Poincaré, allows mathematicians to move from particular instances—$x+1$, $x+2$, etc.—to a general definition. Definition by recursion is taking an infinite set of individual definitions to be one overall general definition.

Third, this procedure is what prevents mathematics from being purely deductive. Moving from the particular to the general gives mathematics the ability to make new truths for Poincaré.

These three aspects secure what mathematical induction does; this topic will be explored thoroughly in §2.3.1. The second focus of the section will be *how* induction is justified. The fundamental position induction holds in Poincaré's system indicates that its epistemic justification can only be psychological. Both what it does and how it is justified, I claim, are the culmination of the three aspects of intuition discussed earlier—mathematical ability, temporal progression, and immediate grasp—which in turn rely on Poincaré's psychologism at base. Let us begin with an examination of how Poincaré views induction as a progression from the particular instances to the general rule.

Induction will provide the solution for Poincaré to the earlier seeming contradiction that begins *Science and Hypothesis*: "The very possibility of mathematical science seems an insoluble contradiction" (p. 17) This contradiction results from two apparent truths about mathematics.

First, mathematics appears to be purely deductive in form. Mathematicians do not argue for a mathematical result as a philosopher does a philosophical result, nor do mathematicians venture out into the world in search of empirical evidence to support their mathematical results, nor are experiments conducted in laboratories to verify geometrical truths. Rather, mathematicians start from some basic principles and follow certain prescribed rules in order to arrive at the intended result. Mathematical proofs can be valid. However, implicit in Poincaré's notion of 'deductive' is that it is non-creative. Deductive reasoning can verify the accuracy of mathematical reasoning, but that is all it is able to do. It is, perhaps, best thought of as an officious copy editor incapable of writing on its own. Deduction is, recall, a complete obedience to a set of rules.

Second, mathematics is capable of generating new truths, by which Poincaré means that mathematics can have results that are new and count as knowledge in a way that deductive consequences do not. Mathematics can create new true statements in a non-trivial way. For instance, the proof of the Heine-Borel theorem taught us something new about closed and bounded sets: every open cover of such a set has a finite subcover. This fact, one of pure mathematics, was genuinely new knowledge that was not known prior to the 19th century. In some way, it was not merely a rearrangement of old knowledge, but genuinely new. Mathematics thus generates new truths.

2.3.1 Induction as Moving from the Particular to the General

The paradox for Poincaré is that rote obedience to rules seems to result in a closed, complete system. For Poincaré, nothing not inherent in the premises. How can such a system generate new knowledge on its own?

To escape the paradox, one might be tempted to claim that logic generates its own new truths, thereby undermining Poincaré's second half of the contradiction, since logic is the epitome for Poincaré of deductive systems. If logic also generates new truths, then the paradox fails. One might argue by considering the basic rule of *Modus Ponens*: p, $p \supset q$, therefore q.

If we then interpret p as 'Plato is an ancient Greek' and q as 'Plato is dead', then we might think that we have "learned" the new fact that 'Plato is dead'.

However, this result comes not from the formal logic *per se*. Rather, it comes from the application of logic to empirical facts; the truth of 'Plato is an ancient Greek'(p) and 'If Plato is an ancient Greek, then Plato is dead' ($p \supset q$). Thus, the new knowledge is generated from this application, not logic *qua* logic. Similarly, mathematics can be applied to the architectural plans and physical information about a building to learn that it would withstand a wind of 100 mph. However, Poincaré would deny that this truth is purely mathematical, but instead applied mathematical learning to empirical facts. "Learning" that q is true independent of any semantic interpretation of q is not *really learning* anything at all to Poincaré. Mathematical theorems, on the other hand, do count genuinely as new knowledge.

Instead, Poincaré disputes the first half of the paradox. He argues that mathematics is not merely deductive at all,

> What is the nature of mathematical reasoning? Is it really deductive, as is commonly supposed? Careful analysis shows us that it is nothing of the kind; that it participates to some extent in the nature of inductive reasoning, and for that reason it is fruitful[46, pxxiv].

Induction is central to Poincaré's conception of mathematics. It is the creative spark in the mathematician that permits new knowledge to be generated even though mathematics is highly abstract, independent of empirical study, and obedient to given rules. Deductive reasoning is subservient to it in that deductions exist in mathematics only to verify that mistakes were not made in the creative mathematical reasoning. Poincaré highlights the primacy of inductive reasoning in his notion of mathematics:

> I asked at the outset why we cannot conceive of a mind powerful enough to see at a glance the whole body of mathematical truth. The answer is now easy... he cannot conceive of [arithmetic's] general truths by direct intuition alone; to prove even the smallest theorem he must use reasoning by recurrence [induction], for that is the only instrument which enables us to pass from the finite to the infinite [43, p.11].

Note that for Poincaré, induction is the instrument—the only one according to his thinking—that allows the mathematician to pass from the particular instances of a deduction to proving it for all possible cases. To repeat, induction is the only tool, according to Poincaré, that permits the mathematician to make universal claims. He continues on to show that induction lies not only at the heart of mathematics, but all of science:

> In this domain of Arithmetic we may think ourselves very far from the infinitesimal analysis, but the idea of mathematical infinity is already playing a preponderating part, and without it there would be no science at all, because there would be nothing general.[43, p.11]

Since arithmetic is necessary for any scientific study, the fundamental role induction plays in arithmetic is similarly played by induction for all sciences. Without the principle of induction, reasoning by recursion, there would be no science at all, because induction is required to make universally general claims.

2.3.2 Induction and Temporal Progression

There are a number of threads to trace here from the earlier discussion of intuition. First, the notion of continuing on indefinitely is made much clearer when viewed through the lens of temporal succession. Recall the comparison of Poincaré's conception of the sequence of moments to the rational numbers embedded in the real line (p. 15). The moments are distinct and isolated from each other, yet there can always be found another moment between two given moments. It is not coincidence, I claim, that Poincaré constructs the rational numbers via the same procedure. That is, he constructs what he calls the first continuum by arguing that the process of incalation between each set of distinct numbers can be carried out indefinitely. Taking the process of intercalating as taking a moment in time (as Poincaré understands it), then a one-to-one correspondence can be imagined between moments in time and the rational numbers. Just as time plods on moment after moment, so too does the incalation process and so too does any inductive process. Induction is the realization that the given operation *could be* carried out forever. Or, alternatively, that there is no good reason to stop at any particular point.

Recall that Poincaré's view of infinity was not as a static quantity, but rather a continually dynamic, ever increasing entity that *would* eventually surpass any quantity, but that *had not* done so yet (p. 27). Again, the notion of infinity continually plodding on its course to ever increasing numbers, undergirded as it is by the intuition of temporal progression, lies behind Poincaré's induction. When a mathematician employs the principle of induction, she merely recognizes that for some property that the same inference from some number n to the next number $n + 1$ could be carried out over and over again. That the mathematician can in principle do this stems from the same intuitive basis for her understanding of infinity: the progression of time. She grasps immediately that each inference can be done in succession in an appropriate way. As such, the principle of mathematical induction simply becomes a recognition that going on so indefinitely would result in concluding that the property *would* be shown to hold for every number. Therefore, the mathematician concludes that it *does* hold. The Principle of Mathematical Induction is a transcendence of this infinite process to its final result.

2.3.3 Induction and Mathematical Ability

The second thread of intuition that must be pulled free is intuition as mathematical ability. The first similarity is to the aesthetic recognition of mathematical unity that Poincaré claimed underlay the ability of the mathematician to create new mathematical proofs (see p. . It is vital that Poincaré, in describing the construction of the real numbers, speaks of "feel[ing] that [a recursive] operation may be continued without limit, and that, so to speak, there is no intrinsic reason for stopping."[46, p.25] This "feeling" echoes the sensorial language used to describe the unconscious recognition of the correct mathematical structure (witness "real aesthetic feeling" (p. 20)[45, p.59]. It is not that we know or comprehend the absence of an intrinsic reason to stop; rather, we *feel* it. We feel the unity that persists throughout the process, that each subsequent step is identical in kind to the one that preceded it and that there is a harmony to how each subsequent step relates to the step before it. This sort of explanation, of course, is warranted only by Poincaré's psychologism: appealing to our feelings as a legitimate indicator of mathematical truth is permitted only if one accepts, as Poincaré does, the primacy of psychological principles.

The second aspect of similarity is the very notion of progressing from the particular to the general. Poincaré, remember, compared mathematical ability to the appreciation of architecture. Someone who only understood the individual deductive steps was someone who appreciated only "the mason's work in the edifices erected by great architects," while those who could appreciate the whole proof—those who possessed mathematical and not just logical ability were able to "understand the general plan of the master"[45, p.126]. So, Poincaré takes mathematical ability as that which allows mathematicians to move from the minor details to the general plan. Similarly, induction simply is the ability to move from the individual instances of an inference to the general truth that underlies it. Progressing from the unending collection of equalities that defined addition recursively to a universal definition of addition *is* a progression from the particular to the general. In this sense, mathematical induction simply *is* mathematical ability.

2.3.4 Induction Justified by Psychological Necessity

The third thread of intuition—immediate psychological necessity—justifies, in the mind of Poincaré, the principle of mathematical induction. The principle is supported only by the fact that we feel it to be unquestionably true.

> Why then is this view [reasoning by recurrence] imposed upon us with such an irresistible weight of evidence? It is because it is only the affirmation of the power of the mind which knows it can conceive of the indefinite repetition of the same act, when the act is once possible. The mind has a direct intuition of this power, and experiment can only be for it an opportunity of using it, and thereby of becoming conscious of it.[43, p.13]

This quotation contains two key interpretive rewards. First, Poincaré equates reasoning by recurrence with a power of the mind. This is perhaps the most explicit rendering of induction as mathematical ability. Second, Poincaré explicitly calls upon intuition to explain our access to this power. Induction is an element of human psychology that enables the application of mathematical induction. Similarly, Poincaré draws an explicit connection when he claims that the rule of induction, "inaccessible to analytical proof and to experiment, is the exact type of the *a priori* synthetic intuition"[43, pp.12-3]. Recall that synthetic *a priori* for Poincaré consists of those things that "can neither be demonstrated analytically, nor reduced to identity, nor established experimentally"[47, p.145]. Classifying induction as synthetic *a priori* simply is, as I argued previously, appealing to the psychological necessity of it in order to provide justification.

Induction then, should be seen as the combination of these three notions of intuition. It presupposes an intuition of the progression of time, consists of a formalization of the mathematical ability to move from particular instances of a deduction to the general truth, and is justified solely by its synthetic *a priori* status.

2.4 Kant's Philosophy of Arithmetic

The obvious background against which to view Poincaré's philosophy of arithmetic is Kant's philosophy of mathematics. This fact should come as no surprise, since Poincaré thought himself to be defending Kant against those who, at the Kant jubilee discourse, he accuses of whispering, "I well see this is the centenary of Kant's *death*"[47, p.146]. And while Poincaré develops a philosophy of geometry wholly independent of Kant, he largely agrees with Kant's conception of arithmetic. However, to dismiss Poincaré's philosophy of arithmetic as purely Kantian—and thus not worth study in its own right—is too hasty.

In particular, I want to highlight two central concepts of Poincaré's philosophy of arithmetic that differ significantly from Kant's position: intuition and syntheticity. In order to witness the difference, we must first explicate Kant's position. However, doing so comes at the risk of becoming overly involved in Kant's philosophy of mathematics. In order to avoid becoming so entangled, I will restrict the sketch of Kant's position to just those two topic and compare them to Poincaré's stance. The goal of this section is simple: to demonstrate that Poincaré, despite being sometimes labeled a pure Kantian, in fact had a different philosophy of mathematics, one worthy of study independent of Kant.

2.4.1 Syntheticity

Kant argues for the syntheticity of mathematical propositions in two steps, first for arithmetic propositions and second for geometric propositions. Taking up the first, he argues that propositions like

$$7 + 5 = 12$$

are synthetic because they are the result of the construction of a new predicate concept not present in the original subject, yet one that is shown to apply to the original subject nonetheless [52, p.98]. In the case of the above equation, 12 is not a predicate concept that is present in the original subject, which is the sum of 7 and 5. Rather, the reasoner comes to exhibit in intuition (more on this in a moment) the concept 12 and that this concept applies to the sum of 7 and 5.

Shabel uses the helpful example of constructing a triangle from its definitive conceptual parts. That is, if I am told that a triangle is a rectilinear, three-sided figure, I construct the concept of the triangle out of the concepts I already have for 'figure', 'rectilinear', and 'three-sided' [52, p.99]. The concept of a triangle is not present in the definition of those terms, and the fact that I have constructed the new concept means that it is synthetic.

Returning to the arithmetic example, the left hand side of the proposition has the concepts of 5, 7, and the concept of summing the two together, but *it does not have the concept of the sum itself*, namely 12. No amount of analysis of these three concepts will result, according to Kant, in the concept of 12. Instead, I must construct that concept via a synthesis of concepts into something new.

The picture for Poincaré, I argue, is different. Poincaré agrees with Kant that all of mathematics is synthetic. However, this syntheticity stems not from the synthetic construction of arithmetic concepts in each and every arithmetic judgment. Rather, it is based on the requirement that fundamental mathematical axioms must be considered synthetic (in fact synthetic *a priori*) propositions because their truth is not determined by their meaning. Poincaré writes that synthetic *a priori* judgments are those "that can neither be demonstrated analytically, nor reduced to identity, nor established experimentally"[46, p.146]. Later, he accuses Russell of asserting the truth of his axioms because of their synthetic character (p. 29). Poincaré's conception of syntheticity is purely epistemic, and flows through the axioms to all later propositions. Moreover, Poincaré's notion of syntheticity is highly psychological in character, resembling psychological necessity, rather than the constructive character of Kant's syntheticity. Whereas "$7 + 5 = 12$" must be apprehended synthetically for Kant, for Poincaré it is a trivial consequence of the axioms, which have the synthetic status.

It is also important that Poincaré's conception of syntheticity necessarily relies on an axiomatic structure. Kant is opposed to axiomatic structures in general.

> As regards magnitude (quantitas), that is, as regards the answer to be given to the question, 'What is the magnitude of a thing? ' there are no axioms in the strict meaning of the term, although there are a number of propositions which are synthetic and immediately certain (indemonstrabilia) [36, A 163-44].

Because of this denial, a Poincaréan style conception of the synthetic is not available to Kant. This divergence marks a key way in which Poincaré's philosophy of mathematics is not purely Kantian.

2.4.2 Intuition

Both Kant's and Poincaré's respective conceptions of syntheticity follow from their respective conceptions of intuition. Given that their understanding of the synthetic is different, it should be no surprise that their notions of intuition differ as well.

For Kant, intuition is "a species of representation (*Vorstellung*) of, in the language of Descartes and Locke, an idea" [41, p.111]. Intuition is a way for the mind to relate to objects. There are criteria that differentiate intuition from concept:[7]

> [Knowledge] is either intuition or concept (intuitus vel conceptus). The former relates immediately to the object and is single, the latter refers to it immediately by means of a feature which several things may have in common [36, A320]

For Kant, intuition is both *singular* and *immediate*. The immediacy criterion

[7] Parsons cites the following passage in his [41].

mirrors Poincaré's conception of intuition as psychological necessity(§2.2.3)—though it lacks the explicitly psychological basis.[8] The singularity criterion, however, is distinct to Kant; Kant holds that intuitions are about particular objects; this is in contrast to concepts that can relate to generalities.

Particularity makes every mathematical judgment a judgment about intuitions for Kant. If intuition is the only method to obtain mathematical knowledge (as Kant claims), then we must come to learn about each mathematical judgment *individually*. Intuition is how we know that $5 + 7 = 12$ is true, as is $3 + 4 = 7$, but not how we know the rules of addition.

Poincaré's notion of intuition fails to have this particularity condition. In fact, I argue in §2.2.1 that a main theme of Poincaré's conception of intuition is as an ability. Poincaré claims that intuition enables us to recognize the uniformity in the procedures of mathematics to gain a full understanding of addition.

Poincaré's philosophy of mathematics was largely Kantian, but two of the central features to his philsophy of arithmetic—intuition and syntheticity—difffer significantly from Kant.

[8]Parsons's interpretation of the immediacy criterion could be read in a psychological manner. He writes, "[The immediacy condition] means that the object of an intuition is in some way directly present to the mind, as in perception, and that intuition is thus a source, ultimately the only source of immediate knowledge of objects"[41, p.112]. However, this discussion is beyond the scope of the current inquiry.

Chapter 3

Hilbert

Poincaré's attack on those who sought to ground mathematics in logic was both widespread as well as indiscriminate. There was little differentiation between Hilbert, Dedekind, Cantor, Russell, or Frege in any given attack. All were simply derided as "logisticians" who had made little progress in the foundations of mathematics. Given that he did not discriminate between wholly distinct philosophers, Poincaré did not distinguish their continually evolving views. Of course, in the case of Hilbert, such differentiations were not possible, as Poincaré died in 1912, before the revision of Hilbert's main views in 1918 and later.

There are three primary aims of this chapter. First, to elucidate the discourse between Hilbert and Poincaré in the first decade of the 20^{th} century, arguing that Poincaré's criticism of Hilbert's nascent program were warranted and effective. Second, to interpret Hilbert's later project by developing four distinctions made by Hilbert: theory vs. metatheory; finite vs. transfinite; idea vs. concrete; and formal vs. contentual. These four contrasts serve to be a substantial, novel interpretation that explains Hilbert's response to Poincaré. Third and finally, to evaluate whether a reformulation of Poincaré critique is viable against Hilbert's fully-fledged system. In so proceeding, the chapter will refer to the Hilbert of 1900 and 1904 as the "early"

Hilbert and the post-1918 Hilbert as the "later" Hilbert.

To achieve the stated goals, I will first outline the early Hilbert's attempt at an axiomitization of arithmetic as outlined in his 1904 *On the foundations of logic and arithmetic*. Second, I will examine how Poincaré's broad *petitio* argument is viable against this early line of thought by Hilbert. Third, I will reconstruct later Hilbert's evolution from a tentative position to a fully formed program and develop a response to Poincaré's objection out of this system. Owing to the sparseness of Hilbert's own reply to this objection—a mere few polemical paragraphs—the discussion will evaluate both Michael Detlefsen's [13] and Marcus Giaquinto's [22] respective reconstructions of Hilbert's argument, ultimately concluding that ultimately both positions have fatal flaws.

3.1 Early Hilbert

Hilbert's work on the foundations of mathematics can be separated into two distinct eras: his early work from 1895-1904 and his later work, which began in 1917 but rounded into shape only in the mid to late twenties. Hilbert's return to a foundational focus in his later period brought about a precisification of the sketches in his early work into a fully developed project to prove the consistency of arithmetic via solely finitistic methods. Understanding this relationship between the early and later work can only be obtained via a thorough grasp of both periods. We begin with the early.

3.1.1 Hilbert's Foundation of Geometry

Hilbert's first project in mathematical foundationalism was to give a novel axiomatic characterization of geometry that was divorced from any interpreted meaning. Hilbert's geometry was unlike Euclid's in that it was not a description of points, lines, and other such geometrical objects. Rather, it was a characterization of the relationships between arbitrary objects. This approach is in contrast to Euclid, whose use of 'point' and 'line' actually refers to those very things. Thus, when Euclid claims that any 2 points define a single line, he is making a claim about the nature of points and lines: for any two we make two (idealized) dots on a piece of (idealized) paper there is a single (idealized) line that connects them.

However, Hilbert's use of 'point' and 'line' does not refer to things commonly understood to be points and lines (or rather he does not speak *merely* of these things). In fact, Hilbert's usage is abstracted from the interpreted meaning of the words.

> In my opinion, a concept can be fixed logically only by its relations to other concepts. These relations, formulated in certain statements, I call axioms, thus arriving at the view that axioms (perhaps together with propositions assigning names to concepts) are the definitions of the concepts.[20, Letter from Hilbert to Frege, 7/11/1903]

The words 'point', 'line', and the rest are meaningless prior to axioms being laid down; they receive meaning only through the system of relations to other concepts. It is only by choosing a given set of axioms that the concepts of 'point', 'line', 'circle', 'distance', and any other geometrical notion gain a definition through their relations to each other.

The procedure of laying down an axiomitization without an explicit interpretation was a radical shift from the mathematics that came prior to Hilbert. That the concepts contained within the axioms are implicitly defined wholly and solely by the axioms themselves—that they do not have a meaning outside this axiomitization—shapes Hilbert's later quasi-metatheoretical approach to proving the consistency of arithmetic.

The best way to witness Hilbert's conception of axioms operating as implicit definitions of the concepts contained therein is to follow an example from Hilbert's text. For exegetical clarity and brevity, it is sometimes advantageous to employ modern terminology, such as 'syntactic', 'interpretation', and the like. Such uses should not imply an attribution of such views to Hilbert himself. Following this exegesis, I will return to Hilbert's lack of these fundamental distinctions in his early work—and how this absence renders him vulnerable to Poincaré.

Taking the elements point, line, plane, etc. as primitive, the axioms of Group II, according to Hilbert, "define the concept of "**between**" and by means of this concept the *ordering* of points on a line, in a plane, and in space is made possible."[29, p. 5] Axioms introduce conditions under which particular relations hold and define their properties. The meaning of 'between' is simply a set of rules under which are licensed certain assertions about points and the relation of between. Axiom II, 1 states,

Axiom II,1: If a point B lies between a point A and a point C then the points A, B, C are three distinct points of a line, and B also lies between C and A.[29]

This axiom determines that 'between' is "symmetric" in the sense that if B lies between A and C, then B also lies between C and A and, moreover, that A, B, and C are all distinct points. However, given only Axiom II, 1, the full, standard meaning of betweenness is absent; betweenness is an impoverished definition, for even the ability to determine, given three points, whether any of them lie between the others is impossible. Only the addition of further rules for the use of the relation 'between' makes it resemble our ordinary definition of betweeness.

In this example, Hilbert has provided a single rule governing the use of the relation 'between', but not defined in the traditional sense, what 'between' means. In modern notion, Hilbert's "definition" defines a three-place relation 'M_{ABC}' as 'B lies between A and C'. Axiom II, 1 can be recast as a statement about the relation M_{ABC} rather than about 'between' in any traditional sense. Axiom II, 1 would be phrased:

> If a point B is such that M_{ABC} holds, then the points A, B, C are three distinct points of a line and M_{CBA} also holds.

Abstracting further away from the "traditional" notions of point and line, we can create a one-place predicate 'P_x' defined as 'x is a point', define a three-place predicate 'L_{xyz}' as 'x, y, and z are colinear', and represent Axiom II, 1 wholly symbolically:

$$(P_A \wedge P_B \wedge P_C \wedge M_{ABC}) \longrightarrow (M_{CBA} \wedge \neg(A = B) \wedge \neg(A = C) \wedge \neg(B = C) \wedge L_{ABC})$$

Understanding Axiom II, 1 thus written divorces the three place relation M from any prior intuition or knowledge of points, lines, or what it means for a point to lie between two others. Under the Hilbertian conception, this axiom (in combination with the others) defines what it is for a point to be 'between' two others.[1] In particular, it depends on the definition of colinearity. By itself, it is unable to permanently fix the definition of 'between'. Hilbert's goal in providing axioms is to give a purely syntactic account that can be interpreted as the ordinary conception of geometry. However, this axiomitization does not rely on any traditional understanding of Euclidian geometry; it does not presuppose, for instance, familiarity with points, lines, and the like. Rather, Hilbert's axioms provide a framework within which one is permitted to work. Hilbert himself says,

> But it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g. the system: love, law, chimney sweep... and then assume all my axioms as relations between these things, then my propositions, e.g. Pythagoras' theorem, are also valid for these things.[20, p. 40]

In short, Hilbert uses axioms to establish a framework within which we can use concepts that are being "defined". Geometry can be about chimney sweeps, laws, cats or whatever on likes insofar as they obey the axioms—geometry has been recast

[1] I want to here acknowledge that this notion of implicit definition via axiom was and remains quite controversial. Frege for one, in his correspondence with Hilbert, objects strenuously to this idea. I want only to highlight this aspect of Hilbert's perspective, and not argue for or against its correctness.

as not the study of points and lines but rather the study of the relationships between arbitrary objects that accord with a set of given axioms. Definitions of terms are not isolated in single lines, as in a dictionary. Rather, Hilbert believes that "only the whole structure of axioms yields a complete definition" and to "try to give a definition of a point in three lines is to [his] mind an impossibility."[20, p. 40] Definitions are only grasped in the context of a system of axioms and are made at (in modern terminology) a syntactic level.

"Grounding" geometry is a matter of lay down a set of purely syntactic axioms that can be interpreted semantically as the truths of traditional Euclidean geometry, and then proving that this set is consistent. Hilbert begins his *Grundlagen der Geometrie* by describing his goal to set forth

> a **complete**, and **as simple as possible** set of axioms and to deduce from them the most important geometric theorems in such a way that the meaning of the various groups of axioms, as well as the significance of the conclusions that can be drawn from the individual axioms, come to light.[29, p. 1, emphasis in the original]

The first desideratum is that the axioms must be complete; they must entail every geometrical result that follows from the Euclidean axioms. Hilbert's axiomatic system must be able to recapture all the standard proofs of the mathematical corpus that were then known as Euclidean geometry. For instance, from Hilbert's axioms, he should be able to prove that "In any triangle two sides taken together in any manner are greater than the remaining one."[2] If Hilbert's axioms fail to produce a Euclidean result, then his characterization will have failed to be "geometry".

Hilbert's second desideratum is that the axioms of his system be "as simple as possible." This has a dual meaning for Hilbert. On a shallow, psychological reading, Hilbert means that the axioms must not appear more complicated than those which they intend to supercede. To provide a more obscure account of a subject whose

[2]Proposition 20, Euclid's Elements, as translated by Heath, 286

original exposition was a paragon of clarity would not be progress. The second more technical and important interpretation of Hilbert's requirement for simplicity is logical independence: no axiom (or its negation) should be derivable via inference rules from a subset of the others. This second interpretation makes Hilbert's project so ambitious and mathematically fruitful. While Euclid had come close to outlining a system of axioms that were complete for (Euclidean) geometry, people had long denied the independence of these axioms and attempted to prove the parallel postulate from the other four axioms. For Hilbert to undertake a proof of the independence of this axiom and succeed was a major step in geometry.

Fundamental to proving the independence of the axioms was to prove their consistency. For if a set of axioms is inconsistent, then one can (trivially) derive any conclusion from that set. Consequently, the negation of any the axioms can be derived from the original set of axioms, violating Hilbert's requirement that the axioms be independent. The axioms must be shown to be consistent in order for them to be shown to be independent. However, proving the consistency of the axioms was more than a simple pre-requisite for Hilbert's independence criterion. Instead, Hilbert saw a proof of their consistency as a proof that established their truth.

> ...if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence[20, p. 39-40]

For Hilbert, a proof of the consistency of his axioms for Euclidean geometry simply is a of their truth.

Hilbert's ultimate aim of grounding and/or justifying geometry can be achieved only through two distinct goals: a reconstitution of Euclidean geometry using his axioms and a proof of their consistency.[3] The former goal is an exercise in for-

[3] A similar two-part strategy is employed by the neo-logicist. The neo-logicist must implicitly introduce terms via abstraction principles and then show a reconstruction of traditional mathematics from these new terms. See §5.2

mal geometry, and thus outside the purview of the current investigation. However, in order to achieve the latter, Hilbert proves the equiconsistency of his axiomitization with arithmetic via a translational procedure that recasts geometrical notions as arithmetical ones. Hilbert provides an arithmetic interpretation of his axioms that proves the equiconsistency of his axiomatic geometry with arithmetic. By this method, Hilbert shows that any proof of the inconsistency of his geometry would likewise prove arithmetic inconsistent as well. Thus, Hilbert's proof of the consistency of geometry depends on the existence of a positive proof of the consistency of arithmetic.[4]

3.1.2 Initial Sketches of the Consistency of Arithmetic

Hilbert first attempted to sketch a positive proof of the consistency of arithmetic in a speech to the Third International Congress of Mathematics in 1904 titled *Foundations of Logic and Arithmetic*. In it, Hilbert laid out his axioms for arithmetic and demonstrated how he thought consistency might be proven via a syntactic proof. Even though the formalization of such a consistency proof by purely arithmetical means is impossible, as Ǵodel later showed, an explicit exegesis of Hilbert's attempt will show the efficacy of Poincaré's naive *petitio*.

Viewed through the lens of a modern logician, Hilbert's attempt at a consistency proof is wholly syntactic and metatheoretical. Hilbert first claims that all contradictions have a certain form—the formal representation of any contradiction follows a particular pattern. The proof argues that applying the axioms and rules of inference always results in a sentence that does not exhibit the pattern characteristic of a contradiction. Therefore, no contradiction can result from Hilbert's axioms conjoined

[4]This model theoretic method is extremely fruitful mathematically and heretofore unknown, so even if the ultimate aim of his project fails, Hilbert made incredible contributions. After Hilbert however, these proofs common (for example, Boolos uses one in §5.1.3).

with the rules of inference, and thus the axioms are consistent.

Though metatheoretic in that it is reasoning about the symbols, it is inappropriate to call it such for Hilbert. Though his axioms are wholly syntactic, there remains minimal differentiation between the syntactic and semantic. At this point in his project, there is formalized metatheory or proof theory. This murky understanding and lack of distinction at this early stage—because everything is ostensibly within the theory—makes Poincaré's *petitio* objection forceful. This objection will be returned to in §3.1.3 after fully examining Hilbert's proof. However, it is important to note that Hilbert qualifies the proof in his address, saying,

> In the brief space of an address I can merely indicate how I conceive of this common construction. I beg to be excused, therefore, if I succeed only in giving you an approximate idea of the direction my researches are taking.[27, p. 131]

In responding to Poincaré's objection in the later period, Hilbert claims that Poincaré's objection is only effective against this nascent position:

> Under these circumstances Poincaré had to reject my theory, which, incidentally, existed at the time only in its completely inadequate early stages.[27, p. 473]

To critique early Hilbert is unfair given that he categorizes his position as provisional and incomplete. However, such a critique is important for two reasons. First, it shows *how* Poincaré's *petitio* argument can be successful. Hilbert's later return to foundationalist questions are significantly refined, no doubt in part because of Poincaré's objection. In particular, Hilbert is forced to distinguish between the object language of the theory and the metatheory. For now, we turn to the early Hilbert and the effectiveness of Poincaré's objection.

Hilbert's Formal System

Hilbert's overall strategy for proving the consistency of his axioms is based on the separation of all propositions into what he calls 'entities' and 'nonentities', the former understood to be true propositions, while the latter are false propositions.[5] Hilbert must prove that it is not possible to derive from his axioms the inclusion of both a and $\bar{a}$ in the class of entities [27, p. 132].

Hilbert's notation in *On The Foundations of Logic and Arithmetic* is quite cumbersome owing to the fact that it is merely a sketch of a more fully fledged notion of proof theory. However, it is vital that full exposition be given in the original notation, because Hilbert's proof relies on the precise syntactic form of his formulae. He argues that application of the rules of inference and the axioms to the primitive symbols results only in formulas that do not have the syntactic form necessary to be a contradiction—within the space of possible combinations of the strings of symbols that count as formulas, there is a limited number of possible combinations that are the result of correct applications of the axioms and all these combinations have a given "shape". All contradictions, Hilbert argues, have a different "shape" and fall outside of possible results of reasoning from the axioms. Specifically, all contradictions have a negated clause. Because the axioms and rules of inference do not result in a formula with a negation, no series of legitimate maneuvers within the system will result in a contradiction.

We now want to see the argument in full detail. Hilbert's first primitive notion is '1', which is merely an object of our thought. It is without meaning and signifies nothing. It is divorced of ordinary notions we ascribe to the symbol '1'. We can combine it with itself to form longer collections of marks, what Hilbert calls *combinations*,

[5] Cf. "We call a a true proposition if a belongs to the class of entities; on the other hand, let $\bar{a}$ be called a true proposition if a belongs to the class of nonentities."[27, p. 132]

or strings. For instance,

11

11111

1111111111111

are all combinations for Hilbert. Differentiation between two strings is achieved by examining them to see if "the combinations deviate in any way from each other with regard to the mode and order of succession in the combinations of the choice and place of the objects 1 and [the second primitive object] = themselves"[27, p. 132]. For two combinations to be the same is simply for them to have the same marks in the same order. Two combinations that are not the same length—for instance the first and second combinations above—cannot possibly be the same. When examining combinations containing only '1', identity is nothing more than having the same length. This sort of reasoning about the "shape" of combinations permeates both Hilbert's axiomitization and his consistency proof.

Any combination of these two primitive thought-objects is likewise considered a thought-object and forms the domain over which Hilbert's variables in his axioms will range. The first two objects are demarcated as *simple* objects to differentiate them from the combinations. This gives us sufficient background to understand the first two of Hilbert's axioms:

$$x = x \tag{1}$$

$$\{x = y \,\text{a.} w(x)\} | w(y)^6 \tag{2}$$

The first of these is the statement of identity. The second axiom is Leibniz's law of the indiscernibility of identicals that states that any two things that are identical

[6]'a.' here denotes the conjunction.

share all of the same properties. In Hilbert's own words, "from $x = y$ and $w(x)$, $w(y)$ follows," meaning that if a predicate 'w' is true of x and x is equal to y, then w is true of y as well. With these two axioms, Hilbert claims that he has formed "the *definition of the notion* = (equals)."[7] Wishing to avoid any controversy on whether such axioms can define implicitly, I will not make such a bold claim and instead say that Hilbert has (at least) governed the use of identity within his formal system.

Before moving on to the next three axioms, the remaining primitive notions must be defined. They are **u**, f, and f' (as well as parentheses, though not mentioned explicitly). '**u**' is called by Hilbert '*infinite set*'.[27, p.133, emphasis in original] The string '$\mathbf{u}x$' is read as the claim that x is a member of the infinite set.

The two functions **f** and **f′** are more obscure. Hilbert calls them the 'following' and 'accompanying' functions respectively. Jointly, they play the role of the modern successor function. However, for Hilbert, they are two different functions with different domains. **f′** has a domain of all possible *thought-objects*, whereas **f** has a domain restricted only to elements of the set **U**. As it turns out, Hilbert's axiom 3 is the statement that the two functions are identical over the domain of the infinite set.

He advances three additional axioms:

$$\mathbf{f}(\mathbf{u}x) = \mathbf{u}(\mathbf{f}'x), \tag{3}$$

$$\mathbf{f}(\mathbf{u}x) = \mathbf{f}(\mathbf{u}y) | \mathbf{u}x = \mathbf{u}y \tag{4}$$

$$\overline{(\mathbf{f}(\mathbf{u}x) = \mathbf{u}1)} \tag{5}$$

Axiom 3 should be read literally as, "The object which follows the element (of

[7]FLA, 132, emphasis in the original

the infinite set) that is x is identical with the element (of the infinite set) that is the result of the accompanying function applied to x." Axiom 4 should be read as "If the element that follows x equals the element that follows y, then the element x equals the element y." Finally, Axiom 5 should be read, "There is no element x such that the element that follows it equals the element 1."

Note that these axioms have been wholly syntactic in nature. They govern the rules by which one may reason within Hilbert's formal system. As such, they have been given no fixed semantic meaning and consequently only produce a structure, which is what Hilbert wants to prove consistent. If the syntactic structure is shown consistent, then any interpretation of the rules will also be consistent. Of course one interpretation—the intended interpretation—is standard arithmetic.

Proof of Consistency of Hilbert's System

The five axioms listed above are not the only axioms Hilbert introduces to his system. Notably, I have omitted the induction axiom, from which the Poincaréan objection of circularity springs. I made this omission because Hilbert's proof of the consistency of the first five axioms is a necessary presupposition for the proof of the consistency of every further axiom. By sketching how Hilbert intends to prove this basis—and later arguing that it presupposes induction—I will show why a Poincaréan objection is a good one.

To prove the consistency of the first five axioms is simply to show syntactically that a contradiction cannot be derived from repeated application of the laws of inference and the axioms. A contradiction in the system can only have the form '$a \wedge \bar{a}$', where a is some combination of the five primitive thought objects. Because Axiom 5 is the only axiom that results in a negation (something of the form '$\bar{a}$'), the only contradiction that can be derived is the conjunction of Axiom 5 with the statement '$f(ux^{(0)}) = u1$', where '$x^{(0)}$' represents some thought-object in the system (e.g.

$\mathbf{f}1 = 1$, $\mathbf{ff}1 = 111$, $\mathbf{fff}1 = 1111$, etc.). In order to show that this sort of contradictory formula cannot arise, all Hilbert must show is that a statement of the form

$$\mathbf{f}(\mathbf{u}x^{(0)}) = \mathbf{u}1 \tag{6}$$

cannot be derived from Axioms 1-4.

In order to prove that no statements of the form of equation 6 can result from Axioms 1-4, Hilbert introduces the notion of "homogeneity of equations", which he defines as follows:

> we call the equation (that is, the thought object) $a = b$ a homogeneous equation if a and b are combinations of two simple objects each, likewise if a and b are combinations of three simple objects each or of four *and so forth*[27, p. 134, my emphasis];

An equation is homogeneous when it has the same number of primitive objects on both sides of the '='; an equation is heterogeneous whenever it has a different number of primitive objects on either side of the '='. Though I have chosen to use 'number' to explain hetero- and homogeneity, Hilbert deliberately avoids doing so via the use of examples and the phrase "and so forth". An example of a homogeneous equation is $\mathbf{f}1 = \mathbf{f}1$. An example of a (false/non-entity) heterogeneous equation is $\mathbf{f}1 = \mathbf{ff}1$, since there are two simple objects on the left of the equation and three on the right.

Hilbert claims that only homogeneous equations follow from applications or reiterations of Axioms 1 through 4. This claim is trivially true for Axiom 1 and for Axiom 2, since the former is the smallest of all homogeneous equations and the latter does not have as its main statement an equation. For Axiom 3, it is also evidently true since $\mathbf{f}(\mathbf{u}x) = \mathbf{uf}'x$ is homogeneous for any object x. Finally, both the premise of Axiom 4 and the conclusion are homogeneous ($\mathbf{f}(\mathbf{u}x) = \mathbf{f}(\mathbf{uy})$ and $\mathbf{u}x = \mathbf{uy}$ respectively)given Hilbert's definition of equality.

Thus, any statement of Axioms 1 or 3 or an application of Axiom 2 or 4 must result in a homogenous equation. Thus, a heterogeneous equation like 6 cannot be a logical consequence of Axioms 1 through 4. To repeat, the left side of an equation of the form of 6 must have at least three simple objects ($\mathbf{f}$, $\mathbf{u}$, and whatever $x^{(0)}$ represents) while the right side has at most two ($\mathbf{u}$ and 1). Therefore, since equations exhibiting the form of equation 6 are the only way for a contradiction to arise, there can be no contradiction derived from Axioms 1-5.

Consistency Results

Hilbert thinks his result successful and from its success, he draws a number of powerful conclusions. First, due to his position that the consistency of a mathematical entity guarantees its existence, the thought objects $\mathbf{u}$, $\mathbf{f}$, and $\mathbf{f}'$ have been proven to exist. This result is particularly important in the case of $\mathbf{u}$, since by demonstrating its existence, Hilbert thinks he has proven the existence of the infinite:

> So far as the notion of the infinite $\mathbf{u}$, in particular is concerned, the assertion of the *existence of the infinite* $\mathbf{u}$ appears to be justified by the argument outlined above; for it now receives a definite meaning and a content that henceforth is always to be employed [27, p. 134, emphasis in original].

The major result, however, is a proof of the consistency (and therefore existence) of mathematics. As noted above (p. 56), the intended interpretation of Hilbert's axioms results in standard arithmetic. Axioms 1 and 2 govern the use of 'equals' within Hilbert's system and mirror how 'equals' is used in arithmetic. Interpreting $\mathbf{u}$ as the set of integers and $\mathbf{f}$ and $\mathbf{f}'$ as the successor function, Axiom 3 states that every integer has a successor that is also an integer. Axiom 4 states that if the successor of x and the successor of y are equal, then $x = y$. Axiom 5 states that 1 is the successor of no integer, or, alternatively, 1 is the first integer. Interpreting the axioms in this way does not wholly prove the consistency of arithmetic, but does establish the foundation from which, proceeding in a similar manner, one can demonstrate the consistency of all of arithmetic. To get to the next step—a proof of the smallest infinite or ω—Hilbert himself says,

> If we translate the well-known axioms for mathematical induction into the language I have chosen, we arrive in a similar way at the consistency of this larger number of axioms, that is, at the proof of the consistent *existence of what we call the smallest infinite* (that is, *of the ordinal type* 1, 2, 3, ...) [27, p. 135, emphasis in original].[8]

By continuing to prove that ever growing sets of axioms are consistent, Hilbert has outlined how he believes the consistency of arithmetic will be proven. The mathematical heavy lifting is present in his initial proof of the consistency of Axioms 1-5.

3.1.3 Poincaré's Petitio against Early Hilbert

Poincaré dedicates a brief section in his *Mathematics and Logic: II* to addressing Hilbert's attempt at proving the consistency of arithmetic [42, pp 166-170]. There is little explication of how Hilbert is supposed to have presupposed mathematics in his derivation of mathematics. Poincaré only offers that Hilbert has used 'two', 'three', or 'several times' repeatedly in defining a combination of primitive symbols [42, p. 167]. This objection is somewhat uncharitable to Hilbert, as these uses of number are only for the purposes of elucidating what is being defined, rather than in the formal definition itself. However, Poincaré's objection can be refocused on another part of Hilbert's program—one in which he implicitly assumes number in a formal definition—and effectively challenge Hilbert's Proof.

In the *petitio*'s simple form, Poincaré accuses Hilbert of presupposing a concept of number in his purely syntactic proof. He argues that Hilbert's consistency proof implicitly assumes the principle of mathematical induction. Since the axiom of math-

[8]Note that this definition of the smallest infinite is a more restrictive notion that that of the previously defined infinite **u**.

ematical induction is one of the things of which Hilbert proves the consistency(and thus also the existence), there is circularity in his reasoning. Due to this circularity, Hilbert fails to prove the consistency of his axioms.

A revised Poincaréan critique attaches itself in two different places in Hilbert's above proof. First, he necessarily uses induction to prove that whenever Axioms 1-4 are applied to a homogeneous equation, it results in a homogeneous equation. When Hilbert says, "Axiom 3 yields only homogeneous equations if in it we take any thought-object for x," he implicitly relies on induction to guarantee the truth of the claim [27, p. 134]. Recall that Axiom 3 states $\mathbf{f}(\mathbf{u}x) = \mathbf{u}(\mathbf{f}'x)$. Hilbert claims that substituting any thought object for x into Axiom 3 results in a homogeneous equation. The proof of this claim must first show that it holds for thought-objects x composed of one simple object, then for x composed of two simple objects, and so forth. This "so forth" is an implicit application of the principle of mathematical induction. Hilbert relies here on an intuitive understanding that if the homogeneity of Axiom 3 holds for a thought-object of length n, then substituting a thought object of length $n + 1$ also results in a homogeneous equation. In order to progress from the truth of this claim to the statement that it holds for all possible thought objects, Hilbert must employ PMI.

The second place that a refocused Poincaréan style objection is pertinent against Hilbert is in the well-definedness of the concept of homogeneity in the first place. Hilbert's definition of homogeneous is recursive. Recall that every recursive definition necessarily relies on an inductive inference to demonstrate that the definition is well formed for definiendum of arbitrary length. Hilbert's definition relies on the reader's intuitive grasp of induction and inductive mathematical reasoning to define homogeneity.

Both of these instances, it is important to note, do not occur in explanatory remarks or otherwise non-essential places. The first instance involves a claim central to Hilbert's proof that only homogeneous equations result from applications or statements of the axioms. The second instance occurs in a formal definition essential for the proof. It is not possible, therefore, for Hilbert to do what Poincaré would call a "patching-up" and remove these instances of mathematical induction.

The intuitive leap that Hilbert relies on in these two instances is precisely what Poincaré argues is inseparable from the foundations of mathematics; it is the crux of his objection (see §2.3.1). By relying on this intuition, Hilbert has failed to give an acceptable, non-circular proof of the consistency of Axioms 1-5. Further consistency proofs for additional axioms rely on similar strategies that could be attacked in a similar fashion, but there is no need to do so, because they necessarily rely on a successful proof of the consistency of Axioms 1-5. Thus, Hilbert has not successfully produced a proof of the consistency of arithmetic.

3.2 The Development of Hilbert's 1920s Program

Each of four major papers throughout the 1920s mark a significant development in Hilbert's program to justify and clarify the foundations of mathematics. The first three papers all solidify a substantial differentiation in Hilbert's proof theory while also providing inklings of the distinction to be drawn in the next one. The current section will impose a partially artificial structure on these papers in order to present a clear account of four vital, yet often implicit contrasts in Hilbert's system. This clear account will then enable the construction of a proper Hilbertian reply to Poincaré's objection.

The new grounding of mathematics: first report [31] introduces a boundary between the theory and the metatheory that was only implicit in Hilbert's earlier papers. *The logical foundations of mathematics*[30] distinguishes between finite and transfinite reasoning, equating the former with the object language and the latter with the metatheory introduced in the prior paper. *On the Infinite* [28] establishes ideal propositions in contrast to concrete propositions. Ideal propositions are axioms adjoined to the concrete propositions of the object language for the purpose of expanding the mathematical power of the object language. Finally, all three distinctions are employed most clearly in *The foundations of mathematics* [26], wherein Hilbert thought his critics answered. Throughout these four papers, one final contrast is implicit and never formally mentioned: purely formal reasoning verses contentual (*inhaltliche*) reasoning.

3.2.1 Theory vs. Metatheory: 1922

Early Hilbert's attempt at a proof of the consistency of mathematics employed a proto-proof theory. The attempt at a positive proof examined the structure of the proofs, defined a purely syntactic property (heterogeneity), and demonstrated first that any contradiction had this property and second that no proposition with this property could result within the proof system (§3.1.2). It was vulnerable to a Poincaréan style objection because the system lacked a formalized differentiation between reasoning done *with* the symbols and reasoning done *about* the symbols themselves. A semantic/syntactic distinction was absent, as was differentiation between the object language and the proof theoretic language.

In 1922, Hilbert made this distinction. He introduced the object level, in which there were only the 'number-signs' that "are themselves the object our consideration, but otherwise they have no *meaning* [*Bedeutung*] of any sort" [31, p. 29]. The formal system is restricted at first to definitions of formulas, signs, and the length of signs. As an example of the sort of proof possible in such a limited system, Hilbert gives a sketch of a proof of the statement '$a + b = b + a$' for any strings a and b. By so restricting the system, Hilbert confidently claims, ". . . no contradictions of any sort are possible. We simply have concrete signs, objects, we operate with them, and we make contentual [inhaltliche] statements about them"[31, p. 31].[9] Though this restricted system cannot prove much, it provides the basis to which the addition of axioms allows Hilbert a derivation of arithmetic.

However, the introduction of any new axioms immediately requires a proof of those axioms' consistency. The way that Hilbert endeavors to perform such a proof is by

> mov[ing] to a higher level of contemplation, from which the axioms, formulae, and proofs of the mathematical theory are themselves the objects of a contentual investigation. But for this purpose the usual contentual ideas of the mathematical theory must be replaced by formulae and rules, and imitated by formalisms. . . and at the same time it becomes possible to draw a sharp and systematic distinction in mathematics between the formulae and formal proofs on the one hand, and the contentual ideas on the other [31, p.33]

Postponing a full discussion of the contentual/formal contrast until §3.2.4, these two terms for the time being should be read as shorthand for 'having some meaning' and 'pure manipulations of symbols'.[10] Hilbert not only differentiates the subject matter of the theory and the metatheory, he also differentiates between the type of reasoning that takes place in each. The subject matter of the theory is solely composed of the basic symbols. It is the instantiation or application of axioms to form proofs.

[9]Note the use of *inhaltliche* even in this early paper. There will be numerous uses of the contrast—particularly in the following quotation—throughout the next three sections, but discussion of this distinction will be postponed until §3.2.4

[10]I do not want to impute to Hilbert the sort of formalism wit which he is often labeled , something akin to thinking that mathematics is simply a game that one plays with symbols devoid of meaning. In fact, the inclarity with which he uses the terms 'contentual' and 'formal' suggests his reluctance to accept that particular view of mathematics. In particular, I think that Hilbert wants to say that the operations in the object language are *purely formal* insofar as they have no given content. However, in reading them and doing the manipulations, I can impute some content or meaning to the proof, even if it lacks that "official" meaning. For example, when I prove that '$a + b = b + a$' in the object language, it has no meaning. What I have shown is merely something about strings. However, I might whisper to a student who is not understanding, "Pssst, what I'm doing here is proving the commutativity of addition of integers." In this way, I can *think or reason* contentually about purely formal notions.

Reasoning within this object language is wholly and completely determined by the axioms and is solely the manipulation of eligible objects according to given rules. In contrast, Hilbert has as the subject matter of the metatheory those very things that are taking place in the object language: proofs, formulae, and axioms. Reasoning in the metatheory has *content* since it is about something, whereas object language manipulations are uninterpreted and purely syntactic. This distinction is brought out explicitly by Hilbert:

> To reach our goal[of proving the consistency of the axioms in the object language], we must make the proofs as such the object of our investigation; we are thus compelled to a sort of *proof theory* which studies operations with the proofs themselves. For concrete-intuitive number theory, which we treated first, the numbers were the objectual and the displayable, and the proofs of theorems about the numbers fell into the domain of the thinkable. In our present investigation, proof itself is something concrete and displayable; the contentual reflections follow the proofs themselves [31, 59, emphasis in original].[11]

The purpose of the metatheory, or as Hilbert calls it the 'metamathematics', is to prove the consistency of the axioms in the object language. Thus, he envision the process of deriving mathematics as a two step cycle. Step 1 is to derive provable formulas within the object language using the existing axioms. This first step provides the desired mathematical results. However, once the fruitful results of the current set of axioms are exhausted, proceed to Step 2, which adds new axioms to the already existing axioms and proves their consistency via metamathematical means. Having proven the new axioms' consistency, Step 1 begins again to obtain more mathematically fruitful results. Once all the necessary axioms are introduced, the whole of arithmetic is a consequence.

By explicitly bifurcating the theory and the metatheory, Hilbert has a potential response to the Poincaréan style objection of circularity. To the charge that he em-

[11]Note also the use of 'concrete'. This subject will be taken up in §3.2.3

ploys induction in the proof of the consistency of the axioms (on of which is induction itself), Hilbert could reply that because the purported consistency proof is done in the metatheory, the principle of induction employed is metatheoretical rather than theoretical. Metatheoretical induction is utterly different from the formal principle of induction. Hence, Poincaré's objection is defanged. On the face of it, this rebuttal seems effective and complete. However, this efficacy will become our topic later in section §3.3.

3.2.2 Finite vs. Transfinite: 1923

Hilbert's next paper clarifies the proof theory of 1922. Of particular importance is his differentiation between finite and transfinite reasoning. Given that the theory is intended to reproduce all existing mathematical results, a limitation to purely finite logic is too restrictive. Elements of mathematics are excluded from this system. However, the epistemic basis for Hilbert's work arises out of its simplicity and surveyability (see p.62) Hilbert writes,

> But in our proof theory we wish to go beyond this domain of finite logic, and we wish to obtain provable formulae that are the images of the transfinite theorems of ordinary mathematics [30, 13].

In an attempt to resolve this dilemma, Hilbert introduces the single transfinite axiom that is to be the sole source of any non-finite conclusion: The axiom reads:

$$A(\tau A) \rightarrow A(a) \qquad \text{(TF Axiom)}$$

In non-technical language, this axiom says that if some predicate A holds of the object τA, then it holds for all objects a. The τ operator picks out a purported counter-example, designating an object of which the predicate A does not hold. Therefore, the axiom states that if the predicate holds even for a purported counter-example—it holds even for τA—then it holds for every object [14, p.1135]. If we imagine the predicate 'A' to mean 'bribable', then τA is a man so morally upright that if he is bribable, then so are the rest of the human rabble [30, 19].

The τ operator is the precursor (and dual) to the more well-known ϵ operator that Hilbert introduces in 1925. Whereas the τ function picks out a purported counter-example, the ϵ operator is intended as a choice function.

This lone axiom "is to be regarded as the original source of all transfinite concepts, principles, and axioms"[30, 20]. By this limitation, Hilbert hopes to preserve the epistemic security garnered from the earlier limitation to purely finite reasoning. The question arises as to what type of reasoning is permitted in the metatheory. Presumably, it might seem that if the theory permits transfinite reasoning, then so should the metatheory. However, the metatheory must guarantee the epistemic basis of the theory, and the only reliable epistemic basis for Hilbert is finite reasoning. The objects of the metatheory are concrete displayable proofs. These objects are all finite, and thus there is no need to involve transfinite reasoning.

This differentiation is borne out in the text, in Hilbert's sketch of a consistency proof of the axioms. This proof proceeds along familiar ground, closely resembling the proof offered in 1904: show via considerations of the form of the proof that a contradiction cannot be derived from the given axioms. In Hilbert's own words,

> The basic idea of such a proof [of consistency] is always as follows: we assume that we are presented with a concrete proof having the end-formula $0 \neq 0$ . . . Then, by considering the matter in a finite and contentual way, we show that this cannot be a proof satisfying our requirements[30, 23].

Hilbert's strategy is to provide a metatheoretical proof schema wherein any purported (finite) derivation of a contradiction in the theory is shown to fail to be a proof. Because all proofs are finite, this proof schema will suffice to demonstrate that the theory is free from contradiction. Moreover, because the metatheory is finite, it attains the secure epistemic footing required to eliminate doubt in the foundations of mathematics.

3.2.3 Ideal vs. Concrete: 1925

There are two apparent problems with Hilbert's project in 1923. First, it is *prima facie* implausible that it is possible to prove the consistency of a set of axioms of a theory using a weaker theory.[12] Particular to Hilbert's program, it seems unlikely that a finitist metatheory has a theorem stating the consistency of a transfinite theory. Second, the transfinite theory stands at odds with Hilbert's attempts to restrict foundational work to the most epistemically simple and clear cases. Hilbert holds the view throughout the 1920s that the only area within which we are able to safely infer and reason are finite domains. How is the introduction of a transfinite axiom not in violation of this prohibition?[13]

These two worries are answered in his 1925 paper, *On the Infinite*, wherein he clarifies the epistemic status of transfinite propositions and axioms within the theory. These transfinite propositions—what he dubs 'ideal'—are distinguished from the finite propositions of the theory—what I will call 'concrete'.[14] Through this differentiation, the hope is that transfinite reasoning within the theory is accessible even though the

[12]Clearly the finite metatheory is weaker than the transfinite theory, as the theory contains all finite results plus all those results obtained by the addition of the lone transfinite axiom.

[13]Gentzen, a student of Hilbert's school, was attempting to do something of this sort even into the 1930s after Ǵodel's result. Gentzen recognizes that an absolute consistency proof is impossible, writing that "A consistency proof can merely *reduce* the correctness of certain forms of inference to the correctness of other forms of inference" [21, p.138]. He then goes on to formulate a proof of the consistency of elementary number theory along Hilbert's lines, wherein he employs induction to show that he can reduce the instances of the only connective that could lead to contradiction. In this way, it is very similar to Hilbert's 1927 approach. Gentzen holds that his proof is a reduction because he does not employ complete induction, but rather only induction up to ϵ_0. This notion of induction, he asserts, can be construed finitarily and therefore is suitably a reduction. However, it is important that this is still induction, and it is still the case that Poincaré would argue that understanding that the proof is universally true presupposes an understanding of complete induction.

[14]Though this usage is not extensive throughout Hilbert's papers, it does agree with his usage and is the most illuminating in terms of contrast with ideal. See, "We simply have concrete signs or objects" [31, 31].

theory itself retains the privileged epistemic status of the purely finite.

The distinction is born out of a method, common in mathematics, of introducing new elements to a system to complete the rules by making them more symmetric, harmonious, or universally applicable. The introduction of these ideal elements create a more powerful system able to address a broader range of topics. It is not surprising, given his axiomitization of geometry, that Hilbert harkens back to the Euclidean plane to introduce this method of ideal elements as a solution to the problems facing his finitistic project.

Consider what Hilbert dubs the "Axiom of Connection":

(AC)Any two points on the plane are connected by one, and only one, line.

The geometric dual of this axiom—the statement obtained by replacing 'point' with 'line', 'line' with 'point', and 'are connected by' with 'intersect at'—however, fails to be true, since it is:

(AC^*) Any two lines on the plane intersect at one, and only one, point.

This dual must be couched with the caveat that (AC^*) fails in the case of parallel lines. The axiomitization of ordinary plane geometry has a fundamental asymmetry: any two points determine a single line without exception, while the same is not true for any two lines determining a single point. Hilbert outlines how to correct this asymmetry.

> But, as is well known, the introduction of ideal elements, namely, points at infinity and a line at infinity, renders the proposition according to which two straight lines always intersect each other in one and only one point universally valid[28, p.372].

The introduction of ideal elements in plane geometry is aimed at a very particular purpose: banishing asymmetries in the system and thereby making the geometrical rules universally applicable without exception. Their advantage—and therefore their *raison d'être*—is that they make "the system of the laws of connection as simple and perspicuous as is at all possible"[28, pp.372-3]. Introducing both a point and a line at infinity makes the Axiom of Connection and its dual true. Let the point at infinity be ω and the line at infinity be λ. We now prove that both (AC) and (AC^*) are true in this new, expanded geometry that includes ideal elements.

(AC) For any two points $\{a, b\}$, if $a, b \neq \omega$, the line l that connects $\{a, b\}$ is connected by the line guaranteed to connect a with b by the axiom of connection in traditional plane geometry. If, without loss of generality, $a = \omega$, then λ connects $a = \omega$ with b.

(AC^*) There are three cases. For any two lines l and m,

1. If $l, m \neq \lambda$ and l is not parallel to m, then l and m intersect at the point guaranteed by the logical consequence of the original axiom of connection in traditional plane geometry.

2. If $l, m \neq \lambda$ and l is parallel to m, then l and m intersect at point ω.

3. If, without loss of generality, $l = \lambda$, then l intersects m at point ω.

The addition of these two ideal elements has made possible the duality principle of geometry—that concepts have a dual with which they may be interchanged producing a new result. A similar strategy appears in the field of analysis. Prior to the adjunction of complex numbers to the reals, polynomial equations such as $y = x^2 + 4$ lack any roots. Any claim made about roots of a polynomial must, in this weaker system, be made with caveats regarding the existence of such roots. However, once the complex numbers are introduced, these caveats may be done away with. All polynomial equations have as many roots as their degree (counting the repetition of roots). The theorems and results of analysis are universally applicable because of the addition of these ideal elements.

However, a discussion 'adjunction', 'addition', and 'introduction' of these ideal elements necessitates an examination of their ontological status. Are they on the same footing as the elements to which they are being introduced? Does the new system replace completely the old system? Is a distinction between these new elements and the old elements worth maintaining?

On this matter, Hilbert draws a sharp line separating the old from the new, between the concrete and the ideal. In discussing the geometric example, Hilbert says, "...the points and straight lines of the plane are initially the only real, actually existing objects"[28, p.372] This quotation suggests that Hilbert holds that these new elements at infinity are different from the original ones. However, he uses the word 'initially', so perhaps the actual existence of the ideal elements is marked by their introduction to the system.

The proper context, however, clarifies that Hilbert is vehemently opposed to assigning the same ontological status of the original elements to the new ideal elements. The introduction of this example comes in the midst of a lengthy discussion of the infinite. More precisely, it comes at the conclusion of a passage wherein Hilbert argues against the actual existence of the infinitely small or infinitely large. For the former, Hilbert argues that considerations of relativity made it necessary to relinquish Euclidean geometry and replace it with elliptical geometry, doing away with ideas of the infinitely large actually existing in nature. As for the latter, Hilbert says,

> ...we do not find anywhere in reality a homogeneous continuum that permits of continued division and hence would realize the infinite in the small. The infinite divisibility of a continuum is an operation that is present only in our thoughts; it is merely an idea, which is refuted by our observations of nature and by the experience gained in physics and chemistry [28, p.371].

Hilbert denies the actual existence of infinite quantities, either infinitely small or large, and concludes that there is nothing to recommend their actual existence in mathematics. Rather, instead of actually existing, Hilbert assigns an essential heuristic role to the infinite in mathematics.

> Yet it could very well be the case that the infinite has a well-justified place *in our thinking* and plays the role of an indispensable notion [28, p.372, emphasis in original].

The ontological justification for the existence of the infinite extends only as far as it helps our reasoning and thinking. Returning to the geometry example, the point at infinity does not exist in the same way as the rest of the points in the plane. Rather, it is a kind of useful fiction—we can talk about it, reason with it mathematically, and even use it to prove interesting results. Ultimately, its existence is justified solely by the extent of its usefulness, whereas ordinary points are ontologically secure and basic. Note that the introduction of the point and the line at infinity does not make the principle of duality *true*. Instead, it clarifies its use by creating a more powerful system within which the principle of duality has no exceptions. It is by making these sorts of calculations simple, the ideal elements are said to exist for Hilbert.[15]

This sort of existence allows Hilbert to introduce ideal elements while keeping the only epistemically relevant elements concrete. Concrete propositions are the ontological simples of the theory; their existence is actual, in contrast to ideal elements that exist only as helpful tools for proving things about the concrete objects. The difference between the ontological status (and thus epistemic status) of ideal and concrete permits Hilbert to address the two potential issues raised at the beginning of this section. First, there is not a finite metatheory proving the consistency of a transfinite theory; instead, the finite metatheory proves that the adjunction of ideal elements to a finite metatheory does not cause contradictions. The theory remains fundamen-

[15]There is resonance here with the role of the infinite and Poincaré's own view of the infinite as a non-existing, mental construct. Compare to §2.3.2

tally finite in nature. Hence, Hilbert can classify the transfinite axiom (page 65) as an ideal element of the system and employ it without sacrificing epistemic certainty. Second, Hilbert's introduction of ideal elements brings the theory in accordance with his prohibition of infinite reasoning, because what appears to be infinite reasoning using ideal propositions is actually indirect reasoning about finite propositions. Ideal propositions are employed only because of convenience of communication —they hold no epistemic status beyond indicating how authentic, formal, finite reasoning should proceed.

3.2.4 Formal vs. Contentual

Unlike the first three, there is no seminal paper that draws this contrast into sharp relief. Instead, the distinction becomes clear over the course of Hilbert's papers. The formal/contentual distinction appears straightforward, but only in light of the ideal/concrete distinction. Once the contrast between ideal and concrete propositions is established, it is clear how to characterize Hilbert's conception of formal versus contentual.

Formal reasoning is reasoning done solely through the manipulation of symbols according to a set of rigid rules. It is defined purely by its form, lacking any interpreted meaning behind the symbols: it is purely syntactic.

Contentual reasoning on the other hand, has content. There is an intended interpreted meaning. For Hilbert, a contentual proposition is one that attempts to communicate an idea to the reader and does so with an intended meaning. The mark '1' is trivially formal, because it is a basic element of the theory, a concrete element devoid of meaning. Just as trivially, '2' is contentual, since it has a meaning: it is intended to communicate to the reader the mark '11'. Similarly, '24' is contentual because it means '111111111111111111111111'. Formal reasoning takes place when and only when concrete objects are involved; contentual reasoning takes place in all other cases, but can be marked out positively by the presence of some identifiable meaning.

An example of the differentiation between contentual and formal reasoning will help illuminate this dichotomy—necessarily, the distinction mirrors the distinction between ideal and concrete. Hilbert [28] distinguishes between propositions like

$$1 + (1 + 1) = (1 + 1) + 1$$

and

$$1 + 2 = 2 + 1$$

and

$$a + b = b + a.$$

The first counts as a concrete, formal proposition. Its epistemic value derives from the fact that it is finite, surveyable, and epistemically secure. The second is also concrete, but strictly speaking, fails to be formal because the '2' means something. It is a method of conveying a formal equation in a shorter, easier to understand manner. Its epistemic value trades on the fact that it communicates a finite, formal proposition, but is not a finite, formal proposition itself. Finally, the third proposition is composed of 'a' and 'b', which are uninterpreted formal objects without finite content. It is an ideal, transfinite proposition. Its epistemic value derives not from immediate grasp of its content but from its derivability from the transfinite axiom (page 65). Formal reasoning must involve only propositions that resemble the first equation. Contentual reasoning takes place involving equations that resemble the second and third equations.

A proof using ideal propositions will necessarily be contentual. It can provide a proof-sketch of properly formal, finitary proofs, but does not formally prove anything. For instance, a proof of the statement $1 + a = a + 1$ can be accomplished through the application of the mathematical induction axiom. We show that $1+1 = 1+1$ is true. Then, we assume that $1 + n = n + 1$ holds and show that $1 + (n + 1) = (n + 1) + 1$ holds. Via the ideal principle of induction, we will have then shown that for all a, the statement '$1 + a = a + 1$' is true. But for Hilbert, the proof just given is not a real proof in the sense that it is not wholly formal and finite. It is merely a useful way of communicating both that there is a finite, purely formal proof of that equation and giving an outline of how to proceed.[16] It is short hand for an infinite collection of real proofs, for example, the proof that shows that '$1 + 57 = 57 + 1$' is true. Hilbert draws the distinction thus:

> If we generalize this conception [of treating ideal propositions as communicating finite propositions], mathematics becomes an inventory of formulas—first, formulas to which contentual communications of finitary propositions [hence in the main, numerical equations and inequalities] correspond and, second, further formulas that mean nothing in themselves and are the *ideal objects of our theory*[28, p.380, emphasis in original]

The first set of formulas Hilbert mentions here is the set of contentual propositions composed of numerals and normal mathematical symbols that serve to communicate formal propositions made up of concrete objects. Their referents are the subject of the theory make the goal of Hilbert's project—the elimination of epistemic doubt in the foundations of mathematics—possible by their secure epistemic status. The second set of objects are the ideal propositions that provide the mathematical machinery to derive arithmetic from Hilbert's foundation.

How do these four distinctions help to clarify Hilbert's project in the 1920s? They do so by providing us a precise vocabulary to describe the details of Hilbert's project.

[16]I should mention here that this interpretation of what Hilbert is doing in terms of the status of proofs using ideal propositions is somewhat controversial. Detlefsen disagrees, but I will argue later that Detlefsen's reconstruction fails to avoid a serious Poincaréan objection, so the question of its faithfulness to Hilbert in this regard becomes moot.

Hilbert must provide an axiomatic foundation and then prove two things about it: 1) that it actually is a foundation for mathematics, and 2) that the foundation is itself without doubt.[17] The foundation that Hilbert provides is a finite, formal axiom system about concrete, purely syntactic marks. Because it deals only with these finite marks, Hilbert holds that it is beyond epistemic doubt.[18] He proves that it is a foundation by reconstructing arithmetic via the introduction of ideal axioms to the theory, thereby allowing a contentual derivation of all mathematical results. However, the introduction of these transfinite ideal elements means that the theory has drifted from the epistemic safety of finitism into the danger of the transfinite. In order to prove that these axioms have not led the project astray, Hilbert creates a proof schema in the purely finite metatheory using solely finitary that the axioms can lead to no contradiction. The proof proceeds by demonstrating that any purported proof of a contradiction is not in obedience with the ideal axioms. Because any such purported proof will be finite, this proof sketch is a contentual demonstration of a set of finite proofs. Because of this finitism, Hilbert thinks the epistemic status of the theory is secure.

3.3 Hilbert's Reply to Poincaré

Hilbert has laid out a complicated structure in refining his initial sketches, yet the seeds of the 1920s project lay in his early work—the metatheory/theory and to a lesser extent finite/transfinite distinctions are present in undeveloped forms; the general structure of the foundation, the commitment to proceeding via the axiomatic method,

[17]This two step process is nearly identical (in general terms) to the project of the neo-logicist, to be examined in chapter 5.

[18]A Poincaréan would seize upon this claim and argue that Hilbert is holding a psychological principle as bedrock (see §2.1.3). To some extent, I think there is a resonance here, but I think the Hilbertian would respond that if finite reasoning is not beyond doubt, then total skepticism about mathematics is the only viable position. Once the complete skeptical position is rejected, there must be something taken as epistemically basic, though not because of psychology.

and the attempt to prove the impossibility of the derivation of a contradiction by purely syntactic means are all present. The pertinent question for current purposes is whether or not this refinement can address a Poincaréan style *petitio* objection.

Hilbert certainly seems to think so when he writes,

> Poincaré already made various statements that conflict with my views; above all he denied from the outset the possibility of a consistency proof for the arithmetic axioms, maintaining that the consistency of the method of mathematical induction could never be proved except through the inductive method itself. But, as my theory shows, two distinct methods that proceed recursively come into play when the foundations of arithmetic are established, namely, on the one hand, the intuitive construction of the integer as numeral. . . that is *contentual* induction, and, on the other hand *formal* induction proper, which is based on the induction axiom and through which alone the mathematical variable can begin to play its role in the formal system.
>
> Poincaré arrives at his mistaken conviction by not distinguishing between these two methods of induction, which are of entirely different kinds. . . [26, p.473, emphasis in original]

However, this quotation is the entirety of Hilbert's response to Poincaré. Though he is right to say that there are two forms of induction—the finite, intuitive induction of the metatheory and the transfinite, formal axiom of induction in the theory—Hilbert fails to show how this differentiation defuses Poincaré's objection.

In order to attempt to answer this question, I will look at two modern defenses of Hilbert against the Poincaréan objection, by Marcus Giaquinto and Michael Detlefsen. Giaquinto appears to misconstrue the heart of the objection, rendering his reply insufficient. However, Detlefsen's response appears effective in rebutting Poincaré—the justification of the Axiom of Induction is, under this interpretation, non-circular. However, despite being the best chance Hilbert has to respond to Poincaré, I will argue that the response comes at too high a cost. The finite metamathematical proof of the axiomatic consistency of the theory implictly invokes non-finitary induction as support for one of his premises. Thus, Hilbert invokes a meta-metatheoretical statement that is non-finitary. This implicit appeal, coupled with Hilbert's stated goal of removing any doubt from all of the foundations of mathematics opens Hilbert to a refocused Poincaréan objection. Thus, while Hilbert can reply to Poincaré's first objection, the goal of his project him to be vulnerable to a second-order Poincaréan objection. First we turn to Giaquinto and then to Detlefsen.

3.3.1 Giaquinto's Hilbert

Giaquinto's reconstruction of Hilbert's program in many ways agrees with the one presented here (summarized on page 74) in most ways, with the differences attributed to different emphases. In particular, his account focuses on the distinction between finite and transfinite propositions and reasoning. A fundamental question for Giaquinto becomes how apparently infinite laws can be recast in solely finitary methods. For example, Giaquinto identifies what he calls "finitary general statements" such as [22, p.146]:

$$c + 1 = 1 + c \tag{FG}$$

The problem with such statements, according to Giaquinto, is that they appear "to involve a tacit reference to an infinite totality," and thus there is reason to believe that they should be excluded from a purely finitistic reasoning [22, p.147]. He addresses this concern by arguing that the meaning of statements like FG can be understood as a general disposition to assent to any instance of FG with a numeral in place of 'c'. Giaquinto highlights that someone can have this disposition without having any reference to an infinite totality or even having been exposed to an infinite object. The point then, is that these sorts of statements are legitimated through this restricted reading.

Proofs of these finitary general statements can be salvaged in a similar way for Giaquinto—a proof of a finitary general statement like FG is considered legitimate whenever the ideal proof of FG gives an outline of how to finitarily prove any specific instance of the generality. Giaquinto phrases this as "when we have a finitary procedure for obtaining from the purported proof a finitary proof of any given instance of the general proposition"[22, p.155].

The principle of induction in Hilbert's theory is salvaged in the same way. Rather than being a quantified principle of induction, Giaquinto considers it as a schema, free-variable induction, or as a rule:

$$\frac{\varphi(1),\ \varphi(y) \longrightarrow \varphi(sy)}{\varphi(y)}$$

Note that this rule, when used in a proof of a generality, can be employed to produce a finite proof of any instance of the rule. For instance, once shown that this rule holds for some predicate φ, then for any given number c, one can construct a proof that $\varphi(c)$ by simply constructing a proof involving $c-1$ instances of *modus ponens*. We (the mathematically educated) recognize that the hypothetical person cognizant of only finite quantities—the idealized person from above never introduced to infinite totalities—is able to construct with this rule a proof for any instance $\varphi(c)$. From this, we, along with Giaquinto, conclude that the finite induction schema in the metatheory is justified and well-grounded.

Giaquinto thinks that this distinction between the unquantified rule of induction in the finite metatheory and the formalized quantified principle of induction resolves the Poincaréan objection. However, he misconstrues the objection, taking the force of the objection to be that the principle of induction is not accessible from the finite point of view:

> The argument is that induction on the natural numbers would be needed in any proof of the consistency of a theory and yet induction is something

that is not evident from the finitist point of view[22, p.162].

Giaquinto thinks his argument that the unquantified rule of induction is acceptable from the finitist point of view rules out this objection. However, the Poincaréan objection is not that induction fails to be finite; rather, the objection's force lies in the claim that Hilbert has presupposed induction—finite or infinite—in proving the reliability of induction, either finite or infinite. A use of finitely acceptable induction to demonstrate the consistency of a finitely acceptable form of induction is just as illicit as using infinite induction to prove the acceptability of infinite induction. Simply showing that Hilbert has a finitistically acceptable form of induction at hand does not answer the question.

Giaquinto's position has painted himself into something of a corner, regarding the actual *petitio* challenge. The emphasis on the finite/transfinite distinction means that Giaquinto is committed to saying that ideal propositions are finitely accessible. In particular, this commitment means that the induction axiom (which is included in the consistency proof that Hilbert needs) is finitely accessible. But then, according to his response to what he took to be Poincaré's objection, so is the principle in the meta-theory—indeed this property hews closely to Hilbert's own language. But then, Giaquinto construes Hilbert as using a finite principle of induction to prove the consistency of a finitely accessible induction axiom. Regardless of the fact that one is in the metatheory and one in the theory, it still remains *prima facie* circular, because they share the same epistemic justification: finitism.

3.3.2 Detlefsen's Hilbert

Michael Detlefsen's reconstruction of Hilbert's response to Poincaré's example clearly differentiates between the formal, transfinite principle in the theory and the contentual, meta-theoretical principle of induction that is employed in its justification.

In fact, Detlefsen's argument that the use of contentual induction to non-circularly justify the formal principle of induction succeeds in rebutting Poincaré. However, though Hilbert is vindicated initially, I will argue that even Detlefsen's reconstruction must admit a meta-metatheoretical statement that employs a use of transfinite induction. This admission, coupled with the Hilbertian goal, again opens the door to another *petitio* type objection.

Detlefsen's argument hinges on a particular notion of circularity. Specifically, he claims that an argument is circular whenever any reason for doubting the conclusion "is an equally strong reason for doubting the premises or inferences which lead up to the conclusion"[13, p.60]. Operating from this fundamental definition of circularity, Detlefsen reconstructs Poincaré's argument as having two premises:

1. Any reason for doubting the real-soundness of I's use of induction is an equally strong reason for doubting the truth (truth-preservingness) of any contentual meta-mathematical induction that might be used to prove I's real-soundness.

2. Any reason for doubting the real-soundness of I is an equally strong reason for doubting the real-soundness of its use of induction.[13, p.60]. [19]

Poincaré's claim, according to Detlefsen, is a combination of these two premises; the strategy to reject a Poincaréan objection is to critique both of these premises. Detlefsen rejects the second premise because any ideal proof I that invokes the induction axiom will also invoke other, non-inductive premises that one might also doubt. One's reason to doubt the non-inductive premises would also count as reason to doubt the soundness of I as a whole, so there are some reasons for doubting the real-soundness that are not doubting the use of induction. Thus premise two is not true.

[19] 'I' denotes any ideal proof that involves induction and "real-soundness" means that the transmission of the epistemic finite status from the premises to the conclusion.

However, this objection misses the mark on a number of fronts. First, it is reliant on a particular notion of circularity, that could just as easily be reconstituted in terms of reasons to believe rather than reasons to doubt. That is, if, instead of Detlefsen's account of circularity, we posit that an argument is circular whenever reasons to believe the premises presuppose reasons to believe the conclusion, then Detlefsen's account of premise two and his objection can be turned away. His account of circularity is contentious and lies at the heart of this objection.

Even granting his account of circularity, he still misconstrues Poincaré's argument in requiring premise two at all. For premise two to be needed, the conclusion of Hilbert's metatheoretical argument must be that ideal proof I is sound, since this is the conclusion to be doubted in Detlefsen's premise. Because the overall soundness of a proof guarantees the soundness of each of its constituents, the use of induction in I is real-sound. That being the case, there is then a parallel meta-theoretical argument with the conclusion that the use of induction is sound. The Poincaréan can direct his attack there. The objection need not accept Detlefsen's second premise since it is trivially true that doubting the conclusion of this parallel argument (that I's use of induction is real-sound) provides good reason to doubt the real-soundness of I's use of induction.

However, Detlefesen's main response to the Poincaréan is an objection to the first premise. His argument can be paraphrased as follows:

1. Finitary logic is a sub-theory of classical reasoning;
2. The closure of any set of propositions in finitary logic is a subset of the closure of the same set of propositions within classical reasoning (from 1);
3. It is conceivable that a use of induction in classical reasoning would result in an inconsistency, while the same use of induction in finitary logic would not;

4. A reason to doubt the soundness of a use of induction in classical reasoning is not necessarily a reason to doubt the soundness of the same use of induction in finitary logic (from 2 and 3)[13, p.62].

Note that this argument is not at all vulnerable to a recasting of the account of circularity, for if we use my suggested alternative above, we could similarly conclude it is not the case that any reason to believe the premise (that finite induction is sound) presupposes a reason to believe the conclusion (that induction within classical reasoning is sound).

One problem with this argument is that it equivocates on a tension that arises from Hilbert's introduction of ideal elements to the theory. On the one hand, the ideal axioms are introduced so that they can extend the results in the concrete theory. The very reason that Hilbert introduces these ideal elements is because the set of conclusions that can be drawn by the concrete theory does not reconstitute all of arithmetic. Hilbert writes,

> And analysis cannot be constructed by a concrete procedure of the sort we have just given for elementary number theory. For we cannot come close to exhausting the essence of analysis merely by using that sort of contentual communication; rather, we need real, actual formulae for its construction [31, 32].

On the other hand, these ideal principles must be included only in the theory—they must in some way be translatable or verifiable by the concrete objects in the theory. Hilbert wants to use the epistemic certainty of the finite, concrete language to guarantee the certainty of the ideal part of the theory. Thus, the results derived by use of the ideal axioms must be finite truths, or at the very least what Giaquinto calls finitely general statements. The tension is that the axioms appear to need to go beyond finite domains in order to achieve their purpose (reconstructing mathematics), while at the same time must be restricted to finite conclusions in order to retain their epistemic certainty.

The way that Detlefsen's response equivocates on this tension is that it hews close to the one side of the tension—that the ideal axioms must introduce or go beyond the purely finitary logic—in order to justify premise 3. What makes this possibility conceivable is that the ideal propositions like the induction axiom extend the finite system; if they did not do so, then the failure envisioned in premise 3 would not be conceivable.

However, Detlefsen's own ascription to an instrumentalist view of the ideal propositions—roughly that they are a tool by which one can derive new finite truths even though the tools themselves are not solely finite—requires that his ideal axioms and non-finitary reasoning be finitarily sound in the finite domain. Detlefsen's own position requires that the ideal axioms must not result in any contradictory finite statements, which means that it should not be conceivable that applications of induction in classical logic lead to a contradiction while failing to lead to a contradiction in finitary logic. This particular commitment is deeply seated in Detlefsen's program.

However, the equivocation on this tension is a problem solely because of Detlefsen's other commitments. One might not subscribe to Hilbertian instrumentalism and have something like Giaquinto's position, wherein ideal proofs are mere proof schemata rather than at the same epistemic level as proofs themselves.[20]. There would need be no equivocation and something like Detlefsen's argument would apply. However, this patch up job ignores a much deeper problem that is present in any response that models itself on Hilbert's while maintaining Hilbert's stated goals.

[20] Detlefsen is committed to this latter position. Cf. Detlefsen [13, pp.54-57]

3.4 Poincaré Redeemed

There is a general problem with any attempt at reconstructing Hilbert's project in an attempt to avoid Poincaré's *petitio* objection. Any such attempt will necessarily run afoul of Hilbert's stated goal for the project: the dismissal of all questions of legitimacy from the foundations of mathematics. First, I will provide some brief evidence expounding on Hilbert's goal and then show how any attempt to prove the consistency of the ideal axioms (which include induction) non-circularly will result in an implicit appeal to a full principle of induction. Such an appeal in the meta-metatheory would not be damning to the project, save that Hilbert's goal precludes such an appeal.

Hilbert's goal remained constant throughout the 1920s—the dismissal of the question of doubting the foundations of mathematics. Early in his 1922 paper, he writes,

> But that is what I require: in mathematical matters there should be in principle no doubt; it should not be possible for half-truths or truths of fundamentally different sorts to exist . . . I am of the opinion that the foundations of mathematics are capable of full clarity and knowledge, and that the problem of grounding our science is difficult but nevertheless conclusively solvable [31, 1].

He begins the 1923 paper,

> My investigations in the new grounding of mathematics have as their goal nothing less than this: to eliminate once and for all, the general doubt about the reliability of mathematical inference [30, 1].

He writes in 1925,

> That, then, is the purpose of my theory. Its aim is to endow mathematical method with the definitive reliability that the critical era of the infinitesimal calculus did not achieve [28, p.370].

And finally, he writes in 1927,

> I pursue a significant goal, for I should like to eliminate once and for all the questions regarding the foundations of mathematics, in the form in which they are now posed, by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science [26, p.464].

The uniting theme in each of these quotations is that Hilbert aims to solidify not just the consistency of his axioms, nor just the status of finitary arithmetic. His goal is to establish the whole of mathematical reasoning. He aims to banish any and all questions of justification and to demonstrate once and for all that it is impossible to have reasonable doubt about the foundations of mathematics.

However, this goal will blunt any attempt to give a non-circular justification of induction in Hilbert's theory. Hilbert attempts a sketch of this proof in 1923; it resembles in many ways his earlier attempts from 1904. Hilbert endeavors to show that the formula $0 \neq 0$ is not derivable within the system. He proceeds by *reductio*, where he first assumes a proof with a conclusion of $0 \neq 0$. Then he performs a number of transformations on the proof to bring it into a logical normal form: he makes every line have only one conclusion, eliminates all variables, and translates it into the object language. From this "refined" proof, he then claims that each formula can be checked "to determine whether it is 'correct' or 'false' in a certain sense that can be precisely stated"[30, 29]. The pertinent quotation immediately follows:

> Now, if the supposed proof were to satisfy all our requirements, then clearly each formula of the proof would have to pass this test in turn. Thus the end-formula $0 \neq 0$ would also have to be 'correct'; but it is not correct [30, 29].

Even setting aside the issue of the definition of 'correct', there remains a controversial claim. Namely, Hilbert claims every proof can be transformed into this linear structure and it can be determined if the formulae are correct. But the fact that this claim is true for *every* proof is based on induction. Its truth is guaranteed by the realization that for any given proof P, with n-many formulas, we are able to check that the first formula is correct, that the second formula is correct, and so on up to the nth formula. That such a procedure is possible for *every* proof—for *all* finite proofs— is then merely an inductive claim. This time, the application of induction is at the meta-metatheoretical level.

Giaquinto has a similar problem in his account. He argues that Hilbert has finitary general statements, and that these statements amount to a general disposition to assent to specific instances. Similarly induction, both theoretical and metatheoretical, becomes an unquantified rule. Giaquinto's argument that this rule can be generalized to provide a finite proof of any particular instance, however, also invokes an implicit inductive inference. Giaquinto gives an example of how to employ the rule to create a finite proof, and on providing such an example claims, "Obviously, the prescription can be generalized to obtain a finitary proof of any instance of $\varphi(n)$" [22, p.155] What he claims is true—we can generalize his procedure and thereby have a finite procedure for producing finite proofs from the finite induction schema. However, this claim itself is metatheoretical (and in this case meta-metatheoretical since Hilbert will employ this induction schema in the metatheory). That we know such a procedure exists is only possible via appeal to a transfinite principle that allows us to conclude that proofs for *all* numbers can be generated from this rule.

Finally, suppose that we grant a form of Detlefsen's reconstruction. That is, suppose that there is a proof in the finite metatheory of the ideal transfinite principle of induction in the theory. We could even concede that this derivation in the metatheory of the theory's consistency is non-circular. However, this proof would have to show that for every proof in the theory, '$0 \neq 0$' is not its conclusion. This amounts to a procedure in the meta-theory of how to show, given a purported proof of '$0 \neq 0$', that it is not actually a proof, that it fails to obey the rules laid down in the theory.

Regardless of how this proof proceeds, it must be that it is applicable to every proof in the theory that purports to show a contradiction. However, the claim that this procedure will work in every case is, again, another meta-metatheoretical claim. Such a procedure would have to be defined recursively in the finite metatheory, along the lines of some rule formulation as in Giaquinto. However, because the procedure is recursively defined, the only way to justify the universal meta-metatheoretic claim would be via induction. In order to know that the proof procedure in the metatheory does, in fact, prove the consistency of the theory, one would have to use, either implicitly or explicitly, induction in the meta-metatheory.

There is good reason to think this problem is endemic. Any consistency proof that purports to increase the epistemic strength of the axioms must use a principle of induction that is at least as weak as the principle within the theory so that the epistemic status is improved. If the metatheoretical principle of induction is not strictly weaker than the theoretical principle, then it is difficult to see how the epistemic security of the theory has been improved.

However, if the metatheory employs a weaker principle of induction than the transfinite principle in the theory, the question arises as to how it is known that the given derivation of consistency is, in fact, a good one. Any explanation of this sort will necessarily appeal to a higher metatheory that employs a stronger principle of induction.

It is Hilbert's stated goal of dismissing *all* of the questions of doubt that makes this appeal unavailable to the Hilbertian. The explication of all the distinctions Hilbert has made and all the progress seemingly gained by the formalization wash away in the face of such an appeal; appealing to a higher level of induction once again makes the project circular. Poincaré's objection, refocused and made meta-metatheoretical, arises again.

In this way, the problem with epistemic justification resembles a Tarski-style regress of metalanguages. The only way to ascertain that the epistemic justification of a meta-theoretical proof is correct is by advancing to a yet higher metatheory.

A Hilbertian reply that such a justification is not needed because finitary reasoning is unquestionable appears dogmatic. Any attempt by the Hilbertian to justify finitary reasoning will lead to a regress, so he must simply accept the epistemic security of finitism. Unless this position is intuitively appealing, the Hilbertian has little to offer as argument for his position.

Chapter 4

Poincare against the Logicists

Poincare's objections to foundationalism in the early 20th century failed to differentiate between Hilbert and bona fide logicists like Frege, Russell, or Dedekind. Moreover, his critiques openly mock the work of those to whom he is objecting, even going so far as to say regarding the *petitio* objection for the logicists, "I have not seen it in the pages I have read, but I do not know whether I should find it in the three hundred pages they have written that I have no wish to read."[48, 183].

Among these objections, the *petitio* objection stands out. Recall that this objection amounts to an accusation of begging the question by employing induction in the derivation of mathematical axioms, one of which is induction itself. In chapter 3 we saw how the original *petitio* was effective against early Hilbert and how it retained its force in a revised form for the later Hilbert. In this chapter, I will show 1) why the original *petitio* is irrelevant to the logicist; 2) give a revised, sophisticated version of the *petitio*; 3) argue, according to Goldfarb [23], why even this sophisticated version fails to touch the logicist.

It should be noted that Goldfarb's argument rests upon a particular interpretation of the logicists' project. More precisely, it relies on a certain interpretation of the logicists' view of logic—that there was no conception of a metatheory in their writings.

Though I find this view compelling, whether or not one objects to it will not affect the purpose of the current chapter. The goal is twofold: first, to show that the logicist is able to wriggle free of Poincaré's grasp only by lights of his peculiar view of logic as inescapably universal—to deny the existence of a metatheory—thereby preventing a revision of the original *petitio* to the sophisticated *petitio*; and second, to show that the neo-logicist cannot respond to Poincaré in the same manner as the logicist. If one disagrees with Goldfarb's interpretation, the sophisticated *petitio* remains a viable objection, both for the logicist and the neo-logicist.

One final terminological note. For this chapter, I will operate under the assumption that Hilbert, regardless of which time period one is talking about, is not a logicist. So, when I speak of the "logicist", I do not mean to include within this group Hilbert. Logicists are those who generally followed a Fregean framework to provide an undoubtable basis in logic for mathematics. Hilbert abandoned this project early to pursue a finitist method of removing doubt from mathematics.

4.1 Original *Petitio* and the Logicist

We have seen the original *petitio* objection previously in reply to early Hilbert (§3.1.3). In it, Poincaré accuses Hilbert of presupposing a principle of induction in his proof of the consistency of induction with the other axioms and given rules of inference. Any justification or proof of such consistency could only be made via an invocation of the principle of induction. As Goldfarb puts it,

> The clearest [*petitio*] is directed against the notion that mathematical induction is not a principle with content, but is just an implicit definition of the natural numbers. Poincaré notes that such a definition must be justified by showing that it does not lead to contradiction; yet any such demonstration would have to rely on mathematical induction [23, 64].

Having to provide a demonstration that given axioms—or implicit definitions—do not lead to a contradiction is a uniquely Hilbertian requirement. The logicist does not give axioms and then guarantee the safety of reasoning with them via a consistency proof. The origin of this difference stems from their underlying assumptions about existence and consistency. We saw earlier that for Hilbert, mathematical consistency guaranteed the existence of the items in question. Recall:

> . . . if the arbitrarily given axioms do not contradict one another with all their consequences, then they are true and the things defined by the axioms exist. This is for me the criterion of truth and existence [20, pp.39-40].

However, this is exactly contrary to the position that Frege and the logicists adopt, wherein the existence of a mathematical object is what guarantees its consistency. Frege writes,

> From the truth of the axioms it follows that they do not contradict one another. There is therefore no need for a further proof. The definitions too, must not contradict one another. If they do, they are faulty. The principles of definition must be such that if we follow them no contradiction can appear[20, p.37].

The fundamental disagreement between logicists and Hilbert is the direction of implication between consistency and existence. Hilbert believes that consistency is the more basic of the two notions, and therefore a proof of the consistency of a set of axioms guarantees their existence; Frege and the logicist hold that the truth of the axioms (their truth means that they exist as axioms) is the more fundamental notion and that it implies that they are consistent. As Goldfarb writes,

> Frege and Russell (and later on, Couturat) insist that existence is not proved by consistency; rather, consistency is vouchsafed by showing existence.[23, 65]

This fundamental difference makes Hilbert vulnerable to the original *petitio* and the logicists not. Hilbert must somehow show that the axioms he gives for arithmetic are consistent because he has no notion of truth of the axioms. There is no way in which Hilbert considers his axioms to be "true". For instance, the parallel postulate could be laid down or not, with each option resulting in a different geometry. But one geometry is not the "true" geometry. Indeed, we saw in the previous chapter that he assiduously avoids even giving his axioms even any meaning so that there is only a purely syntactic axiomatic method as the foundation of mathematics.

However, Frege and the logicists have a fundamentally different conception of axioms. The axioms that they give—the basic laws—have to be true in order to guarantee their consistency. Thus, because they are true as given, they are always interpreted and never the syntactic axioms that Hilbert gives. There is no syntactic demonstration of the consistency of the axioms, and thus no application of mathematical induction. The original *petitio* fails to gain any bite.[1]

4.2 Sophisticated *Petitio*

Poincaré makes a more refined and subtle attack on the logicist program which, following Goldfarb, I will call the sophisticated *petitio*. Similar to the original version, the sophisticated *petitio* accuses the logicist of circularity. However, rather than arguing that the induction is implicitly used in a proof of consistency, it argues that the logicist presupposes induction in establishing the system within which the reduction of mathematics to logic will take place. Poincaré actually makes this argument against Russell's theory of types initially, saying,

> ... the theory of types is incomprehensible, if we do not suppose the theory of ordinal numbers already established. How will it then be possible to base the theory of ordinal numbers on that of types [44, p.52]?

[1]Of course, it should be noted that the axioms chosen by Frege were, in fact, not consistent as shown by Russell's Paradox. This result should be seen, then, as a proof that Basic Law V is not true. There may be very good reasons to doubt the logicist here, but I simply want to show that the original *petitio* is not one of them.

Poincaré alleges that Russell presupposes a theory of ordinal numbers to differentiate between orders of propositions with the goal of structurally restricting propositions of order n to refer only to propositions of order $n-1$. Russell's theory of types presupposes a theory of number even to classify what the order of a proposition is; the presupposition occurs in the formation rules. For Poincaré's temporal understanding of mathematics and intuition, the presupposition happens *before* the formal system of the theory of types is established, and there voids subsequent epistemic simplification.

This objection is generalizable beyond Russell's theory of types to any logicist-type project. Any proof of the reducibility of mathematics to logic will have to give a derivation of the ordinal number theory. This derivation will necessarily employ a formal system of logic. In defining this formal system of logic, the logicist must use induction to define the notions of formula and derivation recursively. Thus, the logicist relies on mathematical induction for the formation rules of the logical system, within which he intends to derive the principle of induction. He has reasoned in a circle.

Important to note about this sophisticated version is that it makes a fundamentally meta-theoretical claim. While the original *petitio* argues from a theoretical standpoint(from within the formal system), the sophisticated *petitio* argues from a metatheoretical standpoint(from outwith the formal system). The former claims that any consistency proof within the system will employ induction; the latter claims that the formation rules for the formal system of logic presuppose induction. The original version is effective against the early Hilbert because his consistency proofs occur within his system (see §3.1.2). However, it fails to touch the logicist, because the *truth* of his fundamental principles guarantees consistency, not a *positive proof* of that consistency. The absence of a consistency proof within the system means that there is no possibility of an implicit application of induction, nor a presupposition of number. The original *petitio* fails, but the status of the sophisticated version remains in question.

Goldfarb provides a defense for the logicists against this sophisticated *petitio* by invoking the logicists' universalist conception of logic to blunt Poincaré's attack before it starts. The defense says roughly that because there is no meta-theoretical perspective from which to mount the objection, Poincaré's sophisticated objection is blocked.

For Poincaré, the objection is made "prior to" the formal system being in place. This sort of formulation should be no surprise, given Poincaré's emphasis on temporal intuition(see 2.2.2). For Poincaré, there is sense in speaking about a time prior to a given formal system. Therefore, he argues that in laying out the rules for any formal system requires a recursive definition and therefore presupposes the principle of induction. Establishing rules for a formal system brings into existence that formal system, for Poincaré. Any justification of this principle from within the system is circular.

However, this objection is not one that the logicist needs to address. It is mounted from a metatheoretical standpoint; it comes prior to the formal system of logic. For the logicist, there is no possibility of such a standpoint; there is, for the logicist, no notion of a metatheory. The conception of logic is all-encompassing insofar as logic is not merely a formal system with funny looking symbols and complicated rules, but rather the space within which any and all reasoning takes place [23, 69]. Logic cannot be decided upon and set forth in formation rules. There is no notion of "prior to" as there is for Poincaré; for the logicists, logic exists outside the framework of time. It is eternal and unchanging, instead of being summoned into being by the formation rules. Logic simply exists and argumentation happens underneath its umbrella. There is no way to get outside of logic and no Archimedean standpoint from which Poincaré can launch his attack.

More than being a potent objection, Poincaré's disagreement with the logicist highlights a fundamental difference between the two conceptions of mathematics. Poincaré's psychologism held that psychological principles are prior to understanding the formal system of mathematics and therefore also epistemically prior (§2.1.3). Thus, the formal system itself trades on this intuitive basis.

However, for Frege, no intuition lies at the bottom of mathematics. Logic exists regardless of our ability to understand it and psychological elements have no place in the epistemic or ontological justification of mathematics. Similarly, formation rules of the system—definitions of derivations, syntax rules, etc.—do not have any bearing on justifying the system. They themselves do not need justification nor examination. They exist merely for our benefit in coming to understand the structure of the system. The rules are similar in kind to laws of physics, which are descriptive rather than prescriptive. These rules may be necessary for any individual to come to understand logic, but this necessity is only a *psychological* one. Unlike for Poincaré, this psychological necessity fails to have any justificatory role for the logicists. The necessity of formation rules has the same epistemic status as the constancy of inkblots on the paper on which we do mathematics. It is necessary that inkblots stay constant throughout our proofs—that they do not jump around the page to change what we had written as an 'x' to a 'y' or worse to nonsense. However, this necessity plays no role whatsoever in the epistemic status of mathematics. Frege says explicitly,

> A delightful example of the way in which even mathematicians can confuse the grounds of proof with the mental or physical conditions to be satisfied if the proof is to be given is to be found in E. Schŕoder. Under the heading "Special Axiom" he produces the following: "The principle I have in mind might well be called the Axiom of Symbolic Stability. It guarantees us that throughout all our arguments and deductions the symbols remain constant in our memory—or preferably on paper."[18, p.viii]

While it is true that symbols must be stable—that their meanings do not change constantly—it is merely a concomitant of proof and not the grounds on which the proof is given. Similarly, the formation rules of logic are required for us to understand logic, but they do not play a role in forming the system.

With this understanding of the formation rules, it becomes easier to understand Frege's belief in the truth of the axioms prior to their consistency. Of course the rules of logic are consistent because logic is the universal language of thought and argument. The rules by which logic operate simply exist and therefore are consistent. There will be a question as to whether a *given* set of axioms is consistent only insofar as there is a question as to whether that given set of axioms accurately reflects the *real* axioms for logic.

These two different conceptions of logic—Poincaré's and the logicists'—also manifest themselves in the notion of the justification of theorems. For Poincaré, derivation can be defined recursively, thereby making sense of the claim that a proposition has been derived. Roughly, pick which rules of inference are allowed and then state that a sequence of $(n+1)$ many proposition is a derivation if and only if the first n propositions are a derivation and the $(n+1)$th proposition in the sequence results from the axioms, some subset of the first n lines, and the proper application of the rules of inference. For Poincaré, a given statement is provable if and only if there exists a derivation with that statement as its last proposition. The statement that sentences are in general provable only if there is a derivation of them is a metatheoretic one.

For Frege and the logicists, appealing to the notion of a derivation in order to determine provability is not possible. There is no assertion to be made that a given sentence is a theorem. The justification of a theorem is the explicit laying out of the derivation. As Goldfarb says, "To give the ultimate basis for a proposition is to give the actual proof inside the system, starting from first principles; that is, it is to assert the proposition with its ground, not to assert the metaproposition 'this sentence is a theorem."'[23, p.69] The syntactic checking of a proof—making sure we have applied the rules correctly—is not a systematic notion for Frege, but rather another concomitant of logical knowledge. When doing logic, we must check to make sure proofs follow the rules correctly, at which point we may say that a given proof is indeed valid. But this checking is only something we must do because of our propensity to make mistakes; it is not epistemically necessary. It is something we have to do only out of psychological necessity, just like our need to write down proofs because we cannot hold their entireties in our heads at once. Our inability to work a proof without aid of paper and pencil does not make paper and pencils epistemically necessary to mathematics. Our need to check proofs does not make such syntactic statements such as "This is a proof," necessary to logic, nor does it legitimize it as an actual statement in the language of logic. Instead, it's something informal meant to indicate that no human mistakes were made. There is, for the logicist, no formalized notion of a metatheory nor an uninterpreted system—there is no syntax absent of semantics.

Poincaré's critique, then, is defused before it gets off the ground. Metatheoretic statements are impossible for the logicist, and Poincaré's attack is launched from a metatheoretical standpoint. As Goldfarb notes, because there is no metatheoretic standpoint from which Poincaré's argument can take hold, the sophisticated version fails to be formulable.

Poincaré's *petitio* objection to the logicist seems to not be a legitimate objection that engages with the logicist, but rather a natural consequence of his fundamental ideas. Similarly, the logicists' reply is not one that shows why Poincaré's objection should be rejected, but rather a demonstration of how their conception of logic does not cede any ground to Poincaré's objection. The problem becomes that there is nothing more for either side to say. The Poincaréan finds this objection to completely convincing, while the logicist thinks no more needs to be said.

Poincaré has made a number of objections to the logicist, of which we have focused on two: the original and sophisticated *petitio*. Both made accusations of presupposing induction: the former within the system and the latter outwith. However, the universality of the logicist's conception of logic allowed both to be turned back and a stalemate reached. The always interpreted, non-syntactic nature of his conception of logic allows the logicist to turn away the original *petitio* and the absence of a metatheory blunts the sophisticated.

For our purposes, this fact is significant because the neo-logicist cannot invoke this universal conception of logic. Besides ignoring more than a century's advance in proof theory and the like, the neo-logicist must make one significant metatheoretical claim. Namely, he must claim that his formulation of the rules in the proof of Frege's theorem—including Hume's principle—does not meet the same fate as Frege's set of axioms; the neo-logicist must show that they are consistent and not susceptible to another "Russellian" paradox. Once they move away from this universal conception, it becomes natural to again raise a Poincaréan objection. It is to this topic that we turn in the next chapter.

Chapter 5

Neo-Logicist

The current chapter looks to revive a Poincaréan *petitio* objection by refocusing it against the neo-logicist. Such a refocusing becomes relevant because neo-logicism has a goal of reviving Fregean approaches to epistemically grounding mathematics while avoiding the antinomies generated by Frege's original attempt.

Though there are many competing conceptions of Neo-logicism, I will focus on the version put forth by Bob Hale and Crispin Wright.[1] This choice is made because the popularity of the view marks it as the *de facto* neo-logicist position. The plethora of responses and objections allows the Poincaréan objection to be located in the geography of other objections, demonstrating that two versions of the *petitio principii* objection can be raised—more specifically, the objection can be made in two different places—both resembling pre-existing objections.

The chapter shall proceed as follows. Section 5.1 will lay out the project of the Neo-logicist, focusing in detail on those areas vulnerable to a Poincaréan objection. Section 5.2 will examine the stated goals of the neo-logicist in comparison to the original goals established by Frege. The final section will examine two distinct places

[1]For alternative constructions see Tennant ([58], [59], [60]), Fine ([16]), Demopoulos ([11], [12]), Hodes ([35], [33], [34]), Zalta ([68], [67]) or Linsky and Zalta ([37])

where a Poincaréan objection threatens—the formation of the formal system and the consistency proof of Hume's Principle with second order logic. These two forms will be compared and contrasted with existing objections put forth by Shapiro ([54], [55]) and by a host of people under the umbrella of the so-called "Bad Company" objections. Bad company objections are so-called because they accuse Hume's Principle of too closely resembling principles that lead to contradiction, for example, Basic Law V. Objections of this sort have been voiced by Field [15] and Boolos [5], among others.

5.1 Neo-Logicism: Reviving Frege

The project of neo-logicism is in one sense easy to summarize. Let Hume's Principle (HP) to be the following:

$$(\forall F)(\forall G)[(Nx : Fx = Nx : Gx) \leftrightarrow (F1-1G)]^2 \qquad \text{(HP)}$$

Neo-logicism is a proof of Frege's theorem,

FT From second order logic and Hume's Principle, Peano's Postulates follow,

with the assumption of a special epistemic status for second order logic and HP. Because of FT, the neo-logicist concludes that mathematics holds the same epistemic status as logic and HP.

However, like all sweeping generalizations, this one obliterates intricate specifics. In particular, it ignores several extensive philosophical theses that the neo-logicist must adopt. The following will give a brief historical background of the failure of the original logicist program, thereby motivating the neo-logicists' replacement of Basic

[2] In giving the formalizations of these principles, I will follow the presentation of MacBride [38]. The formalized version of the principle warrants a brief description of the symbols. 'Nx' is an operator on the extensions of concepts, so '$Nx : Fx$' can be read as 'the Number of xs that are F'. Similarly, '$1-1$' is a relation between concepts, so '$F1-1G$' should be read as 'there exists a one-to-one correspondence between F and G'

Law V by HP; spell out the commitments of Hale and Wright's position; and finally examine the question of HP's consistency with second order logic. For this third section, the chapter will sketch George Boolos's proof of the equiconsistency of HP and second order logic with analysis [3].

5.1.1 The Failure of Frege and Basic Law V

Frege initially introduces what came to be called Hume's Principle as a definition for number in the *Grundlagen.* The first attempt in §§62-4 amounts to Hume's Principle:

However, as MacBride notes, this proposal fails to address the so-called 'Julius Caesar' problem, wherein the definition of number via HP—though adequate to introduce numbers via Frege's Theorem—does not give an acceptable criterion for determining what is and what is not a number. In his analogous case of the direction of parallel lines, Frege highlights the difficulty thus:

> In the proposition "the direction of a is identical with the direction of b" the direction of a plays the part of an object, and our definition affords us a means of recognizing this object as the same again, in case it should happen to crop up in some other guise, say as the direction of b. But this means does not provide for all cases. It will not, for instance, decide for us whether England is the same as the direction of the Earth's axis. [19, §66]. [3]

Returning to the numerical case, the problem is that one is able to determine whether or not a given object b is the same number as a only if it is presented in a certain way: HP provides a way to determine if the number of Fs is the same as the number of Gs, but not a way to determine if the number of Fs is the same as Julius Caesar. One is only able to evaluate the equality of numbers if they are both presented as the number of a concept.

Seemingly in light of this problem, Frege gives a revised definition of number in

[3]Heck [25] has a transposition of this quotation into the context of HP.

§§68-9. He writes,

> My definition is therefore as follows: the Number which belongs to the concept F is the extension of the concept "equal to the concept F." [19, §68]

Frege now reformulates Hume's Principle into what will be generalized into Basic Law V in the *Grundgesetze*:

> But now the proposition: the extension of the concept "equal to the concept F" is identical with the extension of the concept "equal to the concept G" is true if and only if the proposition "the same number belongs to the concept F as to the concept G" is also true.[19, §69][4]

This reformulation of the definition hopes to avoid the Julius Caesar problem by defining numbers as the extensions of concepts, which excludes Julius Caesar as a potential number, since he is not allegedly an extension of a concept but rather a concrete object.[5] However, avoiding the Julius Caesar problem comes at a high cost, for when codified into Basic Law V, this revised definition enables Russell's paradox to rear its head.

Basic Law V states that the objects falling under one concept are the same as the objects falling under another concept if and only if the two concepts are co-extensive. Symbolically—following MacBride [38]—this is written:

$$(\forall F)(\forall G)[(Ext : Fx = Ext : Gx) \leftrightarrow (Fx \leftrightarrow Gx)] \qquad \text{(Basic Law V)}$$

Taking this formulation, Russell defines the predicate Rx as 'x is a predicate which cannot be predicated of itself', and claimed that, "From each answer its opposite follows," for if R is predicated of itself, then it should not be so, and if it is not predicated of itself, then it should be so[51]. This formulation, however, is not

[4] Here, the translation has used 'equal', but a more accurate translation may be 'equinumerous'.

[5] Note that there is a question as to whether or not this step actually solves the problem or merely pushes it back one more step.

formalizable in Frege's system. Frege took this ill-formed objection and produced a formal argument within the system.

First, Frege proves that every concept had an extension in his system:

$$\begin{aligned}
&\text{BLV} = (\forall F)(\forall G)[(\mathit{Ext} : F = \mathit{Ext} : G) \leftrightarrow (Fx \leftrightarrow Gx)] \\
&\text{BLV} \vdash \{x : Fx\} = \{x : Fx\} \leftrightarrow \forall x\, (Fx \leftrightarrow Fx) \\
&\qquad \vdash \forall x\, (Fx \leftrightarrow Fx) \\
&\text{BLV} \vdash \{x : Fx\} = \{x : Fx\} \\
&\text{BLV} \vdash \exists y\, (y = \{x : Fx\})
\end{aligned}$$

Now, from the comprehension principle(CP)—$\exists F \forall Z\, (Fx \leftrightarrow Xx)$, where Xx is any concept not involving F—Frege defined the Russell set:

$$Rx \leftrightarrow \exists F\, (x = \{y : Fy\} \wedge \neg Fx) \qquad \text{(Russell Set)}$$

This is a formalization of the Russell paradox that says that x is in the Russell set if and only if x is the extension of some concept F and x does not fall under that concept. The paradox arises when one considers $r = \{x : Rx\}$, that is the extension of the Russell set itself.

$$\text{Assume } Rr$$

$$\text{iff } \exists F\,(r = \{x : Fx\} \wedge \neg Fr)$$

$$\text{iff } r = \{x : F_0 x\} \wedge \neg F_0 r$$

$$\text{but } r = \{x : Rx\}$$

$$\text{so } \{x : F_0 x\} = \{x : Rx\}$$

$$\text{applying BLV from left to right}$$

$$\forall x\,(F_0 x \leftrightarrow Rx)$$

$$F_0 r \leftrightarrow Rr$$

$$\neg F_0 r$$

$$\neg Rr$$

Therefore, BLV, CP, $Rrvdash\neg Rr$. Thus, BLV, CP $\vdash Rr \to \neg Rr$. Finally, BLF, CP$\vdash \neg Rr$.

$$\text{Now assume } \neg Rr$$

$$\text{iff } \neg\exists F\,(r = \{x : Fx\} \wedge Fr)$$

$$\text{iff } \forall\neg F\,(r = \{x : Fx\} \wedge Fr)$$

$$\text{iff } \forall F\,(r \neq \{x : Fx\} \vee Fr)$$

$$\text{iff } \forall F\,(r = \{x : Fx\} \to Rr)$$

$$\text{iff } \ (r = \{x : Rx\} \to Rr)$$

$$\text{By the definition of } r,\ Rr$$

Therefore, BLV, CP $vdashRr$. BLV and CP together jointly prove $Rr \wedge \neg Rr$, and we have the contradiction.

The historical importance of this story is three-fold. First, Russell's paradox caused Frege to ultimately forsake his project of providing a derivation of mathematics from logic, thereby solidifying the epistemic ground mathematics stood upon. Frege despaired the possibility of demonstrating that mathematics could be derived from logic alone, thereby inheriting the epistemic strength of logic. The neo-logicist position can be seen as an attempt to avoid this despair and resurrect at least part of Frege's program by returning to Frege's initial attempt at defining number via HP.

Second, the only *necessary* application of Basic Law V in Frege's derivation is the derivation of HP. That is, Basic Law V is not used *essentially* in Frege's derivation of mathematics (see Heck [25]), except for proving HP. Once Frege has HP in hand, he is able to continue with the derivation of Peano's Postulates.[6] Important to note here is that whereas Basic Law V conjoined with second order logic is inconsistent, HP conjoined with second order logic is *consistent*: thus, the neo-logicist strategy of resurrecting Frege's project by replacing Basic Law V with HP.[7]

Third, the introduction of Basic Law V was an attempt to avoid the Julius Caesar problem arising from a defining numbers via HP alone. Thus, if the neo-logicist replaces Basic Law V with HP, they need to supply an answer to the Julius Caesar problem that is both adequate to dismiss the objection as well as not run afoul of contradiction.

[6]More precisely, he is able to derive Fregean Arithmetic, within which Peano's postulates can be interpreted.

[7]I will return to this issue in the section 5.1.3.

5.1.2 The Neo-Logicist Project

Fraser MacBride identifies three distinct bodies of doctrine that the neo-logicist is committed to in order to make the proof of Frege's Theorem achieve the epistemic goals of the project:[8]

1. a general conception of language and reality;
2. a particular method for introducing novel expressions into language;
3. a specific understanding of the scope of logic[38, p. 107].[9]

Language and Reality

The neo-logicist's conception of the connection between language and reality is wholly different from the standardly held notion. An ordinary interpretation of the relationship holds that reality is distinct from language and is independently determined. However, for the neo-logicist, language and reality are inextricably and necessarily entwined. As MacBride says, the two "are so related that, if we speak truly, the structure of reality inevitably mirrors the contours of our speech" [38, p. 108]. Therefore, any true statements cannot fail to reflect the way the world actually is.

MacBride further subdivides this doctrine into three components—*Syntactic Decisiveness*, *Referential Minimalism*, and *Linguistic Priority* [38, p.108].[10] The result

[8] These goals will be examined in detail in section 5.2.

[9] The following discussion is based on MacBride[38, pp.108-115]. My presentation is made with an eye to highlighting features of the last two doctrines—the use of abstraction principles and the status of second-order logic—so that objections in §5.3 can be properly placed. As such, my analysis will for the sake of space pass quickly over some distinctions drawn out by MacBride.

[10] These three are described respectively:
Syntactic Decisiveness: if an expression exhibits the characteristic syntactic features of a singular term, then that fact decisively determines that the expression in question has the semantic function of a singular term (reference).
Referential Minimalism: the mere fact that a referring expression figures in a true (extensional) atomic sentence determines that there is an item in the world to respond to the referential probing of that expression.
Linguistic Priority: linguistic categories are prior to ontological ones; an item belongs to the category of objects if it is possible that a singular term refer to it. [38, p.108]

of this doctrine is that “the syntactic form of our (true) sentences cannot deceive us; reality cannot fail to include the objects and concepts which the sentences apparently describe” [38, p. 108]. By adopting this position, the neo-logicist argues in the following manner:

1. Ordinary mathematical practice can be replicated using numerical terms.
2. These numerical terms are parts of true sentences.
3. The Numerical terms have syntactic features characteristic of singular terms, and therefore can be treated as referential in the same way as singular terms. (Syntactic Decisiveness)
4. Because they can be treated (semantically) like singular terms and are parts of true sentences, these numerical terms refer to something in the world. (Referential Minimalism)
5. That “something in the world” to which they refer is an object. (Linguistic Priority)
6. Terms that exhibit the same pattern of use refer to the same objects. (Meaning Supervenes on Use)
7. Therefore, ordinary mathematical objects exist as objects in the world.[11]

The rough synopsis of this argument is that the reconstructed mathematical practice has certain properties that guarantee that the terms refer to existent objects in the world. The argument proceeds from a purely syntactic basis to a claim about the existence of objects in the world. It moves from a statement about the syntactic features of numerical terms to the substantive claim that ordinary numbers exist as objects in the world. Obviously, each of these theses must be defended in detail to complete the neo-logicist project. We will not do so here, because it would take us from our purpose. What is relevant is how the neo-logicist introduces the syntactic terms used in the reconstruction of mathematical practice: the method of abstraction.

[11] Drawn from Macbride [38, pp.108-9]

Abstraction Principles

For the neo-logicist, abstraction principles are the mechanism by which one is able to introduce new terms to a language solely via reference to pre-existing and understood terms. It is the method by which one is able to fix the use and meaning of new operators in a language, bootstrapping from existing resources. For example, in a language that already contains the terms (‘α_1’, . . . , ‘α_k’) and some sort of equivalence relation (‘$\approx$’) that holds between those terms, we can introduce a new operator (‘Σ’) that goes from concepts to objects via the following general principle AP:

$$(\forall\alpha_\varphi)(\forall\alpha_\kappa)[(\Sigma(\alpha_\varphi) = \Sigma(\alpha_\varphi)) \leftrightarrow (\alpha_\varphi \approx \alpha_\kappa)]^{12} \qquad \text{(AP)}$$

The neo-logicist is committed to two claims about principles of the form of AP: *semantic* and *syntactic* novelty. The former states that the method of abstraction introduces genuinely new operators to the language that retain the syntactic features of singular terms, while the latter states that the new terms, provided they actually refer, refer to items to which the α_is do not refer.[13]

Notice that HP is an instance of AP obtained by replacing ‘Σ’ with ‘Nx’, ‘α_φ’ and ‘α_κ’ with ‘Fx’ and ‘Gx’ respectively, and the ‘$\approx$’ with the one-to-one relation ‘1-1’:

$$(\forall F)(\forall G)[(Nx : Fx) = Nx : Gx) \leftrightarrow (Fx1 - 1Gx)]^{14}$$

[12]Taken from MacBride [38, p. 110]

[13]MacBride also identifies *Referential Realism* as a principle to which the neo-logicist is dedicated [38, p. 112]. I omit it here because the potential issues surrounding this position would take me far a field from the objections I wish to examine.

[14]Note that the language within which ‘Fx’ and ‘Gx’ are terms is second order logic. The notion of one-to-one correspondence can be spelled out in a purely second-order logic manner. Rossberg and Ebert [50] give the full statement:

$$\forall F \forall G\, [Nx : Fx = Nx : Gx \leftrightarrow \exists R\, (\forall x\, [Fx \supset \exists y\, (Gy \wedge Rxy \wedge$$

$$\forall z\,(Gz \wedge Rxz \supset z = y) \wedge \forall y\,[Gy \supset \exists x\,(Fx \wedge Rxy \wedge \forall z\,(Fz \wedge \supset z = x))]. \text{ (HP)}$$

Thus, HP inherits all of the properties ascribed to generic abstraction principles. In particular, the stipulation of HP gives numerical terms the syntactic properties of singular terms, which is required for the argument in §5.1.2. Thus, via HP, the neo-logicist claims that one can advance from our knowledge of second-order logic to knowledge of new operators 'Nx'. The patterns of use of the standard mathematical practices mirror the uses of these new operators, and due to their inheritance of the syntactic characteristics of singular terms, the new operators refer to bona fide objects in the world. Thus, via HP and knowledge of second-order logic, one may advance to knowledge of numbers as objects in the world.

Scope of Logic

The final commitment of the neo-logicist is to hold that second-order logic is basically understood and epistemically fundamental. MacBride identifies this as the *Second-Order Logic is Logic* commitment, wherein the neo-logicist is committed to the existence of a class of inferences involving quantification over first-order notions like property and relation whose validity is transparent to us [38, p. 113]. In other words, the neo-logicist is committed to the claim that we have special epistemic abilities to recognize the validity of second-order patterns of inferences, not just first-order patterns.

The reason why such a commitment is necessary is clear: the neo-logicist requires some ability to determine when the right-hand side of HP is satisfied (recall that one-to-oneness is representable only in second-order logic). Only through this special ability are we able to gain knowledge of the right-hand side of HP and thereby gain knowledge of number on the left-hand side. This ability to recognize the truth of second-order statements allows the neo-logicist to claim immediate knowledge of the truth of the following:[15]

$$(x : x \neq x)1-1(x : x \neq x) \tag{A}$$

This statement—the things that fall under the predicate 'is not identical with itself' can be put in one to one correspondence with that selfsame collection of objects—is an instance of the right-hand side of HP, meaning that the following is true by HP:

$$(Nx : x \neq x) = (Nx : x \neq x) \tag{B}$$

This equation essentially "says" that the number of things that are non-identical is equal to the number of things that are non-identical. Then, since it is both true and has the syntactically relevant features (they are the same as singular objects), this means that there is *something* in the world to which '$Nx : x \neq x$' refers, since our true linguistic uses cannot fail but to refer. Because there is such a thing, we are permitted to existentially quantify and say:

$$\exists y\,(y = Nx(x \neq x)) \tag{C}$$

This statement "says" that there exists something that is identical with the number of non-identical things; it "says" that zero exists.

Thus, the neo-logicist is committed to the three positions here outlined. His commitment to the epistemic priority of second-order logic allows the admission of (A) into epistemically warranted judgments. His commitment to the abstraction principle schema AP—in particular the abstraction principle HP—permits the move

[15] The following uses the presentation of MacBride [38, p114], but it replicates Frege's original procedure for defining (if I may be permitted a rough characterization) the number zero as that which is equinumerous with the empty set.

to (B), and his view of the connection between language and reality permits the move to (C). Only by each of these inferences is the neo-logicist able to prove the existence of numbers and obtain the desired result.

5.1.3 Proof of the Consistency of HP with Second-Order Logic

An immediate question arises for the neo-logicist program: how can we know that HP and second-order logic will not lead to another contradiction like Basic Law V and second-order logic did? How is one assured that the resultant system is consistent? As Boolos vividly writes, "How do we know that some Super-Russell of the 22nd Century won't find some ingenious derivation of a contradiction from the number principle, the way our Russell derived a contradiction from the set principle" [7, p. 151]?

Boolos provides a proof that consistency of his Frege Arithmetic(FA)—the system within which Boolos is able to reconstruct Frege's program in the *Grundlagen*—is equiconsistent the strictly weaker theory of analysis as constructed out of the natural numbers via the rationals.[16] That is, Boolos proves that FA is consistent if and only if analysis is consistent. I will focus on the proof in which Boolos provides a constructive account of how a proof of $\bot$ in FA can be directly translated into a proof in analysis of $\bot$, since analysis is not the theory in question.

The proof proceeds by first demonstrating a model $\mathcal{M}$ for HP that also satisfies the principles of second-order logic. We must give a characterization of FA and then interpret those notions within ZF, and then finally translate this interpretation into an interpretation in analysis. This final step must be done by changing the domain from $U = \{0, 1, 2, \ldots, \aleph_0\}$ in ZF to $U = \{0, 1, 2, \ldots\}$ in analysis, while rejiggering

[16]It should be noted that Boolos's proof builds on ideas in Hodes [32] and Burgess [9], but I will focus on Boolos's proof in [3]

some of the related notions.

FA, recall, is the system generated by the conjunction of HP with second-order logic, so the fundamental objects of FA will be nearly identical to the fundamental objects of second-order logic. There are three types of variables: object variables $(a, b, c, \ldots m, n, o, \ldots x, y, z)$, unary-predicate variables $(F, G, H, \ldots, R, S, T, \ldots)$, and binary-predicate variables $(\varphi, \psi, \ldots)$. These variables are simply the standard components of second-order logic, with an added restriction of predicate variables to only two places. The only non-logical component of FA is dubbed by Boolos as 'η', "a two-place predicate letter attaching to a concept variable and object variable. (η is intended to be reminiscent of $\in$ and may be read 'is in the extension.' "[3, p. 185] Atomic formulas are the standard ones in second-order logic, plus formulas of the form '$F\eta x$', and these formulas can be combined in the usual ways by the usual connectives. The final part of FA is the only non-logical axiom:[17]

$$\forall F \exists! x \forall G (G\eta x \leftrightarrow F \text{ eq } G) \qquad \text{(Numbers)}$$

Notice that Numbers is a formal analogue of Hume's Principle. Numbers is the claim that there is the unique object x that is the extension of some higher level concept that G falls under. In the case of Hume's Principle, that higher level concept is marked out by a unary function sign 'N', which has as its domain concept variables and its range object variables. Thus, 'NF' should be read as 'the number of Fs'.

So, to demonstrate a model of Hume's Principle and second-order logic in ZF set theory, let the domain of the model be $U = \{0, 1, 2, \ldots, \aleph_0\}$. We can interpret the concept variables as subsets of U and similarly interpret the binary predicate variables as sets of ordered pairs.[18] η is easily rendered as $\in$. The last to be interpreted is the

[17]'eq' here means equinumerous.

[18]This second interpretation can be generalized for arbitrary n-ary predicate variables.

function letter 'N', which we are able to interpret as the cardinality of a set. Thus, the number of things falling under a concept in FA becomes the cardinality of the set in ZF. The proof of the truth of Hume's Principle in $\mathcal{M}$ follows almost immediately thereafter.[19]

The key to the success of this model is the fact that the cardinality of any of the subsets of U is in U itself. Because this fact fails to be true for the restricted domain of the natural numbers, we were forced to include $\aleph_0$. However, this presents a problem for the attempt to give a model in analysis, as the domain of analysis is only the subsets of the natural numbers. We cannot use $\aleph_0$ as a point in the model. The trick, however, is to reconsider $\aleph_0$ as zero—what Boolos calls "coding"—and the remaining numbers as their successor (that is, n as $n + 1$). Boolos outlines the translative procedure by which to convert any proof of $\perp$ in FA into a proof of $\perp$ in analysis[3, p.190], but he presents a more dialectically clear—if less rigorous—version in "Gottlob Frege and the Foundations of Arithmetic"[7] In §5.3.2, I will present in detail that argument to highlight where a Poincaréan *petitio* seems viable. This objection will be equally applicable to both the more rigorous and less rigorous versions. As such, I want to give a sketch of the rigorous proof of Boolos here.

The basic picture of the coding into analysis is that by interpreting $\aleph_0$ as 0 and every finite n as $n + 1$, one can give an expression of the relation "exactly z natural numbers belong to the set F" through the analysis relation "there exists a one-one correspondence between the natural numbers less than z and the members of F"[3, p. 190]. This expression becomes a way of expressing the FA relation 'η' as a notion of analysis. Then, we are able to form the analysis analogue of Numbers by replacement:

$$\forall F \exists! x \forall G (\text{Eta}(G, x) \leftrightarrow F \text{ eq } G) \qquad \text{(Analogue)}$$

[19] I pass over this proof to look more in detail at the analogous proof in analysis. Boolos performs this proof on p. 188 of [3].

By providing this translation of Numbers into analysis, combined with the fact that the basic comprehension principles of second-order logic are provable in analysis, we have a direct translation procedure for any proof of $\perp$ in FA into analysis.

The contradiction arising from Basic Law V and second-order logic meant that Frege forsaked his ambition of deriving mathematics from logic. The neo-logicist looks to replace Basic Law V with HP and derives in FA an analog to Peano's Postulates. Finally, because of the threat of contradiction, the neo-logicist provides a proof of the equiconsistency of FA with analysis. Having achieved this result and presuming that the original argument of §5.1.2 is supported, the neo-logicist has achieved his goals. We turn not to examine these goals in more detail.

5.2 Respective Goals of Logicism and Neo-Logicism

In the introduction to the collection *Reason's Proper Study*, Hale and Wright briefly describe the neo-logicist program:

> ...our efforts in the service of the neo-Fregean programme have, so far, been concentrated ...on the development and defence of the thesis that arithmetic, as codified in the Dedekind-Peano axioms, does have a basis in logic and definitions of a kind which, although not exactly as called for by Frege's conception of analyticity, coheres with and underwrites a soberly platonistic conception of its ontology combined with an intelligible and relatively inexpensive epistemology.[2, p.23]

This quotation highlights the three main theses for the neo-logicists: what I will call the *mathematical*, the *ontological*, and the *epistemic*. They may each be summarized thus:

Mathematical Thesis(MT) Arithmetic, as codified by the Peano Axioms, is derivable in the system generated by adjoining Hume's Principle to Second-Order Logic.

Ontological Thesis(OT) Numbers are mind independent, abstract objects.

Epistemic Thesis(ET) We possess a privileged knowledge of arithmetic through our privileged access to second-order logic and HP.[20]

The mathematical thesis is so-called because it is a question of a systematic proof within a system. Given that this result is purely technical and proven as a logical consequence within a given system, it can be considered settled (positively) by the work of the neo-logicists and Frege himself.[21] As Wright says, "Frege did at least establish a new *mathematical* foundation of number theory: a subsumption of the laws for finite cardinal numbers under a single principle ... The Dedekind-Peano Famous Five can be reduced to One"[64, 273]. This goal is also the one which Frege himself set out to outline in the *Grundlagen* and later hoped to prove formally in the *Grundgesetze*, yet despaired over after Russell's paradox.

The other two goals are also in accordance with Frege's original intentions. Certainly, he held the ontological thesis that numbers were mind independent objects in a Platonist sense. One need only look at his anti-psychologistic derision in the introduction to the *Grundlagen* to understand his commitment to this ontological status of numbers. Additionally, Frege's project looked to provide an epistemic foundation for mathematics—that the content of mathematics was derivable as a consequence from basic principles of logic. Basic Law V was considered logical in nature, had Frege's derivation not been inconsistent, he would have shown a logical basis for mathematics and the privileged access we have to logic would guarantee privileged access to mathematics.[22]

[20]I purposefully remain vague as to what I mean by "privileged knowledge" here. I will take up the issue in §5.2.2

[21]This result was first noticed by Parsons [40], rediscovered independently by Wright [63], and Boolos gives an in depth discussion in [3] and a rigorous proof in the appendix to [4]

[22]For a discussion of Frege's epistemic and ontological goals, see Ebert and Rossberg [50].

The neo-logicist, unlike Frege, faces an additional epistemic challenge. Even if we grant that there is a special epistemic access to second-order logic, there is no *prima facie* reason to grant the same access to HP; HP is *not* logical for the neo-logicist as it was for Frege. For the logicist, it was a logical consequence of Basic Law V, which was itself logical. . Therefore, the neo-logicist actually has two epistemic theses, one weak and one strong:

Weak Epistemic Thesis (WET): Epistemic justification for HP is transferable to all of arithmetic. The non-logical axioms of mathematics can be reduced to HP.

Strong Epistemic Thesis (SET): HP is analytic and second-order logic is transparent to us in a way different from arithmetic. Thus, knowledge of arithmetic is a result of logical transparency and analyticity.

The weak epistemic thesis is of course, a reclothing of the mathematical thesis in epistemic garb, though nonetheless important. It simplifies the non-logical commitments of mathematics and allows consideration of the strong epistemic thesis in the first place. Supposing that the weak epistemic thesis is settled, we need to address the specifics of the ontological and strong epistemic theses and compare them with the original goals of Frege.

5.2.1 Ontological Thesis

On first glance, it appears that the ontological thesis is proven by the neo-logicist's employment of abstraction principles to introduce number via their conception of language and reality (recall §§5.1.2 – 5.1.2). Using abstraction principles, the neo-logicist introduces new terms to the language that they claim cannot fail but to refer to objects in the world; in some sense, it appears that they have "created" the very things that witness the truth of OT.

Such a claim seems to beg the question, for it looks as if what guarantees the existence of objects is the neo-logicist's own conception of language and reality. Moreover, the endorsement of the use of abstraction principles as a viable method for introducing new terms appears to have the fatal flaw of being ontologically inflationary.[23] This objection has particular bite on the neo-logicist for the dilemma appears to be to either accept that these principles introduce new things into the ontology (and thus should not be regarded as definitional) or that they should be accepted as definitional and legitimately introducing new concepts (and thus not able to guarantee the existence of independent mathematical objects).

The neo-logicist has a response that he takes to be in accordance with Frege's original position. While it is true that *concepts* are created via abstraction principles, it is not the case that *objects* are created in this way. What happens is not an ontological inflation but an *ontological redivision*, called alternatively a "carving" (Hale, [24, 103]) or a "*reconceptualization* of the type of state of affairs depicted on the right [of the abstraction principle]" (Wright, [65, 312]). The stipulation of, for instance, HP does not create new ontologically unique objects but rather re-divides what is already in existence into new arrangements. One is able to talk about the "new" thing called a number because it is "new" in the conceptual sense—having just been organized as such by a new concept—but fails to be new in an ontological sense.[24] One analogy that may be useful is the assembly of furniture pieces. Combining four legs and a flat surface makes a table that did not exist prior to assembly, but no new physical (ontological) matter has been created.

[23]For a more detailed explanation of this objection, see MacBride [38], §5.

[24]MacBride argues that this response from the neo-logicist is not sufficient to address the concern of what he calls the 'rejectionist', who may argue that the neo-logicist is making an un-substantiated, yet significant, claim that two concepts—that on the right-hand side and that on the left-hand side may be "necessarily correlated in the way in which the stipulation demands" [38, 126]. He attributes this disagreement to fundamental differences in the conception of the relation of language to reality, and even more fundamentally on whether metaphysical accounts of reality are tractable.

5.2.2 Strong Epistemic Thesis

Strong Epistemic Thesis (SET): HP is analytic and second-order logic is transparent to us in a way different from arithmetic. Thus, knowledge of arithmetic is a result of logical transparency and analyticity.

Whereas the WET was simplification of epistemic commitments (from five Peano Postulates to one), the SET acts as a bedrock epistemic principle upon which the neo-logicist looks to reliably ground mathematical knowledge. Additionally, SET serves as a means of support for the ontological thesis, as it describes the manner in which we may come to know and have access to mind-independent abstract objects. The argument given in §5.1.2 allows for privileged access to second-order logic to be transmitted through abstraction principles and onto the newly carved out concepts; in this way, the neo-logicist is able to answer how it is that one is able to come to know Platonic numbers.

The neo-logicist owes an account of the form of the transparency of second-order logic and of the analyticity of abstraction principles. For the latter, it is sufficient for the neo-logicist to discuss the epistemic status of HP, rather than the overall status of generalized abstraction principles. As MacBride notes, "And if the neo-logicist is not simply to replace one mystery with another, a grasp of second-order logical truth and consequence must be more epistemologically tractable than a grasp of mathematical truth"[38, p. 136].

Second-order logic Though both second-order logic and HP require an epistemic justification, it appears at first that the epistemic transparency of second-order logic is more plausible.

In fact, the potential problem arises in the claim that *second*-order logic, as compared to first-order logic, is epistemically basic and uncontestable. The neo-logicist needs to argue that "logic", construed as second-order logic, retains the same status as first-order logic. He needs to claim that second-order logic simply is logic.[25] Objections have been raised that there fails to exist a firm distinction that separates second-order logic from mathematics; second-order logic, according to this objection, is inherently mathematical in nature, and alleging epistemic privilege for second-order logic is tantamount to claim the same privilege for arithmetic.[26]

Hume's Principle The second-epistemic claim the neo-logicist must make to satisfy the SET is that HP is somehow epistemically basic. Wright claims that HP is, in fact, *analytic*. However, the definition of analytic must be revised slightly from the traditional one. First, it needs to account for attacks on the notion of analyticity writ large, such as Quine's 'Two Dogmas of Empiricism'. But, leaving this issue to the side, there still needs to be tightening up of HP and its status.

We have already touched on one of Wright's responses to criticisms from Boolos (the notion of recarving, rather than ontologically stipulating p. 117), and Wright responds in kind to Boolos in "Is Hume's Principle Analytic?"([65]).[27] Wright distills the issue to which Boolos objects:

> ... that Hume's Principle may be laid down *without significant epistemological obligation*: that it may simply be stipulated as an explanation of the meaning of statements of numerical identity, and that—beyond the issue of the satisfaction of the truth-conditions it thereby lays down for such statements—no competent demand arises for an independent assurance that there *are* objects whose conditions of identity are as it stipulates.[65, p. 321, emphasis in the original]

The problem that Boolos sees with this claim is that too much is being carried by the notion of explanation.

[25]See §5.1.2

[26]Shapiro argues along these lines in [53], [57], and [56]. I will return to compare this objection to a Poincaréan version in §5.3.1.

[27]We will return to Boolos's 'Bad Company' objection in §5.3.2.

> However, it is hard to avoid the impression that more is meant, that Wright holds that to call a statement an *explanation of a concept* is to assign it an epistemological status importantly similar to the one it was though analytic judgments, including definitions, enjoy. It is to this further suggestion that I wish to demur[6, p. 310, emphasis in original].

The difficulty with such an objection is that it does not seem to meet the neo-logicist head on. Boolos objects that there is some extra epistemological weight being granted HP by calling it an explanation, and that it is upon this basis that HP gains the status that the neo-logicist wants. However, this fails to be an objection insofar as it is a reservation—if one buys into the neo-logicist project, such a concern will not be troubling. Unless one share's Boolos's intuitive worry, one need not bear the same concern. Because the neo-logicist is committed to a conception of how language and reality are connected (§5.1.2), such a worry would have no effect on them. Indeed, Wright even argues that if it is possible to gain an understanding of abstract entities at all, then

> ...it has to be because we have so fixed the use of statements involving reference to and quantification over such entities as to bring the obtaining of their truth conditions somehow within our powers of recognition...something we did by way of *determination of meaning*. [65, p. 323]

Wright argues that unless one wants to dispense with the view of numbers as abstracts, one has to accept something resembling the neo-logicist position.

The disagreement between Boolos and Wright seems to stem from a fundamental disagreement about the status of language and reality. Given that, neither presents a particularly strong reason for the rejection of their opponent or the adoption of their own view. That there are disagreements about the intuitions is hardly surprising and means that this particular point seems at an impasse.

The important point to recognize here is that much of the epistemic status of HP stems largely from whether or not one believes the neo-logicist account of language's inability to fail to mirror reality.

Whether or not one thinks the neo-logicist has done enough to demonstrate SET (and therefore the entirety of his goal) depends on whether or not one shares the neo-logicists' starting intuitions. However, this problem is precisely why the WET exists independent of SET. Regardless of the epistemic status of HP, the proof of WET is a significant result in the foundations of mathematics.

5.3 Poincaréan objections posed to Neo-Logicism

This section shall focus on two potential Poincaréan style objections to be made against the neo-logicist project. The first resembles the sophisticated *petitio* as described by Goldfarb, wherein any formal system necessarily presupposes knowledge of the principle of induction(see §4.2). The objection against the neo-logicist is that second-order logic has a necessary sub-fragment comprised of first-order logic, the foundational rules of which require determination by recursion. Moreover, these rules are not merely descriptive as they were for the logicist but prescriptive in that they establish the logic. Therefore, since induction is used in the establishment of the system, the epistemic goals of the neo-logicist go unrealized. This particular objection is of a kind with Stuart Shapiro's objection that second-order logic is, in fact, mathematical in nature.

The second objection arises in the consistency proof of HP. This proof is required because of the presence of bad-company objections that argue that because HP closely resembles other abstraction principles that lead to contradiction, a verification of HP's legitimacy is required. In the proof of the equiconsistency of FA with analysis, there is an application of induction in order to demonstrate that HP is true in a model of analysis. Given that there is such an application, a mathematically ignorant subject cannot come to be assured of the legitimacy of the stipulation of HP and thereby cannot gain mathematical knowledge out of nothing.

5.3.1 The Sophisticated *Petitio* and the Mathematical Quality of Second-Order Logic

The neo-logicist must claim both that second-order logic simply is logic (§5.1.2) and that logic has a special epistemic status (§5.2.2); from these two claims, the neo-logicist is able to derive mathematics from an assumption of Hume's Principle, thereby endowing mathematics with the epistemic standing of logic and HP.

However, Shapiro counters that there is no boundary between what should be classed as "mathematical" and what should be classed as "logical" within second-order logic. Second-order logic has, under a standard semantics, predicate variables that range over the powerset of the domain of discourse (recall that Boolos's translation of second-order logic into set theory had monadic predicates as analogous to subsets of the domain). Thus, there appear to be set-like objects presupposed in the notion of predication in second-order logic. Thus, Shapiro denies "that there is a sharp boundary between mathematics and logic."[56, p. 59]

If such a division cannot be made, the neo-logicist project is perilously in trouble, as he would be ascribing to logic a special epistemic status and hoping to transmit that warrant to mathematics via Frege's Theorem. However, he would have presupposed that mathematics had such a status when ascribing it to logic. Thus, a circle.

Here is the similarity to a Poincaréan *petitio* objection. The *petitio* at base is an accusation of begging the question. It was an accusation against Hilbert and the logicists that they were presupposing an understanding of number and elementary arithmetic in their derivation of the Peano Postulates. Seen in this general light, Shapiro's objection that second-order logic is mathematical in nature is Poincaréan in spirit.

One can make a slightly more focused objection that better resembles the original *petitio* argument. Recall that the *sophisticated petitio* suggested by Goldfarb in §4.2 accused the logicists of presupposing number in the formation rules of the formal system. This particular objection did not effect the logicists because the logicist explicitly denies a meta-theoretical perspective from which to make the argument.

The same defense cannot be argued for the neo-logicist. Their rules for characterizing the first-order fragment of second-order logic are recursive, and therefore presuppose some notion of induction for a thinker. The employment of induction for the comprehension of the formation rules within the system comes prior to the question of whether second-order logic is fundamentally mathematical, prior even to the question of the epistemic status of logic. In this sense, it is an objection made before the logical system gets off the ground. The Poincaréan objection alleges that the neo-logicist, in his recursive formation rules, has already presupposed knowledge of number.

One particular aspect of the Shapiro objection is phrased by MacBride thus:

Semantics: in order to provide a semantic theory for second-order logic, a considerable body of mathematics must be called upon[38, p. 136]

MacBride suggests a response for the neo-logicist to this semantic objection that seems extendable to defend against the *petitio*.

> More importantly, there is a distinction to be drawn between the tools one employs to investigate a given subject matter and the nature of the subject matter itself. One cannot immediately conclude from the fact that one has to employ tools of such and such a sort that the subject matter itself concerns items of that sort [38, p. 137].

To extend this defense against the *petitio*, one could argue that induction is used as a tool to understand and lay down the formation rules of logic does not mean that logic presupposes mathematics or induction. The tools of comprehension should be strictly separated from the subject matter itself.

This remark, however, does not seem successful on examination. There is a disanalogy between the objection lodged by Shapiro and this new *petitio*; namely, the former is a concern leveled within the system, whereas the latter is made outwith the logical system. Set theory may be used after the system is in place in order to clarify and investigate more fully the theory in question, for instance, in investigating questions of relative consistency. Shapiro emphasizes this practice as what mathematicians commonly do in order to illuminate the nature of structures in question [56]. In this way, the richer theory is used as a tool to examine the sub-theory.

But the employment of induction in the formation rules, however, is not such an investigatory use. The use of induction does not serve as a method by which mathematicians or logicians learn more about second-order logic; rather, it serves as a way to *introduce* a fragment of second-order logic and bring about its proper formation. Thus, it is not a tool used to investigate, but rather a tool used to create. To that extent, when the questions of epistemic foundations are raised, it does beg the question to claim that the neo-logicist can dismiss this concern as being merely a confusion of the tools used to investigate the theory with the theory itself.

A second response given by MacBride to the semantic objection seems more fruitful against both the semantic objection and the *petitio* objection. This response claims that the reconstruction of mathematics does not rely on the totality of second-order logic, which the neo-logicist might concede requires a full blown set-theoretic semantics. Rather it only relies upon the "recursively enumerable fragment relevant to the derivation of Frege's theorem"[38, p. 139]. In this way, the neo-logicist denies that the fragment within which he derives mathematics is necessarily mathematically problematic.

A similar reply appears available to the *petitio* objection. The neo-logicist can claim that the derivation of Frege's Theorem need not rely on a full-blown understanding of first-order or second-order logic. Rather, all that needs to be understood are that the given formulae in the theorem are in fact well-formed, that the actual derivation of the theorem is an acceptable chain of deductions, and so on. In this way, the neo-logicist might be able to claim that induction is not used in the foundational way as envisioned before.

The neo-logicist is even able to frame the *petitio* objection, if stretched and pulled slightly, as a psychologistic objection. The neo-logicist could argue that the force of this objection lies not in the logical presupposition of mathematics, but in the *psychological* presupposition. An intuitive grasp of elementary arithmetical operations is required to understand even what constitutes distinctness between formulae, presupposing recursive definition. The notion of what constitutes a syntactically well-formed formula supervenes on the ability to recognize that the recursive construction of formulae—as laid out by the recursively enumerated fragment of second-order logic—could continue on forever or rather that it need not stop at any given point. This supervenience, the neo-logicist could offer, is psychological rather than logical. Because he does not subscribe to psychological principles as fundamental as does Poincaré, the objection holds no weight.

One need not accept psychological principles as basic in order for the neo-logicist response to fail. Should one hold that the understanding of rules of derivation and syntax in particular cases is parasitic on a generalized grasp of these rules, then the neo-logicist response fails to be successful. In fact, MacBride suggest just such a commitment by the neo-logicist:

> Moreover, the neo-logicist may allow that a systematic understanding of second-order logic requires the exercise of mathematical concepts whilst nevertheless maintaining that a mathematical novice might follow the proof of Frege's theorem even when unable to explicitly formulate or theorise about the specific rules employed [38, p. 139].

Any mathematical novice who is so led through the proof must already have an understanding of recursive rules for the generation of formulae. This "novice" may indeed fail to have a knowledge of the totality of second-order logic, but seems to require having a knowledge of induction in order to understand that the derivation she is being shown is, in fact, a derivation. Recognition that the proof of Frege's theorem is good depends upon a pre-understood notion of recursiveness and thereby of induction. Indeed, the original response from the neo-logicist called upon the "recursively enumerable fragment relevant to Frege's Theorem."

For the novice to recognize that the applications rules in the derivation of Frege's Theorem are instances of a general rule, the novice must first know the general rule. For suppose that there are n many instances of *modus ponens* in the derivation. For the novice to recognize these as n-many applications of the same rule—and not n-many different individual rules—she must already know the general rule. If one thinks that this idealized mathematical novice need not have an understanding of induction prior to being shown the proof of Frege's theorem, then there seems to be no problem with the neo-logicist response. The neo-logicist owes an explanation of how the proof of Frege's Theorem is not circular.

5.3.2 Consistency Proofs and Bad Company Objections

Neo-logicism claims that via a use of HP, an abstraction principle, one is able to derive knowledge of mind-independent abstract objects. Via stipulation of the principle and knowledge of second-order logic, the strong epistemic thesis follows. More generally, the neo-logicist claims that abstraction principles—one of which is HP—are legitimate epistemic means by which to obtain knowledge of new terms on the left-hand side of the principle via a grasp of the right-hand side, which is composed solely of terms antecedently understood (see §5.1.2).

However, abstraction principles are not generally acceptable. For example, Hartry

Field has proposed the following principle:

(G) The God of $x =$ the God of y iff x and y are spatio-temporally related.

Thus, proceeding in the same manner as the neo-logicist, Field claims that knowledge that x and y are temporally related results in knowledge that the God of x is the same as the God of y. Then, given the neo-logicist conception of language and reality, Field can conclude the existence of God via existential generalization. Certainly, such a result is anathema to the neo-logicist.

Though there is a response available for this sort of objection—the neo-logicist can argue that the concept of "God" introduced in (G) is not of the omnipotent deity in the sky but simply a "God" of spatio-temporal relation—there are more subtle objections put forth by Boolos in the form of the Parity Principle (what Wright calls the Nuisance Principle), an abstraction principle that is consistent with second-order logic, yet is true only in finite domains:

(P) $\forall F \forall G\,(\text{Parity}(F) = \text{Parity}(G)$ iff F and G differ evenly $)$

where *differ evenly* means that the objects falling under F and not G or G and not under F is even and finite ([5, pp. 214-5]). Though neo-logicist has another possible response to this objection by requiring conservativeness of abstraction principles (see MacBride [38, p. 145]), the upshot is that the neo-logicist requires some means of differentiating HP from those principles in the neighborhood that lead to contradiction. One of the first requirements, of course, is the differentiation between HP and Basic Law V.

To this end, the neo-logicist proves the consistency of HP with second-order logic. More precisely, he proves the equiconsistency of Frege Arithmetic with analysis. We have already outlined the proof in §5.1.3. The revised Poincaréan objection runs thus:

1. The neo-logicist requires a proof of the consistency of HP with second-order logic in order to differentiate it from the bad-company HP keeps.

2. One of the postulates derived within Frege Arithmetic to reconstruct arithmetic is the principle of induction.

3. The proof given by Boolos on behalf of the neo-logicist employs an application of the principle of induction.

4. Therefore, the neo-logicist fails to reduce the epistemic commitments of arithmetic and fails to show the SET.

Claims 1 and 2 are straightforward descriptions of the status of HP and the outcome of Frege's Theorem. The onus for the Poincaréan objection is to demonstrate 1) that there is an application of induction within the proof of consistency and 2) that such a use of induction is illicit insofar as it undercuts the epistemic simplification that the neo-logicist wants to achieve. In order to accomplish this first goal, we turn to look in detail at Boolos's proof of the equiconsistency.

Induction in the Proof of Equiconsistency

Recall that Boolos provided a constructive account of how a proof of $\perp$ in FA could be translated into a proof of $\perp$ in analysis. This process was done in part by demonstrating how a model of FA could be constructed in ZF set theory, which in turn could be coded into a model in analysis. Recall also that the difficulty in achieving this coding was that in the model in ZF, we employed the notion of cardinality to model the operator 'Nx' in FA, thereby requiring the expansion of the domain of the model to include $\aleph_0$. Such a move is not possible in analysis, where $\aleph_0$ is not in the domain of analysis.

In order to circumvent this difficulty, Boolos defines *number by*:

Number by If the number of Fs is n, then the number *by* Fs is $n + 1$, and if the number of Fs is infinite, then the number *by* Fs is 0.

So, for instance, the number by strikes in an at-bat in baseball is 4, the number by US senators is 101, and the number by the even numbers is 0. We can now reformulate HP into HPa, for analysis:

HPa The number by Fs = the number by Gs iff F 1-1 G

Having interpreted FA in analysis thus, Boolos must prove that HPa is true in analysis. Assume first that the Fs and the Gs are in a one-to-one correspondence. There are either finitely many Fs or infinitely many Fs. If there are finitely many Fs, then *ex hypothesi* there is a mapping from F to G that is finite. Thus, if there are n Fs then there are n Gs. So,

$$\text{the number by } Fs = n + 1 = \text{the number by } Gs$$

If there are infinitely many Fs, then that means there are infinitely many Gs. But this means that

$$\text{the number by} Fs = 0 = \text{the number by} Gs.$$

Thus, the right-to-left direction of the biconditional.

Assume now that the number by Fs = the number by Gs. Either the number by Fs= 0 or the number by Fs= $n + 1$, for some finite n. If the latter, then the number of Fs (in the standard interpretation) equals the number of Gs, and we can create a mapping from a finite set to a finite set. If the former, then the number of Fs and Gs are both infinite. However, because the vocabulary of analysis consists only of the natural numbers, we can establish a function mapping the least element in the Fs to the least element in the Gs, the second least in the Fs to the second least element in the Gs. Since they are both infinite, the process can continue to map every F to only one G and vice versa. we have a one-to-one mapping and a one-to-one correspondence. Therefore, the left to right direction of the biconditional holds.[28]

However, this proof of the model has a number of applications of the principle of induction. In the left to right direction, Boolos writes, "If [the number by Fs = the number by Gs] is positive, $n+1$, say, then there are n Fs and n Gs and the Fs and Gs are in one-one correspondence." Though true, this argument relies on mathematical induction. The claim is that for any finite number n, for two concepts F and G, with the number of Fs = n = the number of Gs, then there is a mapping (one-to-one) from F to G. To prove this claim, we show that there is a mapping in the base case (between two concepts with only one item falling under each). Then, we show that if we assume a mapping for concepts of size n, we can construct a mapping for sets of size $n+1$ simply by taking the pre-existing mapping for the first n objects and extending it by mapping the remaining thing that is F to the remaining thing that is G. Thus, we are able to conclude that a mapping can be made between two concepts that have the same number by. Of course, this claim is merely an application of induction.

Moreover, in addressing the possibility that the number by Fs = the number by Gs is infinite, Boolos writes,

> But if the number [by] is 0, then there are infinitely many Fs and infinitely many Gs. But in this case too, the Fs and Gs are in one-one correspondence: the least F corresponds to the least G, the second least F to the second least G, the third least, etc. [7, p. 153].

But this "etc." is precisely the conception of induction that Poincaré has in mind. It is a recognition that the process described can be carried out indefinitely. One recognizes that the procedure of mapping the least F to the least G, the second-least F to the second-least G, and so on can continue indefinitely only because of an intuitive grasp of induction. It is a grasp of the extensibility of the given procedure.

[28]This proof follows Boolos in [7, pp.152-3]. He writes there that he has "just shown how to interpret analysis in analysis the result of adjoining the number principle to the system of *Begriffsschrift*"

Should a neo-logicist feel compelled to be worried about this sort of inductive procedure? Does he need to buy into Poincaré's psychological conception of induction? It might be that the neo-logicist can dismiss Poincaré's induction in this matter as purely psychological, whereas Boolos' and the neo-logicists' concern is epistemic. The neo-logicist might reply that while it is a psychological fact of our nature that we must understand induction in order to recognize the truth of what Boolos writes, this fact does not bear any epistemic weight. It is a concomitant fact, similar to Frege's view on the constancy of inkblots (p. 95)

That response, however, misses the point. The reason that Boolos's proof has any value at all to the neo-logicist is because it helps to mark out what is epistemically responsible; it demonstrates that HP does not run afoul of any inconsistency. However, part of the epistemic thesis that the neo-logicist wants to advance is that, as MacBride writes,"(HP) may be employed to introduce a mathematically ignorant subject ('Hero') to number theory"[38, p. 146]. Hero's employment of this abstraction principle in some manner in order to verify that he is not actually employing an illicit principle like Basic Law V. Note that this purpose is precisely why Boolos gives his proof.

However, as a mathematically ignorant subject, Hero is not in a position to verify the consistency of HP via Boolos's proof, and therefore unable to even differentiate HP (in terms of its consistency) from Basic Law V. Hero can blindly follow HP, but he cannot assure himself of the safety of its use. The situation echoes the dilemma for Hilbert's program; if the neo-logicist wants to retain his stated goal of the SET—simplifying the epistemic commitments of mathematics—then the prescribed method for achieving that aim appears to fail.

MacBride notes a similar point and phrases it as a dilemma between an internal and external justification; on the one hand, Hero's use of HP cannot be justified internally, as the mathematics required to provide epistemic safety are beyond Hero's ability, while on the other had the attribution of external epistemic warrant to modes of reasoning is implausible. Wright sketches a path between the internal and external horns of the dilemma by noting that even uncontroversial notions of concept formation or sense fixing run into similar problems (see Wright [64, pp.286-8]), so the issues raised by this objection are more universal that first seems. However, he admits that such reflections do "not excuse the Fregean the work of offering some principled restriction where instances of [abstraction] may be excluded"[64, p. 288]. No such internal, principled restrictions are on offer. Therefore, it appears that the dilemma and the Poincaréan objection remains active.

Chapter 6

Conclusion

This project declared a number of goals at the outset. Some were interpretive (e.g. providing an interpretation of Poincaré's 'intuition'); some historical (showing why the *petitio* was not effective against the logicists); and some looked to contribute to ongoing debates (refocusing the *petitio* to the neo-logicist).

Chapter 2 argued for a novel interpretation of Poincaré, identifying four elements of his psychologism that informed his conception of intuition, which in turn, informed his notion of induction. The secondary goal was to demonstrate that Kant's philosophy of mathematics differed from Poincaré significantly, perhaps suggesting that a Poincaréan philosophy of arithmetic is worthy of further investigation.

Chapter 3 addressed both Early and Later Hilbert with the *petitio* objection, arguing that even under multiple interpretations of Hilbert's project—a novel one of which was advanced—a revised version of Poincaré's objection still holds.

Chapter 4 argyed that the only way for the logicist to wriggle free of Poincaré's original objection was to hold a particular view of logic as all-encompassing and universal—to deny any metatheory.

Finally, Chapter 5 argued that two forms of a Poincaréan objection are possible. Moreover, because of particularities in the neo-logicist platform, the responses of the logicists are not available. These two reformulations were compared to extant objections in the literature, and it was argued that the neo-logicist has not supplied a potential response. As such, these objections remain viable.

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