Assouad-type dimensions and the local geometry of fractal sets

Alex Rutar

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Abstract

We study the fine local scaling properties of *rough* or *irregular* subsets of a metric space. In particular, we consider the classical *Assouad dimension* as well as two variants: a *scale-refined* variant called the *Assouad spectrum*, and a *location-refined* variant called the *pointwise Assouad dimension*.

For the Assouad spectrum, we first give a simple characterization of when a function $\varphi\colon (0,1) \to [0,d]$ can be the Assouad spectrum of a general subset of $\mathbb{R}^d$. Using this, we construct examples exhibiting novel exotic behaviour, answering some questions of Fraser & Yu. We then compute the Assouad spectrum of a certain family planar self-affine sets: the class of *Gatzouras–Lalley carpets*. Within this family, we establish an explicit formula as the concave conjugate of a certain "column pressure" combined with simple parameter change. This class of sets exhibits novel behaviour in the setting of dynamically invariant sets, such as strict concavity and differentiability on the whole range $(0,1)$.

We then focus on the interrelated concepts of (weak) tangents, Assouad dimension, and a new localized variant which we call the pointwise Assouad dimension. For general attractors of bi-Lipschitz iterated function systems, we establish that the Assouad dimension is given by the Hausdorff dimension of a tangent at some point in the attractor. Under the additional assumption of self-conformality, we moreover prove that this property holds for a subset of full Hausdorff dimension. We then turn our attention again to planar self-affine sets. For Gatzouras–Lalley carpets, we obtain precise information about tangents which, in particular, shows that points with large tangents are very abundant. However, already for Barański carpets, we see that more complex behaviour is possible.

Acknowledgements

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KR23+; MR24; Rut21+; Rut22+; Rut23+]. In an attempt to present a cohesive thesis, I have chosen to base this thesis primarily on three of these publications:

- A. Rutar. Attainable forms of Assouad spectra. To appear in *Indiana Univ. Math. J.* `https://arxiv.org/abs/2206.06921`
- A. Käenmäki & A. Rutar. Tangents and pointwise Assouad dimension of invariant sets. Preprint (submitted) `https://arxiv.org/abs/2309.11971v1`
- A. Banaji, J. M. Fraser, I. Kolossváry, & A. Rutar. Assouad spectrum of Gatzouras–Lalley carpets. Preprint (submitted) `https://arxiv.org/abs/2401.07168v2`

In order to make this thesis mostly self-contained, a small amount of content has also been included from the following two papers:

- A. Rutar. Multifractal analysis via Lagrange duality. Preprint (submitted) `https://arxiv.org/abs/2312.08974`
- A. Banaji & A. Rutar. Attainable forms of intermediate dimensions. *Ann. Fenn. Math.* **47** (2022), 939–960

All of these papers constitute substantial original research and they have been submitted to or accepted in high quality mathematical journals.

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Declarations

Candidate's declaration

I, Alex Rutar, do hereby certify that this thesis, submitted for the degree of PhD, which is approximately 34,000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for any degree. I confirm that any appendices included in my thesis contain only material permitted by the 'Assessment of Postgraduate Research Students' policy.

I was admitted as a research student at the University of St Andrews in September 2020.

I received funding from an organisation or institution and have acknowledged the funder(s) in the full text of my thesis.

Date: August 8, 2024 **Signature of Candidate:** ____________________

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I hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of PhD in the University of St Andrews and that the candidate is qualified to submit this thesis in application for that degree. I confirm that any appendices included in the thesis contain only material permitted by the 'Assessment of Postgraduate Research Students' policy.

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Contents

List of symbols

Symbol	Description	Page
General		
$\mathbb{R}_{\geq 0}$	non-negative reals $[0, \infty)$	
$\lesssim, \gtrsim, \approx$	(in)equality up to constant	8
O	Landau asymptotic notation	8
$B(x, r)$	closed ball of radius r	4
$N_r(K)$	Number of balls of radius r to cover K	2
$\operatorname{diam} K$	diameter of K	3
$\operatorname{dist}(K_1, K_2)$	shortest distance between points in K_1 and K_2	6
$d_{\mathcal{H}}$	Hausdorff metric	6
$p_{\mathcal{H}}$	one-sided Hausdorff metric	6
$D^{-}f(x)$	upper left Dini derivatives	43
$D^{+}f(x)$	upper right Dini derivatives	43
$\partial f(x)$	subdifferential of f at x	62
$\partial^{-} f(x)$	left derivative of f at x	62
$\partial^{+} f(x)$	right derivative of f at x	62
g^*	concave conjugate of g	62
Dimension theory		
$\dim_{\mathrm{B}} K$	box-counting dimension of K	2
$\mathcal{H}^s(K)$	Hausdorff s-measure of K	3
$\mathcal{H}^s_\infty(K)$	Hausdorff s-content of K	3
$\dim_{\mathrm{H}} K$	Hausdorff dimension of K	3
$\dim_{\mathrm{A}} K$	Assouad dimension of K	4
$\dim_{\mathrm{L}} K$	lower dimension of K	4
$\dim_{\mathrm{A}}^{\theta} K$	Assouad spectrum of K for $\theta \in (0,1)$	5
$\overline{\dim}_{\mathrm{A}}^{\theta} K$	upper Assouad spectrum of K for $\theta \in (0,1)$	5
$\dim_{\mathrm{A}}(K, x)$	pointwise Assouad dimension of K at x	7
Symbolic notation		
$(\Omega, \mathcal{T}, \rho)$	metric tree with boundary Ω, nested partitions $\mathcal{T}$, and cylinder diameter function ρ	8

$\mathcal{G}(\lambda, \alpha)$	box-counting estimate function after double exponential change of parameter	37
$\mathcal{C}_d$	special non-monotonic elements of $\mathcal{A}_d$	47

I. Fractal geometry and the Assouad dimension

1 The geometry of fractal sets

1.1 Fractal geometry

Perhaps one of the oldest branches of mathematics is the study of *geometry*. What is geometry? The Merriam–Webster dictionary [Mer22] defines geometry as

> *a branch of mathematics that deals with the measurement, properties, and relationships of points, lines, angles, surfaces, and solids;*

and more broadly,

> *the study of properties of given elements that remain invariant under specified transformations.*

A classical setting for the study of geometry is the geometry of *smooth* objects, such as circles, lines, or graphs of smooth functions. Such smooth objects look essentially the same everywhere at sufficiently high resolutions: for example, they have well-defined *tangents* at each point, which are real vector spaces with a well-defined *dimension*.

In contrast, many classical examples of sets such as the middle-third Cantor set or the graph of the Weierstrass function do not exhibit the same level of local uniformity. Moreover, concepts such as *dimension*, while easy to define for smooth objects, can be difficult to define more generally: for instance, a definition such as "a continuous path is one-dimensional" is fundamentally flawed, since continuous space-filling curves such as the Peano curve have domain $[0, 1]$ and image $[0, 1]^2$. Furthermore, the occurrence of such irregular sets is not limited to the classical constructions in analysis: widely studied and important examples from the natural world include turbulence in fluid dynamics to diffusion processes and Brownian motion and even to the irregularity of coastlines. Indeed, the study of such irregular sets was termed *fractal geometry* by the pioneering work of Mandelbrot [Man75; Man77; Man82], which also emphasized the many connections to processes occurring in nature.

A general motivating theme in fractal geometry from a pure mathematical perspective is to make some attempt at understanding the following question: *to what extent can we generalize classical concepts from smooth geometry to the geometry of fractal sets?* The study of dimension theory is a fundamental part of fractal geometry and there are many different ways to define sensible notions of dimension for fractal sets. The most well-studied notions of dimensions, such as the *Hausdorff* and *box dimensions*, capture *average* scaling properties of sets. In contrast, in this thesis, we will focus on *localized* notions of dimension (motivated primarily by the

Assouad dimension), which attempt to capture the geometry of the "thickest" parts of a set, or which allow one to focus on scaling properties at individual points. A particular emphasis will be placed on *dimensions* and *tangents* of sets without any *a priori* structure, and we will also attempt to understand how the notions of dimension and tangents are interconnected.

Unfortunately, certain conclusions about general sets are either very challenging to make or simply false, as a result of pathological constructions. In order to impose more structure on our sets under consideration, we will take inspiration from the second half of the definition of geometry: namely that of *invariance*. It will turn out that our conclusions will be the strongest for sets exhibiting various forms of invariance, and moreover certain classes of invariant sets will provide us with an important source of examples.

1.2 BOX AND HAUSDORFF DIMENSIONS

Perhaps the simplest way to define a sensible notion of dimension is through the process of *subdivision* or *covering*.

Suppose we begin with a unit square, which has side-length 1. We can subdivide the unit square into sub-squares each with side-length $1/2$, and we obtain 4 subsquares. On the other hand, if we instead subdivide the unit cube into sub-squares with side length $1/2$, we obtain 8 subcubes. More generally, subdividing the unit square into sub-squares of side length $1/2^k$ will yield $(2^k)^2$ subsquares, and the same process with cubes will yield $(2^k)^3$ subcubes. These exponents 2 and 3, obtained by subdivision, are precisely our intuitive notion of dimension for squares and cubes.

This approach readily generalizes to the case of arbitrary bounded metric spaces (K, d). First, given an $r > 0$ and an arbitrary non-empty set $F \subset K$, let $N_r(F)$ denote the least number of closed balls of radius r required to cover the set F. We then define the *upper box dimension* as the worst-case exponential growth rate of the number $N_r(K)$ relative to r:

$$\overline{\dim}_{\mathrm{B}} K = \limsup_{r \to 0} \frac{\log N_r(K)}{\log(1/r)}. \tag{1.1}$$

We define the *lower box dimension*, denoted $\underline{\dim}_{\mathrm{B}} K$, with a limit infimum in place of the limit supremum. If the upper and lower box dimensions coincide, we denote the common value by $\dim_{\mathrm{B}} K$. Alternatively,

$$\overline{\dim}_{\mathrm{B}} K = \inf\Big\{\alpha \geq 0 : (\exists C > 0)\,(\forall 0 < r < 1)\, N_r(K) \leq C\Big(\frac{1}{r}\Big)^{\alpha}\Big\}.$$

In the case of the unit square and the unit cube, even though we considered subdivision into sub-squares, since we are only interested in the exponential growth rate of the covering number $N_r(F)$, the counts are the same up to a constant factor which does not influence the dimension.

One downside of the box dimension is that it has various inconvenient properties: for example, the box dimension is preserved under closure, which means that the box dimension of a countable set such as $\mathbb{Q} \cap [0, 1]$ is 1. Another notion of dimension, which is more measure-theoretically robust, is the *Hausdorff dimension*.

First, given $r > 0$ and $\delta > 0$, define for $F \subset K$

$$\mathcal{H}^s_\delta(F) = \inf\Big\{\sum_{i=1}^\infty (\operatorname{diam} B_i)^s : A \subset \bigcup_{i=1}^\infty B_i \text{ and } \operatorname{diam} B_i \leq \delta\Big\},$$

with the convention that $\inf \varnothing = 0$. In the above, the infimum is taken over all countable families of sets $\{B_i\}_{i=1}^\infty$. Note that $\mathcal{H}^s_\delta(F)$ is a monotonically increasing as δ converges to 0 from above, so we may set

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F).$$

We refer to $\mathcal{H}^s$ as *s-dimensional Hausdorff (outer) measure*. It can be verified that $\mathcal{H}^s(F)$ is a metric outer measure and therefore defines a Borel measure on K. For completeness, we also define the *Hausdorff content*, which is not a measure, as

$$\mathcal{H}^s_\infty(F) = \inf_{\delta > 0} \mathcal{H}^s_\delta(F). \tag{1.2}$$

In general, $\mathcal{H}^s_\infty(F) \leq \mathcal{H}^s(F)$ and $\mathcal{H}^s_\infty(F) = 0$ if and only if $\mathcal{H}^s(F) = 0$.

Moreover, there is a unique value $s \geq 0$ such that for all $0 \leq t < s$, $\mathcal{H}^t(F) = \infty$, and for all $t > s$, $\mathcal{H}^t(F) = 0$. This value of s is the *Hausdorff dimension* of F, which we denote by $\dim_{\mathrm{H}} F = s$. Alternatively, the Hausdorff dimension satisfies $s = \inf\{t : \mathcal{H}^t_\infty(F) = 0\}$. Note that at the critical exponent s, $\mathcal{H}^s(F)$ may take any value in the interval $[0, \infty]$.

We may also define a relative to the Hausdorff dimension, which is the *packing dimension*. Here, we define the packing dimension to be the countably stabilized upper box dimension:

$$\dim_{\mathrm{P}} F = \inf\left\{\sup_i \overline{\dim}_{\mathrm{B}} E_i : E \subset \bigcup_{i=1}^\infty E_i\right\}.$$

In general, $\dim_{\mathrm{H}} F \leq \dim_{\mathrm{P}} F \leq \overline{\dim}_{\mathrm{B}} F$, and all of these inequalities can be proper inequalities. The packing dimension can also be defined using *packing measures*; however, this formulation will not be required in this thesis and so we omit it here.

To conclude this section, we also recall the definition of *Ahlfors–David regularity*. For an exponent $s \geq 0$, we say that a metric space (K, d) is *Ahlfors–David s-regular* if there is a constant $C \geq 1$ so that for all $x \in K$ and $0 < r < \operatorname{diam} K$

$$C^{-1} r^s \leq \mathcal{H}^s(K \cap B(x, r)) \leq C r^s.$$

Ahlfors–David s-regular sets are very homogeneous in space and in scale and all of the notions of fractal dimension that we will consider in this thesis coincide and take value s for s-regular sets.

1.3 The Assouad dimension

In the previous section, we introduced the notions of the Hausdorff and box dimensions. However, the Hausdorff and box dimensions are *global* measurements of scaling. In contrast, the *Assouad dimension* is a notion of fractal dimension which in some sense captures the "worst-case local scaling" of a set. More precisely, the Assouad dimension of a compact metric space K is defined as

$$\dim_{\mathrm{A}} K = \inf\Big\{ s : (\exists C > 0)\, (\forall 0 < r \leq R < 1)\, (\forall x \in K) \\ N_r(B(x,R) \cap K) \leq C\Big(\frac{R}{r}\Big)^s \Big\}.$$

Here, $B(x,R)$ denotes the closed ball centred at x with radius R. Similarly to the definition of the upper box dimension, we are interested in the exponential growth rate of some covering number N_r, but unlike the box dimension we also localize to a ball $B(x,R)$. Again, similarly to the upper box dimension, the Assouad dimension is unchanged on taking closure. For completeness, and we also briefly mention the dual to the Assouad dimension: the *lower dimension* of K, defined as

$$\dim_{\mathrm{L}} K = \sup\Big\{ s : (\exists C > 0)\, (\forall 0 < r \leq R < 1)\, (\forall x \in K) \\ N_r(B(x,R) \cap K) \geq C\Big(\frac{R}{r}\Big)^s \Big\}.$$

Since $\dim_{\mathrm{L}} K$ is also unchanged under taking a closure and since the Hausdorff dimension is countably stable, this inequality does not hold for general sets (for instance, $1 = \dim_{\mathrm{L}} \mathbb{Q} \cap [0,1] > \dim_{\mathrm{H}} \mathbb{Q} \cap [0,1] = 0$).

It always holds that $\dim_{\mathrm{H}} K \leq \overline{\dim}_{\mathrm{B}} K \leq \dim_{\mathrm{A}} K$, and for compact sets, $\dim_{\mathrm{L}} K \leq \dim_{\mathrm{H}} K$. If K is Ahlfors–David s-regular, then all of the above notions of dimension coincide: $\dim_{\mathrm{L}} K = \dim_{\mathrm{H}} K = \underline{\dim}_{\mathrm{B}} K = \overline{\dim}_{\mathrm{B}} K = \dim_{\mathrm{A}} K = s$.

The term Assouad dimension originates from the work of Patrice Assouad in his PhD thesis [Ass77] to study the embedding theory of metric spaces: indeed, any local obstruction is sufficient to prevent embedding of one space into another. We note that similar dimensional notions were also studied earlier by Larman [Lar67]. Independently, an analogous theory in terms of dynamics on fractals was studied by Furstenberg [Fur70; Fur08], though the explicit relationship with the Assouad dimension was not noticed until more recently; see the discussion surrounding Proposition 1.1. More generally, in recent years, the Assouad dimension has received a significant amount of attention in the literature from a variety of perspectives: see, for example, the books by Mackay & Tyson on conformal geometry [MT10], Robinson on embedding theory [Rob11], and Fraser on Assouad dimension in fractal geometry [Fra20]

One downside of the Assouad dimension is that it can provide scaling information which is *too* coarse. The Assouad dimension captures the worst case scaling at *all locations* and between *all pairs of scales*. In the following two sections, we will consider variants of the Assouad dimension: the first, which imposes some restrictions on the relevant scales, and the second which imposes some restrictions on the relevant locations.

1.4 The Assouad spectrum

If the box dimension and the Assouad dimension of a set agree, this implies that the set has a large amount of spatial regularity. However, it is also very easy for the box dimension and Assouad dimensions to be distinct. In this situation, one way to obtain a more fine-grained understanding of the Assouad dimension is through the *Assouad spectrum*, which was introduced by Fraser & Yu in [FY18b]. For $\theta \in (0,1)$, the Assouad spectrum of a compact metric space K is given by

$$\begin{aligned}\dim_{\mathrm{A}}^{\theta} K = \inf\Big\{\alpha : (\exists C > 0)(\forall 0 < \delta \leq 1)(\forall x \in K)\\ N_{\delta^{1/\theta}}(K \cap B(x,\delta)) \leq C\Big(\frac{\delta}{\delta^{1/\theta}}\Big)^{\alpha}\Big\}.\end{aligned}$$

The definition is very similar to the Assouad dimension, with the only restriction being that the relationship between the upper and lower scales is constrained. Unlike the Assouad dimension, the Assouad spectrum can also be expressed as a limit:

$$\dim_{\mathrm{A}}^{\theta} K = \limsup_{\delta \to 0} \frac{\log \sup_{x \in K} N_{\delta^{1/\theta}}(K \cap B(x,\delta))}{(1/\theta - 1)\log(1/\delta)}. \tag{1.3}$$

In general, $\lim_{\theta \to 0} \dim_{\mathrm{A}}^{\theta} K = \overline{\dim}_{\mathrm{B}} K$, and $\lim_{\theta \to 1} \dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{qA}} K$ by [FHH+19, Theorem 2.1], where $\dim_{\mathrm{qA}} K$ denotes the quasi-Assouad dimension of K as introduced by Lü & Xi [LX16]. Like the Assouad dimension, the Assouad spectrum measures the worst-case local scaling of the set, but the Assouad spectrum specifies the relationship between the small and large scales.

Besides being a useful bi-Lipschitz invariant and an important notion of fractal dimension in its own right, the Assouad spectrum provides more refined information about the Assouad dimension itself. As a result, the Assouad spectrum has been explicitly studied for a wide range of examples (see, for example, [BF23; BFF22; FS23; FT21; FY18a; FY18b]). This relationship has also been useful in applications outside of fractal geometry. For instance, the Assouad spectrum plays an important role in the work by Roos & Seeger [RS23] on L^p bounds for spherical maximal operators (and the analogous work in Heisenberg groups [RSS22+]). The Assouad spectrum has also been used to obtain bounds for quasiconformal distortion in geometric mapping theory [CT22].

We will also consider the *upper Assouad spectrum*, which is defined by bounding the lower scale from above, rather than specifying the relationship precisely:

$$\begin{aligned}\overline{\dim}_{\mathrm{A}}^{\theta} F = \inf\Big\{\alpha : (\exists C > 0)(\forall 0 < \delta \leq 1)(\forall 0 < \delta' \leq \delta^{1/\theta})(\forall x \in F)\\ N_{\delta'}(B(x,\delta)) \leq C\Big(\frac{\delta}{\delta'}\Big)^{\alpha}\Big\}.\end{aligned}$$

The upper Assouad spectrum can alternatively be recovered directly from the Assouad spectrum: by [FHH+19, Theorem 2.1],

$$\overline{\dim}_{\mathrm{A}}^{\theta} F = \sup_{0 < \theta' < \theta} \dim_{\mathrm{A}}^{\theta} F.$$

For context, we note that the Assouad spectrum is part of a more general scheme commonly called *dimension interpolation* [Fra21], of which other notable example include the *lower spectrum* which is analogous to the Assouad spectrum except for lower dimensions, and the *intermediate dimensions* [FFK20], which lie between the upper box and Hausdorff dimensions. A more refined version of the Assouad spectrum, where the relationship between the larger and smaller scales can be specified in essentially an arbitrary way, has also been considered in [BRT23+; GHM21] (see also the many references cited therein).

1.5 Weak tangents, tangents, and pointwise Assouad dimension

In contrast to the previous section, instead of constraining the relevant scales, we will constrain the relevant locations to obtain the *pointwise Assouad dimension*. In order to motivate this connection, we begin by recalling an important relationship between the Assouad dimension and the notion of a *weak tangent*.

Given a set $E \subset \mathbb{R}^d$ and $\delta > 0$, we denote the *open δ-neighbourhood* of E by

$$E^{(\delta)} = \{x \in \mathbb{R}^d : \exists y \in E \text{ such that } |x - y| < \delta\}.$$

Now given a non-empty subset $X \subset \mathbb{R}^d$, we let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X equipped with the *Hausdorff metric*

$$d_{\mathcal{H}}(K_1, K_2) = \max\{p_{\mathcal{H}}(K_1; K_2), p_{\mathcal{H}}(K_2; K_1)\}$$

where

$$p_{\mathcal{H}}(K_1; K_2) = \inf\{\delta > 0 : K_1 \subset K_2^{(\delta)}\}.$$

If X is compact, then well-known arguments show that $\mathcal{K}(X)$ is a compact metric space itself. We also write

$$\operatorname{dist}(E_1, E_2) = \inf\{|x - y| : x \in E_1, y \in E_2\}$$

for non-empty sets $E_1, E_2 \subset \mathbb{R}^d$.

We say that a set $F \in \mathcal{K}(B(0,1))$ is a *weak tangent* of $K \subset \mathbb{R}^d$ if there exists a sequence of similarity maps $(T_k)_{k=1}^\infty$ with $0 \in T_k(K)$ and similarity ratios λ_k diverging to infinity such that

$$F = \lim_{k \to \infty} T_k(K) \cap B(0,1)$$

in $\mathcal{K}(B(0,1))$. We denote the set of weak tangents of K by $\operatorname{Tan}(K)$. A key feature of the Assouad dimension is that it is characterized by Hausdorff dimensions of weak tangents. This result is originally from [KOR17, Proposition 5.7], though the arguments underlying the proof are essentially due to Furstenberg [Fur08]. We refer the reader to [Fra20, Section 5.1] for more discussion on the context and history of this result.

Proposition 1.1 ([Fur08; KOR17]). *We have*

$$\alpha := \dim_{\mathrm{A}} K = \max_{F \in \operatorname{Tan}(K)} \dim_{\mathrm{H}} F.$$

Moreover, the maximizing weak tangent F can be chosen so that $\mathcal{H}^\alpha(F) > 0$.

In a similar flavour, we say that F is a *tangent of K at $x \in K$* if there exists a sequence of similarity ratios $(\lambda_k)_{k=1}^{\infty}$ diverging to infinity such that

$$F = \lim_{k \to \infty} \lambda_k (K - x) \cap B(0,1)$$

in $\mathcal{K}(B(0,1))$. We denote the set of tangents of K at x by $\operatorname{Tan}(K, x)$.

Of course, $\operatorname{Tan}(K, x) \subset \operatorname{Tan}(K)$. Unlike in the case for weak tangents, we require the similarities in the construction of the tangent to in fact be homotheties. This choice is natural since, for example, a function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at x if and only if the set of tangents of the graph of f at $(x, f(x))$ is the singleton $\{B(0,1) \cap \ell\}$ for some non-vertical line ℓ passing through the origin. In practice, compactness of the group of orthogonal transformations in $\mathbb{R}^d$ means this restriction will not cause any technical difficulties.

Continuing the analogy with tangents, we also introduce a localized version of the Assouad dimension which we call the *pointwise Assouad dimension*. Given $x \in K$, we set

$$\begin{aligned}\dim_{\mathrm{A}}(K, x) = \inf\Big\{ s : (\exists C > 0)\, (\exists \rho > 0)\, \forall (0 < r \leq R < \rho) \\ N_r(B(x, R) \cap K) \leq C \Big(\frac{R}{r}\Big)^s \Big\}.\end{aligned}$$

The choice of $\rho > 0$ in the definition of $\dim_{\mathrm{A}}(K, x)$ ensures a sensible form of bi-Lipschitz invariance: if $f\colon K \to K'$ is bi-Lipschitz, then $\dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}}(f(K), f(x))$. It is immediate from the definition that

$$\dim_{\mathrm{A}}(K, x) \leq \dim_{\mathrm{A}} K.$$

Moreover, if for instance K is Ahlfors–David regular, then $\dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}} K$ for all $x \in K$. This holds, for example, with self-similar sets satisfying the open set condition.

1.6 OUTLINE OF THIS THESIS

With all of the key definitions now in place, we give a bird's-eye view of the thesis.

In §2, we introduce some general symbolic terminology and machinery which will be helpful subsequently in the thesis. In that section, we also establish some results on the regularity of the Assouad dimension, and use these results to establish a formula for the Assouad dimension of a certain class of sets: namely *non-autonomous self-similar sets* satisfying certain conditions. Non-autonomous self-similar sets will resurface in §4, and moreover the results on non-autonomous self-similar sets will play an important role in §11 and §12. To conclude the introductory chapter, in §3 we introduce *Gatzouras–Lalley* and *Barański carpets*, which are certain families of invariant sets of which we will provide an in-depth study.

Next, in Chapter II, we provide a detailed study of the Assouad spectrum. In §4 we classify the possible forms of the Assouad spectrum and in §5 we use this classification result to construct some sets exhibiting novel behaviour of the Assouad spectrum. We then turn our attention to Gatzouras–Lalley carpets, and derive an explicit formula for the Assouad spectrum and study its qualitative properties. The derivation of the formula and the study of its properties can be found in §§6–8.

Finally, in Chapter III, we study the pointwise Assouad dimension and its relationship with tangents. After establishing some basic properties of general sets in §9, in §10 we study the pointwise Assouad dimensions of sets satisfying certain forms of invariance, which we call *self-embeddability* and *uniform self-embeddability*. Since these two forms of invariance can be thought of as two extremes, we then turn our attention to intermediate behaviour by again studying Gatzouras–Lalley and Barański carpets in §11 and §12 respectively.

1.7 Asymptotic notation

We will find it useful to use various forms of asymptotic notation. Given functions $f, g\colon A \to \mathbb{R}$, we write $f \lesssim g$ if there is a constant $C > 0$ such that $f(a) \leq Cg(a)$ for all $a \in A$. We write $f \approx g$ if $f \lesssim g$ and $f \gtrsim g$. We will also use Landau's O notation: we say that $f = O(g)$ if there is a constant $C > 0$ so that $|f(a)| \leq C|g(a)|$ for all $a \in A$. The constants in the asymptotic notation will often implicitly depend on the underlying data, such as the parameters defining an iterated function system or a fixed parameter $\theta \in (0, 1)$. If any dependence is explicitly indicated it will be done so by a subscript, such as $\lesssim_\varepsilon$ or O_ε.

2 Symbolic constructions and non-autonomous self-similar sets

In this section, we introduce some general symbolic constructions and define non-autonomous self-similar sets. We will also establish a formula for the Assouad dimension of a non-autonomous self-similar set.

2.1 Metric trees

Throughout this thesis, many of the objects we define will have an underlying "coding space". In this section, we set up some notation to handle such coding spaces by defining a *metric tree*.

First, fix a reference set Ω and write $\mathcal{T}_0 = \{\Omega\}$. Let $\{\mathcal{T}_k\}_{k=1}^\infty$ be a sequence of countable partitions of Ω so that $\mathcal{T}_{k+1}$ is a refinement of the partition $\mathcal{T}_k$. For each $Q \in \mathcal{T}_k$ with $k \in \mathbb{N}$, there is a unique *parent* $\widehat{Q} \in \mathcal{T}_{k-1}$ with $Q \subset \widehat{Q}$. Suppose that for any $\gamma_1 \neq \gamma_2 \in \Omega$ there is a $k \in \mathbb{N}$ such that there are $Q_1 \neq Q_2 \in \mathcal{T}_k$ so that $\gamma_1 \in Q_1$ and $\gamma_2 \in Q_2$. We call such a family $\{\mathcal{T}_k\}_{k=0}^\infty$ a *tree*, and write $\mathcal{T} = \bigcup_{k=0}^\infty \mathcal{T}_k$. We call Ω the *boundary* of the tree and we refer to each element $Q \in \mathcal{T}$ as a *cylinder*. Note that for each $\gamma \in \Omega$ there is a unique sequence $(Q_n)_{n=1}^\infty$ with $Q_n \in \mathcal{T}_n$ where $Q_n = \widehat{Q}_{n+1}$ for each $n \in \mathbb{N}$.

Now, suppose that there is a function $\rho\colon \mathcal{T} \to (0, \infty)$ which satisfies

1. $0 < \rho(Q) < \rho(\widehat{Q})$, and
2. there is a sequence $(r_k)_{k=1}^\infty$ converging to zero from above such that $\rho(Q) \leq r_k$ for all $Q \in \mathcal{T}_k$.

The function ρ induces a metric d on the space Ω by the rule

$$d(\gamma_1, \gamma_2) = \inf\{\rho(Q) : Q \in \mathcal{T} \text{ and } \{\gamma_1, \gamma_2\} \subset Q\}.$$

In particular, $\operatorname{diam}(Q) = \rho(Q)$ with respect to the metric d. We then refer to the data $(\Omega, \{\mathcal{T}_k\}_{k=0}^{\infty}, \rho)$ as a *metric tree*.

We say that a subset $\mathcal{B} \subset \mathcal{T}$ is a *section* if $Q_1 \cap Q_2 = \varnothing$ whenever $Q_1, Q_2 \in \mathcal{B}$ with $Q_1 \neq Q_2$. If $\bigcup_{Q \in \mathcal{B}} Q = Q_0$, we say that $\mathcal{B}$ is a *section relative to* Q_0, and we say that a section is *complete* if it is a section relative to Ω. Note that sections are necessarily countable and, for example, each $\mathcal{T}_k$ for $k \in \mathbb{N} \cup \{0\}$ is a complete section. The set of sections is equipped with a partial order $\mathcal{B}_1 \preccurlyeq \mathcal{B}_2$ if for all $Q_1 \in \mathcal{B}_1$ there is a $Q_2 \in \mathcal{B}_2$ such that $Q_2 \subset Q_1$. In this situation, we say that $\mathcal{B}_1$ is *refined by* $\mathcal{B}_2$. The order is chosen to respect the partial order on inclusion of the cylinder sets $Q \in \mathcal{T}$. This partial order is equipped with a *meet*: that is, given a finite family of sections $\mathcal{B}_1, \ldots, \mathcal{B}_n$, there is a unique section $\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_n$ which is maximal with respect to the partial order such that

$$\mathcal{B}_1 \wedge \cdots \wedge \mathcal{B}_n \preccurlyeq \mathcal{B}_i$$

for all $i = 1, \ldots, n$.

A metric tree is also equipped with a natural family of complete sections which respect the geometry of the metric d. We define

$$\mathcal{T}(r) = \{Q \in \mathcal{T} : \rho(Q) \leq r < \rho(\widehat{Q})\} \tag{2.1}$$

where, abusing notation, we write $\rho(\widehat{\Omega}) = \infty$. Property 1 above ensures that this is indeed a section and property 2 ensures that $\mathcal{T}_k \preccurlyeq \mathcal{T}(r)$ for all k sufficiently large.

2.2 NON-AUTONOMOUS SELF-SIMILAR SETS

The notion of a non-autonomous self-conformal set was introduced and studied in [RU16], where under certain regularity assumptions the authors prove that the Hausdorff and box dimensions are equal and given by the zero of a certain pressure function. In this section, we consider a special case of their construction. For each $n \in \mathbb{N}$, let $\mathcal{J}_n$ be a finite index set and let $\Phi_n = \{S_{n,j}\}_{j \in \mathcal{J}_n}$ be a family of similarity maps $S_{n,j} \colon \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$S_{n,j}(\boldsymbol{x}) = r_{n,j} O_{n,j} \boldsymbol{x} + \boldsymbol{d}_{n,j}$$

where $r_{n,j} \in (0,1)$ and $O_{n,j}$ is an orthogonal matrix. To avoid degenerate situations, we assume that associated with the sequence $(\Phi_n)_{n=1}^{\infty}$ there is an invariant compact set $X \subset \mathbb{R}^d$ (that is $S_{n,j}(X) \subset X$ for all $n \in \mathbb{N}$ and $j \in \mathcal{J}_n$) and moreover that

$$\lim_{n \to \infty} \sup\{r_{1,j_1} \cdots r_{n,j_n} : j_i \in \mathcal{J}_i \text{ for each } i = 1, \ldots, n\} = 0. \tag{2.2}$$

Under these assumptions, associated with the sequence $(\Phi_n)_{n=1}^{\infty}$ is an *attractor*

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(j_1, \ldots, j_n) \in \mathcal{J}_1 \times \cdots \times \mathcal{J}_n} S_{1,j_1} \circ \cdots \circ S_{n,j_n}(X).$$

Since X is compact and invariant under any map $S_{n,j}$ with $j \in \mathcal{J}_n$, finiteness of each $\mathcal{J}_n$ implies that K is the intersection of a nested sequence of compact sets and therefore non-empty and compact. The sequence $(\Phi_n)_{n=1}^{\infty}$ is called a *non-autonomous iterated function system (IFS)* and the attractor K is called the *non-autonomous self-similar set*. We refer the reader to [RU16, §2] for more detail on this construction in a general setting.

Definition 2.1. We say that the non-autonomous IFS $(\Phi_n)_{n=1}^{\infty}$

(i) satisfies the *open set condition* if the invariant compact set X can be chosen to have non-empty interior $U = X^\circ$ so that for each $n \in \mathbb{N}$ and $j, j' \in \mathcal{J}_n$, $S_{n,j}(U) \subset U$ and $S_{n,j}(U) \cap S_{n,j'}(U) = \varnothing$ for $j \neq j' \in \mathcal{J}_n$; and

(ii) has *uniformly bounded contractions* if there is $0 < r_{\min} \leq r_{\max} < 1$ so that $r_{\min} \leq r_{n,j} \leq r_{\max}$ for all $n \in \mathbb{N}$ and $j \in \mathcal{J}_n$.

Since $\operatorname{Leb}\big(\sum_{j \in \mathcal{J}_n} S_{n,j}(U)\big) \leq \operatorname{Leb}(U)$ and $\operatorname{Leb}(S_{n,j}(U)) \geq r_{\min}^d > 0$, the above two conditions combine to give the following additional condition:

(iii) There is an $M \in \mathbb{N}$ so that $\#\mathcal{J}_n \leq M$ for all $n \in \mathbb{N}$.

Our main goal in this section is, assuming the open set condition and uniformly bounded contractions, to establish an explicit formula for $\dim_{\mathrm{A}} K$, depending only on the $r_{n,j}$. This will be done in Theorem 2.12. In order to obtain this result, we first make a reduction to a symbolic representation of the attractor K, which we will denote by Δ. Since this symbolic construction will later be required in §11, we establish this concept in a somewhat more general context.

2.3 REDUCTION TO SYMBOLIC REPRESENTATION

Recalling the definition of a metric tree from §2.1, we now introduce a symbolic representation of the non-autonomous self-similar set K.

Let $\Delta = \prod_{n=1}^{\infty} \mathcal{J}_n$. For $(j_1, \ldots, j_n) \in \mathcal{J}_1 \times \cdots \times \mathcal{J}_n$, we define the cylinder set

$$[j_1, \ldots, j_n] = \{j_1\} \times \cdots \times \{j_n\} \times \prod_{k=n+1}^{\infty} \mathcal{J}_k.$$

We recall that $\mathcal{T}_n$ denotes the set of all cylinders corresponding to finite sequences in $\mathcal{J}_1 \times \cdots \times \mathcal{J}_n$. It is clear that this sequence of partitions, equipped with the valuation ρ (recalling the non-degeneracy assumption (2.2)), induces the structure of a metric tree on Δ, which we denote by $\mathcal{T} = \bigcup_{n=0}^{\infty} \mathcal{T}_n$. We also define a natural projection $\pi \colon \Delta \to K$ by

$$\{\pi((j_n)_{n=1}^{\infty})\} = \bigcap_{n=1}^{\infty} S_{1,j_1} \circ \cdots \circ S_{n,j_n}(X).$$

Again, this map is well-defined by (2.2). A direct argument shows that π is Lipschitz.

We now prove that $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}} \Delta$. The open set condition ensures that the only work in this result is to handle the mild overlaps which occur from adjacent rectangles. In fact, our result will follow from the following standard elementary lemma for metric spaces which are "almost bi-Lipschitz equivalent".

Lemma 2.2. *Let (X, d_1) and (Y, d_2) be non-empty bounded metric spaces and suppose there is a function $f\colon X \to Y$ and constants $M \in \mathbb{N}$ and $c > 0$ so that for all $0 < r < 1$,*

(i) $\operatorname{diam}(f(B(x, r))) \leq cr$ *for all $x \in X$; and*

(ii) for every $y \in Y$ there are $x_1, \ldots, x_M \in X$ such that $B(y, r) \subset \bigcup_{i=1}^{M} f(B(x_i, r))$.

Then $\dim_{\mathrm{A}} X = \dim_{\mathrm{A}} Y$.

Proof. We first prove that $\dim_{\mathrm{A}} X \leq \dim_{\mathrm{A}} Y$. Without loss of generality, we may assume that $c \geq 1$. Throughout the proof, let $\varepsilon > 0$ and $0 < r \leq R < 1$ be arbitrary. First, let $x \in X$ be arbitrary and, writing, $N = N_r(f(B(x, R)))$, get $y_1, \ldots, y_N \in Y$ so that $f(B(x, R)) \subset \bigcup_{i=1}^{N} B(y_i, r)$. Since $\operatorname{diam} f(B(x, R)) \leq cR$,

$$N \lesssim_{\varepsilon} \left(\frac{cR}{r}\right)^{\dim_{\mathrm{A}} Y + \varepsilon} \lesssim \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} Y + \varepsilon}.$$

Moreover, for each $i = 1, \ldots, N$, there are $x_{i,1}, \ldots, x_{i,M} \in X$ such that $B(y_i, r) \subset \bigcup_{j=1}^{M} f(B(x_{i,j}, r))$. Thus since $\{B(x_{i,j}, r) : i = 1, \ldots, N \text{ and } j = 1, \ldots, M\}$ is a cover for $B(x, R)$,

$$N_r(B(x, R)) \leq NM \lesssim_{\varepsilon} \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} Y + \varepsilon}.$$

Since $\varepsilon > 0$ and $0 < r \leq R < 1$ are arbitrary, we see that $\dim_{\mathrm{A}} X \leq \dim_{\mathrm{A}} Y$.

Conversely, let $y \in Y$ be arbitrary and get $x_1, \ldots, x_M \in X$ such that $B(y, R) \subset \bigcup_{i=1}^{M} f(B(x_i, R))$. Moreover, for each $i = 1, \ldots, M$, writing $N_i = N_{c^{-1}r}(B(x_i, R))$, there are $x_{i,1}, \ldots, x_{i,N_i}$ where $B(x_i, R) \subset \bigcup_{j=1}^{N_i} B(x_{i,j}, c^{-1}r)$ and

$$N_i \lesssim_{\varepsilon} \left(\frac{cR}{r}\right)^{\dim_{\mathrm{A}} X + \varepsilon} \lesssim \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} X + \varepsilon}.$$

Thus since $\{f(B(x_{i,j}, c^{-1}r)) : i = 1, \ldots, M \text{ and } j = 1, \ldots, N_i\}$ is a cover for $B(y, R)$ with $\operatorname{diam} f(B(x_{i,j}, c^{-1}r)) \leq r$,

$$N_r(B(y, R)) \lesssim_{\varepsilon} N_1 + \cdots + N_M \lesssim_{\varepsilon} \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} X + \varepsilon}.$$

Again since $\varepsilon > 0$ and $0 < r \leq R < 1$ are arbitrary, we get $\dim_{\mathrm{A}} Y \leq \dim_{\mathrm{A}} X$, completing the proof. $\square$

We now obtain our result on the Assouad dimension as a direct corollary.

Corollary 2.3. *Let $\{\Phi_n\}_{n=1}^{\infty}$ be a sequence of self-similar IFSs with associated non-autonomous self-similar set K and metric tree $\mathcal{T}$ with boundary Δ. Suppose the IFS also satisfies the open set condition and has uniformly bounded contractions. Then* $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}} \Delta$.

Proof. Let $0 < r < 1$. First, recall that the map $\pi\colon \Delta \to K$ is Lipschitz. Moreover, if $[i_1, \ldots, i_m], [j_1, \ldots, j_\ell] \in \mathcal{T}(r)$ are distinct, then

$$S_{1,i_1} \circ \cdots \circ S_{m,i_m}(U) \cap S_{1,j_1} \circ \cdots \circ S_{\ell,j_\ell}(U) = \varnothing$$

and by the uniformly bounded contraction assumption,

$$\mathrm{Leb}\left(S_{1,i_1} \circ \cdots \circ S_{m,i_m}(U)\right) \approx \mathrm{Leb}\left(S_{1,j_1} \circ \cdots \circ S_{\ell,j_\ell}(U)\right) \approx r^d.$$

But for $x \in K$, $\mathrm{Leb}(B(x,r)) \approx r^d$. Thus there is a constant $M \in \mathbb{N}$ not depending on r so that if $x \in K$ is arbitrary, there are cylinders $I_1, \ldots, I_M \in \mathcal{T}(r)$ so that $B(x,r) \subset \pi(I_1) \cup \cdots \cup \pi(I_M)$ so that each $I_j \in \mathcal{T}(r)$ and therefore $\operatorname{diam} I_j \leq r$. Thus the conditions for Lemma 2.2 are satisfied and $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}} \Delta$. □

2.4 Regularity properties of Assouad dimension

In this section, we establish some regularity properties related to the Assouad dimension. We begin with the following subadditivity-type lemma, where we replace the usual subadditivity hypothesis with an "endpoint" hypothesis combined along with a certain asymptotic upper semi-continuity.

Lemma 2.4. *Let $A = \mathbb{R}^+$ or $A = \{\kappa_0 n : n \in \mathbb{N}\}$ for some $\kappa_0 > 0$. Suppose $g\colon A \to \{-\infty\} \cup \mathbb{R}$ is any function satisfying the following two assumptions:*

(i) For all $y, z \in A$,

$$g(y+z) \leq \max\{g(y), g(z)\}.$$

(ii) For all $\varepsilon > 0$, there is a $\delta > 0$ so that for all $y, t \in A$ with $t \leq \delta y$,

$$g(y+t) \leq g(y) + \varepsilon.$$

Then

$$\limsup_{y \to \infty} g(y) = \inf_{y \in A} g(y).$$

Proof. Let $\varepsilon > 0$ be arbitrary, and let $\delta > 0$ be chosen to satisfy the conclusion of (ii). Let $z \in A$ be arbitrary and let $y \geq z\delta^{-1}$ be arbitrary. Write $y = \ell z + t$ for $\ell \in \mathbb{N}$ and $0 \leq t < z$. Then applying (ii) followed by (i) $\ell - 1$ times,

$$g(y) = g(\ell z + t) \leq g(\ell z) + \varepsilon \leq g(z) + \varepsilon.$$

Thus

$$\limsup_{y \to \infty} g(y) \leq g(z) + \varepsilon.$$

But $z \in A$ and $\varepsilon > 0$ were arbitrary, so the desired result follows. □

Remark 2.5. The assumptions of Lemma 2.4 are satisfied by the function $f(x)/x$, where $f\colon A \to \mathbb{R} \cup \{-\infty\}$ is any subadditive function bounded from above. The proof is similar to the proof of Lemma 2.7 below.

Note that assumption (i) of Lemma 2.4 is not sufficient by itself to guarantee the existence of the limit $\lim_{y \to \infty} g(y)$. For example, consider the function $g\colon \mathbb{N} \to \mathbb{R}$ defined by

$$g(n) = \begin{cases} 1 & : n \text{ odd}, \\ 0 & : n \text{ even}. \end{cases}$$

We now observe the following generalization of Lemma 2.4, with an additional varying second parameter. It is this form that will be essential for us in applications.

Lemma 2.6. *Let $A = \mathbb{R}^+$ or $A = \{\kappa_0 n : n \in \mathbb{N}\}$ for some $\kappa_0 > 0$. Suppose $f\colon A \times A \to \{-\infty\} \cup \mathbb{R}$ is any function satisfying the following two assumptions:*

(i) For all $x, y, z \in A$,

$$f(x, y + z) \leq \max\{f(x, y), f(x + y, z)\}.$$

(ii) For all $\varepsilon > 0$, there is a $\delta > 0$ so that for all $x, y, t \in A$ with $t \leq \delta y$ and $x \leq x' \leq x + t$,

$$f(x, y + t) \leq f(x', y) + \varepsilon.$$

Then

$$\begin{aligned} \beta &:= \limsup_{y \to \infty} \limsup_{x \to \infty} f(x, y) \\ &= \lim_{y \to \infty} \limsup_{x \to \infty} f(x, y) \\ &= \lim_{y \to \infty} \sup_{x \in A} f(x, y) \\ &= \inf_{y \in A} \sup_{x \in A} f(x, y). \end{aligned}$$

Moreover, if $B \subset A$ is of the form $B = \{\kappa n : n \in \mathbb{N}\}$ for some $\kappa > 0$, then

$$\beta = \lim_{\substack{y \to \infty \\ y \in B}} \sup_{x \in B} f(x, y).$$

Proof. We assume that $\beta > -\infty$: the proof for $\beta = -\infty$ is similar (and substantially easier).

Write $g(y) = \limsup_{x \to \infty} f(x, y)$. We first show that $\beta = \lim_{y \to \infty} g(y)$. First,

$$\begin{aligned} g(y_1 + y_2) &= \limsup_{x \to \infty} f(x, y_1 + y_2) \\ &\leq \limsup_{x \to \infty} \max\{f(x, y_1), f(x + y_1, y_2)\} \\ &\leq \max\{g(y_1), g(y_2)\}. \end{aligned}$$

Moreover, if $\varepsilon > 0$ is arbitrary, taking $\delta > 0$ to satisfy the conclusions of (ii) and $y, t \in A$ with $t \leq \delta y$, applying (ii),

$$g(y + t) = \limsup_{x \to \infty} g(x, y + t) \leq \limsup_{x \to \infty} f(x, y) + \varepsilon = g(y) + \varepsilon.$$

Thus the function g satisfies the hypotheses of Lemma 2.4, and therefore the limit defining β indeed exists. The same argument with respect to the function $h(y) = \sup_{x \in A} f(x, y)$ gives that

$$\limsup_{y \to \infty} \sup_{x \in A} f(x, y) = \inf_{y \in A} \sup_{x \in A} f(x, y).$$

To complete the proof of the various equalities involving β, it remains to show that

$$\inf_{y\in A}\sup_{x\in A} f(x,y)\leq\beta. \tag{2.3}$$

Let $\varepsilon>0$ be arbitrary and let $\delta>0$ be chosen as in (ii). By definition of β, there are y_0 and K so that for all $x\geq K$, $f(x,y_0)\leq\beta+\varepsilon$. Now let $y\in A$ be arbitrary and write $y=\ell y_0+t$ for some $\ell\in\mathbb{N}\cup\{0\}$ and $0<t\leq y_0$. By (i) and (ii), there is some $M\in A$ depending only on ε and y_0 so that for all $x\geq K$ and $y\geq y_0\delta^{-1}$,

$$\begin{aligned} f(x,y)=f(x,\ell y_0+t)&\leq f(x,\ell y_0)+\varepsilon\\ &\leq\max_{i=0,\dots,\ell-1} f(x+iy_0,y_0)+\varepsilon\leq\beta+2\varepsilon. \end{aligned}\tag{2.4}$$

Now let $x\in(0,K)\cap A$ and $y\geq\max\{y_0\delta^{-1},K\delta^{-1}\}$. Then again applying (ii) followed by (2.4), for all $x\in A$,

$$f(x,y)\leq f(K,y+x-K)+\varepsilon\leq\beta+3\varepsilon.$$

Since $\varepsilon>0$ was arbitrary, this proves (2.3).

Finally, suppose $B\subset A$ is of the form $B=\{\kappa n:n\in\mathbb{N}\}$ for some $\kappa>0$. First, note that since $B\subset A$,

$$\beta\geq\lim_{\substack{y\to\infty\\ y\in B}}\sup_{x\in B} f(x,y)$$

and moreover the limit exists as proven above. Conversely, let $(x,y)\in A\times A$ be arbitrary with $y\geq 2\kappa$ and get $(x_0,y_0)\in B\times B$ such that $x\leq x_0<x+\kappa$ and $x+y-\kappa<x_0+y_0\leq x+y$. Let $\varepsilon>0$ be arbitrary and get $\delta>0$ satisfying the conclusion of (ii). Then for $y\geq\delta^{-1}\kappa$, applying (ii) twice,

$$f(x_0,y_0)\geq f(x,y)-2\varepsilon.$$

Since $\varepsilon>0$ and $(x,y)\in A\times A$ were arbitrary, it follows that

$$\limsup_{\substack{y_0\to\infty\\ y_0\in B}}\sup_{x_0\in B} f(x_0,y_0)\geq\limsup_{y\to\infty}\sup_{x\in A} f(x,y)=\beta$$

as required. □

Finally, we show that the hypotheses of Lemma 2.6 are satisfied by functions satisfying a two-parameter version of subadditivity.

Lemma 2.7. *Let $A=\mathbb{R}^+$ or $A=\{\kappa_0 n:n\in\mathbb{N}\}$ for some $\kappa_0>0$. Suppose $f\colon A\times A\to\{-\infty\}\cup\mathbb{R}$ is any function such that for all $x,y,z\in A$,*

$$f(x,y+z)\leq\frac{yf(x,y)+zf(x+y,z)}{y+z}. \tag{2.5}$$

Then:

(i) *For all $x,y,z\in A$,*

$$f(x,y+z)\leq\max\{f(x,y),f(x+y,z)\}.$$

(ii) *Suppose moreover that f is bounded from above. Then for all $\varepsilon > 0$, there is a $\delta > 0$ so that for all $x, y, t \in A$ with $t \leq \delta y$ and $x \leq x' \leq x + t$,*

$$f(x, y+t) \leq f(x', y) + \varepsilon.$$

Proof. Of course, (i) is immediate. To see (ii), let $C \in \mathbb{R}$ be such that $f(x,y) \leq C$ for all $x, y \in A$. Let $\varepsilon > 0$ be arbitrary and let $\delta = \varepsilon C^{-1}$. Then for all $x, y, t \in A$ with $t \leq \delta y$ and $x \leq x' \leq x + t$, applying (2.5) twice,

$$\begin{aligned} f(x, y+t) &\leq \frac{(x'-x)f(x, x'-x) + (y+t+x-x')f(x', y+t+x-x')}{y+t} \\ &\leq \frac{(x'-x)f(x, x'-x) + yf(x', y) + (t+x-x')f(x'+y, t+x-x')}{y+t} \\ &\leq \frac{y}{y+t}f(x', y) + \frac{tC}{y+t} \\ &\leq f(x', y) + \varepsilon \end{aligned}$$

as claimed. ☐

As an application, we obtain a nice reformulation of the Assouad dimension of an arbitrary set which is reminiscent of a notion of definition first introduced by Larman [Lar67]. Let X be a compact doubling metric space and for $\delta \in (0,1)$ and $r \in (0,1)$, write

$$\psi(r, \delta) = \sup_{x \in X} N_{r\delta}\big(B(x, r) \cap K\big)$$

and then set

$$\Psi(r, \delta) = \frac{\log \psi(r, \delta)}{\log(1/\delta)}.$$

One can think of $\Psi(r, \delta)$ is the best guess for the Assouad dimension of X at scales $0 < r\delta < \delta < 1$. This heuristic is made precise in the following result.

Corollary 2.8. *Let X be a compact doubling metric space. Then*

$$\dim_{\mathrm{A}} X = \limsup_{\delta \to 0} \limsup_{r \to 0} \Psi(r, \delta) = \lim_{\delta \to 0} \sup_{r \in (0,1)} \Psi(r, \delta). \tag{2.6}$$

Proof. Since X is doubling, there is an $M \geq 0$ so that $\Psi(r, \delta) \in [0, M]$. Moreover, given $r, \delta_1, \delta_2 \in (0,1)$, by covering balls $B(x, r\delta_1)$ by balls of radius $r\delta_1\delta_2$,

$$\psi(r, \delta_1\delta_2) \leq \psi(r, \delta_1)\psi(r\delta_1, \delta_2)$$

and therefore

$$\begin{aligned} \Psi(r, \delta_1\delta_2) &= \frac{\log \psi(r, \delta_1\delta_2)}{\log(1/\delta_1\delta_2)} \\ &\leq \frac{\log \psi(r, \delta_1) + \log \psi(r\delta_1, \delta_2)}{\log(1/\delta_1\delta_2)} \\ &= \frac{\log(1/\delta_1)\Psi(r, \delta_1) + \log(1/\delta_2)\Psi(r\delta_1, \delta_2)}{\log(1/\delta_1) + \log(1/\delta_2)}. \end{aligned}$$

Thus with the change of coordinate $g(x,y) = (e^{-x}, e^{-y})$, the second equality in (2.6) follows by applying Lemma 2.7 and Lemma 2.6 to the function $\Psi \circ g$.

To see the first equality in (2.6), it is a direct consequence of the definition of the Assouad dimension that

$$\limsup_{\delta \to 0} \limsup_{r \to 0} \Psi(r, \delta) \leq \dim_{\mathrm{A}} K$$

and that there are sequences $(\delta_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} \delta_n = 0$ such that

$$\lim_{\delta \to 0} \sup_{r \in (0,1)} \Psi(r, \delta) \geq \limsup_{n \to \infty} \Psi(r_n, \delta_n) \geq \dim_{\mathrm{A}} K,$$

as required. □

Finally, we prove that in the definition of the Assouad dimension one may replace the exponent associated to localized coverings of balls of the same size by an exponent coming from localized packings of balls which may have different sizes. This will be useful since the natural covers appearing from the symbolic representation of K consist of cylinders which may have very non-uniform diameters when indexed by length. First, for a metric space X, $x \in X$, and $R \in (0,1)$, denote the family of all localized centred packings by

$$\operatorname{pack}(X, x, R) = \left\{ \{B(x_i, r_i)\}_{i=1}^{\infty} : \begin{array}{c} 0 < r_i \leq R, x_i \in X, B(x_i, r_i) \subset B(x, R), \\ B(x_i, r_i) \cap B(x_j, r_j) = \varnothing \text{ for all } i \neq j \end{array} \right\}.$$

In our proof, we will also require the Assouad dimension of a measure. Given a compact doubling metric space X and a Borel measure μ with $\operatorname{supp} \mu = X$, the *Assouad dimension of* μ is given by

$$\dim_{\mathrm{A}} \mu = \inf \Big\{ \alpha \geq 0 : \forall x \in X \, \forall 0 < r \leq R < \operatorname{diam} X \quad \frac{\mu(B(x,R))}{\mu(B(x,r))} \lesssim_{\alpha} \left(\frac{R}{r}\right)^{\alpha} \Big\}.$$

The main result of [VK88] (the original Russian version can be found in [VK87]) is that for a compact doubling metric space X,

$$\dim_{\mathrm{A}} X = \inf\{\dim_{\mathrm{A}} \mu : \operatorname{supp} \mu = X\}.$$

In the following result, we observe that the existence of good measures provides a convenient way to control the localized disk packing exponent.

Proposition 2.9. *Let X be a bounded metric space. Then*

$$\dim_{\mathrm{A}} X = \inf \Big\{ \alpha : \forall 0 < R < 1 \, \forall x \in X \, \forall \{B(x_i, r_i)\}_{i=1}^{\infty} \in \operatorname{pack}(X, x, R) \quad \sum_{i=1}^{\infty} r_i^{\alpha} \lesssim_{\alpha} R^{\alpha} \Big\}.$$

Proof. That

$$\dim_{\mathrm{A}} X \leq \inf\Big\{\alpha : \forall 0 < R < 1 \,\forall x \in X \,\forall \{B(x_i, r_i)\}_{i=1}^{\infty} \in \operatorname{pack}(X, x, R) \\ \sum_{i=1}^{\infty} r_i^{\alpha} \lesssim_{\alpha} R^{\alpha}\Big\}$$

is immediate by specializing to packings with $r_i = r$ for some $0 < r \leq R$, using the equivalence (up to a constant factor) of covering and packing counts.

Now to show the lower bound, if X is not doubling, then $\dim_{\mathrm{A}} X = \infty$ and the result is trivial. Otherwise, by passing to the completion (which does not change the value of the Assouad dimension) and recalling that a bounded doubling metric space is totally bounded, we may assume that X is also compact. Thus let $\alpha > \dim_{\mathrm{A}} X$ be arbitrary. By [VK88, Theorem 1], there is a probability measure μ with $\operatorname{supp} \mu = X$ and $\dim_{\mathrm{A}} \mu < \alpha$. Then for any $0 < R < 1$, $x \in X$, and $\{B(x_i, r_i)\}_{i=1}^{\infty} \in \operatorname{pack}(X, x, R)$, by disjointness,

$$\mu\big(B(x, R)\big) \geq \sum_{i=1}^{\infty} \mu\big(B(x_i, r_i)\big) \gtrsim \mu\big(B(x, R)\big) \sum_{i=1}^{\infty} \left(\frac{r_i}{R}\right)^{\alpha}.$$

Therefore,

$$\sum_{i=1}^{\infty} r_i^{\alpha} \lesssim R^{\alpha}$$

which, since $\alpha > \dim_{\mathrm{A}} X$ was arbitrary, yields the claimed result. □

2.5 Proof of the Assouad dimension formula

We can now state and prove the desired formula for the Assouad dimension of the non-autonomous self-similar set K. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be arbitrary, and let $\theta(n, m)$ denote the unique value satisfying the equation

$$\sum_{j_1 \in \mathcal{J}_{n+1}} \cdots \sum_{j_m \in \mathcal{J}_{n+m}} \prod_{k=1}^{m} r_{n+k, j_k}^{\theta(n,m)} = 1.$$

Note that $\theta(n, m)$ is precisely the similarity dimension of the IFS

$$\Phi_{n+1} \circ \cdots \circ \Phi_{n+m} = \{f_1 \circ \cdots \circ f_m : f_i \in \Phi_{n+i}\}.$$

For the remainder of this section, we fix a non-autonomous IFS $(\Phi_n)_{n=1}^{\infty}$ satisfying the open set condition and with uniformly bounded contraction ratios. Associated with this IFS is the function θ and the non-autonomous self-similar set K.

We begin by showing that the function θ has a certain continuity properties. To do this, we require some notation. Let $n, m \in \mathbb{N}$ and define $\phi_{n,m} \colon [0, 1] \to \mathbb{R}$ by

$$\phi_{n,m}(s) = \sum_{j_1 \in \mathcal{J}_{n+1}} \cdots \sum_{j_m \in \mathcal{J}_{n+m}} \prod_{k=1}^{m} r_{n+k, j_k}^{s}.$$

Of course, $\phi_{n,m}$ is strictly decreasing with unique zero $\theta(n, m)$.

Lemma 2.10. *For all $n, m, k \in \mathbb{N}$ and $n \leq n' \leq n+k$,*

$$|\theta(n, m+k) - \theta(n', m)| \lesssim \frac{k}{m}.$$

Proof. First, let $n, m \in \mathbb{N}$ be arbitrary. Since the IFS has uniformly bounded contraction ratios and $\sup_{i \in \mathbb{N}} \#\mathcal{J}_i < \infty$, by definition of $\theta(n, m)$,

$$\begin{aligned} \phi_{n,m+1}(\theta(n,m)) &= \sum_{j_1 \in \mathcal{J}_{n+1}} \cdots \sum_{j_m \in \mathcal{J}_{n+m}} \sum_{i \in \mathcal{J}_{n+m+1}} r_{n+1,j_1}^{\theta(n,m)} \cdots r_{n+m,j_m}^{\theta(n,m)} \cdot r_{n+m+1,i}^{\theta(n,m)} \\ &= \sum_{i \in \mathcal{J}_{n+m+1}} r_{n+m+1,i}^{\theta(n,m)} \approx 1. \end{aligned} \tag{2.7}$$

On the other hand,

$$\phi_{n,m+1}(\theta(n,m) + \varepsilon) \leq \phi_{n,m+1}(\theta(n,m)) \cdot r_{\min}^{m\varepsilon}$$

so that, if $\theta(n,m) + \varepsilon \geq \theta(n, m+1)$, applying (2.7), we observe that $r_{\min}^{m\varepsilon} \approx 1$ which forces $\varepsilon \approx 1/m$. The same argument also holds for the lower bound.

Iterating this bound k times,

$$|\theta(n, m+k) - \theta(n, m)| \lesssim \frac{k}{m}.$$

Of course, the same argument also applies when extending to the left, so that

$$|\theta(n, m+k) - \theta(n+k, m)| \lesssim \frac{k}{m}.$$

Thus if $n \leq n' \leq n+k$ is arbitrary,

$$\begin{aligned} |\theta(n, m+k) - \theta(n', m)| &\leq |\theta(n, m+k) - \theta(n, m+n'-n)| \\ &\quad + |\theta(n, m+n'-n) - \theta(n', m)| \\ &\lesssim \frac{k}{m} \end{aligned}$$

as required. □

We now establish the fundamental property of the function θ.

Lemma 2.11. *The function $\theta \colon \mathbb{N} \times \mathbb{N} \to [0, d]$ satisfies the assumptions of Lemma 2.6.*

Proof. Let $n, m_1, m_2 \in \mathbb{N}$ be arbitrary. We first show that

$$\theta(n, m_1 + m_2) \leq s := \max\{\theta(n, m_1), \theta(n + m_1, m_2)\}.$$

Indeed,

$$1 = \sum_{j_1 \in \mathcal{J}_{n+1}} \cdots \sum_{j_{m_1+m_2} \in \mathcal{J}_{n+m_1+m_2}} \left(\prod_{k=1}^{m_1} r_{n+k,j_k} \right)^{\theta(n,m_1)} \left(\prod_{k=m_1+1}^{m_1+m_2} r_{n+k,j_k} \right)^{\theta(n+m_1,m_2)}$$

$$\geq \sum_{j_1 \in \mathcal{J}_{n+1}} \cdots \sum_{j_{m_1+m_2} \in \mathcal{J}_{n+m_1+m_2}} \left(\prod_{k=1}^{m_1+m_2} r_{n+k,j_k} \right)^s$$
$$= \phi_{n,m_1+m_2}(s).$$

Since $\phi_{n,m_1+m_2}(\theta(n, m_1 + m_2)) = 1$ and ϕ_{n,m_1+m_2} is strictly decreasing, it follows that $\theta(n, m_1 + m_2) \leq s$, which is (i).

Next, let $\varepsilon > 0$ be arbitrary, let $n, m, k \in \mathbb{N}$, and let $n \leq n' \leq n + k$. Then by Lemma 2.10, there is a constant $C > 0$ so that

$$|\theta(n', m) - \theta(n, m + k)| \leq C \frac{k}{m}.$$

In particular, taking $\delta = \varepsilon C^{-1}$ yields (ii). □

Finally, we establish the following formula for the Assouad dimension of the non-autonomous self-similar set K.

Theorem 2.12. *Let $(\Phi_n)_{n=1}^{\infty}$ be a non-autonomous IFS satisfying the open set condition and with uniformly bounded contraction ratios. Denote the associated non-autonomous self-similar set by K. Then*

$$\dim_{\mathrm{A}} K = \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \theta(n, m). \tag{2.8}$$

Proof. Recall that the limit in (2.8) exists by Lemma 2.11 and Lemma 2.6, and moreover

$$s := \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \theta(n, m) = \lim_{m \to \infty} \limsup_{n \to \infty} \theta(n, m)$$

We verify the lower and upper bounds separately.

First, recall from Corollary 2.3 that $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}} \Delta$, where Δ denotes the metric tree associated with K. Let $\varepsilon > 0$ be fixed and let M be sufficiently large so that for all $m \geq M$, there is an $n \in \mathbb{N}$ so that

$$|\theta(n, m) - s| \leq \varepsilon.$$

Now fix a cylinder $[j_1, \ldots, j_n] \subset \Delta$ for some $(j_1, \ldots, j_n) \in \mathcal{J}_1 \times \cdots \times \mathcal{J}_n$ and write $R = \operatorname{diam}([j_1, \ldots, j_n]) = r_{1,j_1} \cdots r_{n,j_n}$. Note that if $m \geq M$, by definition of $\theta(n, m)$

$$\sum_{j_{n+1} \in \mathcal{J}_{n+1}} \cdots \sum_{j_{n+m} \in \mathcal{J}_{n+m}} \prod_{k=1}^{n+m} r_{k,j_k}^{\theta(n,m)} = R^{\theta(n,m)}.$$

But the family of cylinders

$$\{[j_1, \ldots, j_{n+m}] : (j_{n+1}, \ldots, j_{n+m}) \in \mathcal{J}_{n+1} \times \cdots \times \mathcal{J}_{n+m}\}$$

forms a packing of $B(x, R)$. Moreover, since $m \geq M$ is arbitrary, by the uniform boundedness assumption, the width of each cylinder in this family relative to $[j_1, \ldots, j_n]$ converges uniformly to 0. Thus by Proposition 2.9, $\dim_{\mathrm{A}} K \geq s - \varepsilon$.

Conversely, let us upper bound $\dim_{\mathrm{A}} K$. Recall that $\varepsilon > 0$ is fixed as above and let $m \geq M$ be fixed. Now let $0 < r \leq R < 1$ and fix a ball $B(x, R) \subset \Delta$. By definition of the metric on Δ, $B(x,R) = [j_1,\ldots,j_n]$ where $r_{1,j_1}\cdots r_{n,j_n} \leq R$. We inductively build a sequence of covers $(\mathcal{B}_k)_{k=1}^\infty$ for $B(x,R)$ such that each $\mathcal{B}_k$ is composed only of cylinder sets and

$$\sum_{[i_1,\ldots,i_\ell]\in\mathcal{B}_k} (r_{1,i_1}\cdots r_{\ell,i_\ell})^{s+\varepsilon} \leq R^{s+\varepsilon}. \tag{2.9}$$

and

$$r_{1,i_1}\cdots r_{\ell,i_\ell} \geq r\cdot r_{\min}^m \qquad \text{for all} \qquad [i_1,\ldots,i_\ell]\in\mathcal{B}_k. \tag{2.10}$$

Begin with $\mathcal{B}_1 = \{[j_1,\ldots,j_n]\}$, which clearly satisfies the requirements.

Now suppose we have constructed $\mathcal{B}_k$ for some $k\in\mathbb{N}$. Let $[i_1,\ldots,i_\ell]\in\mathcal{B}_k$ be an arbitrary cylinder set. If $r_{1,i_1}\cdots r_{\ell,i_\ell}\leq r$, do nothing; this guarantees that (2.10) holds. Otherwise, replace the cylinder $[i_1,\ldots,i_\ell]$ with the family of cylinders

$$\{[i_1,\ldots,i_\ell,j_1,\ldots,j_m] : (j_1,\ldots,j_m)\in\mathcal{J}_{\ell+1}\times\cdots\times\mathcal{J}_{\ell+m}\}.$$

The choice of $m\geq M$ and the definition of $\theta(\ell,m)$ ensures that (2.9) holds.

Repeat this process until every cylinder in $\mathcal{B}_k$ has diameter $\leq r$. That this process terminates at a finite level k is guaranteed by (2.2). Thus replacing each cylinder $[i_1,\ldots,i_\ell]$ with a ball $B(x_{i_1,\ldots,i_\ell}, r)$ for some $x_{i_1,\ldots,i_\ell}\in[i_1,\ldots,i_\ell]$, by (2.9) and (2.10) the corresponding cover has

$$\sum_{[i_1,\ldots,i_\ell]\in\mathcal{B}_k} r^{s+\varepsilon} \leq r_{\min}^{-m(s+\varepsilon)} \sum_{[i_1,\ldots,i_\ell]\in\mathcal{B}_k} (r_{1,i_1}\cdots r_{\ell,i_\ell})^{s+\varepsilon} \lesssim R^{s+\varepsilon}$$

which guarantees that $\dim_{\mathrm{A}} K \leq s+\varepsilon$, as claimed. □

3 Self-affine carpets following Gatzouras–Lalley and Barański

In this section, we introduce the self-affine carpets originally studied by Gatzouras & Lalley [LG92] and Barański [Bar07]. These planar self-affine carpets are examples of *iterated function system attractors* and are invariant under certain families of planar self-affine maps $\{T_i\}_{i\in\mathcal{I}}$. The geometry of self-affine sets has been a highly active area of research going back to the original works of Bedford [Bed84] and McMullen [McM84] under a grid structure; and without grid structure to the "parameter-typical" results of Falconer [Fal88].

Within the geometry of self-affine sets, there are two complementary perspectives. The first "typical" case attempts to find general conditions under which the Hausdorff and box dimensions are equal and are determined by the *affinity dimension* determined by the singular values of the matrix parts of the affine transformations. Especially in the plane, substantial recent progress has been made on this problem; see for instance [BHR19; HR22b] and the many references cited within.

The second "non-typical" case is complementary: the goal is to study self-affine sets for which the box and Hausdorff dimensions may be different and moreover may not be given by the affinity dimension. These exceptional carpets

(A) Gatzouras–Lalley (B) Barański

FIGURE I.1: Some diagonal self-affine sets, which are attractors of the iterated function systems depicted in Figure I.2.

exhibit features such as *alignment* and *a lack of rotation*. The carpets of Gatzouras–Lalley and Barański fall into this category. Outside of some exceptional choices of parameters, the Hausdorff, box, and Assouad dimensions are all distinct and moreover the Assouad spectrum indeed approaches the Assouad dimension as θ converges to 1.

For the remainder of this section, we formally introduce these classes of sets and the notation and constructions which are important for their study.

3.1 DIAGONAL ITERATED FUNCTION SYSTEMS

We begin by introducing a general rotation-free family of iterated function systems: those formed by affine maps for which the linear parts are diagonal matrices.

Fix a non-empty index set $\mathcal{I}$ and for $j = 1, 2$ fix contraction ratios $(\beta_{i,j})_{i\in\mathcal{I}} \subset (0, 1)$ and translations $(d_{i,j})_{i\in\mathcal{I}} \subset \mathbb{R}$. We then call the IFS $\{T_i\}_{i\in\mathcal{I}}$ *diagonal* when

$$T_i(x_1, x_2) = \big(\beta_{i,1}x_1 + d_{i,1}, \beta_{i,2}x_2 + d_{i,2}\big) \quad \text{for each} \quad i \in \mathcal{I}.$$

For $j = 1, 2$, let η_j denote the orthogonal projection onto the j^{th} coordinate axis, i.e. $\eta_j(x_1, x_2) = x_j$. We denote by $\{S_{i,j}\}_{i\in\mathcal{I}}$ the *projected systems*, where $S_{i,j}\colon \mathbb{R} \to \mathbb{R}$ is the unique map satisfying $\eta_j \circ T_j = S_{i,j} \circ \eta_j$. Of course, $S_{i,j}(x) = \beta_{i,j}x + d_{i,j}$ are iterated function systems of similarities. We will sometimes write $\eta = \eta_1$ to denote simply the projection onto the first coordinate axis.

By Hutchinson's application of the contraction mapping principle [Hut81], associated with the IFS $\{T_i\}_{i\in\mathcal{I}}$ is a unique non-empty compact *attractor* K satisfying $K = \bigcup_{i\in\mathcal{I}} T_i(K)$. Of course, for $j = 1, 2$, the projected IFS $\{S_{i,j}\}_{i\in\mathcal{I}}$ has attractor $K_j = \eta_j(K)$ for $j = 1, 2$. Some images of attractors can be found in Figure I.1.

We now introduce some symbolic notation. Set $\mathcal{I}^* = \bigcup_{n=0}^{\infty} \mathcal{I}^n$ equipped with the operation of concatenation. Given $\mathtt{i} \in \mathcal{I}^n$, we denote the *length* of $\mathtt{i}$ by $|\mathtt{i}| = n$. We say that a word $\mathtt{i} \in \mathcal{I}^*$ is a *prefix* of $\mathtt{k}$ if there is a word $\mathtt{j}$ so that $\mathtt{k} = \mathtt{ij}$, and we write $\mathtt{i} \preccurlyeq \mathtt{k}$. If $|\mathtt{i}| \geq 1$, we let $\mathtt{i}^-$ denote the unique prefix of $\mathtt{i}$ of length $|\mathtt{i}| - 1$. We denote the unique word of length 0 by $\varnothing$.

Given $\mathtt{i} = (i_1, \ldots, i_n) \in \mathcal{I}^n$ and $j = 1, 2$, we write

$$\begin{aligned} T_{\mathtt{i}} &= T_{i_1} \circ \cdots \circ T_{i_n}, \\ S_{\mathtt{i},j} &= S_{i_1,j} \circ \cdots \circ S_{i_n,j}, \\ \beta_{\mathtt{i},j} &= \beta_{i_1,j} \cdots \beta_{i_n,j} \end{aligned}$$

Now η_j induces an equivalence relation $\sim_j$ on $\mathcal{I}$ where $i \sim i'$ if $S_{i,j} = S_{i',j}$. Let $\eta_j \colon \mathcal{I} \to \mathcal{I}/\sim_j$ denote the natural projection. Intuitively, $\eta_j(i)$ is the set of indices which lie in the same column or row as the index i. Then η_j extends naturally to a map on Ω by $\eta_j((i_n)_{n=1}^\infty) = (\eta_j(i_n))_{n=1}^\infty \subset \eta_j(\mathcal{I})^{\mathbb{N}} \cong \eta_j(\mathcal{I}^{\mathbb{N}})$; and similarly extends to a map on $\mathcal{I}^*$. For notational clarity, we will refer to words in $\mathcal{I}^*$ using upright indices, such as $\mathtt{i}$, and words in $\eta_j(\mathcal{I}^*)$ using their underlined variants, such as $\underline{\mathtt{i}}$. Note that if $\mathtt{i} \sim_j \mathtt{j}$, then and $S_{\mathtt{i},j} = S_{\mathtt{j},j}$. In particularly, we may unambiguously write $S_{\underline{\mathtt{i}},j}$ and $\beta_{\underline{\mathtt{i}},j}$ for $\underline{\mathtt{i}} \in \eta_j(\mathcal{I}^*)$.

Next, we let $\Omega = \mathcal{I}^{\mathbb{N}}$ denote the space of infinite sequences on $\mathcal{I}$. The concatenation $\mathtt{i}\gamma$ for $\mathtt{i} \in \mathcal{I}^*$ and $\gamma \in \Omega$ is defined similarly, so we also speak of finite prefixes of infinite words. Given $\mathtt{i} \in \mathcal{I}^*$, we denote the *cylinder set*

$$[\mathtt{i}] = \{\gamma \in \Omega : \mathtt{i} \preccurlyeq \gamma\}.$$

Finally, we define the surjective coding map $\pi \colon \Omega \to K$ for $\gamma = (i_n)_{n=1}^\infty$ by

$$\{\pi(\gamma)\} = \bigcap_{n=1}^{\infty} T_{i_1} \circ \cdots \circ T_{i_n}(K).$$

Equivalently, the coding map π is defined by the rule $\pi([\mathtt{i}]) = T_{\mathtt{i}}(K)$ for $\mathtt{i} \in \mathcal{I}^*$.

3.2 INVARIANT MEASURES AND THE SPACE OF PROBABILITY VECTORS

We now introduce notation to handle the space of Bernoulli measures associated with the IFS $\{T_i\}_{i \in \mathcal{I}}$. Let

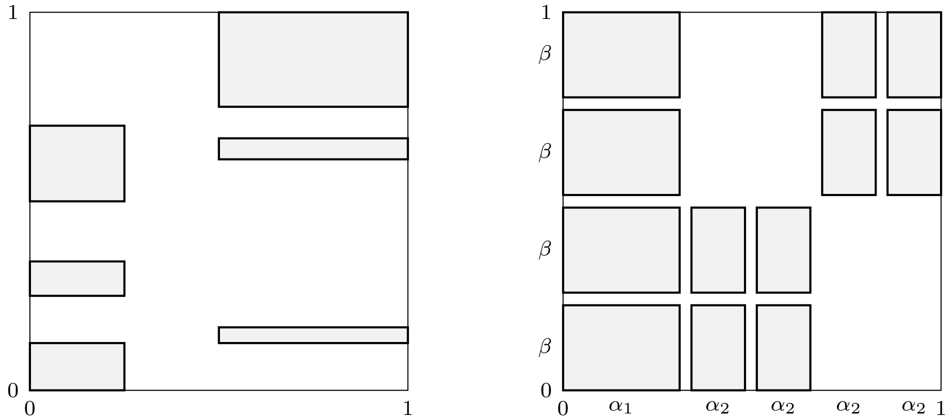
$$\mathcal{P} = \mathcal{P}(\mathcal{I}) = \Big\{(p_i)_{i \in \mathcal{I}} : p_i \geq 0, \sum_{i \in \mathcal{I}} p_i = 1\Big\} \subset \mathbb{R}^{\mathcal{I}},$$

which is a compact metric space with the metric inherited from ambient Euclidean space. We refer to probability vectors in $\mathcal{P}$ using bold-face letters, such as $\boldsymbol{w}$. Recall for $j = 1, 2$ that η_j denotes the orthogonal projection onto the first coordinate axis. In a similar way as before, η_j induces a map $\eta_j \colon \mathcal{P} \to \eta_j(\mathcal{P})$ by the rule

$$\eta_j(\boldsymbol{w}) = \Big(\sum_{i \in \eta_j^{-1}(\underline{j})} w_i\Big)_{\underline{j} \in \eta_j(\mathcal{I})}.$$

Given a probability vector $\boldsymbol{w} = (w_i)_{i \in \mathcal{I}} \in \mathcal{P}$, we define the *entropy*

$$H(\boldsymbol{w}) = \sum_{i \in \mathcal{I}} w_i \log(1/w_i)$$



(A) Gatzouras–Lalley (B) Barański

FIGURE I.2: Generating maps associated with a Gatzouras–Lalley and Barański system. The parameters from the Barański carpet correspond to the example in Corollary 12.7 with $\delta = 1/40$. The corresponding attractors can be found in Figure I.1.

(we extend the value of $x \log(1/x)$ to $x = 0$ by taking the limit) and *Lyapunov exponents*

$$\chi_1(\boldsymbol{w}) = \sum_{i \in \mathcal{I}} w_i \log(1/\beta_{i,1}) \qquad \text{and} \qquad \chi_2(\boldsymbol{w}) = \sum_{i \in \mathcal{I}} w_i \log(1/\beta_{i,2}).$$

For $\boldsymbol{p} \in \eta_j(\mathcal{I})$, we define entropy in the same way, and also note that $\chi_j(\boldsymbol{p})$ is well-defined since $\chi_j(\boldsymbol{w}) = \chi_j(\eta_j(\boldsymbol{w}))$. Note that H, χ_1, and χ_2 are continuous positive functions on $\mathcal{P}$, and moreover χ_1 and χ_2 are uniformly bounded away from 0.

Finally, we denote the *logarithmic eccentricity* of $\boldsymbol{w} \in \mathcal{P}$ by

$$\Gamma(\boldsymbol{w}) = \frac{\chi_2(\boldsymbol{w})}{\chi_1(\boldsymbol{w})}.$$

Note that Γ takes values in a compact interval

$$\Gamma(\mathcal{P}) = [\kappa_{\min}, \kappa_{\max}] \subset (0, \infty),$$

where

$$\kappa_{\min} = \min_{i \in \mathcal{I}} \frac{\log \beta_{i,2}}{\log \beta_{i,1}} \qquad \text{and} \qquad \kappa_{\max} = \max_{i \in \mathcal{I}} \frac{\log \beta_{i,2}}{\log \beta_{i,1}}.$$

We will use Γ to measure the exponential distortion of the rectangle $T_{\mathtt{i}}([0,1]^2)$ in terms of the digit frequencies corresponding to the word $\mathtt{i} \in \mathcal{I}^*$, with $\kappa_{\min}$ corresponding to the cases when the rectangle looks as close as possible to a square.

3.3 GATZOURAS–LALLEY AND BARAŃSKI CARPETS

Recall that an IFS $\{F_i\}_{i \in \mathcal{J}}$ satisfies the *open set condition* with respect to an open set U if $F_i(U) \subset U$ and $F_i(U) \cap F_j(U) = \varnothing$ for all $i \neq j \in \mathcal{J}$. The following definition concerns the main class of self-affine sets studied in [LG92].

Definition 3.1. We say that the IFS $\{T_i\}_{i\in\mathcal{I}}$ is *Gatzouras–Lalley* if:

(i) the original IFS $\{T_i\}_{i\in\mathcal{I}}$ satisfies the open set condition with respect to $(0,1)^2$,

(ii) $\beta_{i,2} < \beta_{i,1}$ for all $i \in \mathcal{I}$, and

(iii) the projected IFS $\{S_{\underline{i},1}\}_{\underline{i}\in\eta(\mathcal{I})}$ satisfies the open set condition with respect to $(0,1)$.

and *Barański* if:

(i) the original IFS $\{T_i\}_{i\in\mathcal{I}}$ satisfies the open set condition with respect to $(0,1)^2$, and

(ii) the projected IFS $\{S_{\underline{i},j}\}_{\underline{i}\in\eta_j(\mathcal{I})}$ satisfies the open set condition with respect to $(0,1)$ for $j = 1,2$.

As depicted in Figure I.2, the key features of a Gatzouras–Lalley IFS are that the rectangles $T_i([0,1]^2)$ are wider than they are tall, they cannot overlap except possibly along edges, they lie in columns which themselves cannot overlap except possibly along edges, and the height of each rectangle is strictly less than its width. For a Barański carpet, the rectangles can also be taller than they are wide, but they must lie in rows and columns which cannot overlap except possibly along edges.

Note that if $\{T_i\}_{i\in\mathcal{I}}$ is Gatzouras–Lalley, then $\chi_1(\boldsymbol{w}) < \chi_2(\boldsymbol{w})$ for all $\boldsymbol{w} \in \mathcal{P}$, and Γ takes values in a compact subset of $(1,\infty)$.

A special class of carpets which are simultaneously Gatzouras–Lalley and Barański are known as *Bedford–McMullen carpets*, introduced in [Bed84; McM84]. In this class, there are numbers natural numbers $2 \leq m < n$ so that $\beta_{i,1} = 1/m$ and $\beta_{i,2} = 1/n$ for all $i \in \mathcal{I}$.

Finally, we recall some standard results on the dimensions of Gatzouras–Lalley carpets. We defer the corresponding results for Barański carpets to §12.1. We state the main results of [LG92]—stated below in (i) and (ii)—as well as the result of [Mac11]—stated below in (iii). We also note that the same proof as given in [Mac11] (which is explained more precisely in [Fra14, Theorem 2.13]) gives the analogous result for the lower dimension.

Proposition 3.2 ([LG92; Mac11]). *Let K be a Gatzouras–Lalley carpet.*

(i) The Hausdorff dimension of K is given by

$$\dim_{\mathrm{H}} K = \sup_{\boldsymbol{p}\in\mathcal{P}} s(\boldsymbol{p})$$

where

$$s(\boldsymbol{p}) := \frac{H(\eta(\boldsymbol{p}))}{\chi_1(\boldsymbol{p})} + \frac{H(\boldsymbol{p}) - H(\eta(\boldsymbol{p}))}{\chi_2(\boldsymbol{p})}.$$

Moreover, the supremum is always attained at an interior point of $\mathcal{P}$ (i.e. at vector $\boldsymbol{w} \in \mathcal{P}$ with $\boldsymbol{w}_i > 0$ for all $i \in \mathcal{I}$).

(ii) The box dimension of K exists and is given by the unique solution to

$$\sum_{i\in\mathcal{I}} \beta_{i,1}^{\dim_{\mathrm{B}}\eta(K)} \beta_{i,2}^{\dim_{\mathrm{B}} K - \dim_{\mathrm{B}}\eta(K)} = 1 \qquad \textit{where} \qquad \sum_{\underline{j}\in\eta(\mathcal{I})} \beta_{\underline{j},1}^{\dim_{\mathrm{B}}\eta(K)} = 1.$$

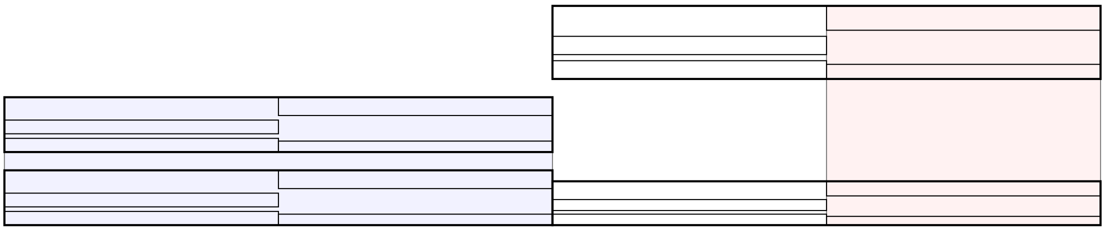


FIGURE I.3: Two iterations of a Gatzouras–Lalley IFS within a cylinder, with a wide pseudo-cylinder in highlighted in blue and a tall pseudo-cylinder in red.

(iii) The Assouad dimension of K is given by

$$\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + \max_{\underline{\ell}\in\eta(\mathcal{I})} t(\underline{\ell})$$

where $t(\underline{\ell})$ is defined as the unique solution to the equations

$$\sum_{j\in\eta^{-1}(\underline{\ell})} \beta_{j,2}^{t(\underline{\ell})} = 1.$$

Similarly, the lower dimension of K is given by

$$\dim_{\mathrm{L}} K = \dim_{\mathrm{B}} \eta(K) + \min_{\underline{\ell}\in\eta(\mathcal{I})} t(\underline{\ell}).$$

3.4 PSEUDO-CYLINDERS, APPROXIMATE SQUARES, AND SYMBOLIC SLICES

A common technique when studying invariant sets for iterated function systems on some index set $\mathcal{I}$ is to first reduce the problem to a symbolic problem on the coding space $\mathcal{I}^*$. However, the main technical complexity in understanding the dimension theory of self-affine carpets is that the cylinder sets $T_{\mathtt{i}}(K)$ are often exponentially distorted "rectangles". As a result, we will keep track of two symbolic systems simultaneously, which together will capture the geometry of the set K. In this section, we focus our attention on Gatzouras–Lalley carpets: these concepts generalize in a natural way to Barański carpets, but we defer the discussion to §12.

Fix a Gatzouras–Lalley IFS $\{T_i\}_{i\in\mathcal{I}}$. We first introduce some notation for handling cylinders. We then associate with the IFS $\{T_i\}_{i\in\mathcal{I}}$, and the related defining data that we introduced in §3, two metric trees: first, the metric tree of *approximate squares*, and second the metric tree of *symbolic slices*.

First, recall that $\Omega = \mathcal{I}^{\mathbb{N}}$ is the space of infinite sequences on $\mathcal{I}$. The family of cylinders $\{[\mathtt{i}] : \mathtt{i} \in \mathcal{I}^k\}_{k=0}^{\infty}$ defined in §3.1 defines a tree (in the sense of §2.1). We sometimes abuse notation and simply refer to $\{\mathcal{I}^k\}_{k=0}^{\infty}$ as a tree. We will associate with this tree a variety of metrics, such as those induced by the maps $\mathtt{i} \mapsto \beta_{\mathtt{i},j}$ for $j = 1, 2$. We will also use the same notation for the projected words $\{\eta(\mathcal{I}^k)\}_{k=0}^{\infty}$.

Next, we define the *metric tree of approximate squares*. Before we do this, we introduce the notion of a *pseudo-cylinder*. Suppose $\mathtt{i} \in \mathcal{I}^k$ and $\underline{\mathtt{j}} \in \eta(\mathcal{I}^{\ell})$. We then write

$$P(\mathtt{i}, \underline{\mathtt{j}}) = \{\gamma = (i_n)_{n=1}^{\infty} \in \Omega : (i_1, \ldots, i_k) = \mathtt{i} \text{ and } \eta(i_{k+1}, \ldots, i_{k+\ell}) = \underline{\mathtt{j}}\}.$$

Note that map $(\mathtt{i}, \underline{\mathtt{j}}) \mapsto P(\mathtt{i}, \underline{\mathtt{j}})$ is injective. Another equivalent way to understand the pseudo-cylinder $P(\mathtt{i}, \underline{\mathtt{j}})$ is as a finite union of cylinders inside the cylinder $[\mathtt{i}]$, all of which lie inside the same column; that is,

$$P(\mathtt{i}, \underline{\mathtt{j}}) = \bigcup_{\mathtt{k} \in \eta^{-1}(\underline{\mathtt{j}})} [\mathtt{ik}]. \tag{3.1}$$

We refer the reader to Figure I.3 for a depiction of the definition of a pseudo-cylinder.

Now given an infinite word $\gamma \in \Omega$, let $L_k(\gamma)$ be the minimal integer so that

$$\beta_{\gamma_1,1} \cdots \beta_{\gamma_{L_k(\gamma)},1} < \beta_{\gamma_1,2} \cdots \beta_{\gamma_k,2}.$$

In other words, $L_k(\gamma)$ is chosen so that the level $L_k(\gamma)$ rectangle has approximately the same width as the height of the level k rectangle. Write $\gamma\upharpoonright_{L_k(\gamma)} = \mathtt{ij}$ where $\mathtt{i} \in \mathcal{I}^k$. We then define the *approximate square* $Q_k(\gamma) \subset \Omega$ by

$$Q_k(\gamma) = P(\mathtt{i}, \eta(\mathtt{j})).$$

While different γ may define the same approximate square, the choice of $\mathtt{i}$ and $\eta(\mathtt{j})$ are unique. For fixed $\mathtt{i}$, let $\mathcal{U}(\mathtt{i}) \subset \eta(\mathcal{I}^*)$ denote the set of $\underline{\mathtt{j}}$ so that $P(\mathtt{i}, \underline{\mathtt{j}})$ is an approximate square. Of course, $Q_{k+1}(\gamma) \subset Q_k(\gamma)$ and moreover for any $\gamma, \gamma' \in \Omega$, either $Q_k(\gamma) = Q_k(\gamma')$ or $Q_k(\gamma) \cap Q_k(\gamma') = \varnothing$. In particular, $\mathcal{U}(\mathtt{i})$ is a complete section and the approximate squares $P(\mathtt{i}, \underline{\mathtt{j}})$ are disjoint in symbolic space for fixed $\mathtt{i}$.

We say that a pseudo-cylinder $P(\mathtt{i}, \underline{\mathtt{j}})$ is *wide* if $\underline{\mathtt{j}} \preccurlyeq \underline{\mathtt{k}}$ for some $\underline{\mathtt{k}} \in \mathcal{U}(\mathtt{i})$; in other words, $P(\mathtt{i}, \underline{\mathtt{j}})$ contains approximate squares of the form $P(\mathtt{i}, \underline{\mathtt{k}})$. Otherwise, we say that $P(\mathtt{i}, \underline{\mathtt{j}})$ is *tall*. In other words, one can think of the wide pseudo-cylinders as "interpolating" between the cylinder $P(\mathtt{i}, \varnothing) = [\mathtt{i}]$ and the approximate square $P(\mathtt{i}, \underline{\mathtt{j}}) = Q_n(\gamma)$.

Denote the tree of all approximate squares by

$$\mathcal{S}_k = \{Q_k(\gamma) : \gamma \in \Omega\} \qquad \text{and} \qquad \mathcal{S} = \bigcup_{k=0}^{\infty} \mathcal{S}_k.$$

As discussed above, every approximate square is uniquely associated with a pair $(\mathtt{i}, \underline{\mathtt{j}})$, so we may therefore define a metric induced by $\rho(Q) = \beta_{\mathtt{i},2}$, which makes the collection of approximate squares into a metric tree. In particular, we recall for $0 < r < 1$ that $\mathcal{S}(r)$ denotes the set of approximate squares with width approximately r; see (2.1) for the precise definition.

To conclude this section, we define the *metric tree of symbolic slices*. Suppose we fix a word $\gamma \in \Omega$. The word $\gamma = (i_n)_{n=1}^{\infty}$ defines for each $n \in \mathbb{N}$ a self-similar IFS $\Phi_n = \{S_{j,2} : j \in \eta^{-1}(\eta(i_n))\}$. This IFS is precisely the IFS corresponding to the column containing the index i_n. Note that there are only finitely many possible choices for the Φ_n, so the sequence $(\Phi_n)_{n=1}^{\infty}$ has as an attractor a non-autonomous self-similar set $K_{\eta(\gamma)}$ and corresponding metric tree $\Omega(\eta(\gamma))$, as defined in §2.1. This non-autonomous IFS has uniformly bounded contractions and satisfies the

OSC with respect to the open interval $(0,1)$. For notational simplicity, we denote the cylinder sets which compose this metric tree as

$$\mathcal{F}_{\eta(\gamma),n} = \{[j_1,\ldots,j_n] : (j_1,\ldots,j_n) \in \Phi_1 \times \cdots \times \Phi_n\} \quad \text{and} \quad \mathcal{F}_{\eta(\gamma)} = \bigcup_{n=0}^{\infty} \mathcal{F}_{\eta(\gamma),n}.$$

We call $K_{\eta(\gamma)}$ the *symbolic slice* associated with the word γ. If the projected IFS $\{S_{\underline{i},1}\}_{\underline{i}\in\eta(\mathcal{I})}$ satisfies the SSC, then if $x = \eta(\pi(\gamma))$,

$$\{x\} \times K_{\eta(\gamma)} = \eta^{-1}(x) \cap K$$

is precisely the vertical slice of K containing x. In general, $K_{\eta(\gamma)}$ is always contained inside a vertical slice of K. The symbolic fibre $K_{\eta(\gamma)}$ (and its associated Assouad dimension) was introduced and studied in [FR24, §1.2] in the more general setting of overlapping diagonal carpets.

3.5 Covering lemmas for Gatzouras–Lalley carpets

As a result of the local inhomogeneity of Gatzouras–Lalley carpets, obtaining sharp bounds on the size of various covers of Gatzouras–Lalley covers requires some care. In this section, we will prove a sequence of lemmas which, morally, provide optimal covers for a variety of symbolic objects: these covering arguments will be essential for the covering arguments in §6 and §11.

We first show that, as a result of the vertical alignment of their component cylinders, pseudo cylinders can essentially be covered by their projection. Recall that $\mathcal{S}$ denotes the set of all approximate squares. Then if $P(\mathtt{i},\underline{\mathtt{j}})$ is any wide pseudo-cylinder, we can write it as a union of the approximate squares in the family

$$\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) = \{Q \in \mathcal{S} : \; Q = P(\mathtt{i},\underline{\mathtt{k}}) \text{ for some } \underline{\mathtt{k}} \in \eta(\mathcal{I}^*) \text{ and } Q \subset P(\mathtt{i},\underline{\mathtt{j}})\}.$$

Since each $Q = P(\mathtt{i},\underline{\mathtt{k}})$ for some $\underline{\mathtt{k}}$, we have $Q \in S(\beta_{\mathtt{i},2})$ so that this family of approximate squares forms a section.

Lemma 3.3. *Let $P(\mathtt{i},\underline{\mathtt{j}})$ be a wide pseudo-cylinder. Then*

$$\#\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) \approx \left(\frac{\beta_{\mathtt{i}\underline{\mathtt{j}},1}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}.$$

Proof. First, enumerate $\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) = \{Q_1,\ldots,Q_m\}$, and for each $i = 1,\ldots,m$, there is a unique $\underline{\mathtt{k}}_i$ so that $Q_i = P(\mathtt{i},\underline{\mathtt{k}}_i)$. Moreover, $\{\underline{\mathtt{k}}_1,\ldots,\underline{\mathtt{k}}_m\}$ forms a section relative to $[\underline{\mathtt{j}}]$, so that writing $s = \dim_{\mathrm{B}} \eta(K)$ and recalling that $\eta(K)$ is the attractor of a self-similar IFS satisfying the open set condition,

$$\sum_{i=1}^{m} \beta_{\underline{\mathtt{k}}_i,1}^s = \beta_{\underline{\mathtt{j}},1}^s. \tag{3.2}$$

But $\beta_{\mathtt{i}\underline{\mathtt{k}}_i,1} \approx \beta_{\mathtt{i},2}$ since each Q_i is an approximate square, which gives the desired result. □

In the next result, we provide good covers for cylinder sets using approximate squares with diameter bounded above by the height of the corresponding rectangle. Heuristically, a cylinder set can first be decomposed into approximate squares using Lemma 3.3, and an "average" approximate square itself has box dimension the same as the box dimension of K. To make this notion precise, we simply reverse the order: we begin with a good cover for the box dimension of K, and take the image under some word $\mathtt{i}$. The image of each approximate square is a wide pseudo-cylinder, so we may apply Lemma 3.3 to complete the bound.

Lemma 3.4. *Suppose* $\mathtt{i} \in \mathcal{I}^*$ *and* $0 < r \leq \beta_{\mathtt{i},2}$. *Then*

$$\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{i}]\} \approx \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\dim_{\mathrm{B}} K} \cdot \left(\frac{\beta_{\mathtt{i},1}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}$$

Proof. Fix $\mathtt{i} \in \mathcal{I}^*$ and $0 < r \leq \beta_{\mathtt{i},2}$. Write $\delta = r/\beta_{\mathtt{i},2}$, so by inspecting the proofs of [LG92, Lemmas 2.1, 2.2, & 2.3], we see that

$$\#\mathcal{S}(\delta) \approx (1/\delta)^{\dim_{\mathrm{B}} K}.$$

Enumerate $\mathcal{S}(\delta) = \{Q_1, \ldots, Q_m\}$ and for each $i = 1, \ldots, m$, we may write $Q_i = P(\mathtt{j}_i, \underline{\mathtt{k}}_i)$ for some $\mathtt{j}_i \in \mathcal{I}^*$ and $\underline{\mathtt{k}}_i \in \eta(\mathcal{I}^*)$. Then for each $i = 1, \ldots, m$,

$$\mathcal{Q}(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i) \subset \mathcal{S}(r) \qquad \text{and} \qquad [\mathtt{i}] = \bigcup_{i=1}^{m} \bigcup_{Q \in \mathcal{Q}(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i)} Q.$$

Thus by Lemma 3.3 applied to each pseudo-cylinder $P(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i)$, since Q_i is an approximate square and $\beta_{\mathtt{j}_i\underline{\mathtt{k}}_i,1} \approx \beta_{\mathtt{j}_i,2}$,

$$\begin{aligned}
\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{i}]\} &= \sum_{i=1}^{m} \#\mathcal{Q}(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i) \\
&\approx \sum_{i=1}^{m} \left(\frac{\beta_{\mathtt{i}\mathtt{j}_i\underline{\mathtt{k}}_i,1}}{\beta_{\mathtt{i}\mathtt{j}_i,2}}\right)^{\dim_{\mathrm{B}} \eta(K)} \\
&\approx \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\dim_{\mathrm{B}} K} \cdot \left(\frac{\beta_{\mathtt{i},1}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}
\end{aligned}$$

as claimed. □

To conclude our collection of preliminary lemmas, we use the Assouad dimension of the symbolic fibre $K_{\eta(\gamma)}$ to control the size of "column sections" of approximate squares. We note that the word $\mathtt{i}$ appears in the hypothesis but not the conclusion: this is simply to clarify the application of this lemma when it is used in Proposition 11.7.

Lemma 3.5. *Let* $\varepsilon > 0$ *and* $\gamma \in \Omega$ *be arbitrary. Suppose* $k \in \mathbb{N}$ *and* $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$. *Let* $\mathcal{B}$ *be any section of* $\mathcal{I}^*$ *such that* $\mathcal{B} \preccurlyeq \eta^{-1}(\underline{\mathtt{j}})$. *Then*

$$\sum_{\mathtt{k} \in \mathcal{B}} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)} + \varepsilon} \lesssim_{\varepsilon,\gamma} 1.$$

Proof. The assumption on the section $\mathcal{B}$ precisely means that $\{\mathtt{ik} : \mathtt{k} \in \mathcal{B}\}$ is a section relative to $\mathtt{i}$ in $\mathcal{F}_{\eta(\gamma)}$. Then by Proposition 2.9 applied to the metric space $\Omega(\eta(\gamma))$ (recalling that $\dim_{\mathrm{A}} \Omega(\eta(\gamma)) = \dim_{\mathrm{A}} K_{\eta(\gamma)}$ from Corollary 2.3), since $\mathcal{B}$ is a section,

$$\sum_{\mathtt{k} \in \mathcal{B}} \left(\frac{\beta_{\mathtt{ik},2}}{\beta_{\mathtt{i},2}} \right)^{\dim_{\mathrm{A}} K_{\eta(\gamma)} + \varepsilon} \lesssim_{\varepsilon,\gamma} 1.$$

Cancelling the $\beta_{\mathtt{i},2}$ gives the desired result. $\square$

II. The Assouad spectrum

In this chapter, we study the first of our two main variants of the Assouad dimension: the Assouad spectrum. We recall that the Assouad spectrum is formally defined in §1.4.

The Assouad spectrum has been explicitly computed for a wide variety of well-studied dynamical sets with distinct box and Assouad dimensions, such as some overlapping self-similar sets and self-affine sets [FR24; FY18a], Bedford–McMullen carpets [FY18a], certain Kleinian limit sets [FS23], parabolic Julia sets [FS22+], elliptical polynomial spirals [BFF22], and random sets such as Mandelbrot percolation [FY18a; Tro20]. In all of these examples, the Assouad spectrum has a very particular form: it is piecewise convex, with pieces of the form $\theta \mapsto a + b \cdot \frac{\theta}{1-\theta}$ for some $a, b \in \mathbb{R}$, and with points of non-differentiability corresponding to certain "geometrically meaningful" values. The Assouad spectrum has also been computed for certain infinitely generated self-conformal sets, where more complicated forms can appear based on the fine structure of the set of fixed points (which may be prescribed in a non-dynamical way) [BF24].

On the other hand, there are fewer results exhibiting more general behaviour. Perhaps the main construction result of note can be found in [FHH+19], which provides some interesting examples which exhibit, for instance, non-differentiability at countably many points. In this chapter, we will consider the problem of general behaviour of Assouad perspective from two perspectives.

First, we focus on the general question of classification: what constraints on a function $\varphi\colon (0,1) \to [0,d]$ guarantee that there is a set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0,1)$? In §4, we will give a precise answer to this question. Using this classification, in §5, we provide some examples exhibiting exceptional behaviour, namely non-monotonicity on every open set and Hölder failure at 1.

A downside of the above results is that the sets constructed in the classification are very far from being general dynamically invariant sets. In the second part of this chapter, we focus on the problem of computing the Assouad spectrum for a particular class of self-affine sets, namely the *Gatzouras–Lalley carpets* first introduced in [LG92] and defined earlier in §3. Roughly speaking, we recall that the difference between Bedford–McMullen and Gatzouras–Lalley carpets is that all rectangles for a Bedford–McMullen carpet have the same widths and heights, while Gatzouras–Lalley carpets allow inhomogeneity across the widths of columns and also of the heights of rectangles within columns. However, we emphasize that the difference between Bedford–McMullen carpets and Gatzouras–Lalley carpets is more than just technical. For instance, it is known that Gatzouras–Lalley carpets need not have a unique measure of maximal dimension [BF11]. More famously, the simplest known example of a self-affine set with no ergodic measure of maximal dimension is defined using a 3-dimensional analogue of the Gatzouras–Lalley construction [DS17]. These features are not exhibited by Bedford–McMullen sponges in any dimension.

Our second main contribution is to establish an explicit formula for the Assouad spectrum of a Gatzouras–Lalley carpet as the concave conjugate of the minimum of a finite family of elementary functions combined with a simple parameter change—see Theorem 7.1 for the precise statement. We observe novel and unexpected behaviour which has not been previously witnessed in any dynamically-invariant examples. More precisely, we obtain the following direct qualitative consequences of our explicit formula:

1. The Assouad spectrum can be a non-trivial differentiable function of θ. In fact, we give a simple characterization of differentiability, and observe that non-differentiability is generic (both topologically and measure theoretically) within the parameter space of Gatzouras–Lalley carpets—see Corollary 8.5.
2. The Assouad spectrum can have a non-trivial interval on which it is strictly concave. Again, this property holds generically—see Corollary 8.7
3. The Assouad spectrum is increasing and piecewise analytic, but with potentially arbitrarily many phase transitions and phase transitions of arbitrarily high odd order—see Remark 8.6.

These properties are direct consequences of the explicit formula, and their proofs can be found in §8. The proof of the explicit formula constitutes the majority of the work.

In contrast to the Bedford–McMullen case (for which the derivation of the Assouad spectrum is, generally speaking, straightforward), the Gatzouras–Lalley case has substantially more technical complications as a result of the inhomogeneity between columns and within each column. Our proof of the general formula uses recent techniques developed in the context of multifractal analysis, which we highlight here. First, in §6 we establish a general *variational formula* for the Assouad spectrum as a certain constrained maximization problem involving information-theoretic quantities evaluated at Bernoulli measures. A key tool here is the *method of types* from large deviations theory, which was used in [Kol23] to compute the L^q-spectrum of self-affine sponges and in [BK21+] to calculate the intermediate dimensions of Bedford–McMullen carpets. The explicit covering arguments build on and refine the fine covering strategies for self-affine carpets used in [Fra14; KR23+; Mac11]. Secondly, in §7 we solve this variational formula to obtain our explicit formula. The main complexity here is that the variational formula is a non-smooth and non-convex optimization problem. Our key technique here is the *geometry of Lagrange duality*, which was used in [Rut23+] in order to elucidate the concave conjugate relationship apparent in the multifractal formalism.

4 ATTAINABLE FORMS OF ASSOUAD SPECTRA

In this section, we establish a complete classification of the possible forms of Assouad spectra.

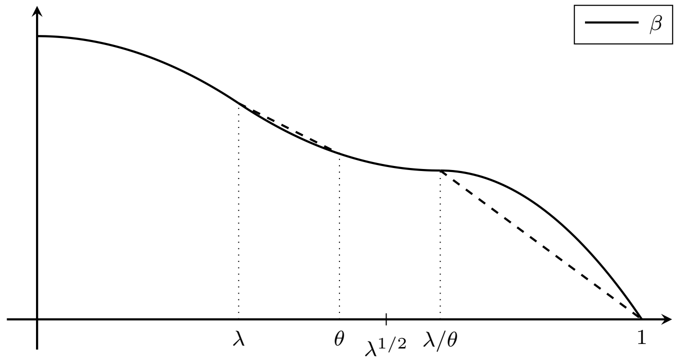


FIGURE II.1: A plot of $\beta(\theta) = (1-\theta)\varphi(\theta)$ where $\varphi \in \mathcal{A}_d$, and the lines with slopes corresponding to (4.2).

4.1 THE FAMILY OF FUNCTIONS $\mathcal{A}_d$

We first define the family of functions $\mathcal{A}_d$, which we will prove are the possible forms of the maps $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ for non-empty bounded sets $F \subset \mathbb{R}^d$.

Definition 4.1. Let $\mathcal{A}_d$ denote the set of functions $\varphi\colon (0,1) \to [0,d]$ where for any $0 < \lambda < \theta < 1$,

$$0 \le (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \le (\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big). \tag{4.1}$$

In Proposition 4.6, we will prove that functions in $\mathcal{A}_d$ are uniformly continuous. Thus, we will embed $\mathcal{A}_d$ in $C([0,1])$ by defining $\varphi(0) = \lim_{\theta\to 0}\varphi(\theta)$ and $\varphi(1) = \lim_{\theta\to 1}\varphi(\theta)$. We will use this notation once we prove uniform continuity.

Given $\varphi \in \mathcal{A}_d$, define $\beta(\theta) = (1-\theta)\varphi(\theta)$. In (4.1), the first inequality implies that $\beta(\theta)$ is decreasing, and the second states that for all $0 < \lambda < \theta < 1$,

$$\frac{\beta(\lambda)-\beta(\theta)}{\theta-\lambda} \le \frac{\beta\big(\frac{\lambda}{\theta}\big)}{1-\frac{\lambda}{\theta}}. \tag{4.2}$$

The left hand side is the negative of the slope of the line passing through $(\lambda, \beta(\lambda))$ and $(\theta, \beta(\theta))$, and the right hand side is the negative of the slope of the line passing through $(\lambda/\theta, \beta(\lambda/\theta))$ and $(1,0)$. The secants in this constraint for a function β are depicted in Figure II.1.

We can now state the main classification.

Theorem 4.2. *Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$ be a function. Then there exists a non-empty bounded $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0,1)$ if and only if $\varphi \in \mathcal{A}_d$.*

The forward implication is well-known (see, for example, [Fra20, Theorem 3.3.1]); the reverse implication is proven in Theorem 4.15.

We can interpret the first inequality in (4.1) as a growth rate constraint, and the second inequality as an oscillation constraint. In fact, the second inequality is always satisfied when φ is increasing (the short argument is given in Lemma 4.18), which yields the following corollary.

Corollary 4.3. *Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$ be an increasing function. Then there exists a non-empty bounded $F \subset \mathbb{R}^d$ with $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ if and only if $\theta \mapsto (1-\theta)\varphi(\theta)$ is decreasing.*

Finally, we obtain a classification of the upper Assouad spectrum. Let

$$M_d = \{(\kappa, c) : 0 \leq \kappa \leq d, 0 < c < 1\}$$

and for $\boldsymbol{i} = (\kappa, c) \in M_d$, we may define

$$f_{\boldsymbol{i}}(\theta) = \begin{cases} \kappa(1-c) & : \theta \in [0,c] \\ \kappa(1-\theta) & : \theta \in [c,1] \end{cases}. \tag{4.3}$$

Then, let

$$\mathcal{M}_d := \Big\{\theta \mapsto \frac{f_{\boldsymbol{i}}(\theta)}{1-\theta} : \boldsymbol{i} \in M_d\Big\}. \tag{4.4}$$

Combining this with Corollary 4.19 gives a full characterization of the upper Assouad spectrum (the details are given in §4.5).

Corollary 4.4. *Let $d \in \mathbb{N}$ and let $\varphi\colon (0,1) \to [0,d]$ be an arbitrary function. Then the following are equivalent:*

(a) There exists a non-empty bounded $F \subset \mathbb{R}^d$ such that $\overline{\dim}_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0,1)$.
(b) $\varphi(\theta)$ is increasing and $\theta \mapsto (1-\theta)\varphi(\theta)$ is decreasing.
(c) φ is the supremum of functions $f \in \mathcal{M}_d$.

Beyond giving a full classification, Theorem 4.2 also clarifies many of the properties of the Assouad spectrum: certain observations which might *a priori* depend on explicit properties of the Assouad spectrum in fact only require the bound (4.1). For instance, the observation that if $\overline{\dim}_{\mathrm{B}} F = 0$ then $\dim_{\mathrm{A}}^{\theta} F = 0$ only requires the fact that $\lim_{\theta \to 0} \dim_{\mathrm{A}}^{\theta} F = \overline{\dim}_{\mathrm{B}} F$ along with the general bound (see Proposition 4.6).

We note that the 2-parameter family of functions $\mathcal{M}_d$ corresponds to the Assouad spectra of sets with upper box dimension $\kappa(1-c)$, quasi-Assouad dimension κ, and Assouad spectrum as large as possible. In [RS23], such sets are called *quasi-Assouad regular*.

4.2 BASIC PROPERTIES OF THE FAMILY $\mathcal{A}_d$

In this section, we collect various properties of the family $\mathcal{A}_d$. First, we observe the following useful lemma which was essentially proven in [FY18b, Remark 3.8]. Here, we obtain it as a direct consequence of (4.1). Heuristically, this lemma states that the function $\varphi(\theta)$ is "almost increasing", up to some possible local oscillations.

Lemma 4.5. *Let $\varphi \in \mathcal{A}_d$. Given $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$,*

$$\varphi(\theta_1) \leq \max\left\{\varphi\Big(\frac{\theta_1}{\theta_2}\Big), \varphi\Big(\frac{\theta_2}{\theta_3}\Big), \ldots, \varphi\Big(\frac{\theta_{n-1}}{\theta_n}\Big), \varphi(\theta_n)\right\}.$$

In particular, for any $n \in \mathbb{N}$ and $\theta \in (0,1)$, $\varphi(\theta) \leq \varphi(\theta^{1/n})$.

Proof. Let $0 < \theta_1 < \theta_2 < \cdots < \theta_n < 1$. Applying (4.1) to each pair θ_i, θ_{i+1},

$$(1-\theta_1)\varphi(\theta_1) \leq (1-\theta_n)\varphi(\theta_n) + \sum_{k=2}^{n}(\theta_k - \theta_{k-1})\varphi\Big(\frac{\theta_{k-1}}{\theta_k}\Big)$$

from which the result follows. Taking $\theta_i = \theta^{\frac{n-i+1}{n}}$ for each $i = 1, \ldots, n$, observe that $\theta_{k-1}/\theta_k = \theta^{1/n}$ and $\theta_n = \theta^{1/n}$ so that $\varphi(\theta) \leq \varphi(\theta^{1/n})$. □

We now have the following essential properties of $\mathcal{A}_d$. All of these properties have been previously observed for the Assouad spectrum, but the main point here is that these properties only depend on the family $\mathcal{A}_d$ and not on other properties of the Assouad spectrum. Some of these properties will be used in the proof of Theorem 4.15, so we cannot formally depend on the corresponding results for the Assouad spectrum. We draw on ideas from [FHH+19; FY18b].

Proposition 4.6. *Let $\varphi \in \mathcal{A}_d$ be arbitrary. Then the following properties hold:*

(i) The limits $\varphi(0) := \lim_{\theta\to 0}\varphi(\theta)$ and $\varphi(1) := \lim_{\theta\to 1}\varphi(\theta)$ exist.
(ii) Each $\varphi \in \mathcal{A}_d$ is uniformly continuous.
(iii) $\varphi(0) = \inf_{\theta\in(0,1)}\varphi(\theta)$ and $\varphi(1) = \sup_{\theta\in(0,1)}\varphi(\theta)$.
(iv) For any $\theta_0 \in (0,1)$, if $\varphi(\theta_0) = \varphi(1)$, then $\varphi(\theta_0) = \varphi(\theta)$ for all $\theta_0 < \theta < 1$.
(v) If $\varphi(0) = 0$, then $\varphi(\theta) = 0$ for all θ.

Proof. First, we show that $\varphi(\theta)$ is continuous on $(0,1)$. For $0 < \theta_1 < \theta_2 < 1$ we have $\theta_1 < \theta_1/\theta_2 < 1$, so applying (4.1) we obtain

$$(1-\theta_2)\varphi(\theta_2) \leq (1-\theta_1)\varphi(\theta_1) \leq \frac{\theta_1}{\theta_2}(1-\theta_2)\varphi(\theta_2) + \Big(1 - \frac{\theta_1}{\theta_2}\Big)\varphi\Big(\frac{\theta_1}{\theta_2}\Big). \tag{4.5}$$

This implies that

$$\varphi(\theta_1) - \varphi(\theta_2) \leq \frac{\varphi(\theta_1/\theta_2)}{\theta_2(1-\theta_1)}(\theta_2 - \theta_1).$$

Similarly, from the first inequality of (4.5),

$$\varphi(\theta_2) - \varphi(\theta_1) \leq \left(\frac{1-\theta_1}{1-\theta_2} - 1\right)\varphi(\theta) = \frac{\theta_2 - \theta_1}{1-\theta_1}\varphi(\theta_1).$$

Since $\varphi(\theta_1/\theta_2) \leq d$ and $\varphi(\theta_1) \leq d$, it follows that $\varphi(\theta)$ is Lipschitz on any closed subinterval of $(0,1)$, and therefore continuous on $(0,1)$.

Now consider (i). Observe that $(1-\theta)\varphi(\theta)$ is a bounded decreasing function of θ, so $\lim_{\theta\to 0}(1-\theta)\varphi(\theta)$ exists so $\lim_{\theta\to 0}\varphi(\theta)$ exists as well. To see that $\lim_{\theta\to 1}\varphi(\theta)$ exists, we use the proof from [FHH+19, Section 3.2]. Set $L = \limsup_{\theta\to 1}\varphi(\theta)$ and let $\varepsilon > 0$. Since $\varphi(\theta)$ is continuous, we can find $0 < u < v < 1$ such that $\varphi(\theta) > L - \varepsilon$ for all $\theta \in [u, v]$. Thus by Lemma 4.5, with

$$X := \bigcup_{n=1}^{\infty}[u^{1/n}, v^{1/n}]$$

we have $\varphi(\theta) > L - \varepsilon$ for all $\theta \in X$. But $v^{1/n} \geq u^{1/(n+1)}$ for all $n \geq n_0$ with $\frac{n_0}{n_0+1} \geq \frac{\log v}{\log u}$, so in fact $(u^{1/n_0}, 1) \subset X$. Thus $\lim_{\theta\to 1}\varphi(\theta)$ exists as well. In particular, combining the existence of endpoint limits with continuity of φ on $(0,1)$, (ii) also follows immediately.

To see (iii), if $\theta_1 \in (0,1)$, then $\theta_n = \theta_1^{1/n}$ is a sequence converging monotonically to 1 with $\varphi(\theta_n) \geq \varphi(\theta_1)$ by Lemma 4.5. Thus $\varphi(1) \geq \varphi(\theta_1)$. Similarly $\varphi(\theta_1^n) \leq \varphi(\theta_1)$ for any $n \in \mathbb{N}$, and $\lim_{n\to\infty} \theta_1^n = 0$. But θ_1 was arbitrary, giving (iii).

Now we see (iv). Suppose $\varphi(1) = \varphi(\theta_1)$ for some $0 < \theta_1 < 1$. By (4.5),

$$(1-\theta_1)\varphi(1) - (1-\theta_2)\varphi(\theta_2) \leq (\theta_2 - \theta_1)\varphi(\theta_1/\theta_2) \leq (\theta_2-\theta_1)\varphi(1)$$

since $\varphi(\theta_1/\theta_2) \leq \varphi(1)$ by (iii). This implies that $\varphi(1) \leq \varphi(\theta_2)$, so (iv) follows.

To see (v), if $\varphi(0) = 0$, then $\lim_{\theta\to 0}(1-\theta)\varphi(\theta) = 0$. But $(1-\theta)\varphi(\theta)$ is a decreasing function of θ, so $(1-\theta)\varphi(\theta) = 0$ for all $\theta \in (0,1)$, i.e. $\varphi(\theta) = 0$ for all $\theta \in (0,1)$. □

4.3 BOUNDING THE ASSOUAD SPECTRUM

We recall the following general bounds, which are given in [FY18b, Proposition 3.4] and [Fra20, Theorem 3.3.1]. We include the details here for completeness.

Proposition 4.7. *For any non-empty bounded $F \subset \mathbb{R}^d$, the function $\varphi(\theta) = \dim_{\mathrm{A}}^\theta F$ is in $\mathcal{A}_d$.*

Proof. Let $0 < \theta_1 < \theta_2 < 1$ and let $\varepsilon > 0$ be arbitrary. For $\delta > 0$ sufficiently small, since $B(x, \delta^{\theta_2}) \subset B(x, \delta^{\theta_1})$ for all $x \in F$,

$$\begin{aligned}\sup_{x\in F} N_\delta(F \cap B(x,\delta^{\theta_1})) &\geq \sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_2}))\\ &\geq \left(\frac{\delta^{\theta_2}}{\delta}\right)^{(\varphi(\theta_2)-\varepsilon)}\\ &= \left(\delta^{\theta_1-1}\right)^{(\varphi(\theta_2)-\varepsilon)\left(\frac{1-\theta_2}{1-\theta_1}\right)}\end{aligned}$$

which proves that $(1-\theta_1)\varphi(\theta_1) \geq (1-\theta_2)(\varphi(\theta_2)-\varepsilon)$. This gives the lower inequality in (4.1).

To obtain the upper inequality, by covering $B(x,\delta^{\theta_1})$ by balls with radius δ^{θ_2},

$$\sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_1})) \leq \sup_{x\in F} N_{\delta^{\theta_2}}(F\cap B(x,\delta^{\theta_1}))\sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_2})).$$

This implies for all $\delta > 0$ sufficiently small

$$\begin{aligned}\sup_{x\in F} N_\delta(F\cap B(x,\delta^{\theta_1})) &\leq \left(\frac{\delta^{\theta_1}}{\delta^{\theta_2}}\right)^{\varphi(\theta_1/\theta_2)+\varepsilon}\left(\frac{\delta^{\theta_2}}{\delta}\right)^{\varphi(\theta_2)+\varepsilon}\\ &= \left(\delta^{\theta_1-1}\right)^{(\varphi(\theta_1/\theta_2)+\varepsilon)\left(\frac{\theta_2-\theta_1}{1-\theta_1}\right)+(\varphi(\theta_2)+\varepsilon)\left(\frac{1-\theta_2}{1-\theta_1}\right)}\end{aligned}$$

which implies that

$$(1-\theta_1)\varphi(\theta_1) \leq (\theta_2-\theta_1)(\varphi(\theta_1/\theta_2)+\varepsilon) + (1-\theta_2)(\varphi(\theta_2)+\varepsilon)$$

as required. □

4.4 CONSTRUCTING SETS WITH PRESCRIBED SPECTRA

Now for any $\varphi \in \mathcal{A}_d$ we construct a homogeneous Moran set C such that $\dim_{\mathrm{A}}^{\theta} C = \varphi(\theta)$ for all $\theta \in (0,1)$. The techniques here are based on ideas first introduced by the author and Banaji used to solve an analogous question for the *intermediate dimensions* [FFK20]. We refer the reader to the paper [BR22] for more details on this general technique.

We first recall the notion of homogeneous Moran sets from [BR22], which is a special class of the general non-autonomous self-similar construction described in §2. The construction is analogous to the usual 2^d-corner Cantor set, except that the subdivision ratios need not be the same at each level.

Let $\mathcal{I} = \{0,1\}^d$, set $\mathcal{I}^* = \bigcup_{n=0}^{\infty} \mathcal{I}^n$, and denote the word of length 0 by $\varnothing$. Let $\boldsymbol{r} = (r_n)_{n=1}^{\infty} \subset (0,1/2]$ and for each n and $\boldsymbol{i} \in \mathcal{I}$, define $S_{\boldsymbol{i}}^n \colon \mathbb{R}^d \to \mathbb{R}^d$ by

$$S_{\boldsymbol{i}}^n(x) := r_n x + b_{\boldsymbol{i}}^n$$

where $b_{\boldsymbol{i}}^n \in \mathbb{R}^d$ has j^{th} coordinate for $j = 1, \dots, d$ given by

$$(b_{\boldsymbol{i}}^n)^{(j)} = \begin{cases} 0 & : \boldsymbol{i}^{(j)} = 0, \\ 1 - r_n & : \boldsymbol{i}^{(j)} = 1. \end{cases}$$

Given $\mathtt{i} = (\boldsymbol{i}_1, \dots, \boldsymbol{i}_n) \in \mathcal{I}^n$, write $S_{\mathtt{i}} = S_{\boldsymbol{i}_1}^1 \circ \dots \circ S_{\boldsymbol{i}_n}^n$. Then set

$$C = C(\boldsymbol{r}) := \bigcap_{n=1}^{\infty} \bigcup_{\mathtt{i} \in \mathcal{I}^n} S_{\mathtt{i}}([0,1]^d).$$

We refer to the set C as a *homogeneous Moran set*. Alternatively, C is precisely the non-autonomous self-similar set associated with the sequence of IFSs $\{\Phi_n\}_{n=1}^{\infty}$ where $\Phi_n = \{S_{\boldsymbol{i}}^n\}_{\boldsymbol{i} \in \mathcal{I}}$.

Given $\delta > 0$, let $k = k(\delta)$ be such that $r_1 \cdots r_k \leq \delta < r_1 \cdots r_{k-1}$. We then define

$$s(\delta) = s_{\boldsymbol{r}}(\delta) := \frac{k(\delta) \cdot d \log 2}{\log(1/\delta)}.$$

Heuristically, $s(\delta)$ is the best candidate for the box dimension of C at scale δ.

We now define a family of functions which we may interpret as a reparametrization of the space of sequences $(0,1/2]^{\mathbb{N}}$. That this forms an alternative representation is described precisely in [BR22, Lemma 3.4].

Definition 4.8. Let $0 \leq \lambda \leq \alpha \leq d$ and let $\mathcal{G}(\lambda, \alpha)$ denote the set of functions $g \colon \mathbb{R} \to [\lambda, \alpha]$ satisfying

$$\lambda - (\lambda - g(y)) \exp(-t) \leq g(y+t) \leq \alpha - (\alpha - g(y)) \exp(-t)$$

for any $y \in \mathbb{R}$ and $t > 0$.

This family is very similar to the family defined in [BR22, Definition 3.1]; see in particular [BR22, Lemma 3.2].

To construct sets with prescribed Assouad spectrum, we will use [BR22, Lemma 3.4]. However, the published version of [BR22, Lemma 3.4] contains some minor inconsistencies. To clarify these inconsistencies, and moreover to be entirely self-contained, we present a corrected version here.

Given a function $g \in \mathcal{G}(\lambda, \alpha)$ and $w \in \mathbb{R}$, we define the *offset* $\kappa_w(g) \in \mathcal{G}(\lambda, \alpha)$ by

$$\kappa_w(g)(x) = \begin{cases} g(x-w) & : x \geq w, \\ g(0) & : x \leq w. \end{cases}$$

We also say that a function $g \in \mathcal{G}(\lambda, \alpha)$ is *rapidly decreasing* if there is a $y \in \mathbb{R}$ and a constant $C > 0$ so that for all $x \geq y$,

$$g(x) \leq g(y)\exp(y-x) + C\exp(-x). \tag{4.6}$$

Note that if g is rapidly decreasing, then $\lim_{x\to\infty} g(x) = 0$. Moreover, for all $w \in \mathbb{R}$, g is not rapidly decreasing if and only if $\kappa_w(g)$ is not rapidly decreasing.

Lemma 4.9. *Let $0 \leq \lambda < \alpha \leq d$ and let $\tilde{g} \in \mathcal{G}(\lambda, \alpha)$. Suppose $\tilde{g}$ is not rapidly decreasing. Then there is a constant $w_0 \in \mathbb{R}$ depending only on $\tilde{g}(0)$ and d such that for all $w \geq w_0$, there exists a sequence $\boldsymbol{r} := (r_j)_{j=1}^\infty \subset (0, 1/2]$ so that $g := \kappa_w(\tilde{g})$ satisfies*

$$|s_{\boldsymbol{r}}(\exp(-\exp(x))) - g(x)| \leq d\log(2)\cdot\exp(-x) \tag{4.7}$$

for all $x \geq w_0$.

Proof. Noting that $\tilde{g}(0) \in (0, d)$, choose r_1 such that $\frac{2d\log(2)}{\log(1/r_1)} = \tilde{g}(0)$. Then let $w_0 = \log\log(1/r_1)$, let $w \geq w_0$ be arbitrary, and let $g = \kappa_w(\tilde{g})$. Since $\tilde{g}$ is not rapidly decreasing, g is also not rapidly decreasing so by (4.6) for every $y \in \mathbb{R}$ there is a minimal $\psi(y) > y$ so that

$$g(y)\exp(y - \psi(y)) = g(\psi(y)) - d\log(2)\cdot\exp(-\psi(y)).$$

Now set $x_1 = w_0$ and, inductively, set $x_{k+1} = \psi(x_k)$ for each $k \in \mathbb{N}$. Let $\rho_k = \exp(-\exp(x_k))$ denote the corresponding scales (note that $\rho_1 = r_1$), and set $r_k := \rho_k/\rho_{k-1}$ for $k \geq 2$. Observe that $r_k \in (0,1)$ for all k. Thus for $0 < \delta \leq r_1$, if k is such that $\rho_k < \delta \leq \rho_{k-1}$, we set

$$\overline{s}(\delta) = \frac{kd\log 2}{\log(1/\delta)}.$$

We will prove by induction that for each $k \in \mathbb{N}$ we have $r_k \in (0, 1/2]$, $\overline{s}(\rho_k) = g(x_k)$, and

$$g(x) - d\log(2)\exp(-x) \leq \overline{s}(\exp(-\exp(x))) \leq g(x) \tag{4.8}$$

holds for all $x \in [x_1, x_k]$. From this, the result follows.

We first note that, by construction, $r_1 \in (0, 1/2]$ and $\overline{s}(\rho_1) = g(x_1) = \tilde{g}(0)$. In general, suppose the hypothesis holds for $k \in \mathbb{N}$. By definition of ψ and the fact that $g(x_k) = \overline{s}(\rho_k)$,

$$\begin{aligned} g(x_{k+1}) &= \overline{s}(\rho_k)\exp(-x_{k+1} + x_k) + d\log(2)\exp(-x_{k+1}) \\ &= \frac{d(k+1)\log 2}{\exp(x_k)}\cdot\exp(-x_{k+1})\exp(x_k) + d\log(2)\exp(-x_{k+1}) \end{aligned}$$

$$= \frac{d(k+2)\log 2}{\exp(x_{k+1})} = \overline{s}(\rho_{k+1}).$$

Moreover, $g(x) \geq g(x_k)\exp(-x+x_k)$ for all $x \geq x_k$ so that (4.8) follows for $x \in [x_k, x_{k+1}]$ by the minimality of x_{k+1} in the definition of ψ. Finally, $g(x_{k+1}) \leq d - (d - g(x_k))\exp(-x_{k+1} + x_k)$. Substituting, this implies that

$$\frac{d(k+2)\log 2}{\log(1/\rho_{k+1})} \leq d - \left(d - \frac{d(k+1)\log 2}{\log(1/\rho_k)}\right) \cdot \frac{\log(1/\rho_k)}{\log(1/\rho_{k+1})}$$

which after simplification gives that $\rho_{k+1} \leq \rho_k/2$, i.e. $r_{k+1} \leq 1/2$. □

Remark 4.10. If instead g is rapidly decreasing, then the function g is eventually bounded above by any function $s_{\boldsymbol{r}}$ for a sequence $\boldsymbol{r} \subset (0, 1/2]$.

Remark 4.11. The bound (4.8) is optimal since $s(\delta)$ has discontinuities of size $\frac{d\log 2}{\log(1/\delta)}$.

Next, we show that triviality of a function $g \in \mathcal{G}(0, d)$ also implies triviality of a certain Assouad spectrum-type limit.

Lemma 4.12. *Let $d \in \mathbb{N}$ and $g \in \mathcal{G}(0, d)$. If $\lim_{x\to\infty} g(x) = 0$, then for all $\theta \in (0, 1)$,*

$$\lim_{x\to\infty} \frac{g\left(x + \log\frac{1}{\theta}\right) - \theta g(x)}{1-\theta} = 0.$$

Proof. We prove the contrapositive. Suppose there is a $\theta > 0$, $\delta > 0$, and sequence $(x_n)_{n=1}^{\infty}$ diverging to infinity such that for all $n \in \mathbb{N}$,

$$\frac{g\left(x_n + \log\frac{1}{\theta}\right) - \theta g(x_n)}{1-\theta} \geq \delta.$$

We may assume that $x_n \geq w$ for all $n \in \mathbb{N}$. Rearranging,

$$g\left(x_n + \log\frac{1}{\theta}\right) \geq \delta(1-\theta) + \theta g(x_n) \geq \delta(1-\theta) > 0.$$

In other words,

$$\limsup_{x\to\infty} g(x) \geq \delta(1-\theta) > 0,$$

so the claim follows. □

Using Lemma 4.9, we establish the following general result which allows us to prescribe Assouad spectra for homogeneous Moran sets.

Proposition 4.13. *Let $d \in \mathbb{N}$ and $g \in \mathcal{G}(0, d)$. Then there exists a homogeneous Moran set C such that*

$$\dim_{\mathrm{A}}^{\theta} C = \limsup_{x\to\infty} \frac{g\left(x + \log\frac{1}{\theta}\right) - \theta g(x)}{1-\theta}. \tag{4.9}$$

Proof. If g is rapidly decreasing, then $\lim_{x\to\infty} g(x) = 0$, so Lemma 4.12 implies that

$$\limsup_{x\to\infty} \frac{g\Big(x + \log\frac{1}{\theta}\Big) - \theta g(x)}{1-\theta} = 0$$

for all $\theta \in (0,1)$. Thus we can define the Moran set $C(\boldsymbol{r})$ where $\boldsymbol{r}$ is a sequence converging monotonically to 0.

Otherwise, performing an appropriate translation of g which does not change (4.9), Lemma 4.9 provides a sequence $\boldsymbol{r} \subset (0, 1/2]$ satisfying (4.7). To obtain (4.9), first recall from (1.3) that

$$\dim_{\mathrm{A}}^{\theta} C = \limsup_{\delta\to 0} \sup_{x\in C} \frac{\log N_{\delta^{1/\theta}}(C\cap B(x,\delta))}{(1-1/\theta)\log\delta}.$$

Observe that there is some constant $M > 0$ such that $B(x,\delta)$ intersects at most M cylinders in level $k(\delta)$. In particular, $C \cap B(x,\delta)$ can be covered by $M \cdot 2^{d(k(\delta^{1/\theta})-k(\delta))}$ balls of radius $\delta^{1/\theta}$. On the other hand, $C\cap B(x,\delta)$ contains an interval in level $k(\delta)$, and therefore contains a δ-separated subset of size $2^{d(k(\delta^{1/\theta})-1-k(\delta))}$. Thus there is a constant $M' > 0$ so that

$$M' \cdot 2^{d(k(\delta^{1/\theta})-k(\delta))} \leq \sup_{x\in C} N_{\delta^{1/\theta}}(C\cap B(x,\delta)) \leq M\cdot 2^{d(k(\delta^{1/\theta})-k(\delta))}$$

and therefore

$$\begin{aligned}\limsup_{\delta\to 0}\sup_{x\in C}\frac{\log N_{\delta^{1/\theta}}(C\cap B(x,\delta))}{(1-1/\theta)\log\delta} &= \limsup_{\delta\to 0}\frac{\theta(k(\delta^{1/\theta})-k(\delta))\cdot d\log 2}{(1-\theta)\cdot(-\log\delta)}\\ &= \limsup_{\delta\to 0}\frac{s(\delta^{1/\theta})-\theta\cdot s(\delta)}{1-\theta}.\end{aligned}$$

Taking $\delta > 0$ small and applying (4.7) yields the desired formula. □

Definition 4.14. Given a sequence of continuous functions $(f_k)_{k=1}^{\infty}$ each defined on some interval $[0, a_k]$, the *concatenation* of $(f_k)_{k=1}^{\infty}$ is the function

$$f\colon (-\infty, \sum_{k=1}^{\infty} a_k) \to \mathbb{R}$$

given as follows: for each $x > 0$ with $\sum_{j=0}^{k-1} a_j < x \leq \sum_{j=0}^{k} a_j$ where $a_0 = 0$ we define

$$f(x) = f_k\left(x - \sum_{j=0}^{k-1} a_j\right).$$

and we define $f(x) = f_1(0)$ for $x \leq 0$. The concatenation of a finite tuple of functions is defined similarly.

We next prove the converse direction of Theorem 4.2. For the convenience of the reader, we also give an explicit description of the construction technique in $\mathbb{R}$. Note that in the proof of Theorem 4.15, the precise choice of the contractions $(r_i)_{i=1}^{\infty} \subset (0, 1/2]$ is concealed in the application of Lemma 4.9 in Proposition 4.13.

Let $\varphi \in \mathcal{A}_1$ be some fixed function. Fix some small constant δ_1. Then we will inductively choose constants $r_1^{(n)}, \dots, r_{m_n}^{(n)}$ in $(0, 1/2]$ for each $n \in \mathbb{N}$ so that for each $1 \leq j \leq m_n$

$$2^j \approx \left(\frac{1}{r_1^{(n)} \cdots r_j^{(n)}}\right)^{\varphi(\theta)} \tag{4.10}$$

where θ is such that $\delta_n^{1/\theta} \approx \delta_n \cdot r_1^{(n)} \cdots r_j^{(n)}$, and m_n satisfies $\delta_n \cdot r_1^{(n)} \cdots r_{m_n}^{(n)} \approx \delta_n^n$. Then take R_n very small, and set $\delta_{n+1} = \delta_n \cdot r_1^{(n)} \cdots r_{m_n}^{(n)} \cdot R_n$. Now let C denote the Moran set corresponding to the sequence

$$(\delta_1, r_1^{(1)}, \dots, r_{m_1}^{(1)}, R_1, r_1^{(2)}, \dots, r_{m_2}^{(2)}, R_2, \dots).$$

For $n \in \mathbb{N}$ and $x \in C$, $N_{\delta_n^{1/\theta}}(C \cap B(x, \delta_n)) \approx 2^j$ where $\delta_n^{1/\theta} \approx \delta_n \cdot r_1^{(n)} \cdots r_j^{(n)}$. In particular, (4.10) guarantees that the Assouad spectrum of C with respect to θ at scale δ_n is precisely $\varphi(\theta)$, for all n sufficiently large so that $1/n \leq \theta$.

The main details of the proof are to show (1) that such a choice of the constants r_i is possible, and (2) that for fixed θ and sufficiently small scales δ not of the form δ_n, the Assouad spectrum at θ of C at scale δ is at most $\varphi(\theta)$.

Theorem 4.15. *Let $\varphi \in \mathcal{A}_d$ be arbitrary. Let α be such that $\varphi(1) \leq \alpha \leq d$. Then there exists a homogeneous Moran set $C \subseteq \mathbb{R}^d$ such that $\dim_{\mathrm{A}} C = \alpha$ and, for all $\theta \in (0,1)$,*

$$\dim_{\mathrm{A}}^{\theta} C = \varphi(\theta).$$

Proof. We may assume $\alpha > 0$, or the result is immediate. We will prove the result for the Assouad spectrum, and then explain how to modify the proof to accommodate the Assouad dimension as well.

First, we apply some convenient rescaling to $\varphi(\theta)$. Given $y \in (0, \infty)$, $\exp(-y) \in (0, 1)$ so we may define

$$\xi(y) = (1 - \exp(-y))\varphi(\exp(-y)).$$

In particular, given $0 < y_1 < y_2 < \infty$, it follows that $0 < \exp(-y_2) < \exp(-y_1) < 1$ so by the definition of $\mathcal{A}_d$ from Definition 4.1,

$$\begin{aligned} 0 &\leq (1 - \exp(-y_2))\varphi(\exp(-y_2)) - (1 - \exp(-y_1))\varphi(\exp(-y_1)) \\ &\leq \exp(-y_1)\left(1 - \exp(-(y_2 - y_1))\right)\varphi(\exp(-(y_2 + y_1))) \end{aligned}$$

or equivalently

$$0 \leq \xi(y_2) - \xi(y_1) \leq \exp(-y_1)\xi(y_2 - y_1). \tag{4.11}$$

Moreover, observe that $\varphi(1) = \lim_{y \to 0} \varphi(\exp(-y))$ so $\lim_{y \to 0} \xi(y) = 0$, and similarly $\lim_{y \to \infty} \xi(y) = \varphi(0)$. In particular ξ is continuous, increasing, and bounded.

Now for $z \in (0, \alpha)$, let ξ_z denote the function

$$\xi_z(y) = \xi(y) + \exp(-y)z$$

and similarly $\Psi_z(y) = \exp(-y)z$. We note that $\xi_z(0) = \Psi_z(0) = z$.

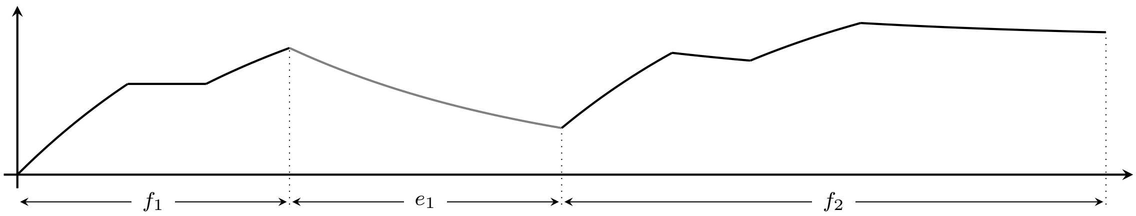


FIGURE II.2: The concatenation of (f_1, e_1, f_2) corresponding to a function $\phi \in \mathcal{C}_d$ defined in §5.2 restricted to the domain $(0, \infty)$.

Now, set $z_1 = 0$ and choose constants w_n, z_n such that the functions $f_n := \xi_{z_n}|_{[0,n]}$ and $e_n := \Psi_{w_n}|_{[0,n]}$ satisfy $f_n(n) = e_n(0)$ and $e_n(n) = f_{n+1}(0)$ for all $n \in \mathbb{N}$. Then, let g be the infinite concatenation of the sequence

$$(f_1, e_1, f_2, e_2, \dots).$$

This construction is illustrated in Figure II.2.

First, let us verify that $g \in \mathcal{G}(0, \varphi(1)) \subset \mathcal{G}(0, \alpha)$. Since membership of $\mathcal{G}$ is equivalent to a pointwise derivative constraint (see [BR22, Lemma 3.2]), it suffices to verify Definition 4.8 piecewise. Let $n \in \mathbb{N}$. Note that $e_n \in \mathcal{G}(0, \varphi(1))$ since the e_n are differentiable with $e_n'(x) = \varphi(0) - e_n(x)$. Next let $0 < y < y + t < \infty$. First observe that

$$\xi(y+t) \leq \xi(t) + \xi(y)\exp(-t) \leq (1 - \exp(-t))\varphi(1) + \xi(y)\exp(-t)$$

by (4.11) and Proposition 4.6 (iii). Thus

$$\begin{aligned} f_n(y+t) &= \xi(y+t) + \exp(-(y+t))z_n \\ &\leq (1 - \exp(-t))\varphi(1) + \xi(y)\exp(-t) + \exp(-(y+t))z_n \\ &= (1 - \exp(-t))\varphi(1) + f_n(y)\exp(-t) \end{aligned}$$

as required. To obtain the other bound, since ξ is increasing,

$$\begin{aligned} f_n(y+t) &= \xi(y+t) + \exp(-(y+t))z_n \\ &\geq \xi(y)\exp(-t) + \exp(-y)\exp(-t)z_n \\ &= f_n(y)\exp(-t). \end{aligned}$$

Now, let C denote the Moran set corresponding to the function g. Let $\theta \in (0, 1)$: we must show that $\dim_{\mathrm{A}}^\theta C = \varphi(\theta)$. Let $\beta = \log(1/\theta)$. By Proposition 4.13, it suffices to show

$$\varphi(\theta) = \limsup_{x \to \infty} \frac{g(x+\beta) - \theta g(x)}{1 - \theta}. \tag{4.12}$$

For $n \in \mathbb{N}$ set $x_n = 2\sum_{i=1}^{n-1} i$ and let $N \in \mathbb{N}$ be sufficiently large so that $N \geq \beta + 1$.

Now if $n \geq N$, $g(x_n + \beta) = f_n(\beta)$ and $g(x_n) = f_n(0) = z_n$ so that

$$\frac{g(x_n + \beta) - \theta g(x_n)}{1 - \theta} = \frac{(1 - \theta)\varphi(\theta) + \theta z_n - \theta z_n}{1 - \theta} = \varphi(\theta).$$

This gives the lower bound in (4.12).

It remains to see the upper bound. We first observe for all $y > 0$ and $z \in \mathbb{R}$ that

$$\frac{\xi_z(y+\beta) - \theta\xi_z(y)}{1-\theta} \leq \varphi(\theta).$$

Indeed, expanding the definition of ξ_z and applying (4.11),

$$\begin{aligned}\xi_z(y+\beta) - \theta\xi_z(y) &= \xi(y+\beta) + \exp(-(y+\beta))z - \exp(-\beta)(\xi(y) + \exp(-y)z) \\ &= \xi(y+\beta) - \exp(-\beta)\xi(y) \\ &\leq \xi(\beta) = (1-\theta)\varphi(\theta).\end{aligned}$$

Now let $x \geq x_N$ be arbitrary and let n be such that $x \in [x_n - (n-1), x_n + n]$. First note that for $y \in [x_n - (n-1), x_n + 2n]$, we have $g(y) = \exp(-(y - x_n))z_n + \phi(y)$ where

$$\phi(y) = \begin{cases} 0 & : x_n - (n-1) \leq y \leq x_n \\ \xi(y - x_n) & : x_n \leq y \leq x_n + n \\ \xi(n)\exp(-(y - x_n + n)) & : x_n + n \leq y \leq x_n + 2n \end{cases}$$

by choice of the constants w_n and z_n. If $x \in [x_n, x_n + n]$, since $x + \log(1/\theta) \leq x_n + 2n$ and $g(y) \leq \xi_{z_n}(y - x_n)$ for all $y \in [x_n, x_n + 2n]$, the prior computation shows that $g(x+\beta) - \theta g(x) \leq (1-\theta)\varphi(\theta)$. Otherwise, $x \in [x_n - (n-1), x_n]$. If $x + \beta \leq x_n$, then $g(x+\beta) - \theta g(x) = 0 \leq (1-\theta)\varphi(\theta)$, and if $x_n < x + \beta \leq x_n + n$, then

$$g(x+\beta) - \theta g(x) = \xi(x + \beta - x_n) \leq \xi(\beta)$$

since ξ is increasing. Thus (4.12) holds, finishing the proof.

In order to obtain the result for the Assouad dimension as well, we modify the construction as follows. Define functions $u_n : [0, 1/n] \to (0, \alpha)$ by the rule $u_n(x) = \alpha - (\alpha - q_n)\exp(-x)$. Choosing the constants q_n appropriately and modifying the constants w_n and z_n, the concatenation $\tilde{g}$ of the sequence

$$(f_1, e_1, u_1, f_2, e_2, u_2, \ldots)$$

is continuous and $\tilde{g} \in G(0, \alpha)$ since $\alpha \geq \varphi(1)$. Since the u_n are supported on intervals with lengths converging to 0, the same arguments as before yield the correct bounds for $\dim_{\mathrm{A}}^{\theta} C$ up to an error decaying to 0 as n goes to infinity. On the other hand, the same arguments as given in [BR22, Lemma 3.7 and Theorem 3.9] give that $\dim_{\mathrm{A}} C = \alpha$. We leave the precise details to the reader. □

4.5 Attainable forms of the upper Assouad spectrum

In this section, we prove Corollary 4.4.

First, we obtain bounds on growth rates of functions in $\mathcal{A}_d$ in terms of their derivatives. Recall that the *upper right Dini derivative* of f at x is defined by

$$D^+ f(x) = \limsup_{\varepsilon \to 0^+} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}.$$

We similarly denote the lower left Dini derivative by $D^- f(x)$ by taking the limit from right left instead of the right.

We obtain the following regularity property for functions $\varphi \in \mathcal{A}_d$.

Proposition 4.16. *Let $\varphi \in \mathcal{A}_d$ be arbitrary and $\theta \in (0,1)$. Then*

$$-\frac{\varphi(1)-\varphi(\theta)}{1-\theta} \leq D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}$$

In particular, φ is d/δ-Lipschitz on $[0,1-\delta]$ for any $\delta > 0$.

Proof. The first inequality in (4.1) is equivalent to saying that $\beta(\theta) = (1-\theta)\varphi(\theta)$ is decreasing. Since φ is continuous by Proposition 4.6 (ii), by [Bru94, Corollary 11.4.2] β is decreasing if and only if $D^+\beta(\theta) = -\varphi(\theta)+(1-\theta)D^+\varphi(\theta) \leq 0$, or equivalently

$$D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}.$$

This gives the upper bound.

To obtain the lower bound, let $0 < \lambda < \theta < 1$ be arbitrary. By (4.1),

$$-\varphi(\lambda/\theta) \leq \frac{\beta(\lambda)-\beta(\theta)}{\lambda-\theta}$$

and taking $\theta \to \lambda$ from the right,

$$-\varphi(1) \leq D^+\beta(\lambda) = -\varphi(\theta) + (1-\theta)D^+\varphi(\theta).$$

Since $0 \leq \varphi(\theta) \leq d$ and $0 \leq \varphi(1) - \varphi(\theta) \leq d$, it follows that φ is d/δ-Lipschitz on $[0, 1-\delta]$ for any $\delta > 0$. □

Remark 4.17. In §5.1, we will see that, in general, elements of $\mathcal{A}_d$ need not be Lipschitz (in fact, not even Hölder) on the entire interval $[0,1]$.

Using this, we can characterize precisely when elements of $\mathcal{A}_d$ are increasing functions.

Lemma 4.18. *If $\varphi\colon (0,1) \to [0,d]$ is increasing, then $\varphi \in \mathcal{A}_d$ if and only if*

$$D^+\varphi(\theta) \leq \frac{\varphi(\theta)}{1-\theta}. \tag{4.13}$$

Proof. The forward direction is Proposition 4.16. To obtain the reverse implication, let $0 < \lambda < \theta < 1$. Since φ is increasing, if $\theta \leq \lambda/\theta$, then $\varphi(\lambda) \leq \varphi(\theta) \leq \varphi(\lambda/\theta)$ and

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta-\lambda)\varphi(\theta) \leq (\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big)$$

and if $\lambda/\theta \leq \theta$,

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (1-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big) - (1-\theta)\varphi\Big(\frac{\lambda}{\theta}\Big) = (\theta-\lambda)\varphi\Big(\frac{\lambda}{\theta}\Big)$$

which is (4.1). □

We obtain the following convenient application, which we use to characterize the upper Assouad spectra.

Corollary 4.19. *Let $\varphi \in \mathcal{A}_d$. Then $\overline{\varphi} \in \mathcal{A}_d$ where*

$$\overline{\varphi}(\theta) = \sup_{0<\theta'\leq\theta} \varphi(\theta').$$

Proof. As proven in Lemma 4.18, since $\overline{\varphi}(\theta)$ is increasing, we only need to verify that $D^+\overline{\varphi}(\theta) \leq \overline{\varphi}(\theta)/(1-\theta)$. Since $\varphi(\theta) \leq \overline{\varphi}(\theta)$, it suffices to show $D^+\overline{\varphi} \leq \max\{D^+\varphi, 0\}$.

Fix θ_0 and let $(\theta_n)_{n=1}^\infty \to \theta_0$ be strictly decreasing. Passing to a subsequence if necessary, we may assume $\overline{\varphi}(\theta_n) > \overline{\varphi}(\theta_0)$ for all n; otherwise $D^+\overline{\varphi}(\theta_0) \leq 0$. Thus for each n there is $\theta_0 < \theta_n' \leq \theta_n$ be such that $\varphi(\theta_n') = \overline{\varphi}(\theta_n)$. Thus

$$\frac{\overline{\varphi}(\theta_n) - \overline{\varphi}(\theta_0)}{\theta_n - \theta_0} \leq \frac{\varphi(\theta_n') - \varphi(\theta_0)}{\theta_n' - \theta_0} \leq D^+\varphi(\theta_0).$$

But $(\theta_n)_{n=1}^\infty$ was an arbitrary sequence, so the result follows. □

Finally, we prove that $\mathcal{A}_d$ is closed under taking suprema. This essentially follows since $\mathcal{A}_d$ is uniformly Lipschitz on $[0, 1-\delta]$ for any $\delta > 0$.

Proposition 4.20. *Let $(\varphi_j)_{j\in\mathcal{J}}$ be some family of elements in $\mathcal{A}_d$. Then $\sup_{j\in\mathcal{J}} \varphi_j \in \mathcal{A}_d$.*

Proof. Let $f = \sup_{j\in\mathcal{J}} \varphi_j$. Get a sequence $J_1 \subset J_2 \subset \cdots \subset \mathcal{J}$ such that each J_n is finite and with

$$f_n := \max\{\varphi_i : i \in J_n\}$$

that $f = \lim_{n\to\infty} f_n$ pointwise. An easy computation shows that if $\varphi_1, \varphi_2 \in \mathcal{A}_d$, then $\max\{\varphi_1, \varphi_2\} \in \mathcal{A}_d$; in particular, each $f_n \in \mathcal{A}_d$.

We first show that $f \in C([0,1])$. Since $(f_n)_{n=1}^\infty$ is monotonically increasing, by the Arzelà–Ascoli Theorem, it suffices to show that $(f_n)_{n=1}^\infty$ is uniformly bounded and uniformly equicontinuous. Uniform boundedness is immediate, so we must verify uniform equicontinuity.

Set $b = \lim_{n\to\infty} f_n(1)$ and let N be sufficiently large so that $f_n(1) > b - \varepsilon/2$ for all $n \geq N$. Since f_N is continuous, get $\delta > 0$ so that $f_N(y) > f_N(1) - \varepsilon/2$ for all $y \in [1-\delta, 1]$. Then $|f_n(x) - f_n(y)| \leq \varepsilon$ whenever $x, y \in [1-\delta, 1]$. Finally, since each $f_n \in \mathcal{A}_d$, the function f_n is uniformly Lipschitz on $[0, 1-\delta]$ as proven in Proposition 4.16. It follows that $(f_n)_{n=1}^\infty$ is uniformly equicontinuous on $[0,1]$.

Thus $f \in C([0,1])$. To verify that $f \in \mathcal{A}_d$, let $0 < \lambda < \theta < 1$ be arbitrary. Then for any $\varepsilon > 0$, get n such that $\|f_n - f\|_\infty \leq \varepsilon$ so that

$$\begin{aligned}(1-\lambda)f(\lambda) - (1-\theta)f(\theta) &\leq (1-\lambda)(f_n(\lambda) + \varepsilon) - (1-\theta)(f_n(\theta) - \varepsilon) \\ &\leq (\theta - \lambda) f_n(\lambda/\theta) + 2\varepsilon \\ &\leq (\theta - \lambda) f(\lambda/\theta) + 3\varepsilon\end{aligned}$$

for any $\varepsilon > 0$, so the inequality holds. The lower inequality follows identically. □

Remark 4.21. Note that $\mathcal{A}_d$ is not compact: for example, consider the functions $\varphi_n(\theta) = \min\{c_n/(1-\theta), 1\}$ with constants $c_n > 0$. If $\lim_{n\to\infty} c_n = 0$, then φ_n converges pointwise to the function which is 0 on $[0,1)$ and 1 at 1, and hence has no uniformly convergent subsequence. However, a simple modification of the above proof gives that for every $\delta > 0$, the restriction of $\mathcal{A}_d$ to $C([0, 1-\delta])$ is compact.

We recall that the family $\mathcal{M}_d$ is defined in (4.4). A direct argument shows that each $\varphi \in \mathcal{M}_d$ is an increasing element of $\mathcal{A}_d$.

Proof (of Corollary 4.4). To see that (a) implies (b), if $F \subset \mathbb{R}^d$ has $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$ and $\overline{\dim}_{\mathrm{A}}^{\theta} F = \overline{\varphi}(\theta)$, by [FHH+19, Theorem 2.1],

$$\overline{\varphi}(\theta) = \sup_{0<\theta'\leq\theta} \varphi(\theta')$$

so by Corollary 4.19, $\overline{\varphi} \in \mathcal{A}_d$ so $\theta \mapsto (1-\theta)\overline{\varphi}(\theta)$ is decreasing. Of course, $\overline{\varphi}$ is increasing as well.

Next, (b) is equivalent to saying that for each $\lambda \in (0,1)$,

$$f_{(\overline{\varphi}(\lambda),\lambda)}(\theta) \leq \overline{\varphi}(\theta) \cdot (1-\theta)$$

for all $\theta \in (0,1)$, with equality at $\theta = \lambda$. Since $\overline{\varphi} \in \mathcal{A}_d$ by Lemma 4.18, $\overline{\varphi}$ is uniformly continuous on $(0,1)$ and therefore $\overline{\varphi} = \sup_{\lambda \in \mathcal{Q}} f_{(\overline{\varphi}(\lambda),\lambda)}$ for any countable dense subset $\mathcal{Q} \subset (0,1)$. This implies (c).

Finally, to see (c) implies (a), since $\mathcal{A}_d$ is closed under suprema by Proposition 4.20, if $\overline{\varphi}(\theta) = \sup_{f\in\mathcal{F}} f(\theta)$ for some $\mathcal{F} \subset \mathcal{M}_d$, then $\overline{\varphi} \in \mathcal{A}_d$. Thus the result follows by Theorem 4.15. □

5 EXCEPTIONAL CONSTRUCTIONS FOR ASSOUAD SPECTRA

Using the classification proven in §4, we now construct some exceptional sets.

5.1 HÖLDER FAILURE AT 1

Our first result concerns Hölder regularity. In fact, we prove the following result which states that there is no control of the rate at which Assouad spectrum approaches the quasi-Assouad dimension. This result is sharp: recall from Proposition 4.16 that $\dim_{\mathrm{A}}^{\theta} F$ is (uniformly) Lipschitz on $(0, 1-\delta)$ for all $\delta > 0$, with constants depending only on δ and the ambient dimension d. This observation provides a complete answer to [FY18b, Question 9.2].

Theorem 5.1. *Let $f\colon [0,1] \to [0,d]$ be an increasing function with $f(0) > 0$. Then there exists a compact set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F \leq f(\theta)$ for all $\theta \in (0,1)$ and $\lim_{\theta\to 1} \dim_{\mathrm{A}}^{\theta} F = f(1)$.*

Proof. For $0 \leq \theta < 1$, let $h(\theta) = \min_{0\leq\theta'\leq\theta}(1-\theta')f(\theta')$ and let $\varphi(\theta) = h(\theta)/(1-\theta)$. By definition, h is decreasing, $\varphi \leq f$, and since $f(\theta) \leq d$, $\lim_{\theta\to 1} h(\theta) = 0$. Next, let us verify that φ is increasing. Let $0 \leq \lambda < \theta < 1$ be arbitrary. If $h(\lambda) = h(\theta)$, then it follows immediately that $\varphi(\theta) > \varphi(\lambda)$. Otherwise, let θ' attain the minimum in the definition of $h(\theta)$. Since $h(\lambda) > h(\theta)$, we must have $\lambda \leq \theta' \leq \theta$ so that

$$\varphi(\lambda) \leq f(\lambda) \leq f(\theta') \leq \frac{h(\theta)}{1-\theta'} \leq \varphi(\theta).$$

Therefore $\varphi \in \mathcal{A}_d$ and $\varphi(\theta) \leq f(\theta)$.

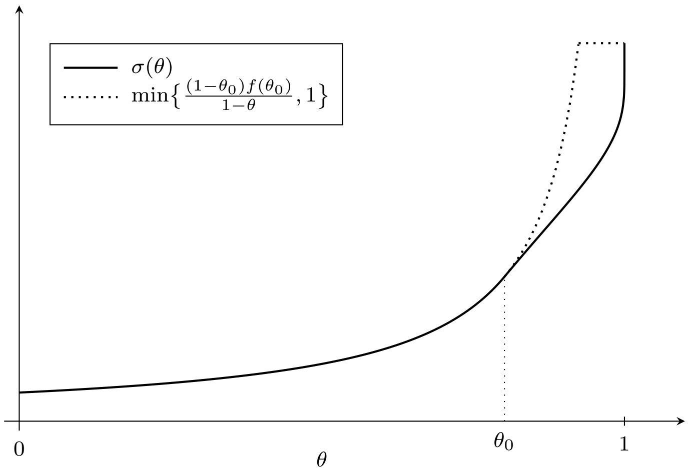


FIGURE II.1: Plot of a spectrum which is not Hölder at 1, along with the general upper bound.

Finally, we verify that $\lim_{\theta\to 1}\varphi(\theta) = f(1)$. Since $(1-\theta)f(\theta) > 0$ for all $0 \leq \theta < 1$ and $\lim_{\theta\to 1} h(\theta) = 0$, for all $0 \leq \lambda < 1$, there is a $\lambda \leq \theta < 1$ so that $h(\theta) = (1-\theta)f(\theta)$. Thus $\varphi(\theta) = f(\theta)$ for a sequence of θ converging to 1; but f and φ are continuous so $\lim_{\theta\to 1}\varphi(\theta) = f(1)$.

To conclude, Theorem 4.15 gives a compact set $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F = \varphi(\theta)$. By the properties of φ established above, the claim follows. $\square$

We also consider an explicit example. Consider the function $f(\theta) = 1 + \frac{1}{\log(1-\theta)}$; note that f is not Hölder at 1. A direct computation shows that there is some minimal $\theta_0 \in (0,1)$ so that $(1-\theta)f(\theta)$ is decreasing on $[\theta_0, 1]$. Thus if we define

$$\sigma(\theta) = \begin{cases} \frac{(1-\theta_0)f(\theta_0)}{1-\theta} & : 0 < \theta \leq \theta_0 \\ f(\theta) & : \theta_0 < \theta < 1 \end{cases}$$

then σ is a continuous increasing function of θ with $(1-\theta)\sigma(\theta)$ decreasing. Thus by Lemma 4.18, $\sigma \in \mathcal{A}_1$. A plot of $\sigma(\theta)$ and the upper bound $\min\{(1-\theta_0)f(\theta_0)/(1-\theta), 1\}$ is given in Figure II.1.

5.2 A FAMILY OF NON-MONOTONIC SPECTRA

This family generalizes the example considered in [Fra20, Theorem 3.4.16]. Let

$$C_d = \left\{(\kappa, c_1, c_2) : 0 \leq \kappa \leq d, 0 \leq c_1 \leq c_2 \leq c_1^{1/2} < 1\right\}.$$

Suppose $\boldsymbol{c} = (\kappa, c_1, c_2) \in C_d$. If $c_1 = 0$, let $h_{\boldsymbol{c}}(\theta) = \kappa(1-\theta)$ for $\theta \in [0,1]$. Otherwise, $c_2 \leq c_1/c_2$. Thus we may define $h = h_{\boldsymbol{c}} \colon [0,1] \to [0,d]$ to be the unique continuous function which has slope 0 on $[0,c_1] \cup [c_2, c_1/c_2]$, has slope $-\kappa$ on $[c_1,c_2] \cup [c_1/c_2, 1]$, and satisfies $h(1) = 0$. Now, let

$$\mathcal{C}_d = \left\{\theta \mapsto \frac{h_{\boldsymbol{c}}(\theta)}{1-\theta} : \boldsymbol{c} \in C_d\right\} \tag{5.1}$$

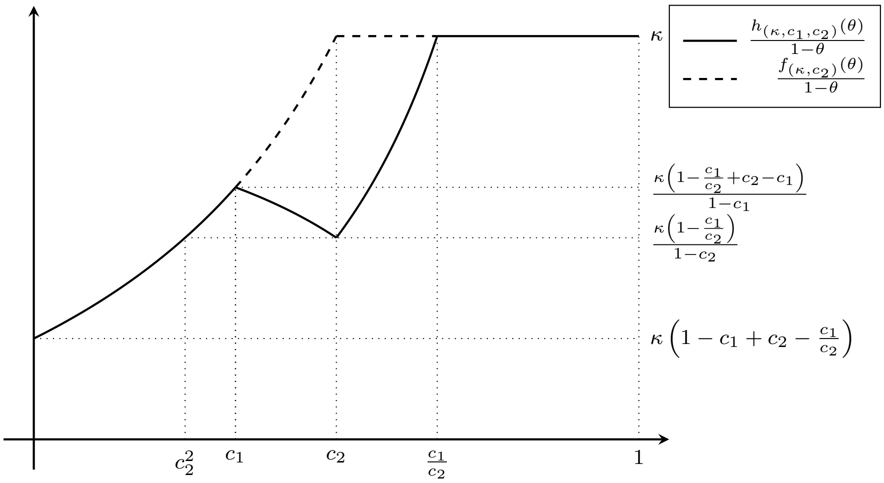


FIGURE II.2: A plot of $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ and $f_{\boldsymbol{i}}(\theta)/(1-\theta)$ where $\boldsymbol{c} = (\kappa, c_1, c_2)$ and $\boldsymbol{i} = (\kappa, c_2)$.

We note that $h_{\boldsymbol{c}}$ satisfies a certain rescaling invariance: for $c_2^2 \leq \theta \leq c_1$,

$$h_{\boldsymbol{c}}(\theta) - h_{\boldsymbol{c}}\Big(\frac{\theta}{c_2}\Big) = \kappa(c_2 - c_1).$$

In particular, $h_{\boldsymbol{c}}(c_2^2)/(1-c_2^2) = h_{\boldsymbol{c}}(c_2)/(1-c_2)$.

There are degenerate cases: if $c_2 = c_1^{1/2}$, then $h_{(\kappa,c_1,c_2)} = f_{(\kappa,c_1)}$ and if $c_1 = c_2$ or $\kappa = 0$, then $h_{\kappa,c_1,c_2} = 0$. Otherwise, $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ is strictly increasing on $[0, c_1]$ and $[c_2, c_1/c_2]$, constant on $[c_1/c_2, 1]$, and strictly decreasing on $[c_1, c_2]$. A plot of the function $h_{\boldsymbol{c}}(\theta)/(1-\theta)$ for non-degenerate parameters is given in Figure II.2.

Proposition 5.2. *For any $d \in \mathbb{N}$, $\mathcal{C}_d \subset \mathcal{A}_d$.*

Proof. Fix $0 < \theta_1, \theta_2 < 1$ and $\boldsymbol{c} = (\kappa, c_1, c_2) \in C_d$. We may assume $0 < c_1 < 1$. Since $h_{\boldsymbol{c}}$ is decreasing, it suffices to show that

$$\frac{h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta)}{\theta} \leq h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big) \tag{5.2}$$

for all $0 < \lambda < \theta < 1$. Since (5.2) is invariant under scaling by a positive factor, we may assume $\kappa = 1$. We prove this result in cases depending on the positions of λ and θ.

If $\lambda \in [0, c_2^2] \cup [c_2, 1]$, then $h_{\boldsymbol{c}}(\lambda)/(1-\lambda) \leq h_{\boldsymbol{c}}(\theta)/(1-\theta)$ for all $\lambda < \theta$. In particular, as argued in Lemma 4.18, (5.2) holds for all such λ. Moreover, suppose that (5.2) holds for the choice $\lambda = c_1$ and all $\lambda < \theta$. Since $h_{\boldsymbol{c}}$ is the constant function on $[c_2^2, c_1]$, this implies the bound on $[c_2^2, c_1]$. Moreover, for $\lambda \in [c_1, c_2]$, since $h_{\boldsymbol{c}}(c_1/\theta) - h_{\boldsymbol{c}}(\lambda/\theta) \leq (c_1 - \lambda)/\theta$,

$$h_{\boldsymbol{c}}(\lambda) - h_{\boldsymbol{c}}(\theta) \leq \theta h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big) - (c_1 - \lambda) \leq \theta h_{\boldsymbol{c}}\Big(\frac{\lambda}{\theta}\Big).$$

Thus it suffices to establish (5.2) for $\lambda = c_1$ and $\theta > \lambda$. Write $g(\theta) = (h_{\boldsymbol{c}}(c_1) - h_{\boldsymbol{c}}(\theta))/\theta$: we must show that $g(\theta) \leq h_{\boldsymbol{c}}(c_1/\theta)$.

1. If $\theta \in [c_1, c_2]$, then $c_1/\theta \geq c_1/c_2$ and
$$g(\theta) = 1 - \frac{c_1}{\theta} = h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big).$$
2. If $\theta \in [c_2, c_1/c_2]$ then $c_1/\theta \in [c_2, c_1/c_2]$ and
$$g(\theta) = \frac{c_2 - c_1}{\theta} \leq 1 - \frac{c_1}{c_2} = h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big).$$
3. If $\theta \in [c_1/c_2, 1]$, then $c_1/\theta \in [c_1, c_2]$ and
$$g(\theta) = (\theta - c_1) + \left(c_2 - \frac{c_1}{c_2}\right) \leq \left(1 - \frac{c_1}{\theta}\right) + \left(c_2 - \frac{c_1}{c_2}\right) = h_{\boldsymbol{c}}\Big(\frac{c_1}{\theta}\Big).$$

This treats all the cases $0 < \lambda < \theta < 1$, as required. □

5.3 Non-monotonicity on any open set

In this section, we prove that Assouad spectra which are non-monotonic on every open subset of $(0, 1)$ are dense in the set of upper Assouad spectra. Throughout this section, we fix a non-zero increasing $\varphi \in \mathcal{A}_d$, and as usual write $\beta(\theta) = (1-\theta)\varphi(\theta)$.

We recall that the functions $h_{(\kappa,c_1,c_2)}$ for $(\kappa, c_1, c_2) \in C_d$ are defined in §5.2. Fix $0 < \lambda < 1$ and for $0 \leq y \leq \beta(\lambda)$, define
$$c(\lambda, y) = \frac{\lambda + y/\varphi(\lambda) - 1 + \sqrt{(\lambda + y/\varphi(\lambda) - 1)^2 + 4\lambda}}{2}.$$
The constraint on y ensures that $\boldsymbol{c}(\lambda, y) \coloneqq (\varphi(\lambda), \lambda, c(\lambda, y)) \in \mathcal{C}_d$. Note that $c(\lambda, y)$ is chosen precisely so that
$$h_{\boldsymbol{c}(\lambda,y)}(\lambda) = y.$$
Observe that $h_{\boldsymbol{c}(\lambda,y)} \leq \beta$ by Corollary 4.4. We also let $L_{\lambda,y}$ denote the unique affine function on $(0, 1)$ passing through the point (λ, y) with slope $-\varphi(\lambda)$. Equivalently,
$$L_{\lambda,y}(\theta) = h_{\boldsymbol{c}(\lambda,y)}(\theta) \qquad \text{for all} \qquad \lambda \leq \theta \leq c(\lambda, y).$$
Note that $L_{\lambda,y}$ has unique zero $\lambda + y/\varphi(\lambda)$. These functions will play a key role in the construction.

Next, we define a useful family of approximations of the function φ. For each $1 \leq t < \infty$, define the functions
$$\varphi_t(\theta) = \varphi(\theta^t) \qquad \text{and} \qquad \beta_t(\theta) = (1-\theta)\varphi_t(\theta).$$
This family of functions uniformly approximates φ from below while also satisfying a key “affine partitioning” property (i).

Lemma 5.3. *Suppose $\varphi \in \mathcal{A}_d$ is strictly increasing. Then the following hold.*

(i) Let $t \in (1, \infty)$ and $\lambda \in (0, 1)$ and let $\ell \coloneqq L_{\lambda^t,\beta(\lambda^t)}$. Then $\ell(\lambda) = \beta_t(\lambda)$. Moreover, $\ell(\theta) > \beta_t(\theta)$ for $0 < \theta < \lambda$ and $\ell(\theta) < \beta_t(\theta)$ for $\lambda < \theta < 1$.

(ii) Let $\lambda \in (0, 1)$. Then $L_{\lambda,\beta_t(\lambda)}(\theta) < \beta_t(\theta)$ for all $\lambda < \theta < 1$.

(iii) *For all* $0 \leq t_1 < t_2$ *and* $0 < \theta < 1$, *we have* $\varphi_{t_1}(\theta) > \varphi_{t_2}(\theta)$.
(iv) *For all* $1 < t < \infty$, β_t *is strictly decreasing.*
(v) *For all* $1 \leq t < \infty$, φ_t *is strictly increasing and an element of* $\mathcal{A}_d$.
(vi) *We have* $\lim_{t\to 1} \|\varphi_t - \varphi\|_\infty = 0$.

Proof. Let $t \in (1, \infty)$ and $\lambda \in (0, 1)$, and let ℓ be defined as in (i). It is a direct computation that $\ell(\lambda) = \beta_t(\lambda)$. Moreover, since φ is strictly increasing, the family of lines $L_{\lambda,\beta_t(\lambda)}$ is strictly increasing in the following sense: $L_{\lambda_1,\beta_t(\lambda_1)}(\theta) < L_{\lambda_2,\beta_t(\lambda_2)}(\theta)$ for all $\lambda_1 < \lambda_2$ and $0 < \theta < 1$. Thus by monotonicity of $\theta \mapsto \theta^t$ and since $L_{\lambda,\beta_t(\lambda)}(\lambda) = \beta_t(\lambda)$, this implies (i) and (iii). Now, (ii) follows from (i) since φ is strictly increasing, so $L_{\lambda,\beta_t(\lambda)}(\theta) < \ell(\theta)$ for all $\lambda < \theta < 1$.

Next, since φ is strictly increasing, it is clear that φ_t is strictly increasing. Moreover, by (i), $L_{\lambda,\beta_t(\lambda)}(\theta) \leq \beta_t(\theta)$ for all $\theta \geq \lambda$, and since $(1-\theta)/(1-\theta^t)$ is decreasing (resp. strictly decreasing for $t > 1$), β_t is also decreasing (resp. strictly decreasing for $t > 1$). This yields (iv). Thus observing that $\ell(\theta) = f_{(\lambda,\varphi_t(\lambda))}(\theta)$ for all $\theta \in [\lambda, 1]$, it follows from Corollary 4.4 that $\varphi_t \in \mathcal{A}_d$, which completes the proof of (v).

And finally, (vi) holds since $\theta \mapsto \theta^t$ uniformly converges to the identity map on $[0, 1]$ as $t \to 1$. □

Remark 5.4. There is nothing particularly special about the function $\theta \mapsto \theta^t$ for $1 \leq t < \infty$. Take any increasing homeomorphism ϕ of $[0, 1]$ such that $\phi(\theta) \leq \theta$ and $\theta \mapsto (1-\theta)/(1-\phi(\theta))$ is decreasing. Then $\varphi \circ \phi$ is an increasing element of $\mathcal{A}_d$ which satisfies (i).

We now introduce the key property for our inductive construction.

Definition 5.5. Let $\mathcal{L} \subset (0, 1)$ be a finite set, let $y \colon \mathcal{L} \to \mathbb{R}$, and let $1 < t < 2$. We say that the triple $(\mathcal{L}, y, t)$ is *monotone* if the following conditions hold:
(a) The function y is strictly decreasing.
(b) For all $\lambda \in \mathcal{L}$, $\beta_t(\lambda) < y(\lambda) < \beta(\lambda)$.
(c) For all $\lambda \in \mathcal{L}$ and $\theta \in \mathcal{L} \setminus \{\lambda\}$, $h_{\boldsymbol{c}(\theta,y(\theta))}(\lambda) < y(\lambda)$.

If $(\mathcal{L}, y, t)$ is monotone, we define the corresponding function

$$\psi = \psi_{\mathcal{L},y,t} = \max\Big\{\max_{\lambda\in\mathcal{L}} h_{\boldsymbol{c}(\lambda,y(\lambda))}, \beta_t\Big\} \tag{5.3}$$

Observe that $\psi \leq \beta$, and moreover $\psi = h_{\boldsymbol{c}(\lambda,y(\lambda))}$ in a neighbourhood of λ.

The key observation is that monotone families can be extended by arbitrary elements not in $\mathcal{L}$ in a way which only changes the definition of ψ locally.

Lemma 5.6. *Let* $\varphi \in \mathcal{A}_d$ *be strictly increasing and let* $(\mathcal{L}, y, t)$ *be monotone with corresponding function* ψ. *Let* $\zeta \in (0, 1) \setminus \mathcal{L}$. *Then for all* $\delta > 0$, *there exists an extension* $y(\zeta) \in \mathbb{R}$ *such that* $(\mathcal{L} \cup \{\zeta\}, y, t)$ *is monotone*, $y(\zeta) \leq \psi(\zeta) + \delta$, *and*

$$h_{\boldsymbol{c}(\zeta,y(\zeta))}(\theta) \leq \psi(\theta) \quad \textit{for all} \quad \theta \geq \lambda + \delta.$$

Proof. Let $(\mathcal{L}, y, t)$ be monotone with corresponding function ψ. Let $\delta > 0$ be fixed. The proof will follow from two key observations.

1. *For all $y \leq \beta(\zeta)$ and $\theta \geq c(\zeta, y)$, $h_{\boldsymbol{c}(\zeta,y)}(\theta) \leq \beta_t(\theta)$.* Recall that $h_{\boldsymbol{c}(\zeta,y)}(\theta) = f_{(\zeta,\varphi(\zeta))}(\theta)$ for all $\theta \geq \zeta/c(\zeta, y) \geq \zeta^{1/2}$. By (i) of Lemma 5.3, $\beta_2(\theta) \geq f_{(\zeta,\varphi(\zeta))}(\theta)$ for all $\theta \geq \zeta^{1/2}$. Moreover, $\beta_t \geq \beta_2$ by Lemma 5.3 (iii). Then the claim follows since β_2 is decreasing and $h_{\boldsymbol{c}(\zeta,y)}$ is constant on the interval $[c(\zeta, y), \zeta/c(\zeta, y)]$.

2. *We have $L_{\zeta,\psi(\zeta)}(\theta) < \psi(\theta)$ for all $\zeta < \theta < 1$.* There are three cases.

   First, if $\psi(\zeta) = \beta_t(\zeta)$, then $L_{\zeta,\psi(\zeta)}(\theta) < \beta_t(\theta) \leq \psi(\theta)$ for all $\zeta < \theta < 1$ by Lemma 5.3 (ii).

   Second, suppose $\psi(\zeta) = h_{\boldsymbol{c}(\lambda,y(\lambda))}$ for some $\lambda > \zeta$. Since φ is strictly increasing and $y(\lambda) < \beta(\lambda) = (1 - \lambda)\varphi(\lambda)$,

$$y(\lambda)\left(\frac{1}{\varphi(\zeta)} - \frac{1}{\varphi(\lambda)}\right) < (1-\lambda)\frac{\varphi(\lambda)}{\varphi(\zeta)} - (1-\lambda) \leq \lambda - \zeta.$$

   The second inequality follows since β is decreasing. Rearranging,

$$\zeta + \frac{y(\lambda)}{\varphi(\zeta)} < \lambda + \frac{y(\lambda)}{\varphi(\lambda)}.$$

   But the left hand side is the unique zero of $L_{\zeta,y(\lambda)}$ and the right hand side is the unique zero of $L_{\lambda,y(\lambda)}$. Thus for all $\theta > \zeta$, recalling that $\psi(\zeta) = y(\lambda)$ by assumption,

$$L_{\zeta,\psi(\zeta)}(\theta) < h_{\boldsymbol{c}(\lambda,y(\lambda))}(\theta) \leq \psi(\theta).$$

   Finally, suppose $\psi(\zeta) = h_{\boldsymbol{c}(\lambda,y(\lambda))}$ for some $\lambda > \zeta$. But $\varphi(\lambda) < \varphi(\zeta)$, so for all $\theta > \zeta$, since $h_{\boldsymbol{c}(\lambda,y(\lambda))}$ has slope either 0 or $-\varphi(\lambda)$,

$$L_{\zeta,\psi(\zeta)}(\theta) < h_{\boldsymbol{c}(\lambda,y(\lambda))} \leq \psi(\theta).$$

   This treats all possible cases, as required.

Now let $\delta > 0$ be arbitrary. We may assume $\zeta + \delta < \lambda$ for all $\lambda \in \mathcal{L}$ with $\lambda > \zeta$. Let $s = \min\{y(\lambda) : \lambda < \zeta\}$; note that $s > \psi(\zeta)$ since ψ is monotone. By 2 and since ψ is continuous, by choosing $\psi(\zeta) < y(\zeta) \leq \min\{s, \psi(\zeta) + \delta\}$ sufficiently small, we may assume that $L_{\zeta,y(\zeta)}(\theta) \leq \psi(\theta)$ for all $\zeta + \delta \leq \theta \leq c(\zeta, y(\zeta))$. But

$$h_{\boldsymbol{c}(\zeta,y(\zeta))}(\theta) \leq \beta_2(\theta) \leq \beta_t(\theta) \leq \psi(\theta)$$

for all $\theta \geq c(\zeta, y(\zeta))$ by 1. Thus choosing $y(\zeta) > \psi(\zeta)$ sufficiently small, the claim follows. □

With this result, we have the following key non-monotonicity result.

Theorem 5.7. *Let $\varphi \in \mathcal{A}_d$ be increasing and let $\mathcal{L} \subset (0, 1)$ be an arbitrary countable set. Then for any $\varepsilon > 0$, there exists $F \subset \mathbb{R}^d$ such that $f(\theta) = \dim_{\mathrm{A}}^{\theta} F$ satisfies $\|f - \varphi\|_\infty \leq \varepsilon$ and for all $\lambda \in \mathcal{L}$, $D^- f(\lambda) > 0$ and $D^+ f(\lambda) < 0$. In particular, there exists an $F \subset \mathbb{R}^d$ such that $\dim_{\mathrm{A}}^{\theta} F$ is non-monotonic on every open interval.*

Proof. First, since every increasing function of $\mathcal{A}_d$ can be uniformly approximated by a strictly increasing function in $\mathcal{A}_d$, we may assume that φ is strictly increasing. Next, by Lemma 5.3 (vi), we may choose $1 < t \leq 2$ sufficiently small so that $\|\varphi - \varphi_t\|_\infty$ is arbitrarily small.

Now, enumerate $\mathcal{L} = \{\lambda_n : n \in \mathbb{N}\}$, and write $\mathcal{L}_n = \{\lambda_1, \ldots, \lambda_n\}$. We inductively define a function $y\colon \mathcal{L} \to \mathbb{R}$ and a decreasing sequence of continuous functions γ_n such that for each $n \in \mathbb{N}$, setting $\psi_n = \psi_{\mathcal{L}_n,y,t}$ as in (5.3), the following hold:

(i) The triple $(\mathcal{L}_n, y, t)$ is monotone.
(ii) The functions γ_n are continuous and decreasing.
(iii) $\gamma_n(\theta) > \psi_n(\theta)$ for all $\theta \in (0,1) \setminus \mathcal{L}_n$, and for $\lambda \in \mathcal{L}$, $\gamma_n(\lambda) = \psi_n(\lambda)$ and

$$D^{\pm}\gamma_n(\lambda) = D^{\pm}h_{\boldsymbol{c}(\lambda,y(\lambda))}(\lambda).$$

We begin by choosing $y(\lambda_1) \in (\beta_t(\lambda_1), \beta(\lambda_1))$ arbitrarily; it is clear that (i) holds and that a function γ_1 satisfying (ii) and (iii) exists.

Now suppose we have defined y on $\mathcal{L}_n$ and a function γ_n such that (i) and (iii) hold. Let $0 < \delta < 1 - \lambda_{n+1}$ be sufficiently small such that $E_\delta = [\lambda_{n+1} - \delta, \lambda_{n+1} + \delta] \cap \mathcal{L}_n = \varnothing$ and moreover $\psi_n < \min\{y(\lambda) : \lambda < \lambda_{n+1}; \lambda \in \mathcal{L}_n\}$ on E_δ. This choice is possible since $D^+\psi_n(\lambda) < 0$ for $\lambda \in \mathcal{L}_n$ since $(\mathcal{L}_n, y, t)$ is monotone. Reducing δ more if necessary, we may also assume that $\gamma_n \geq \psi_n(\lambda_{n+1}) + \delta$ on E_δ. Applying Lemma 5.6 with this choice of δ, get a value $y(\lambda_{n+1})$ such that the triple $(\mathcal{L}_{n+1}, y, t)$ is monotone. Moreover, since γ_n is decreasing, it follows that (iii) holds with ψ_{n+1} in place of ψ_n. Therefore we may choose $\gamma_{n+1} \leq \gamma_n$ such that (ii) and (iii) hold.

Finally, let $\psi = \lim_{n\to\infty} \psi_n$ and let $f(\theta) = \psi(\theta)/(1-\theta)$. Then $f \in \mathcal{A}_d$ by Proposition 4.20, and moreover $\|f - \varphi\|_\infty < \varepsilon$ by choice of t since $\beta_t \leq \psi \leq \beta$ by construction. Moreover for each $\lambda \in \mathcal{D}$, by properties of the inductive construction,

1. $\psi(\lambda) = h_{\boldsymbol{c}(\lambda,y(\lambda))}(\lambda) = y(\lambda) < \beta(\lambda)$,
2. $D^-\psi(\lambda) = D^-h_{\boldsymbol{c}(\lambda,y(\lambda))}$, and
3. $D^+\psi(\lambda) = D^+h_{\boldsymbol{c}(\lambda,y(\lambda))}$.

From this, the non-monotonicity result follows since $y(\lambda) < \beta(\lambda)$ gives that

$$D^-\Big(\frac{h_{\boldsymbol{c}(\lambda,y(\lambda))}(\lambda)}{1-\lambda}\Big) > 0 \qquad \text{and} \qquad D^+\Big(\frac{h_{\boldsymbol{c}(\lambda,y(\lambda))}(\lambda)}{1-\lambda}\Big) < 0$$

so f is non-monotonic at each $\lambda \in \mathcal{D}$. Finally, the existence of the corresponding set follows by Theorem 4.15. □

The construction is quite flexible: the countable set $\mathcal{L}$ can be chosen arbitrarily and moreover the function y at each step of the inductive construction can be chosen from an open set of parameters. This motivates the following question.

Question 5.8. *Are "typical" elements of $\mathcal{A}_d$ non-monotonic? Does the set of functions $\varphi \in \mathcal{A}_d$ where φ is non-monotonic on every open subset of $(0,1)$ form a residual subset of $\mathcal{A}_d$?*

6 A VARIATIONAL FORMULA FOR THE ASSOUAD SPECTRUM OF GATZOURAS–LALLEY CARPETS

We now begin our derivation of the Assouad spectrum of Gatzouras–Lalley carpets. In this section, we establish a formula for the Assouad spectrum as the solution to a continuous but non-smooth and non-convex optimization problem over the compact space of pairs of probability vectors $\mathcal{P} \times \mathcal{P}$.

6.1 STATEMENT OF THE VARIATIONAL FORMULA AND PROOF STRATEGY

We begin by explicitly stating our variational formula. First, define a parameter change for $\boldsymbol{v} \in \mathcal{P}$ by

$$\phi(\theta, \boldsymbol{v}) = \frac{1/\theta - 1}{1 - 1/\Gamma(\boldsymbol{v})} \qquad \text{and} \qquad \phi(\theta) = \inf_{\boldsymbol{v} \in \mathcal{P}} \phi(\theta, \boldsymbol{v}) = \frac{1/\theta - 1}{1 - 1/\kappa_{\max}}.$$

We recall that $\Gamma(\boldsymbol{v})$ is the logarithmic eccentricity; see §3.2. Note that ϕ is strictly decreasing in θ and $\Gamma(\boldsymbol{v})$. We let

$$\begin{aligned} \Delta_{\text{thin}}(\theta) &= \{(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{P} \times \mathcal{P} : \phi(\theta, \boldsymbol{v}) \leq \Gamma(\boldsymbol{w})\}, \\ \Delta_{\text{thick}}(\theta) &= \{(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{P} \times \mathcal{P} : \phi(\theta, \boldsymbol{v}) \geq \Gamma(\boldsymbol{w})\}. \end{aligned} \tag{6.1}$$

Recall that $t_{\min} = \dim_{\mathrm{B}} K - \dim_{\mathrm{B}} \eta(K)$, and set

$$\begin{aligned} f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w}) &= \dim_{\mathrm{B}} \eta(K) + \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})}, \\ f_{\text{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w}) &= \dim_{\mathrm{B}} K + \frac{1}{\phi(\theta, \boldsymbol{v})} \left(\frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w})) - t_{\min}\chi_2(\boldsymbol{w})}{\chi_1(\boldsymbol{w})} \right). \end{aligned}$$

Finally, write

$$f(\theta, \boldsymbol{v}, \boldsymbol{w}) = \begin{cases} f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w}) & : (\boldsymbol{v}, \boldsymbol{w}) \in \Delta_{\text{thin}}(\theta), \\ f_{\text{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w}) & : (\boldsymbol{v}, \boldsymbol{w}) \in \Delta_{\text{thick}}(\theta). \end{cases} \tag{6.2}$$

It is straightforward to check that $f_{\text{thin}} = f_{\text{thick}}$ on $\Delta_{\text{thin}}(\theta) \cap \Delta_{\text{thick}}(\theta)$, so f is indeed well-defined and continuous.

Theorem 6.1. *Let K be a Gatzouras–Lalley carpet. Then for all $\theta \in (0,1)$,*

$$\dim_{\mathrm{A}}^{\theta} K = \max_{(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{P} \times \mathcal{P}} f(\theta, \boldsymbol{v}, \boldsymbol{w}).$$

We now summarise the main idea of the proof. In order to compute $\dim_{\mathrm{A}}^{\theta} K$, it suffices to consider approximate squares. Each approximate square is composed of cylinders, all of which have various heights. For our application there are two cases: either the cylinder is *thin*, i.e. it has height at most $\delta^{1/\theta}$, or the cylinder is *thick*, i.e. it has height at least $\delta^{1/\theta}$. This corresponds to the two cases in the variational formula stated in (6.2), and the covering strategy for each case is different.

1. In the *thin* case (which is handled in §6.3), we can simply group cylinders together (forming a *pseudo-cylinder*) until the heights are approximately $\delta^{1/\theta}$. We cover such cylinders simultaneously, and the count depends on $\dim_B \eta(K)$ (the $\dim_B K$ count does not appear).
2. In the *thick* case (which is handled in §6.4), we must cover each cylinder at a scale $\delta^{1/\theta}$, which is smaller than the height of the cylinder. We note that pseudo-cylinders of height $\delta^{1/\theta}$ and a certain width can be realized as images of approximate squares in K, so we can count the number of such pseudo-cylinders in terms of $\dim_B K$, and then cover each one at scale $\delta^{1/\theta}$ using $\dim_B \eta(K)$.

The key observation is that the covering strategy for the cylinder depends only on the digit frequency (or *type*) of certain indices corresponding to each case. In the thick case, these digit frequencies are precisely a pair $(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{P} \times \mathcal{P}$, where $\boldsymbol{v}$ is the digit frequency of the cylinder defining the approximate square, and $\boldsymbol{w}$ is the digit frequency of the smaller cylinder. In the thin case, $\boldsymbol{v}$ is the same but now $\boldsymbol{w}$ corresponds instead to the pseudo-cylinder. The precise definitions of these types are given in §6.2. The resolutions at which these bounds become relevant depend on the logarithmic eccentricity of both the original cylinder $\boldsymbol{v}$ and the composing cylinders $\boldsymbol{w}$. But now the crucial observation (based on the same strategy underlying the main results in [Kol23]) which allows us to complete this argument is a combination of the following two facts:

3. Since the scales $\delta^{1/\theta}$ and δ are exponentially separated, the set of possible types is much smaller than the number of cylinders with each type; and
4. The covering strategy within each type class is the same, with cost precisely corresponding to the functions f_{thin} and f_{thick}.

This means that the cost to cover the approximate square is dominated by the maximal type, yielding the variational formula for the Assouad spectrum.

6.2 SECTIONS FOR APPROXIMATE SQUARES AND TYPES

In this section, we make the notion of a *type* rigorous, which is the starting point for our covering strategy.

6.2.1 Defining the section $\mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})$

Recall the definition of a *section* from §2.1.

Fix $\theta \in (0,1)$ and an approximate square $Q = P(\mathtt{i}, \underline{\mathtt{j}})$ where $\mathtt{i} \in \mathcal{I}^*$ and $\underline{\mathtt{j}} \in \eta(\mathcal{I}^*)$. Recall that $\eta^{-1}(\underline{\mathtt{j}}) \subset \mathcal{I}^*$ is in bijection with the set of cylinders composing the approximate square Q (see (3.1)), and moreover is a complete section. We now define a section

$$\mathcal{B}_0^\theta(\mathtt{i}, \underline{\mathtt{j}}) = \{\mathtt{k} \in \mathcal{I}^* : \beta_{\mathtt{k},2} < \beta_{\mathtt{i},2}^{1/\theta - 1} \leq \beta_{\mathtt{k}^-,2} \quad \text{and} \quad \eta(\mathtt{k}) \preccurlyeq \underline{\mathtt{j}}\}.$$

In words, this set codes the cylinders intersecting Q with height $(\operatorname{diam} Q)^{1/\theta}$ and width greater than $\operatorname{diam} Q$. However, this set of cylinders may not entirely cover Q, so we add the missing cylinders to form

$$\mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}}) = \mathcal{B}_0^\theta(\mathtt{i}, \underline{\mathtt{j}}) \wedge \eta^{-1}(\underline{\mathtt{j}}).$$

Note that if $\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})$, there is a unique $\underline{\mathtt{l}}(\mathtt{k}) \in \eta(\mathcal{I}^*)$ such that $\underline{\mathtt{j}} = \eta(\mathtt{k})\underline{\mathtt{l}}(\mathtt{k})$. Thus the section $\mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})$ induces a decomposition of Q into wide pseudo-cylinders (recall §3.4 for the definitions):

$$Q = P(\mathtt{i}, \underline{\mathtt{j}}) = \bigcup_{\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})} P(\mathtt{ik}, \underline{\mathtt{l}}(\mathtt{k})).$$

By definition of $\mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})$, if $\underline{\mathtt{l}}(\mathtt{k}) \neq \varnothing$, then the height of the corresponding pseudo-cylinder $P(\mathtt{ik}, \underline{\mathtt{l}}(\mathtt{k}))$ is (up to a constant multiple) $\beta_{\mathtt{i},2}^{1/\theta}$.

Another equivalent way to think about the decomposition is as follows. First, observe that

$$\mathcal{B}^\theta_{\text{thick}}(\mathtt{i}, \underline{\mathtt{j}}) \coloneqq \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}}) \cap \eta^{-1}(\underline{\mathtt{j}}) = \{\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}}) : \underline{\mathtt{l}}(\mathtt{k}) = \varnothing\}.$$

Then set $\mathcal{B}^\theta_{\text{thin}}(\mathtt{i}, \underline{\mathtt{j}}) \coloneqq \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}}) \setminus \mathcal{B}^\theta_{\text{thick}}(\mathtt{i}, \underline{\mathtt{j}})$. The pseudo-cylinders corresponding to the elements of $\mathcal{B}^\theta_{\text{thin}}(\mathtt{i}, \underline{\mathtt{j}})$ are precisely formed by "grouping" the cylinders composing the approximate square Q which have height less than $\delta^{1/\theta}$ into pseudo-cylinders with height approximately $\delta^{1/\theta}$. Moreover, the pseudo-cylinders corresponding to the elements of $\mathcal{B}^\theta_{\text{thick}}(\mathtt{i}, \underline{\mathtt{j}})$ are in fact cylinders, and they have height greater than $\delta^{1/\theta}$.

6.2.2 Defining types and counting type classes

We first define the notion of the *type* corresponding to a word. Suppose $\mathtt{i} = (i_1, \ldots, i_n) \in \mathcal{I}^n$ for some $n \in \mathbb{N}$, and write

$$\boldsymbol{\xi}(\mathtt{i}) = (p_j)_{j \in \mathcal{I}} \qquad \text{where} \qquad p_j = \frac{\#\{k = 1, \ldots, n : i_k = j\}}{n}.$$

Of course, $\boldsymbol{\xi}(\mathtt{i}) \in \mathcal{P}$.

Now, fix $\theta \in (0,1)$, an approximate square $Q = P(\mathtt{i}, \underline{\mathtt{j}})$, and corresponding section $\mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})$ as defined in the previous section. Fix $\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})$ and define the *type* of $\mathtt{k}$ as the pair

$$\boldsymbol{\zeta}(\mathtt{k}) = (\boldsymbol{\xi}(\mathtt{i}), \boldsymbol{\xi}(\mathtt{k}))$$

and denote the set of all types

$$\mathcal{T}^\theta(\mathtt{i}, \underline{\mathtt{j}}) = \{\boldsymbol{\zeta}(\mathtt{k}) : \mathtt{k} \in \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}})\}.$$

Conversely, given a type $(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{T}^\theta(\mathtt{i}, \underline{\mathtt{j}})$, define the corresponding *type class* by

$$\mathcal{C}^\theta(\boldsymbol{v}, \boldsymbol{w}) = \mathcal{C}^\theta_{\mathtt{i},\underline{\mathtt{j}}}(\boldsymbol{v}, \boldsymbol{w}) \coloneqq \left\{\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i}, \underline{\mathtt{j}}) : \boldsymbol{\zeta}(\mathtt{k}) = (\boldsymbol{v}, \boldsymbol{w})\right\}.$$

An important observation is that if $\mathtt{k}, \mathtt{k}' \in \mathcal{C}^\theta_{\mathtt{i},\underline{\mathtt{j}}}(\boldsymbol{v}, \boldsymbol{w})$ have the same type, then

$$|\mathtt{k}| = |\mathtt{k}'|, \quad \eta(\mathtt{k}) = \eta(\mathtt{k}'), \quad \beta_{\mathtt{k},1} = \beta_{\mathtt{k}',1}, \quad \text{and} \quad \beta_{\mathtt{k},2} = \beta_{\mathtt{k}',2}.$$

We will require the following key estimates on the exponential growth rate of the number of possible types and the size of each type class.

Lemma 6.2. *Fix $\theta \in (0,1)$. Then the following hold:*

(i) We have
$$\frac{\log \#\mathcal{T}^\theta(\mathtt{i},\underline{\mathtt{j}})}{\log(1/\beta_{\mathtt{i},2})} = O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right).$$

(ii) Let $\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i},\underline{\mathtt{j}})$ have type $\boldsymbol{\zeta}(\mathtt{k}) = (\boldsymbol{v},\boldsymbol{w})$. Then
$$\frac{\log \#\mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})}{|\mathtt{k}|} = H(\boldsymbol{w}) - H(\eta(\boldsymbol{w})) + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right).$$

Proof. To see (i), there is a constant $M > 0$ so that $|\mathtt{k}| \leq M\cdot|\mathtt{i}|$ for all $\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i},\underline{\mathtt{j}})$. Thus by [DZ10, Lemma 2.1.2],
$$1 \leq \#\mathcal{T}^\theta(\mathtt{i},\underline{\mathtt{j}}) \leq \sum_{n=0}^{M|\mathtt{i}|} \#\{\boldsymbol{\xi}(\mathtt{k}) : \mathtt{k} \in \mathcal{I}^n\} \leq (M|\mathtt{i}|+1)^{\#\mathcal{I}+1}.$$

But $\log(1/\beta_{\mathtt{i},2}) \approx |\mathtt{i}| \approx |\mathtt{k}|$, from which the result follows.

Next, we prove (ii). Let $\mathtt{k} \in \mathcal{B}^\theta(\mathtt{i},\underline{\mathtt{j}})$ have type $\boldsymbol{\zeta}(\mathtt{k}) = (\boldsymbol{v},\boldsymbol{w})$, and recall that for all $\mathtt{k}' \in \mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})$,
$$m := |\mathtt{k}| = |\mathtt{k}'| \qquad \text{and} \qquad \eta(\mathtt{k}) = \eta(\mathtt{k}').$$

Now, by [DZ10, Lemma 2.1.8],
$$(m+1)^{-\#\mathcal{I}}\exp(m\cdot H(\boldsymbol{w})) \leq \#\{\mathtt{j} \in \mathcal{I}^m : \boldsymbol{\xi}(\mathtt{j}) = \boldsymbol{w}\} \leq \exp(m\cdot H(\boldsymbol{w})).$$

However, $\mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})$ consists only of those $\mathtt{j}$ for which $\eta(\mathtt{j}) = \eta(\mathtt{k})$. Moreover, the quantity
$$\#\{\mathtt{j} \in \mathcal{I}^m : \boldsymbol{\xi}(\mathtt{j}) = \boldsymbol{w} \text{ and } \eta(\mathtt{j}) = \underline{\mathtt{h}}\}$$
is independent of the choice of $\underline{\mathtt{h}} \in \eta(\mathcal{I}^m)$ as long as $\boldsymbol{\xi}(\underline{\mathtt{h}}) = \eta(\boldsymbol{w})$, and 0 otherwise. Thus again applying [DZ10, Lemma 2.1.8] to count the number of possible choices for $\underline{\mathtt{h}}$,
$$\begin{aligned}\frac{(m+1)^{-\#\mathcal{I}}\exp(m\cdot H(\boldsymbol{w}))}{\exp(m\cdot H(\eta(\boldsymbol{w})))} &\leq \#\{\mathtt{j} \in \mathcal{I}^m : \boldsymbol{\xi}(\mathtt{j}) = \boldsymbol{w} \text{ and } \eta(\mathtt{j}) = \eta(\mathtt{k})\} \\ &\leq \frac{\exp(m\cdot H(\boldsymbol{w}))}{(m+1)^{-\#\eta(\mathcal{I})}\exp(m\cdot H(\eta(\boldsymbol{w})))}.\end{aligned}$$

Taking logarithms, dividing by m, and recalling again that $|\mathtt{k}| \approx |\mathtt{i}|$ yields the desired result. □

6.3 COVERING THIN CYLINDERS

We now begin our covering arguments for the individual types, beginning with the thin cylinders.

We first require a covering lemma for wide pseudo-cylinders in terms of approximate squares. Recall that $\mathcal{S}$ denotes the set of all approximate squares, and $\mathcal{S}(r)$ denotes the approximate squares with diameter approximately r. Moreover, if $P(\mathtt{i},\underline{\mathtt{j}})$ is a wide pseudo-cylinder, recall that we can write it as a union of the approximate squares in the family

$$\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) \coloneqq \{Q \in \mathcal{S} : Q = P(\mathtt{i},\underline{\mathtt{k}}) \text{ for some } \underline{\mathtt{k}} \in \eta(\mathcal{I}^*) \text{ and } Q \subset P(\mathtt{i},\underline{\mathtt{j}})\}.$$

Note that since each $Q = P(\mathtt{i},\underline{\mathtt{k}})$ for some $\underline{\mathtt{k}}$, we have $\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) \subset \mathcal{S}(\beta_{\mathtt{i},2})$. We can obtain our main covering bound for thin cylinders.

Proposition 6.3. *Fix $\theta \in (0,1)$. Then for all approximate squares $Q = P(\mathtt{i},\underline{\mathtt{j}})$ and type classes $(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{\zeta}(\mathtt{k})$ for some $\mathtt{k} \in \mathcal{B}^\theta_{\mathrm{thin}}(\mathtt{i},\underline{\mathtt{j}})$, the following hold:*

(i) We have $\phi(\theta,\boldsymbol{v}) \le \Gamma(\boldsymbol{w}) + O\big(|\mathtt{i}|^{-1}\big)$.

(ii) Set $E = \bigcup_{\mathtt{k}' \in \mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})} P(\mathtt{i}\mathtt{k}', \mathtt{l}(\mathtt{k}'))$. Then

$$\frac{\log \#\{Q' \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q' \cap E \neq \varnothing\}}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} = f_{\mathrm{thin}}(\theta,\boldsymbol{v},\boldsymbol{w}) + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right).$$

Proof. To see (i), first observe by the definition of $\mathcal{B}^\theta_{\mathrm{thin}}$ that $\beta_{\mathtt{i},2}^{1/\theta - 1} \approx \beta_{\mathtt{k},2}$, and since Q is an approximate square, $\beta_{\mathtt{i},1}\beta_{\mathtt{k},1} \gtrsim \beta_{\mathtt{i},2}$. In particular,

$$\left(\frac{1}{\theta} - 1\right) \cdot |\mathtt{i}| \cdot \chi_2(\boldsymbol{v}) = |\mathtt{k}| \cdot \chi_2(\boldsymbol{w}) + O(1) \tag{6.3}$$

and

$$|\mathtt{i}| \cdot \chi_1(\boldsymbol{v}) + |\mathtt{k}| \cdot \chi_1(\boldsymbol{w}) \le |\mathtt{i}| \cdot \chi_2(\boldsymbol{v}) + O(1). \tag{6.4}$$

Substituting the value of $|\mathtt{k}|$ from (6.4) into (6.3) and dividing through by $|\mathtt{i}|$ yields

$$\frac{1}{\Gamma(\boldsymbol{v})} + \left(\frac{1}{\theta} - 1\right) \cdot \frac{1}{\Gamma(\boldsymbol{w})} \le 1 + O\big(|\mathtt{i}|^{-1}\big).$$

Since Γ takes values in a compact subinterval of $(1,\infty)$, this is a rearrangement of (i).

To see (ii), first note that for each $\mathtt{k}' \in \mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})$,

$$\beta_{\mathtt{i}\mathtt{k}',2} \approx \beta_{\mathtt{i},2}^{1/\theta} \qquad \text{and} \qquad \beta_{\mathtt{i}\mathtt{k}'\underline{\mathtt{l}}(\mathtt{k}'),1} = \beta_{\mathtt{i}\underline{\mathtt{j}},1} \approx \beta_{\mathtt{i},2}$$

and moreover by (6.3) and Lemma 6.2 (ii),

$$\frac{\log \#\mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} = \frac{\log \#\mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})}{|\mathtt{i}| \cdot (1/\theta - 1)\chi_2(\boldsymbol{v})} = \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right). \tag{6.5}$$

Thus by Lemma 3.3,

$$\begin{aligned}\#\{Q' \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q' \cap E \neq \varnothing\} &\approx \sum_{\mathtt{k}' \in \mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w})} \left(\frac{\beta_{\mathtt{i}\mathtt{k}'\underline{\mathtt{l}}(\mathtt{k}'),1}}{\beta_{\mathtt{i}\mathtt{k}',2}}\right)^{\dim_{\mathrm{B}} \eta(K)} \\ &\approx \#\mathcal{C}^\theta(\boldsymbol{v},\boldsymbol{w}) \cdot \beta_{\mathtt{i},2}^{(1-1/\theta)\cdot \dim_{\mathrm{B}} \eta(K)}.\end{aligned}$$

Taking logarithms, dividing through by $(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})$, and applying (6.5) completes the proof. ∎

6.4 COVERING THICK CYLINDERS

We now obtain our main bounds for thick cylinders. First, we require the following covering lemma for cylinders by approximate squares with height smaller than the height of the cylinder. This result is an immediate consequence of Lemma 3.4, with notation modified for ease of usage. Recall that $t_{\min} = \dim_{\mathrm{B}} K - \dim_{\mathrm{B}} \eta(K)$.

Lemma 6.4. *Suppose $\mathtt{i}, \mathtt{k}' \in \mathcal{I}^*$ are such that $\beta_{\mathtt{i},2}^{1/\theta-1} \lesssim \beta_{\mathtt{k}',2}$ and $\beta_{\mathtt{ik}',1} \approx \beta_{\mathtt{i},2}$. Then*

$$\#\{Q \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q \subset [\mathtt{ik}']\} \approx \left(\frac{1}{\beta_{\mathtt{i},2}}\right)^{(1/\theta-1)\cdot \dim_{\mathrm{B}} K} \cdot \left(\frac{1}{\beta_{\mathtt{k}',2}}\right)^{-t_{\min}}.$$

With this lemma in hand, we now obtain our main results concerning thick cylinders.

Proposition 6.5. *Fix $\theta \in (0,1)$. Then for all approximate squares $Q = P(\mathtt{i}, \underline{\mathtt{j}})$ and type classes $(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\zeta}(\mathtt{k})$ for some $\mathtt{k} \in \mathcal{B}^{\theta}_{\text{thick}}(\mathtt{i}, \underline{\mathtt{j}})$, the following hold:*

(i) We have $\phi(\theta, \boldsymbol{v}) \geq \Gamma(\boldsymbol{w}) + O\big(|\mathtt{i}|^{-1}\big)$.

(ii) Set $E = \bigcup_{\mathtt{k}' \in \mathcal{C}^{\theta}(\boldsymbol{v},\boldsymbol{w})} P(\mathtt{ik}', \underline{\mathtt{l}}(\mathtt{k}'))$. Then

$$\frac{\log \#\{Q' \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q' \cap E \neq \varnothing\}}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} = f_{\text{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w}) + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right).$$

Proof. First, by the definition of $\mathcal{B}^{\theta}_{\text{thick}}(\mathtt{i}, \underline{\mathtt{j}})$, $\beta_{\mathtt{i},2}^{1/\theta-1} \lesssim \beta_{\mathtt{k},2}$ and since Q is an approximate square, $\beta_{\mathtt{i},1}\beta_{\mathtt{k},1} \approx \beta_{\mathtt{i},2}$. Therefore the similar as used in the proof of Proposition 6.3 (i) yield (i).

Next, we see (ii). Let $\mathtt{k}' \in \mathcal{C}^{\theta}(\boldsymbol{v}, \boldsymbol{w})$ be arbitrary. Since $\beta_{\mathtt{k}',1} \approx \beta_{\mathtt{i},2}\beta_{\mathtt{i},1}^{-1}$, taking logarithms and rearranging gives that

$$|\mathtt{i}| \cdot (1/\theta - 1)\chi_2(\boldsymbol{v}) = |\mathtt{k}'| \cdot \chi_1(\boldsymbol{w})\phi(\theta, \boldsymbol{v}) + O(1).$$

In particular,

$$\frac{\log\left((1/\beta_{\mathtt{k}',2})^{-t_{\min}}\right)}{(1/\theta-1)\log(1/\beta_{\mathtt{i},2})} = -\frac{t_{\min}}{1/\theta - 1} \cdot \frac{|\mathtt{k}'| \cdot \chi_2(\boldsymbol{w})}{|\mathtt{i}| \cdot \chi_2(\boldsymbol{v})} = -\frac{t_{\min}\chi_2(\boldsymbol{w})}{\phi(\theta, \boldsymbol{v})\chi_1(\boldsymbol{w})} + O(|\mathtt{i}|^{-1}), \tag{6.6}$$

and by Lemma 6.2 (ii),

$$\frac{\log \#\mathcal{C}^{\theta}(\boldsymbol{v}, \boldsymbol{w})}{(1/\theta-1)\log(1/\beta_{\mathtt{i},2})} = \frac{\log \#\mathcal{C}^{\theta}(\boldsymbol{v}, \boldsymbol{w})}{(1/\theta - 1)|\mathtt{i}|\chi_2(\boldsymbol{v})} = \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\phi(\theta, \boldsymbol{v})\chi_1(\boldsymbol{w})} + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right). \tag{6.7}$$

Moreover, since $\mathtt{k}' \in \mathcal{B}^{\theta}_{\text{thick}}$, we have $\underline{\mathtt{l}}(\mathtt{k}') = \varnothing$ so we may apply Lemma 6.4 to each cylinder $P(\mathtt{ik}', \underline{\mathtt{l}}(\mathtt{k}')) = [\mathtt{ik}']$ giving

$$\begin{aligned}\#\{Q' \in \mathcal{S}(\beta_{\mathtt{i},2}^{(1/\theta)}) : Q' \cap E \neq \varnothing\} &\approx \left(\frac{1}{\beta_{\mathtt{i},2}}\right)^{(1/\theta-1)\cdot\dim_{\mathrm{B}} K} \cdot \sum_{\mathtt{k}' \in \mathcal{C}^{\theta}(\boldsymbol{v},\boldsymbol{w})} \left(\frac{1}{\beta_{\mathtt{k}',2}}\right)^{-t_{\min}} \\ &= \left(\frac{1}{\beta_{\mathtt{i},2}}\right)^{(1/\theta-1)\cdot\dim_{\mathrm{B}} K} \cdot \#\mathcal{C}^{\theta}(\boldsymbol{v},\boldsymbol{w}) \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{-t_{\min}}.\end{aligned}$$

Recalling the computations in (6.6) and (6.7), taking logarithms and dividing by $(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})$ gives

$$\begin{aligned}\frac{\log \#\{Q' \in \mathcal{S}(\beta_{\mathtt{i},2}^{(1/\theta)}) : Q' \cap E \neq \varnothing\}}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} \\ &= \dim_{\mathrm{B}} K + \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\phi(\theta, \boldsymbol{v})\chi_1(\boldsymbol{w})} + \frac{-t_{\min}\chi_2(\boldsymbol{w})}{\phi(\theta, \boldsymbol{v})\chi_1(\boldsymbol{w})} + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right) \\ &= f_{\mathrm{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w}) + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right)\end{aligned}$$

as claimed. □

6.5 COMBINING BOUNDS AND COMPLETING THE PROOF

To complete the proof, it simply remains to combine the results established in the previous sections.

Proof (of Theorem 6.1). Let $\theta \in (0,1)$ be fixed. First, there is a constant $M > 0$ such that any ball $B(x,r)$ can be covered by M approximate squares in $\mathcal{S}(r)$ and vice versa. Thus if we set for $n \in \mathbb{N}$

$$\mathcal{D}_n := \left\{P(\mathtt{i}, \underline{\mathtt{j}}) : P(\mathtt{i}, \underline{\mathtt{j}}) \text{ is an approximate square with } |\mathtt{i}| = n\right\},$$

then by (1.3)

$$\dim_{\mathrm{A}}^{\theta} K = \limsup_{n\to\infty} \max_{P(\mathtt{i},\underline{\mathtt{j}})\in\mathcal{D}_n} \frac{\log \#\{Q \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q \cap P(\mathtt{i}, \underline{\mathtt{j}}) \neq \varnothing\}}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})}. \tag{6.8}$$

Now, fix an approximate square $P(\mathtt{i}, \underline{\mathtt{j}})$, which we recall that we can decompose as

$$P(\mathtt{i}, \underline{\mathtt{j}}) = \bigcup_{\mathtt{k}\in\mathcal{B}^{\theta}(\mathtt{i},\underline{\mathtt{j}})} P(\mathtt{ik}, \underline{\mathtt{l}}(\mathtt{k})) = \bigcup_{(\boldsymbol{v},\boldsymbol{w})\in\mathcal{T}^{\theta}(\mathtt{i},\underline{\mathtt{j}})} \bigcup_{\mathtt{k}\in\mathcal{C}^{\theta}_{\mathtt{i},\underline{\mathtt{j}}}(\boldsymbol{v},\boldsymbol{w})} P(\mathtt{ik}, \underline{\mathtt{l}}(\mathtt{k})).$$

For $(\boldsymbol{v}, \boldsymbol{w}) \in \mathcal{T}^{\theta}(\mathtt{i}, \underline{\mathtt{j}})$, write

$$N(\boldsymbol{v}, \boldsymbol{w}) = \#\left\{Q \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q \cap P(\mathtt{ik}, \underline{\mathtt{l}}(\mathtt{k})) \neq \varnothing \text{ for some } \mathtt{k} \in \mathcal{C}^{\theta}_{\mathtt{i},\underline{\mathtt{j}}}(\boldsymbol{v}, \boldsymbol{w})\right\}.$$

Next, let $(\boldsymbol{v}_0, \boldsymbol{w}_0) = \boldsymbol{\zeta}(\mathtt{k})$ be chosen so that $N(\boldsymbol{v}_0, \boldsymbol{w}_0)$ is maximized. Suppose that $\mathtt{k} \in \mathcal{B}^{\theta}_{\mathrm{thin}}(\mathtt{i}, \underline{\mathtt{j}})$. Then by Lemma 6.2 (i) and Proposition 6.3 (ii),

$$\begin{aligned}\frac{\log \#\{Q \in \mathcal{S}(\beta_{\mathtt{i},2}^{1/\theta}) : Q \cap P(\mathtt{i}, \underline{\mathtt{j}}) \neq \varnothing\}}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} &= \frac{\log N(\boldsymbol{v}_0, \boldsymbol{w}_0)}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right) \\ &= f_{\mathrm{thin}}(\theta, \boldsymbol{v}_0, \boldsymbol{w}_0) + O\left(\frac{\log|\mathtt{i}|}{|\mathtt{i}|}\right).\end{aligned}$$

Moreover, since f and f_{thin} are continuous functions on the compact domain $\mathcal{P} \times \mathcal{P}$, they are in fact uniformly continuous. In particular, for all $\varepsilon > 0$ and all $|\mathtt{i}|$

sufficiently large depending on ε, by Proposition 6.3 (i), for all $\mathtt{k} \in \mathcal{B}^{\theta}_{\text{thin}}(\mathtt{i}, \underline{\mathtt{j}})$ with type $(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{\zeta}(\mathtt{k})$,

$$|f(\theta, \boldsymbol{v}, \boldsymbol{w}) - f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w})| \leq \varepsilon.$$

Of course, the analogous results hold with f_{thick} in place of f_{thin} for $\mathtt{k} \in \mathcal{B}^{\theta}_{\text{thick}}(\mathtt{i}, \underline{\mathtt{j}})$ by Proposition 6.5 (i) and (ii).

Combining these observations, for all $\varepsilon > 0$ and n sufficiently large depending on ε, with

$$\mathcal{T}^{\theta}_n \coloneqq \bigcup_{P(\mathtt{i},\underline{\mathtt{j}}) \in \mathcal{D}_n} \mathcal{T}^{\theta}(\mathtt{i}, \underline{\mathtt{j}})$$

denoting the set of all types at level n,

$$\left| \max_{P(\mathtt{i},\underline{\mathtt{j}}) \in \mathcal{D}_n} \frac{\log \#\{Q \in \mathcal{S}(\beta^{1/\theta}_{\mathtt{i},2}) : Q \cap P(\mathtt{i}, \underline{\mathtt{j}}) \neq \varnothing\}}{(1/\theta - 1)\log(1/\beta_{\mathtt{i},2})} - \max_{(\boldsymbol{v},\boldsymbol{w}) \in \mathcal{T}^{\theta}_n} f(\theta, \boldsymbol{v}, \boldsymbol{w}) \right| \leq \varepsilon.$$

But $\mathcal{T}^{\theta}_n$ converges to $\mathcal{P} \times \mathcal{P}$ in the Hausdorff metric as n diverges to infinity, and since $\varepsilon > 0$ was arbitrary, the result follows from (6.8). □

7 AN EXPLICIT FORMULA FOR THE ASSOUAD SPECTRUM OF GATZOURAS–LALLEY CARPETS

In this section, we obtain the explicit formula for the Assouad spectrum as stated in Theorem 7.1. Our main tool to solve the optimization problem which arises in the variational principle in Theorem 6.1 is the duality theory of constrained optimization. This approach is based on the general strategy outlined in [Rut23+, §3.1 and §4], and we recall the main components that we require in §7.2 and §7.3. With these preliminaries out of the way, in §7.4 we obtain explicit solutions to the optimization problems that will be relevant for our derivation of the main formula. The proof of the main result is then completed in §7.5.

7.1 ASSOUAD SPECTRUM OF GATZOURAS–LALLEY CARPETS

We can now state our main result, which is an explicit formula for the Assouad spectrum of a Gatzouras–Lalley carpet.

Let K be a Gatzouras–Lalley carpet. Let $t_{\min}$ denote the unique solution to

$$\sum_{\underline{j} \in \eta(\mathcal{I})} \sum_{i \in \eta^{-1}(\underline{j})} \beta^{\dim_{\mathrm{B}} \eta(K)}_{\underline{j},1} \beta^{t_{\min}}_{i,2} = 1,$$

or equivalently

$$\dim_{\mathrm{B}} K - \dim_{\mathrm{B}} \eta(K) = t_{\min}.$$

We denote this quantity by $t_{\min}$ because of the usage in (7.1) below. We interpret $t_{\min}$ as the "average" column dimension, weighted appropriately using the column widths $\beta_{\underline{j},1}$. Finally, for each $\underline{j} \in \eta(\mathcal{I})$, define $s_{\underline{j}}$ and $t_{\max}$ by the rules

$$\sum_{i \in \eta^{-1}(\underline{j})} \beta^{s_{\underline{j}}}_{i,2} = 1 \qquad \text{and} \qquad t_{\max} = \max_{\underline{j} \in \eta(\mathcal{I})} s_{\underline{j}}.$$

In other words, $s_{\underline{j}}$ is the dimension of the attractor of the IFS consisting only of the maps in column $\underline{j}$, and $t_{\max}$ is the maximal column dimension. Of course,

$$t_{\max} = \dim_{\mathrm{A}} K - \dim_{\mathrm{B}} \eta(K).$$

Note that

$$0 \leq \min_{\underline{j}\in\eta(\mathcal{I})} s_{\underline{j}} \leq t_{\min} \leq \max_{\underline{j}\in\eta(\mathcal{I})} s_{\underline{j}} = t_{\max} \leq 1.$$

Moreover, if either the second or third inequalities are equalities, all notions of dimension for K under consideration in this paper coincide (in fact, K is Ahlfors–David regular).

Now, for $\underline{j} \in \eta(\mathcal{I})$ and $t \in \mathbb{R}$, define

$$\psi_{\underline{j}}(t) = \frac{\log \sum_{i\in\eta^{-1}(\underline{j})} \beta_{i,2}^t}{\log \beta_{\underline{j},1}}.$$

Note that $\psi_{\underline{j}}$ is strictly increasing and concave with unique zero $s_{\underline{j}}$. The Assouad spectrum of K will be described in terms of the *column pressure*

$$\tau(t) = \begin{cases} \min_{\underline{j}\in\eta(\mathcal{I})} \psi_{\underline{j}}(t) & : t \in [t_{\min}, t_{\max}] \\ -\infty & : \text{ otherwise}. \end{cases} \tag{7.1}$$

Observe that τ is strictly increasing on $[t_{\min}, t_{\max}]$ and $\tau(t_{\max}) = 0$ is the unique zero of τ, with $\tau(t) < 0$ for $t_{\min} \leq t < t_{\max}$. Moreover, τ is a minimum of concave functions, and is therefore concave. We denote its concave conjugate by

$$\tau^*(\alpha) = \inf_{t\in\mathbb{R}}(t\alpha - \tau(t)).$$

Finally, define the parameter change

$$\phi(\theta) = \frac{1/\theta - 1}{1 - 1/\kappa_{\max}} \qquad \text{where} \qquad \kappa_{\max} = \max_{i\in\mathcal{I}} \frac{\log \beta_{i,2}}{\log \beta_{i,1}}.$$

Our main result in this section is the following formula for the Assouad spectrum of a Gatzouras–Lalley carpet.

Theorem 7.1. *Let $\{T_i\}_{i\in\mathcal{I}}$ be a Gatzouras–Lalley IFS with attractor K. Then for all $\theta \in (0,1)$,*

$$\dim_{\mathrm{A}}^\theta K = \dim_{\mathrm{B}} \eta(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)}.$$

In §8, we derive a more transparent formula for the Assouad spectrum, and deduce many interesting qualitative features of the spectrum from it.

7.2 THE GEOMETRY OF CONSTRAINED OPTIMIZATION

Suppose Δ is a compact Hausdorff topological space and suppose we are given a continuous function $u\colon \Delta \to \mathbb{R}$ and an upper semicontinuous function $v\colon \Delta \to \mathbb{R}$. We consider the constrained optimization

$$F(\alpha) = \max_{\boldsymbol{w}\in\Delta}\{v(\boldsymbol{w}) : u(\boldsymbol{w}) = \alpha\}$$

with corresponding unconstrained dual

$$T(t) = \min_{\boldsymbol{w}\in\Delta}\{t \cdot u(\boldsymbol{w}) - v(\boldsymbol{w})\}. \tag{7.2}$$

Here, the maximum over the empty set is $-\infty$. Note that both the maximum and minimum are attained on compact sets since v is upper semicontinuous. Of course, T is a concave function of t since it is an infimum of affine functions. On the other hand, F need not be concave.

In some sense, one can think of the function $T(t)$ as encoding the geometry of the Lagrange multiplier problem associated with $F(\alpha)$. However, this is only a motivating heuristic since in our abstract setup, there is no differentiable structure in sight.

Before we continue, let's recall some basic facts from convex optimization. For a more in-depth introduction, we refer the reader to [Roc70]. First, for a general function $g\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, we denote the *concave conjugate* by

$$g^*(\alpha) = \inf_{t\in\mathbb{R}}(t\alpha - g(t)).$$

Note that g^* is always concave, and g^{**} is the concave hull of g.

Now suppose moreover that $g\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a concave function. Then for $t \in \mathbb{R}$, we write $\partial g(t)$ to denote the *subdifferential* of g at t, i.e.

$$\partial g(t) = \{\alpha : \alpha(y - t) + g(t) \geq g(y) \text{ for all } y \in \mathbb{R}\}.$$

Equivalently,

$$g^*(\alpha) + g(t) \leq \alpha t \tag{7.3}$$

with equality if and only if $\alpha \in \partial g(t)$. We also let $\partial^- g(t)$ (resp. $\partial^+ g(t)$) denote the left (resp. right) derivative of g at t. Then $\partial g(t) = [\partial^+ g(t), \partial^- g(t)]$. In particular, g is differentiable at t if and only if $\partial g(t) = \{\alpha\}$, in which case $g'(t) = \alpha$. Since the subdifferentials form an ordered family of intervals of $\mathbb{R}$ which overlap only on their endpoints, there can be at most countably many points with non-singleton subdifferential: in particular, g is differentiable at all but countably many $t \in \mathbb{R}$.

Now, we say that a line

$$\ell(x) = a \cdot x + b$$

is a *supporting line* for T at t if $T(t) = \ell(t)$ and $\ell(x) \geq T(x)$ for all $x \in \mathbb{R}$. Equivalently, $\partial T(t)$ is precisely the set of possible slopes of supporting lines for T at t.

Now consider specifically the function T from (7.2). We say that T is *supported* at (t, α) for $t \in \mathbb{R}$ if there is a $\boldsymbol{w} \in \Delta$ so that $T(t) = t \cdot u(\boldsymbol{w}) - v(\boldsymbol{w})$ and $u(\boldsymbol{w}) = \alpha$. Equivalently, the line $t \mapsto t \cdot u(\boldsymbol{w}) - v(\boldsymbol{w})$ is a supporting line for T at t with slope α. Therefore, the problem of determining the values of α for which T is supported at (t, α) is precisely the problem of determining the slopes of supporting lines which appear from the minimization defining $T(t)$.

For the remainder of this section, we establish some elementary facts concerning the dual problems $T(t)$ and $F(\alpha)$. For each $t \in \mathbb{R}$, denote the set of minimizers by

$$M(t) = \{\boldsymbol{w} \in \Delta : t \cdot u(\boldsymbol{w}) - v(\boldsymbol{w}) = T(t)\}.$$

Since u is continuous and v is upper semicontinuous, $M(t)$ is a compact set for all t.

We begin with the following basic fact about supports of the function T.

Lemma 7.2. *For any $t \in \mathbb{R}$ and $\alpha \in \{\partial^- T(t), \partial^+ T(t)\}$, T is supported at (t, α). In particular,*

$$\min_{\boldsymbol{w} \in M(t)} u(\boldsymbol{w}) = \partial^+ T(t) \qquad \textit{and} \qquad \max_{\boldsymbol{w} \in M(t)} u(\boldsymbol{w}) = \partial^- T(t). \tag{7.4}$$

Proof. Let $(t_n)_{n=1}^\infty$ be a sequence converging to t monotonically from the left. For each $n \in \mathbb{N}$, write $T(t_n) = t_n \cdot u(\boldsymbol{w}_n) - v(\boldsymbol{w}_n)$ for some $\boldsymbol{w}_n \in \Delta$. By compactness of Δ, passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} \boldsymbol{w}_n = \boldsymbol{w}$. Next, observe that the line

$$\ell_n(x) \coloneqq x \cdot u(\boldsymbol{w}_n) - v(\boldsymbol{w}_n)$$

is a supporting line for T at t_n and therefore $u(\boldsymbol{w}_n) \in \partial T(t_n)$. Moreover, since $\lim_{n\to\infty} \partial^- T(t_n) = \partial^- T(t)$, it follows that

$$u(\boldsymbol{w}) = \lim_{n\to\infty} u(\boldsymbol{w}_n) = \partial T^-(t).$$

But then by upper semicontinuity of v and continuity of T,

$$T(t) \leq t \cdot u(\boldsymbol{w}) - v(\boldsymbol{w}) \leq \lim_{n\to\infty} (t \cdot u(\boldsymbol{w}_n) - v(\boldsymbol{w}_n)) = \lim_{n\to\infty} T(t_n) = T(t)$$

so that equality holds and T is supported at $(t, \partial^- T(t))$. The same argument works for $\partial^+ T(t)$, giving the result.

In particular, the above shows that there is a $\boldsymbol{w} \in M(t)$ so that $u(\boldsymbol{w}) = \partial^+ T(t)$, and similarly for $\partial^- T(t)$. Moreover, if T is supported at (t, α), then necessarily $\alpha \in \partial T(t)$, so (7.4) follows. □

Remark 7.3. Since u is continuous and Δ is compact, it follows that the left and right derivatives are uniformly bounded away from $\pm\infty$.

Using Lemma 7.2, we can now characterize concavity of the function F.

Proposition 7.4. *For any $\alpha \in \mathbb{R}$, $F(\alpha) \leq T^*(\alpha)$, and if T is supported at (t, α) for some $t \in \mathbb{R}$, then $F(\alpha) = t\alpha - T(t) = T^*(\alpha)$. Moreover, the following are equivalent:*

(i) For all $t \in \mathbb{R}$ and $\alpha \in \partial T(t)$, T is supported at (t, α).
(ii) $F(\alpha) = T^*(\alpha)$ for all $\alpha \in \mathbb{R}$.
(iii) F is a concave function.

Proof. First, let $\alpha \in \mathbb{R}$ be arbitrary. If $F(\alpha) = -\infty$, we are done; otherwise, since Δ is compact and u and v are continuous, there is some $\boldsymbol{w} \in \Delta$ so that $u(\boldsymbol{w}) = \alpha$ and $v(\boldsymbol{w}) = F(\alpha)$. Then for any $t \in \mathbb{R}$,

$$T(t) \leq t \cdot u(\boldsymbol{w}) - v(\boldsymbol{w}) = t\alpha - F(\alpha).$$

But t was arbitrary, so $F(\alpha) \leq T^*(\alpha)$.

Next, if T is supported at (t,α) for $t\in\mathbb{R}$, get $\boldsymbol{w}$ so that $u(\boldsymbol{w})=\alpha$ and $T(t)=t\alpha-v(\boldsymbol{w})$. But then

$$T^*(\alpha)\geq F(\alpha)\geq v(\boldsymbol{w})=t\alpha-T(t)\geq T^*(\alpha).$$

as claimed.

Now, (i) implies (ii) was proven above, and (ii) immediately implies (iii). It remains to verify that (iii) implies (i). Let $t\in\mathbb{R}$. Then if $\alpha\in\partial T(t)$, write $\alpha=\lambda\partial^+T(t)+(1-\lambda)\partial^-T(t)$ for some $\lambda\in[0,1]$. Then by concavity of F and Lemma 7.2,

$$\begin{aligned}T^*(\alpha)&\geq F(\alpha)\\&\geq \lambda F\big(\partial^+T(t)\big)+(1-\lambda)F\big(\partial^-T(t)\big)\\&=\lambda(t\partial^+T(t)-T(t))+(1-\lambda)(t\partial^-T(t)-T(t))\\&=t\alpha-T(t)\\&\geq T^*(\alpha).\end{aligned}$$

Thus all the inequalities are in fact equalities. In particular, taking $\boldsymbol{w}$ so that $u(\boldsymbol{w})=\alpha$ and $F(\alpha)=v(\boldsymbol{w})$, substituting this into the previous equation implies that $v(\boldsymbol{w})=t\cdot u(\boldsymbol{w})-T(t)$, as required. □

We conclude this section with two explicit situations in which we can establish the concave conjugate relationship. The first situation, even without any knowledge of the underlying optimization, occurs when $T(t)$ is differentiable.

Corollary 7.5. *If $t\in\mathbb{R}$ is such that $\alpha=T'(t)$ exists, then $F(\alpha)=T^*(\alpha)$. In particular, $T(t)=F^*(t)$ for all $t\in\mathbb{R}$.*

Proof. If $t\in\mathbb{R}$ and $T'(t)=\alpha$ exists, by Lemma 7.2, T is supported at t. Thus by Proposition 7.4, $F(\alpha)=T^*(\alpha)$. In particular, $T(t)=F^*(t)$ for all $t\in\mathbb{R}$ for which $T'(t)$ exists. But F^* and T are both continuous functions and $T'(t)$ exists for a dense set of $t\in\mathbb{R}$, so in fact $T(t)=F^*(t)$ for all $t\in\mathbb{R}$. □

As our second (and final) application, we can also use information about the structure of the set on which the optimization is attained to abstractly establish the concave conjugate relationship.

Corollary 7.6. *Suppose $t\in\mathbb{R}$ and $M(t)$ is connected. Then $F(\alpha)=T^*(\alpha)$ for all $\alpha\in\partial T(t)$. Moreover, if $M(t)$ is a singleton, then T is differentiable at t.*

Proof. By Lemma 7.2, T is supported at $(t,\partial^-T(t))$ and $(t,\partial^+T(t))$. Thus get $\boldsymbol{w}_-,\boldsymbol{w}_+\in M(t)$ so that

$$u(\boldsymbol{w}_-)=\partial^-T(t)\qquad\text{and}\qquad u(\boldsymbol{w}_+)=\partial^+T(t).$$

Since $M(t)$ is connected and u is continuous, $u(M(t))\subset\mathbb{R}$ is an interval containing $\partial^-T(t)$ and $\partial^+T(t)$, and therefore $\partial T(t)\subset u(M(t))$. In particular, for any $\alpha\in\partial T(t)$, there is a $\boldsymbol{w}\in M(t)$ so that $u(\boldsymbol{w})=\alpha$, so that T is supported at (t,α) and therefore $F(\alpha)=T^*(\alpha)$ by Proposition 7.4.

If moreover $M(t)$ is a singleton, then we must have $\boldsymbol{w}_-=\boldsymbol{w}_+$, forcing $\partial^-T(t)=\partial^+T(t)$ so that T is differentiable at t. □

7.3 ATTAINING THE OPTIMIZATION ON THE BOUNDARY

We now introduce the concept of an island-free function, and use this to establish some general conditions under which certain constrained optimization problems are attained on the boundary.

Definition 7.7. Let Δ be a topological space and let $f\colon \Delta \to \mathbb{R}$. We say that f is *island-free* if for all $t \in \mathbb{R}$ the upper level set

$$\{x \in \Delta : f(x) \geq t\}$$

is connected.

Here, the empty set is always connected. The following lemma provides some simple conditions to establish island-freeness.

Lemma 7.8. *The following hold.*

(i) Suppose Δ is a convex subset of a linear space. Suppose $f\colon \Delta \to \mathbb{R}$ is concave and $g\colon \Delta \to (0, \infty)$ is affine. Then f/g is island-free.

(ii) Suppose Δ_i is a topological space and $f_i\colon \Delta_i \to \mathbb{R}_{\geq 0}$ is island-free for each $i = 1, \ldots, m$. Equip $\Delta \coloneqq \Delta_1 \times \cdots \times \Delta_m$ with the product topology. Then $f\colon \Delta \to \mathbb{R}_{\geq 0}$ defined by

$$f(x_1, \ldots, x_m) = f_1(x_1) \cdots f_m(x_m)$$

is island-free.

Proof. To see (i), let f be concave and g affine. Then for each $t \in \mathbb{R}$, using positivity of g,

$$\{x \in \Delta : f(x)/g(x) \geq t\} = \{x \in \Delta : f(x) - tg(x) \geq 0\}.$$

This is a convex set (and in particular connected) since $f(x) - tg(x)$ is concave as a function of x.

Next, we see (ii). Let Δ_i and f_i be defined as in the statement of the lemma and let $t \in \mathbb{R}$ be fixed. Set

$$A(t) \coloneqq \{(x_1, \ldots, x_m) : f_1(x_1) \cdots f_m(x_m) \geq t\}.$$

We must show that $A(t)$ is connected. First, let $(y_1, \ldots, y_m) \in A(t)$ be arbitrary. Then

$$E(y_1, \ldots, y_m) \coloneqq \prod_{i=1}^{m} \{x_i : f_i(x_i) \geq f_i(y_i)\} \subset A(t)$$

is a connected set since each f_i is island-free. Moreover, if $(z_1, \ldots, z_m) \in A(t)$ is arbitrary, since the f_i are non-negative,

$$\varnothing \neq \prod_{i=1}^{m} \bigl\{x_i : f_i(x_i) \geq \max\{f_i(y_i), f_i(z_i)\}\bigr\} \subset E(y_1, \ldots, y_m) \cap E(z_1, \ldots, z_m).$$

Since the union of two intersecting connected sets is connected, any two elements of $A(t)$ are contained in a connected subset of $A(t)$. Thus $A(t)$ is connected. $\square$

The preceding lemma, along with concavity of $\boldsymbol{w} \mapsto H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))$ (which follows by the log-sum inequality, see for instance [CT06, §2.7]), yields the following result.

Corollary 7.9. *For all $\theta \in (0,1)$, the functions $(\boldsymbol{v}, \boldsymbol{w}) \mapsto f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w})$ and $(\boldsymbol{v}, \boldsymbol{w}) \mapsto f_{\text{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w})$ are island-free.*

Our main use for island-freeness appears in the following elementary lemma, which provides a partial description of the constrained maximizers of an island-free function in the case that the constrained maximum is not equal to the global maximum.

Lemma 7.10. *Let Δ be a compact Hausdorff topological space and let $f\colon \Delta \to \mathbb{R}$ be continuous and island-free. Then the set of global maximizers*

$$X \coloneqq \left\{ x \in \Delta : f(x) = \max_{y \in \Delta} f(y) \right\}$$

is a non-empty, compact and connected subset of Δ.

Moreover, suppose $E \subset \Delta$ is a non-empty compact set satisfying $X \cap (\Delta \setminus E) \neq \varnothing$. Then the set of constrained maximizers

$$X_E \coloneqq \left\{ x \in E : f(x) = \max_{y \in E} f(y) \right\},$$

intersects the topological boundary of E.

Proof. Since f is continuous, it is immediate that X is well-defined, non-empty and compact. Since f is island-free, X is connected.

Next, suppose $E \subset \Delta$ is compact and $X \cap (\Delta \setminus E) \neq \varnothing$. Consider the set

$$G \coloneqq \left\{ x \in \Delta : f(x) \geq \max_{y \in E} f(y) \right\}.$$

Note that $X_E = G \cap E$ so $G \cap E \neq \varnothing$, and $X \subset G$ so $G \cap (\Delta \setminus E) \neq \varnothing$. But f is island-free, so G is connected and therefore X_E intersects the boundary of E relative to Δ. □

7.4 FIBRED OPTIMIZERS

We now solve a few useful global minimization problems, and also see how these minimization problems encode the Assouad dimension of the Gatzouras–Lalley carpet K.

For $t \in \mathbb{R}$, write

$$g(t) = \min_{\boldsymbol{w} \in \mathcal{P}} \left\{ \frac{t\chi_2(\boldsymbol{w}) - [H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))]}{\chi_1(\boldsymbol{w})} \right\}. \tag{7.5}$$

We first prove in Proposition 7.11 that this definition of $g(t)$ coincides with the alternative definition given in (8.1). Moreover, we will give an explicit description of the set of minimizers.

For each $\underline{j} \in \eta(\mathcal{I})$, recall the definition of $\psi_{\underline{j}}$ from the introduction:

$$\psi_{\underline{j}}(t) = \frac{\log \sum_{i \in \eta^{-1}(\underline{j})} \beta_{i,2}^t}{\log \beta_{\underline{j},1}}.$$

Equivalently, for each $i \in \eta^{-1}(\underline{j})$,

$$\frac{\beta_{i,2}^t}{\sum_{\ell \in \eta^{-1}(\underline{j})} \beta_{\ell,2}^t} = \beta_{\underline{j},1}^{-\psi_{\underline{j}}(t)} \beta_{i,2}^t. \tag{7.6}$$

Therefore given $\boldsymbol{p} \in \eta(\mathcal{P})$, we may define a probability vector

$$\boldsymbol{z}(t, \boldsymbol{p}) := \left(p_{\eta(i)} \cdot \beta_{\eta(i),1}^{-\psi_{\eta(i)}(t)} \beta_{i,2}^t\right)_{i \in \mathcal{I}}.$$

The reason for introducing $\boldsymbol{z}(t, \boldsymbol{p})$ will become clear below.

We also recall the definition of the *Kullback–Leibler divergence* of two probability vectors $\boldsymbol{w}$ and $\boldsymbol{v}$ as

$$D_{\mathrm{KL}}(\boldsymbol{w} \,\|\, \boldsymbol{v}) = \sum_{i \in \mathcal{I}} w_i \log\left(\frac{w_i}{v_i}\right), \tag{7.7}$$

where we set $0 \log(0/v_i) = 0$ regardless of the value of v_i. In general, $D_{\mathrm{KL}}(\boldsymbol{w} \,\|\, \boldsymbol{v}) \geq 0$ with equality if and only if $\boldsymbol{w} = \boldsymbol{v}$.

Finally, we introduce some notation to denote the set of minimizers for $g(t)$. For each $t \in \mathbb{R}$, let

$$\mathcal{J}(t) = \left\{\underline{j} \in \eta(\mathcal{I}) : \psi_{\underline{j}}(t) = \min_{\underline{i} \in \eta(\mathcal{I})} \psi_{\underline{i}}(t)\right\}.$$

We then write

$$\mathcal{R}(t) = \{\boldsymbol{p} \in \eta(\mathcal{P}) : \operatorname{supp} \boldsymbol{p} \subset \mathcal{J}(t)\} \qquad \text{and} \qquad \mathcal{Z}(t) = \{\boldsymbol{z}(t, \boldsymbol{p}) : \boldsymbol{p} \in \mathcal{R}(t)\}.$$

We now have the following formula for the function $g(t)$.

Proposition 7.11. *For each $t \in \mathbb{R}$,*

$$g(t) = \min\{\psi_{\underline{j}}(t) : \underline{j} \in \eta(\mathcal{I})\}.$$

Moreover, the set of minimizing vectors

$$\left\{\boldsymbol{w} \in \mathcal{P} : \frac{t\chi_2(\boldsymbol{w}) - [H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))]}{\chi_1(\boldsymbol{w})} = g(t)\right\} = \mathcal{Z}(t).$$

Proof. Suppose $\boldsymbol{w} \in \mathcal{P}$ is arbitrary and let $\eta(\boldsymbol{w}) = \boldsymbol{p}$. We then compute

$$\begin{aligned}
0 &\leq D_{\mathrm{KL}}(\boldsymbol{w} \,\|\, \boldsymbol{z}(t, \boldsymbol{p})) \\
&= \sum_{\underline{j} \in \eta(\mathcal{I})} \sum_{i \in \eta^{-1}(\underline{j})} w_i \log\left(w_i p_{\underline{j}}^{-1} \beta_{\underline{j},1}^{\psi_{\underline{j}}(t)} \beta_{i,2}^{-t}\right) \\
&= -H(\boldsymbol{w}) + H(\eta(\boldsymbol{w})) + \sum_{\underline{j} \in \eta(\mathcal{I})} p_{\underline{j}} \psi_{\underline{j}}(t) \log \beta_{\underline{j},1} + t\chi_2(\boldsymbol{w}).
\end{aligned}$$

Rearranging, we obtain that

$$\begin{aligned}\frac{t\chi_2(\boldsymbol{w})-[H(\boldsymbol{w})-H(\eta(\boldsymbol{w}))]}{\chi_1(\boldsymbol{w})} &\geq \frac{-\sum_{\underline{j}\in\eta(\mathcal{I})} p_{\underline{j}}\psi_{\underline{j}}(t)\log\beta_{\underline{j},1}}{\chi_1(\boldsymbol{w})}\\ &\geq \min\{\psi_{\underline{j}}(t):\underline{j}\in\eta(\mathcal{I})\}.\end{aligned}$$

Observe that the second equality holds if and only if $\boldsymbol{p}\in\mathcal{R}(t)$. Moreover, for all $\boldsymbol{p}\in\eta(\mathcal{P})$, the first equality holds if and only if $\boldsymbol{w}=\boldsymbol{z}(t,\boldsymbol{p})$. Thus the desired result follows. □

To conclude this section, we show how the function $g(t)$ encodes the Assouad dimension. Recall from the introduction that for $\underline{j}\in\eta(\mathcal{I})$, $s_{\underline{j}}$ is the unique solution to the equation

$$\sum_{i\in\eta^{-1}(\underline{j})}\beta_{i,2}^{s_{\underline{j}}}=1.$$

Moreover, we recall the definition of $t_{\max}$, which by the main result of [Mac11] satisfies

$$t_{\max}=\max_{\underline{j}\in\eta(\mathcal{I})} s_{\underline{j}}=\dim_{\mathrm{A}} K-\dim_{\mathrm{B}}\eta(K).$$

It turns out that $t_{\max}$ is precisely the unique zero of g.

Lemma 7.12. *We have*

$$\max_{\boldsymbol{w}\in\mathcal{P}}\left\{\frac{H(\boldsymbol{w})-H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})}\right\}=t_{\max}.$$

Moreover, $t_{\max}$ is the unique zero of g and the set of maximizing probability vectors is $\mathcal{Z}(t_{\max})$.

Proof. Write

$$\mathcal{R}_{\max}=\Big\{\boldsymbol{p}\in\eta(\mathcal{P}):\operatorname{supp}\boldsymbol{p}\subset\{\underline{j}\in\eta(\mathcal{I}):s_{\underline{j}}=t_{\max}\}\Big\}.$$

Now suppose $\boldsymbol{w}\in\mathcal{P}$ is arbitrary. Write $\boldsymbol{p}=\eta(\boldsymbol{w})$ and $\boldsymbol{v}=\big(p_{\eta(i)}\beta_{i,2}^{s_{\eta(i)}}\big)_{i\in\mathcal{I}}$. Then

$$\begin{aligned}0&\leq D_{\mathrm{KL}}(\boldsymbol{w}\,\|\,\boldsymbol{v})\\ &=\sum_{\underline{j}\in\eta(\mathcal{I})}\sum_{i\in\eta^{-1}(\underline{j})} w_i\log(w_i p_{\underline{j}}^{-1}\beta_{i,2}^{-s_{\underline{j}}})\\ &=-H(\boldsymbol{w})+H(\eta(\boldsymbol{w}))-\sum_{\underline{j}\in\eta(\mathcal{I})} s_{\underline{j}}\sum_{i\in\eta^{-1}(\underline{j})} w_i\log\beta_{i,2}\\ &\leq -H(\boldsymbol{w})+H(\eta(\boldsymbol{w}))+t_{\max}\chi_2(\boldsymbol{w}).\end{aligned}$$

The second inequality is an equality if and only if $\operatorname{supp}\boldsymbol{p}\subset\mathcal{R}_{\max}$, in which case the first inequality is an equality if and only if $\boldsymbol{w}=\boldsymbol{v}$. But if $\boldsymbol{w}=\boldsymbol{v}$ (and $\operatorname{supp}\boldsymbol{p}\subset\mathcal{R}_{\max}$), then $\boldsymbol{w}=\boldsymbol{z}(t_{\max},\boldsymbol{p})$ by the definition of $\boldsymbol{v}$.

We have shown that

$$\frac{t_{\max}\chi_2(\boldsymbol{w}) - [H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))]}{\chi_1(\boldsymbol{w})} \geq 0$$

with equality if and only if $\boldsymbol{w} = \boldsymbol{z}(t_{\max}, \boldsymbol{p})$ for some $\boldsymbol{p} \in \mathcal{R}_{\max}$. But this is precisely the same minimization as the definition of $g(t_{\max})$, so the set of maximizing probability vectors must be $\mathcal{Z}(t_{\max})$ by Proposition 7.11. (Alternatively, observe that $\mathcal{R}_{\max} = \mathcal{R}(t_{\max})$.)

To see that $t_{\max}$ is unique with this property, since the left and right derivatives of $g(t)$ lie in the compact interval $\Gamma(\mathcal{P}) \subset (1, \infty)$ (see Remark 7.3), it follows that $g(t)$ is strictly increasing. □

7.5 Solving the variational formula

Finally, we can establish an explicit formula for the Assouad spectrum of K by solving the maximization process underlying the variational formula. This proof can be subdivided effectively into three parts:

1. First, solve the unconstrained maximization problems corresponding to the functions f_{thick} and f_{thin} and determine the values of θ for which the respective unconstrained and constrained maxima agree (this is Lemma 7.13).
2. Next, recalling that $f_{\text{thick}} = f_{\text{thin}}$ on $\Delta_{\text{thick}}(\theta) \cap \Delta_{\text{thin}}(\theta)$, solve the corresponding boundary maximization (this is Lemma 7.14).
3. Finally, using island-freeness and Lemma 7.10, reduce the general maximization to the above cases.

With this outline in mind, we begin the proof.

Recall that we defined

$$\theta_{\min} = \phi^{-1}\big(\partial^+ g(t_{\min})\big) \qquad \text{and} \qquad \theta_{\max} = \phi^{-1}\big(\partial^- g(t_{\max})\big),$$

and recall the definitions of $\Delta_{\text{thin}}(\theta)$ and $\Delta_{\text{thick}}(\theta)$ from (6.1). We begin by applying the results from §7.4 to solve the unconstrained maximization problems corresponding to f_{thick} and f_{thin}.

Lemma 7.13. *Let $\theta \in (0,1)$ be arbitrary. Given $t \in \mathbb{R}$, let $P_t\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ denote the function defined by $P_t(t) = g(t)$ and $P_t(x) = -\infty$ for $x \neq t$.*

(i) We have

$$\max_{(\boldsymbol{v},\boldsymbol{w}) \in \mathcal{P}\times\mathcal{P}} f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w}) = \dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + \frac{P^*_{t_{\max}}(\phi(\theta))}{\phi(\theta)}$$

and the set of probability vectors for which the maximum is attained is given by

$$E_{\text{thin}} := \mathcal{P} \times \mathcal{Z}(t_{\max}).$$

In particular, $E_{\text{thin}} \cap \Delta_{\text{thin}}(\theta) \neq \varnothing$ if and only if $\theta \geq \theta_{\max}$.

(ii) We have

$$\max_{(\boldsymbol{v},\boldsymbol{w}) \in \mathcal{P}\times\mathcal{P}} f_{\text{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w}) = \dim_{\mathrm{B}} K - \frac{g(t_{\min})}{\phi(\theta)} = \dim_{\mathrm{B}} \eta(K) + \frac{P^*_{t_{\min}}(\phi(\theta))}{\phi(\theta)},$$

and the set of probability vectors for which the maximum is attained is given by

$$E_{\text{thick}} \coloneqq \Gamma^{-1}(\kappa_{\max}) \times \mathcal{Z}(t_{\min}).$$

In particular, $E_{\text{thick}} \cap \Delta_{\text{thick}}(\theta) \neq \varnothing$ if and only if $\theta \leq \theta_{\min}$.

Proof. To see (i), by inspecting the definition of f_{thin}, the first equality and the formula for E_{thin} are immediate consequences of Lemma 7.12, recalling that $\dim_A K = t_{\max} + \dim_B \eta(K)$. To see the second equality, since $\phi(\theta) \in \partial P_{t_{\max}}(t_{\max})$ and since $P_{t_{\max}}(t_{\max}) = 0$ by Proposition 7.4 and Lemma 7.12,

$$\dim_B \eta(K) + \frac{P^*_{t_{\max}}(\phi(\theta))}{\phi(\theta)} = \dim_B \eta(K) + \frac{t_{\max} \cdot \phi(\theta) - P_{t_{\max}}(t_{\max})}{\phi(\theta)} = \dim_A K$$

by the definition of $t_{\max}$. Finally, recalling that $\theta_{\max}$ is defined from (8.2) and using Proposition 7.11 and Lemma 7.2,

$$\max_{\boldsymbol{w} \in \mathcal{Z}(t_{\max})} \Gamma(\boldsymbol{w}) = \partial^- g(t_{\max}) = \phi(\theta_{\max}).$$

Recall that $\phi(\theta) = \inf_{\boldsymbol{v} \in \mathcal{P}} \phi(\theta, \boldsymbol{v})$, and ϕ is decreasing in θ. Therefore $E_{\text{thin}} \cap \Delta_{\text{thin}}(\theta) \neq \varnothing$ if and only if $\theta \geq \theta_{\max}$.

To see (ii), the maximization in $\boldsymbol{v}$ is clearly attained by any $\boldsymbol{v}$ which is supported on indices for which the logarithmic eccentricity Γ is as large as possible. The first equality and the formula for E_{thick} then follows from Proposition 7.11 since the remaining term in $f_{\text{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w})$ is precisely the negative of the reciprocal of the objective function defining $g(t_{\min})$. To see the second inequality, since $\phi(\theta) \in \partial P_{t_{\min}}(t_{\min})$,

$$P_{t_{\min}}(t_{\min}) + P^*_{t_{\min}}(\phi(\theta)) = t_{\min} \cdot \phi(\theta).$$

But recall that $t_{\min} = \dim_B K - \dim_B \eta(K)$, so rearranging gives

$$\dim_B \eta(K) + \frac{P^*_{t_{\min}}(\phi(\theta))}{\phi(\theta)} = \dim_B K - \frac{g(t_{\min})}{\phi(\theta)}$$

as claimed. Finally, let $(\boldsymbol{v}, \boldsymbol{w}) \in \Gamma^{-1}(\kappa_{\max}) \times \mathcal{Z}(t_{\min})$ be arbitrary. By Proposition 7.11 and Lemma 7.2,

$$\min_{\boldsymbol{w} \in \mathcal{Z}(t_{\min})} \Gamma(\boldsymbol{w}) = \partial^+ g(t_{\min}) = \phi(\theta_{\min}).$$

Now, $\phi(\theta, \boldsymbol{v}) = \phi(\theta)$, so $E_{\text{thick}} \cap \Delta_{\text{thick}}(\theta) \neq \varnothing$ if and only if $\theta \leq \theta_{\min}$. $\square$

Next, we solve the maximization problem constrained to the boundary. For notational simplicity, we write

$$\Delta(\theta) = \Delta_{\text{thin}}(\theta) \cap \Delta_{\text{thick}}(\theta).$$

This is the topological boundary of $\Delta_{\text{thin}}(\theta)$ and $\Delta_{\text{thick}}(\theta)$ (relative to $\mathcal{P} \times \mathcal{P}$). We also recall that $f = f_{\text{thin}} = f_{\text{thick}}$ on $\Delta(\theta)$.

Lemma 7.14. *Suppose* $(\boldsymbol{v}, \boldsymbol{w}) \in \Delta(\theta)$*. If* $\theta < \theta_{\max}$*, then*

$$\begin{aligned}\max_{(\boldsymbol{v},\boldsymbol{w})\in\Delta(\theta)} f(\theta, \boldsymbol{v}, \boldsymbol{w}) &= \max_{(\boldsymbol{v},\boldsymbol{w})\in\Delta_{\text{thin}}(\theta)} f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w}) \\ &= \dim_{\mathrm{B}} \eta(K) + \frac{g^*(\phi(\theta))}{\phi(\theta)}.\end{aligned}$$

Proof. In order to prove the desired formula on the boundary, first introduce the auxiliary function

$$F(\alpha) = \max_{\boldsymbol{w}\in\mathcal{P}} \left\{ \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_1(\boldsymbol{w})} : \Gamma(\boldsymbol{w}) = \alpha \right\}.$$

Note that $F(\alpha)$ is the constrained optimization problem corresponding to the unconstrained problem $g(t)$. Moreover, by Proposition 7.11, the minimization defining $g(t)$ is attained precisely on the set $\mathcal{Z}(t)$, which is a connected set. (Alternatively, connectedness of the set of minimizers can be indirectly observed since the negative of the objective function defining $g(t)$ is island-free by Lemma 7.8.) Thus applying Corollary 7.6, $F(\alpha) = g^*(\alpha)$ for all $\alpha \in \Gamma(\mathcal{Z}(t))$.

We now obtain the desired bounds. First,

$$\begin{aligned}&\max_{(\boldsymbol{v},\boldsymbol{w})\in\Delta(\theta)} f(\theta, \boldsymbol{v}, \boldsymbol{w}) - \dim_{\mathrm{B}} \eta(K) \\ &\qquad\leq \max_{(\boldsymbol{v},\boldsymbol{w})\in\Delta_{\text{thin}}(\theta)} f_{\text{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w}) - \dim_{\mathrm{B}} \eta(K) \\ &\qquad= \max_{(\boldsymbol{v},\boldsymbol{w})\in\mathcal{P}\times\mathcal{P}} \left\{ \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} : \Gamma(\boldsymbol{w}) \geq \phi(\theta, \boldsymbol{v}) \right\} \\ &\qquad\leq \max_{\boldsymbol{w}\in\mathcal{P}} \left\{ \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} : \Gamma(\boldsymbol{w}) \geq \phi(\theta) \right\}\end{aligned} \tag{7.8}$$

since $\inf_{\boldsymbol{v}\in\mathcal{P}} \phi(\theta, \boldsymbol{v}) = \phi(\theta)$. Now recalling Lemma 7.13 (i), the unconstrained optimization

$$\max_{\boldsymbol{w}\in\mathcal{P}} \left\{ \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} \right\}$$

is attained precisely on the set $\mathcal{Z}(t_{\max})$ and, since $\theta < \theta_{\max}$, $\Gamma(\boldsymbol{w}) < \phi(\theta)$ for all $\boldsymbol{w} \in \mathcal{Z}(t_{\max})$. Thus by Lemma 7.10 (island-freeness again follows by Lemma 7.8), the maximization is attained on the boundary $\{\boldsymbol{w} \in \mathcal{P} : \Gamma(\boldsymbol{w}) = \phi(\theta)\}$. But

$$\Gamma^{-1}(\kappa_{\max}) \times \{\boldsymbol{w} \in \mathcal{P} : \Gamma(\boldsymbol{w}) = \phi(\theta)\} \subset \Delta(\theta),$$

so in fact there is equality throughout (7.8) and

$$\begin{aligned}&\max_{(\boldsymbol{v},\boldsymbol{w})\in\Delta(\theta)} f(\theta, \boldsymbol{v}, \boldsymbol{w}) - \dim_{\mathrm{B}} \eta(K) \\ &\qquad= \max_{\boldsymbol{w}\in\mathcal{P}} \left\{ \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} : \Gamma(\boldsymbol{w}) = \phi(\theta) \right\} \\ &\qquad= \max_{\boldsymbol{w}\in\mathcal{P}} \left\{ \frac{1}{\phi(\theta)} \cdot \frac{H(\boldsymbol{w}) - H(\eta(\boldsymbol{w}))}{\chi_1(\boldsymbol{w})} : \Gamma(\boldsymbol{w}) = \phi(\theta) \right\} \\ &\qquad= \frac{F(\phi(\theta))}{\phi(\theta)}.\end{aligned}$$

In the second equality, we used the substitution $\chi_2(\boldsymbol{w}) = \Gamma(\boldsymbol{w})\chi_1(\boldsymbol{w})$. Recalling that $F(\phi(\theta)) = g^*(\phi(\theta))$ yields the claimed formula. $\square$

We can finally complete the proof of our main result.

Restatement (of Theorem 7.1). *Let K be a Gatzouras–Lalley carpet and let*

$$\tau(t) = \begin{cases} g(t) & : t \in [t_{\min}, t_{\max}] \\ -\infty & : \textit{otherwise}. \end{cases}$$

Then for all $\theta \in (0,1)$,

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} \eta(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)}.$$

Proof. Let $0 < \theta < 1$. Recall that f_{thin} and f_{thick} are island-free by Corollary 7.9. Moreover, recall from Theorem 6.1 that

$$\dim_{\mathrm{A}}^{\theta} K = \max_{(\boldsymbol{v},\boldsymbol{w}) \in \mathcal{P}\times\mathcal{P}} f(\theta, \boldsymbol{v}, \boldsymbol{w})$$

where $f = f_{\mathrm{thin}}$ on $\Delta_{\mathrm{thin}}(\theta)$ and $f = f_{\mathrm{thick}}$ on $\Delta_{\mathrm{thick}}(\theta)$.

If $\theta \le \theta_{\min}$, then by Lemma 7.13, f_{thin} does not attain its global maximum but f_{thick} does. Therefore by Lemma 7.10 as well as the formula in Lemma 7.13 (ii),

$$\max_{(\boldsymbol{v},\boldsymbol{w}) \in \Delta_{\mathrm{thick}}(\theta)} f_{\mathrm{thick}}(\theta, \boldsymbol{v}, \boldsymbol{w}) = \dim_{\mathrm{B}} \eta(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)} \ge \max_{(\boldsymbol{v},\boldsymbol{w}) \in \Delta_{\mathrm{thin}}(\theta)} f_{\mathrm{thin}}(\theta, \boldsymbol{v}, \boldsymbol{w})$$

yielding the desired formula. The analogous argument provides the result for $\theta \ge \theta_{\max}$.

Otherwise, suppose $\theta_{\min} < \theta < \theta_{\max}$. By Lemma 7.13, the unconstrained maxima are not attained for either f_{thin} or f_{thick}. Therefore by Lemma 7.10,

$$\dim_{\mathrm{A}}^{\theta} K = \max_{(\boldsymbol{v},\boldsymbol{w}) \in \Delta(\theta)} f(\theta, \boldsymbol{v}, \boldsymbol{w}) = \dim_{\mathrm{B}} \eta(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)},$$

where in the last equality we applied Lemma 7.14 and used the fact that $\theta_{\min} < \theta < \theta_{\max}$ so $g^*(\phi(\theta)) = \tau^*(\phi(\theta))$. $\square$

Remark 7.15. The choice of $\boldsymbol{v}_0$ to maximize $f(\theta, \boldsymbol{v}_0, \boldsymbol{w})$ does not depend on the choice of θ; we can simply take $\boldsymbol{v}_0$ to be any probability vector fully supported on the indices i for which $(\log \beta_{i,2})/(\log \beta_{i,1}) = \kappa_{\max}$. This is the reason for the appearance of $\kappa_{\max}$ in the parameter change $\phi(\theta)$.

The dependence of $\boldsymbol{w}$ on θ is more complex. If θ is such that $\phi(\theta) = \partial^+ g(t(\theta))$ (resp. $\partial^- g(\theta)$) for some value $t(\theta) \in (t_{\min}, t_{\max})$, then $\eta(\boldsymbol{z}(\theta))$ can be taken to be supported on a single column. In this case, by the formula for the optimization given in Proposition 7.11, $\boldsymbol{z}(\theta)$ can be given more explicitly as $\boldsymbol{z}(\theta) = \boldsymbol{z}(t(\theta), \boldsymbol{\delta}_{\underline{j}})$, where $\boldsymbol{\delta}_{\underline{j}}$ is the probability vector fully supported on a column satisfying $g(t) = \psi_{\underline{j}}(t)$ for some $\varepsilon > 0$ and $t \in (t(\theta), t(\theta) + \varepsilon)$ (resp. $(t(\theta) - \varepsilon, t(\theta))$). Otherwise, if $\theta < \theta_{\min}$ or $\theta > \theta_{\max}$, then by Lemma 7.13 we can again take $\boldsymbol{z}(\theta)$ to be an explicit vector supported on a single column. Finally, if $\partial g^-(t(\theta)) < \phi(\theta) < \partial g^+(t(\theta))$ for some $t(\theta)$, then $\boldsymbol{z}(\theta)$ can be taken to be a convex combination of the optimizing vectors corresponding to $\partial g^-(t(\theta))$ and $\partial g^+(t(\theta))$, so $\eta(\boldsymbol{z}(\theta))$ can be taken to be supported on at most two columns.

8 Qualitative features of the Assouad spectrum of a Gatzouras–Lalley carpet

Throughout this section, we recall the notation introduced in §7, in particular the notation used in the statement of Theorem 7.1.

8.1 An alternative formula for the Assouad spectrum

We begin by introducing some notation to decompose the function τ in a meaningful way.

First, we distinguish a particular type of column.

Definition 8.1. We say that a column $\underline{j} \in \eta(\mathcal{I})$ is *homogeneous* if there is a $\beta_{\underline{j},2}$ so that $\beta_{i,2} = \beta_{\underline{j},2}$ for all $i \in \eta^{-1}(\underline{j})$.

For example, when K is a Bedford–McMullen carpet, every column is homogeneous.

Homogeneity is characterized by the following elementary lemma, the proof of which follows directly from the definition of $\psi_{\underline{j}}$.

Lemma 8.2. *A column $\underline{j} \in \eta(\mathcal{I})$ is homogeneous if and only if $\psi_{\underline{j}}$ is affine. Moreover:*

(i) If $\psi_{\underline{j}}$ is affine, then

$$\psi_{\underline{j}}(t) = t \cdot \frac{\log \beta_{\underline{j},2}}{\log \beta_{\underline{j},1}} + \frac{\log \#\eta^{-1}(\underline{j})}{\log \beta_{\underline{j},1}} = \kappa_{\underline{j}} \cdot (t - s_{\underline{j}}).$$

(ii) If $\psi_{\underline{j}}$ is not affine, then it is strictly concave.

Now write

$$g(t) = \min_{\underline{j}\in\eta(\mathcal{I})} \psi_{\underline{j}}(t) \tag{8.1}$$

which is a minimum of analytic functions (note the close relationship with the function τ). By analyticity, for all $\underline{i}, \underline{j} \in \eta(\mathcal{I})$, either $\psi_{\underline{i}} = \psi_{\underline{j}}$ or the set $\{t \in \mathbb{R} : \psi_{\underline{i}}(t) = \psi_{\underline{j}}(t)\}$ is finite. Thus there exists a partition

$$t_{\min} = t_0 < t_1 < \cdots < t_m = t_{\max}$$

of the interval $[t_{\min}, t_{\max}]$, with corresponding parts $I_n = [t_{n-1}, t_n]$, such that for each $n \in \mathbb{N}$, there is a $\underline{j}_n$ so that

$$g(t) = \begin{cases} \psi_{\underline{j}_1}(t) & : t \in I_1 \\ \vdots & \\ \psi_{\underline{j}_m}(t) & : t \in I_m \end{cases}$$

and moreover for all $1 \leq n \leq m-1$, $\psi_{\underline{j}_n} \neq \psi_{\underline{j}_{n+1}}$. The latter property ensures that the partition $(I_n)_{n=1}^m$ is uniquely determined. We refer to this partition as the *spectrum partition* associated with the IFS $\{T_i\}_{i\in\mathcal{I}}$. We associate with the spectrum partition the following additional information, all of which depends only on the underlying IFS.

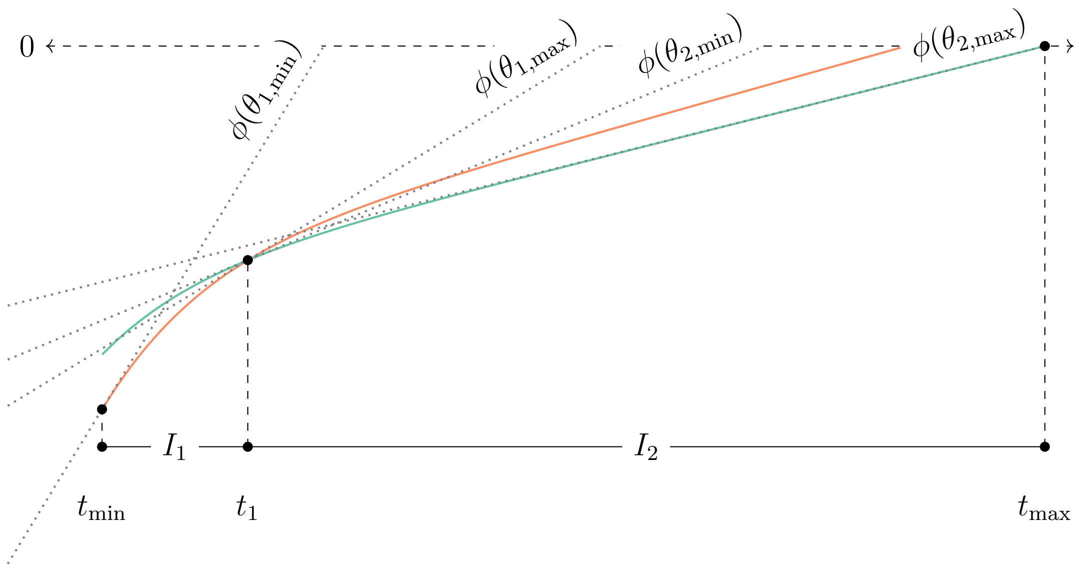


FIGURE II.1: A depiction of the spectrum partition. The dotted lines are tangents to the function $g = \min\{\psi_{\underline{j}_1}, \psi_{\underline{j}_2}\}$ at the points $t_{\min}$, t_1, and $t_{\max}$ corresponding to the left and right derivatives, where appropriate. The labels indicate the slopes of the dotted lines.

1. We say that a part I_n is *homogeneous* if $\underline{j}_n$ is homogeneous, and *inhomogeneous* otherwise.
2. We associate with each part I_n the function $g_n = \psi_{\underline{j}_n}$, which is analytic on the open interval I_n°.
3. We associate with each part $I_n = [t_{n-1}, t_n]$ the endpoint derivatives

$$\theta_{n,\min} = \phi^{-1}(g_n'(t_{n-1})) \qquad \text{and} \qquad \theta_{n,\max} = \phi^{-1}(g_n'(t_n)). \tag{8.2}$$

Expanding the definitions of ϕ^{-1} and g, we may equivalently write

$$\theta_{n,\min} = \frac{1}{\partial^+ g(t_{n-1}) \cdot (1 - 1/\kappa_{\max}) + 1} \quad \text{and} \quad \theta_{n,\max} = \frac{1}{\partial^- g(t_n) \cdot (1 - 1/\kappa_{\max}) + 1}.$$

Here, and elsewhere, for a concave function f, we write $\partial^- f(x)$ and $\partial^+ f(x)$ to denote the left and right derivatives of f at x respectively (when they exist), and set $\partial f(x) = [\partial^+ f(x), \partial^- f(x)]$. In particular,

$$\theta_{1,\min} \leq \theta_{1,\max} \leq \theta_{2,\min} \leq \cdots \leq \theta_{n,\max}$$

and moreover $\theta_{n,\min} = \theta_{n,\max}$ if and only if I_n is homogeneous. We define $\theta_{\min} = \theta_{1,\min}$ and $\theta_{\max} = \theta_{m,\max}$.

Using this notation, Theorem 7.1 yields the following explicit piecewise formula for the Assouad spectrum of K. For clarity in the below formula, note that g is strictly increasing with $g(t_{\max}) = 0$. The cases of this formula are depicted in Figure II.1.

Corollary 8.3. *Fix a Gatzouras–Lalley IFS $\{T_i\}_{i \in \mathcal{I}}$ with spectrum partition $(I_n)_{n=1}^m$, and associated data g_n, $\theta_{n,\min}$, and $\theta_{n,\max}$ as above. If $\dim_{\mathrm{B}} K = \dim_{\mathrm{A}} K$, then K is Ahlfors–David regular and $\dim_{\mathrm{B}} K = \dim_{\mathrm{A}} K = \dim_{\mathrm{A}}^\theta K$ for all $\theta \in (0,1)$.*

Otherwise, one of the following conditions holds for each $\theta \in (0,1)$:

(i) We have $\theta \leq \theta_{\min}$. Then

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} K - \frac{g(t_{\min})}{\phi(\theta)}.$$

(ii) There is an $n \in \{1, \ldots, m\}$ so that $\theta_{n,\min} < \theta < \theta_{n,\max}$. Then

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} \eta(K) + \frac{g_n^*(\phi(\theta))}{\phi(\theta)}.$$

(iii) There is an $n \in \{1, \ldots, m-1\}$ so that $\theta_{n,\max} \leq \theta \leq \theta_{n+1,\min}$. Then

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} \eta(K) + t_n - \frac{g_n(t_n)}{\phi(\theta)}.$$

(iv) We have $\theta \geq \theta_{\max}$. Then

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{A}} K.$$

Moreover,

$$\theta_{\max} = \inf\{\theta \in (0,1) : \dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{A}} K\},$$

and $\theta_{\min} = \theta_{\max}$ if and only if $m = 1$ and I_1 is homogeneous.

Proof. We recall the definition of τ and the main result proven in Theorem 7.1. Note that the formulae for $\theta \leq \theta_{\min}$ and $\theta \geq \theta_{\max}$ follow directly by the definition of the concave conjugate applied at the endpoint. Similarly, for $\theta_{\min} < \theta < \theta_{\max}$, the formulae follow directly since

$$\dim_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{B}} \eta(K) + \frac{g^*(\phi(\theta))}{\phi(\theta)}.$$

The parts in case (ii) correspond to the interiors of I_n, in which case g^* and g_n^* agree, and the parts in case (iii) correspond to the points of non-differentiability of g, which can only occur on the endpoints between adjacent I_n. In the latter case, g^* is an affine function of α with an explicit formula depending only on the value of g_n at t_n.

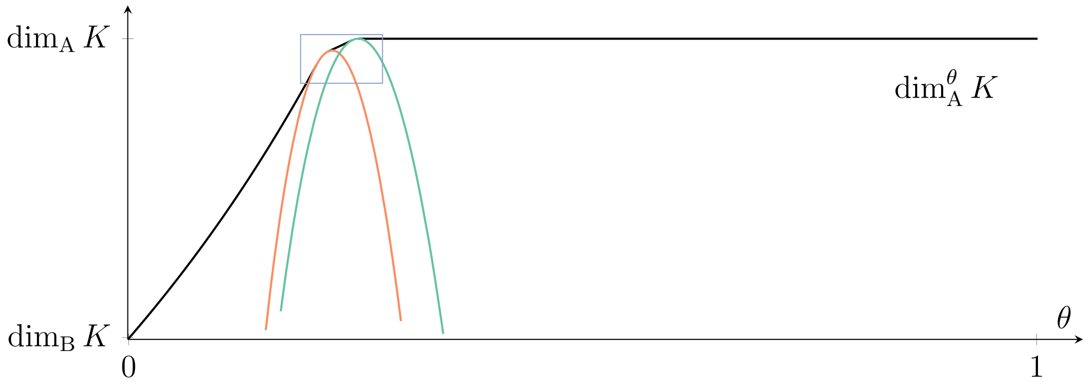
Moreover, suppose $\dim_{\mathrm{B}} K \neq \dim_{\mathrm{A}} K$, so that $t_{\min} < t_{\max}$. First, since we recall that $s_{\underline{j}}$ is the unique zero of $\psi_{\underline{j}}$ for all $\underline{j} \in \eta(\mathcal{I})$ and $t_{\max} = \max s_{\underline{j}}$, it follows that $t_{\max}$ is the unique zero of g. Thus, for $\theta < \theta_{\max}$, let t be such that $\phi(\theta) \in \partial g(t)$, so $t < t_{\max}$ and since g is concave,

$$g^*(\phi(\theta)) = g(t_{\max}) + g^*(\phi(\theta)) < t_{\max}\phi(\theta).$$

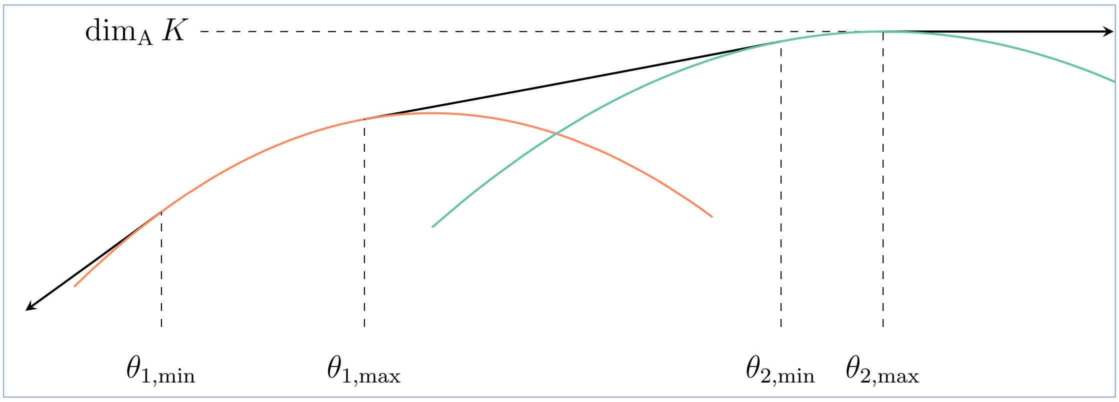
Dividing through by $\phi(\theta)$ gives the claim.

It is also clear directly from the definition that $\theta_{\min} = \theta_{\max}$ if and only if g is a affine function on the interval $[t_{\min}, t_{\max}]$, which occurs if and only if $m = 1$ and I_1 is homogeneous. $\square$

We note that the formulae in (i) and (iv) are in fact special cases of (iii) after substituting the respective value of t_n. In these three cases, expanding the definition of ϕ, the formula is of the form $a + b\frac{\theta}{1-\theta}$, which has occurred previously as discussed



(A) Plot of the original spectrum.



(B) Plot restricted to the rectangular region.

FIGURE II.2: A depiction of decomposition provided by Corollary 8.3. The coloured curves are of the form $\dim_{\mathrm{B}} \eta(K) + g_i^*(\phi(\theta))/\phi(\theta)$ for $i = 1, 2$. The spectrum is differentiable but not twice differentiable at each $\theta_{i,\min}$ and $\theta_{i,\max}$ for $i = 1, 2$.

in the introduction. In contrast, case (ii) is novel and very much *not* of this form. This case occurs only in the presence of an inhomogeneous column.

In case (ii), an implicit formula for the concave conjugate can be obtained from Proposition 7.11 (also see the discussion in Remark 7.15). The spectrum can also be obtained parametrically as a function of t: given a part $I_n = [t_{n-1}, t_n]$ with corresponding column function $\psi = \psi_{\underline{j}_n} = g_n$, the graph of the function $\theta \mapsto \dim_{\mathrm{A}}^{\theta} K$ on $(\theta_{n,\min}, \theta_{n,\max})$ is the same as the image of the interval (t_{n-1}, t_n) under the map

$$t \mapsto \left(\phi^{-1}(\psi'(t)), \dim_{\mathrm{B}} \eta(K) + t - \frac{\psi(t)}{\psi'(t)} \right). \tag{8.3}$$

A graphical depiction of the curve $g(t)$ is given in Figure II.1, and the decomposition provided by Corollary 8.3 is given in Figure II.2. The case $\theta_{\min} = \theta_{\max}$ is satisfied, for example, by every Bedford–McMullen carpet (or more generally by any Gatzouras–Lalley carpet with $\frac{\log \beta_{i,2}}{\log \beta_{i,1}}$ constant for $i \in \mathcal{I}$).

Using this decomposition, we can establish that $\dim_{\mathrm{A}}^{\theta} K$ is an increasing function of K, and in particular by [FHH+19, Theorem 2.1] has equal Assouad and upper Assouad spectra.

Corollary 8.4. *Let K be a Gatzouras–Lalley carpet. Then $\dim_{\mathrm{A}}^{\theta} K$ is strictly increasing on $(0, \theta_{\max}]$ and constant on $[\theta_{\max}, 1)$. In particular, $\overline{\dim}_{\mathrm{A}}^{\theta} K = \dim_{\mathrm{A}}^{\theta} K$ for all $\theta \in (0,1)$.*

Proof. Using the formula in Corollary 8.3, it is clear that $\dim_{\mathrm{A}}^{\theta} K$ is strictly increasing on intervals corresponding to cases (i) and (iii). Moreover, $\dim_{\mathrm{A}}^{\theta} K$ is constant on (iv) (which corresponds to the case when $\theta \in [\theta_{\max}, 1)$). For the remaining case (ii), fix some $n = 1, \dots, m$, write $\psi = g_n$ and let s denote the unique zero of ψ. Now let $\theta \in (\theta_{n,\min}, \theta_{n,\max})$ be arbitrary and let $\alpha = \phi(\theta)$. Note that $\alpha > 0$ and moreover $\alpha > \psi'(s)$ since ψ is strictly concave (for otherwise $\theta_{n,\min} = \theta_{n,\max}$) with $\psi(s) = 0$.

Note that ψ is strictly increasing and strictly concave, so $-\psi^{-1}$ is a well-defined strictly concave function. For notational simplicity, let

$$\varphi = (-\psi^{-1})^* \qquad \text{and} \qquad F(\alpha) = \frac{\psi^*(\alpha)}{\alpha}.$$

In particular, by definition of the concave conjugate, using the change of variable $y = \psi(t)$,

$$F(\alpha) = \inf_{t \in \mathbb{R}} \left\{ t - \frac{\psi(t)}{\alpha} \right\} = \inf_{y \in \mathbb{R}} \left\{ y \cdot \left(-\frac{1}{\alpha} \right) - (-\psi^{-1})(y) \right\} = \varphi\left(-\frac{1}{\alpha} \right).$$

Moreover, φ is maximized at

$$(-\psi^{-1})'(0) = -\frac{1}{\psi'(\psi^{-1}(0))} = -\frac{1}{\psi'(s)} < -\frac{1}{\alpha}.$$

Thus since φ is strictly concave, its derivative is strictly decreasing, so

$$\alpha^2 F'(\alpha) = \varphi'\left(-\frac{1}{\alpha} \right) < \varphi'\left(-\frac{1}{\psi'(s)} \right) = 0.$$

Thus by the chain rule, since $\phi'(\theta) < 0$ for all $\theta \in (0,1)$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \dim_{\mathrm{A}}^{\theta} K = F'\big(\phi(\theta)\big) \cdot \phi'(\theta) > 0.$$

In particular, the spectrum is a strictly increasing function of θ on $(0, \theta_{\max}]$. □

8.2 DIFFERENTIABILITY AND HIGHER-ORDER PHASE TRANSITIONS

We can also obtain precise information concerning differentiability, as well as higher-order phase transitions. We say that a continuous real-valued function h has a *phase transition of k^{th} order at θ* when $k \geq 1$ is the smallest integer such that the k^{th}-derivative $h^{(k)}(\theta)$ does not exist. Recall that we denote the number of parts in the spectrum partition by m.

Corollary 8.5. *The function $\theta \mapsto \dim_{\mathrm{A}}^{\theta} K$ is piecewise analytic and the set of points where the function is not analytic is given precisely by*

$$H = \{\theta : \theta = \theta_{n,\min} \text{ or } \theta = \theta_{n,\max} \quad \text{for some} \quad n = 1, \ldots, m\}.$$

At each $\theta \in H$ there is a phase transition that either has odd order or order 2. Moreover:

(i) The set of $\theta \in (0,1)$ at which $\dim_{\mathrm{A}}^{\theta} K$ has a 1st order phase transition is given by

$$H_1 \coloneqq \{\theta : \theta = \theta_{n,\min} = \theta_{n,\max} \quad \text{for some} \quad n = 1, \ldots, m\}.$$

This implies that $\dim_{\mathrm{A}}^{\theta} K$ has precisely k points of non-differentiability, where k is the number of $i = 1, \ldots, m$ such that I_i is homogeneous, and $k \leq \#\eta(\mathcal{I}) - 1$. In particular, $\dim_{\mathrm{A}}^{\theta} K$ is differentiable if and only if each I_n is inhomogeneous.

(ii) The set of $\theta \in (0,1)$ at which $\dim_{\mathrm{A}}^{\theta} K$ has a k^{th} order phase transition for some odd integer $k \geq 3$ is given by

$$H_{\mathrm{higher}} \coloneqq \{\theta : \theta = \theta_{n,\max} = \theta_{n+1,\min} \quad \text{for some} \quad n = 1, \ldots, m-1\} \setminus H_1.$$

(iii) The set of $\theta \in (0,1)$ at which $\dim_{\mathrm{A}}^{\theta} K$ has a 2nd order phase transition is given by

$$H_2 \coloneqq H \setminus (H_1 \cup H_{\mathrm{higher}}).$$

In particular, at $\theta_{\min}$ and $\theta_{\max}$, $\dim_{\mathrm{A}}^{\theta} K$ either has a 1st or 2nd order phase transition.

Proof. Piecewise analyticity follows directly from the piecewise formula given in Corollary 8.3. Moreover, since the distinct parts correspond to distinct analytic curves and any intersection of distinct analytic curves must have a phase transition of some order, we obtain the formula for H.

First, to see (i), by standard properties of the concave conjugate, the derivative of τ^* at α exists if and only if τ is strictly concave at all t for which $\alpha \in \partial\tau(t)$. But τ fails to be strictly concave at t if and only if $t \in I_n^\circ$ for a homogeneous part I_n, in which case $\theta = \theta_{n,\min} = \theta_{n,\max}$, as claimed. Since the curves $\psi_{\underline{j}}$ are affine for a homogeneous column $\underline{j}$ and otherwise strictly concave, and moreover there must be at least one column $\underline{i}$ with $s_{\underline{i}} \leq t_{\min}$, there are at most $\#\eta(\mathcal{I}) - 1$ points of non-differentiability.

Next, suppose $\theta = \theta_{n,\max} = \theta_{n+1,\min}$ for some $n = 1, \ldots, m-1$. Equivalently, $\theta = g_n'(t_n) = g_{n+1}'(t_n)$, and since $g_n - g_{n+1}$ changes sign at t_n, $g_n - g_{n+1}$ has a saddle point at t_n. Therefore if $k \in \mathbb{N}$ is minimal so that $g_n^{(k)}(t_n) \neq g_{n+1}^{(k)}(t_n)$, then k an odd integer which is at least 3. In particular, if $\theta \notin H_1$, then $\dim_{\mathrm{A}}^{\theta} K$ is differentiable at θ, and therefore has a phase transition of odd order $k \geq 3$.

Otherwise, suppose $\theta \in H_2$ and $\theta = \theta_{n,\max} < \theta_{n+1,\min}$. Since $(g^*)'' = 0$ on $(\phi^{-1}(\theta_{n+1,\min}), \phi^{-1}(\theta_{n,\max}))$ and $(g^*)''$ is uniformly bounded away from 0 on $(\phi^{-1}(\theta_{n,\max}), \phi^{-1}(\theta_{n,\min}))$, and since the parameter change $g^*(\phi(\theta))/\phi(\theta)$ is smooth, the second derivative does not exist at $\theta_{n,\max}$. The case $\theta_{n,\max} < \theta_{n+1,\min} = \theta$ is analogous. Otherwise, $\theta = \theta_{\max}$, but again $\dim_{\mathrm{A}}^{\theta} K$ is constant on $[\theta_{\max}, 1)$ so the same argument yields non-existence of the second derivative at $\theta_{\max}$.

Combining these two observations yields (ii) and (iii), and in particular that every phase transition has either odd order or order 2. □

Remark 8.6. It is straightforward to see that phase transitions of order 1 and order 2 already occur in many of the examples given later in this document. We sketch a construction giving a phase transition of arbitrary odd order $k \geq 3$.

We consider a Gatzouras–Lalley carpet with three columns each of width $1/3$, the first of which contains only one map (with arbitrary height less than $1/3$). Let $N \in \mathbb{N}$ be large and let $B \subset \mathbb{R}^N$ denote the set of all N-tuples $(b_1, \dots, b_N)$ with $0 < b_i < 1/3$ and $b_1 + \cdots + b_N < 1$. Note that to any pair $(b, \tilde{b}) \in B \times B$ there is an attractor $K = K(b, b')$ where the second column has contraction ratios given by b, and the third given by $\tilde{b}$. Fix $b = \tilde{b} = b_0$ for some arbitrary initial choice of b_0 whose entries are not all equal, let $t_{\min}$, $t_{\max}$ be the values corresponding to the carpet $K(b_0, b_0)$, and let t satisfy

$$t_{\min} < t < t_{\max}.$$

Exponentiating, we see that for any given $d \geq 1$, the functions ψ and $\tilde{\psi}$ corresponding to the non-trivial columns have the same d^{th} derivative at t if and only if

$$\sum_{i=1}^{N} \big((\log b_i)^d b_i^t - (\log \tilde{b}_i)^d (\tilde{b}_i)^t \big) = 0.$$

Moreover, by the implicit function theorem, for typical choices of (b_0, b_0) and for sufficiently large N, the set of parameters in $B \times B$ for which at least the first k derivatives match at t is a non-trivial submanifold of $B \times B$ containing the point (b_0, b_0). But given that the first k derivatives match, the parameters for which the first $k + 1$ derivatives match is a proper submanifold, so there must exist parameters $(b, \tilde{b})$ arbitrarily close to (b_0, b_0) such that precisely the first k derivatives of ψ and $\tilde{\psi}$ agree, but the next derivative does not.

Since the parameters $t_{\min}$ and $t_{\max}$ are continuous functions of $(b, \tilde{b})$, and since the sign of $\psi - \tilde{\psi}$ must change at t since k is odd, it follows that the function g corresponding to the carpet $K(b, \tilde{b})$ must have a phase transition of order k at t. Finally, since $k \geq 3$, g^* is differentiable so the derivative of g^* is the inverse of g'. Since the reparameterization in terms of ϕ is smooth, it follows that $\dim_{\mathrm{A}}^{\theta} K(b, \tilde{b})$ has a phase transition of order k at $\phi^{-1}(g'(t))$.

If one instead chooses points $t_{\min} < t_1 < \cdots < t_n < t_{\max}$, a similar argument gives arbitrarily many phase transitions of arbitrary odd orders at least 3.

8.3 CONVEXITY AND CONCAVITY

As the final result of this section, we obtain some information concerning convexity and concavity.

Corollary 8.7. *The spectrum* $\dim_{\mathrm{A}}^{\theta} K$ *is:*

(i) Strictly convex on each interval $(\theta_{n,\max}, \theta_{n+1,\min})$ *for* $n = 1, \dots, m-1$ *as well as the interval* $(0, \theta_{\min})$*;*

(ii) Strictly concave on the interval $(\theta_{\max} - \delta, \theta_{\max})$ *for some* $\delta > 0$ *if and only if* I_m *is inhomogeneous; and*

(iii) Constant on the interval $[\theta_{\max}, 1)$.

In particular, if $\dim_{\mathrm{A}}^{\theta} K$ *is not constant, then* $\dim_{\mathrm{A}}^{\theta} K$ *contains a non-trivial interval of convexity, and if every column is inhomogeneous, then* $\dim_{\mathrm{A}}^{\theta} K$ *also contains a non-trivial interval of concavity.*

Proof. Cases (i) and (iii) follow directly from the piecewise formula for the Assouad spectrum given in Corollary 8.3. The remaining case which requires checking is (ii). If I_m is homogeneous, $\dim_{\mathrm{A}}^{\theta} K$ is strictly convex in a neighbourhood to the left of $\theta_{\max}$ by (i).

Otherwise, assume that I_m is inhomogeneous with corresponding column $\underline{j} \in \eta(\mathcal{I})$. Writing $\psi = \psi_{\underline{j}}$ and continuing the computation from the proof of Corollary 8.4 with the same definitions of F and φ, for all $\theta \in I_m^{\circ}$,

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \dim_{\mathrm{A}}^{\theta} K = F''(\phi(\theta)) \cdot (\phi'(\theta))^2 + F'(\phi(\theta)) \cdot \phi''(\theta).$$

But $F'(\phi(\theta_{\max})) = 0$ and

$$F''(\phi(\theta_{\max})) = \varphi'' \left(-\frac{1}{\phi(\theta_{\max})} \right) \frac{1}{\phi(\theta_{\max})^4} - 2\varphi' \left(-\frac{1}{\phi(\theta_{\max})} \right) \frac{1}{\phi(\theta_{\max})^3} < 0,$$

since we recall that φ is strictly concave so $\varphi'' < 0$, and $\varphi'(-1/\psi'(t_{\max})) = 0$ where $\psi'(t_{\max}) = \phi(\theta_{\max})$. Thus, by continuity of the second derivative,

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} \dim_{\mathrm{A}}^{\theta} K < 0$$

for some $\delta > 0$ and $\theta \in (\theta_{\max} - \delta, \theta_{\max})$, as claimed. □

Remark 8.8. For most of the examples we present in this document, the Assouad spectrum is strictly concave on each non-trivial interval $(\theta_{n,\min}, \theta_{n,\max})$. However, in §8.4.3, we construct an example with a non-trivial inhomogeneous column and a non-trivial open sub-interval of $(\theta_{n,\min}, \theta_{n,\max})$ on which $\dim_{\mathrm{A}}^{\theta} K$ is strictly convex.

8.4 EXAMPLES

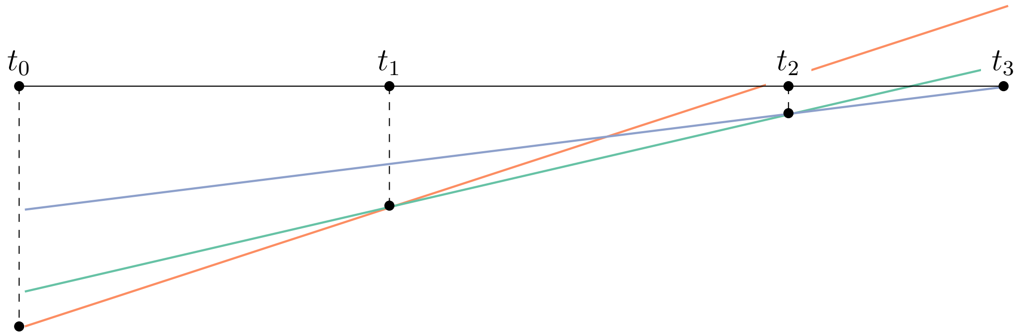
Finally, we consider a few explicit examples to highlight the results in this section.

8.4.1 Homogeneous carpets

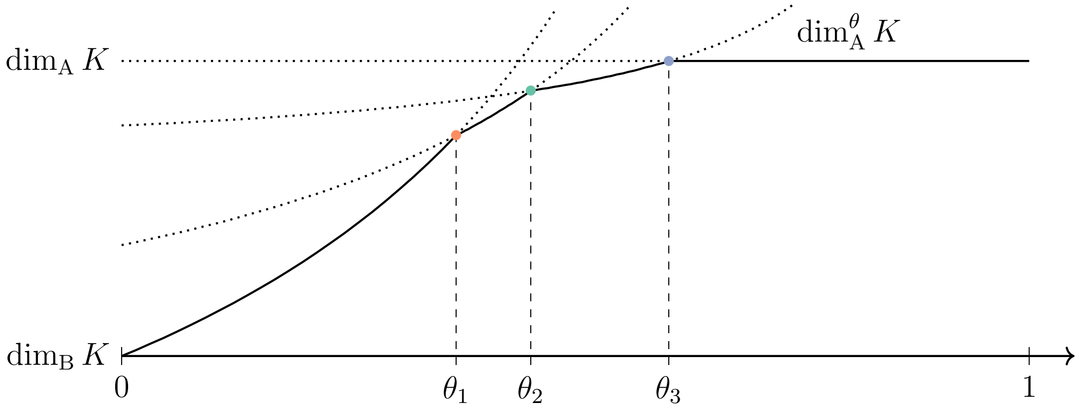
Consider the special case of Gatzouras–Lalley carpets whose columns are all homogeneous, i.e. for each $\underline{j} \in \eta(\mathcal{I})$, there is a unique $\beta_{\underline{j},2}$ so that $\beta_{i,2} = \beta_{\underline{j},2}$ for all $i \in \eta^{-1}(\underline{j})$. Recall from Lemma 8.2 that $\psi_{\underline{j}}$ is the affine function

$$\psi_{\underline{j}}(t) = \kappa_{\underline{j}} \cdot (t - s_{\underline{j}}) \qquad \text{where} \qquad \kappa_{\underline{j}} := \frac{\log \beta_{\underline{j},2}}{\log \beta_{\underline{j},1}}$$

and we recall that $s_{\underline{j}}$ is the dimension of column $\underline{j}$. Hence, the function $g(t)$ is piecewise affine, so for any $\theta \in (0,1)$, $\phi(\theta) \in \partial g(t_n)$ for some $n = 0, \ldots, m$.



(A) Plot of the function $g(t)$.



(B) Plot of the Assouad spectrum.

FIGURE II.3: Plot of the Assouad spectrum corresponding to a system with 4 homogeneous columns, 3 of which are non-trivial. The function $g(t)$ is a minimum of affine lines, so the corresponding spectrum $\dim_{\mathrm{A}}^{\theta} K$ is a piecewise convex function. The slope of $g_i(t)$ corresponds to the value θ_i, for $i = 1, 2, 3$. The dotted lines correspond to the concave conjugates at each t_i for $i = 0, \ldots, 3$, extended beyond the range given by the corresponding affine lines.

Equivalently, $\theta_n := \theta_{n,\min} = \theta_{n,\max}$ for all $n = 1, \ldots, m$, so case (ii) of Corollary 8.3 never occurs.

We can also derive an explicit formula for case (iii) of Corollary 8.3. Since $\psi_{\underline{j}}$ are affine for all $\underline{j} \in \eta(\mathcal{I})$, two functions $\psi_{\underline{i}}$ and $\psi_{\underline{j}}$ either have the same slope (in which case one of them does not appear in the formula at all) or intersect at

$$t_{\underline{i},\underline{j}} := \frac{\kappa_{\underline{i}} s_{\underline{i}} - \kappa_{\underline{j}} s_{\underline{j}}}{\kappa_{\underline{i}} - \kappa_{\underline{j}}} \qquad \text{with value} \qquad \psi_{\underline{i}}(t_{\underline{i},\underline{j}}) = \frac{s_{\underline{i}} - s_{\underline{j}}}{1/\kappa_{\underline{j}} - 1/\kappa_{\underline{i}}}.$$

Given the spectrum partition of the carpet, for every $n = 1, \ldots, m-1$, we have $t_n = t_{\underline{j}_n, \underline{j}_{n+1}}$ and we can express θ_n as

$$\theta_n = \phi^{-1}(\kappa_{\underline{j}_n}) = \frac{1}{1 + \kappa_{\underline{j}_n} \cdot (1 - 1/\kappa_{\max})}.$$

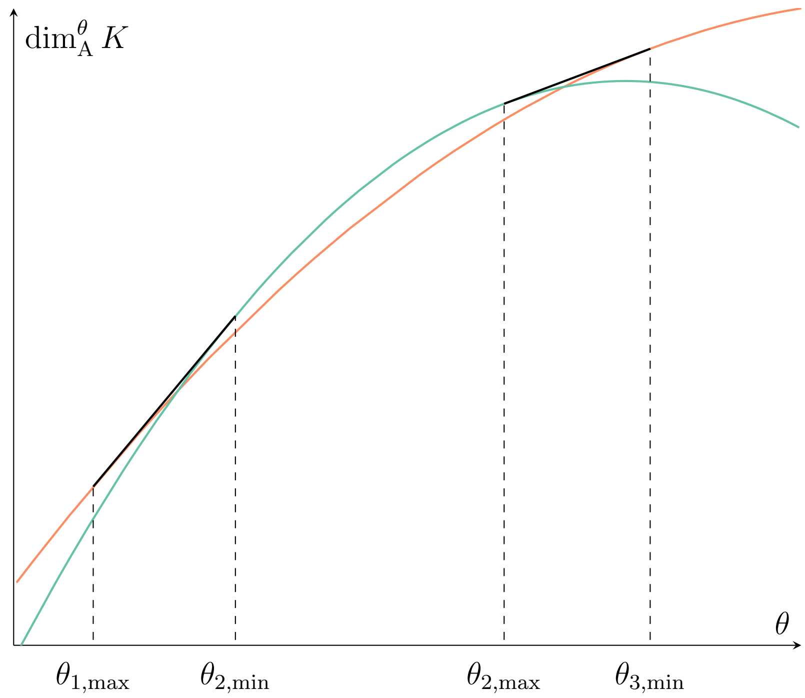


FIGURE II.4: Spectrum plot restricted to a small domain with a given column being relevant on multiple intervals. Note that the values $\theta_{1,\min}$ and $\theta_{3,\max}$ are not in the domain of this image.

To conclude, we obtain the formula

$$\dim_A^\theta K = \dim_B \eta(K) + \frac{\kappa_{\underline{j}_n} s_{\underline{j}_n} - \kappa_{\underline{j}_{n+1}} s_{\underline{j}_{n+1}}}{\kappa_{\underline{j}_n} - \kappa_{\underline{j}_{n+1}}} + \frac{\theta}{1-\theta}(1 - 1/\kappa_{\max}) \frac{s_{\underline{j}_{n+1}} - s_{\underline{j}_n}}{1/\kappa_{\underline{j}_{n+1}} - 1/\kappa_{\underline{j}_n}}.$$

In particular, $\dim_A^{\theta_n} K = \dim_B \eta(K) + s_{\underline{j}_n}$. The spectrum $\dim_A^\theta K$ is piecewise convex with a point of non-differentiability at each θ_n. The number of such points is bounded from above by the number of columns minus one, and this bound is clearly optimal.

A plot of the function g and the Assouad spectrum for a system with four columns (three of which are non-trivial) and three phase transitions is given in Figure II.3.

8.4.2 An example with three columns and six phase transitions

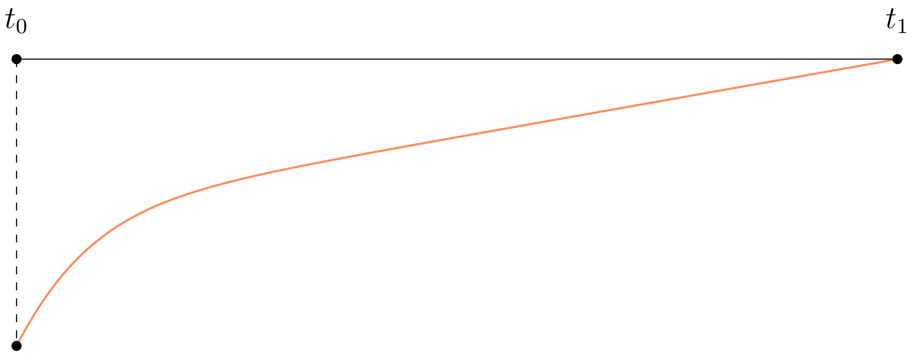
In this section, we provide an explicit example of a Gatzouras–Lalley system with 6 phase transitions, each of order 2. Define maps

$T_{1,1}(x,y) = (0.1 \cdot x, 0.05 \cdot y)$

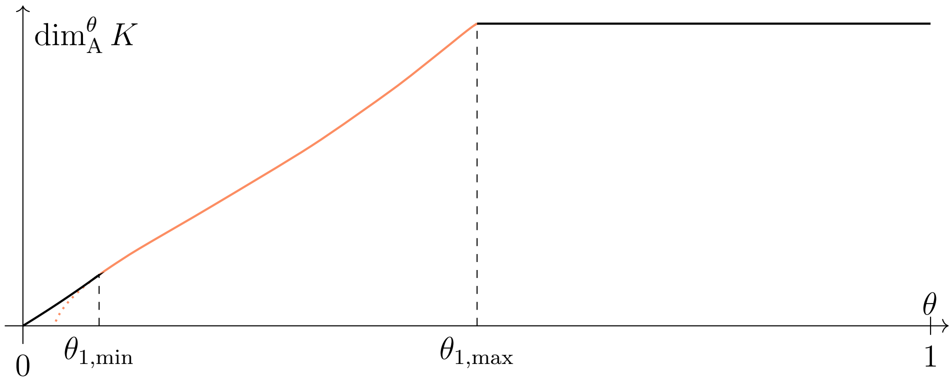
$T_{2,1}(x,y) = (0.4 \cdot x + 0.2, 0.00001 \cdot y) \qquad T_{2,2}(x,y) = (0.4 \cdot x + 0.2, 0.39 \cdot y + 0.61)$

$T_{3,1}(x,y) = (0.31 \cdot x + 0.69, 0.000177 \cdot y) \quad T_{3,1}(x,y) = (0.31 \cdot x + 0.69, 0.2 \cdot y + 0.8)$

This is a system with three columns, consisting of a single map $T_{1,1}$ in the first column, maps $T_{2,1}$ and $T_{2,2}$ in the second, and $T_{3,1}$ and $T_{3,2}$ in the third. This system has the following properties, which can be determined by a straightforward (albeit tedious!) computation:



(A) Plot of the function $g(t)$.



(B) Plot of the Assouad spectrum.

FIGURE II.5: Plot of the Assouad spectrum which has a convex part in the interval $(\theta_{1,\min}, \theta_{1,\max})$. Note that the spectrum is differentiable on $(0, 1)$, including at $\theta_{1,\max}$.

1. The spectrum partition has three parts I_1, I_2, and I_3 where the second column dominates on the parts I_1 and I_3 and the third column dominates on part I_2.
2. The Assouad spectrum has six phase transitions.

A plot of the Assouad spectrum on a restricted domain is given in Figure II.4.

8.4.3 Inhomogeneous column with corresponding part convex

The Assouad spectrum corresponding to case (ii) of Corollary 8.3 is often concave; in particular, this is the case for the other examples given above. However, this is not necessarily the case in general. To give an explicit example, we consider an IFS consisting of two columns. The first column consists of the single map $T_{1,1}(x, y) = (4x/5, y/1000)$. The second column has maps

$$T_{2,j}(x, y) = (x/5, b_j y) + (4/5, t_j) \qquad \text{for} \qquad j = 1, \dots, 52$$

where $b_1 = b_2 = 19/100$ and $b_3 = \cdots = b_{52} = 10^{-20}$, and the t_j are chosen so that the IFS is a Gatzouras–Lalley carpet.

The convexity of the Assouad spectrum on a non-trivial open sub-interval of $(\theta_{1,\min}, \theta_{1,\max})$ can be easily (albeit tediously) verified using the parametric formula (8.3). A plot of the Assouad spectrum is given in Figure II.5.

III. Pointwise Assouad dimension

One of the most fundamental concepts at the intersection of analysis and geometry is the notion of a *tangent*. For sets exhibiting a high degree of local regularity—such as manifolds, or rectifiable sets—at almost every point in the set and at all sufficiently high resolutions, the set looks essentially linear. Moreover, the concept of a tangent is particularly relevant in the study of a different class of sets: those equipped with some form of dynamical invariance. This relationship originates in the pioneering work of Furstenberg, where one associates to a set a certain dynamical system of "zooming in". Especially in the past two decades, the study of tangent measures has played an important role in the resolution of a number of long-standing problems concerning sets which look essentially the same at all small scales; see, for example, [HS12; HS15; KSS15; Shm19; Wu19].

In contrast, (weak) tangents also play an important role in the geometry of metric spaces. We recall from Proposition 1.1 that the Assouad dimension, which bounds the worst-case scaling at all locations and all small scales, is precisely the maximal Hausdorff dimension of weak tangents. Tangents and weak tangents are introduced in §1.5.

In this section, we study the interrelated concepts of tangents and Assouad dimension, with an emphasis on sets with a weak form of dynamical invariance. Our motivating examples include attractors of iterated function systems where the maps are affinities (or even more generally bi-Lipschitz contractions); or the maps are conformal and there are substantial overlaps. In both of these situations, the sets exhibit a large amount of local inhomogeneity. As we will see, these classes of sets exhibits a rich variety of behaviour while still retaining some fundamental properties.

Motivated by the relationship between the Assouad dimension and weak tangents described in Proposition 1.1, our primary questions relevant for the pointwise Assouad dimension and tangents are as follows:

- Does it hold that $\dim_{\mathrm{A}}(K,x) = \max\{\dim_{\mathrm{H}} F : F \in \mathrm{Tan}(K,x)\}$?
- Is there necessarily an $x_0 \in K$ so that $\dim_{\mathrm{A}} K = \dim_{\mathrm{H}} F$ for some $F \in \mathrm{Tan}(K,x_0)$? If not, is there an $x_0 \in K$ so that $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}}(K,x_0)$?
- What is the structure of the level set of pointwise Assouad dimension $\{x \in K : \dim_{\mathrm{A}}(K,x) = \alpha\}$ for some $\alpha \geq 0$?

A particular emphasis will be given sets satisfying some form of dynamical invariance, such as overlapping self-similar sets and self-affine sets. A study of general dynamically invariant sets (through a weak assumption which we call *self-embeddability*) can be found in §10. In that section, we will prove essentially that any attractor of a strictly contracting bi-Lipschitz IFS must contain at least one tangent with Hausdorff dimension equal to the Assouad dimension of the underlying set, and attractors of self-conformal IFSs must contain many such tangents. Here, the IFSs are not required to satisfy any separation conditions.

We then turn our attention to an intermediate class of sets: namely planar self-affine carpets. For Gatzouras–Lalley carpets, we obtain precise information about tangents which, in particular, shows that points with tangents that are as large as possible are very abundant (see §11). However, already for Barański carpets, we see that more complex behaviour is possible §12.

9 BASIC PROPERTIES OF THE POINTWISE ASSOUAD DIMENSION

In this section, we provide a broad introduction to the *pointwise Assouad dimension* and establish some basic results for general sets.

9.1 LEVEL SETS AND MEASURABILITY

We now make some observations concerning the multifractal properties of the function $x \mapsto \dim_{\mathrm{A}}(K, x)$. In particular, we are interested in the following quantities:

$$\mathcal{Y}(K,\alpha) = \{x \in K : \dim_{\mathrm{A}}(K,x) = \alpha\} \qquad \text{and} \qquad \varphi(\alpha) = \dim_{\mathrm{H}} \mathcal{Y}(K,\alpha).$$

We use the convention that $\dim_{\mathrm{H}} \varnothing = -\infty$. Observe that φ is a bi-Lipschitz invariant.

Given a compact set $K \subset \mathbb{R}^d$, we let $N_r^\circ(K)$ denote the minimal number of open sets with diameter r required to cover K, and $N_r^{\mathrm{pack}}(K)$ denote the size of a maximal centred packing of K by closed balls with radius r. Then, for $0 < r_1 \leq r_2$, we write

$$\begin{aligned} \mathcal{N}^\circ_{r_1,r_2}(K,x) &= N^\circ_{r_1}(B(x,r_2) \cap K) \\ \mathcal{N}_{r_1,r_2}(K,x) &= N^{\mathrm{pack}}_{r_1}(B^\circ(x,r_2) \cap K) \end{aligned}$$

The following lemma is standard.

Lemma 9.1. *Fix $0 < r_1 \leq r_2$. Then:*
(i) $\mathcal{N}^\circ_{r_1,r_2} \colon \mathcal{K}(\mathbb{R}^d) \times \mathbb{R}^d \to [0,d]$ is lower semicontinuous.
(ii) $\mathcal{N}_{r_1,r_2} \colon \mathcal{K}(\mathbb{R}^d) \times \mathbb{R}^d \to [0,d]$ is upper semicontinuous.

We can use this lemma to establish the following fundamental measurability results.

Proposition 9.2. *The following measurability properties hold:*
(i) For a fixed compact set K and $t \geq 0$, the set $\{x : \dim_{\mathrm{A}}(K,x) \geq s\}$ is a G_δ set.
(ii) The function $(K,x) \mapsto \dim_{\mathrm{A}}(K,x)$ is Baire class 2.
(iii) $\mathcal{A}(K,\alpha)$ is Borel for any compact set K.

Proof. Since $\mathbb{R}^d$ is doubling,

$$\begin{aligned} \dim_{\mathrm{A}}(K,x) = \inf\Big\{ s : \exists C > 0 \, \exists M \in \mathbb{N} \, \forall M \leq k \leq n \\ \mathcal{N}_{2^{-n},2^{-k}}(K,x) \leq C 2^{(n-k)s} \Big\}. \end{aligned}$$

Equivalently, we may use $\mathcal{N}^\circ_{r_1,r_2}$ in place of $\mathcal{N}_{r_1,r_2}$. In particular,

$$\{(K,x) : \dim_{\mathrm{A}}(K,x) > s\} = \bigcap_{C=1}^{\infty}\bigcap_{M=1}^{\infty}\bigcup_{k=M}^{\infty}\bigcup_{n=k}^{\infty}(\mathcal{N}^\circ_{2^{-n},2^{-k}})^{-1}(C2^{(n-k)s},\infty)$$

is a G_δ set. Thus $\{x : \dim_{\mathrm{A}}(K,x) > s\}$ is also a G_δ set, so

$$\{x : \dim_{\mathrm{A}}(K,x) \geq t\} = \bigcap_{n=1}^{\infty}\{x : \dim_{\mathrm{A}}(K,x) > t - 1/n\}$$

is also a G_δ set, as claimed in (i).

Moreover,

$$\{(K,x) : \dim_{\mathrm{A}}(K,x) < t\} = \bigcup_{C\in\mathbb{Q}\cap(0,\infty)}\bigcup_{M=1}^{\infty}\bigcap_{k=M}^{\infty}\bigcap_{n=k}^{\infty}(\mathcal{N}_{2^{-n},2^{-k}})^{-1}(-\infty, C2^{(n-k)t}).$$

Thus $\{(K,x) : \dim_{\mathrm{A}}(K,x) \in (s,t)\}$ is a $G_{\delta\sigma}$-set, i.e. it is a countable union of sets expressible as a countable intersection of open sets, so $\dim_{\mathrm{A}}$ is Baire class 2.

Of course, the same argument also show that $x \mapsto \dim_{\mathrm{A}}(K,x)$ is Baire class 2 for a fixed compact set K, so that $\mathcal{A}(K,\alpha)$ is $G_{\delta\sigma}$ and, in particular, Borel. □

9.2 Tangents and pointwise dimensions of general sets

We first observe that upper box dimensions of tangents provide a lower bound for the pointwise Assouad dimension.

Proposition 9.3. *For any compact set $K \subset \mathbb{R}^d$ and $x \in K$, $\dim_{\mathrm{A}}(K,x) \geq \overline{\dim}_{\mathrm{B}} F$ for any $F \in \operatorname{Tan}(K,x)$.*

Proof. Let $\alpha > \dim_{\mathrm{A}}(K,x)$ and suppose $F \in \operatorname{Tan}(K,x)$: we will show that $\overline{\dim}_{\mathrm{B}} F \leq \alpha$. First, get $C > 0$ such that for each $0 < r \leq R < 1$,

$$N_r(B(x,R)\cap K) \leq C\Big(\frac{R}{r}\Big)^{\alpha}.$$

Let $\delta > 0$ be arbitrary, and get a similarity T with similarity ratio λ such that $T(x) = 0$ and

$$d_{\mathcal{H}}(T(K)\cap B(0,1), F) \leq \delta.$$

Then there is a uniform constant $M > 0$ so that

$$M \cdot N_\delta(F) \leq N_\delta(T(K)\cap B(0,1)) = N_{\delta\lambda}(K\cap B(x,\lambda)) \leq C\Big(\frac{\lambda}{\delta\lambda}\Big)^{\alpha} = C\delta^{-\alpha}.$$

In other words, $\overline{\dim}_{\mathrm{B}} F \leq \alpha$. □

One should not expect equality to hold in general: in Example 9.10, we construct an example of a compact set $K \subset \mathbb{R}$ and a point $x \in K$ so that $\dim_{\mathrm{A}}(K,x) = 1$ but every $F \in \operatorname{Tan}(K,x)$ consists of at most 2 points.

We now establish some general results on the existence of tangents for general sets. These results will also play an important technical role in the following sections: for many of our applications, it is not enough to have positive Hausdorff α-measure for $\alpha = \dim_{\mathrm{A}} K$, since in general Hausdorff α-measure does not interact well with the Hausdorff metric on $\mathcal{K}\big(B(0,1)\big)$.

We begin with a straightforward preliminary lemma which is proven, for example, in [KR16, Lemma 3.11].

Lemma 9.4. *Let $K \subset \mathbb{R}^d$ be compact. Then* $\operatorname{Tan}(\operatorname{Tan}(K)) \subset \operatorname{Tan}(K)$.

Proof. First suppose $E \in \operatorname{Tan}(K)$ and $F \in \operatorname{Tan}(E)$. Write $E = \lim_{n\to\infty} T_n(K) \cap B(0,1)$ and $F = \lim_{n\to\infty} S_n(E) \cap B(0,1)$ for some sequences of similarities (T_n) and (S_n) with similarity ratios diverging to infinity. For each $\varepsilon > 0$, let N be sufficiently large so that

$$d_{\mathcal{H}}(S_N(E) \cap B(0,1), F) \leq \frac{\varepsilon}{2}.$$

Suppose S_N has similarity ratio λ_N, and let M be sufficiently large so that

$$d_{\mathcal{H}}(T_M(K) \cap B(0,1), E) \leq \frac{\varepsilon}{2\lambda_N}$$

It follows that

$$d_{\mathcal{H}}(S_N \circ T_M(K) \cap B(0,1), F) \leq \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, as required. □

Now, given a set with positive and finite Hausdorff measure, we can always find a tangent with large Hausdorff content.

Lemma 9.5. *Let $K \subseteq \mathbb{R}^d$ be a compact set with $0 < \mathcal{H}^\alpha(K) < \infty$. Then for $\mathcal{H}^\alpha$-almost every $x \in K$, there is an $F \in \operatorname{Tan}(K, x)$ such that $\mathcal{H}^\alpha_\infty(F) \geq 1$.*

Proof. By the same proof as [Mat95, Theorem 6.2], for $\mathcal{H}^\alpha$-almost every $x \in K$, there is a sequence of scales $(r_n)_{n=1}^\infty$ converging to zero such that

$$1 \leq \lim_{n\to\infty} r_n^{-\alpha} \mathcal{H}^\alpha_\infty\big(B(x, r_n) \cap K\big).$$

Then

$$\mathcal{H}^\alpha_\infty\big(r_n^{-1}(K - x) \cap B(0,1)\big) = r_n^{-\alpha}\mathcal{H}^\alpha_\infty\big(B(x, r_n) \cap K\big) \xrightarrow{n\to\infty} 1.$$

But Hausdorff α-content is upper semicontinuous (see, e.g. [MM97, Theorem 2.1]), so passing to a subsequence if necessary,

$$F = \lim_{n\to\infty}\big(r_n^{-1}(K - x) \cap B(0,1)\big)$$

satisfies $\mathcal{H}^\alpha_\infty(F) \geq 1$. □

Of course, we can combine the previous two results to obtain the following improvement of Proposition 1.1.

Corollary 9.6. *Let K be a compact set with $\dim_{\mathrm{A}} K = \alpha$. Then there is a weak tangent $F \in \mathrm{Tan}(K)$ with $\mathcal{H}^\alpha_\infty(F) \geq 1$.*

Proof. By Proposition 1.1, there is $E \in \mathrm{Tan}(K)$ such that $\mathcal{H}^\alpha(E) > 0$. By [Fal90, Theorem 4.10], there is a compact $E' \subset E$ such that $0 < \mathcal{H}^\alpha(E') < \infty$. Then by Lemma 9.5, there is $F' \in \mathrm{Tan}(E')$ with $\mathcal{H}^\alpha_\infty(F') \geq 1$. But $F' \subset F$ for some $F \in \mathrm{Tan}(E)$, and by Lemma 9.4, $F \in \mathrm{Tan}(K)$ with $\mathcal{H}^\alpha_\infty(F) \geq \mathcal{H}^\alpha_\infty(F') \geq 1$. □

The above results allow us to guarantee points with somewhat large tangents or large pointwise Assouad dimension.

Proposition 9.7. *Let $K \subset \mathbb{R}^d$. Then:*

(i) If K is analytic, for any s such that $\mathcal{H}^s(K) > 0$, there is a compact set $E \subset K$ with $\mathcal{H}^s(E) > 0$ so that for each $x \in E$, there is a tangent $F \in \mathrm{Tan}(\overline{K}, x)$ with $\mathcal{H}^s_\infty(F) \geq 1$.

(ii) If K is compact, there is an $x \in K$ such that $\dim_{\mathrm{A}}(K, x) \geq \overline{\dim}_{\mathrm{B}} K$.

Proof. The proof of (i) follows directly from Lemma 9.5, recalling that we can always find a compact subset $E \subset K$ such that $0 < \mathcal{H}^s(E) < \infty$ (combine [Mat95, Theorem 8.19] and [BP17, Corollary B.2.4]).

We now see (ii). Let $\overline{\dim}_{\mathrm{B}} K = t$. We first observe that for any $r > 0$, there is an $x \in K$ so that $\overline{\dim}_{\mathrm{B}} B(x, r) \cap K = t$. In particular, we may inductively construct a nested sequence of balls $B(x_k, r_k)$ with $\lim_{k\to\infty} r_k = 0$ so that $\overline{\dim}_{\mathrm{B}} K \cap B(x_k, r_k) = t$ for all $k \in \mathbb{N}$. Since K is compact, take $x = \lim_{k\to\infty} x_k \in K$. We verify that $\dim_{\mathrm{A}}(K, x) \geq t$. Let $C > 0$ and $\rho > 0$ be arbitrary. Since the x_k converge to x and the r_k converge to 0, get some k so that $B(x_k, r_k) \subset B(x, \rho)$. Thus for all $\varepsilon > 0$ and $r > 0$ sufficiently small depending on ε and ρ, since $\overline{\dim}_{\mathrm{B}} K \cap B(x_k, r_k) = t$,

$$N_r\big(B(x, \rho) \cap K\big) \geq N_r\big(B(x_k, r_k) \cap K\big) \geq C\left(\frac{r_k}{r}\right)^{t-\varepsilon}.$$

Thus $\dim_{\mathrm{A}}(K, x) \geq t$. □

Remark 9.8. Note that compactness is essential in Proposition 9.7 (ii) since there are sets with $\overline{\dim}_{\mathrm{B}} K = 1$ but every point is isolated: consider, for instance, the set $E = \{(\log n)^{-1} : n = 2, 3, \ldots\}$. In this case, $\overline{E} = E \cup \{0\}$ and $\dim_{\mathrm{A}}(\overline{E}, 0) = 1$. This example also shows that (ii) can hold with exactly 1 point.

To summarize, by Proposition 9.3, we have the inequalities

$$\sup\{\overline{\dim}_{\mathrm{B}} F : F \in \mathrm{Tan}(K, x)\} \leq \dim_{\mathrm{A}}(K, x) \leq \dim_{\mathrm{A}} K$$

for all $x \in K$ and, by Proposition 9.7 (ii), there is always an $x \in K$ so that $\overline{\dim}_{\mathrm{B}} K \leq \dim_{\mathrm{A}}(K, x)$. However, in general one cannot hope for more than this: an example in [LR15] already has the property that $K \subset \mathbb{R}$ such that $\dim_{\mathrm{A}} K = 1$ but $\dim_{\mathrm{A}}(K, x) = 0$ for all $x \in K$ (see Example 9.9 for more detail); and moreover, in Example 9.10, we construct a compact set $K \subset \mathbb{R}$ with a point $x \in K$ so that $\dim_{\mathrm{A}}(K, x) = 1$ but each $F \in \mathrm{Tan}(K, x)$ consists of at most two points.

9.3 EXAMPLES EXHIBITING SHARPNESS

Finally, we construct some general examples which go some way to showing that the results for general sets given in this section are sharp.

Example 9.9. In general, the Assouad dimension can only be characterized by weak tangents rather than by tangents. For example, consider the set K from [LR15, Example 2.20], defined by

$$K = \{0\} \cup \left\{2^{-k} + \ell 4^{-k} : k \in \mathbb{N}, \ell \in \{0, 1, \ldots, k\}\right\}$$

Since K contains arithmetic progressions of length k for all $k \in \mathbb{N}$, $\dim_{\mathrm{A}} K = 1$. However, $\dim_{\mathrm{A}}(K, x) = 0$ for all $x \in K$ and, therefore, by Proposition 9.3, $\dim_{\mathrm{H}} F = 0$ for all $F \in \operatorname{Tan}(K, x)$ and $x \in K$.

Example 9.10. We give an example of a compact set K and a point $x \in K$ so that $\dim_{\mathrm{A}}(K, x) = 1$ but each $F \in \operatorname{Tan}(K, x)$ consists of at most finitely many points.

Set $a_k = 4^{-k^2}$ and observe that $k a_{k+1}/a_k \leq 1/k$. For each $k \in \mathbb{N}$, write $\ell_k = \lfloor 2^k/k \rfloor$ and set

$$K = \{0\} \cup \bigcup_{k=1}^{\infty} \left\{ a_k \frac{2^k - \ell_k}{2^k}, a_k \frac{2^k - \ell_k - 1}{2^k}, \ldots, a_k \right\}$$

and consider the point $x = 0$. First observe for all $\varepsilon > 0$ and all k sufficiently small depending on ε,

$$N_{2^{-k} \cdot a_k}\big(B(0, a_k) \cap K\big) \geq \frac{\ell_k}{2} \geq 2^{(1-\varepsilon)k}$$

which gives that $\dim_{\mathrm{A}}(K, 0) = 1$.

On the other hand, for $k \in \mathbb{N}$,

$$a_k^{-1} K \cap B(0, 1) \subset [0, a_{k+1}/a_k] \cup [1/k, 1].$$

Since $k a_{k+1}/a_k \leq 1/k$, it follows that for any $\lambda \geq 1$ and $\lambda K \cap B(0, 1)$ can be contained in a union of two intervals with arbitrarily small length as λ diverges to ∞. Thus any tangent $F \in \operatorname{Tan}(K, 0)$ consists of at most 2 points.

10 TANGENTS OF DYNAMICALLY INVARIANT SETS

10.1 INVARIANCE AND SELF-EMBEDDABILITY

In §9.2, we established some general results concerning the pointwise Assouad dimension but we saw that for general sets we could not say much. However, many commonly studied families of "fractal" sets have a form of dynamical invariance, which is far from the case for general sets. To this end, we make the following definition.

Definition 10.1. We say that a compact set K is *self-embeddable* if for each $z \in K$ and $0 < r \leq \operatorname{diam} K$, there is a constant $a = a(z, r) > 0$ and a function $f : K \to B(z, r) \cap K$ so that

$$a r |x - y| \leq |f(x) - f(y)| \leq a^{-1} r |x - y|. \tag{10.1}$$

for all $x, y \in K$. We say that K is *uniformly self-embeddable* if the constant $a(z, r)$ can be chosen independently of z and r.

The class of self-embeddable sets is very broad and includes, for example, attractors of every possibly overlapping iterated function system $\{f_i\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is a finite index set and f_i is a strictly contracting bi-Lipschitz map from $\mathbb{R}^d$ to $\mathbb{R}^d$.

The class of *uniformly* self-embeddable sets includes the attractors of finite overlapping self-conformal iterated function systems. It is perhaps useful to compare uniform self-embeddability with quasi self-similarity, as introduced by Falconer [Fal89]. Our assumption is somewhat stronger since we also require the upper bound to hold in (10.1). This assumption is critical to our work since, in general, maps satisfying only the lower bound can decrease Assouad dimension. We also note that uniform self-embeddability is the primary assumption in [AKT20, Theorem 2.1].

Within this general class of sets, we establish general results which guarantee the existence of at least one large tangent under self-embeddability, and an abundance of tangents under uniform self-embeddability.

We recall from Example 9.10 and Example 9.9 that the Assouad dimension of K need not be attained as the Assouad dimension of a point, and even the Assouad dimension at a point need not be attained as the upper box dimension of a tangent at that point.

10.2 SELF-EMBEDDABLE SETS HAVE TANGENTS OF MAXIMAL DIMENSION

For self-embeddable sets, we can prove directly that the Assouad dimension of K is attained as the Hausdorff dimension of a tangent. In fact, the tangent can be chosen to have positive $\mathcal{H}^\alpha$-measure for $\alpha = \dim_{\mathrm{A}} K$. Moreover, the set of points with maximal pointwise Assouad dimension is a subset of full packing dimension. However, we will see in Theorem 12.5 that a similar result does not hold with Hausdorff dimension in place of packing dimension.

Theorem 10.2. *Let $K \subseteq \mathbb{R}^d$ be compact and self-embeddable with $\alpha = \dim_{\mathrm{A}} K$. Then there is a dense set of points $x \in K$ for which there exist $F \in \operatorname{Tan}(K, x)$ such that $\mathcal{H}^\alpha_\infty(F) \geq 2^{-\alpha}$. In particular,*

$$\dim_{\mathrm{P}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}} K\} = \dim_{\mathrm{P}} K.$$

Proof. We first note that it suffices to construct a single point x such that $\mathcal{H}^\alpha_\infty(F) \geq 2^{-\alpha}$. By self-embeddability and since $\dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}}(f(K), f(x))$ for a bi-Lipschitz map f, this immediately yields a dense subset of such points. Moreover, recalling Proposition 9.2 (i), since $\dim_{\mathrm{A}}(K, x) \leq \dim_{\mathrm{A}} K$ for all $x \in K$, $\{x \in K : \dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}} K\}$ is a dense G_δ subset of K and therefore has packing dimension equal to the packing dimension of K (see, for instance, [Fal14, Proposition 2.9]).

It therefore remains to construct such a point. Begin with an arbitrary ball $B(x_1, r_1)$ with $x_1 \in K$ and $0 < r_1 \leq 1$. Since K is self-embeddable, get a bi-Lipschitz map $f_1\colon K \to K \cap B(x_1, r_1)$. Since $\dim_{\mathrm{A}} f_1(K) = \alpha$, by Corollary 9.6 there is a weak tangent F_1 of $f_1(K)$ such that $\mathcal{H}^\alpha_\infty(F_1) \geq 1$. Since F_1 is a weak tangent of $f_1(K)$, there is a similarity T_1 with similarity ratio $\lambda_1 \geq 1$ such that $0 \in T_1(K)$ and

$$d_{\mathcal{H}}\big(T_1(f_1(K)) \cap B(0,1), F_1\big) \leq 1.$$

Then choose $x_2 \in K$ and $r_2 \leq 1/2$ so that $B(x_2, r_2) \subset T_1^{-1} B^\circ(0,1)$.

Repeating the above construction, next with the ball $B(x_2, r_2)$, and iterating, we obtain a sequence of similarity maps $(T_n)_{n=1}^\infty$ each with similarity ratio $\lambda_n \geq n$, bi-Lipschitz maps f_n, and compact sets F_n such that

1. $T_{n+1}^{-1} B(0,1) \subseteq T_n^{-1} B(0,1)$,
2. $d_{\mathcal{H}}\big(T_n(f_n(K)) \cap B(0,1), F_n\big) \leq \dfrac{1}{n}$, and
3. $\mathcal{H}^\alpha_\infty(F_n) \geq 1$.

Let $x = \lim_{n\to\infty} T_n^{-1}(0)$ and note by 1 that $x \in T_n^{-1} B(0,1)$ for all $n \in \mathbb{N}$. Let h_n be a similarity with similarity ratio $1/2$ such that

$$d_{\mathcal{H}}\Big(\frac{\lambda_n}{2}(f_n(K) - x) \cap B(0,1), h_n(F_n)\Big) \leq \frac{1}{n}.$$

Observe that $\mathcal{H}^\alpha_\infty(h_n(F_n)) \geq 2^{-\alpha}$. Thus passing to a subsequence if necessary, since $f_n(K) \subseteq K$, we may set

$$F_0 = \lim_{n\to\infty} \frac{\lambda_n}{2}(f_n(K) - x) \cap B(0,1) \qquad \text{and} \qquad F = \lim_{n\to\infty} \frac{\lambda_n}{2}(K - x) \cap B(0,1),$$

and observe that $F_0 \subseteq F$. Again passing to a subsequence if necessary, by compactness of the orthogonal group, 2 and the triangle inequality, there is an isometry h so that $\lim_{n\to\infty} h \circ h_n(F_n) = F_0$. Thus by upper semicontinuity of Hausdorff content,

$$\mathcal{H}^\alpha_\infty(F) \geq \mathcal{H}^\alpha_\infty(F_0) \geq \lim_{n\to\infty} \mathcal{H}^\alpha_\infty(h \circ h_n(F_n)) = 2^{-\alpha}$$

as required. □

We recall from Proposition 9.7 (ii) that, for a general compact set K, the upper box dimension of K provides a lower bound for the pointwise Assouad dimension at *some* point. For self-embeddable sets, we observe that the upper box dimension provides a uniform lower bound for the pointwise Assouad dimension at *every* point. On the other hand, the upper box dimension *does not* lower bound the maximal dimension of a tangent. For an example of this phenomenon, see Theorem 11.8.

Proposition 10.3. *Let $K \subseteq \mathbb{R}^d$ be self-embeddable. Then for any $x \in K$, we have $\dim_{\mathrm{A}}(K, x) \geq \overline{\dim}_{\mathrm{B}} K$.*

Proof. Fix $\alpha < \overline{\dim}_{\mathrm{B}} K$ and $x \in K$. Let $C > 0$ and $\rho > 0$ be arbitrary. Since K is self-embeddable, there is some bi-Lipschitz map $f \colon K \to B(x, \rho)$ so that $f(K) \subseteq K$. Since $\overline{\dim}_{\mathrm{B}} f(K) > \alpha$, there is some $0 < r \leq \rho$ so that

$$N_r(B(x,\rho) \cap K) \geq N_r(f(K)) \geq C\Big(\frac{\rho}{r}\Big)^\alpha.$$

Since $C > 0$ and $\rho > 0$ were arbitrary, $\dim_{\mathrm{A}}(K, x) \geq \alpha$, as required. □

10.3 UNIFORMLY SELF-EMBEDDABLE SETS HAVE MANY TANGENTS OF MAXIMAL DIMENSION

Now assuming uniform self-embeddability, we will see that the set of points with tangents that have positive $\mathcal{H}^\alpha$-measure has full Hausdorff dimension for $\alpha = \dim_A K$. Since uniformly self-embeddable sets satisfy the hypotheses of [Fal89, Theorem 4], it always holds that $\overline{\dim}_B K = \dim_H K$ (see also [Fra14, Theorem 2.10]). On the other hand, it can happen in this class of sets that $\overline{\dim}_B K < \alpha$: for example, this is the situation for self-similar sets in $\mathbb{R}$ with $\overline{\dim}_B K < 1$ which fail the weak separation condition; see [FHO+15, Theorem 1.3]. We provide a subset of full Hausdorff dimension for which each point has a tangent with positive Hausdorff α-measure.

The idea of the proof is essentially as follows. Let F be a weak tangent for K with strictly positive Hausdorff α-content. For each $s < \overline{\dim}_B K$, using the implicit method of [Fal89, Theorem 4], we can construct a well-distributed set of N balls at resolution δ, where $\delta^{-s} \ll N$. Then, inside each ball, using uniform self-embeddability, we can map an image of an approximate tangent $T_\delta^{-1}(B(0,1)) \cap K \approx F$ where T_δ has similarity ratio λ. Choosing N to be large, the resulting collection of images of the approximate tangent F is again a family of well-distributed balls at resolution $\lambda^{-1}\delta$, with $(\lambda^{-1}\delta)^{-s} \approx N$. Repeating this construction along a sequence of tangents converging to F yields a set E with $\dim_H E \geq s$ such that each $x \in E$ has a tangent which is an image of F (up to some negligible distortion), which has positive Hausdorff α-content by upper semicontinuity of content.

We fix a compact set K. To simplify notation, we say that a function $f \colon K \to K$ is in $\mathcal{G}(z,r,c)$ for $z \in K$ and $c, r > 0$ if $f(K) \subset B(z,r)$ and

$$cr|x-y| \leq |f(x)-f(y)| \leq c^{-1}r|x-y|$$

for all $x, y \in K$.

Theorem 10.4. *Let $K \subset \mathbb{R}^d$ be uniformly self-embeddable and let $\alpha = \dim_A K$. Then*

$$\dim_H\{x \in K : \exists F \in \operatorname{Tan}(K,x) \text{ with } \mathcal{H}^\alpha_\infty(F) \gtrsim 1\} = \dim_H K = \overline{\dim}_B K.$$

Proof. Write $\alpha = \dim_A K$. If $\overline{\dim}_B K = 0$ we are done; otherwise, let $0 < s < \overline{\dim}_B K$ be arbitrary. Since K is uniformly self-embeddable, there is a constant $a \in (0,1)$ so that for each $z \in K$ and $0 < r \leq \operatorname{diam} K$ there is a map $f \in \mathcal{G}(z,r,a)$. Next, from Corollary 9.6, there is a compact set $F \subset B(0,1)$ with $\mathcal{H}^\alpha_\infty(F) \geq 1$ and a sequence of similarities $(T_k)_{k=1}^\infty$ with similarity ratios $(\lambda_k)_{k=1}^\infty$ such that

$$F = \lim_{k\to\infty} T_k(K) \cap B(0,1)$$

with respect to the Hausdorff metric. Set $Q_k = T_k^{-1}(B(0,1)) \cap K$. We will construct a Cantor set $E \subset K$ of points each of which has pointwise Assouad dimension at least α and has $\dim_H E \geq s$.

We begin with a preliminary construction. First, since $s < \overline{\dim}_B K$, there is some $r_0 > 0$ and a collection of points $\{y_i\}_{i=1}^{N_0} \subset K$ such that $|y_i - y_j| > 3r_0$ for all $i \neq j$ and $N_0 \geq 2^s a^{-s} r_0^{-s}$. Now for each i, take a map $\phi_i \in \mathcal{G}(y_i, r_0, a)$. Write $\mathcal{I} = \{1, \ldots, N_0\}$, and for $\mathtt{i} = (i_1, \ldots, i_n) \in \mathcal{I}^n$ set

$$\phi_{\mathtt{i}} = \phi_{i_1} \circ \cdots \circ \phi_{i_n},$$

and, having fixed some $x_0 \in K$, write $x_{\mathtt{i}} = \phi_{\mathtt{i}}(x_0) \in \phi_{\mathtt{i}}(K)$. Observe that if the maximal length of a common prefix of $\mathtt{i}$ and $\mathtt{j}$ is m, then

$$\operatorname{dist}(\phi_{\mathtt{i}}(K), \phi_{\mathtt{j}}(K)) \geq r_0 (a r_0)^m.$$

We now begin our inductive construction. Without loss of generality, we may assume that $\lambda_n \geq 12$ for all $n \in \mathbb{N}$ and $r_0 \leq 1$. First, for each $n \in \mathbb{N}$, define constants $(m_n)_{n=1}^\infty \subset \{0\} \cup \mathbb{N}$ and $(\rho_n)_{n=1}^\infty$ converging monotonically to zero from above by the rules

1. $2^{-m_n} \leq \dfrac{a^2 r_0 \lambda_n^{-1}}{3}$,
2. $\rho_0 = \operatorname{diam} K$, and
3. $\rho_n := \rho_{n-1} \cdot \dfrac{a \lambda_n^{-1} \cdot (a r_0)^{m_n}}{3}$.

Next, for $n \in \mathbb{N} \cup \{0\}$ we inductively choose points $y_{n,\mathtt{i}} \in K$ and maps $\Psi_{n,\mathtt{i}} \in \mathcal{G}(y_{n,\mathtt{i}}, \rho_n, a)$ for $\mathtt{i} \in \mathcal{I}^{m_1} \times \cdots \times \mathcal{I}^{m_n}$. Let $\varnothing$ denote the empty word and let $y_{0,\varnothing} \in K$ be arbitrary and let $\Psi_{0,\varnothing}$ denote the identity map. Then for each $\mathtt{k} = \mathtt{ij}$ with $\mathtt{i} \in \mathcal{I}^{m_1} \times \cdots \times \mathcal{I}^{m_{n-1}}$ and $\mathtt{j} \in \mathcal{I}^{m_n}$, sequentially choose:

4. $\psi_{n,\mathtt{k}} \in \mathcal{G}(\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}}), \rho_n \lambda_n a^{-1}, a)$
5. $y_{n,\mathtt{k}} = \psi_{n,\mathtt{k}} \circ T_n^{-1}(0)$
6. $\Psi_{n,\mathtt{k}} \in \mathcal{G}(y_{n,\mathtt{k}}, \rho_n, a)$

Finally, write $\mathcal{J}_0 = \{\varnothing\}$, $\mathcal{J}_n = \mathcal{I}^{m_1} \times \cdots \times \mathcal{I}^{m_n}$ for $n \in \mathbb{N}$, and let

$$E_n = \bigcup_{\mathtt{i} \in \mathcal{J}_n} B(y_{n,\mathtt{ij}}, 3\rho_n) \qquad \text{and} \qquad E = K \cap \bigcap_{n=1}^\infty E_n.$$

Suppose $\mathtt{i} \in \mathcal{J}_{n-1}$ and $\mathtt{j} \in \mathcal{I}^{m_n}$. Since $x_{\mathtt{j}} \in K$, $\Psi_{n-1,\mathtt{i}}(K) \subset B(y_{n-1,\mathtt{i}}, \rho_{n-1})$, and $y_{n,\mathtt{ij}} \in \psi_{n,\mathtt{ij}}(K) \subset B(\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}}), \rho_n)$, we conclude since $\rho_{n-1} \geq 3\rho_n$ that

$$B(y_{n,\mathtt{ij}}, 3\rho_n) \subset B(y_{n-1,\mathtt{i}}, 3\rho_{n-1}).$$

Moreover, $y_{n,\mathtt{ij}} \in K$, so the sets E_n are non-empty nested compact sets and therefore E is non-empty.

We next observe the following fundamental separation properties of the balls in the construction of the sets E_n. Let $n \in \mathbb{N}$ and suppose $\mathtt{j}_1 \neq \mathtt{j}_2$ in $\mathcal{I}^{m_n}$ and $\mathtt{i} \in \mathcal{J}_{n-1}$ (writing $\mathcal{J}_0 = \{\varnothing\}$). Suppose $\mathtt{j}_1$ and $\mathtt{j}_2$ have a common prefix of maximal length m. First recall that $|x_{\mathtt{j}_1} - x_{\mathtt{j}_2}| \geq r_0 (a r_0)^m$, so that

$$|\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}_1}) - \Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}_2})| \geq \rho_{n-1} (a r_0)^{m+1}.$$

Then, since for $j = 1, 2$

$$y_{n,\mathtt{ij}_j} \in \psi_{n,\mathtt{ij}_j}(K) \subset B\Big(\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}_j}), \frac{\rho_{n-1}(a r_0)^{m_n}}{3}\Big)$$

we observe that

$$|y_{n,\mathtt{ij}_1} - y_{n,\mathtt{ij}_2}| \geq \rho_{n-1}(ar_0)^{m+1} - 2\frac{\rho_{n-1}(ar_0)^{m_n}}{3} \geq \frac{\rho_{n-1}(ar_0)^{m+1}}{3}.$$

Since we assumed that $\lambda_n \geq 12$, by the triangle inequality

$$\operatorname{dist}\big(B(y_{n,\mathtt{ij}_1}, 3\rho_n), B(y_{n,\mathtt{ij}_2}, 3\rho_n)\big) \geq \frac{\rho_{n-1}(ar_0)^{m+1}}{3} - 6\rho_n \geq \frac{\rho_{n-1}(ar_0)^{m+1}}{6}. \quad (10.2)$$

We first show that $\dim_{\mathrm{H}} E \geq s$. By the method of repeated subdivision, define a Borel probability measure μ with $\operatorname{supp}\mu = E$ and for $\mathtt{i} \in \mathcal{J}_n$,

$$\mu(B(y_{n,\mathtt{i}}, 3\rho_n) \cap K) = \frac{1}{\#\mathcal{J}_n}.$$

Now suppose U is an arbitrary open set with $U \cap E \neq \varnothing$. Intending to use the mass distribution principle, we estimate $\mu(U)$. Assuming that U has sufficiently small diameter, let $n \in \mathbb{N}$ be maximal so that

$$\operatorname{diam} U \leq \frac{a^{-1}\lambda_n}{2}\rho_n = \frac{\rho_{n-1}(ar_0)^{m_n}}{6}.$$

By (10.2), there is a unique $\mathtt{i} \in \mathcal{J}_n$ such that $U \cap B(y_{n,\mathtt{i}}, 3\rho_n) \neq \varnothing$. We first recall by choice of the constants m_n that

$$\begin{aligned}\rho_n &= (\operatorname{diam} K) \cdot \Big(\frac{a^2 r_0}{3}\Big)^n \lambda_1^{-1} \cdots \lambda_n^{-1} (ar_0)^{m_1+\cdots+m_n} \\ &\geq (\operatorname{diam} K) 2^{-(m_1+\cdots+m_n)} (ar_0)^{m_1+\cdots+m_n}.\end{aligned}$$

There are two cases. First assume $\rho_n/6 < \operatorname{diam} U$. Thus

$$\begin{aligned}\mu(U) &\leq \frac{1}{\#\mathcal{J}_n} \leq \Big(\frac{1}{2}ar_0\Big)^{s(m_1+\cdots+m_n)} \leq (\operatorname{diam} K)^{-s}\rho_n^s \\ &\leq \Big(\frac{6}{\operatorname{diam} K}\Big)^s \cdot (\operatorname{diam} U)^s.\end{aligned}$$

Otherwise, let $k \in \{0, \ldots, m_{n+1} - 1\}$ be so that

$$\frac{\rho_n(ar_0)^{k+1}}{6} < \operatorname{diam} U \leq \frac{\rho_n(ar_0)^k}{6}.$$

By (10.2), U intersects at most $N_0^{m_n - k}$ balls $B(y_{n+1,\omega}, 3\rho_{n+1})$ for $\omega \in \mathcal{J}_{n+1}$, so since $2^{-sk} \leq 1$,

$$\begin{aligned}\mu(U) &\leq \frac{1}{\#\mathcal{J}_n \cdot N_0^k} \leq (\operatorname{diam} K)^{-s}\rho_n^s \cdot (2^{-s}(ar_0)^s)^k \\ &\leq \Big(\frac{6}{ar_0 \operatorname{diam} K}\Big)^s \cdot \Big(\frac{\rho_n(ar_0)^{k+1}}{6}\Big)^s \\ &\leq \Big(\frac{6}{ar_0 \operatorname{diam} K}\Big)^s \cdot \big(\operatorname{diam} U\big)^s.\end{aligned}$$

This treats all possible small values of $\operatorname{diam} U$, so there is a constant $M > 0$ such that $\mu(U) \leq M(\operatorname{diam} U)^s$. Thus $\dim_{\mathrm{H}} E \geq s$ by the mass distribution principle.

Now fix

$$C = (3 + a^{-2})^{-\alpha}.$$

We will show that each $z \in E$ has a tangent with Hausdorff α-content at least C. Let $z \in E$ and define

$$S_n(x) = \frac{x - z}{\rho_n(3 + a^{-2})}.$$

Our tangent will be an accumulation point of the sequence $(S_n(K) \cap B(0,1))_{n=1}^{\infty}$. Now fix $n \in \mathbb{N}$. Since $z \in E$, there is some $\omega \in \mathcal{J}_n$ so that $z \in B(y_{n,\omega}, 3\rho_n)$. By choice of $y_{n,\omega}$, $Q_n = B\big(\psi_{n,\omega}^{-1}(y_{n,\omega}), \lambda_n^{-1}\big) \cap K$ so that

$$\psi_{n,\omega}(Q_n) \subseteq B\big(y_{n,\omega}, \rho_n a^{-2}\big) \cap K \subseteq B\big(z, \rho_n(3 + a^{-2})\big) \cap K$$

and therefore, writing $\Phi_n = S_n \circ \psi_{n,\omega} \circ T_n^{-1}$,

$$\Phi_n(T_n(K) \cap B(0,1)) \subset S_n(K) \cap B(0,1).$$

Then for $x, y \in T_n(K) \cap B(0,1)$, by the choice of ψ in (4),

$$\frac{|x - y|}{3 + a^{-2}} \leq |\Phi_n(x) - \Phi_n(y)| \leq \frac{|x - y|}{a^2(3 + a^{-2})}. \tag{10.3}$$

Now, passing to a subsequence $(n_k)_{k=1}^{\infty}$, we can ensure that

$$\lim_{k \to \infty} \Phi_{n_k}(F) = Z_0 \qquad \text{and} \qquad \lim_{k \to \infty} S_{n_k}(K) \cap B(0,1) = Z.$$

Moreover, recall that $\lim_{k \to \infty} T_{n_k}(K) \cap B(0,1) = F$ and $\mathcal{H}^{\alpha}_{\infty}(F) \geq 1$. Observe by (10.3) that $\mathcal{H}^{\alpha}_{\infty}(\Phi_{n_k}(F)) \geq C$ for each k, so by upper semicontinuity of Hausdorff content, $\mathcal{H}^{\alpha}_{\infty}(Z_0) \geq C$. But again by (10.3),

$$d_{\mathcal{H}}\big(Z_0, \Phi_{n_k}(T_{n_k}(K) \cap B(0,1))\big) \leq d_{\mathcal{H}}(Z_0, \Phi_{n_k}(F)) + \frac{d_{\mathcal{H}}\big(F, T_{n_k}(K) \cap B(0,1)\big)}{a^2(3 + a^{-2})}$$

so in fact $Z_0 \subset Z$ and $\mathcal{H}^{\alpha}_{\infty}(Z) \geq C$, as claimed. □

Remark 10.5. We note that the upper distortion bound in the definition of uniform self-embeddability is used only at the very last step to guarantee that the images $\Phi_{n_k}(T_{n_k}(K) \cap B(0,1))$ converge to a large set whenever the $T_{n_k}(K) \cap B(0,1)$ converge to a large set.

Remark 10.6. As a special case of Theorem 10.4, suppose K is the attractor of a finite self-similar IFS in the real line with Hausdorff dimension $s < 1$. In this case there is a dichotomy: either $\mathcal{H}^s(K) > 0$, in which case K is Ahlfors–David regular, or $\dim_{\mathrm{A}} K = 1$. In particular, the conclusion cannot be improved in general to give a set with positive Hausdorff s-measure.

11 TANGENT STRUCTURE AND DIMENSION OF GATZOURAS–LALLEY CARPETS

Our main goal in this section is to establish the following theorem, which states that Gatzouras–Lalley carpets have an abundance of large tangents. Recall for $x \in \mathbb{R}^2$ that ℓ_x is the vertical line passing through x.

Theorem 11.1. *Let K be a Gatzouras–Lalley carpet. Then*

$$\mathcal{H}^{\dim_{\mathrm{H}} K}\big(\{x \in K : \dim_{\mathrm{A}}(K, x) \neq \dim_{\mathrm{A}} K\}\big) = 0.$$

On the other hand, for any $\dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{A}} K$,

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\} = \dim_{\mathrm{H}} K.$$

Moreover, if $\eta(K)$ satisfies the SSC, then for any $x \in K$,

(i) $\max\{\dim_{\mathrm{H}} F : F \in \mathrm{Tan}(K, x)\} = \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{A}} \ell_x \cap K$,

(ii) $\dim_{\mathrm{A}}(K, x) = \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{A}} \ell_x \cap K\}$.

Of course, if $\alpha \notin [\dim_{\mathrm{B}} K, \dim_{\mathrm{A}} K]$, then $\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\} = \varnothing$. It follows immediately from Theorem 11.1 that

$$\dim_{\mathrm{A}}(K, x) = \max\{\dim_{\mathrm{H}} F : F \in \mathrm{Tan}(K, x)\}$$

if and only if $\dim_{\mathrm{A}} \ell_x \cap K \geq \dim_{\mathrm{B}} K - \dim_{\mathrm{B}} \eta(K)$. Moreover, if $s = \dim_{\mathrm{H}} K$, then $\mathcal{H}^s(K) > 0$ and furthermore $\mathcal{H}^s(K) < \infty$ if and only if K is Ahlfors–David regular (see [LG92]), in which case the results are trivial. We thus see that the majority of points, from the perspective of Hausdorff s-measure, have tangents with Hausdorff dimension attaining the Assouad dimension of K. However, we still have an abundance of points with pointwise Assouad dimension giving any other reasonable value.

The proof of Theorem 11.1 is obtained by combining Theorem 11.8 and Theorem 11.10. The dimensional results given in (i) and (ii) exhibit a precise version of a well-known phenomenon: at small scales, properly self-affine sets and measures look like products of the projection with slices. Note that, in order to obtain (i) and (ii), the strong separation condition in the projection is required or the pointwise Assouad dimension could be incorrect along sequences which are "arbitrarily close together at small scales". The formula holds for more general Gatzouras–Lalley carpets if one restricts attention to points where this does not happen (see Definition 11.2).

11.1 REGULAR POINTS AND INTERIOR WORDS

One difficulty with establishing pointwise results for dimensions is that, especially for carpets with "adjacent" columns, it can happen that a purely symbolic quantity depending only on the coding of a point loses some geometric information when a corresponding metric ball centred at the point contains different points with substantially different coding.

To avoid this issue, we introduce the notion of a regular point and an interior word.

Definition 11.2. We say that a point $x \in K$ is *regular* if for each $r \in (0,1)$, there is an $\mathtt{i} \in \mathcal{I}^*$ with $\beta_{\mathtt{i},1} \lesssim r$ such that $B(\eta(x), r) \cap \eta(K) \subset S_{\mathtt{i},1}(\eta(K))$. Given $\mathtt{i} \in \mathcal{I}^*$, we say that $\mathtt{i}$ is an *interior word* if $S_{\mathtt{i},1}([0,1]) \subset (0,1)$. We let $\mathcal{B}_n \subset \mathcal{I}^n$ denote the set of interior words of length n.

The following lemma is standard. Here, and elsewhere, given an $n \in \mathbb{N}$ and $\mathcal{Y} \subset \mathcal{I}^n$, we embed $\mathcal{Y}^{\mathbb{N}}$ in Ω in the natural way. We will abuse notation and interchangeably refer to elements in the subsystem or in the full system.

Lemma 11.3. *Let K be a Gatzouras–Lalley carpet.*
(i) If $\eta(K)$ satisfies the SSC, then each $x \in K$ is regular.
(ii) Suppose $\gamma \in \mathcal{B}_n^{\mathbb{N}}$ for some $n \in \mathbb{N}$. Then $\pi(\gamma)$ is regular.

We can now guarantee the existence of large subsystems consisting only of regular points. This result is essentially [FJS10, Lemma 4.3].

Proposition 11.4 ([FJS10]). *Let K be a Gatzouras–Lalley carpet corresponding to the IFS $\{T_i\}_{i\in\mathcal{I}}$. Then for every $\varepsilon > 0$, there is an $n \in \mathbb{N}$ and a family $\mathcal{J} \subset \mathcal{I}^n$ so that the IFS $\{T_{\mathtt{j}} : \mathtt{j} \in \mathcal{J}\}$ with attractor K_ε satisfies the following conditions:*
(i) each $\mathtt{i} \in \mathcal{J}$ is an interior word,
(ii) $\dim_{\mathrm{H}} K_\varepsilon \geq \dim_{\mathrm{H}} K - \varepsilon$,
(iii) $\dim_{\mathrm{B}} \eta(K_\varepsilon) \geq \dim_{\mathrm{B}} \eta(K) - \varepsilon$, and
(iv) there are $0 < \rho_2 < \rho_1 < 1$ so that $\beta_{\mathtt{i},1} = \rho_1$ and $\beta_{\mathtt{i},2} = \rho_2$ for all $\mathtt{i} \in \mathcal{I}$ and each column has the same number of maps.
In particular, each $x \in K_\varepsilon$ is a regular point with respect to the IFS $\{T_i\}_{i\in\mathcal{I}}$ and $\dim_{\mathrm{A}} K_\varepsilon = \dim_{\mathrm{H}} K_\varepsilon = \dim_{\mathrm{L}} K_\varepsilon$.

Proof. First, if K is contained in a vertical line, then K is the attractor of a self-similar IFS in $\mathbb{R}$ and the result is substantially easier. Now applying [FJS10, Lemma 4.3], there exists a family $\mathcal{J}_0 \subset \mathcal{I}^{n_0}$ with attractor K_0 satisfying conditions (ii), (iii), and (iv). By condition (iv), there is a $t \in \mathbb{R}$ so that $t(\mathtt{i}) = t$ for all $\mathtt{i} \in \mathcal{J}_0$. Therefore

$$\dim_{\mathrm{H}} K_0 = \dim_{\mathrm{B}} \eta(K) + t$$

and since K is not contained in a vertical line, we may assume that $\dim_{\mathrm{B}} \eta(K_0) > 0$.

Since $\eta(K_0)$ is the attractor of a self-similar IFS, iterating $\mathcal{J}_0$ if necessary and removing the maps in the first and last column, obtain a family $\mathcal{J} \subset \mathcal{J}_0^n$ with corresponding attractor K_ε such that $t(\mathtt{j}) = t$ for any $\mathtt{j} \in \mathcal{J}$, and $\dim_{\mathrm{B}} \eta(K_\varepsilon) \geq \dim_{\mathrm{B}} \eta(K) - \varepsilon$. Since words which correspond to rectangles that do not lie in the first or last column are necessarily interior words, combining this construction with Lemma 11.3 provides a family $\mathcal{J}$ satisfying the desired properties. □

11.2 TANGENTS OF GATZOURAS–LALLEY CARPETS

It turns out that the pointwise Assouad dimension at $x = \pi(\gamma)$ is closely related to the Assouad dimension of the symbolic fibre $K_{\eta(\gamma)}$. In this section, we make this notion precise, and moreover use it to construct large tangents for Gatzouras–Lalley carpets.

In our main result in this section, we prove that approximate squares containing a fixed word $\gamma \in \Omega$ converge in Hausdorff distance to product sets of weak tangents of $K_{\eta(\gamma)}$ with the projection $\eta(K)$, up to some finite distortion and contributions from adjacent approximate squares. First, we define

$$\Phi_{k,\gamma}(x,y) = \big(S^{-1}_{\gamma|_{L_k(\gamma)},1}(x), S^{-1}_{\gamma|_k,2}(y)\big).$$

By choice of $L_k(\gamma)$, the maps $\Phi_{k,\gamma}$ are (up to some constant-size distortion) homotheties. One can think of $\Phi_{k,\gamma}$ as mapping the approximate square $\pi(Q_k(\gamma))$ to the unit square $[0,1]^2$.

Proposition 11.5. *Let K be a Gatzouras–Lalley carpet and let $\gamma \in \Omega$ be arbitrary. Suppose $(\mathtt{i}_n)_{n=1}^{\infty}$ is any sequence such that $\eta(\mathtt{i}_n) = \eta(\gamma|_n)$. Then*

$$p_{\mathcal{H}}\left(\eta(K) \times (S^{-1}_{\mathtt{i}_n,2}(K_{\eta(\gamma)}) \cap [0,1]); \Phi_{n,\gamma}(K) \cap [0,1]^2\right) \lesssim C^n \tag{11.1}$$

where $C = \max\{\frac{\beta_{i,2}}{\beta_{i,1}} : i \in \mathcal{I}\} \in (0,1)$. Moreover, suppose γ is regular. Then for any $\gamma \in \Omega$ and $F \in \operatorname{Tan}(K, \pi(\gamma))$, there is an $E \in \operatorname{Tan}(K_{\eta(\gamma)})$ and a similarity map h so that $h(F) \subset \eta(K) \times E$.

Proof. We first prove that

$$d_{\mathcal{H}}\left(\eta(K) \times (S^{-1}_{\mathtt{i}_n,2}(K_{\eta(\gamma)}) \cap [0,1]), \Phi_{n,\gamma}(\pi(Q_n(\gamma)))\right) \lesssim C^n$$

Fix $n \in \mathbb{N}$ and write $k = L_n(\gamma)$. Let $Q_n(\gamma) = P(\gamma|_n, \underline{\mathtt{j}})$ and enumerate $\eta^{-1}(\underline{\mathtt{j}}) = \{\mathtt{j}_1, \ldots, \mathtt{j}_n\} \subset \mathcal{I}^{k-n}$. Observe that $\eta(T_{\mathtt{j}_i}(K)) = S_{\mathtt{j}_i,1}(K)$ does not depend on the choice of $i = 1, \ldots, m$. Now $\Phi_{n,\gamma}(T_{\gamma|_n \mathtt{j}_i}(K))$ is contained in the rectangle $\eta(K) \times S_{\mathtt{j}_i,2}(K)$. Moreover, the rectangle $\eta(K) \times S_{\mathtt{j}_i,2}(K)$ has height $\lesssim C^n$. Therefore

$$d_{\mathcal{H}}\left(\eta(K) \times \bigcup_{i=1}^{m} S_{\mathtt{j}_i,2}([0,1]), \Phi_{n,\gamma}(Q_n(\gamma))\right) \lesssim C^n. \tag{11.2}$$

But approximating the set $S_{\mathtt{i}_n,2}([0,1]) \cap K_{\eta(\gamma)}$ at level n with cylinders at level $k = L_n(\gamma)$, using the fact that $\eta(\mathtt{i}_n) = \eta(\gamma|_n)$,

$$d_{\mathcal{H}}\left(S^{-1}_{\mathtt{i}_n,2}(K_{\eta(\gamma)}) \cap [0,1], \bigcup_{i=1}^{m} S_{\mathtt{j}_i,2}([0,1])\right) \lesssim C^n. \tag{11.3}$$

Combining (11.2) and (11.3) gives the claim. In particular, noting that $Q_n(\gamma) \subset K$ and $\Phi_{n,\gamma}(Q_n(\gamma)) \subset [0,1]^2$ gives (11.1).

Now suppose in addition that $x = \pi(\gamma)$ is regular and let $r > 0$ be arbitrary. Since x is regular, there is an $n \in \mathbb{N}$ with $r \leq \beta_{\gamma|_n,1} \lesssim r$ such that

$$B(x,r) \cap K \subset \bigcup_{j=1}^{\ell} T_{\mathtt{i}_j}(K)$$

where

$$\{\mathtt{i}_1, \ldots, \mathtt{i}_\ell\} = \{\mathtt{i} \in \mathcal{I}^n : \eta(\mathtt{i}) = \eta(\gamma|_n) \text{ and } T_{\mathtt{i}}(K) \cap B(x,r) \neq \varnothing\}.$$

Now exactly as before, each rectangle $T_{\mathtt{i}_j}(K)$ has width $\approx r$ and height $\lesssim rC^n$. Therefore identifying $x \in K$ with the analogous point $x \in K_{\eta(\gamma)}$, there is a similarity map h_r with contraction ratio in some interval $[1, c]$ for a fixed c depending only on the IFS so that

$$p_{\mathcal{H}}\left(r^{-1}(K - x) \cap B(0,1); h_r(\eta(K)) \times r^{-1}(K_{\eta(\gamma)} - x)\right) \lesssim C^n.$$

Now suppose $F \in \operatorname{Tan}(K, x)$ so that $F = \lim_{n\to\infty} r_n^{-1}(K - x) \cap B(0,1)$. Passing to a subsequence, we may assume that the h_{r_n} have contraction ratios converging to some $r_0 \geq 1$. Thus passing again to a subsequence, let $F_0 = \lim_{n\to\infty}(r_0 r_n)^{-1}(K - x) \cap B(0,1)$. Since $r_0 \geq 1$, we have $F \subset F_0$. Passing again to a subsequence, let

$$\lim_{n\to\infty}(r_0 r_n)^{-1}(K_{\eta(\gamma)} - x) \cap B(0,1) = E \in \operatorname{Tan}(K_{\eta(\gamma)}).$$

Thus $r_0^{-1}F \subset F_0 \subset \eta(K) \times E$, as claimed. □

To conclude this section, we establish our general result which guarantees the existence of product-like tangents for arbitrary points in Gatzouras–Lalley carpets.

Proposition 11.6. *Let K be a Gatzouras–Lalley carpet. Then for each $x \in K$, there is an $F \in \operatorname{Tan}(K, x)$ so that*

$$\mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}(F) \gtrsim 1,$$

where $\gamma \in \Omega$ is such that $\pi(\gamma) = x$. In particular,

$$\dim_{\mathrm{A}}(K, x) \geq \max\{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}, \dim_{\mathrm{B}} K\}.$$

Proof. We will construct the set F essentially as a product $\eta(K) \times E$ where E is a *weak* tangent of $K_{\eta(\gamma)}$. First, recall from Corollary 2.3 that $\dim_{\mathrm{A}} K_{\eta(\gamma)} = \dim_{\mathrm{A}} \Omega(\eta(\gamma))$. Thus by Corollary 9.6, there is a sequence $(n_k)_{k=1}^\infty$ diverging to infinity and words $\mathtt{i}_k \in \mathcal{I}^{n_k}$ with $\eta(\mathtt{i}_k) = \gamma\restriction_{n_k}$ such that

$$E := \lim_{k\to\infty} S_{\mathtt{i}_k,2}^{-1}(K_{\eta(\gamma)}) \cap [0,1]$$

has $\mathcal{H}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}}(E) \gtrsim 1$.

Thus by Proposition 11.5 applied along the sequence $(\mathtt{i}_k)_{k=1}^\infty$, since the images $\Phi_{n,\gamma}^{-1}([0,1]^2)$ are rectangles with bounded eccentricity containing $\pi(\gamma)$, there is a tangent $F \in \operatorname{Tan}(K, x)$ containing an image of $\eta(K) \times E$ under a bi-Lipschitz map with constants depending only on K. But $\eta(K)$ is Ahlfors–David regular so that

$$\mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}(F) \geq \mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}\left(\eta(K) \times E\right) \gtrsim 1$$

as claimed. The result concerning $\dim_{\mathrm{A}}(K, x)$ then follows by Proposition 9.3 and Proposition 10.3. □

11.3 UPPER BOUNDS FOR THE POINTWISE ASSOUAD DIMENSION

We now prove our main upper bound for the pointwise Assouad dimension of Gatzouras–Lalley carpets.

Let us begin with an intuitive explanation for this proof. Since x is regular, we will reduce the problem of computing covers of balls to computing covers for approximate squares. Thus suppose we fix an approximate square $P(\mathtt{i}, \underline{\mathtt{j}})$, which is the union of cylinders $\{\mathtt{ik} : \eta(\mathtt{k}) = \underline{\mathtt{j}}\}$. We wish to cover this set with approximate squares in $\mathcal{S}(r)$. There are two cases. First, the rectangle corresponding to the cylinder $\mathtt{ik}$ has height greater than or equal to r, in which case we simply keep this cylinder and obtain a good bound for the cover using Lemma 3.4: this is the family $\mathcal{B}_1$. Otherwise, the rectangle is shorter, and we instead want to cover groups of cylinders simultaneously. Such groups are precisely wide pseudo-cylinders corresponding to elements of $\mathcal{B}_2$ and have height r, which we can then cover using Lemma 3.3. These covers are then combined using Lemma 3.5. We recall the notation for symbolic slices introduced in §3.4.

Proposition 11.7. *Let K be a Gatzouras–Lalley carpet and suppose $x = \pi(\gamma) \in K$. Then*

$$\dim_{\mathrm{A}}(K, x) \geq \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}\}$$

with equality if x is regular.

Proof. Recalling the general lower bound proven in Proposition 11.6, we must show that

$$\dim_{\mathrm{A}}(K, x) \leq \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}\} =: \zeta$$

when x is regular. We obtain this bound by a direct covering argument. We will prove that for any $k \in \mathbb{N}$ and approximate square $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$, if $0 < r \leq \beta_{\mathtt{i},2}$, then

$$\#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma)\} \lesssim \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta}. \tag{11.4}$$

Assuming this, since x is regular, for any ball $B(x, R)$, there is an $R' \lesssim R$ and at most two approximate squares $Q_1, Q_2 \in \mathcal{S}(R')$ lying in the same column such that $B(x, R) \subset \pi(Q_1) \cup \pi(Q_2)$. Since Q_1, Q_2 lie in the same column, $Q_j = Q_{k_j}(\gamma_j)$ for some $k_j \in \mathbb{N}$ where $\eta(\gamma_j) = \eta(\gamma)$. Moreover, if $0 < r \leq R$ and $Q \in \mathcal{S}(r)$ is arbitrary, then $\operatorname{diam} \pi(Q) \lesssim r$. Thus (11.4) immediately gives the correct bound, up to a constant factor, for $N_r(B(x, R) \cap K)$.

It remains to prove (11.4). Fix an approximate square $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$ and suppose $0 < r \leq \beta_{\mathtt{i},2}$ is arbitrary. Recall the notation $\mathcal{F}_{\eta(\gamma)}$ from §3.4. First, let

$$\mathcal{B}_0 = \eta^{-1}(\underline{\mathtt{j}}) \wedge \mathcal{F}_{\eta(\gamma)}(r/\beta_{\mathtt{i},2}) \qquad \text{and} \qquad \mathcal{B} = \{\mathtt{ik} : \mathtt{k} \in \mathcal{B}_0\}.$$

We then decompose $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 = \mathcal{B} \setminus \mathcal{F}_{\eta(\gamma)}(r) \qquad \text{and} \qquad \mathcal{B}_2 = \mathcal{B} \cap \mathcal{F}_{\eta(\gamma)}(r).$$

First, suppose $\mathtt{ik} \in \mathcal{B}_1$. Then, by definition, $\beta_{\mathtt{ik},2} > r$ which, by definition of $\mathcal{B}_0$, implies that $\eta(\mathtt{k}) = \underline{\mathtt{j}}$. Thus by Lemma 3.4 applied to the cylinder $\mathtt{ik}$ and scale r, since $\dim_{\mathrm{B}} \eta(K) \leq \dim_{\mathrm{B}} K$ and $\beta_{\mathtt{ik},1} \approx \beta_{\mathtt{i},2}$,

$$\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{ik}]\} \approx \left(\frac{\beta_{\mathtt{ik},2}}{r}\right)^{\dim_{\mathrm{B}} K} \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}. \tag{11.5}$$

Otherwise, suppose $\mathtt{ik} \in \mathcal{B}_2 \subset \mathcal{F}_{\eta(\gamma)}(r)$. Since $\mathcal{B}_0 \preccurlyeq \eta^{-1}(\underline{\mathtt{j}})$, there is a $\underline{\mathtt{j}}'$ so that $\eta(\mathtt{k})\underline{\mathtt{j}}' = \underline{\mathtt{j}}$. Thus choice of $\underline{\mathtt{j}}'$ ensures that

$$P(\mathtt{ik}, \underline{\mathtt{j}}') = Q_k(\gamma) \cap [\mathtt{ik}].$$

Thus by Lemma 3.3 and since $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$ is an approximate square,

$$\#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma) \cap [\mathtt{ik}]\} = \#\mathcal{Q}(\mathtt{ik}, \underline{\mathtt{j}}') \approx \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}. \tag{11.6}$$

Thus by applying (11.5) and (11.6) to the respective components and recalling that $\beta_{\mathtt{ik},2} \approx r$ whenever $\mathtt{ik} \in \mathcal{B}_2$,

$$\begin{aligned}
&\#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma)\} \\
&\quad= \sum_{\mathtt{ik} \in \mathcal{B}_1} \#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{ik}]\} + \sum_{\mathtt{ik} \in \mathcal{B}_2} \#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma) \cap [\mathtt{ik}]\} \\
&\quad\approx \sum_{\mathtt{ik} \in \mathcal{B}_1} \left(\frac{\beta_{\mathtt{ik},2}}{r}\right)^{\dim_{\mathrm{B}} K} \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)} + \sum_{\mathtt{ik} \in \mathcal{B}_2} \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)} \\
&\quad\lesssim \sum_{\mathtt{ik} \in \mathcal{B}_1} \left(\frac{\beta_{\mathtt{ik},2}}{r}\right)^{\zeta} \left(\frac{\beta_{\mathtt{i},2}}{\beta_{\mathtt{ik},2}}\right)^{\zeta} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}} + \sum_{\mathtt{ik} \in \mathcal{B}_2} \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}} \\
&\quad= \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta} \sum_{\mathtt{k} \in \mathcal{B}_0} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}} \\
&\quad\lesssim \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta}
\end{aligned}$$

where the last line follows by Lemma 3.5 applied to the section $\mathcal{B}_0$. Thus (11.4) follows, and therefore our desired result. □

11.4 DIMENSIONS OF LEVEL SETS OF POINTWISE ASSOUAD DIMENSION

Given an index $i \in \mathcal{I}$, let $\Phi_{\eta(i)}$ denote the IFS corresponding to the column containing the index i, that is

$$\Phi_{\eta(i)} = \{S_{j,2} : j \in \mathcal{I} \text{ and } \eta(j) = \eta(i)\}.$$

Now given a word $\gamma = (i_n)_{n=1}^{\infty} \in \Omega$, recall that the symbolic slice $K_{\eta(\gamma)}$ is the non-autonomous self-similar set associated with the IFS $\{\Phi_{\eta(i_n)}\}_{n=1}^{\infty}$. Since there are only finitely many choices for the $\Phi_{\eta(i_n)}$, the hypotheses of Theorem 2.12 are automatically satisfied and

$$\dim_{\mathrm{A}} K_{\eta(\gamma)} = \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \theta_{\eta(\gamma)}(n, m)$$

where

$$\sum_{(j_1,\ldots,j_m) \in \eta^{-1}(\eta(i_1,\ldots,i_n))} \prod_{k=1}^{m} \beta_{j_k,2}^{\theta_{\eta(\gamma)}(n,m)} = 1.$$

We now obtain our main formula for the pointwise Assouad dimension of arbitrary points in Gatzouras–Lalley carpets.

Theorem 11.8. *Let K be a Gatzouras–Lalley carpet. Then for every $x \in K$ with $x = \pi(\gamma)$, there is an $F \in \operatorname{Tan}(K, x)$ with $\mathcal{H}^s(F) \gtrsim 1$ where*

$$\begin{aligned} s &:= \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)} \\ &= \dim_{\mathrm{B}} \eta(K) + \lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta(\gamma)}(n, m) \end{aligned}$$

In particular,

$$\max\{\dim_{\mathrm{H}} F : F \in \operatorname{Tan}(K, x)\} \geq s \quad \text{and} \quad \dim_{\mathrm{A}}(K, x) \geq \max\{s, \dim_{\mathrm{B}} K\}$$

where both inequalities are equalities if x is regular. In particular, if $\eta(K)$ satisfies the strong separation condition then equality holds for all $x \in K$.

Proof. By Proposition 11.6, there is an $F \in \operatorname{Tan}(K, x)$ so that

$$\mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}(F) \gtrsim 1.$$

Moreover, $\dim_{\mathrm{A}} K_{\eta(\gamma)} = \lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta(\gamma)}(n, m)$ by Theorem 2.12. The formula for $\dim_{\mathrm{A}}(K, x)$, including the case when x is regular, then follows by Proposition 11.7.

If x is regular, it moreover follows from Proposition 11.5 that for any $F \in \operatorname{Tan}(K, x)$, there is a similarity map h and a weak tangent $E \in \operatorname{Tan}(K_{\eta(\gamma)})$ so that $h(F) \subset \eta(K) \times E$. Since $\dim_{\mathrm{B}} \eta(K) = \dim_{\mathrm{H}} \eta(K)$,

$$\dim_{\mathrm{H}} F = \dim_{\mathrm{H}} h(F) \leq \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{H}} E \leq \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}$$

as required.

Finally, we recall that if $\eta(K)$ satisfies the strong separation condition, then each $x \in K$ is regular by Lemma 11.3 (i). □

Our next goal is to prove that the set of pointwise Assouad dimensions forms an interval. First, for $\mathtt{i} \in \mathcal{I}^n$, let $t(\mathtt{i})$ be chosen so that

$$\sum_{\substack{\mathtt{j}\in\mathcal{I}^n \\ \eta(\mathtt{j})=\eta(\mathtt{i})}} \beta_{\mathtt{j},2}^{t(\mathtt{i})} = 1.$$

Equivalently, the function t is chosen precisely so that

$$\theta_{\eta(\gamma)}(n, m) = t(\gamma_{n+1}, \dots, \gamma_{n+m}).$$

We now have the following result.

Lemma 11.9. *Let K be a Gatzouras–Lalley carpet and suppose $\dim_{\mathrm{L}} K < \alpha < \dim_{\mathrm{A}} K$. Then for all $k_0 \in \mathbb{N}$ sufficiently large, for all $n \in \mathbb{N}$ there is $\mathtt{i}_n \in \mathcal{B}_{k_0}^n \subset \mathcal{I}^{k_0 n}$ satisfying*

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) = \alpha - \dim_{\mathrm{B}} \eta(K).$$

Proof. First, fixing any interior word $\mathtt{j} \in \mathcal{I}^*$ and $i \in \mathcal{I}$ so that $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + t(i)$,

$$\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + \lim_{k \to \infty} t(\mathtt{j} i^k);$$

and similarly for the lower dimension. Thus for all sufficiently large k_0, there are words $\mathtt{j}_L, \mathtt{j}_A \in \mathcal{B}_{k_0}$ so that

$$\dim_{\mathrm{B}} \eta(K) + t(\mathtt{j}_L) < \alpha < \dim_{\mathrm{B}} \eta(K) + t(\mathtt{j}_A).$$

We inductively construct $(\mathtt{j}_{L,k}, \mathtt{j}_{A,k})_{k=1}^{\infty}$ so that, for each $k \in \mathbb{N}$,

1. $\alpha - \dim_{\mathrm{B}} \eta(K) - \frac{1}{k} \leq t(\mathtt{j}_{L,k}) \leq \alpha - \dim_{\mathrm{B}} \eta(K)$,
2. $\alpha - \dim_{\mathrm{B}} \eta(K) \leq t(\mathtt{j}_{A,k}) \leq \dim_{\mathrm{A}} K + \dim_{\mathrm{B}} \eta(K) + \frac{1}{k}$,
3. $\mathtt{j}_{L,k}, \mathtt{j}_{A,k} \in \mathcal{B}_{k_0}^*$ and, for $k \geq 2$, $\mathtt{j}_{L,k}, \mathtt{j}_{A,k} \in \{\mathtt{j}_{L,k-1}, \mathtt{j}_{A,k-1}\}^*$, and
4. $|\mathtt{j}_{L,k}| \geq k$ and $|\mathtt{j}_{A,k}| \geq k$.

First, set $\mathtt{j}_{L,1} = \mathtt{j}_L$ and $\mathtt{j}_{A,1} = \mathtt{j}_A$ which clearly satisfy the desired properties. Now suppose we have constructed $\mathtt{j}_{L,k}$ and $\mathtt{j}_{A,k}$. Since $t(\mathtt{j}_{A,k}) \geq \alpha - \dim_{\mathrm{B}} \eta(K)$, for any $m \in \mathbb{N}$,

$$\lim_{n \to \infty} t(\mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^n) \geq \alpha - \dim_{\mathrm{B}} \eta(K).$$

Moreover, $t(\mathtt{j}_{L,k}^m) \leq \alpha - \dim_{\mathrm{B}} \eta(K)$ and, by taking $m \geq k$ sufficiently large and applying Lemma 2.10, for all $n \in \mathbb{N}$ sufficiently large,

$$|t(\mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^{n+1}) - t(\mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^n)| \leq \frac{1}{k+2} < \frac{1}{k+1}.$$

Combining these two observations, there is a pair m, n so that $\mathtt{j}_{A,k+1} \coloneqq \mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^n \in \mathcal{B}_{k_0}^*$ satisfies conditions 1 and 4. The identical argument gives $\mathtt{j}_{L,k+1} \in \mathcal{B}_{k_0}^*$ satisfying 2, as claimed.

To complete the proof, since $\mathtt{j}_{L,k} \in \mathcal{B}_{k_0}^*$ for all $k \in \mathbb{N}$, we may identify the sequence $(\mathtt{j}_{L,k})_{k=1}^{\infty}$ with a sequence $(\mathtt{i}_n)_{n=1}^{\infty}$ where $\mathtt{i}_n \in \mathcal{B}_{k_0}$ for all $n \in \mathbb{N}$. It immediately follows from 1 and 4 that

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) \geq \alpha - \dim_{\mathrm{B}} \eta(K).$$

To establish the converse bound, it suffices to show for every $k \in \mathbb{N}$ that

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \frac{1}{k}.$$

By 3, for all $k \in \mathbb{N}$, there is a $K \in \mathbb{N}$ so that for all $n \geq K$, $\mathtt{i}_n \in \{\mathtt{j}_{L,k}, \mathtt{j}_{A,k}\}^*$. For each $\ell \in \mathbb{N}$, write $\mathtt{k}_\ell = \mathtt{i}_{K\ell+1} \cdots \mathtt{i}_{K(\ell+1)}$ and note that $\mathtt{k}_\ell \in \{\mathtt{j}_{L,k}, \mathtt{j}_{A,k}\}^*$ for all $\ell \in \mathbb{N}$. Thus for any $n, m \in \mathbb{N}$,

$$t(\mathtt{k}_{\ell+1} \cdots \mathtt{k}_{\ell+m}) \leq \frac{1}{m} \sum_{i=1}^{m} t(\mathtt{k}_{\ell+i}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \frac{1}{k}.$$

But by Lemma 2.6 and the property of t established in Theorem 2.12,

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) = \lim_{m \to \infty} \sup_{n \in \mathbb{N}} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m})$$

which gives the claim. □

To conclude this section, we assemble the results proven in the prior two sections to obtain our main result.

Theorem 11.10. *Let K be a Gatzouras–Lalley carpet. Then for any $\dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{A}} K$,*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\} = \dim_{\mathrm{H}} K. \tag{11.7}$$

Otherwise, if $\alpha \notin [\dim_{\mathrm{B}} K, \dim_{\mathrm{A}} K]$, then $\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\} = \varnothing$. However,

$$\mathcal{H}^{\dim_{\mathrm{H}} K}\big(\{x \in K : \dim_{\mathrm{A}}(K, x) \neq \dim_{\mathrm{A}} K\}\big) = 0. \tag{11.8}$$

Proof. Note that if $\dim_{\mathrm{B}} K = \dim_{\mathrm{A}} K$, then $\dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}} K$ for all $x \in K$ and the results are clearly true. Thus we may assume that $\dim_{\mathrm{H}} K < \dim_{\mathrm{B}} K < \dim_{\mathrm{A}} K$.

We first establish (11.7). Let $\varepsilon > 0$ be arbitrary and $\dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{A}} K$. Apply Proposition 11.4 and get $k \in \mathbb{N}$ and a family $\mathcal{J} \subset \mathcal{B}_k$ with corresponding attractor K_ε satisfying $\dim_{\mathrm{H}} K - \varepsilon \leq \dim_{\mathrm{H}} K_\varepsilon = \dim_{\mathrm{A}} K_\varepsilon$ and $\dim_{\mathrm{B}} \eta(K) - \varepsilon \leq \dim_{\mathrm{B}} \eta(K_\varepsilon)$. If $\alpha < \dim_{\mathrm{A}} K$, iterating the system if necessary, by Lemma 11.9 get a sequence $(\mathtt{i}_n)_{n=1}^\infty$ with $\mathtt{i}_n \in \mathcal{B}_k$ for all $n \in \mathbb{N}$ and moreover

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) = \alpha - \dim_{\mathrm{B}} \eta(K). \tag{11.9}$$

If instead $\alpha = \dim_{\mathrm{A}} K$, instead simply take $\mathtt{i}_n = i_0^k$ where $i_0 \in \mathcal{I}$ is any word such that $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + t(i_0)$. Note that $t(\mathtt{j}) = \dim_{\mathrm{A}} K_\varepsilon - \dim_{\mathrm{B}} \eta(K_\varepsilon)$ for any $\mathtt{j} \in \mathcal{J}$. Thus by taking ε to be sufficiently small, we may assume that $t(\mathtt{j}) \leq \alpha - \dim_{\mathrm{B}} \eta(K)$ for all $\mathtt{j} \in \mathcal{J}$.

Now, let $(N_k)_{k=1}^\infty$ be a sequence of natural numbers satisfying $\lim_{k\to\infty} N_k/k = \infty$ and write

$$\Omega_0 = \prod_{k=1}^\infty \mathcal{J}^{N_k} \times \{\mathtt{i}_1\} \times \cdots \times \{\mathtt{i}_k\}.$$

By taking each N_k to be sufficiently large, we may ensure that $\dim_{\mathrm{H}} \pi(\Omega_0) \geq \dim_{\mathrm{H}} K_\varepsilon - \varepsilon$. Fix $\gamma \in \Omega_0$: it remains to verify that $\dim_{\mathrm{A}}(K, \pi(\gamma)) = \alpha$. Since $\gamma \in \mathcal{B}_k^{\mathbb{N}}$, $\pi(\gamma)$ is a regular point of K by Lemma 11.3 (ii). By passing to the subsystem induced by $\mathcal{B}_k \subset \mathcal{I}^k$, write $\gamma = (\mathtt{k}_k)_{k=1}^\infty$ where $\mathtt{k}_k \in \mathcal{B}_k$. Thus by Theorem 11.8 and Lemma 2.6,

$$\dim_{\mathrm{A}}(K, x) = \max\big\{\dim_{\mathrm{B}} K, \lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m}) + \dim_{\mathrm{B}} \eta(K)\big\}.$$

Since $\mathtt{i}_1 \cdots \mathtt{i}_m$ appears as a subword of γ for arbitrarily large m, by (11.9) and since $\alpha > \dim_{\mathrm{B}} K$, it follows that $\dim_{\mathrm{A}}(K, x) \geq \alpha$.

We now obtain the upper bound. Let $\varepsilon > 0$ be arbitrary. By (11.9), there is an $\ell_0 \in \mathbb{N}$ so that whenever $\ell \geq \ell_0$, we have $t(\mathtt{i}_{j+1} \cdots \mathtt{i}_{j+\ell}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \varepsilon$. Let C be the implicit constant from Lemma 2.10 and let m be sufficiently large so that $C\ell_0/m \leq \varepsilon$. Since $\lim_{k\to\infty} N_k/k = \infty$, for all n sufficiently large, there is a $j \in \mathbb{N}$ so that

$$\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m} = \mathtt{j}_1 \cdots \mathtt{j}_{m-\ell} \mathtt{i}_{j+1} \cdots \mathtt{i}_{j+\ell}.$$

Thus for m, n sufficiently large, if $\ell \geq \ell_0$, by Lemma 2.11,

$$\begin{aligned} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m}) &\leq \max\{t(\mathtt{j}_1 \cdots \mathtt{j}_{m-\ell}), t(\mathtt{i}_{j+1} \cdots \mathtt{i}_{j+\ell})\} \\ &\leq \alpha - \dim_{\mathrm{B}} \eta(K) + \varepsilon \end{aligned}$$

and similarly if $\ell < \ell_0$, recalling that $t(\mathtt{i}_{j+1} \cdots \mathtt{i}_{j+\ell}) \leq 1$, by Lemma 2.10 recalling the definition of C,

$$t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \varepsilon.$$

Therefore

$$\limsup_{m\to\infty} \limsup_{n\to\infty} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \varepsilon$$

and since $\varepsilon > 0$ was arbitrary,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m}) = \limsup_{m\to\infty} \limsup_{n\to\infty} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m}) \leq \alpha - \dim_{\mathrm{B}} \eta(K)$$

so that $\dim_{\mathrm{A}}(K, x) \leq \alpha$, as claimed. Of course, we recall as well that $\dim_{\mathrm{B}} K \leq \dim_{\mathrm{A}}(K, x) \leq \dim_{\mathrm{A}} K$ by Proposition 10.3.

We finally consider the points x such that $\dim_{\mathrm{A}}(K, x) < \dim_{\mathrm{A}} K$. Let $i_0 \in \mathcal{I}$ be such that $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + t(i_0)$. Let

$$\mathcal{J}_M \coloneqq \{(i_1, \ldots, i_M) \in \mathcal{I}^M : (i_1, \ldots, i_M) \neq (i_0, \ldots, i_0)\}$$

have attractor $K_M \subset K$. Since $\mathcal{J}_M$ is a proper subsystem, $\dim_{\mathrm{H}} K_M < \dim_{\mathrm{H}} K$ so that $\mathcal{H}^{\dim_{\mathrm{H}} K}(K_M) = 0$. Now let $x \in K$ have $\dim_{\mathrm{A}}(K, x) < \dim_{\mathrm{A}} K$. Suppose $x = \pi(\gamma)$ where $\gamma = (i_n)_{n=1}^{\infty}$, so that

$$\dim_{\mathrm{A}}(K, x) \geq \max\left\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta(K) + \lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(i_{n+1}, \ldots, i_{n+m})\right\}.$$

Since $\dim_{\mathrm{A}}(K, x) < \dim_{\mathrm{A}} K$,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(i_{n+1}, \ldots, i_{n+m}) < t(i_0).$$

In particular, there is a constant M so that γ does not contain i_0^M as a subword. Thus $x \in K_M$ for some M and therefore

$$\mathcal{H}^{\dim_{\mathrm{H}} K}\left(\{x \in K : \dim_{\mathrm{A}}(K, x) < \dim_{\mathrm{A}} K\}\right) \leq \sum_{M=1}^{\infty} \mathcal{H}^{\dim_{\mathrm{H}} K}(K_M) = 0$$

as required. □

Remark 11.11. We recall that if K is a Gatzouras–Lalley carpet, then $\mathcal{H}^{\dim_{\mathrm{H}} K}(K) > 0$, with $\mathcal{H}^{\dim_{\mathrm{H}} K}(K) < \infty$ if and only if K is Ahlfors regular; see [LG92]. In particular, the positivity of the Hausdorff measure guarantees that the claim (11.8) in Theorem 11.10 is not vacuous; and, if the Hausdorff measure is finite, Theorem 11.10 is trivial.

12 TANGENT STRUCTURE AND DIMENSION OF BARAŃSKI CARPETS

It turns out that the fact that Gatzouras–Lalley carpets have an abundance of large tangents does not extend to the non-dominated setting. More precisely, we prove the following result.

Theorem 12.1. *There exists a Barański carpet K such that*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K.$$

In other words, the conclusion of Theorem 10.4 for uniformly self-embeddable sets does not necessarily extend beyond the uniformly self-embeddable case, even in the at first glance minor generalization consisting only of strictly diagonal self-affine functions acting in $\mathbb{R}^2$. The proof of Theorem 12.1 is given in Corollary 12.7, and it follows from a more general result (Theorem 12.5) describing when Barański carpets satisfying certain separation conditions have a large number of large tangents. This result can be found in Theorem 12.5, and it follows from more general formulas for the pointwise Assouad dimension at points which are coded by sequences which contract uniformly in one direction; see Proposition 12.4 for a precise formulation.

12.1 DIMENSIONS AND DECOMPOSITIONS OF BARAŃSKI CARPETS

Recall the definition of the Barański carpet and basic notation from §3. Suppose K is a Barański carpet and $\gamma \in \Omega$ is arbitrary. For each $k \in \mathbb{N}$, we define a probability vector $\boldsymbol{\xi}_k(\gamma)$ by the rule

$$\boldsymbol{\xi}_k(\gamma)_i = \frac{\#\{1 \leq \ell \leq k : \gamma_\ell = i\}}{k} \quad \text{for each } i \in \mathcal{I}.$$

In other words, $\boldsymbol{\xi}_k(\gamma)$ is the distribution of the letter frequencies in the first k letters of γ. We then define

$$\Gamma_k(\gamma) = \frac{\chi_1(\boldsymbol{\xi}_k(\gamma))}{\chi_2(\boldsymbol{\xi}_k(\gamma))}.$$

The function Γ_k induces a partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ by

$$\begin{aligned}
\Omega_0 &= \{\gamma : \liminf_{k\to\infty} \Gamma_k(\gamma) \leq 1 \leq \limsup_{k\to\infty} \Gamma_k(\gamma)\} \\
\Omega_1 &= \{\gamma : \limsup_{k\to\infty} \Gamma_k(\gamma) < 1\} \\
\Omega_2 &= \{\gamma : 1 < \liminf_{k\to\infty} \Gamma_k(\gamma)\}.
\end{aligned}$$

We now recall the dimensional formula for a general Barański carpet. First, we decompose $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where

$$\mathcal{P}_j = \{\boldsymbol{w} \in \mathcal{P} : \chi_j(\boldsymbol{w}) \leq \chi_{j'}(\boldsymbol{w})\}.$$

Now given a measure $\boldsymbol{w} \in \mathcal{P}_j$, recall [Bar07, Corollary 5.2] which states that

$$\dim_{\mathrm{H}} \pi_* \boldsymbol{w}^{\mathbb{N}} = \frac{H(\eta_j(\boldsymbol{w}))}{\chi_j(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_j(\boldsymbol{w}))}{\chi_{j'}(\boldsymbol{w})}.$$

Here and for the remainder of this document, for notational simplicity, given $j = 1$ we write $j' = 2$ and given $j = 2$ we write $j' = 1$.

We also introduce some notation for symbolic slices both in the horizontal and vertical directions. Given $\gamma \in \Omega$ and $j \in 1, 2$, let $\theta_{\eta_j(\gamma),j}$ be defined by the rule

$$\sum_{(j_1,\dots,j_m)\in\eta_j^{-1}(\eta_j(i_1,\dots,i_n))} \prod_{k=1}^{m} \beta_{j_k,j}^{\theta_{\eta_j(\gamma),j}(n,m)} = 1.$$

The value $\theta_{\eta(\gamma)} = \theta_{\eta_1(\gamma),1}$ was defined previously in the context of a Gatzouras–Lalley carpet. As is the case with a Gatzouras–Lalley carpet, if we denote by $K_{\eta_j(\gamma),j}$ the non-autonomous self-similar set associated with the non-autonomous self-similar IFS $\{S_{i,j} : i \in \eta^{-1}(\eta(\gamma_k))\}_{k=1}^{\infty}$, then

$$\dim_{\mathrm{A}} K_{\eta_j(\gamma),j} = \lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta_j(\gamma),j}(n,m).$$

Assuming $\eta_1(K)$ (resp. $\eta_2(K)$) satisfies the SSC, then $K_{\eta_1(\gamma),1}$ (resp. $K_{\eta_2(\gamma),2}$) is precisely the intersection of K with the vertical (resp. horizontal) line containing $x = \pi(\gamma)$. We now recall [Fra14, Theorem 2.12] concerning the Assouad dimension and the main result of [Bar07] on the Hausdorff dimensions of Barański carpets. While this result is not stated explicitly, the relevant details can be obtained directly by inspecting the proof.

Proposition 12.2 ([Bar07; Fra14]). *Let K be a Barański carpet such that $\Omega_1 \neq \varnothing$ and $\Omega_2 \neq \varnothing$. Then:*

(i) For each $j = 1, 2$,

$$\dim_{\mathrm{H}} \pi(\Omega_0 \cup \Omega_j) \leq d_j$$

where

$$d_j = \max_{\boldsymbol{w}\in\mathcal{P}_j} \left(\frac{H(\eta_j(\boldsymbol{w}))}{\chi_j(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_j(\boldsymbol{w}))}{\chi_{j'}(\boldsymbol{w})} \right).$$

In particular, $\dim_{\mathrm{H}} K = \max\{d_1, d_2\}$.

(ii) We have

$$\dim_{\mathrm{A}} K = \max_{j=1,2} \{\dim_{\mathrm{B}} \eta_j(K) + t_j\}$$

where

$$t_j = \max_{\underline{\ell}\in\eta_j(\mathcal{I})} t_j(\underline{\ell})$$

and $t_j(\underline{\ell})$ is the unique solution to the equation

$$\sum_{j\in\eta_j^{-1}(\underline{\ell})} \beta_{j,2}^{t_j(\underline{\ell})} = 1.$$

12.2 Pointwise Assouad dimension along uniformly contracting sequences

In this section, we state a generalization of our results on Gatzouras–Lalley carpets to Barański carpets, with the caveat that we restrict our attention to points coded by sequences which contract uniformly in one direction. The arguments are similar to the Gatzouras–Lalley case so we only include detail when the proofs diverge. Handling more general sequences would result in a more complicated formula for the pointwise Assouad dimension depending on the scales at which the contraction ratio is greater in one direction than the other, which we will not treat here.

We begin by defining the analogues of pseudo-cylinders and approximate squares. Fix $j = 1, 2$. Suppose $\mathtt{i} \in \mathcal{I}^k$ and $\underline{\mathtt{j}} \in \eta_j(\mathcal{I}^\ell)$. We then write

$$P_j(\mathtt{i}, \underline{\mathtt{j}}) = \{\gamma = (i_n)_{n=1}^\infty \in \Omega : (i_1, \ldots, i_k) = \mathtt{i} \text{ and } \eta_j(i_{k+1}, \ldots, i_{k+l}) = \underline{\mathtt{j}}\}.$$

Now let $\gamma \in \Omega$ be arbitrary and let $k \in \mathbb{N}$. Let j be chosen so that $\beta_{\gamma\upharpoonright_k, j} \geq \beta_{\gamma\upharpoonright_k, j'}$. We then let $L_k(\gamma) \geq k$ be the minimal integer so that

$$\beta_{\gamma\upharpoonright_{L_{k,j}(\gamma)}, j} < \beta_{\gamma\upharpoonright_k, j'}.$$

Write $\gamma\upharpoonright_{L_{k,j}(\gamma)} = \mathtt{i}\mathtt{j}$ and define the approximate square

$$Q_k(\gamma) = P_j(\mathtt{i}, \eta_j(\mathtt{j})).$$

Finally, we call a pseudo-cylinder *wide* if $P_j(\mathtt{i}, \underline{\mathtt{j}})$ contains an approximate square $P_j(\mathtt{i}, \underline{\mathtt{k}})$; otherwise, we call the pseudo-cylinder *tall*.

In the case when the Barański carpet is in fact a Gatzouras–Lalley carpet, these definitions with $j = 1$ coincide with the definitions in the Gatzouras–Lalley case.

Next, the collection of approximate squares forms a metric tree when equipped with the valuation $\rho(P_j(\mathtt{i}, \eta_j(\mathtt{j}))) = \beta_{\mathtt{i}, j'}$. Note that for each approximate square Q, there is a unique choice for j except precisely when $\beta_{\gamma\upharpoonright_k, j} = \beta_{\gamma\upharpoonright_k, j'}$, so indeed ρ is well-defined.

Similarly as in the Gatzouras–Lalley case, given a pseudo-cylinder $P_j(\mathtt{i}, \underline{\mathtt{j}})$, we write

$$\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) = \max\{\mathcal{A} : \mathcal{A} \text{ is a section of } \mathcal{S} \text{ relative to } P_j(\mathtt{i}, \underline{\mathtt{j}})\}$$

where $\mathcal{S}$ is the collection of all approximate squares and the maximum is with respect to the partial ordering on sections. That the maximum always exists follows from the properties of the meet. In the case when the pseudo-cylinder is wide, this coincides precisely with the definition in the Gatzouras–Lalley case.

However, unlike in the Gatzouras–Lalley case, we will also have to handle tall pseudo-cylinders, which have a more complex structure. This additional structure is handled in the following covering lemma.

Lemma 12.3. *(i) Let $P_j(\mathtt{i}, \underline{\mathtt{j}})$ be a wide pseudo-cylinder. Then*

$$\#\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) \approx \left(\frac{\beta_{\mathtt{i}\underline{\mathtt{j}}, j}}{\beta_{\mathtt{i}, j'}}\right)^{\dim_{\mathrm{B}} \eta_j(K)}.$$

(ii) Let $P_j(\mathtt{i}, \underline{\mathtt{j}})$ be a tall pseudo-cylinder. Then

$$\#\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) \lesssim \left(\frac{\beta_{\mathtt{i}, j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}}, j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)}.$$

(iii) *Let $\varepsilon > 0$ be arbitrary. Suppose $\mathtt{i} \in \mathcal{I}^*$ and let j be chosen so that $\beta_{\mathtt{i},j'} \leq \beta_{\mathtt{i},j}$. Let $0 < r \leq \beta_{\mathtt{i},j}$. Then*

$$\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{i}]\} \lesssim_\varepsilon \left(\frac{\beta_{\mathtt{i},j'}}{r}\right)^{\dim_{\mathrm{B}} K+\varepsilon} \cdot \left(\frac{\beta_{\mathtt{i},j}}{\beta_{\mathtt{i},j'}}\right)^{\dim_{\mathrm{B}} \eta_j(K)}.$$

(iv) *Let $\varepsilon > 0$ and $\gamma \in \Omega$ be arbitrary. Suppose $k \in \mathbb{N}$ and $j = 1, 2$ are such that $Q_k(\gamma) = P_j(\mathtt{i}, \underline{\mathtt{j}})$. Let $\mathcal{B}$ be any section of $\mathcal{I}^*$ satisfying $\mathcal{B} \preccurlyeq \eta_j^{-1}(\underline{\mathtt{j}})$. Then*

$$\sum_{\mathtt{k} \in \mathcal{B}} \beta_{\mathtt{k},j'}^{\dim_{\mathrm{A}} K_{\eta_j(\gamma),j}+\varepsilon} \lesssim_{\varepsilon,\gamma} 1.$$

Proof. The proof of (i) is identical to the proof given in Lemma 3.3 and similarly the proof of (iv) is identical to that of Lemma 3.5.

We now prove (ii). In order to do this, we must understand the structure of the pseudo-cylinder $P_j(\mathtt{i}, \underline{\mathtt{j}})$. Heuristically, when (for instance) $j = 1$, $P_j(\mathtt{i}, \underline{\mathtt{j}})$ is a union of cylinders which fall into one of two types: those which are tall, and those which are wide. If a cylinder is tall, we apply (i) in the opposite direction to cover it with approximate squares, and if a cylinder is wide, we group nearby cylinders together to form approximate squares. We then combine these counts using the slice dimension t_j, which is bounded above by $\dim_{\mathrm{B}} \eta_{j'}(K)$.

Write $\mathcal{B} = \eta_j^{-1}(\underline{\mathtt{j}})$ and partition $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where

$$\mathcal{B}_1 = \{\mathtt{k} \in \mathcal{B} : \beta_{\mathtt{ik},j'} \geq \beta_{\mathtt{i}\underline{\mathtt{j}},j}\} \qquad \text{and} \qquad \mathcal{B}_2 = \mathcal{B} \setminus \mathcal{B}_1.$$

First, for $\mathtt{k} \in \mathcal{B}_1$, note that $P_{j'}(\mathtt{ik}, \varnothing)$ is a wide pseudo-cylinder and we set

$$\mathcal{B}_1 = \bigcup_{\mathtt{k} \in \mathcal{B}_1} \mathcal{Q}_{j'}(\mathtt{ik}, \varnothing).$$

By applying (i), since $\beta_{\mathtt{ik},j} \approx \beta_{\mathtt{i}\underline{\mathtt{j}},j}$,

$$\#\mathcal{B}_1 = \sum_{\mathtt{k} \in \mathcal{B}_1} \#\mathcal{Q}_{j'}(\mathtt{ik}, \varnothing) \approx \sum_{\mathtt{k} \in \mathcal{B}_1} \left(\frac{\beta_{\mathtt{ik},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \tag{12.1}$$

Otherwise if $\mathtt{k} \in \mathcal{B}_2$, let $\mathtt{l}_1(\mathtt{k})$ denote the prefix of $\mathtt{k}$ of maximal length so that $\beta_{\mathtt{il}_1(\mathtt{k}),j'} \geq \beta_{\mathtt{i}\underline{\mathtt{j}},j}$. Writing $\mathtt{k} = \mathtt{l}_1(\mathtt{k})\mathtt{l}_2(\mathtt{k})$, this choice guarantees that

$$\mathcal{B}(\mathtt{k}) := P_j(\mathtt{il}_1(\mathtt{k}), \eta_j(\mathtt{l}_2(\mathtt{k})))$$

is the unique approximate square contained in $[\mathtt{i}]$ containing $[\mathtt{ik}]$. Finally, let

$$\mathcal{B}_2' = \{\mathtt{l}_1(\mathtt{k}) : \mathtt{k} \in \mathcal{B}_2\} \qquad \text{and} \qquad \mathcal{B}_2 = \{\mathcal{B}(\mathtt{k}) : \mathtt{k} \in \mathcal{B}_2\}.$$

We then note that, since $\beta_{\mathtt{il},j'} \approx \beta_{\mathtt{i}\underline{\mathtt{j}},j}$ by the choice of $\mathtt{l}_1(\mathtt{k})$,

$$\#\mathcal{B}_2 \approx \sum_{\mathtt{l} \in \mathcal{B}_2'} \left(\frac{\beta_{\mathtt{il},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \tag{12.2}$$

To conclude, observe that $\mathcal{Q}_j(\mathtt{i},\underline{\mathtt{j}}) = \mathcal{B}_1 \cup \mathcal{B}_2$ and applying (12.1) and (12.2),

$$\begin{aligned}
\#\mathcal{Q}_j(\mathtt{i},\underline{\mathtt{j}}) &= \#\mathcal{B}_1 + \#\mathcal{B}_2 \\
&\lesssim \sum_{\mathtt{k}\in\mathcal{B}_1} \left(\frac{\beta_{\mathtt{ik},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} + \sum_{\mathtt{l}\in\mathcal{B}_2'} \left(\frac{\beta_{\mathtt{il},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \\
&= \left(\frac{\beta_{\mathtt{i},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \sum_{\mathtt{k}\in\mathcal{B}_1\cup\mathcal{B}_2'} \beta_{\mathtt{k},j'}^{\dim_{\mathrm{B}} \eta_{j'}(K)} \\
&\leq \left(\frac{\beta_{\mathtt{i},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)}
\end{aligned}$$

where the last line follows since $\mathcal{B}_1 \cup \mathcal{B}_2' \preccurlyeq \eta_j^{-1}(\underline{\mathtt{j}})$ is a section and $\dim_{\mathrm{B}} \eta_{j'}(K) \geq t_j(\underline{\mathtt{j}})$ where

$$\sum_{\mathtt{k}\in\mathcal{B}_1\cup\mathcal{B}_2'} \beta_{\mathtt{k},j'}^{t_j(\underline{\mathtt{j}})} = 1.$$

Finally, we combine the bounds given in (i) and (ii) with a similar argument to the proof of Lemma 3.4 to obtain (iii). Let $\varepsilon > 0$ be arbitrary and fix $\mathtt{i} \in \mathcal{I}^*$ and $j = 0, 1$ so that $0 < r \leq \beta_{\mathtt{i},j'} \leq \beta_{\mathtt{i},j}$. Write $\delta = r/\beta_{\mathtt{i},j'}$ so, recalling the proof of [Bar07, Theorem B],

$$\#\mathcal{S}(\delta) \lesssim_{\varepsilon} (1/\delta)^{\dim_{\mathrm{B}} K+\varepsilon}.$$

Now enumerate

$$\mathcal{S}(\delta) = \{Q_{1,j}, \ldots, Q_{m_j,j}\} \cup \{Q_{1,j'}, \ldots, Q_{m_{j'},j'}\}$$

where for each $z = j, j'$ and $1 \leq i \leq m_z$,

$$Q_{i,z} = P_z(\mathtt{j}_{i,z}, \underline{\mathtt{k}}_{i,z})$$

for some $\mathtt{j}_{i,z} \in \mathcal{I}^*$ and $\underline{\mathtt{k}}_{i,z} \in \eta_z(\mathcal{I}^*)$. Observe that each $P_z(\mathtt{i}\mathtt{j}_{i,z}, \underline{\mathtt{k}}_{i,z})$ is a wide pseudo-cylinder if $z = j$ and a tall pseudo-cylinder if $z = j'$. Thus we may complete the proof in the same way as Lemma 3.4, by applying (i) to the wide pseudo-cylinders and (ii) to the tall pseudo-cylinders. $\square$

We can now prove the following formulas for the pointwise Assouad dimension.

Proposition 12.4. *Let K be a Barański carpet. Then for each $j = 1, 2$, if $\eta_j(K)$ satisfies the SSC, for all $\gamma \in \Omega_j$ and $x = \pi(\gamma)$,*

$$\dim_{\mathrm{A}}(K, x) = \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}\}$$

and

$$\max\{\dim_{\mathrm{H}} F : F \in \mathrm{Tan}(K, x)\} = \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}.$$

Furthermore,

$$\dim_{\mathrm{A}} K_{\eta_j(\gamma),j} = \lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta_j(\gamma),j}(n, m) \leq \max_{\underline{\ell}\in\eta_j(\mathcal{I})} t_j(\underline{\ell}).$$

Proof. If $\gamma \in \Omega_j$, by definition there is a constant $C \in (0,1)$ so that

$$\frac{\beta_{\gamma|_k,j'}}{\beta_{\gamma|_k,j}} \lesssim C^n.$$

In particular, there is a constant $C' \in (0,1)$ so that each maximal cylinder $[\mathtt{i}]$ contained in $Q_k(\gamma)$ has $\beta_{\mathtt{i},j'}/\beta_{\mathtt{i},j} \lesssim (C')^k$, which converges to zero. Thus the same proof as given in Proposition 11.7 but instead applying Lemma 12.3 in place of the analogous bounds for Gatzouras–Lalley carpets gives that

$$\dim_{\mathrm{A}}(K,x) \le \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}\}.$$

Similarly, the same proof as Proposition 11.6 shows that

$$\max\{\dim_{\mathrm{H}} F : F \in \mathrm{Tan}(K,x)\} = \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}.$$

Finally, using Lemma 2.11,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta_j(\gamma),j}(n,m) \le \max_{\underline{\ell}\in\eta_j(\mathcal{I})} t_j(\underline{\ell}).$$

as required. $\square$

12.3 BARAŃSKI CARPETS WITH FEW LARGE TANGENTS

In contrast to Gatzouras–Lalley carpets, the analogue of Theorem 11.10 need not hold for Barański carpets. We first give a precise characterization of when a Barański carpet has few large tangents. Fix the definitions of t_j and d_j from Proposition 12.2.

Theorem 12.5. *Let K be a Barański carpet such that $\eta_j(K)$ satisfies the SSC and $\Omega_j \neq \varnothing$ for $j = 1, 2$. Suppose for one of $j = 1, 2$, $d_j < d_{j'}$ and $\dim_{\mathrm{B}} \eta_j(K) + t_j > \dim_{\mathrm{B}} \eta_{j'}(K) + t_{j'}$. Then*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K.$$

Proof. Suppose $d_1 < d_2$ and $\dim_{\mathrm{B}} \eta_1(K) + t_1 > \dim_{\mathrm{B}} \eta_2(K) + t_2$ (the opposite case follows analogously). By Proposition 12.2, $\dim_{\mathrm{H}} K = d_2$ and $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta_1(K) + t_1$. In particular, by Proposition 12.4, if $\dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta_1(K) + t_1$, then necessarily $x = \pi(\gamma)$ where $\gamma \in \Omega_0 \cup \Omega_1$. But $\dim_{\mathrm{H}} \pi(\Omega_0 \cup \Omega_1) = d_1 < d_2 = \dim_{\mathrm{H}} K$, as required. $\square$

Remark 12.6. In the context of Theorem 12.5, one can in fact prove that the following are equivalent:

(i) $\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K$.
(ii) $\dim_{\mathrm{H}}\{x \in K : \exists F \in \mathrm{Tan}(K,x)$ such that $\dim_{\mathrm{H}} F = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K$.
(iii) For one of $j = 1, 2$, $d_j < d_{j'}$ and $\dim_{\mathrm{B}} \eta_j(K) + t_j > \dim_{\mathrm{B}} \eta_{j'}(K) + t_{j'}$.

Such a proof follows similarly to the Gatzouras–Lalley case with appropriate modifications to restrict attention only to the family Ω_1 or Ω_2. The only additional observation required is that [FJS10, Lemma 4.3] also holds in the Barański case and the uniform subsystem can be chosen so the maps are contracting strictly in direction j and the dimension of the corresponding attractor is arbitrarily close to d_j.

In particular, if one of the above equivalent conditions hold and without loss of generality $d_1 > d_2$ and $\dim_{\mathrm{B}} \eta_1(K) + t_1 < \dim_{\mathrm{B}} \eta_2(K) + t_2$, then the Hausdorff dimension of the level set $\varphi(\alpha) = \dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\}$ is given by the piecewise formula

$$\varphi(\alpha) = \begin{cases} \dim_{\mathrm{H}} K & : \dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{B}} \eta_1(K) + t_1 \\ d_2 & : \dim_{\mathrm{B}} \eta_1(K) + t_1 < \alpha \leq \dim_{\mathrm{A}} K. \end{cases}$$

We leave the remaining details to the curious reader.

With Theorem 12.5 in hand, we can now give an explicit example of a Barański carpet which has few large tangents.

Corollary 12.7. *There is a Barański carpet K such that*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K.$$

Proof. Fix some $\delta \in [0, 1/6)$ and define parameters $\beta = 1/4 - \delta$, $\alpha_1 = 1/3 - \delta$, and $\alpha_2 = 1/6 - \delta$. Now define the families of maps

$$\begin{aligned} \Phi_1 &= \{(x, y) \mapsto (\alpha_1 x, \beta y + i\beta) : i = 0, \ldots, 3\} \\ \Phi_{2,a} &= \{(x, y) \mapsto (\alpha_2 x + \alpha_1 + j\alpha_2, \beta y + i\beta) : j = 0, 1 \text{ and } i = 0, 1\} \\ \Phi_{2,b} &= \{(x, y) \mapsto (\alpha_2 x + \alpha_1 + j\alpha_2, \beta y + i\beta) : j = 3, 4 \text{ and } i = 2, 3\} \end{aligned}$$

and then set

$$\Phi_2 = \Phi_{2,a} \cup \Phi_{2,b} \qquad \text{and} \qquad \Phi = \Phi_1 \cup \Phi_{2,a} \cup \Phi_{2,b}.$$

We abuse notation and use functions and indices interchangeably. Note that Φ is a Barański IFS with five columns; the carpet is conjugate to the carpet generated by the maps depicted in Figure I.1b. Note that if $\delta > 0$, both projected IFSs satisfy the SSC.

We now simplify the dimensional expression in Proposition 12.2 (ii) for our specific system. First, for $\boldsymbol{w} \in \mathcal{P}$, set $p = \sum_{i \in \Phi_2} \boldsymbol{w}_i$. Note that $\chi_1(\boldsymbol{w}) = -p \log \alpha_2 - (1-p) \log \alpha_1$ and $\chi_2(\boldsymbol{w}) = -\log \beta$ depend only on p. But since entropy and projected entropy are maximized uniquely by uniform vectors, defining the vector $\boldsymbol{z}(p) \in \mathcal{P}$ by

$$\boldsymbol{z}(p)_i = \begin{cases} \frac{1-p}{4} : i \in \Phi_1 \\ \frac{p}{8} : i \in \Phi_2 \end{cases}$$

we necessarily have

$$\begin{aligned} \frac{H(\eta_1(\boldsymbol{w}))}{\chi_1(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_1(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} &\leq \frac{H(\eta_1(\boldsymbol{z}(p)))}{\chi_1(\boldsymbol{z}(p))} + \frac{H(\boldsymbol{z}(p)) - H(\eta_1(\boldsymbol{z}(p)))}{\chi_2(\boldsymbol{z}(p))} \\ &= \frac{-p \log p - (1-p) \log(1-p) + p \log 4}{-p \log \alpha_2 - (1-p) \log \alpha_1} \end{aligned}$$

$$+\frac{(2-p)\log 2}{-\log\beta}$$
$$=: D_1(p)$$

and

$$\begin{aligned}\frac{H(\eta_2(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})}+\frac{H(\boldsymbol{w})-H(\eta_2(\boldsymbol{w}))}{\chi_1(\boldsymbol{w})} &\le \frac{H(\eta_2(\boldsymbol{z}(p)))}{\chi_2(\boldsymbol{z}(p))}+\frac{H(\boldsymbol{z}(p))-H(\eta_2(\boldsymbol{z}(p)))}{\chi_1(\boldsymbol{z}(p))}\\ &= \frac{-p\log p-(1-p)\log(1-p)+p\log 2}{-p\log\alpha_2-(1-p)\log\alpha_1}+\frac{\log 4}{-\log\beta}\\ &=: D_2(p).\end{aligned}$$

Moreover, writing $p_0=\frac{\log\alpha_1-\log\beta}{\log\alpha_1-\log\alpha_2}$, $\boldsymbol{z}(p)\in\mathcal{P}_1$ if and only if $p\in[0,p_0]$ and $\boldsymbol{z}(p)\in\mathcal{P}_2$ if and only if $p\in[p_0,1]$. We thus observe that

$$\dim_{\mathrm{H}} K=\sup_{p\in[0,1]} D(p)\qquad\text{where}\qquad D(p)=\begin{cases}D_1(p) & : 0\le p\le p_0\\ D_2(p) & : p_0\le p\le 1\end{cases}.$$

Now, a manual computation directly shows that, substituting $\delta=0$,

$$\sup_{p\in[0,1]} D_1(p)\approx 0.489536\qquad\text{and}\qquad\sup_{p\in[0,1]} D_2(p)\approx 0.529533$$

and moreover the maximum of $D_2(p)$ is attained at a value $p_2\in(p_0,1)$. Thus for all δ sufficiently close to 0, since all the respective quantities are continuous functions of δ, there is a value $p_2\in(p_0,1)$ so that

$$d_1\le\sup_{p\in[0,1]} D_1(p)<\sup_{p\in[0,1]} D(p)=D_2(p_2)=d_2.$$

(In fact, one can show that this is the case for all $\delta\in(0,1/6)$, but this is not required for the proof.)

On the other hand, when $\delta=0$, $t_1=2$ whereas $t_2=1+s<2$ where $s\approx 0.72263$ is the unique solution to

$$\left(\frac{1}{3}\right)^s+2\cdot\left(\frac{1}{6}\right)^s=1.$$

Thus for all δ sufficiently close to 0, the conditions for Theorem 12.5 are satisfied, as required. □

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