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RESEARCH

The Dispersion Relation for Waves in a Magnetic Flux Tube

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Abstract
A recent discussion (Yelagandula, 2023) of waves in a magnetic flux tube questions the use of the normal velocity continuity condition in the derivation of the standard dispersion relation. We re-assert this condition here.

Keywords Magnetic flux-tube waves · Coronal seismology · Coronal oscillations · Boundary conditions · Wave dispersion relation

1. Introduction

A recent discussion by Yelagandula (2023) of the dispersion relation describing linear magnetoacoustic waves in a magnetic flux tube raises for further consideration this important topic. Magnetic waves are now routinely observed in solar structures, and the nature of such waves allows a local seismology of the objects (Roberts, Edwin, and Benz, 1984; Aschwanden et al., 1999; Nakariakov et al., 1999; Nakariakov and Ofman, 2001); for reviews, see Nakariakov and Verwichte (2005), De Moortel and Nakariakov (2012), Nakariakov et al. (2021), Roberts (2008, 2019), and Nakariakov and Kolotkov (2020).

We argue here that Yelagandula (2023) is incorrect in the claim that the normal velocity across the flux-tube boundary is discontinuous but the normal component of the magnetic perturbation is continuous. On the contrary, we re-assert that the dispersion relation governing magnetoacoustic waves in a magnetic flux tube follows appropriately from continuity of the radial velocity component and the total pressure perturbation; the radial component of the perturbed magnetic field is, in general, discontinuous across the boundary of the flux tube.

The dispersion relation for magnetoacoustic waves in a magnetic flux tube was raised in independent studies by Zaitsev and Stepanov (1975) and Edwin and Roberts (1983). The detailed study by Edwin and Roberts (1983) builds on related earlier work by Roberts and Webb (1978), Roberts (1981) and Spruit (1982). A general overview is given in Roberts (2019).

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2. Wave Equations

Consider the small-amplitude (linear) modes of oscillation of a magnetic flux tube, modelled as a straight magnetic field aligned with the z-axis of a cylindrical polar coordinate system r, ϕ, z. The equilibrium magnetic field $\mathbf{B_0}$ is of strength B_0 and is structured radially in r:

$$\mathbf{B_0} = B_0(r)\mathbf{e_z}, \tag{1}$$

where $\mathbf{e_z}$ denotes the unit vector in the z-direction (the longitudinal axis of the flux tube). The equilibrium plasma pressure $p_0(r)$ and density $\rho_0(r)$ are also structured in r; pressure balance in the equilibrium state requires that

$$\frac{d}{dr}\left(p_0(r) + \frac{B_0^2(r)}{2\mu}\right) = 0, \tag{2}$$

where μ denotes the magnetic permeability.

We assume ideal conditions as represented in the equations of ideal magnetohydrodynamics under adiabatic conditions; viscous or ohmic effects are not considered. Small-amplitude motions $\mathbf{u}(r, \phi, z) = (u_r, u_\phi, u_z)$ about the equilibrium 2 may then be shown to satisfy the equation (see, for example, Roberts, 2019)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - c_A^2 \frac{\partial^2 \mathbf{u}}{\partial z^2} = -\mathbf{e_z} c_A^2 \frac{\partial}{\partial z}(\mathrm{div}\,\mathbf{u}) - \frac{1}{\rho_0}\mathrm{grad}\left(\frac{\partial p_T}{\partial t}\right), \tag{3}$$

where $c_A(r)\ (= B_0(r)/(\mu\rho_0(r))^{1/2})$ denotes the Alfvén speed within the field. Equation 3 follows from the time derivative of the momentum equation combined with the induction equation.

Additionally, we need to describe the evolution of the perturbation in the total pressure:

$$p_T(r, \phi, z) = p + B_0 B_z/\mu, \tag{4}$$

the sum of the plasma pressure perturbation p and the magnetic pressure perturbation. The perturbation in the magnetic field is $\mathbf{B} = (B_r, B_\phi, B_z)$, satisfying the solenoidal condition $\mathrm{div}\,\mathbf{B} = 0$. A combination of the isentropic equation and the induction equation leads to the evolution equation

$$\frac{\partial p_T}{\partial t} = \rho_0 c_A^2 \frac{\partial u_z}{\partial z} - \rho_0(c_s^2 + c_A^2)\mathrm{div}\,\mathbf{u}, \tag{5}$$

where $c_s(r)\ (= (\gamma p_0(r)/\rho_0(r))^{1/2})$ is the sound speed within the plasma.

The components of the wave-like Equation 3 give (Roberts, 2019)

$$\frac{\partial^2 u_r}{\partial t^2} - c_A^2(r)\frac{\partial^2 u_r}{\partial z^2} = -\frac{1}{\rho_0(r)}\frac{\partial^2 p_T}{\partial r \partial t}, \tag{6}$$

$$\frac{\partial^2 u_\phi}{\partial t^2} - c_A^2(r)\frac{\partial^2 u_\phi}{\partial z^2} = -\frac{1}{r\rho_0(r)}\frac{\partial^2 p_T}{\partial \phi \partial t}, \tag{7}$$

$$\frac{\partial^2 u_z}{\partial t^2} - c_t^2(r)\frac{\partial^2 u_z}{\partial z^2} = -\left(\frac{c_s^2(r)}{c_s^2(r) + c_A^2(r)}\right)\frac{1}{\rho_0(r)}\frac{\partial^2 p_T}{\partial z \partial t}, \tag{8}$$

where

$$c_{\mathrm{t}}(r)=\left(\frac{c_{\mathrm{s}}^{2}(r)c_{\mathrm{A}}^{2}(r)}{c_{\mathrm{s}}^{2}(r)+c_{\mathrm{A}}^{2}(r)}\right)^{1/2} \tag{9}$$

defines the slow magnetoacoustic speed c_{t}.

Finally, from Equation 5 the evolution of $p_{\mathrm{T}}(r,\phi,z)$ is given by

$$\frac{\partial p_{\mathrm{T}}}{\partial t}=-\left(\rho_0(r)c_{\mathrm{s}}^{2}(r)+\rho_0(r)c_{\mathrm{A}}^{2}(r)\right)\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\phi}{\partial\phi}\right)-\rho_0(r)c_{\mathrm{s}}^{2}(r)\frac{\partial u_z}{\partial z}. \tag{10}$$

3. Ordinary Differential Equations

The wave equations 6 – 8 coupled with the evolution equation 10 describe the linear motions of a cylindrically symmetric magnetic flux tube. We can Fourier analyse these partial differential equations by taking the radial component of the motion to be of the form

$$u_r(r,\phi,z,t)=u_r(r)\exp i(\omega t-\mathrm{m}\phi-k_z z), \tag{11}$$

with similar forms for all other perturbation quantities. The mode number $\mathrm{m}\,(=0,\pm1,\pm2,\ldots)$ describes the geometrical form of the perturbation. The symmetry of the equilibrium state allows us to consider zero or positive integers only. The case $\mathrm{m}=0$ corresponds to *symmetric* motions of the tube, disturbances being independent of ϕ. In addition to torsional Alfvén waves that have $p_{\mathrm{T}}=0$, there are compressible ($p_{\mathrm{T}}\neq 0$) motions that disturb the tube symmetrically; these are the *sausage* modes ($\mathrm{m}=0$). Modes with $\mathrm{m}=1$ are the *kink* modes, giving a global disturbance of the tube, and there are also *fluting* ($\mathrm{m}\geq 2$) modes.

All these motions may be described by the coupled pair of first-order ordinary differential equations (see Roberts (2019) for details)

$$i\omega\frac{dp_{\mathrm{T}}}{dr}=-\rho_0(r)(k_z^2c_{\mathrm{A}}^2(r)-\omega^2)u_r, \tag{12}$$

$$\rho_0(k_z^2c_{\mathrm{A}}^2(r)-\omega^2)\frac{1}{r}\frac{d}{dr}(ru_r)=-\left(m^2(r)+\frac{\mathrm{m}^2}{r^2}\right)i\omega p_{\mathrm{T}}, \tag{13}$$

where $m^2(r)$ is defined by

$$m^2(r)=\frac{(k_z^2c_{\mathrm{s}}^2(r)-\omega^2)(k_z^2c_{\mathrm{A}}^2(r)-\omega^2)}{(c_{\mathrm{s}}^2(r)+c_{\mathrm{A}}^2(r))(k_z^2c_{\mathrm{t}}^2(r)-\omega^2)}. \tag{14}$$

The nature of $m^2(r)$ proves important in determining the physical nature of the modes that the system supports.

The differential equations 12 and 13 are the principal equations governing the behaviour of waves in our system; the behaviour of any of the other quantities, such as the radial magnetic-field component B_r, may be determined once these equations are solved.

We may eliminate one or other variable between the pair of first-order differential equations 12 and 13 to produce a second-order ordinary differential equation. Eliminating p_{T} between these equations yields

$$\frac{d}{dr}\left\{\frac{\rho_0(r)(k_z^2c_{\mathrm{A}}^2(r)-\omega^2)}{\left(m^2(r)+\frac{\mathrm{m}^2}{r^2}\right)}\frac{1}{r}\frac{d}{dr}(ru_r)\right\}=\rho_0(r)(k_z^2c_{\mathrm{A}}^2(r)-\omega^2)u_r, \tag{15}$$

whereas eliminating u_r in favour of the pressure perturbation p_T yields

$$\rho_0(r)(k_z^2c_A^2(r)-\omega^2)\frac{1}{r}\frac{d}{dr}\left\{\frac{1}{\rho_0(r)(k_z^2c_A^2(r)-\omega^2)}r\frac{dp_T}{dr}\right\}=\left(m^2(r)+\frac{\mathrm{m}^2}{r^2}\right)p_T. \tag{16}$$

Equation 15 is a form of the Hain–Lüst equation (Hain and Lüst, 1958), presented here for an untwisted equilibrium magnetic field.

4. Continuity Conditions

We can draw an important observation from Equations 12 – 16. Consider an equilibrium state $\rho_0(r)$, $B_0(r)$ that changes smoothly in r. Suppose that in a specific location the equilibrium varies rapidly in r and in the limit may become discontinuous in r: $\rho_0(r)$ and $B_0(r)$ may jump from one value to another. This describes the boundary of a discrete flux tube. What happens to u_r?

Suppose that u_r is *discontinuous* at some radial location, forming a step function there. Then, at that location du_r/dr contains a *delta* function δ, arising from the derivative of a step function, and so d^2u_r/dr^2 (arising on the left-hand side of Equation 15) consequently contains the *derivative* of a δ-function. Such generalised functions are discussed in detail in, for example, Lighthill (1958). However, this cannot be matched by the term on the right-hand side of Equation 15, which contains at most step functions. Hence, our initial assumption that u_r is discontinuous must be *false*: u_r is continuous at all locations in r. A similar argument applies to p_T, using Equation 16. Hence, we conclude that both u_r and p_T *are continuous across the boundary* of a magnetic flux tube.

It may also be argued on physical grounds that the two quantities u_r and p_T are continuous in r (and in particular across the boundary of a flux tube). The radial *displacement* ξ_r (and hence the radial motion u_r $(=\partial\xi_r/\partial t)$) can be expected to change smoothly from one side of the tube boundary to the other, as otherwise there would be a mismatch between the fluid elements either side of the tube boundary. Furthermore, any jump in the pressure perturbation p_T would imply an *unbalanced* force acting on the tube boundary. Hence, physically, we expect continuity of u_r and p_T across the tube boundary of a flux tube.[1]

Of course, it might also be considered that a flux tube with a discontinuous equilibrium longitudinal magnetic field is unphysical. Certainly, in reality we can expect a smoothly changing profile in such an equilibrium quantity; diffusive processes will ensure that, although the spatial change may occur over a short distance. However, treating an abrupt change in an equilibrium quantity as the limit of a smoothly but rapidly changing equilibrium profile is a legitimate and useful mathematical approach.

We should note that in this discussion we are not considering resonances associated with the singularities of magnetohydrodynamic waves; a recent treatment of resonances in the context of coronal magnetic flux tubes is given, for example, in Soler et al. (2013).

However, not all perturbations are continuous. In particular, the radial component B_r of the perturbed magnetic field $\mathbf{B}$ is discontinuous at any location where the applied magnetic field $\mathbf{B_0}$ is discontinuous. To see this, consider the ideal induction equation for the magnetic

[1] These arguments and any derivation of the dispersion-relation governing waves (see Section 5) are here specific to an equilibrium state of a flux tube in which flows are absent; the case of a tube with a basic flow needs a separate consideration.

perturbation **B**, namely

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}\,(\mathbf{u} \times \mathbf{B_0}) = B_0 \frac{\partial \mathbf{u}}{\partial z} - \mathbf{e_z}\big[B_0 \mathrm{div}\,\mathbf{u} + \mathbf{u} \cdot \mathrm{grad}\, B_0\big]. \tag{17}$$

Note from Equation 17 that $\mathrm{div}\,\mathbf{B} = 0$ is satisfied for all time if it is satisfied initially. Now, we are particularly interested in the radial component B_r, for which

$$\frac{\partial B_r}{\partial t} = B_0(r) \frac{\partial u_r}{\partial z}. \tag{18}$$

It follows from Equation 18 that if u_r is continuous across $r = r_0$, then so also is $B_r/B_0(r)$. However, if the equilibrium magnetic field $B_0(r)$ is a step function in r and u_r is continuous, then it follows that B_r is *discontinuous* in r. Of course, if $B_0(r)$ is a constant, as in a uniform field (as might arise in a low-β plasma), then *both* u_r and B_r are continuous, as noted by Yelagandula (2023).

5. Dispersion Relation

We turn now to the application of the two continuity conditions. Consider a flux tube of radius a that consists of a uniform interior, in which the plasma density and the sound and Alfvén speeds inside the tube are constants, surrounded by a uniform environment. In a uniform medium, the pressure equation reduces to a form of Bessel's equation (Zaitsev and Stepanov, 1975; Edwin and Roberts, 1983)

$$r^2 \frac{d^2 p_{\mathrm{T}}}{dr^2} + r \frac{p_{\mathrm{T}}}{dr} - (m_0^2 r^2 + \mathrm{m}^2) p_{\mathrm{T}} = 0, \qquad m_0^2 = \frac{(k_z^2 c_{\mathrm{s}}^2 - \omega^2)(k_z^2 c_{\mathrm{A}}^2 - \omega^2)}{(c_{\mathrm{s}}^2 + c_{\mathrm{A}}^2)(k_z^2 c_{\mathrm{t}}^2 - \omega^2)}. \tag{19}$$

The Bessel equation 19 has solutions of the modified Bessel functions $I_{\mathrm{m}}(m_0 r)$ and $K_{\mathrm{m}}(m_0 r)$ (see Abramowitz and Stegun, 1965). For a solution that is finite at the centre ($r = 0$) of a flux tube we take

$$p_{\mathrm{T}}(r) = A_0 I_{\mathrm{m}}(m_0 r), \qquad r < a, \tag{20}$$

where A_0 is an arbitrary constant.

In a similar fashion, we may apply Equation 19 to the environment of the flux tube, now selecting

$$p_{\mathrm{T}}(r) = A_e K_{\mathrm{m}}(m_e r), \qquad r > a, \tag{21}$$

where A_e is an arbitrary constant and m_e is the value of m_0 when calculated in the tube's environment ($r > a$) where the sound speed is c_{se} and the Alfvén speed is c_{Ae}; explicitly,

$$m_e^2 = \frac{(k_z^2 c_{\mathrm{se}}^2 - \omega^2)(k_z^2 c_{\mathrm{Ae}}^2 - \omega^2)}{(c_{\mathrm{se}}^2 + c_{\mathrm{Ae}}^2)(k_z^2 c_{\mathrm{te}}^2 - \omega^2)}, \qquad c_{\mathrm{te}}^2 = \frac{c_{\mathrm{se}}^2 c_{\mathrm{Ae}}^2}{c_{\mathrm{se}}^2 + c_{\mathrm{Ae}}^2}. \tag{22}$$

This solution has been selected to give $p_{\mathrm{T}} \to 0$ as $r \to \infty$, requiring that $m_e > 0$; under these conditions, the external pressure disturbance decays exponential fast outside the tube.

Finally, imposing the continuity conditions that p_{T} and u_r are continuous across $r = a$ results in the dispersion relation

$$\frac{1}{\rho_0(k_z^2 c_{\mathrm{A}}^2 - \omega^2)} m_0 a \frac{I_{\mathrm{m}}{}'(m_0 a)}{I_{\mathrm{m}}(m_0 a)} = \frac{1}{\rho_{\mathrm{e}}(k_z^2 c_{\mathrm{Ae}}^2 - \omega^2)} m_e a \frac{K_{\mathrm{m}}{}'(m_e a)}{K_{\mathrm{m}}(m_e a)}, \tag{23}$$

where a prime denotes the derivative of the function (so $I_{\mathrm{m}}{}'(x)$ is the derivative with respect to x of the function $I_{\mathrm{m}}(x)$, and $I_{\mathrm{m}}{}'(x_0)$ denotes $I_{\mathrm{m}}{}'(x)$ evaluated at $x = x_0$). This is the dispersion relation presented by Edwin and Roberts (1983). Other forms are also possible and are discussed in their paper. The nature of the solution of this complicated dispersion relation is discussed in some detail by Edwin and Roberts (1983) and need not be explored further here (see, for example, Nakariakov and Verwichte, 2005; Roberts, 2019; Nakariakov and Kolotkov, 2020; Nakariakov et al., 2021).

6. Conclusion

We have argued why the standard model of waves in a magnetic flux tube, as given by Edwin and Roberts (1983), remains appropriate with the dispersion relation derived therein remaining valid. We have re-established the use of the boundary conditions requiring continuity of the radial velocity and the total pressure perturbation, contrary to a claim by Yelagandula (2023). An independent study by Goedbloed and Poedts (2024), arguing from a different perspective from us, has reached a similar conclusion.

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Author contributions This work is solely that of the author B. Roberts.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

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