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ON THE DIFFERENCE OF THE ENHANCED POWER GRAPH AND THE POWER GRAPH OF A FINITE GROUP

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ABSTRACT. The difference graph $D(G)$ of a finite group G is the difference of the enhanced power graph of G and the power graph of G, where all isolated vertices are removed. In this paper we study the connectedness and perfectness of $D(G)$ with respect to various properties of the underlying group G. We also find several connections between the difference graph of G and the Gruenberg-Kegel graph of G. We also examine the operation of twin reduction on graphs, a technique which produces smaller graphs which may be easier to analyse. Applying this technique to simple groups can have a number of outcomes, not fully understood, but including some graphs with large girth.

1. INTRODUCTION

The study of graphs related to various algebraic structures has been a topic of increasing interest during the last two decades. This kind of study helps us to (1) characterize the resulting graphs, (2) characterize the algebraic structures with isomorphic graphs, and (3) also to realize the interdependence between the algebraic structures and the corresponding graphs. Many different types of graphs, including among many others the commuting graph [6], generating graph [21], power graph [15, 28], enhanced power graph [1, 3, 4], and comaximal subgroup graph [18], have been introduced to explore the properties of algebraic structures using graph theory. The concept of a power graph was introduced in the context of semigroup theory by Kelarev and Quinn [27].

Definition 1.1. Let G be a group. The power graph $\mathsf{Pow}(G)$ is an undirected simple graph defined on G as the set of vertices, in which two distinct vertices a and b are adjacent if a is a power of b or b is a power of a, i.e., $a \in \langle b \rangle$ or $b \in \langle a \rangle$.

The enhanced power graph of a group was introduced by Alipour et al. in [1] as follows.

Definition 1.2. Let G be a group. The enhanced power graph $\mathsf{EPow}(G)$ is an undirected simple graph defined on G as the set of vertices and two distinct vertices a and b are adjacent if there exists $c \in G$ such that both a and b are powers of c, i.e., $a, b \in \langle c \rangle$, i.e., if $\langle a, b \rangle$ is a cyclic group.

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Recently, in a survey, Cameron [9] introduced various open questions on graphs defined on groups. One of them is regarding the difference of enhanced power graph and power graph of a group.

From Proposition 2.6 in [9], we see that both $\mathsf{Pow}(G)$ and $\mathsf{EPow}(G)$ are graphs on same vertex set G and $E(\mathsf{Pow}(G)) \subseteq E(\mathsf{EPow}(G))$. $\mathsf{EPow}(G) - \mathsf{Pow}(G)$ denotes the graph with G as the set of vertices and two vertices a and b are adjacent if they are adjacent in $\mathsf{EPow}(G)$ but not adjacent in $\mathsf{Pow}(G)$. Motivated by Section 3.2 of [9], we define the following graph;

Definition 1.3. Let G be a group. The difference graph $D(G)$ is defined to be the graph $\mathsf{EPow}(G) - \mathsf{Pow}(G)$, with isolated vertices removed.

Given this, we need to understand the set of isolated vertices which are removed, that is, vertices which have the same neighbourhoods in $\mathsf{Pow}(G)$ and $\mathsf{EPow}(G)$. This is done in the next two sections, where we also note a connection between $D(G)$ and the Gruenberg–Kegel graph of G. (This graph, sometimes called the prime graph, is connected with several graphs defined on G, as discussed in the survey [10].)

The next main result of this work shows the universality of the difference graph $D(G)$. Theorem 4.7 shows that given any finite graph Γ, there exists a finite abelian group G such that Γ is an induced subgraph of $D(G)$.

We next concentrate on the connectedness of the difference graph. There have been many works on connectivity of various graphs in recent times. The authors Aalipour et al. in [1, Question 40] asked about the connectivity of power graphs when all the dominating vertices are removed. Later, Cameron and Jafari [11] answered this question for power graphs and Bera *et.al.* [4] answered this question for enhanced power graphs. Regarding connectedness of $D(G)$, our main results are:

- If G is a finite group which is not a p-group and with non-trivial center, then $D(G)$ is connected and $\mathrm{diam}(D(G)) \leq 6$. (This is Theorem 5.1.) When G is nilpotent and not a p-group, we showed that the diameter is ≤ 4.
- The difference graph of the symmetric group S_n is connected if and only if $n \geq 8$ (Theorem 6.1) and the difference graph of the alternating group A_n is connected for $n \geq 12$. (Theorem 6.10)
- If G_1, G_2, G_3 are three finite nontrivial groups such that $G_1 \times G_2 \times G_3$ is not a p-group, then $D(G_1 \times G_2 \times G_3)$ is connected. (Theorem 7.1)
- $D(D_n)$ is a connected graph if and only if n is not a prime power. (This is Theorem 5.5). $D(D_n \times D_m)$ is disconnected if and only if n and m are powers of same odd prime. (Theorem 7.4)

Moving on, we are also interested in the perfectness of the difference graph $D(G)$. The motivation for studying the perfectness stems from the fact that the power graph of a finite group is always perfect (Theorem 5, [2]) but the question that for which finite groups, the enhanced power graph is perfect is still unresolved, although the chromatic number of the enhanced power graph is now known [13], and these graphs are weakly perfect.

In this context we prove the following results:

- $D(\mathbb{Z}_n)$ is perfect. (Theorem 9.2)
- Let G be a group of order pq, p^2q, p^2q^2, p^3q or pqr, where p, q, r are distinct primes. Then $D(G)$ is perfect. (Theorem 9.4)
- We also classify the finite nilpotent groups G for which $D(G)$ is perfect. (Theorem 9.6)
- We give several further examples of groups whose difference graphs are perfect, and others whose difference graphs are imperfect, including a number of finite simple or almost simple groups. Here are some examples.
  Perfect difference graph: S_n ($n \leq 7$), A_n ($n \leq 8$), $\mathrm{PSL}(2, q)$ ($q \geq 4$), $\mathrm{Sz}(q)$ ($q \geq 8$), $\mathrm{PSU}(3, q)$ for $q = 3, 4, 5$, all simple groups smaller than J_1 (the first Janko sporadic group);
  Imperfect difference graph: S_n ($n \geq 8$), A_n ($n \geq 9$), J_1 (and all sporadic groups containing J_1), groups of Lie type of rank at least 3 over the field of q elements if $q - 1$ has at least three distinct prime divisors.

The final section concerns the operation of twin reduction, which can produce smaller graphs while preserving some properties of interest; we apply this to the difference graphs of several simple or almost simple groups.

2. Isolated vertices

It is proved in Aalipour *et al.* [1] that the power graph and enhanced power graph of G are equal if and only if every element of G has prime power order. Groups with these properties are known as EPPO groups. After pioneering work by Higman [25] (who classified the soluble ones in the 1950s) and Suzuki [35] (who classified the simple ones in the 1960s), Brandl [5] gave a description of EPPO groups in 1981. The paper was not well-known, and several authors published similar results. An accessible account appears in the survey [10].

EPPO groups are those groups G for which $D(G)$ has no edges. The next obvious question is: Which are the isolated vertices which are deleted in the construction of $D(G)$? To simplify the discussion, in this section and the next we usually consider the graph $D(G)$ *before* deleting isolated vertices.

For any element a in a group G, $o(a)$ denotes the order of the element a in the group G.

Proposition 2.1. *Let G be a group with order greater than 1. Then the non-identity element g is an isolated vertex in $D(G)$ if and only if either $\langle g\rangle$ is a maximal cyclic subgroup of G, or every cyclic subgroup of G containing g has prime-power order.*

Proof. If $\langle g\rangle$ is a maximal cyclic subgroup, and g and h are joined in $D(G)$, then $\langle g, h\rangle$ is cyclic, so $h \in \langle g\rangle$, so g and h are joined in the power graph, a contradiction.

If every cyclic subgroup of G containing g has prime power order, and g and h are joined in $D(G)$, then $\langle g, h\rangle$ has prime power order, whence g and h are joined in the power graph, again a contradiction.

Conversely suppose that $\langle g\rangle$ is properly contained in a cyclic subgroup $\langle h\rangle$ of G whose order is not a prime power. Then there is a prime p such that the p-part of $o(h)$ is greater than that of $o(g)$. We may also assume that, if $o(g)$ is a power of a prime q, then $p \neq q$. Let k be an element in $\langle h \rangle$ whose order is the p-part of $o(h)$. Then g and k are joined in the enhanced power graph but not in the power graph, so they are joined in $D(G)$. □

Since every finite group has a maximal cyclic subgroup, we see that the set of isolated vertices which are deleted in the definition is non-empty.

3. Connection with the Gruenberg–Kegel graph

The *Gruenberg–Kegel graph* (or GK-graph, for short) of a finite group G is the graph whose vertices are the prime divisors of $|G|$, with primes p and q joined by an edge if and only if G contains an element of order pq.

This was introduced by Gruenberg and Kegel in their study of the integral group ring of a finite group, and showed that the augmentation ideal is decomposable if and only if the GK-graph is disconnected. They proved a structure theorem for graphs with disconnected GK-graph, but did not publish it; a proof was given by Gruenberg's student Williams [36], and refined by subsequent authors. We note a connection with the preceding section:

Assertion 3.1. *The finite group G is an EPPO group if and only if its GK graph has no edges.*

We omit the proof as it is clear.

The GK graph of a finite group has several connections with various graphs defined on groups (some of these are listed in [10]). We can add another one here.

Theorem 3.2. *Let G be a finite group whose order is divisible by the prime p. Then the following are equivalent:*

(a) *every element of order p is an isolated vertex in $D(G)$;*
(b) *every element of p-power order is an isolated vertex in $D(G)$;*
(c) *p is an isolated vertex in the GK graph of G;*
(d) *the centralizer of every element of order p is a p-group.*

Proof. If (a) fails, then some element of order p is contained in a cyclic subgroup whose order is not a power of p, and hence contains an element of order $q \neq p$; so (c) and (d) also fail. If (a) is true, then no element of order p can commute with any element of order coprime to p, so (b), (c) and (d) hold also. Moreover, clearly (b) implies (a). □

Groups satisying (d) of the above theorem have had a lot of attention, especially in the case $p = 2$, where they are called *CIT groups* (the centralizer of an involution is a 2-group). The simple CIT groups were determined by Suzuki [34, 35]. For odd p, when such groups are known as C*pp* groups, Higman and his students have a number of results, for some of which we refer to [26, 10]. In particular, the non-solvable C33 groups were classified in [33], and the C55 groups in [19].

Theorem 3.3. *Let G be a finite group which is not of prime power order, and suppose that $D(G)$ is connected. Then the Gruenberg–Kegel graph of G is connected, apart from possibly some isolated vertices.*

Proof. Let $\{g,h\}$ be an edge of $D(G)$. Then g and h are contained in a cyclic group, and neither of $o(g)$ and $o(h)$ divides the other. So there is a prime p dividing $o(g)$ to a higher power than $o(h)$, and a prime q dividing $o(h)$ to a higher power than $o(g)$. Put $g' = g^{o(g)/p}$ and $h' = h^{o(h)/q}$. Then g' has order p, h' has order q, and $\{p,q\}$ is an edge in the GK graph of G.

Now let (g,h,k) be a path of length 2 in $D(G)$. Then there are elements g' and h' of orders p and q as in the above paragraph, and elements h'' and k'' of prime orders r and s so that $\{h'',k''\}$ is an edge of $D(G)$. If $r = q$ then we can assume that $h' = h''$ and we have a path of length 2; otherwise, $\{h',h''\}$ is an edge of $D(G)$, since $h',h'' \in \langle h \rangle$. Thus we can replace the path of length 2 by a walk of length 2 or 3 all of whose vertices have prime order; their orders are the vertices of a walk in the GK graph.

Now suppose that $D(G)$ is connected. Choose two primes p and q which divide $|G|$, and take elements x and y of orders p and q respectively. We may assume that x and y are not isolated in $D(G)$ (If they are isolated, then their corresponding orders are isolated vertices in GK). By hypothesis, there is a path from x to y, which by the previous construction gives us a walk from p to q in the GK graph. So the GK graph is connected. □

We note that the converse of this result is false. For example, consider the group $G = S_3 \times S_3$. The GK graph has two vertices, the primes 2 and 3, joined by an edge. The vertices of $D(G)$ are the non-identity elements of the two direct factors, and the graph consists of a complete bipartite graph on the elements of order 2 in the first factor and those of order 3 in the second, and another complete bipartite graph where the roles of the two factors are reversed.

4. Independence Number, Clique Covering Number and Universality

We recall a few results and prove some lemmas that will be crucial in the forthcoming sections.

4.1. **Cyclic groups.** Suppose that $G = \mathbb{Z}_n$, the cyclic group of order n. Then the enhanced power graph of G is complete, and so $D(G)$ is the complement of the power graph. We can use this to read off some parameters of $D(G)$, using results from [28, Section 8.2].

- The independence number and clique cover number of $D(G)$ are equal to the clique number and chromatic number of $\mathsf{Pow}(G)$. These numbers are equal, since $\mathsf{Pow}(G)$ is perfect, and the common value is given by $f(n)$, where f is defined by the recursion

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ \phi(n) + f(n/p) & \text{if } n > 1, \end{cases}$$

where ϕ is Euler's totient and p is the smallest prime divisor of n. It satisfies $\phi(n) \leq f(n) \leq c\phi(n)$, where

$$c = \sum_{n \geq 0} \prod_{i=1}^{n} \frac{1}{p_i - 1},$$

where $p_1, p_2, \ldots$ are the primes in order; so $c = 2.6481017597\ldots$. (We note that these values include the isolated vertices; the number of these should be subtracted to get the values in the graph $D(G)$ as we have defined it.)
- The clique number and chromatic number of $D(G)$ are equal to the independence number and clique cover number of $\mathsf{Pow}(G)$, which are equal, and equal to the size of the largest antichain of divisors of n, as we discuss later.
- $D(G)$ is perfect. (This follows from the weak perfect graph theorem of Lovász [29].)

See [28, 13] for proofs and further details.

4.2. **Commuting elements and nilpotency.**

Proposition 4.1. *A finite group G is nilpotent if and only if for all $x, y \in G$ with $\gcd(o(x), o(y)) = 1$ we have $xy = yx$.*

We omit the proof as it is straightforward.

Proposition 4.2 ([4], Lemma 2.5)**.** *Let G be a finite group and let a, b be non-identity elements of G such that $ab = ba$ and $\gcd(o(a), o(b)) = 1$. Then $a \sim b$ in $D(G)$.*

From the above two propositions, we have the following corollary

Corollary 4.3. *In a finite nilpotent group G, if $\gcd(o(x), o(y)) = 1$, then $x \sim y$ in $D(G)$.*

Lemma 4.4. *Let H be a subgroup of a group G. Then $D(H)$ is an induced subgraph of $D(G)$.*

Proof. If the statements "y is a power of x" and "$\langle x, y\rangle$ is cyclic" are true in G then they are true in any subgroup of G containing x and y. □

Proposition 4.5. *Let p be a prime. Then the set of elements of p-power order in $D(G)$ contains no edges.*

Proof. Suppose that $\{x, y\}$ is an edge, where both x and y have prime power order. Then x and y are contained in a cyclic group of prime power order, so one is a power of the other, a contradiction. □

Lemma 4.6. *Let G be a group and $x \in G$ be a (non-isolated) vertex in $D(G)$.*

(a) *If $o(x) = p^\alpha$ for some prime p, then there exists a prime $q (\neq p)$ and $y \in G$ such that $o(y) = q^\beta$ and $x \sim y$ in $D(G)$.*

(b) *If $o(x)$ is not a prime power, then there exists a prime p and $y \in G$ such that $o(y) = p^\alpha$ and $x \sim y$ in $D(G)$.*

Proof. As x is not isolated, from Proposition 2.1 we have that $\langle x\rangle$ is not a maximal cyclic subgroup and not every cyclic subgroup of G containing x is of prime-power order. Therefore there exists a cyclic subgroup H of G containing x and which is not of prime-power order. Hence H must contain an element y of order q^β for some prime q and clearly, $x \sim y$. If $o(x)$ is p^α, we will of course get such a y where $p \neq q$. This completes the proof. □

4.3. Universality.

Theorem 4.7. *The class of difference graphs of groups is universal: that is, given any graph Γ, there exists a finite abelian group G such that Γ is an induced subgraph of $D(G)$.*

Proof. The proof is by induction on the number of vertices of Γ. For a graph with a single vertex, it is obvious. Assume that the result holds for all graphs with $n-1$ vertices. Now let Γ be a graph with vertex set $\{1, 2, \cdots, n\}$, and Γ' be its induced subgraph on the vertices $\{1, 2, \ldots, n-1\}$. From induction hypothesis, let φ be an isomorphism from Γ' to an induced subgraph of $D(G)$, for a finite abelian group G.

Let p be a prime not dividing the order of G, and let $H = \langle a, b\rangle$ be an elementary abelian group of order p^2. Consider the group $G \times H$ and the map $\tilde{\varphi} : \Gamma \to G \times H$ given by

$$\tilde{\varphi}(i) = \begin{cases} (\varphi(i), a) & \text{if } i < n \text{ and } (i,n) \notin E(\Gamma) \\ (\varphi(i), e') & \text{if } i < n \text{ and } (i,n) \in E(\Gamma) \\ (e, b) & \text{if } i = n \end{cases}$$

where e, e' are the identity elements of the groups G and H respectively.

Since $p \nmid |G|$, for any $z \in G$ we have $\langle (z, e'), (e, b)\rangle = \langle (z, e')\rangle \times \langle (e, b)\rangle$, which is cyclic and not generated by either element. So the embedding of $\{1, 2, \cdots, n-1\}$ given by the restriction of $\tilde{\varphi}$ is still an isomorphism to an induced subgraph. Moreover, $\langle (\varphi(i), e'), (e, b)\rangle$ is cyclic while $\langle (\varphi(i), a), (e, b)\rangle$ is not, so we have the correct edges from (e, b) to the other vertices, and the result is proved. □

5. When G has non-trivial center

In this section, we deal with $D(G)$ when G has a non-trivial center. If G is a finite p-group, then its every cyclic subgroup has prime power order and hence by Theorem 3.2, $\mathsf{EPow}(G) = \mathsf{Pow}(G)$, i.e., $D(G)$ is the null graph. So, we focus on non p-groups.

Theorem 5.1. *Let G be a finite group which is not a p-group. If G has non-trivial center, then $D(G)$ is connected and* $\operatorname{diam}(D(G)) \leq 6$*.*

Proof. As $|Z(G)| > 1$, there exists a prime p such that $p \mid |Z(G)|$. Thus there exists $z \in Z(G)$ with $o(z) = p$. We show that any vertex in $D(G)$ is joined by a path to z. Let $x \in V(D(G))$. Then by Lemma 4.6, there exists $y \in V(D(G))$ such that $x \sim y$ and $o(y) = q^\alpha$. If $p \neq q$, we have $x \sim y \sim z$. If $p = q$, as $y \in V(D(G))$, again by Lemma 4.6, there exists a prime $r \neq p$ and an element $y' \in V(D(G))$ such that $y \sim y'$ and $o(y') = r^\beta$. Thus $x \sim y \sim y' \sim z$, i.e.,

$d(x,z) \leq 3$. Similarly, for another vertex x', $d(x',z) \leq 3$. (Note that z is same in both the cases) Hence for two arbitrary vertices x, x', we have $d(x,x') \leq 6$ $\square$

Remark 5.2. The bound in the above theorem is tight. Take $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_{12}$ (GAP ID: (84,1)) [20]. It is a non-nilpotent, super-solvable group with $|Z(G)| = 2$ and $\operatorname{diam}(D(G)) = 6$. If G is a finite nilpotent group which is not a p-group, then G has a non-trivial center, and as a result, by Theorem 5.1, $D(G)$ is connected and $\operatorname{diam}(D(G)) \leq 6$. However, for nilpotent groups this bound can be improved to 4.

Theorem 5.3. *If G is a finite nilpotent group which is not a p-group, then $D(G)$ is connected and* $\operatorname{diam}(D(G)) \leq 4$.

Proof. Let $\pi(G)$ denote the set of prime divisors of $|G|$. As G is a not a p-group, $|\pi(G)| \geq 2$. Again, as G is a finite nilpotent group, G is the direct product of its Sylow subgroups, say $P_1, P_2, \ldots, P_k$, where $k \geq 2$. Also, $Z(G) = Z(P_1) \times Z(P_2) \times \cdots \times Z(P_k)$, i.e., $|Z(G)|$ has atleast two distinct prime factors.

Let $x, x' \in V(D(G))$. Then by Lemma 4.6, there exists $y, y' \in V(D(G))$ such that $x \sim y, x' \sim y'$ and $o(y) = p^\alpha$, $o(y') = q^\beta$ for primes p and q. If $p \neq q$, then by Corollary 4.3, $y \sim y'$ and $d(x,x') \leq 3$. If $p = q$, as $|Z(G)|$ has atleast two distinct prime factors, there exists an element $z \in Z(G)$ of prime order $r (\neq p)$ such that $y \sim z \sim y'$. In this case, we have $d(x,x') \leq 4$. $\square$

Remark 5.4. The bound in the above theorem is strict: Take $G = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_6$. Then using GAP it can be checked that $\operatorname{diam}(D(G)) = 4$ and $(0,2,4), (2,2,4)$ are two antipodal vertices in $D(G)$.

Theorem 5.5. *Let D_n be the dihedral group of order $2n$, then*

- *$D(D_n)$ is an empty graph, if $n = p^r$ for some prime p.*
- *$D(D_n)$ is a connected graph, if n is not a prime power.*

Proof. Let $D_n = \langle a, b : a^n = b^2 = 1; bab = a^{-1} \rangle$. As elements of the form $a^i b$ in D_n are of order 2 and only cyclic subgroup containing $a^i b$ is $\{e, a^i b\}$, $a^i b$ are isolated vertices in $\mathsf{EPow}(D_n)$ and hence they are also isolated vertices in $D(G)$. Thus the remaining vertices in $D(D_n)$ are the elements of the cyclic group of order n generated by a. Thus, by Theorem 5.3, if n is not a prime power, then $D(D_n)$ is connected. On the other hand, if n is a prime power, then by using Theorem 3.2, we have $D(D_n)$ to be the empty graph. $\square$

6. Symmetric group and Alternating group

In the earlier section, we proved that groups with trivial center has connected difference graphs. In this section, we start with an important family of groups with trivial center, namely S_n, the symmetric group on n symbols. The first main result of this section is:

Theorem 6.1. *$D(S_n)$ is connected if and only if $n \geq 8$.*

Let n be a natural number and λ be a partition of n, which we denote as $\lambda \vdash n$. Write a partition in frequency notation as

$$\lambda = \langle 1^{m_1} 2^{m_2} \dots n^{m_n} \rangle. \tag{6.1}$$

Here m_i denotes the number of parts of length i in λ. The following result is well-known which tells about the size of the centralizer of a given permutation.

Lemma 6.2. *Let $\sigma \in S_n$ has cycle type $\lambda = \langle 1^{m_1} 2^{m_2} \dots n^{m_n} \rangle$. Then, the centralizer of σ, denoted as $C(\sigma)$, has the cardinality*

$$|C(\sigma)| = \prod_{j=1}^{n} j^{m_j} m_j! \tag{6.2}$$

Remark 6.3. If $[\sigma] = n-1$ or n, $o(\sigma)$ is a prime power (say p^r), and all the cycles in the decomposition of σ have distinct lengths, then all the m_j's are 0 or 1. Moreover, if j is not a power of p then $m_j = 0$. Thus, from (6.2), we get that the centralizer of σ has cardinality p^t for some t and hence the centralizer is a p-group.

For a permutation $\sigma \in S_n$, σ can be written as $\sigma_1 \cdot \sigma_2 \cdots \sigma_l \cdot f_1 \cdot f_2 \cdots f_r$ where σ_i's are cycles of length > 1 and f_i's are cycles of length 1 or fixed points. In the above case, we say that σ is a product of l disjoint nontrivial cycles and r fixed points. For the ease of writing, we write σ as $\sigma_1 \cdot \sigma_2 \cdots \sigma_l$ ignoring the fixed points and it is understood that the remaining points are fixed.

Let $\sigma \in S_n$. Define $\{\sigma\} = \{i : \sigma(i) \neq i\}$ and $[\sigma] = |\{\sigma\}|$, i.e., $[\sigma]$ is the number of integers in $\{1, 2, \dots, n\}$ which are not fixed by σ.

Lemma 6.4. *Let $\sigma \in S_n$ be a vertex in $D(S_n)$, where $n \geq 8$. If $o(\sigma) = p^\alpha$ for some odd prime p, then there exists a path joining σ and a transposition $(a_1 a_2)$ in $D(S_n)$.*

Proof. If $[\sigma] \leq n-2$, then there exists $a_1, a_2 \in \{1, 2, \dots, n\}$ which are fixed by σ. Then as $o(\sigma)$ is odd, by Proposition 4.2, $\sigma \sim (a_1 a_2)$ in $D(S_n)$. Therefore we assume $[\sigma] = n-1$ or $[\sigma] = n$. By Remark 6.3, we can see that if all the cycles in the decomposition of σ have distinct lengths, the centralizer of σ is a p-group and σ is isolated. Otherwise, there are two cycles of length > 1 in the decomposition of σ with same cycle length. Now, we can construct path. Let $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_l$ where $l \geq 2$. Without loss of generality, let σ_1 and σ_2 be of length p^α. Now, let $\sigma = (a_1 a_2 \cdots a_{p^\alpha})(b_1 b_2 \cdots b_{p^\alpha})\sigma_3 \cdots \sigma_l$. Let $y = (a_1 a_2 \cdots a_{p^\alpha} b_1 b_2 \cdots b_{p^\alpha})$. Clearly y^{p^α} is in the centralizer of σ and is of order 2. Hence $\sigma \sim y^{p^\alpha}$ in $D(S_n)$. As $n \geq 8$, either y^{p^α} is a product of r (≥ 3) disjoint transpositions, or there exists $a, b, c \in \{1, 2, \dots, n\} \setminus \{y^{p^\alpha}\}$.

If y^{p^α} is a product of $r \geq 3$ transpositions, say $y^{p^\alpha} = (c_1 c_2)(c_3 c_4)(c_5 c_6)\tau_4 \cdots \tau_r$, where τ_is are disjoint transpositions, then $z = (c_1 c_3 c_5)(c_2 c_4 c_6)$ is in the centralizer of y^{p^α}. Again, as $n \geq 8$, there exist $a_1, a_2 \in \{1, 2, \dots, n\} \setminus \{z\}$. Combining we get the path $\sigma \sim y^{p^\alpha} \sim z \sim (a_1 a_2)$.

In the other case, there exists $a, b, c \in \{1, 2, \dots, n\} \setminus \{y^{p^\alpha}\}$. Now, we see that $\gcd(o(abc), o(y^{p^\alpha})) = \gcd(3, 2) = 1$ and (abc) commutes with y^{p^α},. By Proposition

4.2, we have $y^{p^\alpha} \sim (abc)$ in $D(S_n)$. Hence, we get the following path $\sigma \sim y^{p^\alpha} \sim (abc) \sim (a_1a_2)$. □

Now, we are in a position to prove Theorem 6.1.

Proof of Theorem 6.1: Let σ_1, σ_2 be two vertices in $D(S_n)$. By Lemma 4.6 there exists σ_1' and $\sigma_2' \in S_n$ having prime power order such that $\sigma_1 \sim \sigma_1'$ and $\sigma_2 \sim \sigma_2'$. If σ_1' or σ_2' has even order, then again by Lemma 4.6, we can find σ_1'' or σ_2'' of odd prime power order such that $\sigma_1' \sim \sigma_1''$ or $\sigma_2' \sim \sigma_2''$. So without loss of generality we can assume that σ_1' and σ_2' are of odd prime power order. Hence by Lemma 6.4, there exist transpositions (a_1b_1) and (a_2b_2) such that there exists a path P_1 joining σ_1 and (a_1b_1) and a path P_2 joining σ_2 and (a_2b_2).

As $n \geq 8$, therefore there exists a 3-cycle $(a_3b_3c_3)$ which is disjoint with both (a_1b_1) and (a_2b_2). Thus we have $(a_1b_1) \sim (a_3b_3c_3) \sim (a_2b_2)$, and hence we get a path joining σ_1 and σ_2 in $D(S_n)$, and $D(S_n)$ is connected for $n \geq 8$.

On the other hand, it can checked by direct computation that $D(S_n)$ is disconnected for $n = 5, 6, 7$ and $D(S_n)$ is empty for $n \leq 4$. For the sake of completeness, one can check that there is no path joining $(1\ 2)$ and $(1\ 3)$ in $D(S_5)$. For $D(S_6)$ and $D(S_7)$, it can be shown that there is no path joining $(1\ 2\ 3)$ and $(1\ 2\ 3)(4\ 5\ 6)$. □

We next concentrate on a family of simple groups A_n, the alternating group on n symbols.

We know that the sign of a permutation σ can be defined from its decomposition into the product of transpositions as $sign(\sigma) = (-1)^m$ where m is the number of transpositions in the decomposition. The following alternative definition of sign is known, see for example Nelson [31]: $sign(\sigma) = (-1)^{n-\mathrm{cyc}(\sigma)}$ where $\mathrm{cyc}(\sigma)$ denotes the number of cycles of σ. That is, if we write σ as $\sigma_1 \cdot \sigma_2 \cdots \sigma_l \cdot f_1 \cdot f_2 \cdots f_r$ (where σ_is are nontrivial cycles and f_is are fixed points), the sign of σ is $(-1)^{n-l-r}$. From this, the following lemma is immediate.

Lemma 6.5. *For a positive integer n and $\sigma = \sigma_1 \cdot \sigma_2 \cdots \sigma_l \in S_n$ where σ_is are nontrivial cycles, the permutation $\sigma \in A_n$ if and only if $[\sigma] - l$ is even.*

Proof. We observe that $\sigma \in A_n$ if and only if $n - l - r$ is even. Using $n - r = [\sigma]$, the proof is complete. □

Remark 6.6. Let $n \geq 10$. Let $\pi = (a_1a_2)(a_3a_4)$ and $\sigma = (a_5a_6a_7)$ where $a_i \neq a_j$ for $i \neq j$. Then $\pi \sim \sigma$ in $D(A_n)$. Thus, any two 2 3-cycles are connected by a path of length 2 in $D(A_n)$. Thus, if a permutation $\pi \in D(A_n)$ is connected to a particular 3-cycle (abc) by a path of length t, then it is connected to any 3 cycle by a path of length $\leq t + 2$.

We now prove some other lemmas which will be crucial for our result on alternating group.

Lemma 6.7. *Let $\pi \in A_n$ be a product of k 2-cycles and $n - 2k$ fixed points where $n \geq 10$ and $k \geq 4$. Then there exists a, b, c, d such that there is a path of length 2 between π and $(ab)(cd)$. Thus, there exists e, f, g such that there is a path of length 3 between π and (efg).*

Proof. Let $\pi = (c_1c_2)(c_3c_4)(c_5c_6)(c_7c_8)\tau_5 \cdots \tau_k$ where τ_is are disjoint transpositions. Clearly, the element $z = (c_1c_3c_5)(c_2c_4c_6)$ is in the centralizer of π. Again, as $n \geq 10$, there exist $a, b, c, d \in \{1, 2, \ldots, n\} \setminus \{z\}$. Hence $z \sim (ab)(cd)$. Combining we get the path $\pi \sim z \sim (ab)(cd)$. This proves the lemma. □

Lemma 6.8. *Let $\pi \in A_n$ be a product of k 3-cycles and $n - 3k$ fixed points where $n \geq 10$ and $k \geq 2$. Then there exists a, b, c such that there is a path between π and (abc).*

Proof. If π is a product of exactly two 3-cycles and $n - 6$ fixed points, say, $\pi = (c_1c_2c_3)(c_4c_5c_6)$, then we of course have $\pi \sim (a, b)(c, d) \sim (e, f, g)$ where $a, b, c, d \in [n] \setminus \{c_1, c_2, c_3, c_4, c_5, c_6\}$ and $\{e, f, g\} \in [n] \setminus \{a, b, c, d\}$.

If π is a product of exactly three 3-cycles and $n - 9$ fixed points, say, $\pi = (c_1c_2c_3)(c_4c_5c_6)(c_7c_8c_9)$, then π is an isolated if $n = 10$ and for $n \geq 11$, we have $\pi \sim (c_1c_4)(c_2c_5)(c_3c_6)(a, b)$ where $a, b \in [n] \setminus \{c_1, c_2, \ldots, c_9\}$.

If π is a product of $k \geq 4$ 3-cycles and $n - 3k$ fixed points, we must have $n \geq 12$ and here we have $\pi = (c_1c_2c_3)(c_4c_5c_6)(c_7c_8c_9)(c_{10}c_{11}c_{12})\tau_5 \cdots \tau_k$ where τ_is are cycles of length 3, then the centralizer of π contains the element $z = (c_1c_4)(c_2c_5)(c_3c_6)(c_7c_{10})(c_8c_{11})(c_9c_{12})$. Thus $\pi \sim z$ in $D(A_n)$. By Lemma 6.7, the proof is complete. □

Lemma 6.9. *Let $\sigma \in A_n$ be a vertex in $D(A_n)$, where $n \geq 10$. If $o(\sigma) = p^\alpha$ for some prime $p \neq 3$, then there exist $a, b, c \in [n]$ such that there is a path joining σ and the 3-cycle (abc). Thus, there is a path joining σ and any 3-cycle.*

Proof. If σ has > 2 fixed points, we get a 3 cycle which is adjacent to σ. So, we assume σ to have at most 2 fixed points. We consider the centralizer of σ in S_n and we have the following cases:

Case 1: Suppose, all the cycles (including the cycles of length 1) in the decomposition of σ have distinct length. By Remark 6.3, we get that the centralizer of σ is a p-group and σ is isolated.

Case 2: Suppose σ has exactly 2 cycles of same length and all the other cycles are of distinct length. If σ has 2 fixed points and all the other cycles of distinct length, then the centralizer of σ in A_n is a p-group and hence σ is isolated. So let σ has exactly 2 cycles of length $l_1 > 1$ and all the other cycles of distinct length. We now claim that this case can not happen if $p = 2$. Let $p = 2$ and $\sigma \sim x$. Then there exists y such that both $\sigma \in \langle y \rangle$ and $x \in \langle y \rangle$. As none of the cycles of σ has been repeated more than twice, each cycle length of y is also a power of 2, and thus the order of x is also a power of 2. Thus the order of any neighbor of σ is a power of 2 which contradicts Lemma 4.6. Hence $p > 3$. In this case, the centralizer of σ in S_n is of cardinality $2p^j$ for some j and hence the centralizer in A_n is a p-group and σ is isolated.

Case 3: Suppose none of the cycles of σ has been repeated more than twice but there are 2 cycles of length l_1 and 2 cycles of length l_2 in the decompostion of σ. If $p = 2$, then by a similar argument as in Case 2, we can show that the order of any neighbor of σ is a power of 2 which contradicts Lemma 4.6. Hence $p > 3$.

If both $l_1 > 1$ and $l_2 > 1$, without loss of generality, we can write $\sigma = (a_1a_2 \cdots a_{p^{\alpha_1}})(b_1b_2 \cdots b_{p^{\alpha_1}})(c_1c_2 \cdots c_{p^{\alpha_2}})(d_1d_2 \cdots d_{p^{\alpha_2}})\sigma_5 \cdots \sigma_l$. The centralizer of σ (in A_n) contains $z = (a_1, b_1)...(a_{p^{\alpha_1}}, b_{p^{\alpha_1}})(c_1, d_1)...(c_{p^{\alpha_2}}, d_{p^{\alpha_2}})$. Now z commutes with $(a_1a_2a_3)(b_1b_2b_3)$ and $(a_1a_2a_3)(b_1b_2b_3)$ commutes with the element $(w,x)(y,z)$ where $w,x,y,z \in [n] \setminus \{a_1,a_2,a_3,b_1,b_2,b_3\}$.

Now, we consider the case when one of l_1 and l_2 is 1 and the other one is > 1. In this case, we have $\sigma = (a_1a_2\cdots a_{p^{\alpha_1}})(b_1b_2\cdots b_{p^{\alpha_1}})\sigma_5\cdots\sigma_l$ and moreover σ has atleast two fixed points say c,d. The centralizer of σ (in A_n) contains $z = (a_1,b_1)...(a_{p^{\alpha_1}},b_{p^{\alpha_1}})(c,d)$. Now z commutes with $(a_1a_2a_3)(b_1b_2b_3)$ and we are done.

Case 4: In the decomposition of σ, there exists some length for which there are ≥ 3 cycles. If σ has three fixed points, we are done. Otherwise, let $\sigma = (a_1a_2\cdots a_{p^\alpha})(b_1b_2\cdots b_{p^\alpha})(c_1c_2\cdots c_{p^\alpha})\sigma_4\cdots\sigma_l$. This commutes with the element $y = (a_1b_1c_1)(a_2b_2c_2)\cdots(a_{p^\alpha}b_{p^\alpha}c_{p^\alpha})$. As y is a product of ≥ 2 disjoint 3 cycles, by Lemma 6.8 we get a path from σ to a 3-cycle (abc).

This completes the proof. □

Theorem 6.10. *If $n \geq 10$, then $D(A_n)$ is connected.*

Proof. Let σ_1,σ_2 be two vertices in $D(A_n)$. By Lemma 4.6, there exists σ_1' and $\sigma_2' \in A_n$ having prime power order such that $\sigma_1 \sim \sigma_1'$ and $\sigma_2 \sim \sigma_2'$. If σ_1' or σ_2' has order 3^{β_1} or 3^{β_2} then again by using Lemma 4.6 we have σ_1'' or $\sigma_2'' \in A_n$ having order $p_1^{\alpha_1}$ or $p_2^{\alpha_2}$ for some prime $p_1,p_2 \neq 3$ such that $\sigma_1' \sim \sigma_1''$ or $\sigma_2' \sim \sigma_2''$. So without loss of generality, we can assume that σ_1' and σ_2' have order $p_1^{\alpha_1}$ and $p_2^{\alpha_2}$ for some primes $p_1,p_2 \neq 3$. By Lemma 6.9, for $n \geq 10$, we have a path joining σ_1' to a 3-cycle (abc) and a path joining σ_2' to the same 3-cycle (abc).

On the other hand, it can checked by direct computation that $D(A_n)$ is disconnected for $n = 7,8,9$ and $D(A_n)$ is empty for $n \leq 6$. For the sake of completeness, one can check that neither of (123) and (456) are isolated vertices in $D(A_7)$ and $D(A_8)$ and there is no path joining (123) and (456) in $D(A_7)$ and $D(A_8)$. Moreover, there is no path joining $(12)(34)(56)(78)$ and (789) in $D(A_9)$ and neither of $(12)(34)(56)(78)$ and (789) are isolated vertices. This completes the proof. □

7. Groups with trivial center

It has been observed that if G is a finite group with non-trivial center, then $D(G)$ is connected. So, the natural question is to enquire about the connectedness of $D(G)$ when $Z(G)$ is trivial. It was shown that $D(S_n)$ is connected if and only if $n \geq 8$. Thus, it is not possible to have a general answer to this question. So, we investigate the connectedness of some special families of groups with trivial center.

Theorem 7.1. *Let G_1, G_2, G_3 be three finite non-trivial groups such that $G_1 \times G_2 \times G_3$ is not a p-group, then $D(G_1 \times G_2 \times G_3)$ is connected.*

Proof. Let $a = (a_1,a_2,a_3), b = (b_1,b_2,b_3)$ be two vertices in $D(G_1 \times G_2 \times G_3)$. As $G_1 \times G_2 \times G_3$ is not a p-group then there exists at least two prime divisers of $|G_1 \times G_2 \times G_3|$. Regardless of whether the orders of a and b are prime powers or not, by Lemma 4.6 there exist elements of prime power order with which a and b are adjacent. So without loss of generality we can start with elements of prime power order and call them a and b.

By Lemma 4.6, there exist $a' = (a'_1, a'_2, a'_3), b' = (b'_1, b'_2, b'_3) \in D(G_1 \times G_2 \times G_3)$ such that $a \sim a'$ with $o(a') = p^\alpha$ (with $\alpha > 0$), and $b \sim b'$ with $o(b') = q^\beta$ (with $\beta > 0$), for some primes p and q. Note that a and a' are of coprime order as well as b and b'). If $p = q$ then there exists another prime divisor r, so we can procced similarly. Therefore we can consider $p \neq q$.

Since $o(a') = p^\alpha$, therefore one of $o(a'_1), o(a'_2), o(a'_3)$ is p^α. Without loss of generality, let $o(a'_1) = p^\alpha$. As (a_1, a_2, a_3) and (a'_1, e_2, e_3) are of coprime order and $a \sim a'$, we have that (a_1, a_2, a_3) and (a'_1, e_2, e_3) are adjacent.

Again, one of $o(b'_1), o(b'_2), o(b'_3)$ is q^β. If $o(b'_2) = q^\beta$ or $o(b'_3) = q^\beta$, say $o(b'_2) = q^\beta$, then we have $(e_1, b'_2, e_3) \sim (b_1, b_2, b_3)$ in $D(G_1 \times G_2 \times G_3)$. Again, as $o((a'_1, e_2, e_3)) = p^\alpha$ and $o((e_1, b'_2, e_3)) = q^\beta$ and they commute, we have $(a'_1, e_2, e_3) \sim (e_1, b'_2, e_3)$. Thus, we get a path:

$$(a_1, a_2, a_3) \sim (a'_1, e_2, e_3) \sim (e_1, b'_2, e_3) \sim (b_1, b_2, b_3) \text{ in } D(G_1 \times G_2 \times G_3).$$

If none of $o(b'_2)$ or $o(b'_3)$ is q^β, then we must have $o(b'_1) = q^\beta$. Then $(b_1, b_2, b_3) \sim (b'_1, e_2, e_3)$. If there exists a prime r other than p and q which divides $|G_2|$ or $|G_3|$ then we get a path joining a and b.

Now consider the case when $|G_2|$ and $|G_3|$ have no prime factors other than p and q. Without loss of generality, let q divides $|G_2|$. Then there exists $c_2 \in G_2$ with $o(c_2) = q$. Thus we get a path $(a_1, a_2, a_3) \sim (a'_1, e_2, e_3) \sim (e_1, c_2, e_3)$. As (b'_1, e_2, e_3) is not isolated and $o((b'_1, e_2, e_3)) = q^\beta$, by Lemma 4.6, there exists a prime $r(\neq q)$, i.e., $r = p$, and $(d_1, d_2, d_3) \in G_1 \times G_2 \times G_3$ such that $o((d_1, d_2, d_3)) = p^\gamma$ and $(b'_1, e_2, e_3) \sim (d_1, d_2, d_3)$ in $D(G_1 \times G_2 \times G_3)$.

If $o(d_1)$ or $o(d_3)$ is p^γ, say $o(d_1) = p^\gamma$, then we get the following path in $D(G_1 \times G_2 \times G_3)$:

$$(a_1, a_2, a_3) \sim (a'_1, e_2, e_3) \sim (e_1, c_2, e_3) \sim (d_1, e_2, e_3) \sim (b'_1, e_2, e_3) \sim (b_1, b_2, b_3).$$

If $o(d_1), o(d_3) \neq p^\gamma$, we must have $o(d_2) = p^\gamma$.

As p and q are the only possible distinct prime factors of $|G_3|$, G_3 must contain an element f_3 of order p or q.

- If $o(f_3) = p$, then we get the following path in $D(G_1 \times G_2 \times G_3)$:

$$(a_1, a_2, a_3) \sim (a'_1, e_2, e_3) \sim (e_1, c_2, e_3) \sim (e_1, e_2, f_3) \sim (b'_1, e_2, e_3) \sim (b_1, b_2, b_3).$$

- If $o(f_3) = q$, then we get the following path in $D(G_1 \times G_2 \times G_3)$:

$$(a_1, a_2, a_3) \sim (a'_1, e_2, e_3) \sim (e_1, e_2, f_3) \sim (e_1, d_2, e_3) \sim (b'_1, e_2, e_3) \sim (b_1, b_2, b_3).$$

Thus, summing up all the cases, we have shown that $D(G_1 \times G_2 \times G_3)$ is connected. □

Remark 7.2. In light of Theorem 7.1, if G_i's are groups with trivial centers, then the difference graph of their direct product is connected. So, Theorem 7.1 gives us a natural way to construct infinitely many finite groups G with trivial center such that $D(G)$ is connected.

Remark 7.3. It is to be noted that Theorem 7.1 can be generalized to direct product of n groups where $n \geq 3$ in a similar way. However, we add an word of caution that Theorem 7.1 may not be true for direct product of two groups.

For example, $D(S_3 \times S_3)$ is a disconnected graph of order 10 with 2 isomorphic components of order 5 each.

In the next theorem, we identify some families of graphs which can be expressed as direct product of two groups G_1 and G_2 such that $Z(G_1)$, $Z(G_2)$ are trivial but $D(G_1 \times G_2)$ is connected.

Theorem 7.4. *$D(D_n \times D_m)$ is disconnected if and only if n and m are powers of same odd prime.*

Proof. If any of n or m is even, then $Z(D_n \times D_m)$ is non-trivial and as a result $D(D_n \times D_m)$ is connected. Note that if both n and m are powers of 2, i.e., $D_n \times D_m$ is a 2-group, then $D(D_n \times D_m)$ is empty, which is vacuously connected. Thus we deal only with the case when both m and n are odd. Let $D_n = \langle r_1, s_1 : r_1^n = s_1^2 = e, s_1 r_1 s_1 = r_1^{-1}\rangle$ and $D_m = \langle r_2, s_2 : r_2^m = s_2^2 = e, s_2 r_2 s_2 = r_2^{-1}\rangle$. Clearly any element of the form $(r_1^i s_1, r_2^j s_2)$ is not a vertex of $D(D_n \times D_m)$. Therefore any vertex (a, b) of $D(D_n \times D_m)$ is one of the three forms: $(r_1^i, r_2^j s_2), (r_1^i s_1, r_2^j)$ or (r_1^i, r_2^j).

Case 1: Let $(a, b) = (r_1^i, r_2^j s_2) \in D(D_n \times D_m)$. By Lemma 4.6, $(r_1^i, r_2^j s_2)$ is adjacent to some (a_1, b_1) of order p^α in $D(D_n \times D_m)$. Hence $(a_1, b_1) \in \langle (r_1^i, r_2^j s_2)\rangle$. If p is odd then $b_1 = e$ and therefore $(a, b) \sim (r_1^x, e)$ for some $r_1^x \in \langle r_1^i\rangle$. If $p = 2$, then similarly we can have $(a, b) \sim (a_1, b_1) \sim (r_1^x, e)$ where $o(r_1^x) = q^\beta$, q being an odd prime. If $(a, b) = (r_1^i s_1, r_2^j)$ then we proceed similarly to get a path joining (a, b) and a vertex of the form (e, r_2^y).

Case 2: Let $(a, b) = (r_1^i, r_2^j) \in D(D_n \times D_m)$. By Lemma 4.6, there exists an element $(a_1, b_1) \in D(D_n \times D_m)$ such that $(a, b) \sim (a_1, b_1)$ and $o((a_1, b_1)) = p^\alpha$. As n and m are not powers of same prime, there exists a prime $q(\neq p)$ such that $q|n$ or $q|m$. Let $q|n$. Then there exists an element $r_1^x \in D_n$ of order q and we get the following path $(a, b) \sim (a_1, b_1) \sim (r_1^x, e)$.

Thus combining the above two cases, we have shown that any vertex $(a, b) \in D(D_n \times D_m)$ is joined by a path to either an element of the form (r_1^x, e) or (e, r_2^y) in $D(D_n \times D_m)$. Now, we construct a path between two vertices of the form (r_1^x, e) and (e, r_2^y) in $D(D_n \times D_m)$ as follows:

$$(r_1^x, e) \sim (e, s_2) \sim (r_1^z, e) \sim (e, r_2^w) \sim (s_1, e) \sim (e, r_2^y),$$

where we choose w and z such that $o(r_1^z)$ and $o(r_2^w)$ are powers of different primes. Thus, if n and m are not powers of same odd prime, $D(D_n \times D_m)$ is connected.

Now if n and m are powers of same odd prime, we can easily show that there is no path joining (r_1, e) and (e, r_2) in $D(D_n \times D_m)$, completing the proof. □

8. The clique number of $D(G)$

If S is a set of elements of a group G such that every two elements of S generate a cyclic group, then S generates a cyclic group (see [1, Lemma 32]). Hence every maximal clique in $\mathsf{EPow}(G)$ is a maximal cyclic subgroup; thus every maximal clique in $D(G)$ is contained in a maximal cyclic subgroup of G. So we first need to find the clique number of $D(\mathbb{Z}_n)$ for an integer n.

Proposition 8.1. *The clique number of $D(\mathbb{Z}_n)$ is equal to the maximum size of an antichain in the lattice of divisors of n.*

Proof. Two elements of $\mathbb{Z}_n$ which are joined in $D(\mathbb{Z}_n)$ must have different orders, and neither divides the other; conversely, two elements with this property are joined in $D(\mathbb{Z}_n)$. $\square$

The maximum size of an antichain in the lattice of divisors of n was found by de Bruijn *et al.* [8]; this is a generalization of Sperner's lemma. Define the *degree* of n to be the number of prime divisors of n, counted with multiplicity. Let m be the degree of n. Then an antichain of maximal size consists of all the divisors of n of degree $m/2$, if m is even; or either all divisors of degree $(m-1)/2$ or all divisors of degree $(m+1)/2$, if m is odd.

For example, a clique of maximal size in C_{360} is obtained by choosing elements of orders 8, 12, 18, 20, 30, 45.

Proposition 8.2. *The clique number of $D(G)$ is equal to the maximum clique number of a cyclic subgroup of G, so is determined by the set of orders of elements of G.*

This is now clear from our earlier remarks.

Corollary 8.3. *The graph $D(G)$ is triangle-free if and only if the order of every element of G is of the form $p^k q$ where p, q are primes.*

Proof. Let g be an element of G of order n. If n is not of the forms in the Corollary, either it has three prime divisors p, q, r, or it is divisible by p^2q^2 for some prime q. In the first case, elements of orders p, q, r in $\langle g \rangle$ form a triangle; in the second, elements of orders p^2, pq, q^2 form a triangle. $\square$

Note that a triangle requires at least three colours.

The chromatic number of $D(G)$ may be larger than the clique number. This example is taken from [13]. In the symmetric group S_8, the orders of elements are 1, 2, 3, 4, 5, 6, 7, 8, 10, 12 and 15. By the Corollary above, the clique number of $D(S_8)$ is 2. But $D(S_8)$ is not bipartite, since it contains a 5-cycle

$$\{(1,2),(3,4,5),(6,7),(1,2,3),(4,5,6,7,8)\}.$$

9. Perfectness and other properties

It is known that the power graph of a finite group is perfect (see [2]). On the other hand, the question "For which groups is the enhanced power graph perfect?" is still unresolved. In this section, we discuss perfectness of $D(G)$. We also say something about the related problem of when $D(G)$ is a cograph.

9.1. **Graph classes, induced subgraphs and twin reduction.** The *clique number* of a graph is the size of the largest complete subgraph, and the *chromatic number* is the smallest number of colours required to colour the vertices so that adjacent vertices are given different colours. The clique number does not exceed the chromatic number since, in a proper colouring, all vertices of a clique are given different colours. A graph Γ is *perfect* if every induced subgraph of Γ has clique number equal to chromatic number. The *strong perfect graph theorem*, conjectured by Berge and proved by Chudnovsky *et al.* [16], states that a graph is perfect if and only if it does not contain either an odd cycle or the complement of an odd cycle as an induced subgraph. It follows that a graph is perfect if and only if its complement is perfect. This statement, known as the *weak perfect graph theorem*, was proved earlier by Lovász. A number of graph classes are known to be perfect, including bipartite graphs and comparability graphs of partial orders.

Several other classes of graphs also have characterizations in terms of forbidden induced subgraphs. Among these, we will only consider the class of *cographs*, graphs which contain no induced subgraph isomorphic to the 4-vertex path P_4. Since P_4 is isomorphic to its complement and any cycle of length greater than 4 contains an induced P_4, we see that cographs are perfect. Cographs form the smallest class of graphs which can be built from the 1-vertex graph by the operations of complementation and disjoint union.

Any class of graphs defined by forbidden induced subgraphs is subgraph-closed. We will use two tools to investigate when difference graphs are perfect or belong to one of the other classes:

(a) If H is a subgroup of G, then the induced subgraph of $D(G)$ on the set H, after removing isolated vertices, is $D(H)$. So the class of groups for which the difference graph belongs to one of the above classes is subgroup-closed.
(b) Two vertices of a graph are *twins* if they have the same neighbours (possibly excluding each other). The process of *twin reduction* involves finding a pair of twins and identifying them, and continuing until no further twins remain. The result of twin reduction is (up to isomorphism) independent of the process of reduction, and is called the *cokernel* of the graph (since Γ is a cograph if and only if its cokernel is the 1-vertex graph). The cokernel is an induced subgraph of the original graph and (in the cases we consider) is often much smaller and more amenable to analysis. The important fact is given in the next result.

Proposition 9.1. *(*[9, Theorem 7.5]*) Let $\mathcal{F}$ be a class of finite graphs, and suppose that no graph in $\mathcal{F}$ possesses a pair of twin vertices. Then a graph Γ has no induced subgraph in $\mathcal{F}$ if and only if the same applies to the cokernel of Γ.*

This is because twin reduction cannot destroy an induced subgraph isomorphic to a graph in $\mathcal{F}$, whereas the cokernel is an induced subgraph of the original graph. This result applies, for example, to perfect graphs. Since the cokernel of $D(G)$ may be much smaller than G, it is easier to show that the cokernels are perfect. The groups G at the end of Section 9 are examples; the cokernel of $D(G)$ is bipartite, so $D(G)$ is perfect, for these groups (including M_{11} and M_{12}).

9.2. **Perfect difference graphs.** Since an induced subgraph of a perfect graph is perfect, but difference graphs of groups are universal, it is clear that there are groups whose difference graph is not perfect. In this section we give some explicit examples of perfect and imperfect difference graphs.

Proposition 9.2. *The difference graph of a cyclic group is perfect.*

Proof. The enhanced power graph of a cyclic group is complete, so the difference graph is simply the complement of the power graph, which as we saw is perfect. Now we can invoke the Weak Perfect Graph Theorem. □

Moreover, we can determine the cyclic groups for which the difference graph is a cograph: for the class of cographs is self-complementary, and the nilpotent groups whose power graph is a cograph were determined in [30, Theorem 12]. The result is:

Proposition 9.3. *The difference graph of a cyclic group $\mathbb{Z}_n$ is a cograph if and only if either n is a prime power or n is the product of two distinct primes.*

This result does not extend to abelian groups. We saw that the difference graphs of abelian groups, even those which are direct products of two isomorphic cyclic groups, are universal, and in particular, we can find one which embeds a 5-cycle.

Theorem 9.4. *Let G be a group of order pq, p^2q, p^3q, p^2q^2, or pqr, where p, q, r are distinct primes. Then $D(G)$ is perfect.*

Proof. By Theorem 9.2, we may assume that G is not cyclic. Also, we may assume that G is not an EPPO group (one with all elements of prime power order), since for such a group G the graph $D(G)$ has no edges.

Let G be a group of order pq. Then either G is cyclic or it is an EPPO group, and the result follows.

Let G be a group of order p^2q. We may assume that G is non-cyclic but has elements of order pq. We claim that elements of order pq are isolated in $D(G)$. If not, let x be an element of order pq which is adjacent to a vertex y. Then from the adjacency condition of difference graph, $o(y) = p^2$. But this implies $\langle x, y\rangle$ is a cyclic group of order p^2q in G, a contradiction. Thus any edge in $E(G)$ must join elements of orders p and q. So $E(G)$ is bipartite (with the sets of elements of these orders as bipartite sets) and hence perfect.

Next let G be a group of order p^3q, and suppose that G is not cyclic. The possible orders of elements of G are p, p^2, p^3, q, pq, or p^2q. Elements of order p^2q cannot be adjacent to elements of order dividing p^2q, or to elements of order p^3; so they are isolated. Thus any edge of $E(G)$ must join a vertex of order a power of p with one of order q or pq; so the graph is bipartite, and hence perfect.

Now let G be a non-cyclic group of order p^2q^2, where $p > q$. There is no element of order p^2q^2, and arguing as above we see that elements of orders p^2q or pq^2 are isolated, and elements of orders p^2 and q^2 cannot be adjacent. We can assume that G is not an EPPO group; so it contains elements of order pq. Moreover, we can assume there are elements of orders p^2 and q^2. For, if there are no elements of order q^2, then all edges join elements with order divisible by q to elements with order a power of p, and the graph is bipartite, and hence perfect. Hence the Sylow subgroups of G are cyclic. Now there is a normal q-complement, which is cyclic of order p^2. Now an element of order q or q^2 which acts nontrivially on a cyclic group of order p^2 must have trivial centralizer there. So either the group G is cyclic, or there is no element of order pq^2. So elements of order q^2 are isolated, and $D(G)$ is bipartite by the same argument as before.

Finally, let G be a non-cyclic group of order pqr. As in the previous cases, we can show that elements of order pq, pr and qr are not in $D(G)$. (If an $o(x) = pq$ and x is joined to y, then if $r \mid o(y)$ then $\langle x, y\rangle$ is cyclic of order pqr, while if $r \nmid o(y)$ then y is a power of x.) So all edges join elements of distinct prime orders. If $D(G)$ contains elements of order at most two of p, q, r then the graph is bipartite, and hence perfect; so we assume that elements of order all three primes exist in $D(G)$. Now if $p > q > r$, then G contains a unique cyclic Sylow p-subgroup P. Let C be an induced odd cycle in $D(G)$. Clearly, neither the vertices of C are of same prime order nor their orders alternate among any two primes. Thus, C contains vertices of orders p, q and r each. As P is a unique subgroup of order p, C has a unique vertex of order p and orders of rest of the vertices in C alternates between q and r. Thus we get four consecutive vertices $a_1, a_2, a_3, a_4 \in C$ such that $o(a_1) = r$, $o(a_2) = p$, $o(a_3) = q$ and $o(a_4) = r$, where $o(g)$ denotes the order of g. Thus $\langle a_1, a_2\rangle$ is a cyclic group of order pr, $\langle a_2, a_3\rangle$ is a cyclic group of order pq and $\langle a_3, a_4\rangle$ is a cyclic group of order qr. Now, G, being a group of order pqr with cyclic subgroups of orders pq, qr and pr, must be cyclic, a contradiction. □

9.3. **Imperfect difference graphs.** Now we show that some groups have imperfect difference graphs.

Proposition 9.5. (a) *For any three distinct primes p, q, r, the group $\mathbb{Z}_{pqr} \times \mathbb{Z}_p$ has imperfect difference graph.*

(b) *For two distinct odd primes p and q, the group $Q_8 \times \mathbb{Z}_{pq}$, where Q_8 is the quaternion group of order 8, has imperfect difference graph.*

(Note that all proper subgroups of these groups have perfect difference graphs.)

Proof. (a) Let a, b, c be elements of $\mathbb{Z}_{pqr}$ with orders p, q, r respectively, and a' an element of $\mathbb{Z}_p$. Then the set

$$\{(c, e), (b, e), (ac, e), (bc, e), (b, a')\}$$

induces a 5-cycle in $D(G)$.

(b) Let a, a' be non-commuting elements of order 4 in Q_8, and b and c elements of orders p and q. Then the set

$$\{x_1 = (e, c), x_2 = (e, b), x_3 = (a, c), x_4 = (e, bc), x_5 = (a', b)\}$$

induces a 5-cycle in $D(G)$, where $o(x_1) = q$, $o(x_2) = p$, $o(x_3) = 4q$, $o(x_4) = pq$, and $o(x_5) = 4p$. Note that, as $o(x_1)$ divides $o(x_3)$ and $o(x_4)$, so x_1 is not adjacent to x_3 or x_4. Other adjacencies and non-adjacencies follow in the similar manner. □

Now we can determine which nilpotent groups have perfect difference graph. Let $\pi(G)$ be the number of distinct prime divisors of G. We may assume that $\pi(G) > 1$.

Theorem 9.6. *Let G be a finite nilpotent group.*

(a) *If $\pi(G) \geq 3$, then $D(G)$ is perfect if and only if G is cyclic.*
(b) *If $\pi(G) = 2$, then $D(G)$ is a comparability graph, and hence perfect.*

Proof. Suppose that $\pi(G) \geq 3$. If G is cyclic then the theorem follows from Theorem 9.2. If G has a generalized quaternion Sylow 2-subgroup, then it contains a subgroup $Q_8 \times \mathbb{Z}_{pq}$ for odd primes p and q. Otherwise, by [23, Theorem 4.10(ii)], at least one Sylow subgroup contains commuting elements of prime order (say p), and G contains a subgroup $\mathbb{Z}_{pqr} \times \mathbb{Z}_p$. In either of the last two cases, $D(G)$ is imperfect, by the preceding Proposition.

If $\pi(G) = 2$, then $G \cong H \times K$ where H is the Sylow p-subgroup and K is the Sylow q-subgroup of G. Thus any element of $g \in G$ can be uniquely expressed as ab, where $a \in H$ and $b \in K$. Also, order of any element in G is either a power of p or a power of q or product of powers of p and q. In the first two cases, we call it a element or vertex of type-I and the last case is denoted by type-II.

Define a relation $\to$ on $V(D(G))$ as follows:

- If x_1, x_2 are of type-I, then $x_1 \to x_2$ if $\circ(x_1)$ is a power of p and $\circ(x_2)$ is a power of q.
- If $x_1 = a_1b_1, x_2 = a_2b_2$ are of type-II, then $x_1 \to x_2$ if $\langle a_2 \rangle \leq \langle a_1 \rangle$ and $\langle b_1 \rangle \leq \langle b_2 \rangle$ and at least one of the inequality is strict.
- If x_1 is of type-I with $o(x_1)$ is a power of p and $x_2 = a_2b_2$ is of type-II, then $x_1 \to x_2$ if $\langle a_2 \rangle < \langle a_1 \rangle$.
- If x_1 is of type-I with $\circ(x_1)$ is a power of q and $x_2 = a_2b_2$ is of type-II, then $x_2 \to x_1$ if $\langle b_2 \rangle < \langle b_1 \rangle$.

It is easy to check that $\to$ is anti-symmetric and transitive on $V(D(G))$. The comparability graph of $\to$ on $V(D(G))$ is given by $x \sim y$ if and only if $x \to y$ or $y \to x$. It can be checked that $D(G)$ coincides with the comparability graph of $\to$ on $V(D(G))$. Hence the theorem follows. □

By arguments similar to those already given, the following results can be shown. Suppose that q and r are primes with $r \mid q-1$. Then $\mathbb{Z}_q$ has an automorphism α of order r. We define an action φ of $\mathbb{Z}_{r^2}$ on $\mathbb{Z}_q$ where the generator of $\mathbb{Z}_{r^2}$ induces the automorphism α. We say that a group G is *minimal imperfect* if its difference graph is imperfect but, for all proper subgroups H of G, $D(H)$ is perfect.

Theorem 9.7. (a) *Let p, q, r be three distinct primes such that $r|q-1$. Then $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{r^2})$, where φ is defined as above, is a minimal imperfect group.*
(b) *Let q, r be two distinct primes such that $r \mid q-1$. Then $\mathbb{Z}_{q^2} \times (\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{r^2})$, where φ is defined as above, is a minimal imperfect group.*

Proof. (a) Let a, b, c be elements of order p, q, r^2 in $\mathbb{Z}_p, \mathbb{Z}_q$ and $\mathbb{Z}_{r^2}$ respectively. Then $C : (e,e,c) \sim (a,e,c^r) \sim (a,b,e) \sim (e,e,c^r) \sim (a,e,e) \sim (e,e,c)$ is an induced five cycle in $D(\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{r^2}))$, and hence it is not perfect.

Now, any proper subgroup of H of $\mathbb{Z}_p \times (\mathbb{Z}_q \rtimes_\varphi \mathbb{Z}_{r^2})$ is of order p, q, r, r^2, pq, pr, qr, pqr, pr^2 or qr^2. Again, by Theorem 9.4, $D(H)$ is perfect.
(b) The proof is similar as above.

□

Now we turn to finite simple groups, and first show:

Theorem 9.8. *The symmetric group S_8, the alternating group A_9, and the Janko group J_1 all have imperfect difference graphs.*

Proof. In the first two cases we can give an explicit induced 5-cycle:

- in S_8, the set $\{(1,2),(3,4,5),(6,7),(1,2,3),(4,5,6,7,8)\}$ induces a 5-cycle;
- in A_9, the set

$$\{(1,2,3),(4,5)(6,7),(8,9,1),(2,3)(4,5),(6,7,8),(9,1)(2,3),(4,5,6,7,8)\}$$

induces a 5-cycle.

For J_1 we proceed as follows. All information we require is given in the $\mathbb{ATLAS}$ of Finite Groups [17].

The order of the group is $2^3.3.5.7.11.19$, and the Sylow 2-subgroup is elementary abelian. Elements of orders 7, 11 and 19 commute only with their powers, so are isolated, and deleted in the difference graph; so all the vertices have orders 2, 3 and 5, and vertices which are joined must have different orders. So the graph is tripartite (that is, has a 3-colouring). Moreover, there is no element of order 30, so the clique number is 2. To show it is not perfect, we just have to show that it is not bipartite.

The group contains a subgroup $D_3 \times D_5$, and so has a path of length 3 joining two commuting involutions, with the elements having orders $2,3,5,2$. Note that any two subgroups of J_1 isomorphic to Klein groups are conjugate, since they are contained in a Sylow 2-subgroup $(\mathbb{Z}_2)^3$ whose normaliser acts trivially on such subgroups; and since J_1 contains a subgroup A_4, the three pairs of involutions in a Klein group are permuted transitively by its normaliser. Take a subgroup isomorphic to the Klein group, and join each pair of involutions by such a path. This produces a closed walk of length 9, so indeed the graph is non-bipartite. □

9.4. **Simple groups.** We have some partial results on the question "Which finite simple groups have perfect difference graphs?" According to the preceding section, any simple group which contains the symmetric group S_8, the alternating group A_9 or the Janko group J_1 has imperfect difference graph. Among the sporadic groups, this list includes the Fischer groups, the Baby Monster and the Monster, the Harada–Norton group, the Conway group Co_1, the Thompson group, the Lyons group, the Higman–Sims group, and the O'Nan group. Moreover, of course, the alternating groups A_n for $n \geq 9$, and the groups of Lie type of rank at least 9 also contain S_8 or A_9 and so have imperfect difference graph.

On the other hand, we have the following theorem. (All quoted information about subgroup structure and centralisers in these groups can be found in the $\mathbb{ATLAS}$ [17], or in [7].)

Theorem 9.9. *Let G be the simple group $\mathrm{PSL}(2,q)$ (for prime power $q \geq 4$) or $\mathrm{Sz}(q)$ (for q an odd power of 2). Then G has perfect difference graph.*

Proof. The simplest cases to deal with are $\mathrm{PSL}(2,q)$ for q a power of 2 and $\mathrm{Sz}(q)$. These groups have the property that the centralizer of any element is either cyclic or a 2-group; and, moreover, distinct centralizers meet only in the identity. So the difference graph consists of isolated vertices together with a disjoint union of difference graphs of cyclic groups, and so (by Theorem 9.2) it is perfect.

For the record, the maximal cyclic subgroups of $\mathrm{PSL}(2,q)$ have orders p (the prime divisor of q) and $(q\pm1)/\gcd(q\pm1,2)$; those of $\mathrm{Sz}(q)$ have orders 4, $q-1$, and $q\pm\sqrt{2q}+1$.

Indeed, we also see that for these groups, the difference graph is a cograph (or a threshold or split graph) if and only if the difference graphs of all the cyclic subgroups are. So, if q is a power of 2, then

- $D(\mathrm{PSL}(2,q))$ is a cograph if and only if each of $q-1$ and $q+1$ is a prime power or the product of two distinct primes;
- For q an odd power of 2, $D(\mathrm{Sz}(q))$ is a cograph if and only if each of $q-1$, $q+\sqrt{2q}+1$ and $q-\sqrt{2q}+1$ is either a prime power or the product of two primes.

Consider $\mathrm{PSL}(2,q)$ with q odd. In this group, if elements x and y have different centralizers, then they are not adjacent in the difference graph; for, if they commute but have different centralizers, they must both be involutions. Thus it is again true that $D(G)$ is the disjoint union of isolated vertices and the difference graphs of element centralizers (which are cyclic, dihedral, or elementary abelian). Moreover, we have a similar characterization of the case where the difference graph is a cograph: both $(q+1)/2$ and $(q-1)/2$ must be prime powers or products of two distinct primes. □

The result does not extend to all groups of Lie type of rank 1. For example, let G be the Ree group ${}^2G_2(q)=R_1(q)$, where q is an odd power of 3. Then $(q+1)/2$ is twice an odd number. Suppose that $(q+1)/2$ has two distinct prime divisors p and r. The centralizer of an involution t in G is isomorphic to $\mathbb{Z}_2\times\mathrm{PSL}(2,q)$, and so contains a subgroup $\mathbb{Z}_2\times\mathbb{Z}_{2pr}$, whose difference graph is not perfect, by Theorem 9.4. (It can be shown that, if $(q+1)/2$ is twice an odd prime power, then $D(R_1(q))$ is perfect; but we do not give the argument here.)

Further, the difference graphs of the Ree groups are never cographs. For let t be an involution in $G=R_1(q)$, so that $C_G(t)\cong\langle t\rangle\times\mathrm{PSL}(2,q)$. As noted above, q is an odd power of 3, and $(q-1)/2$ (the order of a cyclic subgroup of $\mathrm{PSL}(2,q)$) is twice an odd number. Let p be an odd prime dividing $(q+1)/2$, and let u and v be elements of order p and 2 in a cyclic group of order $(q+1)/2$; let s be an element of order 3 in $\mathrm{PSL}(2,q)$. Then $\{s,t,u,v\}$ induces a path of length 3.

This can be seen another way. The (non-simple) smallest Ree group $R_1(3)$ is isomorphic to $\mathrm{P\Gamma L}(2,8)$, and is contained in all other Ree groups. Computation shows that the cokernel of its difference graph is non-trivial. It has 147 vertices and is bipartite, with bipartite sets of sizes 63 and 84; it has diameter 6 and girth 10.

We have not examined the fourth type of rank 1 group, the unitary groups $\mathrm{PSU}(3,q)$, except to note that computation shows that their difference graphs are perfect for $q=3,4,5$.

Turning to groups of Lie type of rank greater than 1, we have the following:

Proposition 9.10. *Let q be a prime power, and assume that $q-1$ has at least three distinct prime divisors. Then $\mathrm{PSL}(3,q)$ has imperfect difference graph.*

Proof. Let F be the field of q elements. The subgroup of diagonal matrices in $\mathrm{SL}(3,q)$ is isomorphic to $F^\times \times F^\times$, under the map

$$\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} \mapsto (a,b)$$

(since $abc = 1$). This is also a subgroup of $\mathrm{PSL}(3,q)$ if $3 \nmid q-1$, whereas if $3 \mid q-1$ then we take the quotient by the cyclic group of order 3. With our hypothesis, in either case we have a subgroup $\mathbb{Z}_{plr} \times \mathbb{Z}_p$, where p, l, r are distinct primes. The result now follows from Theorem 9.4. □

We note that the argument shows also that, under the same hypothesis on q, $\mathrm{SL}(3,q)$ has imperfect difference graph. Now many groups of Lie type contain either $\mathrm{PSL}(3,q)$ or $\mathrm{SL}(3,q)$ as a subgroup; in particular, all those of rank at least 3, as we may see by looking at the Coxeter–Dynkin diagrams. The relevant information can be found in [14, Section 8.5]. A group of Lie type has a Levi factor corresponding to any sub-diagram of its Coxeter–Dynkin diagram, which is itself a group of Lie type or a central extension of one. So, if the diagram contains two vertices with a single edge joining them, the corresponding Levi factor is $\mathrm{PSL}(2,q)$ or $\mathrm{SL}(2,q)$. In addition, the group $G_2(q)$ contains $\mathrm{SL}(3,q)$. So, under the same hypothesis on q, the difference graphs of these groups are imperfect.

We summarise the situation for small simple groups. All simple groups of order smaller than $|J_1|$ have perfect difference graphs. These results were obtained using GAP [20], with the package `GRAPE` [32] for handling graphs. All these groups are described in the $\mathbb{ATLAS}$ [17].

- **Simple groups G for which $D(G)$ has no edges:** These are the simple EPPO groups: $\mathrm{PSL}(2,q)$ for $q = 4, 7, 8, 9, 17$, $\mathrm{Sz}(q)$ for $q = 8, 32$, and $\mathrm{PSL}(3,4)$.
- **Simple groups G for which $D(G)$ has edges but is a cograph, so that the cokernel has a single vertex:** Some further $\mathrm{PSL}(2,q)$ and $\mathrm{Sz}(q)$, depending on number-theoretic properties of q (e.g. in our range $\mathrm{PSL}(2,q)$ for $q = 11, 13, 16$).
- **Simple groups for which $D(G)$ is not a cograph but its cokernel is bipartite:** some further $\mathrm{PSL}(2,q)$ (e.g. $q = 23, 25$), $\mathrm{PSL}(3,3)$, $\mathrm{PSU}(3,3)$, M_{11}, A_8, $\mathrm{PSU}(4,2)$, $\mathrm{PSU}(3,4)$, M_{12}, $\mathrm{PSU}(3,5)$.

10. Twin reduction

We now describe a technique which should be useful in the study of various types of graphs on large groups.

Two vertices v and w of a graph Γ are called *twins* if they have the same neighbours except possibly for one another. (We sometimes distinguish *closed* and *open* twins according as they are joined or not.) The process of *twin reduction* on Γ consists of deleting one of a pair of twins repeatedly until no more twins remain. Recall from Section 9 that the resulting graph is called the cokernel of Γ; up to isomorphism, it is independent of the way the reduction is performed.

Recall that a graph Γ is a *cograph* if it does not contain the 4-vertex path P_4 as an induced subgraph. A graph is a cograph if and only if its cokernel has a single vertex. See [9] for discussion.

Twin reduction leaves some properties of a graph invariant.

Proposition 10.1. *Twin reduction of a graph Γ leaves the following properties unchanged:*

(a) *the number of connected components of a graph which are not cographs;*
(b) *the diameter of a connected component (if this is greater than 2);*
(c) *the girth of a connected component (if this is greater than 4);*
(d) *the property of containing an induced subgraph isomorphic to a fixed graph Δ (which itself contains no twins);*
(e) *perfectness.*

Proof. Consider a step in twin reduction, which identifies two vertices v and w. Suppose that v and w are not isolated. Then they belong to the same connected component, and their distances to any other vertex in this component are the same. So the connected component simply loses a vertex but remains connected. Moreover, if its diameter is at least 2, then we see that the diameter is unchanged by the reduction. (If the diameter is 1, then the component is complete, and so is a cograph; we have excluded this case.)

If Γ has diameter at least 3, or girth at least 5, then it contains a 4-vertex path, and so it is not a cograph.

If Δ is an induced subgraph of $\Gamma \setminus \{w\}$, then clearly it is an induced subgraph of Γ. Conversely, if Δ is an induced subgraph of Γ containing no pair of twins, then twin reduction will have no effect on Δ.

A graph is perfect if and only if it contains no induced odd cycle or complement of one. These graphs have no twins so their absence is preserved by twin reduction. (Note that a cograph is perfect.) □

Other properties may change. For example, if a graph Γ has girth $g > 4$, then its cokernel also has girth g; this may fail if $g = 4$. The list below shows several graphs with girth 4 whose cokernels have girth 6.

10.1. **Twin reduction of difference graphs.** Like most types of graphs defined on groups, difference graphs tend to have many pairs of twins: if x and y generate the same cyclic subgroup, then they are twins in the power graph, enhanced power graph (and hence the difference graph), as well as commuting graph, generating graph, and others. We have applied it to the difference graphs of various simple or almost simple groups. The list below gives some results on the number, diameters and girths of the connected components after removing isolated vertices, and whether they are cographs.

- $\mathrm{PSL}(3,3)$: connected, with 221 vertices, diameter 5 and girth 4; cokernel with 169 vertices and girth 6.
- $\mathrm{PSU}(3,3)$: connected, with 749 vertices, diameter 6 and girth 4; cokernel with 217 vertices, same diameter and girth.
- $\mathrm{PSL}(3,4)$: an EPPO group, no edges.

- PSU(3, 4): connected with 5187 vertices, diameter 6 and girth 4; cokernel with 481 vertices and girth 6.
- A_7: a cograph with 35 components of size 5 and diameter 2.
- S_7: one component with 1120 vertices, diameter 8 and cokernel with 322 vertices; seven isomorphic components with 55 vertices, diameter 6, and cokernels with 35 vertices and girth 6.
- A_8: two components, 1666 and 1225 vertices, diameters 8 and 6; cokernels wth 182 and 665 vertices, each of girth 6.
- S_8: connected, 5439 vertices, diameter 6; cokernel 1715 vertices.
- M_{11}: connected, 605 vertices, diameter and girth 10, cokernel has 385 vertices.
- M_{12}: two components, 2225 and 12540 vertices, diameters 6 and 8, both with girth 4; cokernels 1375 and 2112 vertices, girths 4 and 6.
- J_1: one component, 19019 vertices, diameter 6 and girth 4, cokernel with 7315 vertices and girth 6.

The examples show that the cokernel may be considerably smaller than the original graph. The computations reported here were performed in the computing system GAP [20], using the package GRAPE [32] for handling graphs. Generators for large groups were obtained from the On-Line Atlas of Finite Groups [37].

The difference graph of M_{11} has diameter 10; this is the largest value we have found for the diameter of the difference graph of any group. It is perhaps tempting to speculate that 10 is an upper bound. (It is perhaps worth mentioning here the surprising result of Giudici and Parker [22] that there is no upper bound for the diameters of commuting graphs of groups; but the examples with large diameter were of prime power order, and so their difference graphs are empty.)

Some interesting graphs arise as cokernels of difference graphs of simple groups. Here are two examples.

- **The Mathieu group M_{11}:** In this case, removal of isolated vertices and twin reduction brings the number of vertices down from 7920 to 385. The resulting graph is bipartite, with bipartite sets of sizes 165 and 220, and the vertices in the two sets have valencies 4 and 3 respectively. The graph has diameter 10 and girth 10; the girth is rather large for a graph of this size. The automorphism group of the graph is just M_{11}.
- **The group PSL(3, 3):** In this case, we found a very natural graph which has not been studied, as far as we are aware. The vertices are the ordered pairs (P, L), where P is a point and L a line of the projective plane of order 3 (so 169 vertices). The pairs fall into two types, *flags* (P incident with L) and *antiflags* (P not incident with L). The graph is bipartite: each edge joins a flag to an antiflag. The rule for adjacency is as follows: the flag (P, L) is incident with the antiflag (Q, M) if $Q \in L$ and $P \in M$. The automorphism of the graph is Aut(PSL(3, 3)).
- **The Janko group J_1:** We saw that $D(J_1)$ is imperfect. But its cokernel has a very interesting structure. As noted earlier, this graph has 7315 vertices, falling into three orbits O_1, O_2, O_3 under the automorphism group, which have sizes 2926, 1463, 2926 respectively. Each of O_1, O_2, O_3 is a

coclique. The induced subgraph on $O_1 \cup O_2$ is connected bipartite with diameter 7, girth 6 and valencies 5 (for vertices in O_1) and 10 (for vertices in O_2). The induced subgraph on $O_2 \cup O_3$ is connected bipartite with diameter and girth 10 and valencies 3 and 6. Both of these bipartite graphs have automorphism group J_1. The induced subgraph on $O_1 \cup O_3$ is a matching. So we can find an induced odd cycle in the graph as follows. Let $\{v, w\}$ be an edge of the matching between O_1 and O_3. Take an edge from w to x in O_2, and then a shortest path from x to v (necessarily of odd length). We see that the smallest odd cycle in the graph has length 7 or 9, though we have not decided which is the case.

11. Conclusion and Open Issues

In this paper, we studied the difference graph $D(G)$ of a finite group G. The study was mainly based on connectedness and perfectness of such graphs. Some of the problems which arise from this work can be interesting topics of further research.

For a finite group G with non-trivial center, it was shown that $D(G)$ is connected and with diameter less or equal to 6. However for groups G with trivial center, $D(G)$ may or may not be connected. So the question arises:

Question 11.1. If G has trivial center and $D(G)$ is connected, can $\mathrm{diam}(D(G))$ be greater than 10? More generally, does can any component of such a graph have diameter greater than 10?

Question 11.2. Complete the classification of finite groups whose difference graph is perfect.

Question 11.3. Find necessary and sufficient conditions on a complete graph with edges coloured red, green and blue for it to be embeddable in a finite group G such that

(a) red edges are adjacent in the power graph of G;
(b) green edges are adjacent in the difference graph (that is, in the enhanced power graph but not in the power graph); and
(c) blue edges are non-adjacent in the enhanced power graph.

Necessary conditions are that the red edges form the comparability graph of a partial order, and if x and y are joined by a green edge then there is a point z joined to both by green edges. Are these conditions sufficient? (A similar result for the enhanced power graph and commuting graph was proved in [9].)

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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