Volume rigidity of simplicial complexes

Jack Oliver Southgate

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Abstract

This thesis develops a theory of rigidity of frameworks of simplicial complexes subject to maximal-simplex-volume constraints inspired by the well-studied theory of rigidity of frameworks of graphs subject to edge-length constraints. We take three main approaches: (simplicial) combinatorial, proving combinatorial conditions for generic local and global rigidity in all dimensions; algebro-combinatorial, exploring techniques of Bulavka et al. [2022] and conjecturing a lower bound on the rank of a simplicial complex in the volume rigidity matroid; and geometric, giving bounds on the number of embeddings of generic frameworks of bipyramids and showing that global rigidity is not a generic property of simplicial complexes in general. We additionally provide notes on ongoing and potential future research areas in volume rigidity theory that are, as of yet, underdeveloped, such as forbidden sign patterns and the rigidity with respect to volume constraints on non-maximal simplices.

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I, Jack Oliver Southgate, do hereby certify that this thesis, submitted for the degree of PhD, which is approximately 50,000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for any degree. I confirm that any appendices included in my thesis contain only material permitted by the 'Assessment of Postgraduate Research Students' policy.

I was admitted as a research student at the University of St Andrews in October 2020.

I received funding from an organisation or institution and have acknowledged the funder(s) in the full text of my thesis.

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of PhD in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree. I confirm that any appendices included in the thesis contain only material permitted by the 'Assessment of Postgraduate Research Students' policy.

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I attended the *Thematic Program on Geometric Constraint Systems, Framework Rigidity and Distance Geometry* (virtually) at the Fields Institute at the start of 2021.
Thank you to the organisers of the Program and to Tony Nixon and Meera Sitharam for running graduate courses which, coincidentally, served as my introduction to rigidity theory.
Through the Thematic Program, I became involved with a research group featuring Daniel Bernstein, Georg Grasegger, Fatemeh Mohammadi, Tony Nixon, William Sims Jr. and Meera Sitharam, although no published work came as a result of the work we undertook, through this group I was introduced to many of the results referenced in Sections 4.3 and 5.3.

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Contents

Chapter 1

Introduction

Consider a building, bridge or some other physical structure. In such cases, it is useful to know whether our structure moves, and if it does, how it moves. Structural engineers have long been interested in the *rigidity* and *flexibility* of their constructions and more recently, the universe of types of structures considered has expanded to include biological molecules such as proteins, non-physical information structures such as sensor networks and large datasets as well as many more examples. In sum, the world is full of objects that are rigid or flexible and studying these properties is becoming an increasingly valuable pursuit.

In each case above, we may model our structure as a *framework*, consisting of a description of the arrangement of the rigid components in the object and a description of the object's placement in space. Mathematically speaking, we can define a graph, G, whose edges represent rigid bars and whose vertices represent universal joints. Subgraphs of G may be built up to describe larger rigid components of G, which flex around vertices connecting them to each other. The second part of a framework is a configuration, p, a vector which lists out the coordinates of each vertex in space (usually this will be a real space, such as $\mathbb{R}^3$, where distance is measured by the familiar Euclidean metric, but we will see later that this is not the only option). Together, (G, p) denotes a framework, a fundamental object of study in *rigidity theory*.

We will work through definitions more rigorously in the subsequent chapters. For now, we will call a framework (G, p) rigid, in some space containing it, if we can not continuously move the vertices of G around, starting at their initial positions as described by p, in that space, whilst keeping all edge-lengths constant, without also keeping all non-edge lengths constant, i.e. unless we are moving the whole framework around in an isometric motion of the entire space. If (G, p) is not rigid, then we say it is flexible. Figure 1.1 illustrates an example of a flexible and of a rigid framework.

The mathematical study of the rigidity of polygons and polyhedra goes back to Cauchy in 1813, who approached the subject from a more planar-geometric point of view in Cauchy [1813/2009]. It was Maxwell, fifty years later, who

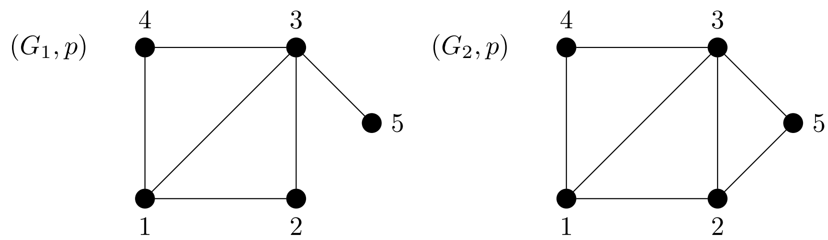

Figure 1.1: We can continuously move vertex 5 while keeping the rest of (G_1, p) still by having it trace out a circle with centre vertex 3. This changes the lengths of the non-edges 15, 25 and 45, therefore (G_1, p) is flexible. We then add the edge 25 to G_1 to get G_2, and, in doing so, made such a motion impossible in (G_2, p). Therefore (G_2, p) is rigid.

introduced the language of frames (frameworks) and and early analogues to graph theory in order to study rigidity in Maxwell [1864]. In particular, Maxwell described the necessary conditions, in terms of the numbers of points (vertices) and connexions (edges), for a frame to be rigid in 1-, 2- and 3-dimensional space.

Theorem 1.0.1. *Maxwell [1864] Let G be a graph on n vertices and e edges and let (G, p) be a typical framework in $\mathbb{R}^d$, for $d \in \{1, 2, 3\}$. If (G, p) is rigid in $\mathbb{R}^d$, then*

$$e \geq dn - \frac{d(d+1)}{2}.$$

As an example of how this theorem is applied, G_1 in fig. 1.1 has five vertices and six edges,

$$6 < 7 = 2(5) - \frac{2(3)}{2},$$

so every typical framework of G_1 in $\mathbb{R}^2$ will fail to be rigid. However, adding the edge 25, as in G_2 in fig. 1.1, increases the left hand side of the inequality above by 1, achieving equality, so that frameworks of G_2 in $\mathbb{R}^2$ may be rigid.

Note that, when $d = 1$, the inequality becomes

$$e \geq n - 1,$$

so every connected graph will admit a rigid framework in $\mathbb{R}^1$.

We will highlight two points from Maxwell's writing on rigidity theory that remain influential to this day. Firstly, he introduced the notion of degrees of freedom of frameworks arising from the polynomial equations that measure edge-lengths. A framework's degrees of freedom may be visualised as the number of *independent* ways it can flex, and as Maxwell described, is roughly related to the number of edges compared to that of vertices in the underlying graph. This leads on to our second point, that he thought about rigidity *combinatorially*, i.e. in terms that descend from the underlying graph. His necessary condition was later proven for all dimensions, the converse was shown to be true in the line and a slightly more involved version of the converse was shown to be true in the plane.

The converse to Maxwell's condition in the plane was proved by Pollaczek-Geiringer in the 1920s before being lost - Pollaczek-Geiringer was a Polish Jewish woman living in pre-World War 2 Germany, she later fled to Turkey and then the United States (O'Connor and Robertson) - it was proved again by Laman, who received sole credit for the result in subsequent decades, half a century later. The condition is as follows.

Theorem 1.0.2. *Pollaczek-Geiringer [1927], Laman [1970] Let G be a graph on n vertices and let (G, p) be a framework of G in $\mathbb{R}^2$. If (G, p) is a generic framework, then (G, p) is rigid in $\mathbb{R}^2$ if and only if G admits a spanning subgraph H on e edges such that*

$$e = 2n - 3,$$

and, for any subgraph H' of H on n' vertices and e' edges,

$$e' \leq 2n' - 3.$$

If H satisfies the inequality above, we say that it is $(2, 3)$-sparse, if, moreover, it satisfies the preceding equality, we say it is $(2, 3)$-tight. Figure 1.2 illustrates this condition.

We note that the analogous result holds in $\mathbb{R}^2$, i.e. that if G is a graph on n vertices and if (G, p) is a generic framework of G in $\mathbb{R}^1$, then (G, p) is rigid in $\mathbb{R}^1$ if and only if G admits a spanning subgraph H on e edges such that

$$e = n - 1$$

and, for any subgraph H' of H on n' vertices and e' edges,

$$e' \leq n - 1.$$

This condition, known as $(1, 1)$-tightness, is equivalent to G being connected, and has long been considered a *folkloric* result in rigidity theory (see [Graver et al., 1993, p. 4]).

Notice that it is not true for all frameworks of G in $\mathbb{R}^2$, just *generic* ones. Genericity will be formally defined later, but roughly means that the framework does not experience any geometric dependencies, such as having coincident vertices, parallel edges or lying on a certain curve, which might induce a degree of freedom where there usually would not be one. It is a more mathematically rigorous version of the *typical* condition from Maxwell's theorem. Asimow and Roth showed that if one generic framework of a graph is rigid, then all of them are (Asimow and Roth [1978]). Therefore, while we gain a remarkable amount from speaking in a combinatorial language in terms of making statements such as those above, we lose the ability to account for specific, tricky cases.

Since Laman, Asimow, Roth and others laid (or re-laid) the foundations of modern rigidity theory in the 1970s, the subject has dramatically expanded in scope. Relevant to us, are three main developments: rigidity under different geometric conditions, global rigidity and algebro-combinatorial tools for rigidity.

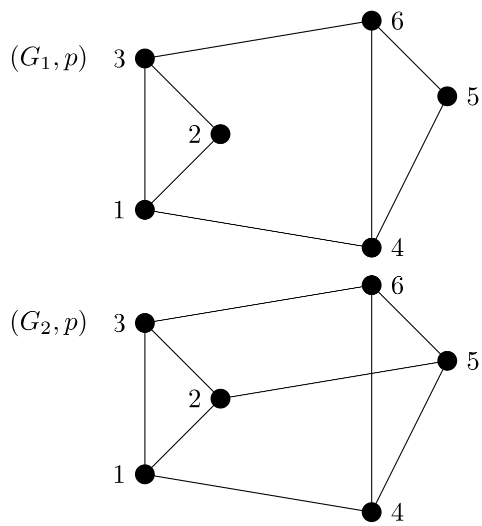

Figure 1.2: The graph G_1 fails to admit a $(2,3)$-tight spanning subgraph: the subgraph induced by vertices 1, 3, 4 and 6 has $4 < 5 = 2(4) - 3$ edges. The framework (G_1, p) is flexible, as we may continuously deform the central quadrilateral, squeezing vertices 3 and 4 together and pulling vertices 1 and 6 apart. Adding the edge 25 however yields the $(2,3)$-tight graph G_2, and its given framework (G_2, p) is rigid.

Firstly, and most importantly for us, is the notion of taking rigidity under different geometric constraints than maintaining the distances between pairs of points. There are several ways of formulating this, such as changing the norm of the space to a non-Euclidean norm (see Kitson and Power [2014]), however we will focus instead on taking d-dimensional volumes between sets of $(d+1)$ points in d-dimensional space ($d+1$ is the smallest number of points that may enclose a d-dimensional hull in d-dimensional space - think of, for example, two points enclosing a line-segment, or three points enclosing a triangle). This formulation of rigidity is known as *volume rigidity*. Clearly, the formulation of a framework as a pair consisting of a graph and a configuration is no longer valid, as graph edges are of size 2, not $d+1$. However, d-dimensional simplicial complexes satisfy our requirements (and come with their own well-developed theory), so we will work with frameworks in $\mathbb{R}^d$ consisting of a d-dimensional simplicial complex Σ and a configuration p. While it is hard to imagine that no-one has ever thought of this problem before, the first meaningful look at this problem from a rigidity-adjacent perspective was by Tay, White and Whiteley in the 1990s (Tay et al. [1995a]), however both their precise formulations of rigidity and their goal (to study the g-conjecture) were slightly different to ours. Since then, Whiteley has studied matroid arising in, again, a similar but not identical setting to us. The first rigidity-theoretic formulation of our problem specifically, that I could find, was by Whiteley in a presentation given at Oberwolfach in 2006 (the slides are available at time of writing Whiteley [2006], but do not appear to have been officially published). This presentation was shortly followed by a manuscript by Streinu and Theran [2007] which proposed an algorithm to determine (d, d^2+d-1)-*sparsity* in $(d+1)$-*uniform hypergraphs* (or in our language, pure d-dimensional simplicial complexes). While the algorithm is correct, the manuscript's assertion that admitting a (d, d^2+d-1)-tight spanning sub-hypergraph was equivalent to being d-volume rigid in $\mathbb{R}^d$ was not, as we will see in chapter 4. This was followed by two conference papers by Borcea and Streinu: Borcea and Streinu [2013] and Borcea and Streinu [2015] applying techniques from algebraic geometry to volume rigidity theory.

Since Borcea and Streinu's work, there have been at least two concurrent attempts to study volume rigidity theory. The first is by the author, with results outlined in this thesis and in the two pre-prints Southgate [2023a] and Southgate [2023b]. The second is by Bulavka, Nevo and Peled, who have approached the subject from an algebro-combinatorial point of view that we will discuss shortly in Bulavka et al. [2022]. We should also note that Cruickshank, Mohammadi, Nixon and Tanigawa have also studied a generalisation of rigidity theory, g-*rigidity*, that encompasses rigidity with both Euclidean and non-Euclidean norms as well as volume rigidity (Cruickshank et al. [2023]).

Next, returning to the setting of graph rigidity where edge-lengths are preserved, is global rigidity. The reader might have wondered, while reading about transformations that preserve rigidity, why we had not mentioned reflections. That is because such a transformation can not be achieved by *continuously* a framework, reflections are *discontinuous motions*. Roughly, one may think of a continuous motion as being traceable by a pen on paper (or in space) in one

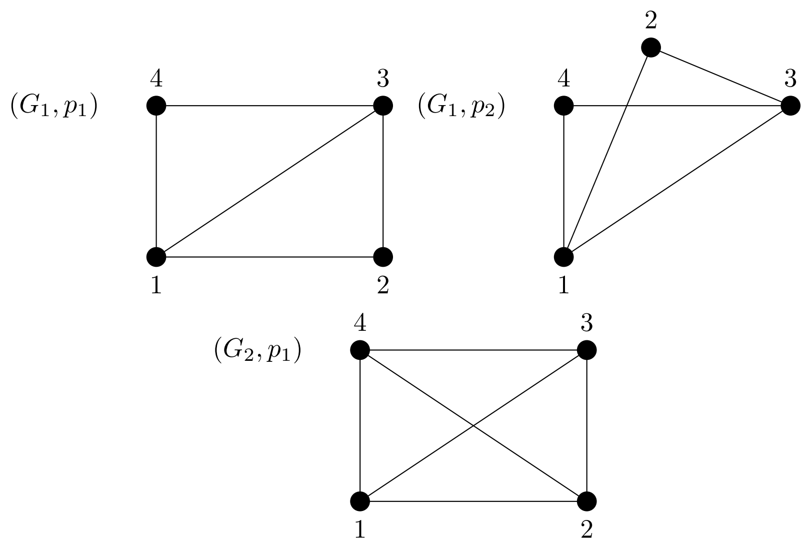

Figure 1.3: Vertex 2 in (G_1, p_1) may be reflected in the line between vertices 1 and 3, as shown in (G_1, p_2). This is a discontinuous transformation, and there is no way to continuous move vertex 2 between these two position while remaining in $\mathbb{R}^2$. Therefore both (G_1, p_1) and (G_1, p_2) are rigid in $\mathbb{R}^2$, as they are $(2, 3)$-tight, but are not globally rigid in $\mathbb{R}^2$. The framework (G_2, p_1) is globally rigid in $\mathbb{R}^2$, the reflection allowed by the on the left is not possible, as it would change the length of the now-present edge 24. As G_2 has one more edge than is required to be rigid, it is redundantly rigid, it is also 3-vertex connected, so it satisfies Hendrickson's conditions for global rigidity in $\mathbb{R}^2$.

motion, while a discontinuous one requires jumps or has kinks in it. Therefore, our *local* formulation of rigidity, does not need to account for pairs of frameworks that have the same edge-lengths that are related by a reflection. This is where the study of *global rigidity* comes in. A framework (G, p) in $\mathbb{R}^d$ is globally rigid if and only if every framework (G, q) that has all the same edge-lengths as (G, p) is related to (G, p) by a Euclidean isometry of $\mathbb{R}^d$ (including reflections).

Global rigidity is a harder problem than what we have seen so far, *local* rigidity, indeed many of the combinatorial and geometric tools used to study local rigidity fail completely in the global case. Hendrickson did make some progress on this front in the 1990s, giving a combinatorial characterisation of global rigidity in 1- and 2-dimensional space.

Theorem 1.0.3. *Hendrickson [1992] Let G be a graph, then any generic framework of (G, p) is globally rigid in $\mathbb{R}^d$, for $d \in \{1, 2\}$ if and only if G is $(d+1)$-vertex connected and G is redundantly rigid in $\mathbb{R}^d$.*

Figure 1.3 demonstrates global rigidity.

For dimensions 3 and up Hendrickson's condition is necessary but not sufficient.

Whether or not global rigidity is a generic property of graphs, i.e. whether, like local rigidity, either all generic frameworks of a graph are globally rigid or not globally rigid, was not known for much longer than the local case. It was not until 2005 when Connelly showed a necessary generic condition for global rigidity and later 2010 when Gortler, Healy and Thurston showed that it was sufficient, that it was shown to be generic (Connelly and Back [1998], Gortler et al. [2010]). Gortler, Healy and Thurston's proof of the sufficiency of Connelly's condition used the same core idea behind Hendrickson's proof of his conditions nearly 20 years earlier, however going up in dimensions required accounting for a much more complicated geometric and topological situation. Now that genericity of global rigidity has been established, efforts to combinatorially characterise global rigidity (which had been underway anyway, Jackson et al. [2006]) were on surer footing, as a full combinatorial characterisation may only exist if rigidity/global rigidity is generic. Indeed, promising progress has been made on that front, since Gortler, Healy and Thurston's result (Garamvölgyi and Jordán [2023]).

Finally, we turn to an algebro-combinatorial tool that has been applied to rigidity theory. We will introduce *algebraic shifting* more thoroughly in chapter 5, however for now, suffice it to say that it is a way of relating to a simplicial complex a *simpler* simplicial complex that retains many combinatorial and topological properties that are of interest to combinatorists. Kalai wrote a good expository article that we refer to as a source throughout, Kalai et al. [2002], in which he gives a combinatorial characterisation of local rigidity of a different flavour to those we have seen before.

Theorem 1.0.4. *Let G be a graph on n vertices, indexed $\{1, \ldots, n\}$, G is rigid in $\mathbb{R}^d$ if and only if a certain shifted version of G contains the edge dn.*

Many of the applications of algebraic shifting have been to problems in algebraic combinatorics. Since shifting a simplicial complex preserves its *f-vector*, it was a tool of interest in better understanding the g-conjecture, a conjecture concerning the f-vectors of simplicial spheres, which was proven by Adiprasito [2018], for example in Nevo [2007].

It was Nevo again, who along with Bulavka and Peled, proved an analogous result to Kalai's but for volume rigidity.

Theorem 1.0.5. *Bulavka et al. [2022] Let Σ be a simplicial compelx on n vertices, indexed $\{1, \ldots, n\}$, Σ is rigid in $\mathbb{R}^d$ if and only if a certain shifted version of Σ contains the d-simplex $([d+1] \setminus \{2\}) \cup \{n\}$.*

The extendability of Kalai's result to other forms of rigidity suggests that there may be a deeper connection between algebraic shifting and rigidity. Exploring this topic further will be the subject of future research by the author.

In summary, this thesis aims to develop a theory of volume rigidity that is analogous to Euclidean rigidity theory. As noted above, Borcea, Streinu, Theran and Whiteley have all laid the foundations for this (see Borcea and Streinu [2013], Streinu and Theran [2007]). However, Borcea and Streinu's results were quite niche, while Streinu and Theran's assertion was not correct.

In this thesis, we begin by developing a theory of local and *infinitesimal* volume rigidity theory in a way analogous to Euclidean rigidity theory. We then proceed to prove some combinatorial results, in particular, we extend a theorem of Borcea and Streinu to prove a combinatorial necessary condition à la Maxwell. We also study the work of Bulavka, Nevo and Peled, linking algebraic shifting and volume rigidity and end by exploring global volume rigidity.

1.1 Guide to Reading

Chapter 2 features preliminary results from algebraic geometry, matroid theory, simplicial topology and Euclidean rigidity theory which will either be required or useful to know for later on in the thesis and, in the case of Euclidean rigidity, will highlight the areas that we will try to extend to volume rigidity theory. Readers familiar with rigidity theory may choose to skip this chapter.

Chapters 3 and 4 take a more traditional rigidity-theoretic approach to volume rigidity, recreating familiar tools and results, and proving the rigidity of some classes of simplicial complexes. Those familiar with rigidity theory may take for granted the equivalence of the different definitions of rigidity in section 3.2, as is done, for example, in Borcea and Streinu [2013], or that Asimow and Roth's theorem (theorem 2.4.5) has a volume rigidity analogue. C4.3 links volume rigidity in $\mathbb{R}^2$ to the study of phylogenetic trees and graph orientations through a result of Bernstein, extending this connection is a potential future research direction, see section 5.3. Section 4.5 improves a necessary combinatorial condition for volume rigidity in $\mathbb{R}^d$.

Chapter 5 builds on the work of Bulavka, Nevo and Peled in applying Kalai's algebraic combinatorics to volume rigidity. In particular, in section 5.2 we are able to use these techniques to give a lower bound on the rank of a simplicial complex in its appropriate rigidity matroid.

Chapter 6 studies the third question posed in the previous section, and improves upon a bound on the number of *congruence classes* of a typical framework of a bipyramid. We also discuss at length methods used in Borcea and Streinu [2013].

Chapter 7 studies the global rigidity of frameworks of simplicial complexes. In particular, we show that, unlike global Euclidean rigidity in $\mathbb{R}^d$, global volume rigidity in $\mathbb{R}^d$ is not a generic property of simplicial complexes when $d \geq 2$. We are able to, however, define infinitely large classes of simplicial complexes that are *generically globally rigid* in $\mathbb{R}^d$, for any d. We end by reflecting on the differences in working over $\mathbb{R}$ and $\mathbb{C}$, highlighting how our results differ from those in Cruickshank et al. [2023].

This thesis uses code written in Python with the *SageMath 9.0* package to calculate rigidity matrices and perform exterior shifting. The code used may be found at https://github.com/josouthgate/PhDThesis.

Finally, appendix A deals with a smaller side project undertaken with Bulavka at the Fields Institute in 2023, while appendix B lists out cases used in proofs in chapter 4.

Chapter 2

Preliminaries

In this chapter, we outline the definitions and results from various areas of maths that we will use over the course of this thesis. We also outline in section 2.4 the points of the theory of Euclidean rigidity that we wish to develop in the volume measurement setting.

Before focusing on any one area, we note some conventional and notational points:

Terms being defined will be highlighted in *blue*, when introducing terms without definition, or using them before definition, we will often use *italics*.

The natural numbers $\mathbb{N}$ begin at 1, i.e. $\mathbb{N} = \{1, 2, 3, \dots\}$, where the non-negative integers are considered, we will write $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$ (and analogously for the non-positive integers). For any $N \in \mathbb{N}$, we will define $[N]$ to be the set $\{1, 2, \dots, N\}$.

Given a vector space V over some field $\mathbf{k}$ with entries indexed by the set I, we may write $V \subseteq \mathbf{k}^I$, all vector spaces encountered in this thesis will be finite-dimensional. The same indexing convention applies to both affine and projective spaces (so, for example $\mathbb{CP}^{\binom{[N]}{2}}$ is the $\left(\binom{N}{2} - 1\right)$-dimensional complex projective space with entries indexed by unordered pairs $\{i, j\}$, for $i, j \in [N]$ distinct).

An relation $\preceq$ on a set I is *partial* if, for any $i, j, k \in I$,

PO1 $i \preceq i$;

PO2 if $i \preceq j$ and $j \preceq i$, then $i = j$;

PO3 if $i \preceq j$ and $j \preceq k$, then $i \preceq k$.

A *linear order* on I is a relation $\preceq$ on I that satisfies PO1, PO2, PO3 and such that, for any $i, j \in I$,

LO $i \preceq j$ or $j \preceq i$.

A linear order $\preceq'$ of I is a *linear extension* of a partial ordering $\preceq$ of I if $i \preceq j$ implies $i \preceq' j$.

There is a natural linear ordering $\preceq$ on $\mathbb{N}$, as well as on $[N]$, for any $N \in \mathbb{N}$, given by $i \prec j$ if and only if $i < j$ and $i = j$, with respect to $\preceq$, if and only if $i = j$. Given any subset I of $\binom{\mathbb{N}}{k}$, and writing elements of I $\{i_1, \ldots, i_k\}$, where $i_1 < \cdots < i_k$ in $\mathbb{N}$, as $i_1 \ldots i_k$, the natural linear ordering (which we shall call $\leq$ for convenience) of $\mathbb{N}$ induces a *partial lexicographic ordering* $\preceq$ of I, given by $i_1 \ldots i_k \preceq j_1 \ldots j_k$ if and only if $i_l \leq j_l$, for all $1 \leq l \leq k$. The *(linear) lexicographic ordering* $\prec_{\text{lex}}$ of I is an example of a linear extension of this partial ordering and is defined by $i_1 \ldots i_k \prec_{\text{lex}} j_1 \ldots j_k$ if and only if

$$\min\{i_1 \ldots i_k \Delta j_1 \ldots j_k\} \in i_1 \ldots i_k,$$

where Δ denotes the *symmetric difference*, defined

$$S \Delta T = (S \setminus T) \cup (T \setminus S),$$

for any sets S, T.

2.1 Real and Complex Algebraic Geometry

In this thesis, we will encounter real and complex *algebraic varieties*, usually representing the measurements of frameworks, or the frameworks with a specific measurement. Algebraic varieties are either affine or projective, the former being subsets of a N-dimensional, where $N \in \mathbb{N}$, affine space over some field $\mathbf{k}$ (in our case, either $\mathbb{R}$ or $\mathbb{C}$), which we identify with $\mathbf{k}^N$, the latter being subsets of the projective space $\mathbf{k}\mathbb{P}^N$.

Throughout this thesis, we will mostly refer to Bochnak et al. [2013] and Harris [2013] as sources for real and complex algebraic geometry respectively. In this Section, we will introduce the most fundamental objects from each setting that we will encounter as we progress.

Definition 2.1.1. Let $\mathbf{k}$ be either $\mathbb{R}$ or $\mathbb{C}$. An *algebraic set* X in $\mathbf{k}^N$ with respect to some set $I \subseteq \mathbf{k}[x_1, \ldots, x_N]$ is defined

$$X = \mathbb{V}(I) := \{p \in \mathbf{k}^N : f(p) = 0,\ \forall f \in I\}.$$

A *semi-algebraic set* S in $\mathbb{R}^N$ with respect to some sets $I, J \subseteq \mathbb{R}[x_1, \ldots, x_N]$ is defined

$$S = \{p \in \mathbb{R}^N : f(p) = 0,\ g(p) > 0,\ \forall f \in I,\ g \in J\}.$$

Every algebraic set in $\mathbb{R}^N$ is a semi-algebraic set in $\mathbb{R}^N$, however, not every semi-algebraic set in $\mathbb{R}^N$ is algebraic, take for example $S = \{p \in \mathbb{R}^1 : p > 0\}$.

Let V be a vector space over the field $\mathbf{k}$, with coordinates indexed by I, an *orthogonal projection* of V onto a subset $J \subseteq I$ is a map π_J with kernel $\text{Span}_{\mathbf{k}}\{u_i : i \in I \setminus J,\ u \text{ a basis vector of } \mathbf{k}^I\}$.

Theorem 2.1.2. *[Bochnak et al., 2013, p. 26] Let S be a semi-algebraic subset of $\mathbb{R}^N$, with entries indexed by $[N]$, let $I \subseteq [N]$, then $\pi_I(S)$ is a semi-algebraic subset of $\mathbb{R}^I$.*

Definition 2.1.3. Let $S \subseteq \mathbb{R}^M$ and $T \subseteq \mathbb{R}^N$ be two semi-algebraic sets. A map $f : S \to T$ is *semi-algebraic* if its graph $\Gamma(f) = \{(p, f(p)) : p \in S\} \subseteq \mathbb{R}^{M+N}$ is a semi-algebraic set.

The following theorem is fundamental to questions of *local rigidity*.

Theorem 2.1.4. *[p. 33]Bochnak et al. [2013] Let $S \subseteq \mathbb{R}^N$ be a semi-algebraic set, then $S = S_1 \sqcup \cdots \sqcup S_t$, for some $t \in \mathbb{N}$, with each S_i semi-algebraically homeomorphic to the open hypercube $(0,1)^{D_i} \subseteq \mathbb{R}^{D_i}$.*

Where $A \sqcup B$ denotes the disjoint product of sets A and B.

We note that, since every algebraic set is semi-algebraic, theorem 2.1.4 states that every algebraic set has finitely many connected components.

Definition 2.1.5. Let $\mathbf{k}$ be either $\mathbb{R}$ or $\mathbb{C}$, the *Zariski topology* on $\mathbf{k}^N$ is the topology whose closed sets are the algebraic subsets of $\mathbf{k}^N$.

Let S be a semi-algebraic subset of $\mathbb{R}^N$, the closure of S under the Zariski topology, denoted $\text{clos}_Z(S)$, is the smallest, with respect to the Zariski topology, algebraic subset of $\mathbb{R}^N$ containing S.

An algebraic set X is *irreducible* if there do not exist distinct proper algebraic sets $X_1, X_2 \subset X$, so that $X_1 \cup X_2 = X$.

Definition 2.1.6. Let $\mathbf{k}$ be either $\mathbb{R}$ or $\mathbb{C}$, an *(affine) algebraic variety* $X \subseteq \mathbf{k}^N$ is an irreducible algebraic set in $\mathbf{k}^N$.

If $\mathbf{k}$ is $\mathbb{C}$, the *coordinate ring* of X is the quotient ring

$$\mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_N] / \mathbb{I}(X),$$

where $\mathbb{I}(X) = \{f \in \mathbb{C}[x_1, \ldots, x_N] : f(p) = 0,\ \forall p \in X\}$ is the *ideal of vanishing* of X.

Given $\mathbf{k}$, the *projective space* $\mathbf{k}\mathbb{P}^N$ is the N-dimensional quotient space

$$\mathbf{k}\mathbb{P}^N = (\mathbf{k}^{N+1} \setminus \{0\}) / \sim,$$

where $p \sim \lambda p$, for any $\lambda \in \mathbf{k} \setminus \{0\}$. A polynomial $f \in \mathbf{k}[x_0, \ldots, x_N]$ is *homogeneous (of degree D)* if

$$f(\lambda p) = \lambda^D f(p).$$

Definition 2.1.7. A *projective algebraic set* X in $\mathbf{k}\mathbb{P}^N$ with respect to some set $I \subseteq \mathbf{k}[x_0, \ldots, x_N]$ of homogeneous polynomials is defined

$$X = \mathbb{V}(I) := \{p \in \mathbf{k}\mathbb{P}^N : f(p) = 0,\ \forall f \in I\}.$$

A *projective algebraic variety* X in $\mathbf{k}\mathbb{P}^N$ is an irreducible projective algebraic set.

The points of a projective algebraic variety $X \subseteq \mathbf{k}\mathbb{P}^N$ may be written in *homogeneous coordinates*, $p = [p_0 : \cdots : p_N]$, an equivalence class representative under $\sim$.

Let $0 \leq i \leq N$, the i^{th} *affine chart* of the projective space $\mathbf{k}\mathbb{P}^N$, defined as the set

$$U_i = \left\{ \left(\frac{p_0}{p_i}, \dots, \frac{p_{i-1}}{p_i}, \frac{p_{i+1}}{p_i}, \dots, \frac{p_N}{p_i} \right) : p = [p_1 : \cdots : p_N] \in \mathbf{k}\mathbb{P}^N,\, p_i \neq 0 \right\},$$

is isomorphic to the N-dimensional affine space over $\mathbf{k}$ (and therefore, canonically to $\mathbf{k}^N$). Given a projective variety $X \subseteq \mathbf{k}\mathbb{P}^N$, we may consider its affine charts by intersecting it with the affine charts of $\mathbf{k}\mathbb{P}^N$, each affine chart is itself an affine variety.

Given two algebraic varieties, there are multiple notions of their *same-ness*, one that will be particularly relevant to us is that of *birational equivalence*, i.e. that their *coordinate rings* are isomorphic.

Definition 2.1.8. Let $X \subseteq \mathbb{C}^M$ and $Y \subseteq \mathbb{C}^N$ be two affine algebraic varieties. A *rational map* φ from X to Y, denoted $\varphi : X \dashrightarrow Y$ is a map of the form $\varphi = (\varphi_1, \dots, \varphi_N)$, where each φ_i is of the form $\frac{f_i}{g_i}$, where $f_i \in \mathbb{C}[X]$ and $g_i \in \mathbb{C}[X]$.

Let $X \subseteq \mathbb{C}\mathbb{P}^M$ and $Y \subseteq \mathbb{C}\mathbb{P}^M$ be projective algebraic varieties. A *rational map* φ from X to Y, denoted $\varphi : X \dashrightarrow Y$ is a rational map between some affine chart of X and some affine chart of Y.

Two varieties X and Y are *birationally equivalent* if there exist rational maps $\varphi : X \dashrightarrow Y$ and $\psi : Y \dashrightarrow X$ so that $\psi \circ \varphi$ and $\varphi \circ \psi$ are respectively the identity map on an open subset of X and an open subset of Y in the affine case, or an open subset of an affine chart of X and an open subset of an affine chart of Y in the projective case.

Proposition 2.1.9. *Suppose that X and Y are two affine algebraic varieties over $\mathbb{C}$. If X and Y are birationally equivalent, then $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are isomorphic as rings.*

Associated to a projective variety X defined over $\mathbb{C}$ are notions of *dimension* and *degree*. Dimension has an algebraic definition as one less than the *Krull dimension* of $\mathbb{C}[U]$, for an affine chart U of X, however within this thesis, we will either take the dimension of an algebraic variety as given, having left others to do the hard work, or it will suffice to use the intuitive notion present in the setting of vector spaces of counting degrees of freedom. Degree tells us how many points of a *general* hyperplane L of complementary dimension lie in the intersection of L and X, we will give this a more formal definition.

Definition 2.1.10. Let $X \subseteq \mathbb{C}\mathbb{P}^N$ be a D-dimensional variety and let L be a general $(N - D)$-dimensional hyperplane in $\mathbb{C}\mathbb{P}^N$ (i.e. L is defined by D linear homogeneous equations with sufficiently generic coefficients). The *degree* of X, denoted $\deg(X)$, is the number of points in the intersection of L and X, i.e. $\deg(X) = |L \cap X|$.

The degree of an algebraic variety is a birational invariant.

Proposition 2.1.11. *Let $X \subseteq \mathbb{CP}^N$ be a projective algebraic variety and let U be an affine chart of X, then $\deg(X) = [\mathbb{C}[U] : \mathbb{C}[x_1, \dots, x_N]]$, i.e. the degree of X is the transcendence degree of its function field over that of its ambient space.*

A point $p \in \mathbb{R}^D$ (or in $p \in \mathbb{C}^D$) is *generic* if $f(p) \neq 0$ for all $f \in \mathbb{Q}[X_1, \dots, X_D]$.

Finally, we state a version of a classical result: Bézout's theorem; which expresses the number of intersections of two plane curves in terms of their degrees. The original statement is non-generic and counts points of intersection with multiplicity, derived from the order of vanishing of the rational curve locally defined at the intersection.

Theorem 2.1.12. *Bézout's theorem (generic case)*
Let $C_1, \dots, C_N$ be N generic curves in $\mathbb{CP}^N$ of degrees $c_1, \dots, c_N$ respectively, then, when counted with multiplicity,

$$|C_1 \cap \cdots \cap C_N| = c_1 \dots c_N < \infty.$$

2.2 Matroids

A *matroid* is, roughly speaking, an assignment of a notion of independence to a finite set system, first developed by Nakasawa and Whitney (independently) in the 1930's (Nishimura and Kuroda [2009], Whitney [1935]), they have been used to characterise many combinatorial and geometric properties such as cycles in graphs, dependence in algebraic varieties and, most relevant to us, Euclidean rigidity of graphs.

Several equivalent definitions of matroids exist, the four most relevant to us define them in terms of their *independent sets*, their *bases*, their *rank function* and their *circuits*.

A standard reference for matroid theory is Oxley [2011], all the standard definitions may be found in its first chapter, for algebraic matroids, see the expository article Rosen et al. [2020], finally, Graver et al. [1993] introduces many of the matroidal concepts required for the understanding of rigidity theory in that particular setting.

In the tradition of matroid theory, we provide four separate definitions of a matroid, all of which are equivalent. We will not show the equivalence of these definitions in this thesis.

Definition 2.2.1. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{I})$, where E is a finite *ground set* and $\mathcal{I} \subset 2^E$ is the set of *independent sets* of $\mathcal{M}$, satisfying

I1 $\varnothing \in \mathcal{I}$;

I2 If $T \in \mathcal{I}$ and $S \subseteq T$, then $S \in \mathcal{I}$;

I3 The *augmentation criterion*: if $S, T \in \mathcal{I}$ and $|S| < |T|$, then there exists $x \in T \setminus S$ so that $S \cup \{x\} \in \mathcal{I}$.

If $S \in \mathcal{I}$, then we say that S is *independent* in $\mathcal{M}$

Where 2^E denotes the set of all subsets of E (including $\varnothing$ and E itself).

Definition 2.2.2. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{B})$, where E is a finite ground set and $\mathcal{B} \subseteq 2^E$ is the set of *bases* of $\mathcal{M}$, satisfying

B1 $\mathcal{B} \neq \varnothing$;

B2 The *exchange property*: if $S, T \in \mathcal{B}$ and $x \in S \setminus T$, then there exists $y \in T \setminus S$ so that $(S \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

If $S \in \mathcal{B}$, then we say that S is a *basis* of $\mathcal{M}$.

In order to define the rank function of a matroid, we first need to introduce the concepts of a *submodular* and *monotonic* functions of sets.

Definition 2.2.3. Let E be a finite set and $f : 2^E \to \mathbb{R}$ a function.
We say that f is *submodular* if, for all $S, T \subseteq E$,

$$f(S \cup T) + f(S \cap T) \leq f(S) + f(T).$$

We say that f is *monotonic (increasing)* if, for all $S \subseteq T \subseteq E$,

$$f(S) \leq f(T).$$

Definition 2.2.4. A *matroid* $\mathcal{M}$ is a pair (E, r), where E is a finite ground set and $r : 2^E \to \mathbb{Z}_{\geq 0}$ is a monotonic submodular function satisfying $r(S) \leq |S|$, for all $S \subseteq E$, called the *rank function* of $\mathcal{M}$.
We say that $\mathcal{M}$ has rank $r(E)$.

If $S \subseteq E$, then the *closure* of S is the unique maximal (with respect to inclusion) subset $S \subseteq T \subseteq E$ such that

$$\operatorname{rank}(T) = \operatorname{rank}(S).$$

Definition 2.2.5. A *matroid* $\mathcal{M}$ is a pair $(E, \mathcal{C})$, where E is a finite ground set and $\mathcal{C} \subseteq 2^E$ is the set of subsets of E such that

Z1 $\varnothing \notin \mathcal{C}$;

Z2 If $S, T \in \mathcal{C}$ and $S \subseteq T$, then $S = T$;

Z3 If $S, T \in \mathcal{C}$, $S \neq T$ and $x \in S \cap T$, then there exists $R \in \mathcal{C}$ such that $R \subseteq (S \cap T) \setminus \{e\}$.

If $S \in \mathcal{C}$, then we say that S is a *circuit* of $\mathcal{M}$.

Next we introduce *linear* matroids and *algebraic* matroids. The former is the matroid arising from linear independence of rows or columns of a matrix and the latter is the matroid arising from the orthogonal projections of affine algebraic varieties onto coordinate axes.

We will show that algebraic matroids may be thought of as linear matroids of the Jacobian of the variety in question, which will be helpful since, as is often the case, studying the linear algebraic setting is more straightforward than studying the higher-order algebraic setting. This restriction to studying the linear matroid is precisely the one we will encounter in terms of simplicial complexes in chapter 3.

Definition 2.2.6. Let $\mathbf{k}$ be a field and let M be a matrix with entries in $\mathbf{k}$. The *linear matroid* of M is the matroid $(E, \mathcal{I})$, where E indexes the rows of M and $S \subseteq E$ is independent if and only if the set of rows of M is linearly independent over $\mathbf{k}$.

A similar definition exists for the linear matroid defined in terms of the columns of M. However, unless specified otherwise, we are not be interested in this formulation (in fact, it is just the linear matroid of the transpose of the matrix).

Definition 2.2.7. Let $\mathbf{k}$ be an algebraically closed field and let $X \subseteq \mathbf{k}^N$ be a D-dimensional affine algebraic variety. The *algebraic matroid* of X, denoted $\mathcal{M}(X, \mathbf{k})$, is the matroid $([N], \mathcal{I})$, where

$$\mathcal{I} = \{S \subseteq E : \dim_{\mathbf{k}}(\pi_S(X)) = |S|\}.$$

Algebraic matroids of algebraic varieties (as well as in terms of other algebraic objects, such as field extensions) are thoroughly introduced in Rosen et al. [2020].

In order to show the relation between the algebraic matroid of a variety and the linear matroid of its Jacobian, we need a notion of *sameness* of matroids.

Definition 2.2.8. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matroids on ground sets E_1 and E_2 respectively. We say that $\mathcal{M}_1$ and $\mathcal{M}_2$ are *isomorphic* if there is a bijection $F : E_1 \to E_2$ and one of the following equivalent conditions hold:

1. The set $S \subseteq E_1$ is independent in $\mathcal{M}_1$ if and only if $F(S) \subseteq E_2$ is independent in $\mathcal{M}_2$;
2. The set $S \subseteq E_1$ is a basis of $\mathcal{M}_1$ if and only if $F(S) \subseteq E_2$ is a basis of $\mathcal{M}_2$;
3. The rank of S is equal to the rank of $F(S)$, for all $S \subseteq E_1$;
4. The set $S \subseteq E_1$ is a circuit in $\mathcal{M}_1$ if and only if $F(S) \subseteq E_2$ is a circuit of $\mathcal{M}_2$.

Proposition 2.2.9. *Let $X \subseteq \mathbb{C}^N$ be an affine algebraic variety defined $X = \mathrm{clos}_Z(f(\mathbb{C}^M))$, for some polyomial map f. Then $\mathcal{M}(X, \mathbb{C})$ and the linear matroid of $\mathrm{d}f_p$ over $\mathbb{C}$, the differential of f at some generic point $p \in \mathbb{C}^M$, are equal.*

Here, $p \in \mathbb{C}^M$ is generic if $f(p) \neq 0$, for all $f \in \mathbb{Q}[x_1, \ldots, x_M] \setminus \{0\}$.

Proof. Index the rows of $\mathrm{d}f_p$ by $[N]$, then the matrix defined by the restriction to a subset of rows indexed by $S \subseteq [N]$ is the differential $\mathrm{d}(\pi_S \circ f)_p$. The column space of the differential of a map at some point is the tangent space at the image of that point of the image of the map, therefore, the rank of that differential is the dimension of the corresponding tangent space. Since p is generic, the tangent space to $\mathrm{clos}_Z((\pi_S \circ f)(\mathbb{C}^M))$ at $(\pi_S \circ f)(p)$ has dimension $\dim_{\mathbb{C}}(\mathrm{clos}_Z((\pi_S \circ f)(\mathbb{C}^M)))$.

Therefore a set $S \subseteq [N]$ is independent in the linear matroid of the differential $\mathrm{d}f_p$ over $\mathbb{C}$ if and only if the projection $\pi_S(X)$ has dimension $|S|$, which holds if and only if the set S is independent in $\mathcal{M}(X, \mathbb{C})$. □

2.3 Simplicial Topology

The objects we will use throughout this thesis to encode the combinatorial data of our frameworks are simplicial complexes.

Definition 2.3.1. A *simplicial complex* is a finite set Σ such that

SC1 If $\sigma \in \Sigma$ and $\tau \subseteq \sigma$, then $\tau \in \Sigma$;

SC2 If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a sub-simplex of both σ and τ.

If $|\sigma| = k + 1$, for some $k \in \mathbb{Z}$, then we call σ a *k-simplex* (of Σ), we denote the set of k-simplices of Σ by $\Sigma^{(k)} \subseteq \Sigma$. If $d = \max_{k \in \mathbb{Z}}\{\Sigma^{(k)} \neq \varnothing\}$, then we say that Σ is *d-dimensional*, or that $\dim(\Sigma) = d$. If Σ is d-dimensional and, for all $\tau \in \Sigma$, there exists a d-simplex $\sigma \in \Sigma$ so that $\tau \subseteq \sigma$, then we say that Σ is *pure (d-dimensional)*.

There are some more interesting/relevant dimensions simplices of a simplicial complex may have, we have specific terms to refer to such simplices.

Definition 2.3.2. Let Σ be a pure d-dimensional simplicial complex.

- Call $\Sigma^{(0)}$ the *vertex set* of Σ and call its elements *vertices*;
- Call $\Sigma^{(1)}$ the *edge set* of Σ and call its elements *edges*;
- Call $\Sigma^{(d)}$ the *face set* or *set of maximal simplices* of Σ and call its elements *faces* or *maximal simplices.*

If Σ is not pure, then $\Sigma^{(0)}$ and $\Sigma^{(1)}$ keep their names, but the *set of maximal simplices*, denoted $\Sigma^{(\max)}$, refers instead to the set

$$\Sigma^{(\max)} = \{\sigma \in \Sigma : \sigma \subseteq \tau \implies \sigma = \tau\}.$$

The *f-vector* of Σ, $f(\Sigma) \in \mathbb{Z}^{-1,\dots,d}$, enumerates the numbers of simplices of each dimension:

$$f(\Sigma)_k = \begin{cases} 1, & \text{if } k = -1, \\ |\Sigma^{(k)}|, & \text{else.} \end{cases}$$

If A is a subset of 2^E, for some finite set E, then let

$$\overline{A} = A \cup \{\tau : \exists \sigma \in \Sigma \text{ such that } \tau \subseteq \sigma\}.$$

We note that a graph is a 1-dimensional simplicial complex. We may therefore generalise a few graph-theoretic concepts to simplicial complexes.

Definition 2.3.3. Let Σ be a d-dimensional simplicial complex and let $0 \leq k \leq d$. We say that Σ is *connected (through codimension k)* if, for any $u, v \in \Sigma^{(0)}$, there exists a sequence of $(d-k+1)$-simplices $\sigma_1, \ldots, \sigma_N$ so that $u \in \sigma_1$, $v \in \sigma_N$ and, for every $2 \leq i \leq N$, $(\sigma_1 \cap \sigma_2) \in \Sigma^{(d-k)}$.

If there exists some $0 \leq k \leq d$ so that Σ is connected through codimension k, then we say that Σ is *connected*.

In order to generalise the concept of a graph-theoretic cycle, we must introduce the language of homology.

A *chain* over the ring R of k-simplices of Σ is a formal sum

$$c = \sum_{\sigma \in \Sigma^{(k)}} c_\sigma \sigma,$$

where the coefficients c_σ are in R and the summands are the k-simplices of Σ. The k simplices are oriented, and negatively oriented k-simplices induce multiplication by -1. Denote by $C_k(\Sigma, R)$ the group of such chains under addition, C_k is a free-abelian group. There exists a long exact sequence

$$0 \to C_d(\Sigma, R) \xrightarrow{\partial_d} C_{d-1}(\Sigma, R) \xrightarrow{\partial_{d-1}} \ldots \xrightarrow{\partial_1} C_0(\Sigma, R) \to 0,$$

which we call the *chain complex* associated to Σ and R. The *boundary operators* ∂_k are linear maps defined in terms of their actions on the generators of $C_k(\Sigma, R)$ (i.e. the k-simplices of Σ) as

$$\partial_k(\sigma) = \sum_{i \in \sigma^{(0)}} \text{sign}(i, \sigma)(\sigma - i),$$

where $\sigma - i$ denotes the $(k-1)$-simplex consisting of the vertices of σ not equal to i. These linear maps may be represented by matrices $D_k \in R^{\Sigma^{(k)} \times \Sigma^{(k+1)}}$. Then *$k$-cycles* in the chain complex are kernel vectors of ∂_k while *k-boundaries* are image vectors of ∂_{k+1}. The *k^{th} homology group* of Σ with respect to R is the quotient group

$$H_k(\Sigma, R) = {\text{Ker}(\partial_k)}\big/{\text{Im}(\partial_{k+1})}.$$

A graph theoretical cycle may be thought of as a 1-cycle of the 1-dimensional simplicial complex (i.e. graph) G with respect to the ring $\mathbb{Z}$, we will therefore refer to k-cycles of Σ as the elements of $\text{Ker}(\partial_k)$ in the chain complex associated to Σ and $\mathbb{Z}$.

A dual setting to that of chain complexes and homology exists, that of cochain complexes and cohomology. Cohomology theory is, in general, very deep, but we give it a cursory treatment here, defining the cochain complex associated to Σ and R as

$$0 \leftarrow C_d(\Sigma, R) \xleftarrow{\delta^d} C_{d-1}(\Sigma, R) \xleftarrow{\delta^{d-1}} \dots \xleftarrow{\delta^1} C_0(\Sigma, R) \leftarrow 0,$$

where the *coboundary operator* δ^k is the transpose of the boundary operator ∂_k (represented by the matrix $D^k = (D_d)^t$). We may define the cochains and coboundaries, and therefore the cohomology groups of the cochain complex, however we will not encounter these in this thesis, so will settle for mentioning that we may define them analogously to the prior setting.

2.4 Euclidean Rigidity Theory

The most well studied form of rigidity theory is that of the rigidity of *graph frameworks* in $\mathbb{R}^d$ - pairs (G, p), where $G = (V, E)$ is an undirected graph on n vertices and $p = (p(i) : i \in V) \in (\mathbb{R}^d)^n$ is a configuration of those vertices in $\mathbb{R}^d$ - under the Euclidean isometries of $\mathbb{R}^d$, which form the group $\mathrm{Euc}(d, \mathbb{R})$.

This section exists to give context to the later work and highlight the results we will try to recreate.

Definition 2.4.1. The *(complete) Euclidean measurement map* of a set of n points, $[n]$, in $\mathbb{R}^d$ is the following polynomial

$$\ell_n^d : (\mathbb{R}^d)^n \to \mathbb{R}^{\binom{[n]}{2}}; p \mapsto \left(\|p(i) - p(j)\| : ij \in \binom{[n]}{2} \right).$$

Then, for any graph $G = (V, E)$ on n vertices, the *Euclidean measurement map* of G in $\mathbb{R}^d$ is defined $\ell_G^d = \pi_E \circ \ell_n^d$.

Definition 2.4.2. Let (G, p) and (G, q) be two frameworks of the graph $G = (V, E)$ in $\mathbb{R}^d$.

We say that (G, p) and (G, q) are *Euclidean equivalent* if one of the two following equivalent statements hold:

1. $\ell_G^d(p) = \ell_G^d(q)$;
2. For each $ij \in E$, there exists $f_{ij} \in \mathrm{Euc}(d, \mathbb{R})$ so that $f_{ij}(p(i)) = q(i)$ and $f_{ij}(p(j)) = q(j)$.

We say that (G, p) and (G, q) are *Euclidean congruent* if one of the two following equivalent statements hold

1. $\ell_n^d(p) = \ell_n^d(q)$;
2. There exists $f \in \mathrm{Euc}(d, \mathbb{R})$ so that $f(p(i)) = q(i)$, for all $i \in V$.

The equivalence of these, and later statements going between the measurement map and isometry formulations of Euclidean rigidity are taken for granted, but the analogous equivalences will need to be shown in the volume setting.

Definition 2.4.3. Let (G, p) be a framework in $\mathbb{R}^d$ of the graph $G = (V, E)$ on n vertices. A *finite Euclidean flex* of (G, p) is a continuous map $\gamma : [0, 1] \to (\mathbb{R}^d)^n$ so that $\gamma(0) = p$ and

$$\ell_G^d(\gamma(t)) = \ell_G^d(p),$$

for all $t \in [0, 1]$. A finite Euclidean flex is *trivial* if

$$\ell_n^d(\gamma(t)) = \ell_n^d(p),$$

for all $t \in [0, 1]$.

We are now in a place to state the three most common formulations of Euclidean rigidity in $\mathbb{R}^d$.

Definition 2.4.4. Let (G, p) be a graph framework in $\mathbb{R}^d$, we say that (G, p) is *(locally) Euclidean rigid* in $\mathbb{R}^d$ if one of the three following equivalent statements hold:

1. There exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ so that, if $q \in U$ and (G, p) and (G, q) are Euclidean equivalent, then they are Euclidean congruent;
2. There exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ so that

$$(\ell_G^d)^{-1}(\ell_G^d(p)) \cap U = (\ell_n^d)^{-1}(\ell_n^d(p)) \cap U;$$

3. Every finite flex of (G, p) is trivial.

A goal of rigidity is to classify the Euclidean rigid frameworks of a graph. Determining whether or not a graph framework is Euclidean rigid in $\mathbb{R}^d$ is `NP-hard` (Saxe [1979]), as atypical placements of points may induce rigidity in a framework of a graph that is typically flexible (such as two vertices of a 4-cycle being coincident). Therefore rigidity theorists often consider the rigidity of a *generic* framework, where such degeneracies do not occur, what's more, it suffices to determine whether or not one generic framework of a graph G is Euclidean rigid in $\mathbb{R}^d$ in order to say whether or not any generic framework of G is.

Here, a *generic framework* in $\mathbb{R}^d$ is a pair (G, p) where $p \in (\mathbb{R}^d)^n$ is a *generic configuration*, i.e. if $f \in \mathbb{Q}[x_1, \dots, x_{dn}] \setminus \{0\}$, then $f(p) \neq 0$.

Theorem 2.4.5. *[Asimow and Roth [1978]] Let (G, p) be a generic framework in $\mathbb{R}^d$. Suppose that (G, p) is Euclidean rigid in $\mathbb{R}^d$, then (G, q) is Euclidean rigid in $\mathbb{R}^d$, for any generic framework (G, q) in $\mathbb{R}^d$.*

If G is a graph such that the generic framework in $\mathbb{R}^d$, (G, p), is Euclidean rigid in $\mathbb{R}^d$, then we say that G is *(generically) Euclidean rigid* in $\mathbb{R}^d$. A refinement of our earlier goal may therefore be to classify the Euclidean rigid graphs in $\mathbb{R}^d$. Since we have done away with all but the most fundamental geometric dependencies, we believe that the Euclidean rigidity of a graph in $\mathbb{R}^d$ is determined exclusively by the combinatorics.

Maxwell gave a necessary condition for Euclidean rigidity in $\mathbb{R}^3$ in Maxwell [1864]. Since then, his condition was refined and generalised into what are now referred to as *Maxwell counts*, these have strong links to *graph sparsity*.

Definition 2.4.6. Let $G = (V, E)$ be a graph, we say that G is *(k, l)-sparse* if, for any $X \subseteq V$,

$$i_G(X) \leq k|X| - l,$$

where $i_G(X) = |E(G[X])|$ is the number of edges present in the subgraph of G induced by X.

We say that G is *(k, l)-tight* is it is (k, l)-sparse and $|E| = k|V| - l$.

Then the Maxwell counts necessary for rigidity are as follows

Theorem 2.4.7. *Suppose that the graph G is Euclidean rigid in $\mathbb{R}^d$, then G admits a $\left(d, \binom{d+1}{2}\right)$-tight spanning subgraph.*

Roughly speaking, the $\left(d, \binom{d+1}{2}\right)$ term comes from the fact that each completely unconstrained vertex has d degrees of freedom in $\mathbb{R}^d$, and each completely constrained vertex has $\binom{d+1}{2}$ degrees of freedom, corresponding to isometric transformations of $\mathbb{R}^d$.

If G is generically Euclidean rigid in $\mathbb{R}^d$ and $\left(d, \binom{d+1}{2}\right)$-tight, then we say that G is *minimally (generically) Euclidean rigid* in $\mathbb{R}^d$ as removing any edge from G will yield a graph that is generically Euclidean flexible in $\mathbb{R}^d$. On the other hand, if G is generically Euclidean rigid in $\mathbb{R}^d$ and remains so after removing any edge, then we say that G is *redundantly (generically) Euclidean rigid* in $\mathbb{R}^d$.

In dimensions 1 and 2, these conditions are sufficient. The 1-dimensional case is folkloric and the 2-dimensional case was initially proved by Pollaczek-Geiringer in 1927 but faded into obscurity and was independently re-proved by Laman in 1970.

Theorem 2.4.8. *[Folklore; Pollaczek-Geiringer [1927], Laman [1970]] Let G be a graph.*

1. *G is Euclidean rigid in $\mathbb{R}^1$ if and only if G admits a $(1, 1)$-tight spanning subgraph (i.e. G is connected);*

2. *G is Euclidean rigid in $\mathbb{R}^2$ if and only if G admits a $(2, 3)$-tight spanning subgraph.*

Definition 2.4.9. The Zariski-closure of the Euclidean measurement map ℓ_n^d is the *(real) Euclidean measurement variety*, denoted $EM_n^d \subseteq \mathbb{R}^{\binom{[n]}{2}}$.

If we consider the Zariski-closure of the image under ℓ_n^d of $(\mathbb{C}^d)^n$ we obtain the *complex Euclidean measurement variety*, denoted $CEM_n^d \subseteq \mathbb{C}^{\binom{[n]}{2}}$.

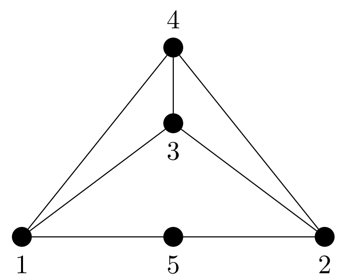

Figure 2.1: The graph framework (G, p) from example 2.4.11.

2.4.1 Infinitesimal Rigidity

Checking whether or not any given graph framework in $\mathbb{R}^d$ is Euclidean rigid is `NP-Complete` (Saxe [1979]), however, by considering a linear approximation of the higher-order equations defining Euclidean rigidity, we can check, with high probability, rigidity in $\mathbb{R}^d$ *geometrically*.

Definition 2.4.10. The *(complete) Euclidean rigidity matrix* of the configuration of n vertices, p in $\mathbb{R}^d$, denoted $ER(p)$, is the $\binom{n}{2} \times dn$-matrix representing the differential of ℓ_n^d evaluated at p, $(\mathrm{d}\ell_n^d)_p : T_p(\mathbb{R}^d)^n \to T_{\ell_n^d(p)} EM_n^d$. The rows of $ER(p)$ are indexed by pairs of vertices while the columns are grouped into d-tuples, each indexed by a vertex.

Let $G = (V, E)$ be a spanning subgraph of K_n^1, the *Euclidean matrix of* (G, p), denoted $ER(G, p)$, is the sub-matrix of $ER(p)$ defined by restricting to the rows indexed by E.

Within a neighbourhood of p, $(\ell_G^d)^{-1}(\ell_G^d(p))$ and $T_p(\ell_G^d)^{-1}(\ell_G^d(p))$ are isomorphic and the infinitesimal velocities of paths originating at p are in one-to-one correspondences with the tangent vectors in the latter. These infinitesimal velocities are *infinitesimal Euclidean flexes* of (G, p), corresponding to its finite Euclidean flexes. If an infinitesimal Euclidean flex is the infinitesimal velocity of a trivial finite Euclidean flex, then it is *trivial*. Moreover, by a result from differential geometry, the trivial infinitesimal Euclidean flexes and infinitesimal Euclidean flexes of (G, p) are precisely the kernel vectors of $ER(p)$ and $ER(G, p)$ respectively. A framework may admit non-trivial infinitesimal Euclidean flexes that arise from geometric dependencies and are not the infinitesimal velocities of any finite Euclidean flexes.

Example 2.4.11. Consider the graph framework (G, p) shown in fig. 2.1. We may not perform a non-trivial finite flex of vertex 5, as doing so would yield a non-trivial finite flex of the rigid sub-framework induced by vertices 1 to 4. However, (G, p) admits a non-trivial infinitesimal flex at vertex 5, as we may infinitesimally perturb it perpendicularly to the edges connecting it to vertices 1 and 2. ◇

The *flex* admitted framework in fig. 2.1 is infinitesimal since it may not actually move vertex 5 a finite distance, this is in comparison to the flexes admitted by the flexible frameworks in figs. 1.1 and 1.2, where vertices in the flex do move in space.

Following from the above discussion is the fact that the linear matroid defined by the rows of $ER(p)$, for some generic $p \in (\mathbb{R}^d)^n$, is precisely the generic Euclidean rigidity matroid $\mathcal{ER}_n^d$.

It is by considering the Euclidean rigidity matrices of generic frameworks in $\mathbb{R}^d$ that theorem 2.4.5 is proved. Indeed, a generic framework (G, p) is rigid if and only if it admits a non-trivial finite flex. Since all non-trivial infinitesimal Euclidean flexes that are not infinitesimal velocities of finite Euclidean flexes arise from geometric dependencies, (G, p) is rigid if and only if it admits a non-trivial finite flex. This latter condition is equivalent to $ER(G, p)$ not attaining its maximum rank of $dn - \binom{d+1}{2}$, which is itself equivalent to p being the solution of a polynomial in $\mathbb{Q}[x_1, \dots, x_{dn}] \setminus \{0\}$.

The dual picture to *kinematics*, the study of infinitesimal rigidity in terms of infinitesimal flexes, is that of *statics*, where we consider the *stresses* of frameworks.

Definition 2.4.12. Let $G = (V, E)$ be a graph and let (G, p) be a framework in $\mathbb{R}^d$. A *stress* of (G, p) is a vector $\omega \in \mathbb{R}^E$ assigning a weight to each edge. If

$$\sum_{j \in N_G(i)} \omega_{ij}(p(j) - p(i)) = 0 \in \mathbb{R}^d, \tag{2.1}$$

for each $i \in V$, then we call ω an *equilibrium Euclidean stress*.

We note that $\omega \in \mathbb{R}^E$ satisfies eq. (2.1) if and only if ω is a cokernel vector of $ER(G, p)$ (hence how the static picture is dual to the kinematic picture). This correspondence also allows us to state the following theorem, a corollary of the Rank-Nullity and Corank-Conullity theorems applied to $ER(G, p)$.

Theorem 2.4.13. *[Euclidean index theorem] Let G be a graph on n vertices and m edges and let (G, p) be a framework in $\mathbb{R}^d$, then*

$$\begin{aligned}&|\{\textit{independent non-trivial infinitesimal Euclidean flexes of } (G,p)\}|\\&\quad - |\{\textit{independent equilibrium Euclidean stresses of } (G,p)\}|\\&\quad = dn - \binom{d+1}{2} - m.\end{aligned}$$

2.4.2 The Euclidean Rigidity Matroid

Given a graph G and a generic framework (G, p) in $\mathbb{R}^d$, the rigidity Euclidean matrix $R_E(G, p)$ and the image of $(\mathbb{C}^d)^n$ under ℓ_G^d may be thought of as a submatrix of the complete Euclidean rigidity matrix $R_E(p)$ and a orthogonal projection onto a coordinate subset of the complex Euclidean measurement variety CEM_n^d respectively. These views represent two equivalent (see section 4.2 for an explanation of why in terms of volume rigidity) ways of formulating the

Euclidean rigidity matroid. In this subsection, we synthesise these with an abstract combinatorial definition introduced by Graver (see Graver et al. [1993]).

Definition 2.4.14. For any $n, d \geq 1$, the *Euclidean rigidity matroid* of n points in $\mathbb{R}^d$, denoted $\mathcal{ER}_n^d$ is the matroid on ground set $\binom{[n]}{2} = E(K_n)$ such that

ER1 If $|V(E_1) \cap V(E_2)| \geq d$, $\text{cl}(E_1) = K(E_1)$ and $\text{cl}(E_2) = K(E_2)$, then

$$\text{cl}(E_1 \cup E_2) = K(E_1 \cup E_2);$$

ER2 If $|V(E_1) \cap V(E_2)| \leq d - 1$, then

$$\text{cl}(E_1 \cup E_2) \subseteq K(E_1) \cup K(E_2).$$

Where, given $E \subseteq \binom{[n]}{2}$, $K(E)$ is the smallest complete graph with edge set containing E.

Theorem 2.4.15. *The matroid $\mathcal{ER}_n^d$ is the linear matroid of $R_E(p)$, for any generic $p \in (\mathbb{R}^d)^n$ and the algebraic matroid of CEM_n^d.*

We also note that, given any (not necessarily generic) $p \in (\mathbb{R}^d)^n$, we can define a linear matroid by $R_E(p)$ called the *infinitesimal Euclidean rigidity matroid*, denoted $\mathcal{EF}(p)$. When p is generic, $\mathcal{EF}(p) = \mathcal{R}_n^d$.

We may use existing combinatorial necessary and sufficient conditions for rigidity from the prequel to characterise $\mathcal{ER}_n^1$ and $\mathcal{ER}_n^2$.

Theorem 2.4.16. *Let $n \geq 1$, then*

1. *$\mathcal{ER}_n^1$ is the graphic matroid on ground set $\binom{[n]}{2}$;*
2. *$\mathcal{ER}_n^2$ is the $(2,3)$-sparsity matroid on ground set $\binom{[n]}{2}$.*

Where the *$(2,3)$-sparsity matroid* on ground set $\binom{[n]}{2}$ has as its independent sets $E \subseteq \binom{[n]}{2}$ where the graph $(V(E), E)$ is $(2,3)$-sparse.

As of yet, such characterisations are unknown when $d \geq 3$. Determining a combinatorial characterisation of rigidity in $\mathbb{R}^3$ and above is the subject of ongoing research (see, for example Clinch et al. [2019]).

2.4.3 Global Euclidean Rigidity

Global Euclidean rigidity is a stronger version of Euclidean rigidity, where instead of just disallowing continuous deformations, we disallow all deformations of the vertices.

Definition 2.4.17. Let G be a graph on n vertices and let (G, p) be a framework in $\mathbb{R}^d$. We say that (G, p) is *globally Euclidean rigid* in $\mathbb{R}^d$ if either of the two following equivalent statements hold:

1. If (G, p) and (G, q) are Euclidean equivalent, for any $q \in (\mathbb{R}^d)^n$, then they are Euclidean congruent;

2. $(\ell_G^d)^{-1}(\ell_G^d(p)) = (\ell_n^d)^{-1}(\ell_n^d(p))$;

Again, it has been a goal of rigidity theorists to determine whether or not global Euclidean rigidity in $\mathbb{R}^d$ is a generic property of graphs, and if so, to combinatorially characterise the globally Euclidean rigidity graphs in $\mathbb{R}^d$.

Hendrickson proved these claims in $\mathbb{R}^2$ (the proof of the $\mathbb{R}^1$ claim again being again folkloric - that the graph is 2-vertex-connected) by offering two conditions, known as *Hendrickson's criteria*, whose satisfaction is necessary and sufficient for global Euclidean rigidity in $\mathbb{R}^1$ and $\mathbb{R}^2$.

Theorem 2.4.18. *[Hendrickson [1992]]*
Let $d \in [2]$.

1. *Global rigidity in $\mathbb{R}^d$ is a generic property of graphs;*
2. *A graph G is generically globally rigid in $\mathbb{R}^d$ if and only if*
   (a) *G is $(d+1)$-vertex-connected;*
   (b) *G is redundantly rigid in $\mathbb{R}^d$.*

Hendrickson also conjectured that his criteria for generic global Euclidean rigidity were necessary and sufficient in all dimensions, however they are not sufficient in dimensions 3 and up.

The genericity of global Euclidean rigidity in $\mathbb{R}^d$ had not yet been proved, however Connelly and later Gortler, Healy and Thurston devised a geometric condition that was both generic and necessary and sufficient for global Euclidean rigidity.

Definition 2.4.19. Let $G = (V, E)$ be a graph on n vertices and let (G, p) be a framework in $\mathbb{R}^d$. Let $\omega \in \operatorname{Coker} ER(G, p)$ be an equilibrium Euclidean stress vector of (G, p). Define the *equilibrium Euclidean stress matrix* of (G, p) with respect to ω, denoted $E\Omega(\omega) \in \mathbb{R}^{V \times V}$, entry-wise as follows:

$$E\Omega(\omega)_{i,j} = \begin{cases} \omega_{ij}, & \text{if } ij \in E, \\ -\sum\limits_{k \in N_G(i)} \omega_{ik}, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases}$$

I.e. $E\Omega(\omega)$ is the matrix encoding the linear system balancing equilibrium Euclidean stresses where the stress is constant and the configuration is variable, so $E\Omega(\omega)p = 0$ if and only if ω is an equilibrium Euclidean stress of (G, p).

Theorem 2.4.20. *[Connelly [2005], Gortler et al. [2010]]*
Let G be a graph on n vertices, let (G, p) be a framework in $\mathbb{R}^d$ and let ω be an equilibrium stress of (G, p), then (G, p) is globally Euclidean rigid in $\mathbb{R}^d$ if and only if $\operatorname{nullity}(E\Omega(\omega)) = n - d - 1$. Moreover, this condition is generic.

The necessity of the above condition was shown by Connelly whilst its sufficiency was shown by Gortler, Healy and Thurston. Roughly speaking, equilibrium Euclidean stresses of (G, p) are invariant under affine transformations of (G, p), and $q \in \mathrm{Ker}(E\Omega(\omega))$ if and only if $C(q) = TC(p)$, where T is a change-of-basis matrix that preserves the all 1s row of the configuration matrix. As Euclidean equivalence implies that the stress spaces of (G, p) and (G, q) are equal, it suffices to show that $\mathrm{Ker}(E\Omega(\omega))$ consists only of affine images of (G, p).

Chapter 3

Volume Rigidity

In this chapter we introduce the foundational definitions and results in volume rigidity theory. Concepts we make explicit here, such as the equivalent definitions of rigidity and aspects of infinitesimal rigidity, are taken for largely granted for the remainder of this thesis.

3.1 Frameworks and Volume Measurement

Let Σ be a d-dimensional simplicial complex with $n = f(\Sigma)_0$ and with $\Sigma^{(0)}$ bijectively identified with $[n]$.

Definition 3.1.1. A *configuration* of the set of vertices $[n]$ in $\mathbb{R}^d$ is a vector $p = (p(1), \ldots, p(n)) \in (\mathbb{R}^d)^n$ where $p(i)$ is the position of vertex i in $\mathbb{R}^d$ in the configuration, for each $i \in [n]$.
A *framework* of Σ in $\mathbb{R}^d$ is a pair (Σ, p), where p is a configuration of $\Sigma^{(0)}$ in $\mathbb{R}^d$.

If a configuration $p \in (\mathbb{R}^d)^n$ of $[n]$ is such that no $k+1$ points $p(i_1), \ldots, p(i_{k+1})$ lie on any k-dimensional linear subspace of $\mathbb{R}^d$, then we say that p is in *general position*. If (Σ, p) is a framework in $\mathbb{R}^d$ and p is in general position, then we say that (Σ, p) is a *general position framework* in $\mathbb{R}^d$.

If $p \in (\mathbb{R}^d)^n$ is algebraically independent over $\mathbb{Q}$, i.e. $f(p) \neq 0$, for all $f \in \mathbb{Q}[x_1, \ldots, x_{dn}] \setminus \{0\}$, then we say that p is a *generic configuration* in $\mathbb{R}^d$. If (Σ, p) is a framework in $\mathbb{R}^d$ and p is generic, then we say that (Σ, p) is a *generic framework* in $\mathbb{R}^d$.

As we will be considering affine transformations of configurations, it is important to situate them in affine space. In order to do so, we will work with their homogeneous coordinates, defined by adding a 1 entry to each vector $p(i)$ to get $(1, p(i))$ in a configuration p of $[n]$. A convenient way of storing the data of the homogeneous coordinates in of p is in a *configuration matrix*, a $(d+1) \times n$ matrix defined

$$C(p) = \begin{bmatrix} 1 & \ldots & 1 \\ p(1) & \ldots & p(n) \end{bmatrix}.$$

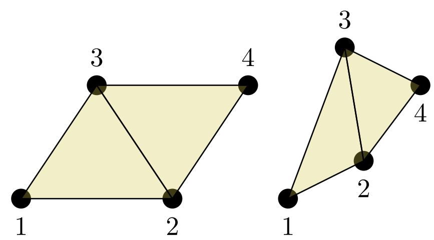

Figure 3.1: Frameworks (Σ, p) (left) and (Σ, q) (right) in $\mathbb{R}^2$.

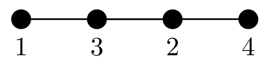

Figure 3.2: A 1-dimensional framework of Σ in $\mathbb{R}^1$

For the sake of clean notation, we will define $C(\sigma, p)$ to be the matrix whose i^{th} column is the σ_i^{th} column of $C(p)$.

Example 3.1.2. Let Σ be the 2-dimensional simplicial complex with maximal simplices $\{123, 234\}$. Figure 3.1 shows two frameworks of Σ in $\mathbb{R}^2$, (Σ, p) on the left and (Σ, q) on the right. If

$$p = ((0,0),(2,0),(1,1.5),(3,1.5)) \text{ and}$$
$$q = ((0,0),(1,0.5),(0.75,2),(1.75,1.5)),$$

then their respective configuration matrices are

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1.5 & 1.5 \end{bmatrix},$$
$$C(q) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0.75 & 1.75 \\ 0 & 0.5 & 2 & 1.5 \end{bmatrix}.$$

◇

Let (Σ, p) be a framework in $\mathbb{R}^d$, the *dimension* of (Σ, p) is the maximum over the dimensions of the affine spans of the maximal simplices of Σ in the framework.

Example 3.1.3. Let Σ be as in example 3.1.2 and consider the configuration

$$p = ((0,0),(2,0),(1,0),(3,0)) \in (\mathbb{R}^2)^4,$$

then (Σ, p) is a 1-dimensional framework in $\mathbb{R}^2$, lying entirely on the subspace $\{y = 0\} \subset \mathbb{R}^2_{(x,y)}$. ◇

Since we want to measure the d-dimensional volume enclosed by maximal simplices in frameworks algebraically, we will introduce a polynomial map to do so.

Definition 3.1.4. Define the *(complete) (d-volume) measurement map* of a set of n points, $[n]$, in $\mathbb{R}^d$ as follows:

$$\alpha_n^d : (\mathbb{R}^d)^n \to \mathbb{R}^{\binom{[n]}{d+1}}; p \mapsto \left(\frac{1}{d!} \det(C(\sigma, p)) : \sigma \in \binom{[n]}{d+1} \right),$$

where, as noted above, $C(\sigma, p)$ is the matrix with columns the homogeneous coordinates of the $p(i)$, for $i \in \sigma$, ordered in increasing order. Then, for any d-dimensional simplicial complex Σ, the *(d-volume) measurement map* of Σ is defined $\alpha_\Sigma^d = (\pi_{\Sigma^{(d)}} \circ \alpha_n^d) : (\mathbb{R}^d)^n \to \mathbb{R}^{\Sigma^{(d)}}$.

Notice that $\alpha_{K_n^d}^d = \alpha_n^d$ by definition.

We have made a choice here to consider signed d-volumes. We will point out where relevant how signed d-volume rigidity results yield analogous results in squared or absolute d-volume rigidity. In general, local behaviour is the same between all three, but globally signed d-volume rigidity is the stronger of the three formulations.

Notice here that, even if Σ is not pure, only the d-dimensional simplices are measured by α_n^d. Therefore, from now on we will assume Σ to be pure d-dimensional, pointing out where non-pure simplicial complexes are considered.

Since we have a polynomial map α_n^d, we may consider its image $\alpha_n^d((\mathbb{R}^d)^n) \subseteq \mathbb{R}^{\binom{[n]}{d+1}}$ as an object to perform real, and subsequently complex, algebraic geometry on.

Definition 3.1.5. Define the *(real) (d-volume) measurement variety* of n points in $\mathbb{R}^d$, denoted M_n^d, to be the Zariski-closure of $\alpha_n^d((\mathbb{R}^d)^n)$ in $\mathbb{R}^{\binom{[n]}{d+1}}$.
The (real) (d-volume) measurement variety of a pure d-dimensional simplicial complex Σ in $\mathbb{R}^d$, denoted M_Σ^d, is the Zariski-closure of $\alpha_\Sigma^d((\mathbb{R}^d)^n)$ in $\mathbb{R}^{\Sigma^{(d)}}$.

We prove some basic properties of M_n^d.

Proposition 3.1.6. *Let M_n^d be the real d-dimensional measurement variety of n points in $\mathbb{R}^d$ and let Σ be a pure d-dimensional simplicial complex.*

1. *The set M_Σ^d is the Zariski-closure of the orthogonal projection of M_n^d onto coordinates indexed by $\Sigma^{(d)}$: $M_\Sigma^d = \overline{\pi_{\Sigma^{(d)}}(M_n^d)}$;*

2. *The sets M_n^d and M_Σ^d are affine real algebraic varieties defined over $\mathbb{Q}$;*

3. *The sets M_n^d and M_Σ^d are real loci of affine cones in $\mathbb{R}^{\binom{[n]}{d+1}}$ and $\mathbb{R}^{\Sigma^{(d)}}$ respectively.*

4. *The dimension of M_n^d is $dn - (d^2 + d - 1)$ and the dimension of M_Σ is less than or equal to $dn - (d^2 + d - 1)$.*

Proof. 1. By definition,

$$M_\Sigma^d = \overline{\alpha_\Sigma^d((\mathbb{R}^d)^n)} = \overline{(\pi_{\Sigma^{(d)}} \circ \alpha_n^d)((\mathbb{R}^d)^n)} = \overline{\pi_{\Sigma^{(d)}}(\alpha_n^d((\mathbb{R}^d)^n))}.$$

2. The sets M_n^d and M_Σ^d are affine real algebraic varieties by definition - they are contained in $\mathbb{R}^{\binom{[n]}{d+1}}$ and $\mathbb{R}^{\Sigma^{(d)}}$ respectively, both affine spaces, and they are defined as the vanishing of real polynomials. Since they are the images of $(\mathbb{R}^d)^n$ under α_n^d and α_Σ^d respectively, both of which are in $\mathbb{Q}[\underline{x}]$, both are defined over $\mathbb{Q}$.

3. Let CM_n^d denote the Zariski closure of $\alpha_n^d((\mathbb{C}^d)^n)$, then we may think of the subset of CM_n^d with the coordinate indexed by $[d+1]$ non-zero as an affine chart of a *projective measurement variety* (this will be formally introduced later). Then CM_n^d is an affine cone, with *vertical axis* indexed by $[d+1]$. We note that M_n^d is the real locus of CM_n^d as $M_n^d = \alpha_n^d((\mathbb{R}^d)^n)$. Finally the M_Σ case follows from the first two parts.

4. This follows from definition 3.3.1.

□

3.2 Definitions of Volume Rigidity

As in the case of Euclidean bar-joint rigidity, there are multiple ways of visualising volume rigidity. In this section, we will introduce four formulations of volume rigidity independently and show that they are equivalent.

3.2.1 Volume Rigidity 1

Our first formulation of volume rigidity is the first in terms of the volume measurement map we defined in section 3.1, and the first to use the equivalence and congruence paradigm of rigidity.

Definition 3.2.1. Let Σ be a d-dimensional simplicial complex, let $n = f(\Sigma)_0$ and let (Σ, p) and (Σ, q) be two frameworks in $\mathbb{R}^d$.
We say that (Σ, p) and (Σ, q) are *equivalent 1* if

$$\alpha_\Sigma^d(p) = \alpha_\Sigma^d(q).$$

We say that (Σ, p) and (Σ, q) are *congruent 1* if

$$\alpha_n^d(p) = \alpha_n^d(q).$$

So two frameworks of Σ are equivalent 1 if their configurations map to the same point on M_Σ^d and congruent 1 if, moreover, they pull back to the same point in M_n^d under $\pi_{\Sigma^{(d)}}$.

Example 3.2.2. Let Σ_1 and Σ_2 be defined by their respective maximal simplices $\Sigma_1^{(2)} = \{123, 234\}$ and $\Sigma_2^{(2)} = \{123, 124, 234\}$ as in fig. 3.3.

Let (Σ_1, p) be a general position framework in $\mathbb{R}^d$ and suppose that $q \in (\mathbb{R}^2)^4$ is a configuration satisfying

$$(q(1), q(2), q(3)) = (p(1), p(2), p(3)).$$

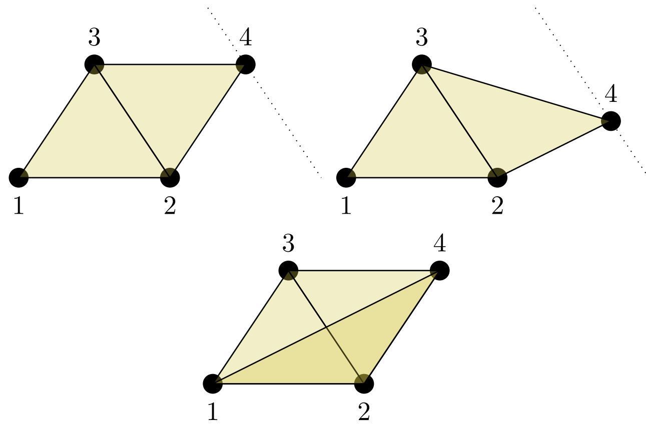

Figure 3.3: Two equivalent frameworks of Σ_1 and a unique (up to congruence) framework of Σ_2.

Then, if $q(4)$ lies on the line parallel to the affine span of $p(2)$ and $p(3)$ in $\mathbb{R}^2$ that passes through $p(4)$, then (Σ_1, q) is equivalent 1 to (Σ_1, p), indeed

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ p(1)_1 & p(2)_1 & p(3)_1 & p(4)_1 \\ p(1)_2 & p(2)_2 & p(3)_2 & p(4)_2 \end{bmatrix},$$

$$C(q) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ p(1)_1 & p(2)_1 & p(3)_1 & q(4)_1 \\ p(1)_2 & p(2)_2 & p(3)_2 & \frac{p(3)_2 - p(2)_2}{p(3)_1 - p(2)_1}(q(4)_1 - p(4)_1) + p(4)_2 \end{bmatrix},$$

and so

$$\frac{1}{2!}\det(C(123, q)) = \frac{1}{2!}\det(C(123, p)) \text{ and}$$
$$\frac{1}{2!}\det(C(234, q)) = \frac{1}{2!}\det(C(234, p)),$$

the second equality follows from expanding the determinant and cancelling terms.

Let (Σ_2, p) be a general position framework in $\mathbb{R}^2$ and suppose that $q \in (\mathbb{R}^2)^4$ is such that $q(1) = p(1)$, $q(2) = p(2)$ and $q(3) = p(3)$. Then, (Σ_2, p) and (Σ_2, q) are equivalent 1 if and only if $q(4)$ lies on the intersection of two lines, parallel to the affine spans of $p(1)$ and $p(2)$ and of $p(2)$ and $p(3)$ respectively, and both passing through $p(4)$. Since p is in general position, $q(4) = p(4)$. Since $q = p$, it is immediate that $\alpha_4^2(q) = \alpha_4^2(p)$, hence (Σ_2, p) and (Σ_2, q) are congruent 1. $\diamond$

Definition 3.2.3. Let (Σ, p) be a framework in $\mathbb{R}^d$. We say that (Σ, p) is *(d-volume) rigid 1* in $\mathbb{R}^d$ if there exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ so that, for all $q \in U$, if (Σ, p) and (Σ, q) are equivalent 1, then (Σ, p) and (Σ, q)

are congruent 1. If (Σ, p) fails to be rigid 1 in $\mathbb{R}^d$, then we say it is *(d-volume) flexible 1* in $\mathbb{R}^2$

The idea behind rigidity 1 is that (Σ, p) is rigid 1 if the only frameworks with the same measurement as (Σ, p) that may be reached by continuously deforming the vertices of (Σ, p) are congruent 1 to it (i.e. these continuous deformations are trivial).

Example 3.2.4. Consider the frameworks from example 3.2.2. The framework (Σ_1, p) is flexible 1 in $\mathbb{R}^2$ since we may choose q with $q(4)$ arbitrarily close, but not equal, to $p(4)$ and obtain a framework (Σ_1, q) that is equivalent 1, but not congruent 1 to (Σ_1, p). The framework (Σ_2, q) is rigid 1 in $\mathbb{R}^2$ since the only choice of q within an open neighbourhood of p so that (Σ_2, p) is equivalent 1 to (Σ_2, q) is $q = p$, and therefore, (Σ_1, q) is congruent 1 to (Σ_2, p). ⋄

3.2.2 Volume Rigidity 2

This is the second formulation of volume rigidity in terms of the volume rigidity measurement map. Here, however, we forego thinking in terms of equivalence and congruence.

Definition 3.2.5. Let (Σ, p) be a framework in $\mathbb{R}^d$. We say (Σ, p) is *(d-volume) rigid 2* in $\mathbb{R}^d$ if there exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ so that

$$(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U = (\alpha^d_n)^{-1}(\alpha^d_n(p)) \cap U.$$

Note that we always have

$$(\alpha^d_n)^{-1}(\alpha^d_n(p)) \cap U \subseteq (\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U,$$

indeed,

$$(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) = (\alpha^d_n)^{-1}(\pi_\Sigma^{-1}(\pi_\Sigma(\alpha^d_n(p)))) \supseteq (\alpha^d_n)^{-1}(\alpha^d_n(p)),$$

for any $p \in (\mathbb{R}^d)^n$. Therefore, it suffices to show the reverse containment to show that a framework is rigid.

Whereas rigidity 1 frames rigidity in the sense of continuously deforming the vertices of a framework in $\mathbb{R}^d$, rigidity 2 observes that this is the same as perturbing the configuration in $(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \subseteq (\mathbb{R}^d)^n$.

Example 3.2.6. Take the frameworks (Σ_1, p) and (Σ_2, p) from example 3.2.2. We note that, for every open neighbourhood U of p in $(\mathbb{R}^2)^4$, there exists

$$q \in (\alpha^2_{\Sigma_1})^{-1}(\alpha^2_{\Sigma_1}(p)) \cap U \setminus (\alpha^2_4)^{-1}(\alpha^2_4(p)) \cap U,$$

but that the two sets are equal in the case of Σ_2. ⋄

3.2.3 Volume Rigidity 3

This third formulation of volume rigidity is in terms of finite flexes of frameworks.

Definition 3.2.7. Let (Σ, p) be a framework in $\mathbb{R}^d$, where $p \in (\mathbb{R}^d)^n$, with $n = f(\Sigma)_0$.
A *finite flex* of (Σ, p) is a continuous map $\gamma : [0,1] \to (\mathbb{R}^d)^n$ so that $\gamma(0) = p$ and

$$\alpha^d_\Sigma(\gamma(t)) = \alpha^d_\Sigma(p),$$

for all $t \in [0,1]$.
A finite flex γ of (Σ, p) is *trivial* if

$$\alpha^d_n(\gamma(t)) = \alpha^d_n(p),$$

for all $t \in [0,1]$.

A finite flex represents a continuous deformation of the vertices of (Σ, p) that preserves the d-volumes of the maximal simplices of (Σ, p). A trivial finite flex represents a continuous deformation of the vertices of (Σ, p) that preserves the d-volumes of all d-tuples of vertices. As we will see towards the end of this section, this corresponds to a *rigid motion* of $\mathbb{R}^d$.

Definition 3.2.8. Let (Σ, p) be a framework in $\mathbb{R}^d$, where $p \in (\mathbb{R}^d)^n$, with $n = f(\Sigma)_0$. We say that (Σ, p) is *(d-volume) rigid 3* if every finite flex of (Σ, p) is trivial.

Rigidity 3 makes explicit the continuous deformation of a configuration in $(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U$ as a path originating at p.

Example 3.2.9. Take again the frameworks (Σ_1, p) and (Σ_2, p) from example 3.2.2. A continuous deformation of the (Σ_1, p) corresponding to sending $p(4)$ to some $q(4)$ within a neighbourhood of $p(4)$ lying on the line defining equivalent 1 frameworks to (Σ_1, p) corresponds to a path in $(\alpha^2_{\Sigma_1})^{-1}(\alpha^2_{\Sigma_1}(p)) \cap U$ originating at p and ending up at $q \notin (\alpha^2_4)^{-1}(\alpha^2_4(p)) \cap U$. Therefore this path is a non-trivial finite flex of (Σ_1, p).

Since $(\alpha^2_{\Sigma_2})^{-1}(\alpha^2_{\Sigma_2}(p)) \cap U = (\alpha^2_4)^{-1}(\alpha^2_4(p)) \cap U$, any finite flex of (Σ_2, p) is trivial. ◇

3.2.4 Volume Rigidity 4

Euclidean bar-joint rigidity has a nice formulation in terms of Euclidean, or isometric, transformations of $\mathbb{R}^d$, volume rigidity 4 is the analogous formulation of d-volume rigidity.

We begin by introducing our transformations.

Definition 3.2.10. An *affine transformation* of $\mathbb{R}^d$ is a map $f : \mathbb{R}^d \to \mathbb{R}^d$ defined

$$f(x) = Ax + b,$$

for all $x \in \mathbb{R}^d$, where $A \in \mathbb{R}^{d\times d}$ and $b \in \mathbb{R}^d$.
The set of affine transformations of $\mathbb{R}^d$ where $\det(A) = 1$ form a group under composition, denoted $\mathcal{SA}(d, \mathbb{R})$ and are called *special affine transformations* of $\mathbb{R}^d$.

Affine transformations may be thought of as transformations of affine space, and therefore may be represented as acting on the homogeneous coordinates of points by writing them in *augment matrix form*. Let $f : x \mapsto Ax + b$, for all $x \in \mathbb{R}^d$, then the augmented matrix form of this transformation is

$$\begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ b + Ax \end{bmatrix}.$$

Through this formulation, we may express the special affine group as the set

$$\left\{ \begin{bmatrix} 1 & 0 \\ b & A \end{bmatrix} \in \mathbb{R}^{(d+1)\times(d+1)} : \det(A) = 1 \right\}$$

under matrix multiplication.

Proposition 3.2.11. *The special affine group $\mathcal{SA}(d, \mathbb{R})$ is isomorphic to the semidirect product $\mathbb{R}^d \rtimes_\phi \mathrm{SL}(d, \mathbb{R})$, where*

$$\phi : \mathrm{SL}(d, \mathbb{R}) \to \mathrm{Aut}(d, \mathbb{R}); A \mapsto (x \mapsto Ax).$$

Proof. First, we examine the group structure of $\mathbb{R}^d \rtimes_\phi \mathrm{SL}(d, \mathbb{R})$. The vector space $\mathbb{R}^d$ is a group under addition, while $\mathrm{SL}(d, \mathbb{R})$ is a group under matrix multiplication, so, for $(b, A), (d, C) \in \mathbb{R}^d \rtimes_\phi \mathrm{SL}(d, \mathbb{R})$,

$$(b, A) * (d, C) = (b + Ad, AC).$$

Now, let $\psi : \mathcal{SA}(d, \mathbb{R}) \to \mathbb{R}^d \rtimes_\phi \mathrm{SL}(d, \mathbb{R})$ be defined

$$\psi(x \mapsto Ax + b) = (b, A).$$

Then, for $f, g \in \mathcal{SA}(d, \mathbb{R})$ with $f(x) = Ax + b$ and $g(x) = Cx + d$

$$\begin{aligned} \psi(f \circ g) &= \psi(x \mapsto ACx + Ad + b) \\ &= (Ad + b, AC) \\ &= (b, A) * (d, C) = \psi(f) * \psi(g), \end{aligned}$$

so ψ is a group homomorphism. Define $\eta : \mathbb{R}^d \rtimes_\phi \mathrm{SL}(d, \mathbb{R}) \to \mathcal{SA}(d, \mathbb{R})$ by

$$\eta(b, A) = f,$$

where $f(x) = Ax + b$. Then, by the same argument as above, η is a group homomorphism. Finally, let $f \in \mathcal{SA}(d, \mathbb{R})$ be defined

$$f(x) = Ax + b$$

and let $(d, C) \in \mathbb{R}^d \rtimes_\phi \mathrm{SL}(d, \mathbb{R})$, then

$$(\eta \circ \psi)(f) = \eta(b, A) = (x \mapsto Ax + b) = f \text{ and}$$
$$(\psi \circ \eta)(d, C) = \eta(x \mapsto Cx + d) = (d, C),$$

so ψ is an isomorphism with inverse η. ☐

Proposition 3.2.12. *The special affine group $\mathcal{SA}(d, \mathbb{R})$ is an algebraic group of dimension $d^2 + d - 1$.*

Proof. By proposition 3.2.11, $\mathcal{SA}(d, \mathbb{R})$ is a group, and by definition, it is a hypersurface as an algebraic variety, being cut out of $\mathbb{R}^{d\times(d+1)}$ by a single equation (specifically, $\det(A) = 1$). Since it is a hypersurface in this ambient space, it has codimension 1, and therefore dimension $d^2 + d - 1$. ☐

We end this discussion of the special affine group by noting that special affine transformations preserve d-volume:

Proposition 3.2.13. *The action of the special affine group $\mathcal{SA}(d, \mathbb{R})$ on $\mathbb{R}^d$ preserves d-volumes enclosed by $(d + 1)$-tuples of points.*

Proof. Suppose that $p \in (\mathbb{R}^d)^{d+1}$ is a configuration of $d + 1$ points in $\mathbb{R}^d$, let q be the configuration

$$q = (f(p(1)), \ldots, f(p(d + 1))),$$

for some $f \in \mathcal{SA}(d, \mathbb{R})$. Then the d-volume of the $(d + 1)$-simplex defined by the points of q in $\mathbb{R}^d$ is

$$\begin{aligned} \alpha^d_{d+1}(q) &= \left(\frac{1}{d!}\det(C(q))\right) \\ &= \left(\frac{1}{d!}\det(fC(p))\right) \\ &= \left(\frac{1}{d!}\det(C(p))\right) = \alpha^d_{d+1}(p), \end{aligned}$$

noting that

$$C(p) = \begin{bmatrix} 1 & \ldots & 1 \\ p(1) & \ldots & p(d+1) \end{bmatrix}, \; C(q) = \begin{bmatrix} 1 & \ldots & 1 \\ q(1) & \ldots & q(d+1) \end{bmatrix}$$

are both square. ☐

Before we continue, we note that a configuration $p \in (\mathbb{R}^d)^n$, for some $n \geq d + 2$, is *affinely dependent* if there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, not all zero, so that

$$\lambda_1 \begin{bmatrix} 1 \\ p(1) \end{bmatrix} + \cdots + \lambda_n \begin{bmatrix} 1 \\ p(n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If p is an affinely dependent configuration and $f \in \mathcal{SA}(d, \mathbb{R})$, then $f(p(j))$ is uniquely determined by the positions of $f(p(i_1)), \ldots, f(p(i_{d+1}))$, for any affinely independent sub-configuration $(p(i_1), \ldots, p(i_{d+1}))$ of p, with $j \notin \{i_1, \ldots, i_{d+1}\}$.

We return to discussing rigidity by again introducing concepts of equivalence and congruence:

Definition 3.2.14. Let (Σ, p) and (Σ, q) be two frameworks in $\mathbb{R}^d$ of the d-dimensional simplicial complex Σ, with $n = f(\Sigma)_0$.
We say that (Σ, p) and (Σ, q) are *equivalent 4* if, for each $\sigma \in \Sigma^{(d)}$, there exists $f_\sigma \in \mathcal{SA}(d, \mathbb{R})$ so that

$$f_\sigma(p(i)) = q(i),$$

for all $i \in \sigma^{(0)}$.
We say that (Σ, p) and (Σ, q) are *congruent 4* if there exists $f \in \mathcal{SA}(d, \mathbb{R})$ so that

$$f(p(i)) = q(i),$$

for all $i \in \Sigma^{(0)}$.

Definition 3.2.15. Let (Σ, p) be a framework in $\mathbb{R}^d$. We say that (Σ, p) is *(d-volume) rigid 4* is there exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ so that, for all $q \in U$, if (Σ, p) and (Σ, q) are equivalent 4, then (Σ, p) and (Σ, q) are congruent 4.

Example 3.2.16. Take (Σ_1, p) and (Σ_2, p) from example 3.2.2.

Suppose that $q \in (\mathbb{R}^2)^4$ is a configuration satisfying

$$(q(1), q(2), q(3)) = (p(1), p(2), p(3)),$$

so the identity transformation in $\mathcal{SA}(2, \mathbb{R})$ sends $(123, p)$ to $(123, q)$. Then, if $q(4)$ lies on the line parallel to the affine span of $p(2)$ and $p(3)$ in $\mathbb{R}^2$ that passes through $p(4)$, then there exists a shear transformation f_{234} sending $(234, p)$ to $(234, q)$. These two transformations do not agree, so (Σ_1, p) is equivalent 4 but not congruent 4 to (Σ_1, q), and since q may lie within an arbitrarily small neighbourhood of p in $(\mathbb{R}^2)^4$, (Σ_1, p) is not rigid 4.

Next, if (Σ_2, p) is equivalent 4 to some (Σ_2, q), where $q \in (\mathbb{R}^2)^4$ is a configuration satisfying

$$(q(1), q(2), q(3)) = (p(1), p(2), p(3)),$$

then the unique affine transformation between this first 2-simplex is the identity in $\mathcal{SA}(2, \mathbb{R})$. Next, there are unique affine transformations $f_{124}, f_{234} \in \mathcal{SA}(2, \mathbb{R})$ sending $(124, p)$ and $(234, p)$ to $(124, q)$ and $(234, q)$ respectively. Since each must be a shear but also must agree on their action on $p(4)$, they are uniquely identified as the identity in $\mathcal{SA}(2, \mathbb{R})$. Therefore, for any $q \in (\mathbb{R}^2)^4$, there is a unique affine transformation sending (Σ_2, p) to (Σ_2, q), namely the one sending $(123, p)$ to $(123, q)$. ◇

3.2.5 Equivalence of Definitions

Here we show the equivalence of definitions 1 to 4 of volume rigidity.

Theorem 3.2.17. *Let (Σ, p) be a d-dimensional framework in $\mathbb{R}^d$, where $p \in (\mathbb{R}^d)^n$, with $n = f(\Sigma)_0$. Assume further that every d-simplex in (Σ, p) has d-dimensional affine span. The following are equivalent:*

1. *The framework (Σ, p) is d-volume rigid 1 in $\mathbb{R}^d$;*
2. *The framework (Σ, p) is d-volume rigid 2 in $\mathbb{R}^d$;*
3. *The framework (Σ, p) is d-volume rigid 3 in $\mathbb{R}^d$;*
4. *The framework (Σ, p) is d-volume rigid 4 in $\mathbb{R}^d$.*

Before proving this theorem, we recall a lemma from analysis:

Lemma 3.2.18 (Curve selection lemma)**.** *[Milnor, 1969, p. 25] Let $S \subseteq \mathbb{R}^D$ be a real algebraic set, $p \in S$, and let U be an open neighbourhood of p in $\mathbb{R}^D$. For any $q \in S \cap U$, there exists a real analytic curve $\gamma : [0, 1] \to S \cap U$ so that $\gamma(0) = p$ and $\gamma(1) = q$.*

Proof of theorem 3.2.17. Suppose that (Σ, p) is rigid 1. Therefore there exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ such that, for all $q \in U$, if

$$\alpha^d_\Sigma(p) = \alpha^d_\Sigma(q),$$

then

$$\alpha^d_n(p) = \alpha^d_n(q).$$

This is equivalent to saying that if

$$q \in (\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)),$$

then

$$q \in (\alpha^d_n)^{-1}(\alpha^d_n(p)).$$

Since this holds for all $q \in U$, this is equivalent to saying that

$$(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U = (\alpha^d_n)^{-1}(\alpha^d_n(p)) \cap U.$$

Therefore rigidity 1 and rigidity 2 are equivalent.

Suppose that (Σ, p) is rigid 2. Therefore, there exists an open neighbourhood U of p in $(\mathbb{R}^d)^n$ such that

$$(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U = (\alpha^d_n)^{-1}(\alpha^d_n(p)) \cap U.$$

Suppose that $\gamma : [0, 1] \to (\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p))$ is a finite flex of (Σ, p), then γ is trivial since $\gamma([0, 1]) \subseteq (\alpha^d_n)^{-1}(\alpha^d_n(p))$.

Suppose that (Σ, p) is rigid 3. Therefore every finite flex of (Σ, p) is trivial. Suppose that $q \in (\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U$, for some open neighbourhood U of p.

By the curve selection lemma, there exists a real analytic curve $\gamma : [0,1] \to (\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U$ such that $\gamma(0) = p$ and $\gamma(1) = q$. By rigidity 3, $\gamma([0,1]) \subseteq (\alpha^d_n)^{-1}(\alpha^d_n(p))$, and since this holds for all $q \in (\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U$, we have that

$$(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p)) \cap U = (\alpha^d_n)^{-1}(\alpha^d_n(p)) \cap U.$$

Therefore rigidity 1, rigidity 2 and rigidity 3 are equivalent. We now show that equivalence 1 and equivalence 4 are equivalent and that congruence 1 and congruence 4 are equivalent, and therefore that rigidity 1 and rigidity 4 are equivalent.

Suppose that (Σ, p) is equivalent 1 to (Σ, q). Therefore, for any $i_1 \dots i_{d+1} \in \Sigma^{(d)}$,

$$\alpha^d_\Sigma(p)_{i_1 \dots i_{d+1}} = \alpha^d_\Sigma(q)_{i_1 \dots i_{d+1}}. \tag{3.1}$$

There exists an affine transformation $f_{i_1 \dots i_{d+1}}$ sending $p(i_j)$ to $q(i_j)$, for each $j \in [d+1]$, and by eq. (3.1), $f \in \mathcal{SA}(d, \mathbb{R})$. Hence (Σ, p) is equivalent 4 to (Σ, q). The converse follows from the definition of $\mathcal{SA}(d, \mathbb{R})$.

Suppose that (Σ, p) is congruent 1 to (Σ, q). Therefore, by the above, there exists $f_{[d+1]} \in \mathcal{SA}(d, \mathbb{R})$ sending $p(i)$ to $q(i)$ for each $i \in [d+1]$. Now, $p(j)$ is affinely dependent on $p(1), \dots, p(d+1)$ for each $d+2 \leq j \leq n$, therefore the position of $f(p(j))$ in $(\Sigma, f \circ p)$ is uniquely defined. Suppose that $f(p(j)) \neq q(j)$, for some $d+2 \leq j \leq n$, then $q(j)$ must not lie on the intersection of the $d+1$ hyperplanes preserving the d-volumes of the d-simplices $([d+1] \setminus \{i\}) \cup \{j\}$ in the framework. However this would contradict

$$\alpha^d_n(p) = \alpha^d_n(q),$$

so we congruence 1 implies congruence 4. The converse again follows from the definition of $\mathcal{SA}(d, \mathbb{R})$. □

To see why it is necessary for all the d-simplices (Σ, p) to have d-dimensional affine span, consider the following example.

Example 3.2.19. Let Σ be the 2-dimensional simplicial complex consisting of just the 2-simplex 123. Suppose that (Σ, p) and (Σ, q) are two 1-dimensional frameworks with configuration matrices

$$C(p) = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ 0 & 0 & 0 \end{bmatrix}, \; C(q) = \begin{bmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ \lambda y_1 & \lambda y_2 & \lambda y_3 \end{bmatrix},$$

where $\lambda \in \mathbb{R}$. Then

$$\alpha^2_\Sigma(p) = \alpha^2_3(p) = \alpha^2_3(q) = \alpha^2_\Sigma(q).$$

Now suppose $f_{123} \in \mathcal{SA}(2, \mathbb{R})$ is the special affine transformation of $\mathbb{R}^2$ taking $(123, p)$ to $(123, q)$ respectively. Then, if

$$f_{123} = \begin{bmatrix} 1 & 0 & 0 \\ b_1 & a_{1,1} & a_{1,2} \\ b_2 & a_{2,1} & a_{2,2} \end{bmatrix},$$

we have that

$$f_{123}C(p) = \begin{bmatrix} 1 & 1 & 1 \\ b_1 + a_{1,1}x_1 & b_1 + a_{1,1}x_2 & b_1 + a_{1,1}x_3 \\ b_2 + a_{2,1}x_1 & b_2 + a_{2,1}x_2 & b_2 + a_{2,1}x_3 \end{bmatrix} = C(q),$$

so $\frac{b_2+a_{2,1}x_i}{b_1+a_{1,1}x_i} = \lambda$, for all $i \in 3$, which, after rearranging, gives $x_i = \frac{\lambda b_1 - b_2}{a_{2,1}-\lambda a_{1,1}}$, for all $i \in 3$, a contradiction. $\diamond$

For the sake of brevity, if (Σ, p) is a framework in $\mathbb{R}^d$, and (Σ, p) has dimension strictly less than d, then we will say that (Σ, p) is *flat*; if for some $\sigma \in \Sigma^{(d)}$, the affine span of σ in (Σ, p) has dimension strictly less than d, then we will say that (Σ, p) is *degenerate*. If, however, every d-simplex in (Σ, p) has d-dimensional affine span, then we will say that (Σ, p) is *non-degenerate*.

We are now in a position to define the d-volume rigidity of non-degenerate frameworks in $\mathbb{R}^d$.

Definition 3.2.20. Let (Σ, p) be a non-degenerate framework in $\mathbb{R}^d$, where $p \in (\mathbb{R}^d)^n$, with $n = f(\Sigma)_0$. We say that (Σ, p) is *(d-volume) rigid* in $\mathbb{R}^d$ if (Σ, p) is d-volume rigid i, for any $i \in [4]$.
If (Σ, p) is not d-volume rigid in $\mathbb{R}^d$, then we say that (Σ, p) is *(d-volume) flexible* in $\mathbb{R}^d$.

We finish this section with a few concepts and results that will be useful later, particularly in chapter 6.

Proposition 3.2.21. *Let Σ_1 and Σ_2 be two simplicial complexes, each on at least $d+1$ vertices, and let (Σ_1, p_1) and (Σ_2, p_2) be two volume rigid frameworks in $\mathbb{R}^d$. If $|\Sigma_1^{(0)} \cap \Sigma_2^{(0)}| \geq d+1$, then the framework (Σ, p), where Σ is the simplicial complex obtained by gluing Σ_1 and Σ_2 together at their common vertices and where $p|_{\Sigma_1^{(0)}} = p_1$ and $p|_{\Sigma_2^{(0)}} = p_2$, is volume rigid in $\mathbb{R}^d$.*

Proof. Suppose that (Σ, p) is equivalent to (Σ, q). By the volume rigidity of (Σ_1, p_1) and (Σ_2, p_2), there exist special affine transformations $f, g \in \mathcal{SA}(d, \mathbb{R})$ so that

$$f(p_1(i)) = q|_{\Sigma_1^{(0)}}(i),$$

for all $i \in \Sigma_1^{(0)}$, and

$$g(p_2(j)) = q|_{\Sigma_2^{(0)}}(j),$$

for all $j \in \Sigma_2^{(0)}$. Since f and g agree on their actions on the at least $d+1$ points in both (Σ_1, p_1) and (Σ_2, p_2), they define a unique special affine transformation h, so that

$$h(p(i)) = q(i),$$

for all $i \in \Sigma^{(0)}$. Therefore, (Σ, p) is volume rigid in $\mathbb{R}^d$. $\square$

Definition 3.2.22. Let Σ be a d-dimensional simplicial complex on n vertices, let $\sigma \in \binom{[n]}{d+1}$ be a $(d+1)$-tuple that is not necessarily a d-simplex in Σ. We say that σ is *generically globally linked* if, for any generic framework (Σ, p) in $\mathbb{R}^d$, if (Σ, p) and (Σ, q) are equivalent, then

$$\alpha_n^d(p)_\sigma = \alpha_n^d(q)_\sigma.$$

Clearly all d-simplices of any d-dimensional simplicial complex are generically globally linked, however, we can have generically globally linked $(d+1)$-tuples that are not d-simplices of Σ.

Example 3.2.23. Let Σ be the 2-dimensional simplicial complex defined by its maximal simplices

$$\Sigma^{(2)} = \{123, 124, 134\}.$$

Then 234 is generically globally linked in Σ. Indeed, suppose that (Σ, p) is a generic framework in $\mathbb{R}^2$ and that (Σ, p) and (Σ, q) are equivalent. Then, $p(4)$ and $q(4)$ are affinely dependent on $\{p(1), p(2), p(3)\}$ and $\{q(1), q(2), q(3)\}$ respectively, therefore, $q(4)$ is uniquely determined. Moreover, by rearranging the equations, we notice that for any configuration $r \in (\mathbb{R}^2)^4$,

$$\alpha_4^2(r)_{234} = \alpha_4^2(r)_{123} - \alpha_4^2(r)_{124} + \alpha_4^2(r)_{134},$$

and therefore

$$\begin{aligned}\alpha_4^2(q)_{234} &= \alpha_4^2(q)_{123} - \alpha_4^2(q)_{124} + \alpha_4^2(q)_{134}\\ &= \alpha_4^2(p)_{123} - \alpha_4^2(p)_{124} + \alpha_4^2(p)_{134}\\ &= \alpha_4^2(p)_{234}.\end{aligned}$$

◇

3.3 Infinitesimal Volume Rigidity

Although we gave several *cryptomorphic* definitions of volume rigidity in section 3.2, in practice, given a framework (Σ, p) in $\mathbb{R}^d$, it is not straightforward to say whether or not (Σ, p) is rigid. Moreover, given a framework, even a generic one, (Σ, p), we can say very little about whether other frameworks of Σ are rigid or not. In this section we introduce *infinitesimal* volume rigidity as a means by which to solve both of these problems, as well as introduce the volume rigidity version of kinematics and statics as ways of approaching rigidity.

The basic idea behind infinitesimal rigidity, both in the Euclidean and volume settings, is to study the first order linear approximation of rigidity (this is why infinitesimal rigidity is sometimes know as first order rigidity). This approximation preserves much of the algebro-geometric picture of rigidity theory and is *simpler* than the non-linear picture, moreover, the polynomial maps are simplified to linear maps, whose matrix properties may be studied.

Given a framework (Σ, p) in $\mathbb{R}^d$, we may linearly approximate its volume rigidity by considering the differential of α^d_Σ evaluated at p, a linear map between the tangent spaces $T_p(\mathbb{R}^d)^n$ and $T_{\alpha^d_\Sigma(p)} M^d_\Sigma$ (we shall define tangent spaces formally later). On open neighbourhoods of p and α^d_Σ, these spaces are isomorphic to $(\mathbb{R}^d)^n$ and M^d_Σ respectively, and continuous paths in the latter two spaces are approximated by linear paths in the former two. Meanwhile, there exist internal *stresses* within a framework that correspond to vectors in $N_{\alpha^d_\Sigma(p)} M^d_\Sigma$, which we will cover later on.

In the rest of this section, we give names to these approximations and show how they can be used to prove rigidity theoretic properties of frameworks and simplicial complexes.

We begin by considering the differential of the measurement map.

Definition 3.3.1. Let $p \in (\mathbb{R}^d)^n$ be a configuration of n points in $\mathbb{R}^d$, the *(complete) (d-volume) rigidity matrix* of p, denoted $R(p)$, is the $\binom{n}{d+1} \times dn$-matrix representing the differential of α^d_n evaluated at p.
Let (Σ, p) be a framework in $\mathbb{R}^d$, the *(d-volume) rigidity matrix* of (Σ, p), denoted $R(\Sigma, p)$, is the $f(\Sigma)_d \times dn$-matrix representing the differential of α^d_Σ evaluated at p.

Proposition 3.3.2. *The rigidity matrix $R(\Sigma, p)$ is the restriction of $R(p)$ to rows indexed by $\Sigma^{(d)}$.*

Proof. Recall that $\alpha^d_\Sigma = \pi_{\Sigma^{(d)}} \circ \alpha^d_n$, so

$$R(\Sigma, p) = \mathrm{d}(\pi_{\Sigma^{(d)}} \circ \alpha^d_n)_p = \mathrm{d}(\pi_{\Sigma^{(d)}})_p R(p),$$

and $\mathrm{d}(\pi_{\Sigma^{(d)}})_p$ is simply the restriction of the identity matrix $I_{\binom{[n]}{d+1}}$ to rows indexed by $\Sigma^{(d)}$. $\square$

The rigidity matrix $R(\Sigma, p)$ represents a map from $T_p(\mathbb{R}^d)^n$ to $T_{\alpha^d_\Sigma(p)} M^d_\Sigma$, therefore we will index its columns by n groups of d-tuples, each one corresponding to the coordinates of a vertex of Σ, and its columns by the d-simplices of Σ. Then the d-tuple given at row σ and column group i is defined to be

$$R(\Sigma, p)^i_\sigma = \begin{cases} \left(\frac{\mathrm{d}}{\mathrm{d}x_{i,j}} \begin{vmatrix} 1 & \dots & 1 \\ x_{\sigma_1,1} & \dots & x_{\sigma_{d+1},1} \\ \vdots & \ddots & \vdots \\ x_{\sigma_1,d} & \dots & x_{\sigma_{d+1},d} \end{vmatrix} : j \in [d+1] \right), & \text{if } i \in \sigma^{(0)}, \\ 0, & \text{else}, \end{cases}$$
$$= \begin{cases} \mathbf{n}(\sigma - i, p), & \text{if } i \in \sigma^{(0)}, \\ 0, & \text{else}, \end{cases}$$

where $\mathbf{n}(\sigma - i, p)$ is the vector in $\mathbb{R}^d$ orthogonal to the affine span of the $(d-1)$-simplex $\sigma - i$ in (Σ, p).

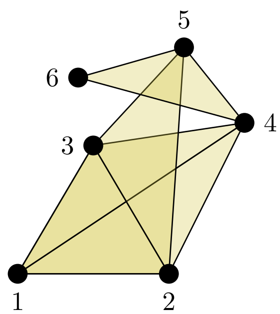

Figure 3.4: The framework (Σ, p) from example 3.3.3.

Example 3.3.3. Consider the framework (Σ, p) in fig. 3.4, where Σ is the 2-dimensional simplicial complex with maximal simplices

$$\Sigma^{(2)} = \{123, 124, 134, 235, 456\}$$

and where p has configuration matrix

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 1 & 3 & 2.2 & 0.8 \\ 0 & 0 & 1.7 & 2 & 3 & 2.6 \end{bmatrix}.$$

Then the rigidity matrix of (Σ, p) is

$$R(\Sigma, p) = \begin{bmatrix} -1.7 & -1 & 1.7 & -1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 1 & 2 & -3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -0.3 & 2 & 0 & 0 & 2 & -3 & -1.7 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.3 & 1.2 & 3 & -0.2 & 0 & 0 & -1.7 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & -1.4 & 0.6 & 2.2 & -1 & -0.8 \end{bmatrix}.$$

◇

In order to proceed, we state a standard result from differential geometry.

Proposition 3.3.4 (Lee [2022])**.** *Let M_1 and M_2 be real manifolds, let $p \in M_1$ and let $F : M_1 \to M_2$ be a differentiable map, then* $\ker(\mathrm{d}F_p) \cong T_p F^{-1}(F(p))$ *and* $\mathrm{Coker}(\mathrm{d}F_p) \cong N_{F(p)} M_2$.

In the statement of proposition 3.3.4, $N_{F(p)} M_2$ corresponds to the normal space to M_2 at $F(p)$.

Moreover, tangent vectors to M_1 at p, i.e. kernel vectors of $\mathrm{d}F_p$, are, by definition, the infinitesimal velocities of continuous paths in M_1 based at p:

$$T_p M_1 = \left\{ \left. \frac{d\gamma}{dt} \right|_{t=0} : \gamma : [0,1] \to M_1 \text{ is a continuous path} \right\}.$$

Now, noting that $R(\Sigma, p)$ represents $\mathrm{d}(\alpha^d_\Sigma)_p$ and that continuous paths in $(\mathbb{R}^d)^n$ based at p are flexes if they remain within $(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(p))$, we arrive at the following definition.

Definition 3.3.5. A vector $\eta \in (\mathbb{R}^d)^n$ is a *infinitesimal flex* of (Σ, p) if it is the infinitesimal velocity of some finite flex γ of (Σ, p).
An infinitesimal flex is *trivial* if it is the infinitesimal velocity of a trivial finite flex.

Which allows us to define infinitesimal volume rigidity analogously to rigidity 3:

Definition 3.3.6. Let (Σ, p) be a framework in $\mathbb{R}^d$. We say that (Σ, p) is *infinitesimally (d-volume) rigid* in $\mathbb{R}^d$ if the only infinitesimal flexes of (Σ, p) are trivial.

If we denote the subspace of $\mathrm{IF}(\Sigma, p) = \ker(R(\Sigma, p))$ consisting of trivial infinitesimal flexes by $\mathrm{IF}(p)$, then $\mathrm{NTIF}(\Sigma, p) = \mathrm{IF}(p)^{\perp} \cap \mathrm{IF}(\Sigma, p)$ is the subspace of non-trivial infinitesimal flexes, and (Σ, p) is infinitesimally rigid if $\ker(R(\Sigma, p)) = \mathrm{IF}(p)$, or alternatively, if $\mathrm{NTIF}(\Sigma, p) = \{0\}$ (note here that 0 is counted as a non-trivial infinitesimal flex for the sake of $\mathrm{NTIF}(\Sigma, p)$ being a linear space, however in practice, we will only refer to non-zero elements of $\mathrm{NTIF}(\Sigma, p)$ when we talk about non-trivial infinitesimal flexes of (Σ, p)). Write $\mathrm{if}(\Sigma, p) = \dim(\mathrm{IF}(\Sigma, p))$, $\mathrm{if}(p) = \dim(\mathrm{IF}(p))$ and $\mathrm{ntif}(\Sigma, p) = \dim(\mathrm{NTIF}(\Sigma, p))$.

This is summed up in the following theorem.

Theorem 3.3.7. *Let (Σ, p) be a framework in $\mathbb{R}^d$, then (Σ, p) is infinitesimally rigid if and only if $\mathrm{rank}(R(\Sigma, p)) = dn - (d^2 + d - 1)$.*

Our proof uses similar tactics to those employed in Asimow and Roth [1978] to show a similar result for Euclidean rigidity.

Proof. Recall from proposition 3.2.12 that $\dim(\mathcal{SA}(d, \mathbb{R})) = d^2 + d - 1$ and from theorem 3.2.17 that for each $q \in (\alpha_n^d)^{-1}(\alpha_n^d(p))$, there exists a unique $f \in \mathcal{SA}(d, \mathbb{R}^d)$ so that

$$f(p(i)) = q(i),$$

for all $i \in \Sigma^{(0)}$.

Suppose that (Σ, p) is infinitesimally rigid, then the only infinitesimal flexes it admits are trivial infinitesimal flexes. Therefore, $\mathrm{Ker}(R(\Sigma, p)) = IF(p)$. By the above discussion, the dimension of $IF(p)$ is $\dim(\mathcal{SA}(d, \mathbb{R})) = d^2 + d - 1$, so by rank-nullity,

$$\mathrm{rank}(R(\Sigma, p)) = dn - (d^2 + d - 1).$$

To the contrary, suppose that

$$\mathrm{rank}(R(\Sigma, p)) = dn - (d^2 + d - 1),$$

i.e. that

$$\mathrm{nullity}(R(\Sigma, p)) = d^2 + d - 1,$$

since $IF(p) \subseteq \mathrm{Ker}(R(\Sigma, p))$ and $\dim(IF(p)) = d^2 + d - 1$, we have that

$$\mathrm{Ker}(R(\Sigma, p)) = IF(p),$$

hence (Σ, p) is infinitesimally rigid. □

A framework being infinitesimally rigid is not equivalent to it being rigid, indeed consider the following example.

Example 3.3.8. Consider the 2-dimensional simplicial complex Σ, defined by its maximal simplices

$$\Sigma^{(2)} = \{123, 125, 134, 145, 256, 346, 456\},$$

and consider the two frameworks (Σ, p) and (Σ, q) in $\mathbb{R}^2$ (Σ, p) and (Σ, q), defined by configurations p and q with configurations matrices

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{11} & \frac{1}{7} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{13} & \frac{1}{17} & \frac{1}{19} \end{bmatrix} \text{ and}$$
$$C(q) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{11} & \frac{1}{7} & \frac{1}{5} \\ 0 & 0 & 1 & \frac{1}{7} & \frac{1}{11} & \frac{1}{5} \end{bmatrix}.$$

Then the two frameworks have respective rigidity matrices

$$R(\Sigma, p) = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{17} & -\frac{6}{7} & \frac{1}{17} & -\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{12}{13} & \frac{1}{11} & 0 & 0 & \frac{1}{13} & -\frac{1}{11} & -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{221} & \frac{4}{77} & 0 & 0 & 0 & 0 & \frac{1}{17} & -\frac{1}{7} & -\frac{1}{13} & \frac{1}{11} & 0 & 0 \\ 0 & 0 & \frac{2}{323} & \frac{2}{35} & 0 & 0 & 0 & 0 & \frac{1}{19} & \frac{4}{5} & -\frac{1}{17} & -\frac{6}{7} \\ 0 & 0 & 0 & 0 & \frac{6}{247} & \frac{6}{55} & -\frac{18}{19} & -\frac{1}{5} & 0 & 0 & \frac{12}{13} & \frac{1}{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{323} & \frac{2}{35} & -\frac{6}{247} & -\frac{6}{55} & \frac{4}{221} & \frac{4}{77} \end{bmatrix},$$
$$R(\Sigma, q) = \begin{bmatrix} -1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{11} & -\frac{6}{7} & \frac{1}{11} & -\frac{1}{7} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{6}{7} & \frac{1}{11} & 0 & 0 & \frac{1}{7} & -\frac{1}{11} & -1 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{77} & \frac{4}{77} & 0 & 0 & 0 & 0 & \frac{1}{11} & -\frac{1}{7} & -\frac{1}{7} & \frac{1}{11} & 0 & 0 \\ 0 & 0 & -\frac{6}{55} & \frac{2}{35} & 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{4}{5} & -\frac{1}{11} & -\frac{6}{7} \\ 0 & 0 & 0 & 0 & -\frac{2}{35} & \frac{6}{55} & -\frac{4}{5} & -\frac{1}{5} & 0 & 0 & \frac{6}{7} & \frac{1}{11} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{6}{55} & \frac{2}{35} & \frac{2}{35} & -\frac{6}{55} & \frac{4}{77} & \frac{4}{77} \end{bmatrix}.$$

Notably,

$$\operatorname{rank}(R(\Sigma, p)) = 7 = 2.6 - 5, \text{ but } \operatorname{rank}(R(\Sigma, q)) = 6 = 2.7 - 6,$$

so (Σ, q) admits a non-trivial infinitesimal flex while (Σ, p) does not. Since, moreover, (Σ, p) and (Σ, q) are both rigid frameworks in $\mathbb{R}^2$ (i.e. neither admits a non-trivial finite flex), the non-trivial infinitesimal flex admitted by (Σ, q) may be explained by the symmetry of the framework inducing a dependence in the columns of the rigidity matrix. ◇

In order for a rigidity matrix of some framework of a simplicial complex to be rank-deficient whilst the framework is rigid, then there must be a dependency induced by the non-zero entries of the matrix, which are obtained from the geometry of the framework. That is to say, infinitesimally flexible rigid, but locally rigid, frameworks have some geometric dependencies within their configuration vectors.

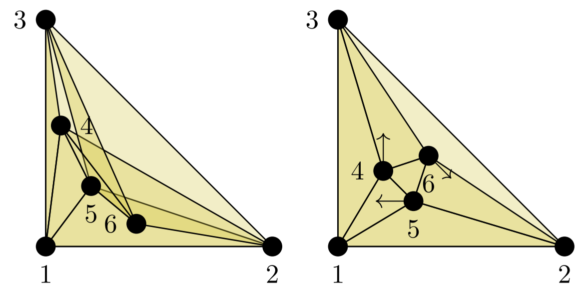

Figure 3.5: The frameworks (Σ, q) (left) and (Σ, p) (right), with the non-trivial infinitesimal flex of (Σ, p) denoted by arrows.

Theorem 3.3.9. *The maximum rank of $R(\Sigma, p)$ as p varies over $(\mathbb{R}^d)^n$ is achieved at every generic point of $(\mathbb{R}^d)^n$.*

Proof. Write $r = \max\{\text{rank}(R(\Sigma, p)) : p \in (\mathbb{R}^d)^n\}$, then $\text{rank}(R(\Sigma, q)) < r$ if and only if every $r \times r$ minor of $R(\Sigma, q)$ vanishes. These vanishing-of-minor equations are polynomials with coefficients in $\mathbb{Q}$ evaluated at configurations, indeed the vanishing of a minor of any matrix is a polynomial with coefficients in $\mathbb{Q}$ evaluated at some entries of the matrix and the entries of the rigidity matrix are polynomials with coefficients in $\mathbb{Q}$ evaluated at configurations. Therefore, $\text{rank}(R(\Sigma, q)) < r$ if and only if q lies in the vanishing of these polynomials, as r is achieved by some configuration, these polynomials are not all zero, and hence q is not generic. □

As a corollary, infinitesimal rigidity is a generic property of simplicial complexes.

Corollary 3.3.10. *If (Σ, p) is infinitesimally rigid at some $p \in (\mathbb{R}^d)^n$, then (Σ, q) is infinitesimally rigid for every generic $q \in (\mathbb{R}^d)^n$.*

Proof. Suppose that $\text{rank}(R(\Sigma, p)) = dn - (d^2 + d - 1)$, then, for any generic $q \in (\mathbb{R}^d)^n$, $\text{rank}(R(\Sigma, q)) \geq \text{rank}(R(\Sigma, p))$, and so $\text{rank}(R(\Sigma, q)) = dn - (d^2 + d - 1)$. □

Finally, we see that rigidity itself is a generic property of simplicial complexes.

Theorem 3.3.11. *If (Σ, p) is rigid in $\mathbb{R}^d$, for some generic $p \in (\mathbb{R}^d)^n$, then (Σ, q) is rigid in $\mathbb{R}^d$ for every generic $q \in (\mathbb{R}^d)^n$*

Proof. Suppose that (Σ, p) is rigid, then (Σ, p) is infinitesimally rigid, so every generic framework of Σ is infinitesimally rigid. Suppose that (Σ, q) is an infinitesimally rigid generic framework of Σ, then the neighbourhood of $\alpha^d_\Sigma(q)$ in M^d_Σ is $(dn - (d^2 + d - 1))$-dimensional, and hence the neighbourhood of q in $(\alpha^d_\Sigma)^{-1}(\alpha^d_\Sigma(q))$ is $(d^2 + d - 1)$-dimensional, so (Σ, q) is rigid in $\mathbb{R}^d$. □

Having studied the infinitesimal rigidity of complexes in terms of infinitesimal flexes, we now turn our attention to the dual setting, that of *equilibrium stresses*. Our prior discussion has made extensive use of the rank-nullity theorem applied to the rigidity matrices of generic frameworks, we will now develop the theory arising from the corank-conullity theorem of these same matrices.

Physically, equilibrium stresses are easy to intuit in the Euclidean case - when continuously deforming a framework in $\mathbb{R}^d$, we may induce internal stresses in some bars (edges), these correspond to sub-frameworks that are *overconstrained*, and when enough force is applied these bars will become stressed and eventually snap. In the mathematical world, edges of a framework do not snap, but these equilibrium stresses may increase greatly when external forces are applied.

In the d-volume case, we may similarly interpret equilibrium stresses to represent resistance to deformation, physical analogies are less straightforward in this setting, however, so we will content ourselves with thinking abstractly.

Definition 3.3.12. Let (Σ, p) be a framework in $\mathbb{R}^d$, an *equilibrium stress* of (Σ, p) is a vector $\omega \in \mathbb{R}^{\Sigma^{(d)}}$, satisfying the following balancing condition:

$$\sum_{\sigma \in \mathrm{Star}_\Sigma(i)} \mathbf{n}(\sigma - i, p)\omega_\sigma = 0 \in \mathbb{R}^d,$$

for all $i \in \Sigma^{(0)}$. Denote by $\mathrm{S}(\Sigma, p)$ the space of equilibrium stresses of (Σ, p) and let $\mathrm{s}(\Sigma, p) = \dim(\mathrm{S}(\Sigma, p))$.

We notice that the above system equation of equations is equivalent to the system $R(\Sigma, p)\omega^t = 0$, i.e. the space of equilibrium stresses of (Σ, p) is precisely the cokernel of $R(\Sigma, p)$. Then, by proposition 3.3.4, these equilibrium stresses are normal vectors to M^d_Σ at $\alpha^d_\Sigma(p)$.

This gives rise to the following proposition, an immediate consequence of the corank-conullity theorem, the dual to the rank-nullity theorem:

Proposition 3.3.13. *Let (Σ, p) be a framework in $\mathbb{R}^d$, where $n = f(\Sigma)_0$, then (Σ, p) is infinitesimally rigid in $\mathbb{R}^d$ if and only if*

$$f(\Sigma)_d - \mathrm{conullity}(R(\Sigma, p)) = dn - (d^2 + d - 1).$$

Moreover, Σ is generically rigid in $\mathbb{R}^d$ if, for some $p \in (\mathbb{R}^d)^n$ generic,

$$f(\Sigma)_d - \mathrm{conullity}(R(\Sigma, p)) = dn - (d^2 + d - 1).$$

We will continue our discussion of stresses in section 3.4, however before doing so, we give an theorem encompassing both the geometry of infinitesimal rigidity and the combinatorics of simplicial complexes.

Theorem 3.3.14 (Index theorem)**.** *Let Σ be a d-dimensional simplex on n vertices, then for any framework (Σ, p) in $\mathbb{R}^d$,*

$$dn - (d^2 + d - 1) - f(\Sigma)_d = \mathrm{ntif}(\Sigma, p) - \mathrm{s}(\Sigma, p).$$

Proof. Note first that $\operatorname{rank}(R(\Sigma,p)) = \operatorname{corank}(R(\Sigma,p))$, so

$$\begin{aligned}\operatorname{rank}(R(\Sigma,p)) &= dn - \operatorname{if}(\Sigma,p)\\ &= dn - \operatorname{if}(p) - \operatorname{ntif}(\Sigma,p)\\ &= dn - (d^2+d-1) - \operatorname{ntif}(\Sigma,p),\\ \operatorname{corank}(R(\Sigma,p)) &= f(\Sigma)_d - \operatorname{s}(\Sigma,p).\end{aligned}$$

Combining these two equations yields

$$dn - (d^2+d-1) - f(\Sigma)_d = \operatorname{ntif}(\Sigma,p) - \operatorname{s}(\Sigma,p).$$

□

We end with two classes of simplicial complex arising as consequences of the index theorem.

Definition 3.3.15. Let Σ be a d-dimensional simplicial complex on n vertices. We say that Σ is *minimally (d-volume) rigid* in $\mathbb{R}^d$ if Σ is rigid in $\mathbb{R}^d$ and $\Sigma - \sigma$ is flexible, for any $\sigma \in \Sigma^{(d)}$, or equivalently, if Σ is rigid in $\mathbb{R}^d$ and $f(\Sigma)_d = dn - (d^2+d-1)$.
We say that Σ is *redundantly (d-volume) rigid* in $\mathbb{R}^d$ if Σ is rigid in $\mathbb{R}^d$ and $\Sigma-\sigma$ is rigid in $\mathbb{R}^d$, for any $\sigma \in \Sigma^{(d)}$.

Proposition 3.3.16. *The two definitions of minimal rigidity in $\mathbb{R}^d$ given in definition 3.3.15 are equivalent.*

Proof. Clearly, if $f(\Sigma)_d = dn-(d^2+d-1)$, then $\Sigma-\sigma$ will be flexible, for any $\sigma \in \Sigma^{(d)}$.

Suppose that Σ is rigid, but $\Sigma-\sigma$ is flexible for every $\sigma \in \Sigma^{(d)}$. Suppose, for the sake of contradiction, that $f(\Sigma)_d = dn-(d^2+d-1)$, then, for some generic $p \in (\mathbb{R}^d)^n$, some row, indexed by $\tau \in \Sigma^{(d)}$, say, of the rigidity matrix $R(\Sigma,p)$ is linearly dependent on some subset of the others. Therefore, removing this row, which corresponds to removing τ from Σ, yields a full-rank matrix. Therefore

$$\operatorname{rank}(R(\Sigma,p))_{\Sigma^{(d)}\setminus\{\tau\}} = \operatorname{rank}(R(\Sigma-\tau,p)) = dn-(d^2+d-1),$$

but $\Sigma-\tau$ is flexible. This contradicts theorem 3.3.9. □

Minimally rigid simplicial complexes are necessarily unstressed (i.e. they generically admit no non-zero infinitesimal flexes), indeed, for $p \in (\mathbb{R}^d)^n$ generic,

$$dn-(d^2+d-1)-(dn-(d^2+d-1)) = 0 - \operatorname{s}(\Sigma,p) = 0 \implies \operatorname{s}(\Sigma,p) = 0,$$

and redundantly rigidity complexes are necessarily stressed (i.e. they generically admit a non-zero subspace of stresses), indeed, for $p \in (\mathbb{R}^d)^n$ generic,

$$dn-(d^2+d-1)-f(\Sigma)_d = 0 - \operatorname{s}(\Sigma,p) < 0 \implies \operatorname{s}(\Sigma,p) > 0.$$

3.4 Further Discussion on Stresses

In this short section, we draw attention to a subspace of topologically derived equilibrium stresses present over all frameworks of a given d-dimensional simplicial complex in $\mathbb{R}^d$.

Definition 3.4.1. Let Σ be a d-dimensional simplicial complex and let (Σ, p) be a framework in $\mathbb{R}^d$. An equilibrium stress ω of (Σ, p) is a *topological stress* of Σ if ω lists the coefficients of a homological d-cycle of Σ. A stress of (Σ, p) that is not a topological stress of Σ is *non-topological stress* of (Σ, p).

Note the minor subtlety in the above definition - topological stresses are invariants of the simplicial complex, whilst non-topological stresses are determined by specific frameworks.

The dimension of the space of stresses of a simplicial complex is therefore at least the sum of the dimension of the space of topological stresses (which is the d^{th} Betti number of our complex) and the dimension of the space of non-topological stresses of a generic framework of our complex.

Proposition 3.4.2. *Let Σ be a d-dimensional simplicial complex, suppose that $\beta_d(\Sigma) > 0$ and that ω lists the coefficients of some homological d-cycle in Σ. Then $\omega \in \mathrm{Stress}(\Sigma, p)$, for any framework (Σ, p) in $\mathbb{R}^d$.*

Proof. Suppose that ω lists the coefficients of a d-circuit, i.e. a d-cycle, no subset of which forms a d-cycle, then

$$\partial_d \left(\sum_{\sigma \in \Sigma^{(d)}} \omega_\sigma \sigma \right) = \sum_{\sigma \in \Sigma^{(d)}} \omega_\sigma \sum_{\tau \in \sigma^{(d-1)}} \mathrm{sign}(\tau, \sigma)\tau = 0.$$

We rearrange the right hand side of the first equality above

$$\begin{aligned} \sum_{\sigma \in \Sigma^{(d)}} \omega_\sigma \sum_{\tau \in \sigma^{(d-1)}} \mathrm{sign}(\tau, \sigma)\tau &= \sum_{\tau \in \Sigma^{(d-1)}} \tau \sum_{\sigma^{(d-1)} \ni \tau} \mathrm{sign}(\tau, \sigma)\omega_\sigma \\ &= \sum_{i \in \Sigma^{(0)}} \sum_{\tau i \in \Sigma^{(d)}} \tau \, \mathrm{sign}(\tau, \tau i)\omega_{\tau i} \end{aligned}$$

where τi denotes the d-simplex consisting of the d elements of τ as well as i. As $\mathrm{Lk}_\Sigma(i)$ is homeomorphic to $\mathbb{S}^{d-1}$, a suitable choice of $\omega_{\tau i} \in \{-1, 1\}$, for each i contained in the support of ω, will yield

$$\sum_{i \in \Sigma^{(0)}} 0 = 0,$$

and so such an ω must be the unique list of coefficients (up to scalar multiplication).

Now, let (Σ, p) be a framework in $\mathbb{R}^d$ and consider $\omega^t R(\Sigma, p)$, for the column group corresponding to vertex i,

$$\omega^t R(\Sigma, p)^i = \sum_{\sigma^{(0)} \ni i} \omega_\sigma \, \mathrm{sign}(i, \sigma)\mathbf{n}(\sigma \setminus i, p) = \sum_{\tau i \in \Sigma^{(d)}} \mathbf{n}(\tau, p) \, \mathrm{sign}(\tau, \tau i)\omega_{\tau i}.$$

This equals $0 \in \mathbb{R}^d$, since, in each cooordinate slice, we have a sum of the same form as in the prior equation (going from a symbolic τ to a numerical one). $\square$

3.5 Pinning and the Configuration Space of a Framework

Pinning is a well-known method in Euclidean rigidity of considering deformations of graph frameworks in $\mathbb{R}^d$ without having to account for the rigid motions of $\mathbb{R}^d$ (i.e. the Euclidean transformations of $\mathbb{R}^d$). Considering pinned frameworks also simplifies the analysis of the fibre of the framework's measurement, as we effectively mod out the space of configurations yielding congruent frameworks to obtain a smaller *configuration space*. In this section, we define pinning and configuration spaces in the volume rigidity setting, note some proof-of-concept results and their relation to the other spaces we have considered so far, as well as give examples of its usefulness.

Let Σ be a d-dimensional simplicial complex and assume $[d+1] \in \Sigma^{(d)}$ (this is true for all Σ with at least 1 d-simplex after an appropriate relabelling). Let (Σ, p) be a framework of $\mathbb{R}^d$ so that $\alpha^d_\Sigma(p)_{[d+1]} \neq 0$, then we may *pin* (Σ, p) to $(\Sigma, \overline{p})$ by applying the unique special affine transformation $f \in \mathcal{SA}(d, \mathbb{R})$ defined in terms of its action on the first $[d+1]$ vertices of Σ as

$$\overline{p}(i) = f(p(i)) = \begin{cases} e_i, & \text{if } 1 \leq i \leq d; \\ (0, \dots, 0, d!\alpha^d_\Sigma(p)_{[d+1]}), & \text{if } i = d+1. \end{cases}$$

In fact, we may choose any simplex in $\mathbb{R}^d$ of the same volume as $[d+1]$ in (Σ, p) to pin $[d+1]$ to, however, we will see that choosing this scaling of the unit simplex proves convenient in practice. The configuration matrix of $(\Sigma, \overline{p})$ is therefore

$$C(\overline{p}) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & \overline{p}(d+2)_1 & \dots & \overline{p}(n)_1 \\ 0 & 0 & 1 & \dots & 0 & 0 & \overline{p}(d+2)_2 & \dots & \overline{p}(n)_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \overline{p}(d+2)_{d-1} & \dots & \overline{p}(n)_{d-1} \\ 0 & 0 & 0 & \dots & 0 & d!\alpha^d_\Sigma(p)_{[d+1]} & \overline{p}(d+2)_d & \dots & \overline{p}(n)_d \end{bmatrix}$$

Call $(\Sigma, \overline{p})$ the *pinned framework* or *pinning* of (Σ, p).

Any two congruent frameworks have the same pinned framework in $\mathbb{R}^d$. Indeed, suppose that (Σ, p) and (Σ, q) are congruent, with $f \in \mathcal{SA}(d, \mathbb{R})$ so that

$$f(p(i)) = q(i),$$

for all $i \in \Sigma^{(0)}$. If $g \in \mathcal{SA}(d, \mathbb{R})$ sends (Σ, p) to $(\Sigma, \overline{p})$, then $(g \circ f^{-1}) \in \mathcal{SA}(d, \mathbb{R})$ sends (Σ, q) to $(\Sigma, \overline{p})$ - a corollary to this is that (Σ, p) and (Σ, q) are congruent if and only if $(\Sigma, \overline{p}) = (\Sigma, \overline{q})$.

Consider the fibre $(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p))$ consisting of all $q \in (\mathbb{R}^d)^n$ yielding frameworks equivalent to (Σ, p). Lying within $(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d)(p)$, we always find the space of configurations $q \in (\mathbb{R}^d)^n$ yielding frameworks congruent to (Σ, p), $(\alpha_n^d)^{-1}(\alpha_n^d(p))$.

Proposition 3.5.1. *The quotient space*

$$\mathcal{C}(\Sigma, p) = {}^{(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p))}\Big/_{(\alpha_n^d)^{-1}(\alpha_n^d(p))}$$

is homeomorphic to the space of pinned frameworks equivalent to $(\Sigma, \overline{p})$.

Proof. Note that elements of $\mathcal{C}(\Sigma, p)$ are equivalence classes of configurations in $\mathbb{R}^d$, with equivalence defined by congruence of the frameworks of Σ yielded by these configurations. Furthermore, if $[q] \in \mathcal{C}(\Sigma, p)$, then $(\Sigma, \overline{q})$ is an equivalence class representative of $[q]$, by the above discussion. Therefore, $\mathcal{C}(\Sigma, p)$ is homeomorphic to the space $\mathrm{PinRep}(\Sigma, p)$ of equivalence class representatives that are pinnings, by the mutually inverse maps

$$F : \mathcal{C}(\Sigma, p) \to \mathrm{PinRep}(\Sigma, p); [q] \mapsto \overline{q} \text{ and}$$
$$G : \mathrm{PinRep}(\Sigma, p) \to \mathcal{C}(\Sigma, p); \overline{q} \mapsto [q] = \{r \in (\mathbb{R}^d)^n : (\Sigma, \overline{r}) = (\Sigma, \overline{q})\}.$$

Both of these maps are continuous, with respect to the Euclidean topology on $\mathrm{PinRep}(\Sigma, p)$ and the quotiented Euclidean topology on $\mathcal{C}(\Sigma, p)$, so the two spaces are homeomorphic. □

Definition 3.5.2. Call the space $\mathcal{C}(\Sigma, p)$, as defined in proposition 3.5.1, the *configuration space* of (Σ, p).

The fibre of every measurement of a framework is higher-dimensional, testing for rigidity comes down to determining whether or not the dimension is greater than $dn - (d^2 + d - 1)$. However, not every configuration space has dimension greater than zero, this occurs precisely when the framework fails to be rigid. Therefore, determining rigidity becomes the simpler problem of determining whether or not the dimension is 0. In fact, by theorem 2.1.4 it is an even simpler problem than that.

Proposition 3.5.3. *Let* (Σ, p) *be a framework in* $\mathbb{R}^d$*, then* (Σ, p) *is rigid in* $\mathbb{R}^d$ *if and only if* $\mathcal{C}(\Sigma, p)$ *is 0-dimensional, and therefore if and only if* $\mathcal{C}(\Sigma, p)$ *is finite.*

Proof. Suppose that (Σ, p) is rigid in $\mathbb{R}^d$, then, by rigidity 2,

$$(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) \cap U = (\alpha_n^d)^{-1}(\alpha_n^d(p)) \cap U,$$

for an open neighbourhood U of p in $(\mathbb{R}^d)^n$. Suppose that $q \in (\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) \setminus U$ is generic, then by the genericity of rigidity,

$$(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma(q)) \cap V = (\alpha_n^d)^{-1}(\alpha_n^d(q)) \cap V,$$

for any open neighbourhood V of q. Therefore, nowhere do we have a generic point in the fibre of $\alpha^d_\Sigma(p)$ whose neighbourhood has dimension greater than $d^2+d-1=\dim((\alpha^d_n)^{-1}(\alpha^d_n(p)))$, so

$$\dim(\mathcal{C}(\Sigma,p))=0.$$

Suppose that $\mathcal{C}(\Sigma,p)$ is 0-dimensional, then by theorem 2.1.4, $|\mathcal{C}(\Sigma,p)|<\infty$.

Suppose that $|\mathcal{C}(\Sigma,p)|<\infty$, then there are finitely many frameworks (Σ,q) so that (Σ,p) is equivalent to (Σ,q), but (Σ,p) is not congruent to (Σ,q). Therefore, (Σ,p) is rigid, indeed if (Σ,p) were flexible there would be a continuum of configurations $\gamma((0,1])$ so that (Σ,p) is equivalent, but not congruent to each $(\Sigma,\gamma(t))$ as $t\in(0,1]$ varies. □

Given $p\in(\mathbb{R}^d)^n$, the measurements $\alpha^d_\Sigma(p)$ and $\alpha^d_\Sigma(\overline{p})$ are equal, so we have $M^d_\Sigma=\overline{\alpha^d_\Sigma(\{\overline{q}:q\in(\mathbb{R}^d)^n\})}$. This sets us up to consider the projective case. Recall that the projectivisation of the measurement variety, denoted $\mathbb{P}(M^d_n)$, identifies measurements that are scale multiples of each other. In terms of frameworks, this is equivalent to identifying the measurements of frameworks that are uniform non-zero scalings of each other (indeed, if

$$\alpha^d_\Sigma(q)=\lambda\alpha^d_\Sigma(p),$$

for some $\lambda\neq 0$, then $p=\lambda^{\frac{1}{d}}q$, as vectors in $(\mathbb{R}^d)^n$). Therefore, given a point $y\in\mathbb{P}(M^d_\Sigma)$ (i.e. in the projective measurement variety, which we will introduce formally in chapter 4), we have that $y=\{\alpha^d_\Sigma(\lambda p):\lambda\neq 0\}$, for some $p\in(\mathbb{R}^d)^n$, for each $\lambda\neq 0$,

$$\overline{\lambda p}(d+1)_d=\lambda\overline{p}(d+1)_d.$$

We may choose as the representative of this equivalence class the pinning of λp, where $\lambda=\frac{1}{d!\alpha^d_\Sigma(p)}$, i.e. the pinned configuration whose first $d+1$ vertices form the unit simplex in $\mathbb{R}^d$. Call this pinned framework the *standard pinning* of (Σ,p).

Proposition 3.5.4. *The rigidity properties of* (Σ,p) *and its standard pinning* $(\Sigma,\overline{p})$ *are identical.*

Proof. The configurations p and $\overline{p}$ are related by a unique non-degenerate affine transformation $f\in\mathcal{A}(d,\mathbb{R})$ so that $f=gh$ where $h\in\mathcal{SA}(d,\mathbb{R})$ and $h(p)$ is the configuration of the pinning of (Σ,p) and g is a uniform scaling of $\mathbb{R}^d$ by $\frac{1}{d!\alpha^d_\Sigma(p)}$ along the x_d-axis. Working through volume rigidity 4, every special transformation may be composed by either f or f^{-1}, with respect to this unique factorisation, to go uniquely between these settings. □

Example 3.5.5. Let Σ be the boundary of the octahedron with a face removed, a 2-dimensional simplicial complex with maximal simplices

$$\Sigma^{(2)}=\{123,126,156,234,246,345,456\}.$$

Let (Σ, p) be the framework in $\mathbb{R}^2$ with configuration matrix

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1.5 & 0.8 & 0.1 & -0.5 \\ 0 & 0 & 1.7 & 3.2 & 2.8 & 1.3 \end{bmatrix}.$$

In order to pin (Σ, p) to $(\Sigma, \tilde{p})$, we transform the whole framework by the shear $f \in \mathcal{SA}(2, \mathbb{R})$ defined uniquely by

$$(f(0,0), f(1,0), f(1.5,1.7)) = ((0,0),(1,0),(0,1.7))$$

The resulting framework has configuration matrix

$$C(\tilde{p}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2.03\ldots & -2.37\ldots & -1.65\ldots \\ 0 & 0 & 1.7 & 3.2 & 2.8 & 1.3 \end{bmatrix}.$$

Finally, in order to standard pin (Σ, p), we scale $(\Sigma, \tilde{p})$ by $\frac{1}{1.7}$ to get the framework $(\Sigma, \overline{p})$ with configuration matrix

$$C(\overline{p}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -2.03\ldots & -2.37\ldots & -1.65\ldots \\ 0 & 0 & 1 & 1.88\ldots & 1.65\ldots & 0.76\ldots \end{bmatrix}.$$

Next, we will take a look at the configuration space of each of (Σ, p), $(\Sigma, \tilde{p})$ and $(\Sigma, \overline{p})$. Then

$$\alpha^2_\Sigma(\tilde{p}) = \alpha^2_\Sigma(p) = (0.85, 0.65, 0.765, 0.97, 1.6, 0.665, 0.405)$$

and $\alpha^2_6(\tilde{p}) = \alpha^2_6(p) \in \mathbb{R}^{20}$,

$$\alpha^2_\Sigma(\overline{p}) = (0.5, 0.38\ldots, 0.45, 0.57\ldots, 0.94\ldots, 0.39\ldots, 0.23\ldots)$$

and $\alpha^2_6(\overline{p}) \in \mathbb{R}^{20}$. Then

$$\mathcal{C}(\Sigma, \tilde{p}) = \mathcal{C}(\Sigma, p) = \left\{ [q] \in {}^{(\mathbb{R}^2)^6}\!\big/_{\alpha^2_6(p)} : \alpha^2_\Sigma(r) = \alpha^2_\Sigma(p),\ \forall r \in [q] \right\},$$
$$\mathcal{C}(\Sigma, \overline{p}) = \left\{ [q] \in {}^{(\mathbb{R}^2)^6}\!\big/_{\alpha^2_6(\overline{p})} : \alpha^2_\Sigma(r) = \alpha^2_\Sigma(\overline{p}),\ \forall r \in [q] \right\}.$$

We notice that these form affine semi-algebraic subvarieties of ${}^{(\mathbb{R}^2)^6}\!\big/_{\alpha^2_6(p)}$ and ${}^{(\mathbb{R}^2)^6}\!\big/_{\alpha^2_6(\overline{p})}$ respectively. By computing them in `SageMath`, we determine that they are finite, and hence all three frameworks of Σ are rigid, as we expect. ◇

3.6 The Volume Rigidity Matroid

In this section we introduce the volume rigidity matroid. We introduced matroids in section 2.2 and the Euclidean rigidity matroid in section 2.4.2. In this

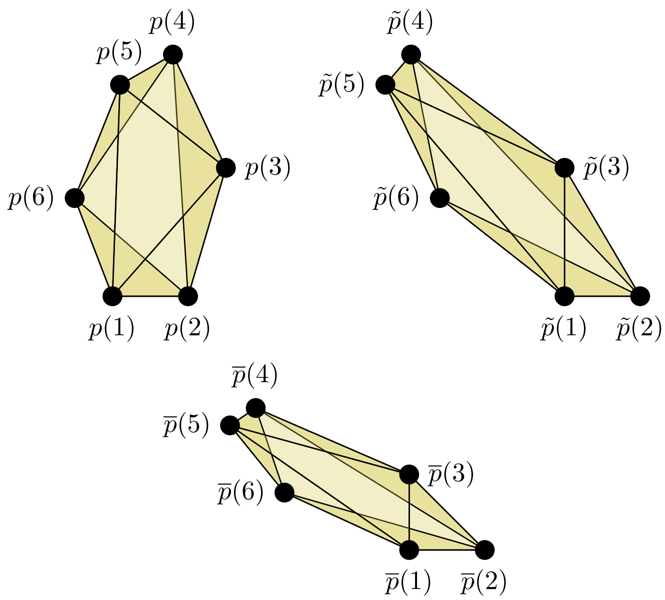

Figure 3.6: The top left framework is unpinned and in general position, the pinning on the top right preserves all 2-volumes, meanwhile the standard pinning below does not preserve 2-volumes. All have the same configuration space (up to homeomorphism).

Section, we will define geometric volume rigidity matroids that are inspired by the observed behaviour of frameworks. In chapter 4, we spend time identifying the latter matroids as well as some of its properties.

We will refer to d-dimensional simplicial complexes and their sets of maximal simplices as dependent, independent, circuits, etc. interchangeably throughout this thesis.

Definition 3.6.1. Let $p \in (\mathbb{R}^d)^n$ be a configuration of n points in $\mathbb{R}^d$. The *infinitesimal (volume) rigidity matroid*, denoted $\mathcal{F}(p)$, of p in $\mathbb{R}^d$ is the linear matroid of $R(p)$. When p is generic, call the infinitesimal volume rigidity matroid the *generic (volume) rigidity matroid*, denoted $\mathcal{R}_n^d$.

We need to show that our matroid $\mathcal{R}_n^d$ is well-defined.

Lemma 3.6.2. *Suppose that p and q are two generic configurations of $[n]$ in $\mathbb{R}^d$, then $\mathcal{F}(p) = \mathcal{F}(q)$.*

Proof. The ground sets of $\mathcal{F}(p)$ and $\mathcal{F}(q)$ are both $\binom{[n]}{d+1}$. Let Σ be a d-dimensional simplicial complex on n vertices, denote the rank of $\Sigma^{(d)} \subseteq \binom{[n]}{d+1}$ in $\mathcal{F}(p)$ and $\mathcal{F}(q)$ by $r_p(\Sigma^{(d)})$ and $r_q(\Sigma^{(d)})$ respectively. Then

$$r_p = \operatorname{rank}(R(\Sigma, p)),$$
$$r_q = \operatorname{rank}(R(\Sigma, q)),$$

so by the same argument as in the proof of theorem 3.3.9, $r_p = r_q$. □

Through $\mathcal{R}_n^d$, we may link the combinatorial and geometric definitions of rigidity.

Proposition 3.6.3. *Let Σ be a d-dimensional simplicial complex on n vertices, then Σ is rigid in $\mathbb{R}^d$ if and only if Σ contains, as a sub-complex, a basis of $\mathcal{R}_n^d$.*

Proof. This proof is immediate from the definition since we notice that $\Sigma^{(d)}$ indexes a basis of $\operatorname{RowSpace}(R(p))$, for $p \in (\mathbb{R}^d)^n$ generic, if and only if (Σ, p), and therefore Σ, is rigid in $\mathbb{R}^d$. □

We may also define the *rank* function of $\mathcal{R}_n^d$ as

$$\operatorname{rank} : \binom{[n]}{d+1} \to \mathbb{N}; S \mapsto \operatorname{rank}(R(p)_S),$$

for $p \in (\mathbb{R}^d)^n$ so given Σ,

$$\operatorname{rank}(\Sigma) := \operatorname{rank}(R(p)_{\Sigma^{(d)}}) = \operatorname{rank}(R(\Sigma, p)).$$

Therefore an equivalent formulation of proposition 3.6.3 is that Σ is rigid if and only if $\operatorname{rank}(\Sigma) = dn - (d^2 + d - 1)$.

The goal of the volume rigidity matroid is to separate the question of the rigidity of Σ from the rigidity of any framework of Σ (even a generic one), instead relating it to the combinatorics of the abstract simplicial complex. As we will see in chapter 4, this is easier said than done.

A reformulation of theorem 2.4.8 is that the 1- and 2-dimensional Euclidean rigidity matroids of graphs on n vertices are the graphic matroid (on n vertices) and the $(2,3)$-sparsity matroid $\mathcal{S}_n(2,3)$, i.e. the matroid on ground set $\binom{[n]}{2}$ with independent sets

$$\mathcal{I}(\mathcal{S}_n(2,3)) = \left\{ S \subseteq \binom{[n]}{2} : S \leq 2|V(S)| - 3 \right\},$$

respectively

We note that the analogous identification does not hold for volume rigidity in $\mathbb{R}^2$, indeed consider the following well-known counterexample:

Example 3.6.4. Consider the simplicial complex Σ defined by maximal simplices

$$\Sigma^{(2)} = \{123, 124, 125, 126, 345, 346, 356\}.$$

Pin the first two points of a generic framework (Σ, p) in $\mathbb{R}^2$ to $(0,0)$, $(1,0)$ to get $(\Sigma, \overline{p})$ as in fig. 3.7 with configuration matrix

$$C(\overline{p}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & \overline{p}(3)_1 & \overline{p}(4)_1 & \overline{p}(5)_1 & \overline{p}(6)_1 \\ 0 & 0 & \overline{p}(3)_2 & \overline{p}(4)_2 & \overline{p}(5)_2 & \overline{p}(6)_2 \end{bmatrix}.$$

Then, as well as the trivial infinitesimal flex corresponding to shearing $(\Sigma, \overline{p})$ parallel to the x-axis, there is a non-trivial infinitesimal flex corresponding to translating $\overline{p}(3), \ldots, \overline{p}(6)$ parallel to the x-axis. Therefore (Σ, p) is flexible, and since

$$f(\Sigma)_2 = 7 = 2.f(\Sigma)_0 - 5,$$

Σ must be dependent in $\mathbb{R}^2_6$. However, $\Sigma^{(2)}$ is independent in $\mathcal{S}_{(2,5)}$. Indeed, for $X \subseteq \Sigma^{(0)}$,

$$\begin{aligned} &i(X) = 0, \text{ if } |X| \leq 2; \\ &i(X) \leq 1 = 2|X| - 5, \text{ if } |X| = 3; \\ &i(X) \leq 3 = 2|X| - 5, \text{ if } |X| = 4; \\ &i(X) \leq 4 < 2|X| - 5, \text{ if } |X| = 5; \\ &i(X) = 7 = 2|X| - 5, \text{ if } |X| = 6. \end{aligned}$$

◇

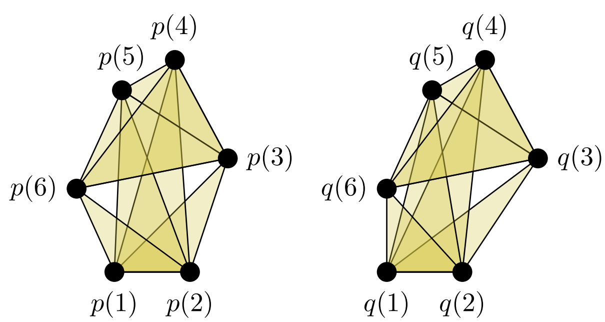

Figure 3.7: Two equivalent but not congruent frameworks of a *slider complex* Σ; (Σ, q) is obtained from (Σ, p) by translating vertices 3,4,5,6 parallel to the span of vertices 1,2 by a fixed amount.

Chapter 4

The Combinatorics of Volume Rigidity

In this Chapter, we discuss several places where combinatorics and simplicial topology come up in the study of volume rigidity. We have already seen some small instances of this, such as with gluing frameworks at a certain number of points and with topological stresses. The volume rigidity matroid, introduced in section 3.6 will also be explored more in this Chapter.

4.1 Volume Rigidity in $\mathbb{R}^1$

We begin this Chapter with the simplest combinatorial characterisation of volume rigidity. The following theorem is folkloric, and follows the same way as the analogous characterisation of Euclidean rigidity [Graver et al., 1993, p. 4].

Theorem 4.1.1. *Let Σ be a 1-dimensional simplicial complex, then Σ is rigid in $\mathbb{R}^1$ if and only if Σ is connected.*

An immediate consequence of this is that the 1-dimensional volume rigidity matroid $\mathcal{R}_n^1$ is the graphic matroid on ground set $\binom{[n]}{2}$, with independent sets forests on the vertex set $[n]$.

Proof. Suppose that Σ is connected, let (Σ, p) be a generic framework in $\mathbb{R}^1$. Pin vertices 1 and 2 to 0 and $\det(C(12), p)$ in $\mathbb{R}^1$ respectively and call the resulting framework $(\Sigma, \overline{p})$. Let $i \in [n] \setminus [2]$ be adjacent to 1 or 2 in any graph-theoretic path in Σ, then we cannot continuously perturb $\overline{p}(i)$ without changing the length of 1-simplex $1i$ or $2i$ respectively. Suppose that for all j in such a path, we cannot perturb $\overline{p}(j)$ without changing the length of some 1-simplex in that path, then consider $k \in [n] \setminus [2]$, the next vertex along any path we have considered so far, we cannot perturb $\overline{p}(k)$ without changing the length of either the 1-simplex kl, where l is the vertex below k in any such path, or without perturbing $\overline{p}(l)$. Therefore every vertex in $(\Sigma, \overline{p})$ is uniquely defined, and so (Σ, p) is rigid in $\mathbb{R}^1$.

Suppose that Σ is not connected, let (Σ, p) be a generic framework in $\mathbb{R}^1$. Denote by Σ_1 and Σ_2 two connected components of Σ, then they have corresponding sub-frameworks (Σ_1, p_1) and (Σ_2, p_2) of (Σ, p), where $p_1 = p|_{\Sigma_1^{(0)}}$ and $p_2 = p|_{\Sigma_2^{(0)}}$. Now (Σ, p) is equivalent to any (Σ, q), where $(\Sigma_1, q_1) = (\Sigma, p_1)$ but $(\Sigma_2, q_2) = (\Sigma_2, p_2 + \underline{t})$, where

$$p_2 + \underline{t} = (p_2(i) + t : i \in \Sigma_2^{(0)}),$$

for any $t \in \mathbb{R}$. Clearly, when $t \neq 0$, (Σ, p) and (Σ, q) are not congruent, so (Σ, p) is not rigid in $\mathbb{R}^1$ and hence Σ is not rigid in $\mathbb{R}^1$. □

As noted above, this is exactly the same argument and result as in the Euclidean rigidity case, so in dimension 1, volume and Euclidean rigidity are identical. This is not surprising, as 1-volumes are signed lengths, which are, locally in $\mathbb{R}^1$, identical to distances, and 1-dimensional simplicial complexes are graphs.

We now move on to more interesting combinatorics.

4.2 The Algebraic Matroid of the Measurement Variety

In order to give an upper bound for the number of equivalent, but not congruent, frameworks of a rigid simplicial complex, Borcea and Streinu [2013] birationally identify the complex measurement variety CM_n^d with the $(d, n-1)$-Grassmannian variety. We will discuss their motivations for doing so further in chapter 6, but for now we will focus on their identification and its implications for the rigidity matroid $\mathcal{R}_n^d$.

First we need to define some objects.

Definition 4.2.1. The *(complete) complex (d-volume) measurement variety* of n points in $\mathbb{C}^d$, denoted CM_n^d, is the Zariski-closure of the image of the map

$$\alpha_n^d : (\mathbb{C}^d)^n \to \mathbb{C}^{\binom{[n]}{d+1}}; p \mapsto \left(\frac{1}{d!}\det(C(\sigma, p)) : \sigma \in \binom{[n]}{d+1}\right).$$

The *complex (d-volume) measurement variety* of Σ in $\mathbb{C}^d$, denoted CM_Σ^d, is the Zariski-closure of the image of the map

$$\alpha_\Sigma^d : (\mathbb{C}^d)^n \to \mathbb{C}^{\Sigma^{(d)}}; p \mapsto \left(\frac{1}{d!}\det(C(\sigma, p)) : \sigma \in \Sigma^{(d)}\right).$$

Definition 4.2.2. Let $1 \leq k \leq n$, the *(k, n)-Grassmannian variety*, denoted $\mathrm{Gr}(k, n)$, is the projective complex algebraic variety parameterising k-dimensional

linear subspaces of $\mathbb{C}^n$.
We may embed $\mathrm{Gr}(k,n)$ in $\mathbb{CP}^{\binom{[n]}{k}}$ via the *Plücker embedding*:

$$\mathrm{Pl} : \mathrm{Gr}(k,n) \hookrightarrow \mathbb{CP}^{\binom{[n]}{k}}; U \mapsto \left(\det(B(U)^{\underline{i}}) : \underline{i} \in \binom{[n]}{k} \right),$$

where $B(U)$ is the $k \times n$ reduced row-echelon basis matrix of U.

The Plücker embedding is sometimes defined as a map into the exterior product space:

$$\tilde{\mathrm{Pl}} : \mathrm{Gr}(k,n) \hookrightarrow \bigwedge^k \mathbb{C}^n; \mathrm{Span}_{\mathbb{C}}\{u_1, \dots, u_n\} \mapsto \left(u_{i_1} \wedge \cdots \wedge u_{i_k} : \underline{i} \in \binom{[n]}{k} \right),$$

An identification of the images of the two maps follows by identifying k-step tensors in $\bigwedge^k \mathbb{C}^n$ with vectors in $\mathbb{C}^{\binom{n}{k}}$ as follows

$$\sum_{i_1 \dots i_k} a_{i_1 \dots i_k} e_{i_1} \wedge \dots e_{i_k} \leftrightarrow \left(a_{i_1 \dots i_k} : i_1 \dots i_k \in \binom{[n]}{k} \right)$$

We prove Borcea and Streinu's identification here, the proof given was developed independently of Borcea and Streinu, but follows the same argument.

Theorem 4.2.3. *There is a birational equivalence between CM_n^d and $\mathrm{Gr}(d, n-1)$.*

Proof. We will construct mutually inverse rational maps between open subsets of CM_n^d and $\mathrm{Gr}(d, n-1)$. Fix some $u \in [n]$ to be a distinguished vertex.

Let U be the Zariski-open subset of the open subset $\alpha_n^d((\mathbb{C}^d)^n$ of $CM_n^d \subseteq \mathbb{C}^{\binom{[n]}{d+1}}$ defined by $y_\sigma \neq 0$, for some σ containing u as a vertex. Then define $\psi : U \to \mathrm{Pl}(\mathrm{Gr}(d, n-1))$ by

$$\psi(y) = (y_\tau : u \in \tau^{(0)}).$$

Since, as with each element of $\alpha_n^d((\mathbb{R}^d)^n)$, the fibre of each $\alpha_n^d(p) \in \alpha_n^d((\mathbb{C}^d)^n)$ has a representative obtained by pinning p to $\overline{p}$ so that σ is sent to the unit-simplex, $\alpha_n^d(p)$ is simply the list of $(d+1) \times (d+1)$ minors of $C(\overline{p})$. However, since $\overline{p}(u) = 0$, each $(d+1) \times (d+1)$ minor featuring a column indexed by a d-simplex containing u is in fact a $d \times d$ minor of $C(p)^{[n]\setminus\{u\}}_{[d+1]\setminus\{1\}} \in \mathbb{C}^{d \times (n-1)}$, which is itself a row-echelon basis matrix of some d-dimensional linear subspace of $\mathbb{C}^n$. Therefore, $\psi(U)$ is simply the lists of $d \times d$ minors of such subspaces, which are precisely elements of $\mathrm{Pl}(\mathrm{Gr}(d, n-1))$.

Next, let $V \subseteq \mathrm{Pl}(\mathrm{Gr}(d, n-1))$ be the Zariski-open subset defined by $x_\sigma \neq 0$. Define $\varphi : \mathrm{Pl}(\mathrm{Gr}(d, n-1)) \to U$ by

$$\varphi(x) = \left(x_\tau, \sum_{\pi \in \rho^{(d-1)}} (-1)^{\mathrm{sign}(\pi,\rho)} x_\pi : \tau \in \binom{[n] \setminus \{u\}}{d}, \rho \in \binom{[n] \setminus \{u\}}{d+1} \right),$$

(up to potentially rearranging entries). Then $\varphi(V)$ lists $(d+1)\times(d+1)$ minors of pinned configuration matrices (with respect to the same pinning as above) $C(\overline{p})$ as $p \in (\mathbb{C}^d)^n$ varies, i.e. elements of CM_n^d.

Both ψ and φ are rational maps, so if we let $x \in V$ and $y \in U$, then

$$\begin{aligned}(\psi \circ \varphi)(x) &= \psi\left(x_\tau, \sum_{\pi \in \rho^{(d-1)}} (-1)^{\text{sign}(\pi,\rho)} x_\pi : \tau \in \binom{[n]\setminus\{u\}}{d}, \rho \in \binom{[n]\setminus\{u\}}{d+1}\right) \\ &= \left(x_\tau : \tau \in \binom{[n]\setminus\{u\}}{d+1}\right) \\ &= x\end{aligned}$$

and

$$\begin{aligned}(\varphi \circ \psi)(y) &= \varphi(y_\tau : u \in \tau^{(0)}) \\ &= \left(y_\tau, \sum_{\pi \in \rho^{(d-1)}} (-1)^{\text{sign}(\pi,\rho)} y_\pi : \tau \in \binom{[n]\setminus\{u\}}{d}, \rho \in \binom{[n]\setminus\{u\}}{d+1}\right) \\ &= y,\end{aligned}$$

and we see that they are mutually inverse, completing the proof. □

Given this equivalence between CM_n^d and $\text{Gr}(d, n-1)$, can we make any equivalences between their corresponding algebraic matroids?

Recall that, given a D-dimensional complex affine variety $X \subseteq \mathbb{C}^[N]$ with algebraic matroid $\mathcal{M}(X)$, a set $I \subseteq [N]$ is independent in $\mathcal{M}(X)$ if and only if

$$\overline{\pi_I(X)} = \mathbb{C}^I,$$

or equivalently $\dim(\pi_I(X)) = |I|$.

The following technical lemma relates the independent sets of algebraic matroids of varieties related by a *linear* map.

Lemma 4.2.4. *Let $X \subseteq \mathbb{C}^{[N]}$ and $Y \subseteq \mathbb{C}^{[M]}$ be two D-dimensional complex affine varieties, let $F : U \to Y$ be a degree-1 regular morphism, where $U \subseteq X$ is open. Let Φ be the $(M \times N)$-matrix describing the action of F on $\mathbb{C}^{[N]}$, then $T \subseteq [M]$ is independent in $\mathcal{M}(Y)$ if $S \subseteq [N]$ independent in $\mathcal{M}(X)$ is, where $[S]$ indexes the leading 1s of* $\text{RREF}(\Phi_T)$.

Proof. Let $T = \{T_1, \ldots, T_t\} \subseteq [M]$ and consider $\pi_T(f(U))$, then

$$\pi_T(f(U)) = \left\{(x_{S_1} + a_{S_1}, \ldots, x_{S_s} + a_{S_s}, 0, \ldots, 0) : a_{S_i} = \sum_{j=s}^{n} \lambda_j x_j,\, x \in U\right\},$$

if

$$\text{RREF}(\Phi_T) = \begin{bmatrix} I_S & A \\ 0 & 0 \end{bmatrix},$$

so if S, the indexing set of the leading 1s of $\text{RREF}(\Phi_T)$, is independent in $\mathcal{M}(X)$, then the set corresponding to the first $|S|$ entries of T will be independent in $\mathcal{M}(Y)$. □

Theorem 4.2.5. *Let $1 \leq d \leq n-1$, let $u \in [n]$ and fix an ordering of $\binom{[n]\setminus\{u\}}{d}$, then a set $S \subseteq \binom{[n]}{d+1}$ is independent in $\mathcal{M}(CM_n^d)$ if the indexing set $T \subseteq \binom{[n]\setminus\{u\}}{d}$ of the leading 1s of the matrix Φ describing the rational map φ from theorem 4.2.3, with respect to the distinguished vertex u, is independent in the algebraic matroid of $\mathrm{Gr}(d, n-1) \subseteq \mathbb{CP}^{\binom{[n]\setminus\{u\}}{d}}$.*

Proof. Every generic framework of a simplicial sub-complex of K_n^d admits no k-simplices contained in a $(k-1)$-dimensional affine subspace of $\mathbb{C}^d$, for any $1 \leq k \leq d$. Therefore, we consider affine charts of CM_n^d and $\mathrm{Gr}(d, n-1)$, defined respectively by setting some coordinate indexed by $u \in \sigma \in \binom{[n]}{d+1}$ to be non-zero and by setting the coordinate indexed by $\sigma - u$ to be non-zero, for a suitable choice of σ. We then have a birational equivalence between these two varieties as described in theorem 4.2.3, and we may apply lemma 4.2.4 to the map φ in that equivalence to get the desired result. $\square$

We note that, given any distinguished vertex $u \in [n]$, the $\binom{[n]}{d+1} \times \binom{[n]\setminus\{u\}}{d}$-matrix Φ is the restriction of the matrix of d-coboundary operator

$$\delta^d : C_{d-1}\left(\binom{[n]}{d}, \mathbb{C}\right) \to C_d\left(\binom{[n]}{d}, \mathbb{C}\right)$$

to the columns not indexed by a $(d-1)$-simplex containing vertex u. Moroever, if Σ is a d-dimensional simplicial complex on vertex set $[n]$ and we again distinguish $u \in [n]$, then the restriction of the matrix Φ to the rows indexed by $\Sigma^{(d)}$, $\Phi_{\Sigma^{(d)}}$, has column space isomorphic to that of the matrix of the d-coboundary operator

$$\delta^d : C_{d-1}(\Sigma, \mathbb{C}) \to C_d(\Sigma, \mathbb{C})$$

under the same restriction as above. If we remove the zero-columns in $\Phi_{\Sigma^{(d)}}$ introduced by restricting to the rows indexed by $\Sigma^{(d)}$, the resulting matrix is precisely the matrix describing δ^d.

We have thus obtained a characterisation of the algebraic matroid of the complex measurement variety CM_n^d in terms of the algebraic matroid of the $(d, n-1)$-Grassmannian variety $\mathrm{Gr}(d, n-1)$. In order to relate this to the volume rigidity setting set out in the first Section of this Chapter and in chapter 3, we need to related $\mathcal{M}(CM_n^d)$ to $\mathcal{R}_n^d$.

In order to do this, we show that the neighbourhood of the measurement of any generic framework of a simplicial complex in M_n^d has the same dimension of the measurement of that same framework in CM_n^d.

We introduce some terms and notation: If $A \subseteq \mathbb{R}^N$ is a real set, then the *complexification* of A, denoted $A^{\mathbb{C}}$, is defined

$$A^{\mathbb{C}} = A \otimes_{\mathbb{R}} \mathbb{C} = \{a_1 \otimes 1 + a_2 \otimes i : a_1, a_2 \in A,\ i^2 = -1\} = \{a_1 + a_2 i : a_1, a_2 \in A\}.$$

Given two sets $A \subseteq \mathbb{R}^N$ and $B \subseteq \mathbb{C}^M$, denote the real and complex dimensions of A and B by $\dim_{\mathbb{R}}(A)$ and $\dim_{\mathbb{C}}(B)$ respectively (we will not use this notation throughout other chapters of this Thesis unless we are in a situation where we want to differentiate between the two).

Lemma 4.2.6. *Let $L = \mathbb{V}_{\mathbb{R}}(l_1, \ldots, l_s) \subseteq \mathbb{R}^N$ be an algebraic set, and suppose that* $\deg(l_1), \ldots, \deg(l_s) = 1$*, then*

1. $L^{\mathbb{C}} = \mathbb{V}_{\mathbb{C}}(l_1, \ldots, l_s)$*;*
2. $\dim_{\mathbb{C}}(L^{\mathbb{C}}) = \dim_{\mathbb{R}}(L)$*.*

Proof. Proceeding directly,

$$\begin{aligned}\mathbb{V}_{\mathbb{C}}(l_1, \ldots, l_s) &= \{x + yi \in \mathbb{C}^N : l_j(x + yi) = 0,\ j \in [s]\}\\ &= \{x + yi \in \mathbb{C}^N : l_j(x) = l_j(y) = 0,\ j \in [s]\}\\ &= \mathbb{V}_{\mathbb{R}}(l_1, \ldots, l_s) \otimes_{\mathbb{R}} \mathbb{C}\\ &= L^{\mathbb{C}},\end{aligned}$$

proving claim 1.
Claim 2 follows, since $L^{\mathbb{C}}$ and L are linear spaces in $\mathbb{C}^N$ and $\mathbb{R}^N$ respectively, defined by the same number of independent equations, t say, (noting that dependences are linear and therefore in one-to-one correspondence between the settings), and therefore have complex and real dimensions both equal to $N - t$. ☐

We note that the coordinate ring of the projectivisation of the linear space $\mathbb{C}^D$ is $\mathbb{C}[X_0, \ldots, X_D]$ and has Krull dimension $D+1$, therefore $\mathbb{C}^D$ has dimension D.

We need some results from real algebraic geometry to show that, for any d-dimensional simplicial complex on n vertices, Σ, M^d_Σ is irreducible, and therefore that, if p is a generic point in $(\mathbb{R}^d)^n$, $\alpha^d_n(p)$ is generic in M^d_Σ, and therefore has a maximum-dimensional image.

Lemma 4.2.7. *Let $S \subseteq \mathbb{R}^N$ be an irreducible semi-algebraic set defined over* $\mathbf{k} \subseteq \mathbb{R}$*, let $\psi : S \to \mathbb{R}^M$ be an algebraic map, then $\psi(S)$ is an irreducible semi-algebraic set defined over* $\mathbf{k}$*.*

Proof. Let $T = \overline{\psi(S)}$ and suppose, for the sake of contradiction, that $T = T_1 \cup T_2$ for non-empty algebraic sets T_1 and T_2. Then

$$\begin{aligned}\psi^{-1}(\psi(S)) &= \psi^{-1}((T_1 \cap \psi(S)) \cup (T_2 \cap \psi(S)))\\ &= (S \cap \psi^{-1}(T \cap \psi(S))) \cup (S \cap \psi^{-1}(T \cap \psi(S)))\\ &= S_1 \cup S_2,\end{aligned}$$

where S_1 and S_2 are non-empty semi-algebraic sets with $\overline{S_1 \cup S_2} = S$, indeed

$$\begin{aligned}S_\alpha &= \{x \in S \subseteq \mathbb{R}^N : \psi(x) \in T \cap \psi(S_\alpha)\}\\ &= \{x \in S \subseteq \mathbb{R}^N : f_i(x) = h_k(\psi(x)) = 0,\ g_j(x), r_l(\psi(x)) > 0,\\ &\qquad f_i \in F_\alpha,\ g_j \in G_\alpha,\ h_k \in H_\alpha,\ r_l \in R_\alpha\},\end{aligned}$$

where F_α and G_α are the sets of equalities defining S_α and H_α and R_α those defining T_α, for each $\alpha \in [2]$. Then, since, for any h_k and r_l in H_α and R_α

respectively, $h_k \circ \psi$ and $r_l \circ \psi$ are both polynomials over $\mathbf{k}$, we have that S_α is semi-algebraic and non-empty, for each $\alpha \in [2]$.

Next, suppose that $x \in S \setminus (\overline{S_1 \cup S_2})$, then $\psi^{-1}(\psi(x)) \in S_1 \cup S_2$, a contradiction. Meanwhile, S is an algebraic set containing $S_1 \cup S_2$, hence

$$S = \overline{S_1 \cup S_2}.$$

Finally, since $S = \overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$, and S is irreducible, one of S_1 and S_2 must be empty, a contradiction. □

Lemma 4.2.8. *Let $S \subseteq \mathbb{R}^N$ and $T \subseteq \mathbb{R}^M$ be irreducible semi-algebraic sets defined over some finite field extension $\mathbf{k}$ of $\mathbb{Q}$, $\psi : S \to T$ a surjective algebraic map. If $x \in S$ is generic, then $\psi(z)$ is generic in T.*

Proof. Let $f \in \mathbb{Q}[y_1, \dots, y_M]$ be a polynomial that does not vanish on T and suppose that

$$f(\psi(x)) = 0.$$

Then $g = (f \circ \psi) \in \mathbb{Q}[x_1, \dots, x_N]$ and $g(x) = 0$. Finally, since f does not vanish on all of $T = \psi(S)$, there exists $y \in S$ so that

$$g(y) = f(\psi(y)) \neq 0.$$

Therefore g does not vanish on all of S, contradicting the genericity of x. □

Theorem 4.2.9. *Let $1 \leq d \leq n-1$, the rigidity matroid $\mathcal{R}_n^d$ and the algebraic matroid of the complex measurement variety $\mathcal{M}(CM_n^d)$ are equal.*

Proof. Let $S \subseteq \binom{[n]}{d+1}$ and suppose that S is independent in $\mathcal{R}_n^d$, this is equivalent to $R(p)_S$ having rank $|S|$, for any $p \in (\mathbb{R}^d)^n$ generic. This is, in turn, equivalent to the tangent space to $\overline{\pi_S(M_n^d)}$ at $\pi_S(\alpha_n^d(p))$ being $|S|$-dimensional. Since M_n^d and CM_n^d are the Zariski-closures of $(\mathbb{R}^d)^n$ and $(\mathbb{C}^d)^n$ under the algebraic-over-$\mathbb{Q}$ map $\pi_S \circ \alpha_n^d$, the tangent spaces to each at the generic (by lemma 4.2.8, which we may apply by lemma 4.2.7) point $(\pi_S \circ \alpha_n^d)(p)$ have maximal real- and complex-dimension respectively, and are defined as the vanishings of the same set of linear polynomials, so by lemma 4.2.6,

$$\dim_{\mathbb{C}}(\overline{\pi_S(CM_n^d)}) = \dim_{\mathbb{R}}(\overline{\pi_S(M_n^d)}) = \dim(T_{(\pi_s \circ \alpha_n^d)(p)} \overline{\pi_S(M_n^d)}),$$

completing the proof. □

We have thus shown the equality of three matroids (or the equality of two matroids and one family of matroids), allowing for us to go back and forth between the settings of simplicial combinatorics, real algebraic geometry and complex algebraic geometry in order to solve rigidity problems.

The following example outlines this in the only complete combinatorial characterisation of volume rigidity we have so far encountered.

Example 4.2.10. We showed in section 4.1 that $\mathcal{R}_n^1$ is the graphic matroid on base set $\binom{[n]}{2}$. Fix a linear extension of the lexicographic order on $\binom{[n]}{2}$ and distinguish vertex $1 \in [n]$ (without loss of generality). Let $S \subseteq \binom{[n]}{2}$ be an independent set in $\mathcal{R}_n^1$, then $([n], S)$ is a forest with connected components $(V_1, S_1), \ldots, (V_C, S_C)$, where $V_1 = [n_1]$, $|V_i| = n_i$, for all $2 \leq i \leq C$ and if $v \in V_j$ and $w \in V_k$, $v < w \iff j < k$, . Then, the matrix Φ_S has block form

$$\Phi_S = \begin{bmatrix} I_{[n_1]} & 0 & \ldots & 0 \\ 0 & \delta^d_{S_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \delta^d_{S_C} \end{bmatrix}.$$

Focusing on the first connected component $([n_1], S_1)$, we see that $[n_1] \setminus \{1\}$ indexes the leading 1s in this block of the RREF of Φ_S. Next, S_1 is an independent set in $\mathcal{M}(\mathrm{Gr}(1, [n_1] \setminus \{1\})) = \mathcal{M}(\mathbb{CP}^{[n_1]\setminus\{1\}})$. Indeed, the affine chart of $\mathbb{CP}^{[n_1]\setminus\{1\}}$ we are considering is a copy of $\mathbb{C}^{n_1-1}$, and thus every orthogonal projection onto coordinates has the maximum dimension.

Now let $2 \leq i \leq C$ and consider the subset S_i. The indexes of the block of $\mathrm{RREF}(\Phi_S)$ defined by rows S_i are precisely the first $n_i - 1$ vertices of V_i, since (V_i, S_i) is acyclic connected. This gives us $n_i - 1$ lines in $\mathbb{C}^{[n_i]-1}$, which are independent by the argument above. $\diamond$

4.3 Volume Rigidity in $\mathbb{R}^2$

In this section, we outline a combinatorial characterisation of volume rigidity in $\mathbb{R}^2$ first proposed as a combinatorial characterisation of the algebraic matroid of the $(2, N)$-Grassmannian in Bernstein [2017].

Theorem 4.3.1. *Let Σ be a 2-dimensional simplicial complex on vertex set $[n]$, fix a linear extension of the lexicographic ordering of $\binom{[n]}{3}$ and let $u \in [n]$ be a distinguished vertex. Then Σ is volume rigid in $\mathbb{R}^2$ if and only if the graph $([n]\setminus\{u\}, S)$, where S indexes the leading 1s of* $\mathrm{RREF}(\Phi_{\Sigma^{(d)}})$*, admits an acyclic, alternating-closed-trail-free orientation and $|S| = 2n - 5$.*

We first outline Bernstein's characterisation of $\mathcal{M}(\mathrm{Gr}(2, N))$:

Theorem 4.3.2. *Bernstein*
A subset S of $\binom{[N]}{2}$ is independent in the matroid $\mathcal{M}(\mathrm{Gr}(2, N))$ if and only if the graph $(V(S), S)$ admits an acyclic, alternating-closed-trail-free orientation.

Bernstein's proof of theorem 4.3.2 uses the identification of $\mathrm{Gr}(2, N)$ with the space of phylogenetic trees described in Billera et al. [2001] and is therefore not extendable to higher dimensions (i.e. characterising $\mathcal{M}(\mathrm{Gr}(3, N))$ and so on). We note that a combinatorial characterisation of $\mathcal{M}(\mathrm{Gr}(k, N))$ is of interest to researchers from many different areas of algebraic geometry and combinatorics, for example Clarke and Tanigawa are currently approaching this topic from the point of view of splitting fields and toric geometry. We will see a very limited formulation of dependencies as forbidden sign-patterns in section 5.3.

Proof of theorem 4.3.1. Suppose that Σ is volume rigid in $\mathbb{R}^2$, then $\Sigma^{(2)}$ is a basis in $\mathcal{R}_n^2$. Therefore $f(\Sigma)_2 = 2n-5$ and $\Sigma^{(2)}$ is independent in $\mathcal{R}_n^2$, so, by theorems 4.2.5 and 4.2.9 the result follows.

Suppose that $([n]\setminus\{u\}, T)$ admits such an orientation and $|T| = 2n-5$. Since $|\Sigma^{(2)}| \geq 2n-5$, and $\Sigma^{(2)}$ admits a subset of size $2n-5$ that is independent in $\mathcal{R}_n^2$, therefore Σ is volume rigid in $\mathbb{R}^2$. □

Example 4.3.3. Consider the 2-dimensional simplicial complex Σ on vertex set [6] defined by its maximal simplices

$$\Sigma^{(2)} = \{123, 124, 125, 126, 345, 346, 356\}.$$

Although

$$f(\Sigma)_2 = 2f(\Sigma)_0 - (2^2 + 2 - 1) = 2f(\Sigma)_0 - 5,$$

Σ is flexible in $\mathbb{R}^2$, therefore, by the index theorem, Σ must be dependent in $\mathbb{R}^2$.

Let 1 be the distinguished vertex of Σ and consider the ordering of $\Sigma^{(2)}$ written above and the linear lexicographic ordering of $\Sigma^{(1)} \setminus \{1\}$. Then $\Phi_{\Sigma^{(2)}}$ is as below

$$\Phi_{\Sigma^{(2)}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix},$$

the columns indexing leading 1s of $\mathrm{RREF}(\Phi_{\Sigma^{(2)}})$ with respect to this ordering are $T = \{23, 24, 25, 26, 34, 35, 45\}$. When considering the graph $([6]\setminus\{1\}, T)$, we see that it contains a copy of K_4^1 as a subgraph, no acyclic orientation of K_4^1 is alternating-closed-trail-free and so T is dependent in $\mathcal{M}(\mathrm{Gr}(2, [6]\setminus\{1\}))$. Since T is dependent in $\mathcal{M}(\mathrm{Gr}(2, [6]\setminus\{1\}))$, Σ is dependent in $\mathcal{R}_6^2$, and since

$$f(\Sigma)_2 = 7 = 2f(\Sigma)_0 - 5,$$

Σ is flexible. ◇

We note two points about this characterisation. The first is that there does not seem to be an intuitive link between alternating-closed-trails in induced graphs and stresses in frameworks, the combinatorial characterisation does not give a satisfying reason for a complex to be either dependent or independent. The second is that, although we do not necessarily expect determining independence to be fast (see Streinu and Theran [2007] for an unsuccessful attempt at finding such a fast algorithm), checking for alternating-closed-trails over all acyclic orientations of the induced graph is slow - a current best estimate being that it is double-exponential and we believe this to be the best possible in general with the machinery currently available to us.

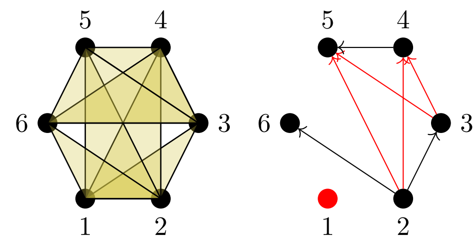

Figure 4.1: An ACT (highlighted in *red*) in an acyclic orientation of the graph induced by Σ. Every acyclic orientation of such a graph (with respect to any distinguished vertex) will admit an ACT, hence Σ is flexible in $\mathbb{R}^2$.

This last point is due to Bernstein's formulation of the acyclic ACT-free condition for independence. By Yu [2017], independence in $\mathcal{M}(\mathrm{Gr}(2, n-1))$ is equivalent to independence in the algebraic matroid of the tropicalisation of $\mathrm{Gr}(2, n-1)$. Therefore, elimination theory may be used to determine independence, the standard being Gröbner basis methods which are double-exponential in complexity. On top of this, there is also the question of determining the leading 1s of $\mathrm{RREF}(\Phi_{\Sigma^{(2)}})$, which is polynomial in complexity.

4.4 The Lexicographically Greedy Rigid Complex

We take a detour from discussions about arbitrary (pure d-dimensional) simplicial complexes to consider a special example in the setting of volume rigidity, the *lexicographically greedy rigid complex*, defined as follows.

Definition 4.4.1. Let $0 \leq n \leq d-1$, the d-dimensional *lexicographically greedy rigid complex (LGRC)* on n-vertices, denoted Λ_n^d, is the simplicial complex defined in terms of its maximal simplices

$$(\Lambda_n^d)^{(d)} = \{[d+1], ([d+1] \setminus \{i\}) \cup \{j\} : 2 \leq i \leq d+1 < j \leq n\}.$$

We call Λ_n^d *lexicographically greedy* because it is the indexing set of the rowspace of $R(p)$, for some generic configuration $p \in (\mathbb{R}^d)^n$, with respect to the linear lexicographic ordering of $\binom{[n]}{d+1}$.

Proposition 4.4.2. *Let $\prec_{lex}$ be the linear lexicographic ordering of $\binom{[n]}{d+1}$. Let $p \in (\mathbb{R}^d)^n$ be a generic configuration of $[n]$. Then $(\Lambda_n^d)^{(d)}$ indexes the rows of $R(p)$ that form a $\prec_{lex}$-greedy basis of the row space of $R(p)$.*

Proof. Let

$$L_1 = \{[d] \cup \{j\} : d+1 \leq j \leq n\} \text{ and}$$
$$L_i = \{([d+1] \setminus \{i\}) \cup \{j\} : d+2 \leq j \leq n\},$$

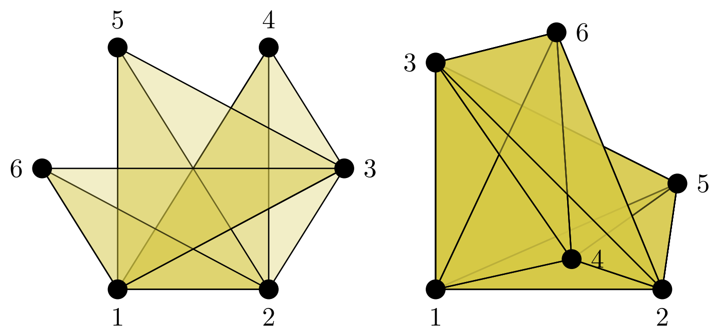

Figure 4.2: The LGRCs Λ_6^2 on the left and Λ_6^3 on the right.

for $2 \leq i \leq d$. Then $L_1 \cup \cdots \cup L_d = (\Lambda_n^d)^{(d)}$ and, for each i, the elements of L_i are $\prec_{\text{lex}}$-consecutive.

We know that Λ_n^d is a basis of $\mathcal{R}_n^d$, since it is minimally rigid in $\mathbb{R}^d$, so in order to show that $(\Lambda_n^d)^{(d)}$ is a $\prec_{\text{lex}}$-greedy basis of the row space of $R(p)$, it suffices to show that $L_1 \cup \ldots L_i \cup \{\sigma\}$ is dependent for every

$$\max\{L_i\} \prec_{\text{lex}} \sigma \prec_{\text{lex}} \min\{L_{i+1}\},$$

for each $2 \leq i \leq d$ (noting that $\max\{L_1\}$ and $\min\{L_2\}$ are $\prec_{\text{lex}}$-consecutive).

Let $2 \leq i \leq d$ and let

$$\max\{L_i\} \prec_{\text{lex}} \sigma \prec_{\text{lex}} \min\{L_{i+1}\}.$$

Noting that the row matroid of $R(p)$ is the algebraic matroid of the affine complex measurement variety CM_n^d (see theorem 4.2.9), we will show that the projection of CM_n^d on to entries indexed by $L = L_1 \cup \cdots \cup L_i \cup \{\sigma\}$ is dependent in CM_n^d. The dimension of $\pi_L(CM_n^d)$ is the number of variables present in the restriction of the pinned configuration matrix

$$C(\overline{p}) = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \ldots & 0 & \overline{p}(d+2)_1 & \ldots & \overline{p}(n)_1 \\ 0 & 0 & 1 & \ldots & 0 & \overline{p}(d+2)_2 & \ldots & \overline{p}(n)_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \overline{p}(d+1)_d & \overline{p}(d+2)_d & \ldots & \overline{p}(n)_2 \end{bmatrix}$$

to its last i rows, which is precisely $i(n-d-1)+1$. Meanwhile, the number of d-simplices present in L is $(n-d)+(i-1)(n-d-1)+1$, one greater than the former. Therefore,

$$\dim(\overline{\pi_L(CM_n^d)}) = i(n-d-1)+1 < (n-d)+i(n-1)(n-d-1) = |L|,$$

so $L \notin \mathcal{I}(\mathcal{R}_n^d)$, completing the proof. □

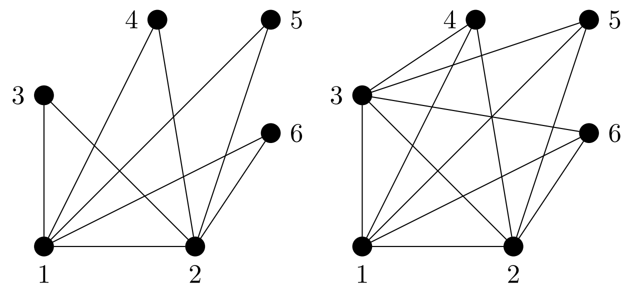

Figure 4.3: The graph on the left is obtained by performing repeated 0-extensions to the complete graph on vertex set $\{1, 2, 3\}$, meanwhile the graph on the right is a trilateration, i.e. each vertex greater than 3 is attached to the graph on the vertices smaller than it by three edges.

The LGRC may be thought of as a volume rigidity analogue to the d-dimensional generalisations of both graphs obtained by performing repeated 0-extensions to the complete graph on three vertices and trilateration graphs in Euclidean rigidity, the link being that we repeatedly add vertices to K^d_{d+1}, connecting them to K^d_{d+1} with the fewest d-simplices to guarantee both rigidity and unique placement (we will discuss this latter point and its implications further in chapter 7).

4.5 Minimal Face Numbers for Volume Rigidity

In this section, we prove one of the main results of this thesis: a combinatorial necessary condition for a d-dimensional simplicial complex on n vertices to be volume rigid in $\mathbb{R}^d$ which holds for all $d \in \mathbb{N}$. Moreover, this necessary condition, a lower bound on the number of k-simplices of a d-volume rigid complex, for all $0 \leq k \leq d$, is met by the LGRC, so the LGRC may be thought of as *minimal* amongst bases of $\mathcal{R}^d_n$.

Finding necessary lower bounds for the number of constraints for rigidity has a long history - indeed one of the first papers in what would become rigidity theory, Maxwell [1864], laid out the necessary lower bound for the number of edges required for the Euclidean rigidity of a (graph) framework in $\mathbb{R}^3$. As a result, these lower bounds are often called Maxwell counts.

Borcea and Streinu gave a Maxwell count in terms of the maximal simplices of a d-dimensional simplicial complex, using arguments seen in section 4.2.

Theorem 4.5.1. *[Borcea and Streinu [2013]] Let Σ be a d-dimensional simplicial complex. If Σ is volume rigid in $\mathbb{R}^d$, then*

$$f(\Sigma)_d \geq dn - (d^2 + d - 1).$$

However, we see in fig. 4.4 two 2-dimensional complexes that satisfy the count in theorem 4.5.1 but are flexible in $\mathbb{R}^2$. We notice that the complex on

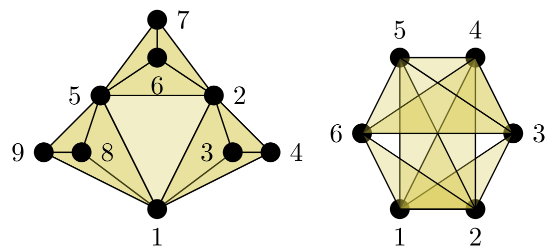

Figure 4.4: These complexes meet Borcea and Streinu's count but are flexible in $\mathbb{R}^2$.

the right has a complete 1-skeleton, and therefore we cannot hope to show that it is flexible due to having too few 1-simplices, however we will show precisely that for the complex on the left.

Our bound, given in theorem 4.5.2 was first proved by the generic methods outlined in chapter 5.

Theorem 4.5.2. *Let Σ be a pure d-dimensional simplicial complex on n vertices. If Σ is volume rigid in $\mathbb{R}^d$, then*

$$f(\Sigma)_k \geq \begin{cases} dn - (d^2 + d - 1), & \text{when } k = d; \\ \binom{d+1}{k+1} + (n - d - 1)\binom{d+1}{k}, & \text{when } k < d. \end{cases}$$

We begin by showing that the LGRC will always meet this count.

Lemma 4.5.3. *Let Λ_n^d be the d-dimensional LGRC on n vertices, then*

$$f(\Lambda_n^d)_k = \begin{cases} dn - (d^2 + d - 1), & \text{when } k = d; \\ \binom{d+1}{k+1} + (n - d - 1)\binom{d+1}{k}, & \text{when } k < d. \end{cases}$$

Proof. To see this, we recall that

$$(\Lambda_n^d)^{(d)} = \{[d+1], ([d+1] \setminus \{i\}) \cup \{j\} : 2 \leq i \leq d+1 < j \leq n\},$$

so

$$f(\Lambda_n^d)_d = dn - (d^2 + d - 1).$$

Let $0 \leq k \leq d$. There are $\binom{d+1}{k+1}$ distinct d-simplices contained within d-simplex $[d+1] \in (\Lambda_n^d)^{(d)}$. For each of the $n - d - 1$ vertices $d + 2 \leq j \leq n$, there are $\binom{d+1}{k}$ distinct k-simplices containing j in Λ_n^d. □

In order to prove theorem 4.5.2, we must define the *covering number* of a k-simplex by l-simplices. Given a k-simplex τ of some d-dimensional simplicial complex Σ and some $k \leq l \leq d$, the *l covering number* of τ is

$$\kappa_k^l(\tau) = |\{\sigma \in \Sigma^{(l)} : \tau \subseteq \sigma\}|.$$

The *(expected) (k,l) covering number* of the whole of Σ is the average of $\kappa_k^l(\tau)$ as τ varies over $\Sigma^{(k)}$,

$$\mathbb{E}[\kappa_k^l(\Sigma)] = \frac{\sum\limits_{\tau \in \Sigma^{(k)}} \kappa_k^l(\tau)}{f(\Sigma)_k}.$$

The following lemma is the simplicial complex analogue of a classical graph theory result.

Lemma 4.5.4 (Handshaking lemma)**.** *Let Σ be a pure d-dimensional simplicial complex and let $0 \leq k \leq l \leq d$, then*

$$\binom{l+1}{k+1} f(\Sigma)_l = \sum_{\tau \in \Sigma^{(k)}} \kappa_k^l(\tau) = \mathbb{E}[\kappa_k^l(\Sigma)] f(\Sigma)_k.$$

Proof. Each l-simplex of Σ contains precisely $\binom{l+1}{k+1}$ k-simplices, proving the first equality. The second equality follows from the definition of $\mathbb{E}[\kappa_k^l(\Sigma)]$. □

In particular, we notice that, fixing n, the expected $(0,k)$-covering number of a simplicial complex is directly proportional to its number of k-simplices:

$$\mathbb{E}[\kappa_0^k(\Sigma)] = \frac{(k+1) f(\Sigma)_k}{n}.$$

Moreover, we notice that the covering number of a vertex $j \in (\Lambda_n^d)^{(0)} \setminus [d+1]$ is constant, equal to $\binom{d+1}{k}$. We will show that, as n tends to infinity, this yields a well-defined (minimal) limit for the $(0,k)$ covering number of a basis of $\mathcal{R}_d^n$.

Lemma 4.5.5.

$$\lim_{n \to \infty} \min_{\Sigma \in \mathcal{B}(\mathcal{R}_n^d)} \{\mathbb{E}[\kappa_0^k(\Sigma)]\} = (k+1)\binom{d+1}{k}. \tag{4.1}$$

Proof. Assume that $d \geq 2$, note that the case of $d = 1$ is immediate, following from theorem 4.1.1.

Firstly, the limit in eq. (4.1) is well-defined. Indeed, for each $n \geq d+1$, each simplicial complex in $\mathcal{B}(\mathcal{R}_n^d)$ has a well-defined, finite expected $(0,k)$-covering number.

For each $n \geq d+1$, $\Lambda_n^d \in \mathcal{B}(\mathcal{R}_n^d)$, by the above discussion, fix $1 \leq k \leq d-1$, then

$$\lim_{n \to \infty} \mathbb{E}[\kappa_0^k(\Lambda_n^d)] = \lim_{n \to \infty} \frac{(k+1)\left(\binom{d+1}{k+1} + (n-d-1)\binom{d+1}{k}\right)}{n} = (k+1)\binom{d+1}{k}. \tag{4.2}$$

We will now show that we cannot do better than this.

Suppose that $\Sigma \in \mathcal{B}(\mathcal{R}_n^d)$ and that $j \in \Sigma^{(0)}$, with $\mathrm{Star}_\Sigma(j) = \{\sigma_1, \ldots, \sigma_t\}$. Firstly, $t \geq d$, since otherwise u would be underconstrained. Let $L_1, \ldots, L_s$ be the components of $\mathrm{Lk}_\Sigma(j)$ that are connected through dimension k (thought of as a $(d-1)$-dimensional simplicial complex. Let N_a denote the number of d-simplices containing j that define L_a, for each $1 \leq a \leq s$. Then,

$$f(L_a)_k \geq \binom{d-1}{k} + N_a \binom{d-1}{k-1}$$

for each $1 \leq a \leq s$. Indeed, this is the case where all vertices of each d-simplex defining each L_a intersect on all but one vertex. Then, summing over all connected components yields

$$\begin{aligned} f(\mathrm{Lk}_\Sigma(j))_k &\geq \sum_{a=1}^{s} \binom{d-1}{k-1} + N_a \binom{d-1}{k-1} \\ &= s\binom{d-1}{k} + \binom{d-1}{k-1} \sum_{a=1}^{s} N_a \\ &\geq s\binom{d-1}{k} + d\binom{d-1}{k-1}. \end{aligned}$$

Then,

$$\begin{aligned} \binom{d+1}{k} - f(\mathrm{Lk}_\Sigma(j))_k &\leq \binom{d+1}{k} - s\binom{d-1}{k} - d\binom{d-1}{k-1} \\ &= \frac{d!(1+(d-k)(1-k))}{(d+1-k)!k!} - s\binom{d-1}{k} \\ &\leq 0, \end{aligned}$$

with the final inequality being immediate if $k \geq 2$, and otherwise following from the fact that

$$s(d-1) \geq 1.$$

Hence,

$$\min_{\Sigma \in \mathcal{B}(\mathcal{R}_n^d)} \{\kappa_0^k(j) : j \in \Sigma^{(0)}\} = (k+1)\binom{d+1}{k}.$$

□

A consequence of this is that there does not exist an infinite family of pure d-dimensional simplicial complexes $\{\Sigma_n : \Sigma_n \in \mathcal{B}(\mathcal{R}_n^d),\ n \geq d+1\}$, whose expected $(0,k)$-covering number converges to something strictly less than $(k+1)\binom{d+1}{k}$ as n tends to infinity.

Proof of theorem 4.5.2. Let Σ be as in the statement of the theorem, then, Borcea and Streinu [2013] showed that

$$f(\Sigma)_d \geq dn - (d^2 + d - 1).$$

It remains to show that

$$f(\Sigma)_k \geq \binom{d+1}{k+1} + (n-d-1)\binom{d+1}{k},$$

for all $0 \leq k < d$. We note that

$$f(\Sigma)_0 = n = (d+1) + (n-d-1) = \binom{d+1}{0+1} + (n-d-1)\binom{d+1}{0},$$

so our claim holds with equality when $k = 0$ (by definition).

Now, suppose that $0 < k < d$ and, for the sake of contradiction, that

$$f(\Sigma)_k < \binom{d+1}{k+1} + (n-d-1)\binom{d+1}{k} = f(\Lambda_n^d)_k.$$

We will use Σ to construct an infinite family, as described above, of successively larger minimally rigid simplicial complexes whose expected $(0,k)$-covering numbers converge to less than the limit defined in lemma 4.5.5.

Let $\sigma \in \Sigma^{(d)}$ and let Σ^m be the d-dimensional simplicial complex on $N(m) = mn - (m-1)(d+1)$ vertices defined by gluing together m copies of Σ with σ as their common face, with $\Sigma^1 = \Sigma$. By proposition 3.2.21, Σ^m is rigid, and

$$f(\Sigma^m)_d = m(dn - (d^2+d-1)) + 1 = dN(m) - (d^2+d-1),$$

for each $m \geq 1$. Let

$$\rho_m = \frac{f(\Sigma^m)_k}{f(\Lambda_{N(m)}^d)_k},$$

for each $m \geq 1$, noting that $0 < \rho_1 < 1$, by assumption, and so

$$\begin{aligned} 0 < \rho_m &= \frac{mf(\Sigma)_k - (m-1)\binom{d+1}{k+1}}{\binom{d+1}{k+1} + (mn - (m-1)(d+1) - d - 1)\binom{d+1}{k}} \\ &= \frac{m\rho_1\left(\binom{d+1}{k+1} + (n-d-1)\binom{d+1}{k}\right) - (m-1)\binom{d+1}{k+1}}{\binom{d+1}{k+1} + m(n-d-1)\binom{d+1}{k}} < 1, \end{aligned}$$

for each $m \geq 1$. Then,

$$\begin{aligned} \rho &= \lim_{\substack{N(m)\to\infty \\ m\in\mathbb{N}(m)}} \rho_m = \lim_{m\to\infty} \rho_m \\ &= \frac{f(\Sigma)_k - \binom{d+1}{k+1}}{(n-d-1)\binom{d+1}{k}}. \end{aligned} \tag{4.3}$$

Since, $f(\Sigma)_d \geq 1$, $\rho > 0$ and since $f(\Sigma)_k < f(\Lambda_n^d)_k$, $\rho < 1$. Moreover, $(\rho_m)_{m\in\mathbb{N}}$ is decreasing, indeed, for any $m \geq 1$,

$$\begin{aligned} \rho_{m+1} - \rho_m &= \frac{f(\Sigma^{m+1})_k}{f(\Lambda_{N(m+1)}^d)_k} - \frac{f(\Sigma^m)_k}{f(\Lambda_{N(m)})_k} \\ &= \frac{(m+1)\rho_1 f(\Lambda_n^d)_k - m\binom{d+1}{k+1}}{f(\Lambda_{N(m+1)}^d)_k} - \frac{m\rho_1 f(\Lambda_n^d)_k - (m-1)\binom{d+1}{k+1}}{f(\Lambda_{N(m)}^d)_k} \\ &= \frac{(\rho_1 - 1)f(\Lambda_n^d)_k}{f(\Lambda_{N(m)}^d)_k f(\Lambda_{N(m+1)}^d)_k} \\ &< 0. \end{aligned}$$

Therefore, $\rho < \rho_1 < 1$, and hence

$$\begin{aligned}
\lim_{m\to\infty} \mathbb{E}[\kappa_0^k(\Sigma^m)] &= \lim_{m\to\infty} \frac{(k+1)f(\Sigma^m)_k}{N} \\
&= \lim_{m\to\infty} \rho_m \frac{(k+1)f(\Lambda^d_{N(m)})_k}{N} \\
&= \rho(k+1)\binom{d+1}{k} \\
&< \rho_1(k+1)\binom{d+1}{k} \\
&< (k+1)\binom{d+1}{k},
\end{aligned}$$

with the two limits separated by an open interval in $(0,1)$, a contradiction, which completes our proof. □

Example 4.5.6. Take the two bases of $\mathcal{R}^2_5$, Λ^2_5 and C^2_5, with

$$(C^2_5)^{(2)} = 123, 145, 125, 234, 345$$

and set $\sigma_1 = 123$ and $\sigma_2 = 125$ in each. We will measure how the above argument fares with respect to their 1-simplices.

Gluing together m copies of Λ^2_5 at σ_1 produces successively larger LGRCs, meanwhile gluing them at σ_2 produces successively larger bases (of rigidity matroids on successively larger vertex sets). However, by definition, ρ_m is constantly 1 for Λ^2_5.

Meanwhile, in the case of C^2_5, for both σ_1 and σ_2,

$$(\rho_1, \rho_2, \rho_3, \dots) = \left(\frac{10}{9}, \frac{17}{15}, \frac{24}{21}, \dots\right).$$

Gluing each additional copy of C^2_5 adds 2 new vertices and 7 new 1-simplices, as opposed to 2 new vertices and 6 new 1-simplices for Λ^2_5, hence

$$\rho_m = \frac{7m+3}{6m+3} \xrightarrow{m\to\infty} \frac{7}{6}.$$

◇

Example 4.5.7. Let Σ be a triangulation of $\mathbb{S}^2$. Then every 1-simplex of Σ is contained in precisely two 2-simplices, while every 2-simplex of Σ contains precisely three 1-simplices, i.e. $2f(\Sigma)_1 = 3f(\Sigma)_2$. Moreover, Σ has Euler characteristic 2, i.e.

$$\chi(\Sigma) = f(\Sigma)_2 - f(\Sigma)_1 + f(\Sigma)_0 = 2,$$

putting these two equations together, we see that

$$f(\Sigma) = (n, 3n-6, 2n-4),$$

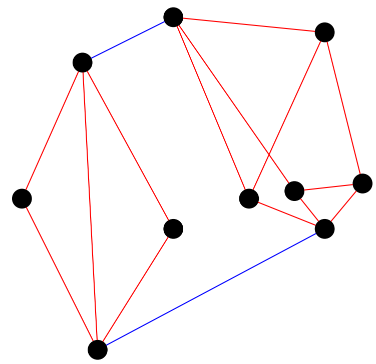

Figure 4.5: The two *red* rigid components on four and six vertices respectively are connected by two *blue* rigid components, each on two vertices.

for all $n \geq 4$. Now, we will prove in section 4.7 that every triangulation of $\mathbb{S}^2$ is rigid in $\mathbb{R}^2$ (theorem 4.7.8) and that every triangulation of $\mathbb{S}^2$ remains rigid after a 2-simplex is removed (proposition 4.7.10).

Therefore, for every triangulation of $\mathbb{S}^2$ on n vertices, there is at least one distinct (up to combinatorial isomorphism) basis Σ' (where $(\Sigma')^{(2)} = \Sigma^{(2)} \setminus \{\sigma\}$) of $\mathcal{R}^2_n$ with f-vector

$$f(\Sigma') = (n, 3n-6, 2n-5) = f(\Lambda^2_n).$$

◇

4.6 LGRC Decompositions

One way of simplifying the study of a flexible generic framework (Σ, p) is to look first at the decomposition of Σ into its *rigid components* (maximal rigid sub-complexes). These induce sub-frameworks of (Σ, p) that flex trivially in any flex of (Σ, p). We can simplify further by replacing each rigid component with a copy of the LGRC on as many vertices, therefore giving ourselves a minimal rigid component, i.e. one with the "simplest" combinatorics, and see what we can deduce from there.

Similar (identical in the 1-dimensional case) decompositions into rigid components exist for Euclidean rigidity - in $\mathbb{R}^1$ (the rigid components here are precisely the connected components, by the same argument as in section 4.1). In $\mathbb{R}^2$, the decomposition of a graph into its rigid components is precisely the decomposition into its $(2,3)$-tight subgraphs (see fig. 4.5 and Lovász [2019] for a discussion of the result)

However, such decompositions do not exist in general (see the *double banana* graph, fig. 4.6, a flexible basis for the 3-dimensional Euclidean rigidity matroid on 8 vertices).

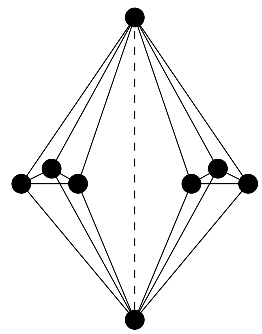

Figure 4.6: The two *bananas* may rotate independently around the dashed axis, so this graph is not Euclidean rigid in $\mathbb{R}^3$

Such decompositions represent the partitioning of sets in the (Euclidean or volume) rigidity matroid into their unique maximal independent sets.

Lemma 4.6.1. *Let Σ be a d-dimensional simplicial complex on n vertices, there exists a unique maximal decomposition of Σ into its rigid components $\Sigma_1, \ldots, \Sigma_N$, where*

$$\bigsqcup_{i=1}^{N} \Sigma_i^{(d)} = \Sigma^{(d)}.$$

In the statement of the lemma, *maximal* refers to there being no decomposition of size strictly less than N.

Proof. Let $\Sigma_1^{(d)}, \ldots, \Sigma_N^{(d)}$ and $\Pi_1^{(d)}, \ldots, \Pi_n^{(d)}$ be two distinct partitionings of $\Sigma^{(d)}$ so that

$$\operatorname{rank}(R(\Sigma, p)_{\Sigma_i^{(d)}}) = df(\overline{\Sigma_i^{(d)}})_0 - (d^2 + d - 1),$$

for each $1 \leq i \leq N$ and

$$\operatorname{rank}(R(\Sigma, p)_{\Pi_j^{(d)}}) = df(\overline{\Pi_j^{(d)}})_0 - (d^2 + d - 1),$$

for each $1 \leq j \leq N$, for some generic $p \in (\mathbb{R}^d)^n$.

By the pigeonhole principle, there must exist $\Sigma_i^{(d)}$, $\Pi_j^{(d)}$ so that

$$\Sigma_i^{(d)} \cap \Pi_i^{(d)} \neq \varnothing.$$

Writing $\Sigma_i = \overline{\Sigma_i^{(d)}}$ and $\Pi_j = \overline{\Pi_j^{(d)}}$, we see that

$$|\Sigma_i^{(0)} \cap \Pi_j^{(0)}| = d + 1.$$

By maximality of this partition, there is no set of rows of $R(\Sigma, p)$ inducing a max-rank submatrix containing both $\Sigma_i^{(d)}$ and $\Pi_j^{(d)}$. Therefore, by the gluing lemma, proposition 3.2.21, $\Sigma_i = \Pi_j$, and hence $\Sigma_i^{(d)} = \Pi_j^{(d)}$.

Since pairs of sets with one from each partition may either be equal or distinct, we see that the two partitions are the same, up to relabelling. Now, any maximal decomposition of Σ into rigid components will induce such a partitioning of the rigidity matrix of a generic framework of Σ in $\mathbb{R}^d$, so any maximal decomposition of Σ into rigid components is unique, up to relabelling. □

The following proposition follows immediately from section 4.1.

Proposition 4.6.2. *Let Σ be a 1-dimensional simplicial complex. The rigid components of Σ are precisely the connected components of Σ.*

Finding the rigid components of 2- and higher-dimensional simplicial complexes is more involved.

Example 4.6.3. Let Σ be the 2-dimensional simplicial complex on six vertices from example 3.6.4, so

$$\Sigma^{(2)} = \{123, 124, 125, 126, 345, 346, 356\}.$$

The rigid components of Σ are the simplicial complexes defined by

$$\Sigma_1^{(2)} = \{123\},\ \Sigma_2^{(2)} = \{124\},\ \Sigma_3^{(2)} = \{125\},\ \Sigma_4^{(2)} = \{126\} \text{ and}$$
$$\Sigma_5^{(2)} = \{345, 346, 356\}.$$

Indeed, Σ_5 is precisely a copy of Λ_3^2 and is therefore rigid, moreover, adding one simplex $12i$, where $i \in \{3, 4, 5, 6\}$ would yield a flexible complex, as $12i$ could rotate freely in any framework, meanwhile adding two or more simplices from $\{123, \ldots, 126\}$ would yield a complex that slides in the same way as Σ in any framework, therefore Σ_5 is the largest rigid component containing $\{345, 346, 356\}$. Meanwhile, no rigid component of size strictly greater than 1 exists containing 2-simplex from $\{123, \ldots, 126\}$, since any such component would consist of a number of 2-simplices joined at a mutual 1-simplex.

Therefore, $\Sigma_1, \ldots, \Sigma_5$ are indeed the maximal rigid sub-complexes of Σ, i.e. they are its rigid components. ◇

Recall that, by the corank-conullity theorem, if Σ is a d-dimensional simplicial complex with rigid component $\Sigma_1, \ldots, \Sigma_N$, then

$$\text{rank}(\Sigma) = \text{corank}(\Sigma) = \sum_{i=1}^{N} f(\Sigma_i)_d - \text{stress}(\Sigma), \tag{4.4}$$

next, stress(Σ) may be partitioned as follows:

$$\text{stress}(\Sigma) = \sum_{i=1}^{N} \text{istress}(\Sigma_i) + \text{estress}(\Sigma), \tag{4.5}$$

where istress(Σ_i) is the dimension of the space of *internal stresses* of Σ_i, i.e. stresses supported by $\Sigma_i^{(d)}$ in any framework of Σ in $\mathbb{R}^d$, and estress(Σ) is the

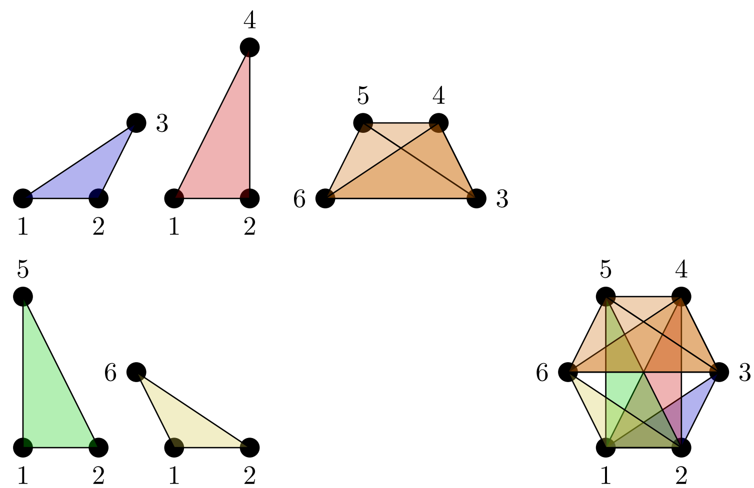

Figure 4.7: The 2-simplices are coloured according to what rigid component they lie in.

dimension of the space of *external stresses* of Σ, i.e. stresses not supported by any single rigid components d-simplices in every framework of Σ in $\mathbb{R}^d$. Combining eqs. (4.4) and (4.5) yields

$$\begin{aligned}\operatorname{rank}(\Sigma) &= \sum_{i=1}^{N}(f(\Sigma_i)_d - \operatorname{stress}(\Sigma_i)) - \operatorname{estress}(\Sigma) \\ &= \sum_{i=1}^{N}\operatorname{rank}(\Sigma_i) - \operatorname{estress}(\Sigma) \\ &= \sum_{i=1}(df(\Sigma_i)_0 - (d^2 + d - 1)) - \operatorname{estress}(\Sigma).\end{aligned} \tag{4.6}$$

Remark. This is close to Lovasz and Yemini's rank formula in the 2-dimensional Euclidean case (see Lovász [2019]): Let G be a graph with 2-dimensional Euclidean rigid components $G_1, \ldots, G_N$, then

$$\operatorname{rank}_E(G) = \sum_{i=1}^{N}(2|V(G)| - 3).$$

Therefore, the rank of the graph in fig. 4.5 is

$$(2(4) - 3) + 2(2(2) - 3) + (2(6) - 3) = 5 + 2 + 9 = 16,$$

since

$$2|V(G)| - 3 = 2(10) - 3 = 17 > 16 = \operatorname{rank}_E(G)\ (= |E(G)|),$$

G is flexible in $\mathbb{R}^2$.

The main technical result of this Section demonstrates that we may replace each rigid component Σ_i of the d-dimensional complex Σ with a copy of $\Lambda^d_{f(\Sigma_i)_0}$.

Definition 4.6.4. Let Σ be a d-dimensional simplicial complex with rigid components $\Sigma_1, \ldots, \Sigma_N$. Define the d-dimensional complex $\Lambda(\Sigma)$ as follows:

1. For each Σ_i, let Λ_i be the LGRC on vertex set Σ_0 (with respect to the lexicographic ordering of the vertices inherited from Σ_i);
2. Glue together the Λ_i to form $\Lambda(\Sigma)$ by making the same identifications of vertices as are made when gluing together the Σ_i to form Σ.

Call the Λ_is and their recombination information the *LGRC decomposition* of Σ.

Lemma 4.6.5. *With Σ and $\Lambda(\Sigma)$ as in definition 4.6.4,* $\mathrm{rank}(\Lambda(\Sigma)) = \mathrm{rank}(\Sigma)$.

Proof. Note that

$$\begin{aligned}\mathrm{rank}(\Sigma) &= df(\Sigma)_0 - (d^2 + d - 1) - \mathrm{ntif}(\Sigma),\\ \mathrm{rank}(\Lambda(\Sigma)) &= df(\Lambda(\Sigma))_0 - (d^2 + d - 1) - \mathrm{ntif}(\Lambda(\Sigma))\\ &= df(\Sigma)_0 - (d^2 + d - 1) - \mathrm{ntif}(\Lambda(\Sigma)).\end{aligned}$$

Suppose that $\eta \in \mathrm{NTIF}(\Sigma, p)$, for some $p \in (\mathbb{R}^d)^{f(\Sigma)_0}$ generic, then $\eta \in \mathrm{NTIF}(\Lambda(\Sigma, p))$. Indeed, η acts as a rigid motion on the vertices of any given rigid component of (Σ, p), and therefore $(\Lambda(\Sigma), p)$. Since the rigid components of Σ and $\Lambda(\Sigma)$ are joined together at the same sets vertices (regardless of whether or not they are simplices in either complex), η is also in $\mathrm{NTIF}(\Lambda(\Sigma), p)$, so $\mathrm{NTIF}(\Sigma, p) \subseteq \mathrm{NTIF}(\Lambda(\Sigma), p)$.

The exact same argument applies in reverse, so $\mathrm{NTIF}(\Lambda(\Sigma), p) \subseteq \mathrm{NTIF}(\Sigma, p)$. □

This is an aspect of volume rigidity that is ripe for further investigation. In particular, can a complete combinatorial characterisation of external stresses be deduced? When $d = 2$, for example, we know that such a characterisation would include slider-like complexes, and, although some small effort has been expended testing whether or not any other combinatorial patterns induce external stresses, a satisfying answer has not yet been found.

4.7 Ball Splitting and Vertex Splitting

In the setting of Euclidean rigidity of graphs, vertex splitting is a combinatorial operation that preserves rigidity (for sufficiently large vertex splits with respect to dimension). It was developed by Tay and Whiteley [1985], primarily to study Euclidean rigidity in $\mathbb{R}^2$, as a well-defined analogue to Henneberg operations, a set of similarly motivated operations originating in engineering. We will develop an analogous operation in the context of volume rigidity of simplicial complexes in $\mathbb{R}^d$ and use it to prove that, amongst other families of simplicial complexes, those defined by triangulating the 2-dimensional sphere are volume rigid in $\mathbb{R}^2$.

Definition 4.7.1. Let Σ be a d-dimensional simplicial complex and let $\sigma_1, \ldots,$ σ_k be d-simplices connected through codimension 1 so that

$$\overline{\{\sigma_1, \ldots, \sigma_k\}}^{(0)} = (\partial(\overline{\{\sigma_1, \ldots, \sigma_k\}}))^{(0)},$$

i.e. $\sigma_1, \ldots, \sigma_k$ contain no internal vertices. Then an *m-ball split*, denoted Σ^*, at $\sigma_1, \ldots, \sigma_k$ is defined by the following steps:

1. Remove $\sigma_1, \ldots, \sigma_k$ to get Σ';
2. Add a new vertex u and new edges $\tau_1, \ldots, \tau_m$ to get Σ^* in such a way that $u \in \tau_1^{(0)}, \ldots, \tau_m^{(0)}$, $u \notin \rho^{(0)}$, for any $\rho \in \Sigma^{(d)} \setminus \{\tau_1, \ldots, \tau_m\}$, and
$$\mathrm{Lk}_{\Sigma^*}(u) = \partial(\overline{\{\sigma_1, \ldots, \sigma_k\}}).$$

Ball splits are so called since the simplicial complex $\overline{\{\sigma_1, \ldots, \sigma_k\}}$ is homeomorphic to a d-dimensional ball.

The following definition is a more direct analogue of Tay and Whiteley's original construction.

Definition 4.7.2. Let Σ be a d-dimensional simplicial complex, let i be a vertex and let $\sigma_1, \ldots, \sigma_k$ be d-simplices connected through codimension 1, each containing i as a vertex. A *m-vertex split* at i is a m-ball split at $\sigma_1, \ldots, \sigma_k$.

Both m-ball and m-vertex splitting have (not unique) inverse operations: *m-ball* and m-vertex contractions.

Definition 4.7.3. Let Σ be a d-dimensional simplicial complex and let $u \in \Sigma^{(0)}$ be such that $\mathrm{Star}_\Sigma(u)$ is a triangulation of the d-dimensional ball and $f(\mathrm{Star}_\Sigma(u))_d = m$.

An *m-ball contraction* at u, denoted $\Sigma^\vee$, is defined by the following steps:

1. Remove $\mathrm{Star}_\Sigma(u)$ to get Σ'';
2. Add new d-simplices $\tau_1, \ldots, \tau_k$, connected through codimension 1, so that
$$(\partial(\overline{\{\tau_1, \ldots, \tau_k\}}))^{(0)} = \mathrm{Lk}_\Sigma(u)^{(0)}$$
and
$$f(\overline{\{\tau_1, \ldots, \tau_k\}})_0 = 0$$
to get $\Sigma^\vee$.

An *m-vertex contraction* to $v \in \mathrm{Lk}_\Sigma(u)^{(0)}$ at u, denoted $\Sigma^\vee$, is defined by the following steps:

1. Remove $\mathrm{Star}_\Sigma(u)$ to get Σ'';

2. Add new d-simplices $\tau_1, \ldots, \tau_k$, connected through codimension 1, so that $v \in \tau_1^{(0)}, \ldots, \tau_k^{(0)}$,

$$(\partial(\overline{\{\tau_1, \ldots, \tau_k\}}))^{(0)} = \mathrm{Lk}_\Sigma(u)^{(0)}$$

and

$$f(\overline{\{\tau_1, \ldots, \tau_k\}})_0 = 0$$

to get $\Sigma^\vee$.

The only difference between an m-ball contraction at u and an m-vertex contraction to v at u is that the new d-simplices we add in the former case are homeomorphic to a disc and do not necessarily all share a common vertex, whilst in the latter case they are homeomorphic to a disc and all do share a common vertex.

The following lemma shows that these operations are in fact inverse.

Lemma 4.7.4. *Let Σ be a d-dimensional simplicial complex.*

If Σ^* is obtained from Σ by performing an m-ball split, then we may perform a k-ball contraction to Σ^* to obtain Σ (i.e. $\Sigma = (\Sigma^*)^\vee$).

If Σ^* is obtained from Σ by performing an m-vertex split at v, then we may perform a k-vertex contraction to v to obtain Σ (i.e. $\Sigma = (\Sigma^*)^\vee$).

Proof. Suppose that Σ^* is obtained from Σ by performing an m-ball split. Then we may perform an m-ball contraction to Σ^* by deleting those 0- and d-simplices added by the m-ball split and adding back in those d-simplices that were deleted by the m-ball split.

The second case follows by exactly the same argument. □

The main technical lemma of this section is that m-ball splitting (and therefore m-vertex splitting) preserves rigidity when $m \geq d+1$ under appropriate conditions.

Lemma 4.7.5. *Let Σ be a d-dimensional simplicial complex and let Σ^* be obtained by performing an m-ball split to Σ. Suppose that Σ is rigid in $\mathbb{R}^d$ and that $(\Sigma^*)^\vee$ is rigid in $\mathbb{R}^d$, for all $(\Sigma^*)^\vee$ obtained by performing an k-vertex contraction to Σ^*. Then Σ^* is rigid in $\mathbb{R}^d$.*

We quickly state and prove the following lemma.

Lemma 4.7.6. *Let Σ be a d-dimensional simplicial complex with $f(\Sigma)_d \geq d$, suppose that there exists $u \in \Sigma^{(0)}$ so that $\Sigma = \mathrm{Star}_\Sigma(u)$ (i.e. Σ is the star complex of u). If (Σ, p) and (Σ, q) are equivalent and in general position, then the position of u in (Σ, q) is uniquely determined by p and $\{q(i) : i \in \mathrm{Lk}_\Sigma(u)^{(0)}\}$.*

Proof. Let $\sigma_1, \ldots, \sigma_d \in \Sigma^{(d)}$, then

$$\det(C(\sigma_s, p)) = \det(C(\sigma_s, q)),$$

for all $1 \leq s \leq d$. Therefore, $q(u)$ lies on the transverse intersection (by the general position of q) of d hyperplanes in $\mathbb{R}^d$, precisely the planes parallel to the

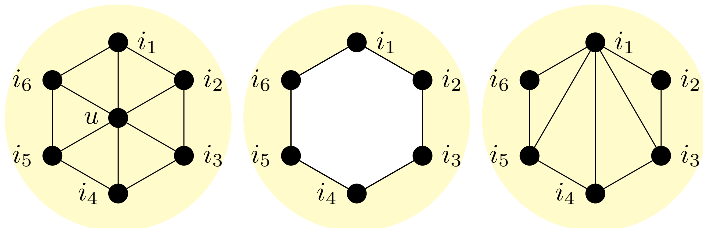

Figure 4.8: An example of going from Σ^* to Σ' to $(\Sigma^*)^\vee$

affine span of $\{q(j) : j \in \sigma_s^{(0)} \setminus \{u\}\}$ a distance $\frac{1}{d!}\det(C(\sigma_s, p))$ away, for each $1 \leq s \leq d$. □

Proof of lemma 4.7.5. Let $n = f(\Sigma)_0$, suppose that we remove k d-simplices, $\tau_1, \ldots, \tau_k$, so that $\overline{\{\tau_1, \ldots, \tau_k\}}$ is the triangulation of a d-dimensional ball with boundary vertex set $U = \{i_1, \ldots, i_l\}$, from Σ to obtain Σ', and we add m d-simplices $\sigma_1, \ldots, \sigma_m$, so that

$$\overline{\{\sigma_1, \ldots, \sigma_m\}} = \mathrm{Star}_{\Sigma^*}(u)$$

and

$$\overline{\{\sigma_1, \ldots, \sigma_m\}}^{(0)} = \{i_1, \ldots, i_l, u\},$$

to Σ' to obtain Σ^* (see fig. 4.8 for an illustration).

Let $p^* = (p, p(u)) \in (\mathbb{R}^d)^{n+1}$ be a generic configuration of $(\Sigma^*)^{(0)}$ (and therefore $p \in (\mathbb{R}^d)^n$ is a generic configuration of $\Sigma^{(0)}$). Let γ be a finite flex of (Σ^*, p^*), then by lemma 4.7.6, for all $t \in [0, 1]$, $\gamma(t)$ is uniquely defined by $\pi_U(\gamma(t))$ and $p^*(u)$. Therefore γ and $p^*(u)$ uniquely correspond to the flex $\pi_U(\gamma)$ of $((\Sigma^*)^\vee, p)$, for any k-vertex contraction $(\Sigma^*)^\vee$ of u of Σ^*

Suppose, for a sake of contradiction, that γ is a non-trivial flex of (Σ^*, p^*), then there exists $\rho \in \binom{[n+1]}{d+1} \setminus (\Sigma^*)^{(d)}$ so that, for some $t \in (0, 1]$, we have

$$\det(C(\rho, \gamma(t))) \neq \det(C(\rho, p^*)).$$

First of all, $\rho \notin (K_n^d)^{(d)}$, since otherwise it would correspond to some non-trivial flex of some $((\Sigma^*)^\vee, p)$, all of which we have assumed to be rigid in $\mathbb{R}^d$. Secondly, $u \notin \rho^{(0)}$, since the position of u in the flex is uniquely determined by the position of all the other vertices, of which there are at least $d + 1$, hence the flex is a rigid motion of $\mathbb{R}^d$. As these constitute all choices of ρ, we conclude that γ may not be trivial, and hence (Σ^*, p^*) is rigid.

□

Corollary 4.7.7. *Let Σ be a d-dimensional simplicial complex with $f(\Sigma)_d > d$ and suppose that Σ^* is obtained from Σ by performing a $(d + 1)$-vertex split. If (Σ, p) and (Σ^*, p^*) are generic frameworks in $\mathbb{R}^d$, with $p^* = (p, p(u))$, where $u \in (\Sigma^*)^{(d)} \setminus \Sigma$, then the rigidity properties of (Σ, p) and (Σ^*, p^*) are identical.*

Proof. Suppose that we obtain Σ^* from Σ by splitting vertex i_1, incident to d-simplex $i_1 \dots i_{d+1}$, to get the vertex u and the $d+1$ d-simplices $i_1 \dots \hat{i_j} \dots i_d u$ for $1 \leq j \leq d+1$.

Suppose that (Σ, p) and (Σ, q) are frameworks in $\mathbb{R}^d$ of Σ and suppose that (Σ^*, p^*) and (Σ^*, q^*), where $q* = (q, q(u))$, are frameworks in $\mathbb{R}^d$ of Σ^*.

If (Σ, p) and (Σ, q) are equivalent, then all d-simplices of (Σ^*, p^*) and (Σ^*, q^*) have the same d-volumes, except for those containing u. The position of u is uniquely defined, as it lies on the intersection of $d+1$ non-degenerate hyperplanes in $\mathbb{R}^d$, hence (Σ^*, p^*) and (Σ^*, q^*) are equivalent.

If (Σ^*, p^*) and (Σ^*, q^*) are equivalent, then all d-simplices of (Σ, p) and (Σ, q) have the same d-volumes, except for $i_1 \dots i_{d+1}$. However, there exists $\omega \in \mathbb{R}^{\mathrm{Star}_{\Sigma^*}(u)}$ so that

$$\begin{aligned}
\alpha^d_\Sigma(p)_{i_1 \dots i_{d+1}} &= \sum_{j \in [d+1]} \omega_{i_1 \dots \hat{i_j} \dots i_{d+1} u} \alpha(p)_{i_1 \dots \hat{i_j} \dots i_{d+1}} \\
&= \sum_{j \in [d+1]} \omega_{i_1 \dots \hat{i_j} \dots i_{d+1} u} \alpha(q)_{i_1 \dots \hat{i_j} \dots i_{d+1}} \\
&= \alpha^d_\Sigma(q)_{i_1 \dots i_{d+1}},
\end{aligned}$$

and so (Σ, p) and (Σ, q) are equivalent.

If (Σ, p) and (Σ, q) are congruent, then, since $p^*(u)$ is affinely dependent on

$$(p^*(i_1), \dots, p^*(i_{d+1})) = (p(i_1), \dots, p(i_{d+1})),$$

the frameworks (Σ^*, p^*) and (Σ^*, q^*) are congruent.

If (Σ^*, p^*) and (Σ^*, q^*) are congruent, then so are (Σ, p) and (Σ, q), since $\binom{\Sigma^{(0)}}{d+1} \subset \binom{(\Sigma^*)^{(0)}}{d+1}$. ☐

Let $\mathbb{S}^2$ denote the 2-dimensional sphere. We may think of $\mathbb{S}^2$ as the collection of points in $\mathbb{R}^3$ a distance of 1 away from the origin:

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3 : d(x, 0) = 1\}, \tag{4.7}$$

then $\mathbb{S}^2$ is a closed manifold in $\mathbb{R}^3$.

Theorem 4.7.8. *Let Σ be a triangulation of $\mathbb{S}^2$. Then Σ is rigid in $\mathbb{R}^2$.*

In order to prove theorem 4.7.8, we need the following lemma due to Steinitz.

Lemma 4.7.9. *Steinitz and Rademacher [1934]*
Let Σ be a triangulation of $\mathbb{S}^2$ then there exists a (not-necessarily unique) sequence of simplicial complexes

$$K_4^2 = \Sigma_0, \dots, \Sigma_N = \Sigma,$$

so that Σ_i is obtained from Σ_{i-1} by performing an m_i-vertex split, for some $m_i \geq 3$, for each $1 \leq i \leq N$.

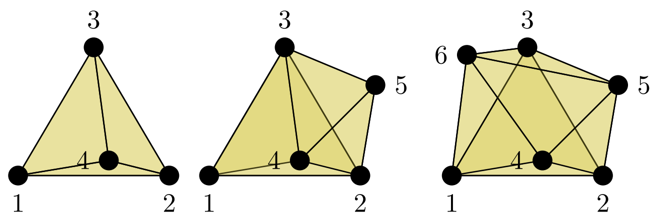

Figure 4.9: Beginning at K_4^2, we perform a 3-vertex split at vertex 4 to get the unique (up to combinatorial isomorphism) triangulation of $\mathbb{S}^2$ on 5 vertices and then a 4-vertex split at vertex 3 to get a triangulation of $\mathbb{S}^2$ on 6 vertices.

Proof. Proceed by induction on $n = f(\Sigma)_0$. Note that $n \geq 4$, since the only 2-dimensional simplicial complex on strictly fewer than 4 vertices with any 2-simplices consists of a single 2-simplex and is therefore not homeomorphic to $\mathbb{S}^2$, i.e. is not a triangulation of $\mathbb{S}^2$. Moreover, the only simplicial complex on 4 vertices that is a triangulation of $\mathbb{S}^2$ is K_4^2. Indeed, there are 3 distinct (up to combinatorial isomorphism) simplicial complexes with more than 1 2-simplex on 4 vertices, they have 2, 3 and 4 2-simplices respectively, the first two may be ruled out immediately as they are homeomorphic to the disc. A homeomorphism may be constructed sending the final simplicial complex above, K_4^2, to $\mathbb{S}^2$, therefore K_4^2 is the only triangulation of $\mathbb{S}^2$ on 4 vertices.

Assume that all triangulations of $\mathbb{S}^2$ on $n \geq 4$ vertices admit a sequence of the form in the statement of the lemma. Let Σ be a triangulation of $\mathbb{S}^2$ on $n+1$ vertices. We claim that we may perform an m-vertex contraction to Σ to obtain a triangulation of $\mathbb{S}^2$, $\Sigma^\vee$, on n vertices. Suppose, for the sake of contradiction, that there is no vertex that may be contracted to obtain a triangulation of $\mathbb{S}^2$, i.e. that, for all $u \in \Sigma^{(0)}$, performing an m-vertex contraction to Σ at u would yield a triangulation of some topological space not homeomorphic to $\mathbb{S}^2$. Such a Σ must have every vertex contained in some separating triangle, i.e. some set of 1-simplices $\{\rho_1, \rho_2, \rho_3\}$ that form a graph-theoretic cycle so that $\Sigma \setminus \{\rho_1, \rho_2, \rho_3\}$ is disconnected. The only such Σ is K_4^3, contradicting our assumption that $f(\Sigma)_0 > 4$. Therefore we may perform an m-vertex contraction to Σ to obtain some triangulation of $\mathbb{S}^2$, $\Sigma^\vee$, and therefore Σ admits the sequence defined as the sequence yielding $\Sigma^\vee$ appended by Σ, of the form in the statement of the lemma. □

Proof of theorem 4.7.8. Proceed by induction on $n = f(\Sigma)_0$. In the base case, $n = 4$, the only possibly triangulation is K_4^2, which is rigid in $\mathbb{R}^2$ by definition. Suppose that $n \geq 4$ and that every triangulation of $\mathbb{S}^2$ on n vertices is rigid in $\mathbb{R}^2$. Now perform an m-vertex split to obtain the simplicial complex Σ, a triangulation of $\mathbb{S}^2$ on n vertices. Any k-vertex contraction, for $k \geq 3$, of Σ would yield a triangulation of $\mathbb{S}^2$, hence by our inductive hypothesis and lemma 4.7.5, Σ is rigid in $\mathbb{R}^2$. □

Moreover, every triangulation of $\mathbb{S}^2$ is redundantly rigid in $\mathbb{R}^2$.

Proposition 4.7.10. *Let Σ be a triangulation of $\mathbb{S}^2$, let $\sigma \in \Sigma^{(2)}$. Then $\Sigma - \sigma$ is rigid in $\mathbb{R}^2$.*

Proof. Since Σ is a triangulation of $\mathbb{S}^2$, the graph $(\Sigma^{(0)}, \Sigma^{(1)})$ is planar. Let $(\Sigma - \sigma, p)$ be a generic configuration that yields a planar embedding of $(\Sigma^{(0)}, \Sigma^{(1)})$ in $\mathbb{R}^2$. Then there exists a signed sum of the areas of all $\tau \in (\Sigma - \sigma)^{(2)}$ so that

$$\alpha_n^2(p)_\sigma = \sum_{\tau \in (\Sigma - \sigma)^{(2)}} (-1)^{a_\tau} \alpha_n^2(p)_\tau.$$

Moreover, this sum holds for all configurations q so that $(\Sigma - \sigma, p)$ and $(\Sigma - \sigma, q)$ are equivalent. So, for such a configuration q, (Σ, p) and (Σ, q) are also equivalent. Since redundant rigidity is a generic property, this then holds for all generic frameworks of Σ in $\mathbb{R}^2$. $\square$

And, the simplex removed from any triangulation of $\mathbb{S}^2$ is generically globally linked in the resulting triangulation of the disc.

Proposition 4.7.11. *Let Σ be a triangulation of $\mathbb{S}^2$ on n vertices and let $\sigma \in \Sigma^{(2)}$, then σ is generically globally linked in $\Sigma - \sigma$.*

Proof. Let $\omega \in \{\pm 1\}^{\Sigma^{(2)}}$ be the unique topological stress of Σ. Then

$$\sum_{\sigma \in \Sigma^{(2)}} \omega_\sigma \alpha_\Sigma^2(p)_\sigma = 0, \tag{4.8}$$

for any framework (Σ, p). This follows since ω indexes a cycle in the second chain group $C_2(\Sigma, \mathbb{R})$.

Let $\sigma \in \Sigma^{(2)}$ and let $(\Sigma - \sigma, p)$ be a generic framework in $\mathbb{R}^2$. Suppose that (Σ, p) is equivalent to (Σ, q). Then, rearranging eeq. (4.8) gives us

$$\alpha_n^2(q)_\sigma = -\omega_\sigma \sum_{\tau \in \Sigma^{(2)} \setminus \{\sigma\}} \alpha_{\Sigma-\sigma}^2(q)_\tau = -\omega_\sigma \sum_{\tau \in \Sigma^{(2)} \setminus \{\sigma\}} \alpha_{\Sigma-\sigma}^2(p)_\tau = \alpha_n^2(p)_\sigma.$$

Hence, σ is generically globally linked in $\Sigma - \sigma$. $\square$

Therefore, for every triangulation Σ of $\mathbb{S}^2$ on n vertices, which we note has $2n - 4$ 2-simplices, there are $2n - 4$ minimally rigid simplicial complexes $\Sigma - \sigma$, where $\sigma \in \Sigma^{(2)}$ (counting potentially combinatorially isomorphic ones) that have the same rigidity properties as Σ.

In the 1980s, Barnette and Edelson showed that for any 2-dimensional manifold M there exists a finite number of *minimal triangulations* of M so that any triangulation of M may be obtained from one of these triangulations by performing a sequence of vertex splits à la Steinitz (Barnette and Edelson [1988] and Barnette and Edelson [1989]).

Therefore, for any such M, we may deduce whether or not all triangulations of M are rigid as in theorem 4.7.8, by checking the minimal triangulations of M

and applying lemma 4.7.5. In Bulavka et al. [2022], a computer-check is used to determine that all minimal triangulations (and therefore all triangulations) of the projective plane, 2-dimensional torus and Klein bottle are rigid in $\mathbb{R}^2$, by methods outlined in chapter 5. We will use our methods to prove explicitly that triangulations of the projective plane is rigid in $\mathbb{R}^2$, since it admits 2 minimal triangulations (a more manageable number than the 21 for the 2-dimensional torus and 29 for the Klein bottle, which we verify via our own computer-checks in appendix B).

Theorem 4.7.12. *Let Σ be a triangulation real projective plane, $\mathbb{RP}^2$. Then Σ is rigid in $\mathbb{R}^2$.*

We state Barnette's minimal triangulation lemma.

Lemma 4.7.13 (Barnette [1982])**.** *Let Σ be a triangulation of $\mathbb{RP}^2$, then there exists a (not-necessarily unique) sequence of simplicial complexes*

$$\Sigma_0, \ldots, \Sigma_N = \Sigma,$$

so that Σ_i is obtained from Σ_{i-1} by performing an m_i-vertex split, for some $m_i \geq 3$, for each $1 \leq i \leq N$, and where Σ_0 is either K_6^2 or is defined by the following set of maximal simplices

$$\Sigma_0^{(2)} = \{123, 127, 134, 145, 156, 167, 235, 257, 347, 356, 367, 457\}.$$

Proof of theorem 4.7.12. Proceed by induction on $n = f(\Sigma)_0$. There are two base cases to consider, one when $n = 6$, which holds trivially since the only possible triangulation is K_6^2, and one when $n = 7$, which we will prove to be rigid.

Let Σ minimal triangulation of $\mathbb{RP}^2$ on 7 vertices given in lemma 4.7.13 and let (Σ, p) be a generic framework of Σ in $\mathbb{R}^2$. Pin vertices 1, 2 and 3 in (Σ, p) to get $(\Sigma, \overline{p})$, with configuration matrix

$$C(\overline{p}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \overline{p}(4)_1 & \overline{p}(5)_1 & \overline{p}(6)_1 & \overline{p}(7)_1 \\ 0 & 0 & 1 & \overline{p}(4)_2 & \overline{p}(5)_2 & \overline{p}(6)_2 & \overline{p}(7)_2 \end{bmatrix}.$$

Suppose that $(\Sigma_0, \overline{p})$ and $(\Sigma_0, \overline{q})$ are equivalent pinned frameworks. Firstly, the subframework $(\mathrm{Star}_{\Sigma_0}(1), \overline{q})$ has four degrees of freedom. The constraint imposed by 2-simplex 235 reduces this to three degrees of freedom, since $\overline{q}(5)$ must now be restricted to the intersection of the lines parallel to

$$y = 1 - x$$

passing through $\overline{p}(5)$. Next, the constraint imposed by 2-simplex 257 reduces this to two degrees of freedom, since $\overline{q}(7)$ must be restricted to the line parallel to

$$y = \frac{\overline{q}(5)_2}{\overline{q}(5)_1 - 1}(x - 1)$$

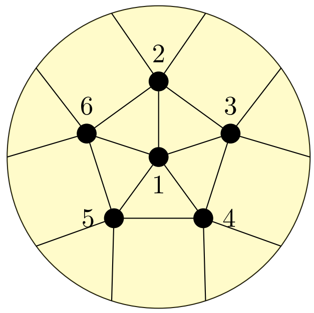

Figure 4.10: The two minimal triangulations of $\mathbb{RP}^2$ with K_6^2 on the left, represented as planar triangulations of $\mathbb{RP}^2$, so antipodal points on each boundary circle are identified.

passing through $\overline{p}(5)$. Finally, the constraints imposed by 2-simplices 347 and 367 reduces this to one and zero degrees of freedom in turn, since $\overline{q}(7)$ must be restricted to the lines parallel to

$$y - 1 = \frac{\overline{q}(4)_2 - 1}{\overline{q}(4)_1} x$$

and

$$y - 1 = \frac{\overline{q}(6)_2 - 1}{\overline{q}(6)_1} x.$$

Then the inductive step used in the proof of theorem 4.7.8 completes this proof. □

Chapter 5

Algebraic Shifting and Volume Rigidity

Algebraic shifting is an umbrella term for algebro-combinatorial operations that take a (not necessarily pure, as we will assume all simplicial complexes encountered in this section to be) simplicial complex Σ and return a *combinatorially shifted* simplicial complex $\Delta(\Sigma)$. A d-dimensional simplicial complex Δ is *combinatorially shifted* with respect to some linear ordering $\prec$ of $\Delta^{(0)}$, which induces linear orderings of $\Delta^{(1)}, \ldots, \Delta^{(d)}$, all denoted $\prec$ in a minor abuse of notation, if, for every $\sigma \in \Delta$, we have that $\tau \in \Delta$, for every $\tau \prec \sigma$.

Our main reference for algebraic shifting is Kalai et al. [2002], which introduces algebraic shifting as well as the two explicit algebraic shifting operations that are most relevant to Euclidean and volume rigidity respectively: *symmetric shifting* and *exterior shifting*.

5.1 Exterior Shifting

Let Σ be a d-dimensional simplicial complex on n vertices and let $\prec$ be a linear ordering of $\Sigma^{(0)} = [n]$ which induces linear orderings of all tuples of $[n]$ We may associate an exterior algebra over $\mathbb{R}$ to Σ, denoted $\bigwedge(\Sigma)$, as follows:

$$\bigwedge(\Sigma) = \bigwedge(\mathbb{R}^n) \big/ I_\Sigma,$$

where I_Σ is an *exterior Stanley-Reisner subspace*, defined

$$I_\Sigma = \{e_\sigma : \sigma \notin \Sigma\},$$

where $e_{i_1 \ldots i_{k+1}} = e_{i_1} \wedge \cdots \wedge e_{i_{k+1}}$, for any k-simplex $i_1 \ldots i_{k+1}$, and where $\{e_1, \ldots, e_n\}$ is the standard basis for $\mathbb{R}^n$ (and hence $\{e_\sigma : \sigma \in 2^{[n]}\}$ is the standard basis for $\bigwedge(\mathbb{R}^n)$).

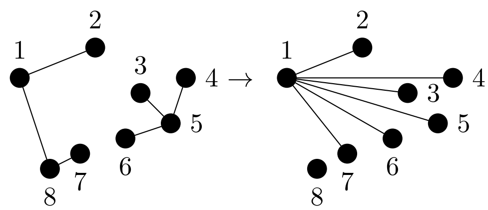

Figure 5.1: Shifting a forest with two connected components to a large star and one isolated vertex.

Let $\{f_1, \dots, f_n\}$ be a generic basis for $\mathbb{R}^n$ (often f_1 will be set to $\underline{1} \in \mathbb{R}^n$ to yield a *quasi-generic* basis, which is practically the same for our purposes), and let $q : \bigwedge(\mathbb{R}^n) \to \bigwedge(\Sigma)$ be the natural quotient map.

Definition 5.1.1. The *exterior shifted complex* of Σ, with respect to $\prec$, denoted $\Delta^{\text{ext}}_{\prec}(\Sigma)$ is the d-dimensional simplicial complex defined in terms of its maximal simplices $\{\sigma \in 2^{[n]} : q(f_\sigma) \notin \mathrm{Span}_{\mathbb{R}}\{q(f_\tau) : |\tau| = |\sigma|, \tau \prec \sigma\}$.

I.e. The k-simplices of $\Delta^{\text{ext}}_{\prec}(\Sigma)$ index the generic $\prec$-greedy basis of $q\left(\bigwedge(\mathbb{R}^n)\right)$.

An alternative algorithm that yields the exterior shifted complex is as follows.

1. Define $F = (f_{i,j})_{i,j\in[n]}$, where $\{f_1, \dots, f_n\}$ is our generic (or quasi-generic) basis for $\mathbb{R}^n$;
2. For each $0 \le k \le d$, let
$$\bigwedge^{k+1} F = \left(\begin{vmatrix} f_{i_1,j_1} & \cdots & f_{i_1,j_{k+1}} \\ \vdots & \ddots & \vdots \\ f_{i_{k+1},j_1} & \cdots & f_{i_{k+1},j_{k+1}} \end{vmatrix} \right)_{\underline{i},\underline{j} \in \binom{[n]}{k+1}} ;$$
3. Define $\Delta^{\text{ext}}_{\prec}(\Sigma)$ in terms of its k-simplices, for all $0 \le k \le d$, so that $\Delta^{\text{ext}}_{\prec}(\Sigma)^{(k)}$ denotes a $\prec$-greedy basis for the column space of $\bigwedge^{k+1} F_{\Sigma^{(k)}}$.

Example 5.1.2. The simplest example of exterior algebraic shifting is that, for graphs, trees are shifted to stars, and forests are shifted to one big star and several isolated vertices, as in fig. 5.1. ◇

We make an observation about the choice of extension of the ordering of vertices.

Example 5.1.3. Let Σ be the pure 3-dimensional simplicial complex on 7 vertices defined by its maximal simplices

$$\Sigma^{(3)} = \{1234, 1235, 1236, 1245, 1246, 1345, 1567, 2347, 3457, 3467\}.$$

Let $\prec_{\text{lex}}$ be the linear lexicographic ordering of $\binom{[7]}{4}$ and let $\prec'$ be a linear extension of the lexicographic ordering of $\binom{[7]}{4}$ in which the 3-simplices of Λ^3_7

are the smallest 10 entries. Then $\Delta_1 = \Delta^{\text{ext}}_{\prec_{\text{lex}}}(\Sigma)$ and $\Delta_2 = \Delta^{\text{ext}}_{\prec'}(\Sigma)$ are defined in terms of their maximal simplices as follows:

$$\begin{aligned}\Delta_1^{(\max)} &= \{67, 346, 347, 356, 1234, 1235, 1236, 1237, 1245, 1246, 1247, 1256,\\ &\quad 1257, 1345\},\\ \Delta_2^{(\max)} &= \{167, 267, 346, 1234, 1235, 1236, 1237, 1245, 1246, 1247, 1256,\\ &\quad 1257, 1345\}.\end{aligned}$$

◇

We make a general observation about the rigidity of a combinatorially shifted pure d-dimensional complex.

Proposition 5.1.4. *Let Δ be a shifted pure d-dimensional simplicial complex on n vertices. The number of independent equilibrium stresses of Δ,* $\mathrm{s}(\Delta)$*, is equal to the number of d-simplices of Δ not present in Λ^d_n:*

$$s(\Delta) = |\Delta^{(d)} \setminus (\Lambda^d_n)^{(d)}|.$$

Proof. Suppose that $\sigma \in \Delta^{(d)} \setminus (\Lambda^d_n)^{(d)}$, if $\sigma = i_1 \dots i_{d+1}$, where $i_1 < \cdots < i_{d+1}$, then $\Lambda^d_{i_{d+1}} \cup \{\sigma\} \subseteq \Delta$. Now, the rank of $\Lambda^d_{i_{d+1}}$ in the volume rigidity matroid is $di_{d+1} - (d^2 + d - 1)$, and so the addition of σ induces a dependency, and therefore an equilibrium stress in any generic framework in $\mathbb{R}^d$. Since the distinct vertices induce independent stresses, $\mathrm{s}(\Delta) = |\Delta^{(d)} \setminus (\Lambda^d_n)^{(d)}|$ as required. □

Corollary 5.1.5. *Let Δ be a shifted pure d-dimensional simplicial complex on n vertices. The rank of Δ is equal to the number of d-simplices of Δ in common with Λ^d_n:*

$$\mathrm{rank}(\Delta) = |\Delta^{(d)} \cap (\Lambda^d_n)^{(d)}|.$$

Proof. We apply the corank-conullity theorem, taking note before that

$$\mathrm{s}(\Delta) = \mathrm{conullity}(R(\Delta, p)),$$

for any $p \in (\mathbb{R}^d)^n$ generic:

$$\mathrm{corank}(R(\Delta, p)) + \mathrm{conullity}(R(\Delta, p)) = f(\Delta)_d,$$

so, by proposition 5.1.4,

$$\begin{aligned}\mathrm{rank}(\Delta) &= \mathrm{corank}(\Delta)\\ &= f(\Delta)_d - \mathrm{s}(\Delta)\\ &= f(\Delta)_d - |\Delta^{(d)} \setminus (\Lambda^d_n)^{(d)}|\\ &= f(\Delta)_d - (f(\Delta)_d - |\Delta^{(d)} \cap (\Lambda^d_n)^{(d)}|\\ &= |\Delta^{(d)} \cap (\Lambda^d_n)^{(d)}|,\end{aligned}$$

as required. □

5.2 Exterior Shifting and Volume Rigidity

In Kalai et al. [2002], a necessary and sufficient condition for the Euclidean rigidity of a graph is given in terms of the combinatorics of its *symmetric shifted* complex:

Theorem 5.2.1. *Kalai et al. [2002]*
Let G be a graph on n vertices and let $\prec_{lex}$ be the linear lexicographic ordering of $\binom{[n]}{2}$. Then G is Euclidean rigid in $\mathbb{R}^d$ if and only if the edge dn is present in its symmetric shifted complex (with respect to $\prec_{lex}$).

We will not spend any time discussing this result or symmetric shifting, but we state it in order to motivate the following result of Bulavka et al. [2022], giving an analogous result for volume rigidity of d-dimensional simplicial complexes in $\mathbb{R}^d$. The fact that different specialisations of algebraic shifting algorithms yield similar results in different rigidity settings suggests that there could be a more fundamental link between the combinatorics of shifted complexes and rigidity under different algebraic constraints and would likely be a fruitful topic for future research.

Theorem 5.2.2. *Bulavka et al. [2022]*
Let Σ be a pure d-dimensional simplicial complex on n-vertices, let $\prec$ be the lexicographic ordering of $[n]$. Then Σ is volume rigid in $\mathbb{R}^d$ if and only if $(\Lambda_n^d)^{(d)} \subseteq \Delta_{\prec}^{ext}(\Sigma)^{(d)}$, for some linear extension of $\prec$ to $\binom{[n]}{d+1}$.

An immediate consequence of this, noted in Southgate [2023b], is an alternative proof to theorem 4.5.2. In order to do so, we note the following lemma:

Lemma 5.2.3. *Björner and Kalai [1989]*
Let Σ be a d-dimensional simplicial complex, and let Δ be the exterior shifted complex of Σ with respect to a suitable $\prec$. Then

1. *$f(\Delta)_k = f(\Sigma)_k$, for all $0 \leq k \leq d$;*
2. *If $\Sigma' \subseteq \Sigma$, then $\Delta_{\prec}^{ext}(\Sigma') \subseteq \Delta$;*
3. *If Σ is shifted, then $\Delta = \Sigma$;*
4. *$\beta(\Delta)_k = \beta(\Sigma)_k$, for all $0 \leq k \leq d$;*
5. *$\beta(\Delta)_d = |\{\sigma \in \Delta^{(d)} : 1 \notin \sigma^{(0)}\}|$.*

Alternative Proof of theorem 4.5.2. The pure d-dimensional simplicial complex on n vertices Σ is volume rigid in $\mathbb{R}^d$ if and only if $(\Lambda_n^d)^{(d)} \subseteq \Delta_{\prec}^{\text{ext}}(\Sigma)^{(d)}$, for a suitable $\prec$. Then

$$f(\Lambda_n^d)_k \leq f(\Delta_{\prec}^{\text{ext}}(\Sigma))_k = f(\Sigma)_k,$$

for all $0 \leq k \leq d$. □

The proof of theorem 5.2.2 involves reformulating the complete rigidity matrix $C(p)$, for $p \in (\mathbb{R}^d)^n$ generic, as a linear map between exterior algebraic spaces

$$\psi : \bigoplus_{i=2}^{d+1} \bigwedge^{1} \mathbb{R}^n \to \bigwedge^{d+1} \mathbb{R}^n; (m_2, \ldots, m_{d+1}) \mapsto \sum_{i=2}^{d+1} f_{[d+1]\setminus\{i\}} \wedge m_i, \tag{5.1}$$

where

$$f_1 = \underline{1} \text{ and}$$
$$f_j = (p(1)_{j-1}, \ldots, p(n)_{j-1}),$$

for all $2 \leq j \leq d+1$ and $f_{d+2}, \ldots, f_n$ complete the quasi-generic basis. The image of ψ is spanned by $\{f_\sigma : \sigma \in (\Lambda_n^d)^{(d)}\}$. Next, the rigidity matrix $R(\Sigma, p)$, for any pure d-dimensional simplicial Σ complex on n vertices is reformulated in exterior-algebraic terms as

$$\psi_\Sigma = q_\Sigma \circ \psi,$$

where $q_\Sigma : \bigwedge(\mathbb{R}^n) \to \bigwedge(\Sigma)$ is the natural quotient map.

We make the following observation relating the rank of a shifted complex to the rank of the original complex in dimension 1.

Proposition 5.2.4. *Let Σ be a 2-uniform simplicial complex on n vertices. Then*

$$\text{rank}(\Sigma) = \text{rank}(\Delta_\prec^{ext}(\Sigma)),$$

for any linear extension $\prec$ of the partial ordering of $\binom{[n]}{2}$.

Proof. Combining theorem 4.1.1 and lemma 5.2.3 yields

$$\text{rank}(\Sigma) = f(\Sigma)_0 - \beta(\Sigma)_0 = f(\Delta_\prec^{\text{ext}}(\Sigma))_0 - \beta(\Delta_\prec^{\text{ext}}(\Sigma))_0 = \text{rank}(\Delta_\prec^{\text{ext}}(\Sigma)),$$

for any $\prec$ as in the statement of the proposition. □

Theorem 5.2.2 states that, for any rigid d-dimensional simplicial complex Σ on n vertices,

$$\text{rank}(\Sigma) = \max_\prec\{\text{rank}(\Delta_\prec^{\text{ext}}(\Sigma))\},$$

as $\prec$ varies over linear extensions of the partial ordering of $\binom{[n]}{d+1}$. We conjecture that when Σ is not necessarily rigid,

$$\text{rank}(\Sigma) \geq \max_\prec\{\text{rank}\, \Delta_\prec^{\text{ext}}(\Sigma))\},$$

as $\prec$ varies over linear extensions of the partial ordering of $\binom{[n]}{d+1}$.

Combining this with corollary 5.1.5 would give a nice bound of the rank of a simplicial complex. Although not quicker than calculating the rank of a symbolic rigidity matrix, results such as lemma 5.2.3 would reveal some underlying necessary conditions to achieve certain ranks, such as our alternative proof to theorem 4.5.2.

5.3 Forbidden Sign Patterns

In this section, we attempt to recreate a weaker version of the acyclic ACT-free formulation of independence developed by Bernstein [2017] by using the LGRC-containment condition we encountered in section 5.2. In doing so we get an idea of what a generalisation of this formulation to higher dimensions.

We begin with a simple example that illustrates how this correspondence arises.

Example 5.3.1. A shifted d-dimensional complex on n vertices Δ is dependent in $\mathcal{R}_n^d$ if and only if it contains a copy of Λ_n^d as a proper sub-complex. Now consider the graph $G(\Delta, 1) = ([n] \setminus \{1\}, D)$, for Δ dependent, where $D \subseteq \Delta^{(1)}$ is the set of 1-simplices of Δ not containing vertex 1. This graph contains copy of K_4^1. While Δ is a-2-cyclic, there is a bijective correspondence between the 2-simplices of Δ and the edges of $G(\Delta, 1)$, therefore, we will define an ordering of the edges of $G(\Delta, 1)$ in terms of the signs of the areas of their defining 2-simplices in some framework of Δ.

Let (Δ, p) be a general position framework in $\mathbb{R}^2$, then the *sign pattern* of the face-areas of (Δ, p), is the vector

$$\zeta_{(\Delta,p)} = (\operatorname{sign}(\det(C(\sigma, p))) : \sigma \in \Delta^{(2)}).$$

Pinning (Δ, p) is equivalent to multiplying $\zeta_{(\Sigma,p)}$ by $\operatorname{sign}(\det(C(123, p)))$ to get

$$\begin{aligned}\zeta_{(\Delta,\overline{p})} = (1, &\operatorname{sign}(\overline{p}(4)_2), \dots, \operatorname{sign}(\overline{p}(n)_2), -\operatorname{sign}(\overline{p}(4)_1), \dots, -\operatorname{sign}(\overline{p}(n)_1,) \\ &\operatorname{sign}(\overline{p}(4)_1\overline{p}(5)_2 - \overline{p}(4)_2, \overline{p}(5)_1), \dots).\end{aligned}$$

Note that $\zeta_{(\Delta,\overline{p})}$ (and therefore $\zeta_{(\Delta,p)}$) does not necessarily induce an acyclic orientation of $G(\Delta, 1)$, indeed general position pinned frameworks of Δ exist so that

$$(\operatorname{sign}(\overline{p}(4)_1), \operatorname{sign}(\overline{p}(4)_2)) = (-1, 1),$$

inducing the directed cycle $(2, 3)$, $(3, 4)$, $(4, 2)$ in $G(\Delta, 1)$. However, for every general position p that induces an acyclic orientation of $G(\Delta, 1)$, there is an ACT in edges $23, 24, 25, 34, 35, 45$. $\diamond$

We might note that example 5.3.1 is somewhat trivial - we will always induce a K_4^1 in making a shifted complex dependent, and the edge set of K_4^1 is the smallest dependent set in $\mathcal{M}(\operatorname{Gr}(2, N))$. In fact, for all dependent shifted complexes Δ, there is an analogous sub-complex of $G(\Delta)$.

Lemma 5.3.2. *Let Δ be a pure d-dimensional simplicial complex on $n \geq d+4$ vertices that is shifted with respect to some linear extension $\prec$ of the lexicographic ordering. If $\Lambda_n^d \subseteq \Delta$ and $\Delta^{(d)} = (\Lambda_n^d)^{(d)} \cup \{\sigma\}$, then either $\sigma = ([d+1] \setminus \{1\}) \cup \{d+2\}$ and Δ admits a d-cycle, or $\sigma = [d-1] \cup \{d+2, d+3\}$.*

Proof. Let $\prec_p$ denote the partial lexicographic ordering of $\binom{[n]}{d+1}$ and let $\prec$ denote a linear extension of $\prec_p$. Suppose that $\sigma \in \Delta^{(d)} \setminus (\Lambda_n^d)^{(d)}$, then $\sigma \neq ([d+1] \setminus \{i\}) \cup \{j\}$, for $2 \leq i \leq d+1 < j \leq n$. We therefore have several options for σ:

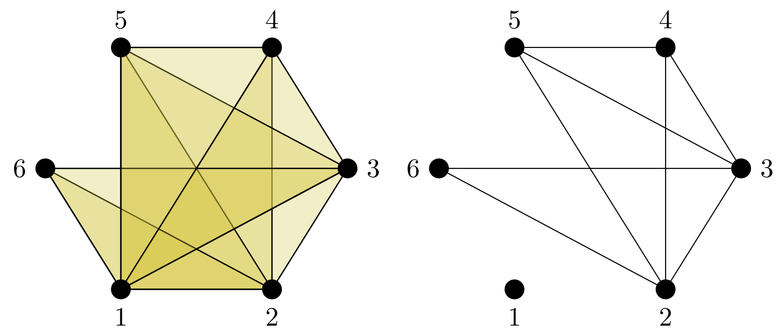

Figure 5.2: The 2-simplex 145 in Δ induces the edge 45 in $G(\Delta, 1)$, which in turn completes a copy of K_4^1, yielding a dependency in $\mathcal{M}(\mathrm{Gr}(2, [6] \setminus \{1\}))$

.

1. Suppose that $\sigma = ([d+1] \setminus \{1\}) \cup \{j\}$, so by lemma 5.2.3, Δ is d-cyclic. Then, by the shifted-ness of Δ, $j = d+2$;
2. Suppose that $1 \in \sigma$. Then, σ must be one $\prec_p$-greater than some d-simplex $([d+1] \setminus \{i\}) \cup \{j\} = [i-1] \cup \{i+1, \ldots, d+1, j\}$, so

$$\begin{aligned}\sigma \in \{&[i-2] \cup \{i, \ldots, d+1, j\}, [i-1] \cup \{i+1, \ldots, d, d+2, j\}, \\ &[i-1] \cup \{i+1, \ldots, d+1, j+1\}, \\ &\{([d+1] \setminus \{i-1\}) \cup \{j\}, ([d] \setminus \{i\}) \cup \{d+2, j\}, \\ &([d+1] \setminus \{i\})\{j+1\}\},\end{aligned}$$

so $\sigma = ([d] \setminus \{i\}) \cup \{d+2, j\}$, for some $2 \leq i \leq d+1 < j \leq n$. Since $([d] \setminus \{d\}) \cup \{d+2, d+3\} \prec_p \sigma$, for any $2 \leq i \leq d+1 < j \leq n$, shifted-ness tells us that $i = d$ and $j = d+3$

□

Now, suppose that Δ is an a-d-cyclic dependent pure d-dimensional shifted complex on n vertices, then $\Delta^{(d)} \supseteq ((\Lambda^d_{d+3})^{(d)} \cup ([d-1] \cup \{d+2, d+3\}))$. If $G(\Delta)$ is the induced pure $(d-1)$-dimensional simplicial complex with respect to distinguished vertex 1, then $G(\Delta)$ contains the $(d-1)$-dimensional sub-complex G', defined by its maximal simplices

$$\begin{aligned}(G')^{(d-1)} = \{&([d+1] \setminus \{1, i\}) \cup \{j\}, ([d-1] \setminus \{1\}) \cup \{d+2, d+3\} \\ &2 \leq i \leq d+1 < j \leq d+3\}.\end{aligned}$$

As it happens, given a generic framework (Δ, p), pinning it to $(\Delta, \overline{p})$ yields

$$\begin{aligned}\zeta_{(\Delta,\overline{p})} = (&1, \mathrm{sign}(\overline{p}(d+2)_d), \mathrm{sign}(\overline{p}(d+3)_d), -\mathrm{sign}(\overline{p}(d+2)_{d-1}), \\ &-\mathrm{sign}(\overline{p}(d+3)_{d-1}), \ldots, (-1)^d \mathrm{sign}(\overline{p}(d+2)_1), (-1)^d \mathrm{sign}(\overline{p}(d+3)_1), \\ &\mathrm{sign}(\overline{p}(d+2)_{d-1}\overline{p}(d+3)_d - \overline{p}(d+2)_d\overline{p}(d+3)_{d-1})).\end{aligned}$$

This yields a set of sign patterns that imply dependence in the rigidity matroid.

Chapter 6

Counting Frameworks of Triangulations of Spheres

Given a framework, either of a graph or of a d-dimensional simplicial complex, in $\mathbb{R}^d$, either that framework is flexible, in which case it admits a continuum of congruence classes, or it is rigid, in which case it admits finitely many congruence classes. In the latter case, it is natural to ask "can we bound the number of congruence classes in terms of properties of the graph or simplicial complex?"

This question was first posed and answered in the Euclidean graph rigidity setting in Borcea and Streinu [2002], where the authors used two techniques from intersection theory - Bézout's theorem and the degree of the measurement variety - to obtain successively stronger upper bounds on the number of congruence classes of a generic framework in $\mathbb{R}^d$ of a graph on n vertices in terms of d and n.

Since then, improved upper bounds were found for a class of graphs satisfying the *M-connectedness* property in Jackson and Jordán [2005] and a group based at RICAM in Austria has studied lower bounds for minimally Euclidean rigid graphs (see Georg Grasegger and Tsigaridas [2020]).

The same question was then posed in the volume rigidity setting in Borcea and Streinu [2013], where the authors again used the degree of the measurement variety to obtain a upper bound on the number of congruence classes of a generic framework in $\mathbb{R}^d$ of a d-dimensional simplicial complex on n vertices in terms of d and n.

Throughout this section we will denote by u_n^d the upper bound and by l_n^d the lower bound on the number of congruence classes of a generic framework in $\mathbb{R}^d$ of a rigid d-dimensional simplicial complex on n vertices. Given a specific framework (Σ, p) in $\mathbb{R}^d$, we will denote the number of its congruence classes by $c(\Sigma, p)$, and we will denote the upper and lower bounds on the number of congruence classes of generic frameworks of Σ in $\mathbb{R}^d$ by $u^d(\Sigma)$ and $l^d(\Sigma)$ respectively. Therefore, if $f(\Sigma)_0 = n$,

$$l_n^d \leq l^d(\Sigma) \leq c(\Sigma, p) \leq u^d(\Sigma) \leq u_n^d,$$

for any $p \in (\mathbb{R}^d)^n$ generic.

As we will see in this chapter and chapter 7, $c(\Sigma, p)$ is not constant as p varies amongst valid generic configurations, nor are $l_n^d(\Sigma)$ and $u_n^d(\Sigma)$ constant as Σ varies amongst d-dimensional simplicial complexes on n vertices, hence the distinction between these different quantities.

The main results of this Chapter appear in Southgate [2023a].

6.1 Bounds From Intersection Theory

All the bounds we will consider arise in some way from the algebraic geometry of the objects we are considering - the framework, the measurement variety and the configuration space.

In this section we consider two upper bounds on the number of congruence classes of simplicial complexes arising from the algebraic geometry of the system. In particular, we make use of Bézout's theorem and the definition of the degree of an algebraic variety to obtain bounds for any rigid simplicial complex and minimally rigid simplicial complexes respectively. The latter technique was used by Borcea and Streinu [2013], we recreate their proof in this section.

Notably, we do not take advantage of the combinatorics of the simplicial complex in deducing these bounds, they are such that they would apply to the most generic hypersurfaces or algebraic varieties of the same degree as the measurement variety. Therefore, potential dependencies that could lead to smaller upper bounds arising from the fact that α_n^d has rational coefficients, or the combinatorics of certain families of simplicial complexes are overlooked. We endeavour, in sections 6.2 and 6.3 to utilise these to obtain some better, albeit less general, upper bounds.

6.1.1 Bézout's theorem

Suppose that Σ is a rigid d-dimensional simplicial complex with $[d+1] \in \Sigma^{(d)}$ and suppose that (Σ, p) is a generic framework in $\mathbb{C}^d$. Then (Σ, p) and (Σ, q) are equivalent if and only if $q \in (\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p))$. We note that this is equivalent to $\alpha_\Sigma^d(q) \in \mathbb{C}^d$ satisfying the following set of equations

$$z_\sigma - \alpha_\Sigma^d(p)_\sigma = 0, \tag{6.1}$$

for all $\sigma \in \Sigma^{(d)}$. These equations determine the equality of the measurement of each simplex with the corresponding simplex in (Σ, p). In order to projectivise, we homogenise these equations with the variable $z_{[d+1]}$, in order to maintain scale, we multiply z_σ by $\alpha_\Sigma^d(p)_{[d+1]}$ to obtain the set of homogeneous equations

$$\alpha_\Sigma^d(p)_{[d+1]} z_\sigma - \alpha_\Sigma^d(p)_\sigma z_{[d+1]} = 0, \tag{6.2}$$

for all $\sigma \in \Sigma^{(d)} \setminus \{[d+1]\}$. These define $f(\Sigma)_d - 1$ projective hypersurfaces in $\mathbb{CP}^{f(\Sigma)_d - 1}$, we may therefore apply Bézout's theorem (theorem 2.1.12) to obtain the following upper bound for the number of congruence classes admitted generically.

Theorem 6.1.1. *The upper bound on the number of congruence classes of a generic framework in $\mathbb{R}^d$ of a rigid d-dimensional simplicial complex on n vertices and m maximal simplices satisfies the following inequality:*

$$u^d_{n,m} \leq d^{m-1}.$$

Proof. Suppose Σ is a rigid d-dimensional simplicial complex on m maximal simplices with $[d+1] \in \Sigma^{(d)}$ and suppose that (Σ, p) is a generic framework in $\mathbb{C}^d$. Then (Σ, p) and (Σ, q) are equivalent if and only if $[\alpha^d_\Sigma(q)] \in \mathbb{CP}^{m-1}$ satisfies eq. (6.2), for each $\sigma \in \Sigma^{(d)} \setminus \{[d+1]\}$. This is equivalent to $[\alpha^d_\Sigma(q)]$ lying in the intersection of the $m-1$ hypersurfaces in $\mathbb{CP}^{m-1}_z$

$$H_\sigma = \mathbb{V}(\alpha^d_\Sigma(p)_{[d+1]} z_\sigma - \alpha^d_\Sigma(p)_\sigma z_{[d+1]}),$$

for all $\sigma \in \Sigma^{(d)} \setminus \{[d+1]\}$. Therefore, by Bézout's theorem, there are at most

$$\prod_{\sigma \in \Sigma^{(d)} \setminus \{[d+1]\}} \deg(\alpha^d_\Sigma(p)_{[d+1]} z_\sigma - \alpha^d_\Sigma(p)_\sigma z_{[d+1]}) = \prod_{\sigma \in \Sigma^{(d)} \setminus \{[d+1]\}} d = d^{m-1}$$

possible intersection points, and hence, possible values of $[q]$, modulo congruent images and uniform scalings.

The number of frameworks in $\mathbb{R}^d$ equivalent (but not congruent) to a framework of Σ in $\mathbb{R}^d$ is at most equal to the number of frameworks in $\mathbb{C}^d$ equivalent (but not congruent) to a framework of Σ in $\mathbb{C}^d$. □

In order for Σ to be a rigid d-dimensional simplicial complex on n vertices, Σ admits a spanning minimally rigid sub-complex Π with $f(\Pi)_d = dn-(d^2+d-1)$, then

$$d^{dn-(d^2+d-1)-1} = d^{dn-d^2-d} \xrightarrow{d,n\to\infty} d^{d(n-d-1)}, \tag{6.3}$$

and each additional d-simplex in Σ restricts the number of congruence classes further, so the number of congruence classes of a d-dimensional simplicial complex on n vertices, u^d_n, satisfies

$$u^d_n \leq d^{d(n-d-1)}. \tag{6.4}$$

As we will see, this is a vast overestimate in a lot of cases.

6.1.2 Degree of the Measurement Variety

Suppose that Σ is a rigid d-dimensional simplicial complex on n vertices with $[d+1] \in \Sigma^{(d)}$, then Σ admits a spanning minimally rigid sub-complex Π also with $[d+1] \in \Pi^{(d)}$. If (Σ, p) is a generic framework in $\mathbb{C}^d$, then (Σ, p) will have at most as many congruence classes as (Π, p).

Then (Π, p) and (Π, q) are equivalent if and only if

$$\alpha^d_\Pi(q) = \pi_{\Pi^{(d)}} \circ \alpha^d_n(q) = \pi_{\Pi^{(d)}} \circ \alpha^d_n(p) = \alpha_{\Pi^{(d)}}(p). \tag{6.5}$$

The middle equality is equivalent to $\alpha_n^d(q)$ lying on the intersection of the complex measurement variety CM_n^d with the set of hyperplanes

$$z_\sigma - \alpha_n^d(p)_\sigma = 0, \tag{6.6}$$

for all $\sigma \in \Pi^{(d)}$. In order to projectivise, we consider the projective complex measurement variety $\mathbb{P}(CM_n^d) \subseteq \mathbb{CP}^{\binom{[n]}{d+1}-1}$ and homogenise the hyperplanes as in section 6.1.1 to get defining equations

$$\alpha_n^d(p)_{[d+1]} z_\sigma - \alpha_n^d(p)_\sigma z_{[d+1]} = 0, \tag{6.7}$$

for all $\sigma \in \Pi^{(d)} \setminus \{[d+1]\}$. These define $dn - d^2 - d$ projective hyperplanes in $\left(\binom{n}{d+1} - d\right)$-dimensional projective space, by definition, they intersect the $(dn - d^2 - d)$-dimensional variety $\mathbb{P}(CM_n^d)$ in either infinitely many or (when counted with multiplicity) $\deg(\mathbb{P}(CM_n^d))$ points.

Theorem 6.1.2. *The upper bound on the number of congruence classes of a generic framework in $\mathbb{R}^d$ of a rigid d-dimensional simplicial complex on n vertices satisfies the following inequality:*

$$u_n^d \leq (d(n-d-1))! \prod_{i=0}^{d-1} \frac{i!}{(n-d-1+i)!}.$$

The inequality here arises from potential generic dependencies in $\mathbb{P}(M_n^d)$. Our proof relies on the following lemma.

Lemma 6.1.3. *[Harris, 2013, ch. 18] The degree of the $(d, n-1)$-Grassmannian variety $\mathrm{Gr}(d, n-1)$ is*

$$(d(n-d-1))! \prod_{i=0}^{d-1} \frac{i!}{(n-d-1+i)!}.$$

Proof of theorem 6.1.2. Suppose that Σ is a rigid d-dimensional simplicial complex on n vertices with $[d+1] \in \Sigma^{(d)}$, then Σ admits a rigid d-dimensional simplicial complex Π on n vertices and $dn - (d^2 + d - 1)$ maximal simplices with $[d+1] \in \Pi^{(d)}$.

Suppose that (Σ, p) is a generic framework in $\mathbb{C}^d$ and that (Σ, p) and (Σ, q) are equivalent, then (Π, p) and (Π, q) are equivalent, then $[\alpha_n^d(q)]$ lies on the intersection of $\mathbb{P}(CM_n^d)$ with the $dn - d^2 - d$ hyperplanes in $\mathbb{CP}_z^{\binom{[n]}{d+1}-1}$

$$H_\sigma = \mathbb{V}(\alpha_n^d(p)_{[d+1]} z_\sigma - \alpha_n^d(p)_\sigma z_{[d+1]}),$$

for all $\sigma \in \Pi^{(d)} \setminus \{[d+1]\}$. This intersection contains either infinitely many or, when counted with multiplicity, $\deg(\mathbb{P}(CM_n^d))$ points. In the former case, $\alpha_\Sigma^d(p)$ will be greater than $(d^2 + d - 1)$-dimensional, contradicting our assumption of rigidity, therefore we are in the latter case.

Since, by theorem 4.2.3, $\mathbb{P}(CM_n^d)$ is birationally equivalent to $\mathrm{Gr}(d, n-1)$, and by proposition 2.1.11, degree is birationally invariant, $[\alpha_n^d(q)]$ may lie in one of, at most, $\deg(\mathrm{Gr}(d, n-1))$ intersections of our set of hyperplanes with $\mathbb{P}(CM_n^d)$, each corresponding to a distinct congruence class of (Σ, p). lemma 6.1.3 gives us an explicit formula for this number.

The number of frameworks in $\mathbb{R}^d$ equivalent (but not congruent) to a framework of Σ in $\mathbb{R}^d$ is at most equal to the number of frameworks in $\mathbb{C}^d$ equivalent (but not congruent) to a framework of Σ in $\mathbb{C}^d$. □

Finally, we observe the behaviour of this bound as n grows, using Stirling's approximation, to obtain

$$(d(n-d-1))! \prod_{i=0}^{d-1} \frac{i!}{(n-d-1+i)!} \xrightarrow{d,n\to\infty} d^{d(n-d-1)}. \tag{6.8}$$

Although Stirling's approximation does tend to its limit from below, we have not performed computer checks to verify that this is always the case, especially for simplicial complexes with n a small multiple of d, where the number of congruence classes can be high, but their computation is difficult. Therefore,

$$u_n^d \leq d^{d(n-d-1)}. \tag{6.9}$$

6.1.3 Review

By considering the algebraic geometry of volume rigidity, we observe that the asymptotic behaviour of the number of congruence classes in $\mathbb{R}^d$ of a rigid d-dimensional simplicial complex on n vertices is

$$u_n^d \leq d^{d(n-d-1)},$$

therefore, this upper bound of Borcea and Streinu [2015], although using more intricate intersection theory methods, does not perform better asymptotically than one achieved using Bézout's theorem.

We try some examples to test the tightness of some of these bounds:

Example 6.1.4. Both rigid 2-dimensional simplicial complexes on 4 vertices (up to combinatorial isomorphism) admit 1 congruence class, i.e.

$$l_4^2 = u_4^2 = 1.$$

This is strictly less than the bounds arising from Bézout's theorem:

$$u_{4,m}^2 < \begin{cases} 4 = 2^{3-1}, & \text{if } m = 3, \\ 8 = 2^{4-1}, & \text{if } m = 4, \end{cases}$$

however it is equal to Borcea and Streinu's upper bound:

$$u_4^2 = 1 = (2(4-2-1))! \left(\frac{0!}{(4-2-1)!}\right)\left(\frac{1!}{(4-2-1+1)!}\right).$$

◇

Example 6.1.5. Consider the rigid 3-dimensional simplicial complex on 7 vertices Σ, defined by its maximal simplices

$$\Sigma^{(3)} = \{1234, 1235, 1236, 1245, 1346, 1356, 1467, 2345, 3457, 4567\}.$$

A generic framework of this simplicial complex admits just two congruence classes, while Borcea and Streinu's upper bound is 42. ◇

6.2 Counting Congruence Classes of Bipyramids

Next we move on to a family of 2-dimensional simplicial complexes for which we can prove improved upper bounds. Let $n \geq 5$, an *$(n-2)$-gonal bipyramid*, denoted B_{n-2} is a triangulation of $\mathbb{S}^2$ on n vertices with maximal simplices

$$\begin{aligned} B_{n-2}^{(2)} = \{&123, 12(n-1), 134, \ldots, 1(n-2)(n-1), \\ &23n, 2(n-1)n, 34n, \ldots, (n-2)(n-1)n\}. \end{aligned}$$

As outlined in lemma 6.2.1, there is a simple recipe for constructing bipyramids using vertex-splits: after an initial 3-vertex split taking K_4^2 to B_3, any B_{n-2} may be constructed by performing a sequence of 4-vertex splits.

We make use of the following two lemmas in proving this:

Lemma 6.2.1. *Let $n \geq 6$, then there exists a (unique, up to combinatorial isomorphism) sequence of simplicial complexes*

$$B_3, B_4, \ldots, B_{n-2},$$

so that B_{i-2} is obtained from B_{i-3} by performing a 4-vertex split, for each $6 \leq i \leq n$.

Proof. Let $4 \leq i \leq n$, then B_{i-3} is the 2-dimensional simplicial complex on vertex set $[i-1]$ and maximal simplex set

$$\begin{aligned} B_{i-3}^{(2)} = \{&123, 12(i-2), \ldots, 1(i-3)(i-2), \\ &23(i-1), 2(i-2)(i-1), \ldots, (i-3)(i-2)(i-1)\} \end{aligned}$$

Meanwhile, subject to relabelling to match our original definition, B_{i-2} is the 2-dimensional simplicial complex on vertex set $[i]$ and maximal simplex set

$$B_{i-2}^{(2)} = (B_{i-3}^{(2)} \setminus \{12(i-2), 2(i-2)(i-1)\}) \cup \{12i, 1(i-2)i, 2(i-2)i, (i-2)(i-1)i\}.$$

We notice that this is exactly the result of 4-splitting vertex $i-2$ to introduce the new vertex i along the equator.

We also notice that, as long as we perform out 4-split along the equator, the resulting complex will be the desired bipyramid, its vertices a permutation of those given in the definition. □

With this in mind, we prove the following lemma regarding the effect 4-splitting has on the number of congruence classes of a simplicial complex in $\mathbb{R}^2$.

Lemma 6.2.2. *Let Σ be a triangulation of $\mathbb{S}^2$ and let Σ^* be a triangulation of $\mathbb{S}^2$ obtained from Σ by performing a 4-vertex split. Then $u^2(\Sigma^*) = u^2(\Sigma)+a(\Sigma)$, where $a(\Sigma)$ is a calculable geometric property of any generic framework of Σ in $\mathbb{R}^2$.*

Proof. We will proceed by constructing two polynomials, $f, g \in \mathbf{k}[s]$, where $\mathbf{k}$ is a suitable field extension of $\mathbb{Q}$, so that the roots of f and g are in one-to-one correspondence with congruence classes of some generic framework of Σ and Σ^* respectively. We will then show that

$$\deg(g) = \deg(f) + 1.$$

After potentially relabelling the vertices of Σ, so that $123 \in \Sigma^{(2)}$, we will perform our 4-split at vertex 1, deleting either the pair of 2-simplices $12i, 1ij$ or the pair $12j, 2ij$ to get Σ' and adding vertex u and 2-simplices $12u, 1ju, 2iu, iju$.

Choose generic frameworks (Σ, p), (Σ, q), (Σ^*, p^*) and (Σ^*, q^*), where

$$p^* = (p^*, p^*(u)) = (p, p^*(u)) \text{ and } q^* = (q^*, q^*(u)) = (q, q^*(u)),$$

so that (Σ, p) and (Σ, q) are equivalent and (Σ^*, p^*) and (Σ^*, q^*) are equivalent. Pin each framework to get $(\Sigma, \overline{p})$, $(\Sigma, \overline{q})$, $(\Sigma^*, \overline{p^*})$ and $(\Sigma^*, \overline{q^*})$ respectively.

The systems of equations that define the pinned equivalences above may be written as follows:

$$\begin{aligned} R =& R_1 \cup R_2 \\ =& \{\alpha^2_\Sigma(\overline{p})_\sigma = \alpha^2_\Sigma(\overline{q})_\sigma : \sigma \in (\Sigma')^{(2)}\} \\ & \cup \{\alpha^2_\Sigma(\overline{p})_\sigma = \alpha^2_\Sigma(\overline{q})_\sigma : \sigma \in \{12j, 2ij\}\}, \end{aligned} \tag{6.10}$$

if $12j, 2ij \in \Sigma^{(2)}$,

$$\begin{aligned} S =& S_1 \cup S_2 \\ =& \{\alpha^2_\Sigma(\overline{p})_\sigma = \alpha^2_\Sigma(\overline{q})_\sigma : \sigma \in (\Sigma')^{(2)}\} \\ & \cup \{\alpha^2_\Sigma(\overline{p})_\sigma = \alpha^2_\Sigma(\overline{q})_\sigma : \sigma \in \{12i, 1ij\}\}, \end{aligned} \tag{6.11}$$

if $12i, 1ij \in \Sigma^{(2)}$, and

$$\begin{aligned} T =& T_1 \cup T_2 \\ =& \{\alpha^2_{\Sigma^*}(\overline{p^*})_\sigma = \alpha^2_{\Sigma^*}(\overline{q^*})_\sigma : \sigma \in (\Sigma')^{(2)}\} \\ & \cup \{\alpha^2_{\Sigma^*}(\overline{p^*})_\sigma = \alpha^2_{\Sigma^*}(\overline{q^*})_\sigma : \sigma \in \{12u, 1ju, 2iu, iju\}\}. \end{aligned} \tag{6.12}$$

Since $\pi_{\Sigma^{(0)}}(\overline{p^*}) = p$ and $\pi_{\Sigma^{(0)}}(\overline{q^*}) = q$, we have that $R_1 = S_1 = T_1$. We want to find $\overline{q^*}$ that satisfies T_1 and T_2 in terms of one that satisfies either R_1 and R_2 or S_1 and S_2.

Solving R_1 to get $\tilde{q}$ leaves one degree of freedom, we solve it in terms of the variable $\tilde{q}(i)_2$, when solving S_1, we do so in terms of $\tilde{q}(j)_2$.

Meanwhile solving R_2 in terms of $\hat{q}(i)_2$ yields

$$\hat{q}(i) = \left(\frac{\begin{vmatrix} \overline{p}(i)_1 & \overline{p}(j)_1 \\ \overline{p}(i)_2 & \overline{p}(j)_2 \end{vmatrix} + \overline{p}(i)_2 + (\overline{p}(j)_1 + r - 1)\overline{q}(i)_2}{\overline{p}(j)_2}, \hat{q}(i)_2 \right) \tag{6.13}$$
$$\hat{q}(j) = (\overline{p}(j)_1 + r, \overline{p}(j)_2),$$

and solving S_2 in terms of $\hat{q}(j)_2$ yields

$$\begin{aligned} \hat{q}(i) &= (\overline{p}(i)_1 + s, \overline{p}(i)_2), \\ \hat{q}(j) &= \left(\frac{\begin{vmatrix} \overline{p}(i)_1 & \overline{p}(j)_1 \\ \overline{p}(i)_2 & \overline{p}(j)_2 \end{vmatrix} + (\overline{p}(j)_1 + s)\hat{q}(j)_2}{\overline{p}(i)_2}, \hat{q}(j)_2 \right). \end{aligned} \tag{6.14}$$

Next, solving T_2 in terms of r, $\hat{q}(i)_2$ yields

$$\begin{aligned} \hat{q^*}(i) &= \left(\frac{\begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(u)_2 \end{vmatrix} + \overline{p^*}(i)_2 + \hat{q^*}(i)_2(\overline{p^*}(u)_1 + r - 1)}{\overline{p^*}(u)_2}, \hat{q^*}(i)_2 \right), \\ \hat{q^*}(j) &= \left(\frac{\begin{vmatrix} \overline{p^*}(j)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(j)_2 & \overline{p^*}(u)_2 \end{vmatrix} + \hat{q^*}(j)_2(\overline{p^*}(u)_1 + r)}{\overline{p^*}(u)_2}, \right. \\ &\left. \frac{\left(\overline{p^*}(i)_2 - \begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(j)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(j)_2 \end{vmatrix}\right)\overline{p^*}(u)_2 + \left(\overline{p^*}(u)_2 - \begin{vmatrix} \overline{p^*}(j)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(j)_2 & \overline{p^*}(u)_2 \end{vmatrix}\right)\hat{q^*}(i)_2}{\hat{q^*}(i)_2 - \overline{p^*}(i)_2 - \begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(u)_2 \end{vmatrix}} \right), \\ \hat{q^*} &= (\overline{p^*}(u)_1 + r, \overline{p^*}(u)_2) \end{aligned} \tag{6.15}$$

and solving T_2 in terms of s, $\hat{q^*}(j)_2$ yields

$$
\begin{aligned}
\hat{q^*}(i) =& \left(\frac{-\left(\begin{array}{c} \overline{p^*}(i)_2\overline{p^*}(j)_2\overline{p^*}(u)_1(\overline{p^*}(u)_1 - 1) + \overline{p^*}(i)_1(\overline{p^*}(j)_1 - 1)\overline{p^*}(u)_2^2 + \\ (\overline{p^*}(i)_1\overline{p^*}(j)_2 + \left(\begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(j)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(j)_2 \end{vmatrix} + \overline{p^*}(i)_2 \right) \overline{p^*}(u)_1)\overline{p^*}(u)_2 + \\ \left(-\left(\begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(j)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(j)_2 \end{vmatrix} + \overline{p^*}(i)_2 \right) \overline{p^*}(u)_2 + \left(\begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(u)_2 \end{vmatrix} + \overline{p^*}(i)_2 \right) \hat{q^*}(j)_2 \right) (\overline{p^*}(u)_1 + s) \end{array} \right)}{\left(\overline{p^*}(u)_2 - \begin{vmatrix} \overline{p^*}(j)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(j)_2 & \overline{p^*}(u)_2 \end{vmatrix} - \hat{q^*}(j)_2 \right) \overline{p^*}(u)_2}, \right. \\
& \left. \frac{\left(\begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(j)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(j)_2 \end{vmatrix} + \overline{p^*}(i)_2 \right) \overline{p^*}(u)_2 - \left(\begin{vmatrix} \overline{p^*}(i)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(i)_2 & \overline{p^*}(u)_2 \end{vmatrix} - \overline{p^*}(i)_2 \right) \hat{q^*}(j)_2}{\overline{p^*}(u)_2 - \begin{vmatrix} \overline{p^*}(j)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(j)_2 & \overline{p^*}(u)_2 \end{vmatrix} - \hat{q^*}(j)_2} \right), \\
\hat{q^*}(j) =& \left(\frac{\begin{vmatrix} \overline{p^*}(j)_1 & \overline{p^*}(u)_1 \\ \overline{p^*}(j)_2 & \overline{p^*}(u)_2 \end{vmatrix} + \hat{q^*}(j)_2(\overline{p^*}(u)_1 + s)}{\overline{p^*}(u)_2}, \hat{q}(j)_2 \right), \\
\hat{q^*}(u) =& (\overline{p^*}(u)_1 + s, \overline{p^*}(u)_2).
\end{aligned} \tag{6.16}
$$

We have a well defined solution to R (resp. S) if and only if

$$\tilde{q}(i)_2 = \overline{p}(i)_2 \text{ (resp. } \tilde{q}(j)_2 = \overline{p}(j)_2). \tag{6.17}$$

These equations each have degree 1, so the number of solutions to R (resp. S), generically, is the degree of R (resp. the degree of S).

Next, we have a well defined solution to T if and only if

$$\hat{q^*}(i) = \tilde{q}(i) \text{ (resp. } \hat{q^*}(j) = \tilde{q}(j)). \tag{6.18}$$

The maximum degree of these equations (each expressed as a polynomial, by multiplying both sides by the denominators) is

$$\deg(\operatorname{num}(\hat{q^*}(i)_2) = 0) + 1 \text{ (resp. } \deg(\operatorname{num}(\hat{q^*}(j)_2) = 0) + 1). \tag{6.19}$$

Since each framework $(\Sigma^*, \overline{q^*})$ equivalent to $(\Sigma^*, \overline{p})$ yields a solution to one of the equations in eq. (6.18), the number of distinct such frameworks is at most the quantities in eq. (6.19). □

Theorem 6.2.3. *Let $n \geq 5$, then $u^2(B_{n-2}) = n - 4$.*

Proof. Suppose that (B_{n-2}, p) is a generic framework in $\mathbb{R}^2$ and that (B_{n-2}, p) and (B_{n-2}, q) are equivalent.

We use the same notation as in the proof of lemma 6.2.2, with vertices $n-2$ and $n-1$ of B_{n-3} as j and i respectively. We will split B_{n-3} by deleting 2-simplices $12(n-2)$ and $2(n-2)(n-1)$, relabelling vertex $n-1$ to vertex n, adding

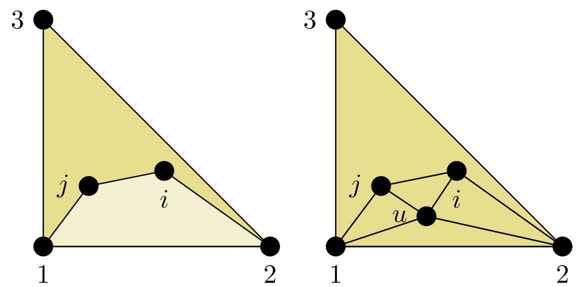

Figure 6.1: An example of how 4-splitting may be viewed in a pinned framework.

a new vertex $n-1$ (which fills the role of vertex u from the proof of lemma 6.2.2) and adding 2-simplices $12(n-1), 1(n-1)(n-2), 2(n-1)n, (n-2)(n-1)n$. Our goal is now to find the degree of $\text{num}(\hat{q^*}(n-2))_2$.

Since

$$\alpha^2_{B_{n-2}}(\overline{p})_{134} = \alpha^2_{B_{n-2}}(\overline{q})_{134},$$
$$\alpha^2_{B_{n-2}}(\overline{p})_{23n} = \alpha^2_{B_{n-2}}(\overline{q})_{23n},$$

we may write

$$\overline{q}(4) = (\overline{p}(4)_1, \overline{p}(4)_2 + t),$$
$$\overline{q}(n) = (\overline{p}(n)_1 + r, \overline{p}(n)_2 - r).$$

And since

$$\alpha^2_{B_{n-2}}(\overline{p})_{34n} = \alpha^2_{B_{n-2}}(\overline{q})_{34n},$$

we have

$$r = \frac{\overline{p}(n)_1 t}{1 - \overline{p}(4)_1 - \overline{p}(4)_2 - t}.$$

Then, for each $4 \leq i \leq n$, $\overline{q}(4)_1$ lies on the intersection of the two lines

$$\alpha^2_{B_{n-2}}(\overline{p})_{1(i-1)i} = \alpha^2_{B_{n-2}}(\overline{q})_{1(i-1)i},$$
$$\alpha^2_{B_{n-2}}(\overline{p})_{(i-1)in} = \alpha^2_{B_{n-2}}(\overline{q})_{(i-1)in},$$

in $\mathbf{k}^2$. Hence

$$\tilde{q}(i) = \left(\frac{\left(\begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix} - \begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(n)_1 \\ \overline{p}(i-1)_2 & \overline{p}(n)_2 \end{vmatrix} + \begin{vmatrix} \overline{p}(i)_1 & \overline{p}(n)_1 \\ \overline{p}(i)_2 & \overline{p}(n)_2 \end{vmatrix} \right) \tilde{q}(i-1)_1}{\begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix}} \right.$$
$$+ \frac{\begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(i)_1 \\ \overline{p}(i-1)_2 & \overline{p}(i)_2 \end{vmatrix} (\overline{p}(n)_1 + r)}{\begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix}},$$
$$\frac{\left(\begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix} - \begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(n)_1 \\ \overline{p}(i-1)_2 & \overline{p}(n)_2 \end{vmatrix} + \begin{vmatrix} \overline{p}(i)_1 & \overline{p}(n)_1 \\ \overline{p}(i)_2 & \overline{p}(n)_2 \end{vmatrix} \right) \tilde{q}(i-1)_2}{\begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix}}$$
$$\left. + \frac{\begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(i)_1 \\ \overline{p}(i-1)_2 & \overline{p}(i)_2 \end{vmatrix} (\overline{p}(n)_2 - r)}{\begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix}} \right), \tag{6.20}$$

for each $4 \leq i \leq n-1$.

Next, for each $4 \leq i \leq n-1$, we have the identity

$$\begin{vmatrix} \tilde{q}(i)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i)_2 & \overline{p}(n)_2 - r \end{vmatrix} - \begin{vmatrix} \tilde{q}(i-1)_1 & \overline{p}(n)_1 + r \\ \tilde{q}(i-1)_2 & \overline{p}(n)_2 - r \end{vmatrix} = \begin{vmatrix} \overline{p}(i)_1 & \overline{p}(n)_1 \\ \overline{p}(i)_2 & \overline{p}(n)_2 \end{vmatrix} - \begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(n)_1 \\ \overline{p}(i-1)_2 & \overline{p}(n)_2 \end{vmatrix}. \tag{6.21}$$

Plugging this identity into the formula for $\tilde{q}(i)$ gives us

$$\tilde{q}(i) = \left(\frac{\left(\begin{vmatrix} \overline{p}(i)_1 & \overline{p}(n)_1 \\ \overline{p}(i)_2 & \overline{p}(n)_2 \end{vmatrix} - r \right) \tilde{q}(i-1)_1 + \begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(i)_1 \\ \overline{p}(i-1)_2 & \overline{p} \end{vmatrix} (\overline{p}(n)_1 + r)}{\begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(n)_1 \\ \overline{p}(i-1)_2 & \overline{p}(n)_2 \end{vmatrix} - r}, \right.$$
$$\left. \frac{\left(\begin{vmatrix} \overline{p}(i)_1 & \overline{p}(n)_1 \\ \overline{p}(i)_2 & \overline{p}(n)_2 \end{vmatrix} - r \right) \tilde{q}(i-1)_2 + \begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(i)_1 \\ \overline{p}(i-1)_2 & \overline{p} \end{vmatrix} (\overline{p}(n)_2 - r)}{\begin{vmatrix} \overline{p}(i-1)_1 & \overline{p}(n)_1 \\ \overline{p}(i-1)_2 & \overline{p}(n)_2 \end{vmatrix} - r} \right). \tag{6.22}$$

Therefore,

$$\deg(\tilde{q}(n-2)_2) = \deg(\tilde{q}(n-3)_2) + 1 = \cdots = \deg(\tilde{q}(4)_2) + n - 6 = n - 5.$$

So by lemma 6.2.2,

$$\deg(\mathrm{num}(\hat{q^*}(n-1)_2) = 0) = n - 5 + 1 = n - 4.$$

Then,

$$\hat{q^*}(n-1)_2 = \overline{p}(n-1)_2$$

is the highest-degree equation defining $(B_{n-2}, \overline{q})$, so there are, generically, $n-4$ solutions in $\mathbb{C}^2$, completing the proof. □

6.3 Gluing Frameworks

We have already encountered some instances of gluing frameworks, for example we glue a star complex framework to a simplicial complex with a number of d-dimensional simplices removed when we perform a vertex split. In this section, we make explicit how gluing affects the number of congruence classes of a framework and use it to prove results about, in particular, triangulations of spheres.

Proposition 6.3.1. *Let Σ be a d-dimensional simplicial complex obtained by gluing Σ_1 and Σ_2 together at vertices $S = \Sigma_1^{(0)} \cap \Sigma_2^{(0)}$. Let (Σ, p) be a generic framework of Σ in $\mathbb{R}^d$. Then*

$$c^d(\Sigma, p) = \begin{cases} \infty, & \text{if } |S| < d+1; \\ c^d(\Sigma_1, p|_{\Sigma_1^{(0)}})c(\Sigma_2, p|_{\Sigma_2^{(0)}}), & \text{if } |S| = d+1. \end{cases}$$

Proof. Suppose that $|S| < d+1$. Then we can flex the sub-framework of (Σ, p) induced by Σ_2 by a $(d-1)$-dimensional subspace of the space of trivial flexes of $(\Sigma_2, p|_{\Sigma_2^{(0)}})$ whilst keeping the sub-framework induced by Σ_1 constant. Therefore

$$c(\Sigma, p) = \infty.$$

Suppose that $|S| = d+1$. Then, for each (Σ, q) where $q|_{\Sigma_1^{(0)}}$ lies in a given congruence class of $(\Sigma_1, p|_{\Sigma_1^{(0)}})$, we pin the elements of $S \subseteq \Sigma_2^{(0)}$ to their position, which leaves $c(\Sigma_2, p|_{\Sigma_2^{(0)}})$ possible congruence classes for $(\Sigma_2, p|_{\Sigma_2^{(0)}})$ to lie in. □

For the final part of this section, concerning general triangulations of the sphere, we rely on the following lemma.

Lemma 6.3.2. *Nakamoto and Negami [2002] Every triangulation of $\mathbb{S}^2$, whose 1-skeleton has minimum degree at least 4, may be generated from B_4 by a sequence of 4-vertex splits and gluing of B_4.*

Let Σ be a triangulation of $\mathbb{S}^2$ whose 1-skeleton has minimum degree at least 4, then call a sequence of 4-splits and B_4-gluings

$$B_4 = \Sigma_0, \Sigma_1, \ldots, \Sigma_N = \Sigma$$

an *NN-sequence* of Σ.

Lemma 6.3.3. *Every triangulation Σ of $\mathbb{S}^2$ may be generated from a triangulation of $\mathbb{S}^2$ whose 1-skeleton has minimum degree at least 4 by a sequence of 3-vertex splits.*

Proof. Let u be a vertex of Σ whose degree in its 1-skeleton is 3, then, since its star is homeomorphic to a disc (since Σ is a triangulation of $\mathbb{S}^2$), they are each incident to three 2-simplices of the form iju, iku, jku. This is precisely the result of performing a 3-vertex split to the complex $\Sigma^\vee$ at one of i, j or k to get the new vertex u and the above new 2-simplices.

We may perform a similar operation for every such vertex of Σ, since, by necessity, each one is contained in a unique triangle (ijk). □

Let Σ be a triangulation of $\mathbb{S}^2$, then call the sequence

$$B_4 = \Sigma_0, \Sigma_1, \dots, \Sigma_M, \dots, \Sigma_N,$$

where Σ_i is obtained from Σ_{i-1} by either a 4-split or a B_4-gluing, for each $1 \le i \le M$, and Σ_j is obtained from Σ_{j-1} by a 3-split, for each $M+1 \le j \le N$, an *extended NN-sequence* of Σ.

Lemma 6.3.4. *Let Σ be a triangulation of $\mathbb{S}^2$ on n vertices with extended NN-sequence*

$$B_4 = \Sigma_0, \Sigma_1, \dots, \Sigma_M, \dots, \Sigma_N = \Sigma,$$

where Σ_{M+1} being obtained from Σ_M is the first instance of a 3-split, then

$$c(\Sigma, p) = c(\Sigma_M, p|_{\Sigma_M^{(0)}})$$

for any $p \in (\mathbb{R}^2)^n$ generic.

Proof. This follows from corollary 4.7.7. □

By combining lemma 6.2.2 and proposition 6.3.1, we obtain the following bound on the number of congruence classes of a triangulation Σ of $\mathbb{S}^2$, on n vertices, with extended NN-sequence

$$B_4 = \Sigma_0, \Sigma_1, \dots, \Sigma_M, \dots, \Sigma_N,$$

in which the blocks of consecutively added vertices $i_{1,1}, \dots, i_{1,s_1}, i_{2,1}, \dots, i_{t,s_t}$ are the results of 4-splitting:

$$\begin{aligned} u^2(\Sigma) \le & 2(2(\dots(2(2 + \deg(\overline{q}(i_{1,1})_2) + \dots \deg(\overline{q}(i_{1,s_1})_2)) \\ & + \deg(\overline{q}(i_{2,1})_2) + \dots + \deg(\overline{q}(i_{2,s_2})_2)) \dots) \\ & + \deg(\overline{q}(i_{t-1,1})_2) + \dots + \deg(\overline{q}(i_{t-1,s_{t-1}})_2)) \\ & + \deg(\overline{q}(i_{t,1})_2) + \dots + \deg(\overline{q}(i_{t,s_t})_2) \\ = & 2^t + 2^{t-1}(\deg(\overline{q}(i_{1,1})_2) + \dots + \deg(\overline{q}(i_{1,s_1})_2)) + \dots \\ & + 2(\deg(\overline{q}(i_{t-1,1})_2) + \dots + \deg(\overline{q}(i_{t-1}, s_{t-1})) \\ & + \deg(\overline{q}(i_{t,1})_2) + \dots + \deg(\overline{q}(i_{t-1}, s_{t-1})_2), \end{aligned} \tag{6.23}$$

where $\overline{q}$ is as in the proof of lemma 6.2.1. We note that

$$n \ge 3t + \sum_{j=1}^{t} s_j,$$

with the difference counting vertices added by 3-splits, which do not affect $u^2(\Sigma)$. It is currently unclear whether or not the degree always increases by 1 when 4-splitting, we have reason to believe that this would be the case before performing B_4-additions, since before and after the 4-split we take the intersection of two lines, but after the split they are a distance 1 further away from the pinned vertices. If degree did always increase by one, eq. (6.23) would become

$$u^2(\Sigma) \leq 2^t + 2^{t-1}s_1 + \cdots + 2s_{t-1} + s_t, \tag{6.24}$$

and since $t < n$ and $s_j < n$, for each $1 \leq j \leq t$,

$$u^2(\Sigma) < 2^n, \tag{6.25}$$

which would be a vast improvement on known bounds. We would not go as far as to conjecture this since it is likely that gluing copies of B_4 to the triangulation would increase the (geometric) degree of the vertices i_j faster than arithmetically.

6.4 Lower Bounds

We end with a very short section to state and prove the following theorem:

Theorem 6.4.1. *Let $1 \leq d \leq n-1$, then there exists a d-dimensional simplicial complex, Σ, on n vertices so that*

$$c^d(\Sigma) = 1.$$

Proof. Fix $d \geq 1$. When $n = d+1$, take Σ to be a single d-simplex, this admits a single congruence class by definition. We may then repeatedly apply $(d+1)$-vertex splits to Σ to obtain a d-dimensional simplicial complex on any number of vertices greater than $d+1$ that admits (by corollary 4.7.7) a single congruence class. □

A Corollary to our discussion at the end of the previous section is that there is also a sharp lower bound for the number of congruence classes of triangulations of $\mathbb{S}^2$:

Corollary 6.4.2. *Let $4 \leq n$, then there exists a triangulation of $\mathbb{S}^2$ on n vertices so that*

$$c^2(\Sigma) = 1.$$

Proof. Let Σ be the triangulation of $\mathbb{S}^2$ obtained from K_4^2 by performing $n-5$ 3-vertex splits. By corollary 4.7.7,

$$c^2(\Sigma) = c^2(K_4^2) = 1.$$

□

6.5 Summary

We summarise the findings of this Chapter and their relations to existing results in the following table

Σ	$l^d(\Sigma)$	Reference	$u^d(\Sigma)$	Reference
Rigid d-dimensional	1	Theorem 6.4.1	$d^{d(f(\Sigma)_0-d-1)}$	Theorem 4.5.1
Triangulation of $\mathbb{S}^2$	1	Corollary 6.4.2	?	
Bipyramid	?		$f(\Sigma)_0 - 4$	Theorem 6.2.3

.

Chapter 7

Global Volume Rigidity

So far we have been concerned with whether or not a framework, or all generic frameworks, of a d-dimensional simplicial complex is rigid in $\mathbb{R}^d$, i.e. whether or not we may continuously deform the vertices in such a framework in a non-trivial manner to obtain an equivalent framework. However, we may be stricter and ask whether or not we may deform the vertices non-trivially at all - continuously or discontinuously - to obtain an equivalent framework. If the answer is no, we call our framework *globally rigid*.

In general, studying global rigidity theories is *trickier* than studying *local* rigidity theories (as we have been doing in the volume rigidity setting so far). Indeed, some of the fundamental tools we take advantage of, such as rigidity matrices, only give us local information. Meanwhile, in the language of configuration spaces, in the local setting we need only determine whether or not the configuration space is 0-dimensional, whilst in order to determine global rigidity, we must also count its connected components. For these reasons and more, interest in global rigidity is more of a recent phenomenon in rigidity theory, moreover, to the author's knowledge, there had not been any published work in global volume rigidity before Southgate [2023a].

7.1 Global Volume Rigidity

We begin by formally defining global rigidity in terms of equivalence and congruence.

Definition 7.1.1. Let Σ be a d-dimensional simplicial complex on n vertices and let (Σ, p) be a framework in $\mathbb{R}^d$. We say that (Σ, p) is *globally (d-volume) rigid 1* in $\mathbb{R}^d$ if, for all $q \in (\mathbb{R}^d)^n$, if (Σ, p) and (Σ, q) are equivalent, then (Σ, p) and (Σ, q) are congruent.

Note that, by the equivalence of definitions of equivalence 1 and 4 and congruence 1 and 4, the terms equivalence and congruence in definition 7.1.1 may refer to either.

Definition 7.1.2. Let Σ and (Σ, p) be as in definition 7.1.1. We say that (Σ, p) is *globally (d-volume) rigid 2* in $\mathbb{R}^d$ if

$$(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) = (\alpha_n^d)^{-1}(\alpha_n^d(p)).$$

Proposition 7.1.3. *Definitions 7.1.1 and 7.1.2 are equivalent.*

Proof. Let Σ and (Σ, p) be as in definition 7.1.1 and suppose that (Σ, p) is globally rigid 1 in $\mathbb{R}^d$. Suppose that $(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) \supsetneq (\alpha_n^d)^{-1}(\alpha_n^d(p))$, i.e. that there exists $q \in (\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) \setminus (\alpha_n^d)^{-1}(\alpha_n^d(p))$. Taking the d-volume measurements of (Σ, p) and (Σ, q) yields $\alpha_\Sigma^d(p) = \alpha_\Sigma^d(q)$, so (Σ, p) and (Σ, q) are equivalent, but taking the complete d-volume measurements yields

$$\alpha_n^d(p) \neq \alpha_n^d(q),$$

so (Σ, p) and (Σ, q) are not congruent, contradicting our assumption of global rigidity 1.

Let Σ and (Σ, p) be as in definition 7.1.1 and suppose that (Σ, p) is globally rigid 2 in $\mathbb{R}^d$. Suppose that there exists $q \in (\mathbb{R}^d)^n$ so that (Σ, p) and (Σ, q) are equivalent but not congruent. Then

$$\alpha_\Sigma^d(p) = \alpha_\Sigma^d(q) \text{ but } \alpha_n^d(p) \neq \alpha_n^d(q),$$

however, taking the preimages under both maps yields $q \in (\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) \setminus (\alpha_n^d)^{-1}(\alpha_n^d(p))$, so (Σ, p) is not globally volume rigid 2. □

From now on, we will say that (Σ, p) is *globally (d-volume) rigid* in $\mathbb{R}^d$ if it is *globally (d-volume) rigid 1* or equivalently *globally (d-volume) rigid 2* in $\mathbb{R}^d$.

Note that implicit in each definition of global rigidity in $\mathbb{R}^d$ is the fact that (K_n^d, p) is globally volume rigid for all $p \in (\mathbb{R}^d)^n$.

Corollary 7.1.4. *The framework (Σ, p) in $\mathbb{R}^d$ is globally rigid in $\mathbb{R}^d$ if and only if $c(\Sigma, p) = 1$.*

Proof. This follows immediately from the definitions of global rigidity and configuration space, indeed

$$(\alpha_\Sigma^d)^{-1}(\alpha_\Sigma^d(p)) \big/ (\alpha_n^d)^{-1}(\alpha_n^d(p)) = (\alpha_n^d)^{-1}(\alpha_n^d(p)) \big/ (\alpha_n^d)^{-1}(\alpha_n^d(p)) = \{[p]\}.$$

□

Corollary 7.1.5. *The framework (Σ, p) is globally rigid if and only if the only pinned framework equivalent to $(\Sigma, \overline{p})$ is $(\Sigma, \overline{p})$.*

Proof. This follows immediately from corollary 7.1.4, since pinned configurations equivalent to $(\Sigma, \overline{p})$ are precisely the representatives of elements of $\mathcal{C}(\Sigma, p)$. □

In order to differentiate the rigidity we have been talking about so far in this thesis from global rigidity, we will refer to rigidity in $\mathbb{R}^d$ as *local rigidity* in $\mathbb{R}^d$ All globally rigid frameworks are necessarily *locally* rigid in $\mathbb{R}^d$, however the converse is not true, take, for example, the octahedron, which we showed in chapter 6 generically admitted two congruence classes. If (Σ, p) is locally rigid in $\mathbb{R}^d$, but not globally rigid in $\mathbb{R}^d$, then we say that (Σ, p) is *locally, but not globally, rigid* in $\mathbb{R}^d$, which we will abbreviate as *LNGR*.

This gives us the following characterisation of the rigidity of (Σ, p) in $\mathbb{R}^d$ in terms of $\mathcal{C}(\Sigma, p)$:

- (Σ, p) is flexible if $|\mathcal{C}(\Sigma, p)| = \infty$ (or equivalently, if $\dim(\mathcal{C}(\Sigma, p)) > 0$);
- (Σ, p) is rigid if $|\mathcal{C}(\Sigma, p)| < \infty$ (or equivalently, if $\dim(\mathcal{C}(\Sigma, p)) = 0$);
  - (Σ, p) is globally rigid if $|\mathcal{C}(\Sigma, p)| = 1$;
  - (Σ, p) is LNGR if $|\mathcal{C}(\Sigma, p)| > 1$.

Hendrickson's conditions (see Hendrickson [1992]) are necessary conditions for the global Euclidean rigidity of a generic framework of a graph G in $\mathbb{R}^d$. It states that if (G, p) is a globally Euclidean rigid generic framework of G in $\mathbb{R}^d$, then G is generically redundantly Euclidean rigid and G is $(d+2)$-vertex connected as a graph.

An analogous condition does not hold in the volume rigidity setting, furthermore, we have already encountered a family of pure d-dimensional simplicial complexes that admit globally rigid generic frameworks, fail to be $(d+2)$-vertex connected and are bases in $\mathcal{R}_n^d$ (and therefore can not be redundantly rigid).

Proposition 7.1.6. *Let (Λ_n^d, p) be a generic framework in $\mathbb{R}^d$ of the LGRC on n vertices, then (Λ_n^d, p) is globally rigid in $\mathbb{R}^d$.*

Proof. Pin the d-simplex $[d+1]$ in (Λ_n^d, p) to the unit simplex to get $(\Lambda_n^d, \overline{p})$. Suppose that the pinned framework $(\Lambda_n^d, \overline{q})$ is equivalent to $(\Lambda_n^d, \overline{p})$. Then

$$\overline{q}(i) = \overline{p}(i),$$

for all $i \in [d+1]$, by definition. Now, suppose that $d+1 < i \leq n$, then $\overline{q}(i)$ is restricted to the intersection of d general-position hyperplanes in $\mathbb{R}^d$, which consists of a single point, which we know must equal $\overline{p}(i)$. Since

$$\overline{q}(i) = \overline{p}(i),$$

for all $1 \leq i \leq d+1$ and for all $d+2 \leq i \leq n$, we have that

$$(\Lambda, \overline{p}) = (\Lambda, \overline{q}).$$

By corollary 7.1.5, (Λ_n^d, p) is globally rigid. ☐

7.2 Non-Genericity of Global Volume Rigidity and Stress Matrices

So far we have considered the global rigidity of frameworks of simplicial complexes. Theorem 3.3.9 showed that local rigidity in $\mathbb{R}^d$ is a generic property of the underlying simplicial complex, and hence we may consider the local rigidity of that simplicial complex as the *typical behaviour* of a framework of that complex.

Connelly [2005] and Gortler et al. [2010] showed that global Euclidean rigidity in $\mathbb{R}^d$ is a generic property of the underlying graph by constructing a geometrically-inspired linear-algebraic object, the stress matrix of a framework, and equating global rigidity to this matrix achieving maximum rank (the former showing that this was a necessary condition and the latter that it was sufficient). Notably, this condition is geometric, not combinatorial, i.e. global Euclidean rigidity in $\mathbb{R}^d$ cannot yet be read off the combinatorial structure of the graph, however this is a topic of ongoing research by Jackson, Jordán and others (see, for example, Jackson and Jordán [2005], Garamvölgyi and Jordán [2023]).

Cruickshank et al. [2023] showed that global volume rigidity in $\mathbb{C}^d$ is a generic property (in fact, they show that this is true for a much broader array of measurements than just d-dimensional volume), however in this section, we will show that this fails to be true when we restrict ourselves to $\mathbb{R}^d$. Cruickshank et al. construct similar objects to Connelly and Gortler, Healy and Thurston's (C-GHT) stress matrices and their characterisation of *global g-rigidity* is partly in terms of the dimensions of spaces associated to these matrices.

In this section, we give a counterexample to the genericity of global volume rigidity in $\mathbb{R}^2$ and then discuss the difficulties of recreating a C-GHT-style argument in the case of volume rigidity in $\mathbb{R}^d$.

Theorem 7.2.1. *The pentagonal bipyramid B_5 admits both a globally rigid and an LNGR generic framework in $\mathbb{R}^2$.*

Proof. We showed in theorem 6.2.3 that distinct congruence classes of generic frameworks (B_5, p) in $\mathbb{R}^2$ are in one-to-one correspondence with the roots of the degree 3 polynomial $f(p) \in (\mathbb{Q}[\overline{p}(4), \dots, \overline{p}(7)])[t]$. Conveniently, the number of real roots of a cubic equation in one variable is given by its discriminant, a polynomial with rational coefficients with entries the coefficients of that polynomial. Therefore,

$$|\mathcal{C}(B_5, p)| = \begin{cases} 1, & \text{if and only if } \operatorname{disc}(f(p)) < 0, \\ 3, & \text{if and only if } \operatorname{disc}(f(p)) > 0. \end{cases}$$

It suffices to choose non-generic frameworks q_1 and q_2 so that $\operatorname{disc}(f(q_1)) < 0$ and $\operatorname{disc}(f(q_2)) > 0$ as we may perturb them to obtain generic frameworks p_1 and p_2 respectively so that $\operatorname{disc}(f(p_1)) < 0$ and $\operatorname{disc}(f(p_2)) > 0$. Indeed, the following are the configuration matrices of two such frameworks:

$$\begin{aligned} C(q_1) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{7} & \frac{1}{11} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{13} & \frac{1}{19} & \frac{1}{17} & \frac{1}{2} \end{bmatrix}, \\ C(q_2) &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{7} & \frac{1}{5} & \frac{1}{41} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{19} & \frac{1}{17} & \frac{1}{13} & 2 \end{bmatrix}, \end{aligned} \tag{7.1}$$

thus completing the proof. □

Let (Σ, p) be a framework in $\mathbb{R}^d$ and let $\omega \in \operatorname{Stress}(\Sigma, p)$, then define the *volume stress matrix* with respect to ω, denoted $\Omega(\omega) \in \mathbb{R}^{\Sigma^{(0)} \times \Sigma^{(d-1)}}$, by

$$\Omega(\omega)_{i,\tau} = \begin{cases} \operatorname{sign}(i,\tau)\omega_{i\tau}, & \text{if } i\tau \in \Sigma^{(d)}, \\ 0, & \text{if } i\tau \notin \Sigma^{(d)}. \end{cases}$$

Then, representing the linear transformation $\omega^t R(\Sigma, p)$ by multiplying $\Omega(\omega)$ on the right by the *facet-normal* vectors of (Σ, p) in each direction, i.e. the columns of the matrix

$$N(\Sigma, p) = \begin{bmatrix} \mathbf{n}([d], p)_1 & \dots & \mathbf{n}([d], p)_d \\ \vdots & \ddots & \vdots \\ \mathbf{n}([n] \setminus [n-d], p)_1 & \dots & \mathbf{n}([n] \setminus [n-d], p)_d \end{bmatrix} \in \mathbb{R}^{\Sigma^{(d-1)} \times d}.$$

Then, the orbit of $N(\Sigma, p)$ under right-multiplication by elements of $\mathrm{GL}(d, \mathbb{R})$ corresponds to the facet-normal vectors of frameworks related to (Σ, p) by affine transformations of $\mathbb{R}^d$ (indeed translations are cancelled out, leaving the d^2-dimensional space of $\mathcal{SA}(d, \mathbb{R})$ and non-zero uniform scalings). These correspond to changes of basis of the component of $\operatorname{Ker} \Omega(\omega)$ spanned by the columns of $N(\Sigma, p)$.

The following examples highlight some of the issues we run into with this approach:

Example 7.2.2. Take the pentagonal bipyramid B_5, generically B_5 admits only one stress, the topological stress

$$\omega = (1, -1, 1, 1, 1, -1, 1, -1, -1, -1).$$

The stress matrix $\Omega(\omega)$ is a 7×15-matrix with an 11-dimensional kernel, far greater than the 2-dimensional space of facet-normals of affine images of any (B_5, p). ⋄

Example 7.2.3. Take the globally rigid framework (Σ, p) in $\mathbb{R}^2$, with Σ defined in terms of its maximal simplices

$$\Sigma^{(2)} = \{123, 124, 125, 126, 234, 235, 236, 456\}$$

and with p having configuration matrix

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{7} & \frac{1}{11} \\ 0 & 0 & 1 & \frac{1}{13} & \frac{1}{19} & \frac{1}{17} \end{bmatrix},$$

then the unique stress of (Σ, p) (up to scaling) is

$$\omega = (2612, -73970, 205618, -131648, 10010, 29260, -39270, -1616615).$$

However, we obtain a stress matrix $\Omega(\omega)$ with a 9-dimensional kernel, again far greater than the space of facet-normals of affine images of (Σ, p). ⋄

Example 7.2.4. Finally, the worst-case-scenario for stress matrices is realised by LGRCs which are independent globally rigid d-dimensional simplicial complexes. Therefore, the only equilibrium stress of and generic framework (Λ_n^d, p) in $\mathbb{R}^d$ is the zero vector, and hence the only equilibrium stress matrix of (Λ_n^d, p) is the zero matrix in $\mathbb{R}^{(\Lambda_n^d)^{(0)} \times (\Lambda_n^d)^{(d-1)}}$, which has a $\frac{1}{2}(d(d+1)n - (d^3 + 2d^2 - d - 2))$-dimensional kernel. ⋄

Although we cannot hope for a necessary condition in terms of the stress matrix, Cruickshank et al. [2023] give the following sufficient condition (in slightly more general notation which we omit).

Theorem 7.2.5. *Cruickshank et al. [2023] Suppose that* (Σ, p) *is infinitesimally rigid in* $\mathbb{C}^d$ *(resp.* $\mathbb{R}^d$*) and*

$$\dim \left(\bigcap_{\omega \in \mathrm{Stress}(\Sigma, p)} \mathrm{Ker}(\Omega(\omega)) \right) = d,$$

then (Σ, p) *is globally rigid in* $\mathbb{C}^d$ *(resp.* $\mathbb{R}^d$*).*

They give the following necessary condition.

Proposition 7.2.6. *Cruickshank et al. [2023] Let* Σ *be a* d*-dimensional simplicial complex with* $f(\Sigma)_0 \geq d(d+1)$*. If* Σ *is generically globally rigid, then* Σ *is* k*-vertex connected, where* $k = 1$*, when* $d = 1$ *and* $k = d + 1$ *for all* $d \geq 2$*.*

Noting that a $(d+2)$-vertex connectivity is not a best-case necessary condition (look at the LGRC, for example).

7.3 Constructing Families of Generically Globally Rigid Complexes

Whilst, in general, global rigidity in $\mathbb{R}^d$ is not a generic property of d-dimensional simplicial complexes, there are families of d-dimensional simplicial complexes for whom every generic framework in $\mathbb{R}^d$ is globally rigid.

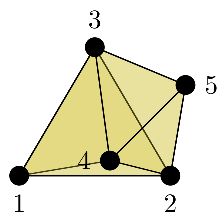

Figure 7.1: This is a 2-lateration ordering of the vertices of B_3^3. Indeed, 2-simplex 123 is present, and so are 124 and 134, 235 and 245.

Definition 7.3.1. Let Σ be a d-dimensional simplicial complex on n vertices. We say that Σ is *generically globally (d-volume) rigid (GGR)* in $\mathbb{R}^d$ if, for all $p \in (\mathbb{R}^d)^n$ generic, (Σ, p) is globally rigid in $\mathbb{R}^d$.

For every $d \geq 1$, two families of simplicial complexes stand out as being GGR in $\mathbb{R}^d$: complete d-dimensional simplicial complexes and d-dimensional LGRCs.

The following definition is inspired by trilateration orderings of graph vertices, a useful tool in computational aspects of distance geometry.

Definition 7.3.2. Let Σ be a d-dimensional simplicial complex on n vertices and let $k \in \mathbb{N}$. A *k-lateration ordering* of $\Sigma^{(0)}$ is a linear ordering $(\Sigma^{(0)}, \prec)$ so that the $\prec$-smallest $d+1$ vertices are contained in a d-simplex of Σ and, for every $d+1 \prec i \preceq n$, there exist k d-simplices of the form $j_1 \dots j_d i$, where $j_1, \dots, j_d \prec i$.

Compare the graph trilateration ordering in fig. 4.3 with simplicial complex setting in fig. 7.1.

Proposition 7.3.3. *Let Σ be a d-dimensional simplicial complex on n vertices, if Σ admits a d-lateration ordering (of its vertices), then Σ is generically globally rigid in $\mathbb{R}^d$.*

Proof. The proof of this proposition follows the same argument as the proof of proposition 7.1.6, except in this case, the vertices $\prec$-greater than $d+1$ are connected to the unique realisation of the pinned graph $\prec$-smaller than them, rather than to just the pinned simplex. □

Recall also that $(d+1)$-vertex splitting also preserved generic global rigidity in $\mathbb{R}^d$, see corollary 4.7.7.

Proposition 7.3.4.

1. *For every $n \geq 4$, there exists a triangulation of $\mathbb{S}^2$ that is generically globally rigid in $\mathbb{R}^2$*
2. *For every $n \geq 6$, there exists a triangulation of $\mathbb{RP}^2$ that is generically globally rigid in $\mathbb{R}^2$*

Proof. For each surface M, we need to show that there exists a triangulation of M on the minimal number of vertices given in the statement that is generically globally rigid in $\mathbb{R}^2$. From that point, we may apply repeated 3-vertex splits to that triangulation to obtain triangulations of M that are generically globally rigid in $\mathbb{R}^2$ on successively larger vertex sets, increasing in increments of 1.

1. The boundary of the tetrahedron, K_4^2, is generically globally rigid in $\mathbb{R}^2$ by definition, so statement 1 holds;

2. One of the minimal triangulations of $\mathbb{RP}^2$ is K_6^2 (Barnette [1982]), which is generically globally rigid in $\mathbb{R}^2$ by definition, so statement 2 holds.

□

The next constructive argument is inspired by the following theorem of Tanigawa.

Theorem 7.3.5. *ichi Tanigawa [2015] Let $G = (V, E)$ be a graph and let $i \in V$ be a vertex of degree at least $d \geq 1$. If $G - i$ is Euclidean rigid in $\mathbb{R}^d$ and if $G - i + K(N_G(i))$ is globally Euclidean rigid in $\mathbb{R}^d$, then G is globally Euclidean rigid in $\mathbb{R}^d$.*

Here $G - i$ and $G - i + K(N_G(i))$ denote the subgraph of G induced by $V \setminus \{i\}$ and the graph obtained from $G - i$ by gluing to it a complete graph whose vertices are the neighbours of i in G respectively. Theorem 7.3.5 is often referred to as the *vertex-removal lemma* because we infer the global rigidity of G from the rigidity and global rigidity of two graphs obtained from $G - i$. Note that this theorem takes for granted the genericity of global rigidity of graphs

Proposition 7.3.6. *Let Σ be a d-dimensional simplicial complex and let $i \in \Sigma^{(0)}$ be a vertex with $|\operatorname{Lk}_\Sigma(i)| \geq d$. Suppose that $\Sigma - i + K^d(\operatorname{Lk}_\Sigma(i))$ is generically globally rigid in $\mathbb{R}^d$, then Σ is generically globally rigid in $\mathbb{R}^d$.*

Here, for any $U \subseteq \Sigma^{(k)}$, $K^d(U)$ denotes the smallest complete graph whose k-skeleton contains U. Meanwhile, $\Sigma - i$ denotes the d-dimensional simplicial sub-complex of Σ induced by $\Sigma^{(0)} \setminus \{i\}$ and $\Sigma - i + K^d(\operatorname{Lk}_\Sigma(i))$ denotes the d-dimensional simplicial complex obtained by gluing to $\Sigma - i$ the complete graph $K^d(\operatorname{Lk}_\Sigma(i))$ at their common $(d-1)$-simplices.

We prove the following lemma to make the proof of proposition 7.3.6 more straightforward.

Lemma 7.3.7. *Let Σ be a d-dimensional simplicial complex and let $i \in \Sigma^{(0)}$ be a vertex with $|\operatorname{Lk}_\Sigma(i)| \geq d$. Let (Σ, p) be a generic framework in $\mathbb{R}^d$, and let $p' = p|_{\Sigma^{(0)} \setminus \{i\}}$. If (Σ, p) and (Σ, q) are equivalent, then $(\Sigma - i + K^d(\operatorname{Lk}_\Sigma(i)), p')$ and $(\Sigma - i + K^d(\operatorname{Lk}_\Sigma(i)), q')$ are equivalent, where $q' = q|_{\Sigma^{(0)} \setminus \{i\}}$. Moreover, (Σ, p) and (Σ, q) are congruent if and only if $(\Sigma - i + K^d(\operatorname{Lk}(i)), p')$ and $(\Sigma - i + K^d(\operatorname{Lk}_\Sigma(i)), q')$ are congruent.*

Proof. Suppose that (Σ, p) and (Σ, q) are equivalent. Then, for each d-simplex $\sigma \in (\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)))^{(d)} \setminus \Sigma^{(d)}$,

$$\begin{aligned} \alpha^d_{\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i))}(p)_\sigma &= \sum_{\tau \in \mathrm{Lk}_\Sigma(i)} \mathrm{sign}(i, \sigma) \alpha^d_\Sigma(p)_{i\tau} \\ &= \sum_{\tau \in \mathrm{Lk}_\Sigma(i)} \mathrm{sign}(i, \sigma) \alpha^d_\Sigma(q)_{i\tau} \\ &= \alpha^d_{\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i))}(q)_\sigma, \end{aligned} \tag{7.2}$$

with the first and third equalities following from the fact that σ and the $(i\tau)$s form the d-simplices of K^d_{d+1}, for each σ, and the second equality coming from the facts that each $i\tau$ is present in $\Sigma^{(d)}$ and that (Σ, p) and (Σ, q) are equivalent. Therefore $(\Sigma - i + K^d(\mathrm{Lk}(i)), p')$ and $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), q')$ are equivalent.

Suppose that (Σ, p) and (Σ, q) are congruent. Then there exists $f \in \mathcal{SA}(d, \mathbb{R})$ so that

$$q(j) = f(p(j)),$$

for all $j \in \Sigma^{(0)}$. Since $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)))^{(0)} \subseteq \Sigma^{(0)}$,

$$q(j) = f(p(j)),$$

for all $j \in \Sigma^{(0)}$.

Suppose that $(\Sigma - i + K^d(\mathrm{Lk}(i)), p')$ and $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), q')$ are congruent. Then there exists $g \in \mathcal{SA}(d, \mathbb{R})$ so that

$$q(j) = g(p(j)),$$

for all $j \in \Sigma^{(0)}$. By the genericity of $p(i)$ and the fact that it has at least $d+1$ neighbours, all with generic placements in $\mathbb{R}^d$, $p(i)$ is affinely dependent on its neighbours and so

$$q(i) = g(p(i)).$$

Therefore (Σ, p) and (Σ, q) are congruent. □

Proof of proposition 7.3.6. Suppose that $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), p)$ is a generic framework in $\mathbb{R}^d$ and that there exists a generic framework (Σ, p^*), where $p^* = (p, p(i))$, that is not globally rigid. Then there exists (Σ, q^*) in $\mathbb{R}^d$ so that (Σ, p^*) and (Σ, q^*) are equivalent, but not congruent. Since (Σ, p^*) and (Σ, q^*) are equivalent, $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), p)$ and $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), q)$, where $q = q^*|_{\Sigma^{(0)} \setminus \{i\}}$, are equivalent, by lemma 7.3.7. Next, by the global rigidity of $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), p)$, we have that $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), p)$ and $(\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i)), q)$ are congruent. Then, by lemma 7.3.7, (Σ, p^*) and (Σ, q^*) must be congruent, a contradiction, and so the proposition is proved. □

Although proposition 7.3.6 allows us to construct successively larger d-dimensional simplicial complexes that are generically globally rigid in $\mathbb{R}^d$, the condition that $\Sigma - i + K^d(\mathrm{Lk}_\Sigma(i))$ must be generically globally rigid is quite restrictive. Note that when $K^d(\mathrm{Lk}_\Sigma(i))$ is the complete graph on $(d+1)$-vertices, our operation is identical to $(d+1)$-vertex splitting. In general, proposition 7.3.6 is quite redundant, indeed, since $|\mathrm{Lk}_\Sigma(i)| \geq d$, we can adapt the argument from trilateration orderings to prove it.

7.4 Global Squared Volume Rigidity

Local generic signed and squared (and absolute) volume rigidity are equivalent (indeed, one may always find a neighbourhood of a generic configuration in $(\mathbb{R}^d)^n$ small enough that perturbing the configuration would not cause any $d+1$ points to lie on the same hyperplane in $\mathbb{R}^d$). This is not the case for global rigidity, as reflections in facets of simplices - discontinuous deformations of the vertices of the framework - preserve squared (and absolute) volume but not signed volume.

We make all the analogous definitions (equivalence, congruence, rigidity) for squared global rigidity with respect to the *(complete) squared (d-volume) measurement map*

$$\beta_n^d : (\mathbb{R}^d)^n \to \mathbb{R}^{\binom{[n]}{d}}; p \mapsto (\alpha_n^d(p))^2 = \left((\alpha_n^d(p)_\sigma)^2 : \sigma \in \binom{[n]}{d}\right),$$

and the group of squared-area preserving affine transformations

$$\left\{\begin{bmatrix} 1 & 0^t \\ b & A \end{bmatrix} : b \in \mathbb{R}^d,\ A \in \mathbb{R}^{d\times d},\ \det(A)^2 = 1\right\} \cong \mathcal{SA}(d, \mathbb{R}) \times \{\pm 1\}.$$

We notice that squared volume rigidity in $\mathbb{R}^1$ and Euclidean graph rigidity in $\mathbb{R}^1$ are completely equivalent, since

$$\beta_n^1(p) = \left((p(j) - p(i))^2 : ij \in \binom{[n]}{2}\right),$$

for all $p \in \mathbb{R}^n$ defines the complete Euclidean-distance measurement map in $\mathbb{R}^1$. Therefore, like Euclidean graph rigidity in $\mathbb{R}^1$, global squared rigidity in $\mathbb{R}^1$ is equivalent to 2-vertex connectedness (i.e. every vertex being contained in a 1-cycle).

Proposition 7.4.1. *Global squared volume rigidity in $\mathbb{R}^d$ is not a generic property of simplicial complexes in general.*

Proof. Take the pentagonal bipyramid from the proof of theorem 7.2.1, since it admits both globally rigid and LNGR generic frameworks in $\mathbb{R}^2$, it admits globally squared rigid and locally, but not globally, squared rigid generic frameworks in $\mathbb{R}^2$. □

Let $\mathcal{C}_S(\Sigma, p)$ denote the configuration space of the framework (Σ, p) with respect to the squared volume measurement map, i.e.

$$\mathcal{C}_S(\Sigma, p) = {(\beta_\Sigma^d)^{-1}(\beta_\Sigma^d(p))}\big/{(\beta_n^d)^{-1}(\beta_n^d(p))}.$$

Recall that the LGRC Λ_n^d is generically globally rigid in $\mathbb{R}^d$, the same does not hold with respect to squared rigidity.

Theorem 7.4.2. *The LGRC Λ_n^d admits $2^d(n-d-1)$ squared volume congruence classes when in general position.*

Proof. Pin the general position framework in $\mathbb{R}^d$ (Λ_n^d, p) at $[d+1]$ to get $(\Lambda_n^d, \overline{p})$ and suppose that there exists a pinned framework $(\Lambda_n^d, \overline{q})$ so that $(\Lambda_n^d, \overline{p})$ and $(\Lambda_n^d, \overline{q})$ are squared equivalent. Let $i \geq d+2$, then $\overline{q}(i)$ lies on one of the intersections of the d pairs of hyperplanes, each parallel to $([d+1] \setminus \{j\}, \overline{p})$, for some $2 \leq j \leq d+1$. Since we are in general position, there are 2^d such intersections, one in each orthant of $\mathbb{R}^d$. There are $n-d-1$ vertices i, for which $\overline{q}(i)$ may lie in any choice of intersection independently, therefore each corresponds to a congruence class, so $c_S(\Lambda_n^d) \geq 2^d(n-d-1)$. Moreover, these for all congruence classes of $(\Lambda_n^d, \overline{p})$, as we have accounted for every equation defining the equivalence of $(\Lambda_n^d, \overline{p})$ and $(\Lambda_n^d, \overline{q})$. □

Inspired by the failure of the LGRC to be generically globally squared volume rigid in $\mathbb{R}^d$, we draw the following link to global Euclidean graph rigidity.

Theorem 7.4.3. *Let Σ be a d-dimensional simplicial complex. If Σ is generically globally squared volume rigid in $\mathbb{R}^d$, then*

1. *Σ is redundantly squared d-volume rigid;*
2. *Σ is (at least) $(d+1)$-vertex connected.*

The proof of this follows in much the same way as a typical proof of Hendrickson's criteria for global Euclidean graph rigidity does (see Gortler et al. [2010] for an example). Indeed, this is just a squared volume rigidity analogue of Hendrickson's criteria.

Proof. Condition 2 follows by noticing that if there is a cut set $S \subseteq \Sigma^{(0)}$ of size d, i.e. $\Sigma^{(0)} = \Sigma_a^{(0)} \sqcup S \sqcup \Sigma_b^{(0)}$ so that no d-simplex contains vertices in both $\Sigma_a^{(0)}$ and $\Sigma_b^{(0)}$ Then, in any generic framework (Σ, p) in $\mathbb{R}^d$ we may reflect $p(i)$ in the affine span of $\{p(j) : j \in S\}$ to get a squared equivalent, but not squared congruent framework (Σ, q).

Suppose that Σ is not redundantly squared rigid in $\mathbb{R}^d$, then there exists $\sigma \in \Sigma^{(d)}$ so that, for any generic configuration $p \in (\mathbb{R}^d)^n$, $(\Sigma - \sigma, p)$ is flexible in $\mathbb{R}^d$. Consider $\mathcal{C}_S(\Sigma - \sigma, p)$, it is a curve in $\mathbb{R}^{d(n-d-1)}$ (as we may consider the pinned congruence class representatives as the elements of $\mathcal{C}_S(\Sigma - \sigma, p)$.

Define the map $f_\sigma : \mathcal{C}_S(\Sigma - \sigma, p) \to \mathbb{R}$ by

$$f_\sigma(p) = \beta_n^d(p)_\sigma = \frac{1}{d!} \begin{vmatrix} 1 & \dots & 1 \\ p(\sigma_1) & \dots & p(\sigma_{d+1}) \end{vmatrix}^2 .$$

Since $\mathcal{C}_S(\Sigma - \sigma, p)$ is 1-dimensional, so is $\mathrm{Im}(f_\sigma)$. Since f_σ is even-degree in any choice of parameterisation of $\mathcal{C}(\Sigma - \sigma, p)$ (due to its squaring), the preimage of any $f_\sigma(p)$, for $p \in (\mathbb{R}^d)^n$ generic, has size $2m$, for some $m \geq 1$. Therefore, (Σ, p) is a generic LNGR framework in $\mathbb{R}^d$, and this holds for all generic frameworks of Σ in $\mathbb{R}^d$. □

Note that $(d+1)$ is the highest vertex connectivity we can say is necessary for global squared volume rigidity in $\mathbb{R}^d$, since the triangular bipyramid is 3-, but not 4-vertex connected and generically globally rigid in $\mathbb{R}^2$.

Appendix A

Volume Rigidity of Lower Dimensional Faces

So far we have considered the d-volumes of the (maximal) d-simplices in frameworks of Σ in $\mathbb{R}^d$. It is natural, however, to ask whether or not the k-volumes of the k-simplices of Σ are also preserved under some special affine transformations, for some $1 \leq k < d$.

This is a topic inspired by James East's work on counting integer-length polygons in $\mathbb{R}^2$ and tetrahedra in $\mathbb{R}^3$ whose edge lengths sum to some prescribed values (see East and Niles [2019] and East et al. [2023]) as well as unfinished work undertaken with Denys Bulavka at the *Focus Program on Geometric Constraint Systems* at the Fields Institute in the Summer of 2023.

Previous work in similar areas include a 2-part paper by Tay, White and Whiteley (Tay et al. [1995a], Tay et al. [1995b]), where the infinitesimal rigidity of the $(k-1)$-simplices of a simplicial complex under the volume constraints of its k-simplices is studied. Their approach defines rigidity matrices and classifies flexes and stresses for multiple formulations of such a rigidity theory, with the ultimate goal of further understanding the g-conjecture. Very recently, a paper was uploaded to the `arXiv` studying the algebraic matroid of the Heron variety (Asante et al. [2024]), this variety parameterises the d-volumes of the maximal faces of a simplex in terms of its edge-lengths.

Many of the concepts outlined have not been explored to their full potential yet, and therefore do not warrant a chapter of their own, but we choose to include them in this Appendix.

We begin with an example that is illustrative of the difficulty of this problem.

Example A.0.1. Consider the pure 2-dimensional simplicial complex Σ defined by its maximal simplices

$$\Sigma^{(2)} = \{123, 124, 134\}$$

Then Σ is 2-volume rigid in $\mathbb{R}^2$.

Let (Σ, p) be a generic framework of Σ in $\mathbb{R}^2$, a shear $f \in \mathcal{SA}(2, \mathbb{R})$ of (Σ, p) preserves its 2-volumes, but not the 1-volumes of its 1-skeleton. Indeed, since $((\Sigma^{(0)}, \Sigma^{(1)}), p)$ is a Euclidean rigid graph framework in $\mathbb{R}^2$, its only 1-volume preserving transformations are precisely the transformations in the Euclidean group $\mathrm{Euc}(2, \mathbb{R})$. ⋄

Indeed, this is true more generally, as is pointed out by the following exercise, noting that the k-volume of a k-simplex in $\mathbb{R}^d$ is $\frac{1}{k!}$ times the k-volume of the k-dimensional hyperparallelepiped in $\mathbb{R}^d$ with the same spanning vectors.

Exercise 1 (Chapter 8, Question 33, Shilov [2012]). Let f be a linear operator acting in an n-dimensional Euclidean space $\mathbb{R}^d$, and suppose that f does not change the volume of any k-dimensional hyperparallelepiped ($k < d$). Show that $f \in \mathrm{LinEuc}(d, \mathbb{R})$.

We note again that any affine transformation consists of a linear part and a translation, which is an affine isometry.

Proof. As in Chapter 8 of Shilov [2012], if P is a hyperparallelepiped in $\mathbb{R}^d$ spanned by vectors $x_1, \ldots, x_k \in \mathbb{R}^d$, then

$$|\operatorname{Vol}(P)| = |x_1||h_1|\ldots|h_{k-1}|,$$

$$\operatorname{Vol}(P)^2 = \begin{vmatrix} x_1.x_1 & x_1.x_2 & \ldots & x_1.x_k \\ x_2.x_1 & x_2.x_2 & \ldots & x_2.x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_k.x_1 & x_2.x_k & \ldots & x_k.x_k \end{vmatrix}, \tag{A.1}$$

where h_j is the length of the perpendicular from x_{j+1} to $\mathrm{Span}_{\mathbb{R}}\{x_1, \ldots, x_j\}$, for all $1 \le j \le k-1$. Now suppose that f preserves $\operatorname{Vol}(P)$, then, if Q is the $d \times d$

matrix representing f,

$$\begin{aligned}
\mathrm{Vol}(f(P))^2 &= \begin{vmatrix} (Qx_1).(Qx_1) & (Qx_1).(Qx_2) & \dots & (Qx_1).(Qx_k) \\ (Qx_2).(Qx_1) & (Qx_2).(Qx_2) & \dots & (Qx_2).(Qx_k) \\ \vdots & \vdots & \ddots & \vdots \\ (Qx_k).(Qx_1) & (Qx_k).(Qx_2) & \dots & (Qx_k).(Qx_k) \end{vmatrix} \\
&= \begin{vmatrix} x_1^tQ^tQx_1 & x_1^tQ^tQx_2 & \dots & x_1^tQ^tQx_k \\ x_2^tQ^tQx_1 & x_2^tQ^tQx_2 & \dots & x_2^tQ^tQx_k \\ \vdots & \vdots & \ddots & \vdots \\ x_k^tQ^tQx_1 & x_k^tQ^tQx_2 & \dots & x_k^tQ^tQx_k \end{vmatrix} \\
&= \begin{vmatrix} x_1^tx_1 & x_1^tx_2 & \dots & x_1^tx_k \\ x_2^tx_1 & x_2^tx_2 & \dots & x_2^tx_k \\ \vdots & \vdots & \ddots & \vdots \\ x_k^tx_1 & x_k^tx_2 & \dots & x_k^tx_k \end{vmatrix} \\
&= \begin{vmatrix} x_1.x_1 & x_1.x_2 & \dots & x_1.x_k \\ x_2.x_1 & x_2.x_2 & \dots & x_2.x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_k.x_1 & x_k.x_2 & \dots & x_k.x_k \end{vmatrix} = \mathrm{Vol}(P)^2.
\end{aligned}$$

Now, $f \in \mathrm{LinEuc}(d, \mathbb{R})$ if and only if $Q^tQ = I$. For the sake of contradiction, suppose that $Q^tQ = A^t \neq I$, then, for any $x \in \mathbb{R}^d$

$$x^tQ^tQx = x^tA^tx = Ax.x,$$

so

$$\begin{aligned}
\mathrm{Vol}(f(P))^2 &= \begin{vmatrix} Ax_1.x_1 & Ax_1.x_2 & \dots & Ax_1.x_k \\ Ax_2.x_1 & Ax_2.x_2 & \dots & Ax_2.x_k \\ \vdots & \vdots & \ddots & \vdots \\ Ax_k.x_1 & Ax_k.x_2 & \dots & Ax_k.x_k \end{vmatrix} \\
&= \det(A) \begin{vmatrix} x_1.x_1 & x_1.x_2 & \dots & x_1.x_k \\ x_2.x_1 & x_2.x_2 & \dots & x_2.x_k \\ \vdots & \vdots & \ddots & \vdots \\ x_k.x_1 & x_k.x_2 & \dots & x_k.x_k \end{vmatrix} = \mathrm{Vol}(P)^2.
\end{aligned}$$

Then, since $\mathrm{Vol}(P) = \mathrm{Vol}(f(P))$, we must have that

$$\det(A) = \det(Q^tQ) = \det(Q)^2 = 1 \Rightarrow \det(Q) = \pm 1.$$

Therefore, either $Q \in \mathrm{SL}(d, \mathbb{R})$ or $\tilde{Q} \in \mathrm{SL}(d, \mathbb{R})$, where $\tilde{Q}$ is obtained from Q by multiplying the first row of Q by -1.

1. Suppose that $Q \in \mathrm{SL}(d,\mathbb{R}) \setminus \mathrm{SO}(d,\mathbb{R})$, where $\mathrm{SO}(d,\mathbb{R})$ is the matrix-group of linear isometries of $\mathbb{R}^d$, then

$$Q = \begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \dots & \lambda_{1,d} \\ 0 & \lambda_{2,2} & \dots & \lambda_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{d,d} \end{bmatrix},$$

where

$$\lambda_{1,1}\lambda_{2,2}\dots\lambda_{d,d} = 1. \tag{A.2}$$

We can choose vectors $x_1, \dots, x_k$ whose entries are transcendental over $\mathbb{Q}[\lambda_{1,1}, \lambda_{1,2}, \dots, \lambda_{d,d}]$. If we let $y_j = f(x_j)$, $y_k = x_k$ and let i_j denote the length of the perpendicular from y_{j+1} to $\mathrm{Span}_{\mathbb{R}}\{y_1, \dots, y_j\}$, for all $1 \le j \le k-1$ then

$$|x_1||h_1|\dots|h_{k-1}| = |y_1||i_1|\dots|i_{k-1}|$$

if and only if $(x_1, \dots, x_k)$ is the solution to a non-zero polynomial with coefficients in $\mathbb{Q}[\lambda_{1,1}, \lambda_{1,2}, \dots, \lambda_{d,d}]$, a contradiction.

2. Repeat the argument from part 1 with the modification of equation A.2 to

$$(-\lambda_{1,1})\lambda_{2,2}\dots\lambda_{d,d} = 1.$$

Therefore, for every non-isometric d-volume preserving linear transformation f of $\mathbb{R}^d$, there exists a k-dimensional hyperparallelepiped, where $k < d$, whose k-dimensional volume is not preserved by f. □

Therefore the space of transformations defining *rigidity 4* in this scenario are the Euclidean isometries of $\mathbb{R}^d$, meaning the study of the rigid transformations of lower dimensional faces of a simplicial framework reduces to the study of the Euclidean rigid transformations of its 1-skeleton.

Appendix B

Checking the Rigidity of Minimal Triangulations of Some Surfaces

Lawrencenko proved that there are 21 generating triangulations of triangulations of the 2-dimensional torus in Lavrencenko [1990]. They are on between 7 and 10 vertices, so it suffices to check that they admit rigid frameworks with respect to projections of the configuration $p \in (\mathbb{R}^2)^{10}$ with configuration matrix

$$C(p) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} & \frac{1}{11} & \frac{1}{17} & \frac{1}{23} & \frac{1}{31} & \frac{1}{41} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{7} & \frac{1}{13} & \frac{1}{19} & \frac{1}{29} & \frac{1}{37} & \frac{1}{43} \end{bmatrix},$$

which they do.

He later proved that there are 25 generating triangulations of triangulations of the Klein bottle in Lawrencenko and Negami [1997]. They are on between 8 and 11 vertices, so it suffices to check that they admit rigid frameworks with respect to projections of the configuration $q \in (\mathbb{R}^2)^{11}$ with configuration matrix

$$C(q) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{5} & \frac{1}{11} & \frac{1}{17} & \frac{1}{23} & \frac{1}{31} & \frac{1}{41} & \frac{1}{47} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{7} & \frac{1}{13} & \frac{1}{19} & \frac{1}{29} & \frac{1}{37} & \frac{1}{43} & \frac{1}{53} \end{bmatrix},$$

which they do.

Therefore, by our results in section 4.7, triangulations of the 2-dimensional torus and Klein bottle are rigid in $\mathbb{R}^2$.

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