MAXIMUM LIKELIHOOD THRESHOLDS VIA GRAPH RIGIDITY

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The maximum likelihood threshold (MLT) of a graph G is the minimum number of samples to almost surely guarantee existence of the maximum likelihood estimate in the corresponding Gaussian graphical model. We give a new characterization of the MLT in terms of rigidity-theoretic properties of G and use this characterization to give new combinatorial lower bounds on the MLT of any graph.

We use the new lower bounds to give high-probability guarantees on the maximum likelihood thresholds of sparse Erdös-Rényi random graphs in terms of their average density. These examples show that the new lower bounds are within a polylog factor of tight, where, on the same graph families, all known lower bounds are trivial.

Based on computational experiments made possible by our methods, we conjecture that the MLT of an Erdös-Rényi random graph is equal to its generic completion rank with high probability. Using structural results on rigid graphs in low dimension, we can prove the conjecture for graphs with MLT at most 4 and describe the threshold probability for the MLT to switch from 3 to 4.

We also give a geometric characterization of the MLT of a graph in terms of a new "lifting" problem for frameworks that is interesting in its own right. The lifting perspective yields a new connection between the weak MLT (where the maximum likelihood estimate exists only with positive probability) and the classical Hadwiger-Nelson problem.

1. Introduction. Modern statistical applications often require researchers to make inferences about a large number of variables from few observations (see e.g. [31, Chapter 18]). For example, certain biological network modeling problems, including those related to gene regulation [23, 54, 64] and metabolic pathways [40], can be approached by fitting a Gaussian graphical model to a dataset that has fewer datapoints than variables. This invites one to ask the motivating question of this paper, which was previously explored by Uhler [60], who attributes recent interest in it to Lauritzen: *given a fixed Gaussian graphical model, what is the minimum number of datapoints so that the maximum likelihood estimate exists*? We now define some terms and state the question more precisely.

Let G be a graph with n vertices. The Gaussian graphical model associated with G is the set of n-variate normal distributions $\mathcal{N}(0, \Sigma)$ so that if ij is *not* an edge of G, then

$(\Sigma^{-1})_{ij} = 0$, i.e. the corresponding random variables are conditionally independent given all of the other random variables. Suppose now that we have iid samples $X_1, \ldots, X_d$ from a Gaussian graphical model. The MLE of the covariance is, then, the inverse of the matrix K that solves the following optimization problem (see, e.g., [31, p. 632])

(1)
$$\underset{K}{\text{minimize}} \quad \text{Trace}(SK) - \log \det K$$
$$\text{subject to} \quad K \in \mathcal{S}^n_{++} \text{ and } K_{ij} = 0 \text{ if } ij \notin E(G)$$

where S is the sample covariance[1] and $\mathcal{S}^n_{++}$ is the set of positive definite $n \times n$ matrices. This is a convex problem that can be solved efficiently in practice [61]. Computing the MLE is a common way to fit a Gaussian graphical model to data. If $d \geq n$ and G is complete then the MLE of the covariance is $K = S^{-1}$. Indeed, almost surely S^{-1} exists and then

$$\frac{d}{dK}\left(\text{Trace}(SK) - \log \det K\right) = S - K^{-1}$$

vanishes at S^{-1}. As a warmup for some of the ideas in Section 3, now consider the case when $d < n$ and G is complete. Since S has rank at most d, we can find a non-zero vector v in the kernel of S. For all $t \geq 0$, $I + tvv^T$ is positive definite and

$$\text{Trace}(S(I + tvv^T)) - \log \det(I + tvv^T) = \text{Trace}(S) - \log \det(I + tvv^T) \to -\infty$$

as $t \to \infty$. Hence the MLE of the covariance does not exist. What goes wrong is that the sampled datapoints lie in a proper linear subspace of $\mathbb{R}^n$, so we can "overfit" the sample by Gaussian densities that have, as their level sets, increasingly flat ellipsoids.

If G is not complete, however, the MLE might exist almost surely even when $d < n$. This prompts the following definition.

DEFINITION 1.1. The *maximum likelihood threshold* (MLT) of a graph G, denoted $\text{mlt}(G)$, is the smallest number of samples[2] required for the MLE of the Gaussian graphical model associated with G to exist almost surely.

REMARK 1.2. The maximum likelihood threshold can be similarly defined for any class of Gaussian models. See e.g. [3, 24, 45]

1.1. *Existing bounds on the MLT.* The discussion above implies that $\text{mlt}(K_n) = n$. For any G, $1 \leq \text{mlt}(G) \leq n$, since if H is a subgraph of G, then $\text{mlt}(H) \leq \text{mlt}(G)$. Heuristically, if G is very sparse, we could hope that $\text{mlt}(G)$ is much less than n. However, counting edges is not enough to get good bounds, since, as we will see, small or sparse subgraphs can push the MLT up.

Ideally, one would like an efficient algorithm to compute $\text{mlt}(G)$, but this seems difficult and the complexity of computing $\text{mlt}(G)$ remains open.[3] Instead, the literature, which we now review, focuses on finding combinatorial properties that bound the MLT, a problem first raised by Dempster [22] and, more recently, popularized by Lauritzen (see [60, 5]). The first nontrivial bounds on the MLT are due to Buhl [12].

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[1] The sample covariance is $S = \frac{1}{d}XX^T$.

[2] Here, we are assuming that the samples are i.i.d. from a distribution whose probability measure is mutually absolutely continuous with respect to Lebesgue measure.

[3] It follows from Dempster's work [22] that one can compute $\text{mlt}(G)$ using, e.g., cylindrical decomposition of a semi-algebraic set, but the algorithms for this task are not fast enough to be of practical interest.

THEOREM 1.3 ([12]). *Let G be a graph with clique number $\omega(G)$ and treewidth $\tau(G)$. Then*

$$\omega(G) \leq \mathrm{mlt}(G) \leq \tau(G) + 1.$$

We will see presently that both of these estimates are unsatisfactory: computing clique number and treewidth are NP-hard problems and both inequalities are extremely weak. As a running example to compare inequalties, we will use the complete bipartite graph $K_{m,m}$. Theorem 1.3 implies that

$$2 = \omega(K_{m,m}) \leq \mathrm{mlt}(K_{m,m}) \leq \tau(K_{m,m}) + 1 = m + 1.$$

In a landmark paper that studied the (semi-) algebraic geometry of Gaussian maximum likelihood estimation, Uhler [60] used tools from algebraic geometry to bound the MLT.

DEFINITION 1.4. Let G be a graph with n vertices and m edges. Let $\mathcal{S}^{d+1}$ be the set of symmetric matrices of rank $d + 1$. The *generic completion rank of G*[4], denoted $\mathrm{gcr}(G)$, is the smallest $d + 1$ so that the orthogonal projection of $\mathcal{S}^{d+1}$ onto the diagonal entries and the entries corresponding to the edges of G is $(m + n)$-dimensional.

THEOREM 1.5 ([60]). *Let G be a graph. Then $\mathrm{mlt}(G) \leq \mathrm{gcr}(G)$.*

Uhler formulated the generic completion rank in terms of a certain elimination ideal being empty, but one can compute $\mathrm{gcr}(G)$ with a randomized algorithm and linear algebra (see [30]). The upper bound from Theorem 1.5 is very much tighter than the one from Theorem 1.3. It can also be used to extract other combinatorial bounds on the MLT, for example the presence of a k-core (the maximum induced subgraph of minimum degree k). Via [7, Corollary 4.5], Uhler's bound implies that if k is the minimum integer such that the k-core of G is empty, then $\mathrm{mlt}(G) \leq k - 1$.

In our running example, we have

$$\mathrm{mlt}(K_{m,m}) \leq \mathrm{gcr}(K_{m,m}) = m - 2$$

(see Theorem 1.6 below for the GCR of $K_{m,m}$). Thus, on our running example, Uhler's bound is better than Buhl's *and* it is much easier to compute.

For some time, it was open whether, in fact, $\mathrm{mlt}(G) = \mathrm{gcr}(G)$ for every graph G. Blekherman and Sinn [8] provided a negative answer as part of a detailed study of complete bipartite graphs. We will give a more detailed account of [8], but here is one summary result.

THEOREM 1.6 ([8]). *Let $m, D \in \mathbb{N}$ so that $m > 2$ and D is largest number satisfying $2m > \binom{D+1}{2}$. Then*

$$\mathrm{gcr}(K_{m,m}) = m \qquad and \qquad \mathrm{mlt}(K_{m,m}) = D.$$

Comparing with Theorem 1.3, we see that $K_{m,m}$ has clique number 2 and MLT $\Theta(\sqrt{m})$. Comparing with Theorem 1.5, we see that the upper bound from generic completion rank is also off by an $O(\sqrt{m})$ factor, making $\mathrm{gcr}(G)$ far from tight as an upper bound.

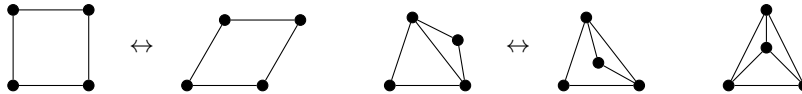[4]The term "generic completion rank" is due to Blekherman and Sinn [8].

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FIG 1. *Above are some frameworks in $\mathbb{R}^2$. The framework on the left fails to be rigid because there exist arbitrarily close frameworks that are equivalent but not congruent - one can deform it an arbitrarily small amount as indicated. The frameworks in the middle fail to be globally rigid since they are equivalent but not congruent. They are, however, rigid. Indeed, neither can be perturbed an infinitesimally small amount without changing edge lengths. Finally, the framework on the right is globally rigid and therefore also rigid.*

1.2. *MLT and rigidity.* In this paper, we give new lower bounds on the MLT, which are more general and sharper than those mentioned above. Our methods are based on a connection to graph rigidity theory, which we briefly introduce. Figure 1 illustrates the following definitions for $d = 2$.

DEFINITION 1.7. Let $d \in \mathbb{N}$ be a dimension. A *framework in $\mathbb{R}^d$* is a pair (G, p) where G is a graph with n vertices $\{1, \ldots, n\}$ and $p = (p(1), \ldots, p(n))$ is a configuration of n points in $\mathbb{R}^d$. Two frameworks (G, p) and (G, q) are *equivalent* if

$$\|p(j) - p(i)\| = \|q(j) - q(i)\| \qquad \text{for all edges } ij \text{ of } G$$

and *congruent* if p and q are related by a Euclidean isometry, i.e. if there exists a Euclidean isometry $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $q(i) = T(p(i))$ for $i = 1, \ldots, n$. If two frameworks are congruent, then they are also equivalent but the converse need not hold. Frameworks for which the converse *does* hold are called *globally rigid* in dimension d, i.e. (G, p) is globally rigid if all equivalent d-dimensional frameworks are congruent. If this happens only for some neighborhood U around p, i.e. if (G, p) and (G, q) are congruent whenever $q \in U$ and (G, q) and (G, p) are equivalent, then (G, p) is said to be *rigid* in dimension d.

On an intuitive level, rigidity of a d-dimensional framework (G, p) means that if one were to physically build G in $\mathbb{R}^d$ using rigid bars for the edges and universal joints for the vertices, placed according to p, then the resulting structure could not deform. Rigidity of a specific framework is difficult to check [1], but for each dimension d, every graph has a generic behavior. Following [60, 30], we use the following notion of generic, which comes from algebraic geometry.

DEFINITION 1.8. Let p be a configuration of n points in $\mathbb{R}^d$. We say that p is *generic* if the coordinates of p do not satisfy any polynomial with rational coefficients.

The following theorem is fundamental in combinatorial or graph rigidity theory. It tells us that by invoking a genericity assumption, we can treat rigidity and global rigidity as properties of a graph rather than as properties of a framework.

THEOREM 1.9 ([4, 27]). *Let d be a fixed dimension and G a graph. Then either every generic d-dimensional framework (G, p) is (globally) rigid or every generic d-dimensional framework (G, p) is not (globally) rigid.*

DEFINITION 1.10. Let d be a fixed dimension and G be a graph with m edges. We call G *(globally) d-rigid* if its generic d-dimensional frameworks are (globally) rigid. We call G *d-independent* if there is an m-dimensional space of differential changes to the edge lengths of a (or any) generic framework (G, p).

In Figure 1, the graphs underlying the frameworks in the middle and on the left are 2-independent, whereas the graph of the framework on the right is not. To see this, note that in frameworks in the middle and left, it is possible to increase or decrease the length of any edge a small amount without changing any other edge lengths. This is not the case for the framework on the right.

An important fact in rigidity theory is that the d-independent graphs form the independent sets of a matroid. Gross and Sullivant [30] reformulated Theorem 1.5 in the language of algebraic matroids (see [52] for an introduction) and proved the following.

THEOREM 1.11 ([30]). *Let G be a graph. Then the generic completion rank of G is $d+1$ if and only if d is the smallest dimension in which G is d-independent.*

This result does not improve Uhler's upper bound on the MLT, but it does open up the possibility of employing graph rigidity-theoretic ideas to understand it better. An interesting example is:

THEOREM 1.12 ([30]). *If G is a planar graph, then $\mathrm{mlt}(G) \leq 4$.*

The proof uses the Cauchy–Dehn–Alexandrov theorem (see [26]) which implies that any planar graph is 3-independent. One can immediately deduce the same bound for the wider class of K_5-minor free graphs using a result of Nevo [47].

Graph rigidity theory also makes it easier to compare treewidth to the generic completion rank. The following shows just how far away from tight Buhl's upper bound can be. In the sequel, we will make statements about sequences of events E_n involving random graph families indexed by the number of vertices n that hold *with high probability* (whp). This means that $\mathbb{P}(E_n) \to 1$ as $n \to \infty$.

It is well-known that, for $d \geq 2$, a random $(d+1)$-regular graph G with n vertices has, whp, treewidth $\tau(G) > \beta n$, for some $\beta > 0$ (see, e.g, [39]). Since, for $d \geq 2$, the only $(d+1)$-regular graph that is not d-independent is K_{d+2} [33], Theorem 1.11 implies that

$$\mathrm{gcr}(G) \leq d+1 < \beta n < \tau(G) + 1$$

whp for a random $(d+1)$-regular graph G.

1.3. *Results and guide to reading.* In this paper, we will reformulate the MLT of a graph in terms of equilibrium stresses, a graph rigidity-theoretic concept that plays an important role in global rigidity. Given vertices i and j of a graph G, we write $i \sim j$ to indicate that G has an edge between i and j.

DEFINITION 1.13. Let G be a graph with n vertices and let (G, p) be a framework. An *equilibrium stress* ω of (G, p) is an assignment of weights ω_{ij} to the edges of G so that, for all vertices i

$$\sum_{j \sim i} \omega_{ij}(p(j) - p(i)) = 0 \qquad \text{(sum over neighbors } j \text{ of } i).$$

The *equilibrium stress matrix* associated to an equilibrium stress ω is the matrix Ω obtained by setting $\Omega_{ji} = \Omega_{ij} = -\omega_{ij}$ for all edges ij of G, $\Omega_{ii} = \sum_j \omega_{ij}$ and all other entries zero. The *rank* and *signature* of ω are defined to be the rank and signature of Ω, and ω is said to be *PSD* if Ω is positive semi-definite.

FIG 2. *The framework in $\mathbb{R}^1$ on the left is equivalent to a framework in $\mathbb{R}^3$ with full-dimensional affine span. To see this, first note that it is equivalent to the framework in $\mathbb{R}^2$ to the right of it. Then note that this two-dimensional framework is equivalent to a framework in $\mathbb{R}^3$ with full-dimensional affine span since we can lift one of the vertices into the third dimension without changing edge-lengths. However, the maximum likelihood threshold of the underlying graph, the four-cycle, is* not *two since every framework equivalent to the framework in the middle has a one-dimensional affine span. On the other hand, the path with four vertices has an MLT of 2 because any generic one-dimensional framework on it can be folded out to three dimensions. On the right, we see such a one-dimensional framework folding out into two dimensions. We can further fold it into three by bringing the vertex on the left out of the affine plane spanned by the other vertices.*

A fact going back to Maxwell [46] is that a framework (G, p) in dimension d is independent if and only of it has no non-zero equilibrium stress. Similarly, a graph is d-independent if and only if no generic framework (G, p) has a non-zero equilibrium stress.

To see the relation with Uhler's bound (Theorem 1.5), we can use Theorem 1.11 and the discussion above to get the following formulation.

THEOREM 1.14 ([60, 30]). *Let G be a graph with n vertices. Suppose that no generic framework in dimension d supports a non-zero equilibrium stress. Then the MLT of G is at most $d + 1$.*

To obtain a lower bound on the MLT, we will need to consider the signature of the equilibrium stress. Our central new tool will be the following theorem, proved in Section 3.

THEOREM 1.15. *Let G be a graph with n vertices. Then the MLT of G is $d + 1$ if and only if d is the smallest dimension in which no generic d-dimensional framework supports a non-zero PSD equilibrium stress.*

This technical theorem along with some known graph rigidity-theoretic results and arguments will allow us to significantly expand our understanding of the MLT (as well as directly reproduce most of what is already understood).

To return to our running example, Theorem 1.6 implies that $K_{m,m}$ generically supports equilibrium stresses in dimension $m - 2$, but that they are all indefinite. In Section 5, we will re-derive Theorem 1.6 by first understanding the equilibrium stresses of complete bipartite graphs.

There is a geometric counterpart to Theorem 1.15, originally conjectured by Gross and Sullivant [30]. A d-dimensional framework (G, p) *has full affine span* if p affinely spans $\mathbb{R}^d$.

THEOREM 1.16. *Let G be a graph with n vertices. Then the MLT of G is $d + 1$ if and only if d is the smallest dimension in which every generic d-dimensional framework (G, p) is equivalent to an $(n - 1)$-dimensional framework $(G, \tilde{p})$ with full affine span.*

See Figure 2 for an illustration of Theorem 1.16.

To give combinatorial bounds, we use a connection between globally rigid graphs and PSD equilibrium stresses from [18]. Our bounds are in terms of a new graph parameter based on globally rigid subgraphs of G.

DEFINITION 1.17. The *global rigidity number* of G, denoted $\text{grn}(G)$, is the maximum d such that G is globally d-rigid and has at least $d + 2$ vertices. The *globally rigid subgraph number* of G, denoted $\text{grn}^*(G)$, is the maximum d so that G contains a subgraph H on at least $d + 2$ vertices that is globally rigid.

We obtain the following new lower bound on the MLT of a graph.

THEOREM 1.18. *Let G be a graph. Then* $\mathrm{grn}(G) + 2 \leq \mathrm{grn}^*(G) + 2 \leq \mathrm{mlt}(G)$.

Since complete graphs are globally d-rigid for all d, Theorem 1.18 generalizes the lower bound of Theorem 1.3. To our knowledge, this is the first unconditional improvement of Buhl's lower bound from 1993 (Theorem 1.3).

In our running example of $K_{m,m}$, the lower bound from Theorem 1.18 gives the right answer. The complete bipartite graph $K_{m,m}$ has global rigidity number $m-3$ [19] and, by Theorem 1.6, MLT $m-1$. Hence $\mathrm{mlt}(K_{m,m}) = \mathrm{grn}(K_{m,m}) + 2$.

In Section 7, we combine Theorem 1.18 with results on graph rigidity in dimension 2 to completely solve the MLT problem for small values of $\mathrm{mlt}(G)$ and $\mathrm{gcr}(G)$. The main result (which is best possible – see Remark 7.5) is as follows.

THEOREM 1.19. *If G is a graph and* $\mathrm{mlt}(G) \leq 3$ *or* $\mathrm{gcr}(G) \leq 4$*, then* $\mathrm{mlt}(G) = \mathrm{gcr}(G)$.

Theorem 1.18 is also strong enough to give a quick proof of the results in [8]. Section 5 explores the connection.

1.4. *Improvement in MLT bounds.* The following table summarizes the MLT bounds for various families of graphs given in this paper and compares with the best known previous bounds. Let $d > 1$.

Type of Graph	Best Previous MLT interval	This Paper
Minimally d-rigid	$[2, d+1]$ Thm 1.3	$d+1$ Cor 4.8
Globally d-rigid d-circuit	$[2, d+2]$ Thm 1.3	$d+2$ Cor 4.10
$G(n, c/n)$ $0 < c < c_2'$	$[3, 4]$ Thm 1.5 [7]	3 whp Thm 2.3
$G(n, c/n)$ $c_2' < c < c_3$	$[3, 4]$ Thm 1.5 [7]	4 whp Thm 2.3
$G(n, M_d \log n/n)$	$\exists k[3, (\log n)^k]$ whp Thm 1.3,1.5	$\exists k[d, (\log n)^k]$ whp Cor 2.9

2. Examples and conjectures. The families of graphs for which the MLT has been computed exactly are quite limited in the literature. As mentioned in the introduction, Buhl [12] computes $\mathrm{mlt}(K_{d+2}) = d+1$ and Bleckhermann and Sinn [8] compute $\mathrm{mlt}(K_{m,n})$ (see Section 5). Uhler [60] provides, in addition, that the MLT of a cycle is 3.

This sections contains examples illustrating improvements that can be obtained from our methods and some remaining conjectures.

2.1. *4-regular graphs.* As discussed above, any connected 4-regular graph G on $n > 5$ vertices is 3-independent and, hence, by Theorem 1.11, $\mathrm{gcr}(G) \leq 4$. Since G has $2n > 2n - 3$ edges, the Laman–Pollaczek-Geiringer Theorem (Theorem 7.2, below) implies that G is 2-dependent so $\mathrm{gcr}(G) > 3$, again via Theorem 1.11. On the other hand, it is well-known that, whp, a random 4-regular graph has clique number at most 3[5]. We now have that, whp, for a random 4-regular graph G:

$$\omega(G) \leq 3 < 4 = \mathrm{mlt}(G).$$

This gap, while not terribly impressive, was easy to get.

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[5]One way to see this is that a 4-regular graph containing a subgraph isomorphic to K_4 is not essentially 6-connected. Random 4-regular graphs are essentially 6-connected with high probability [63].
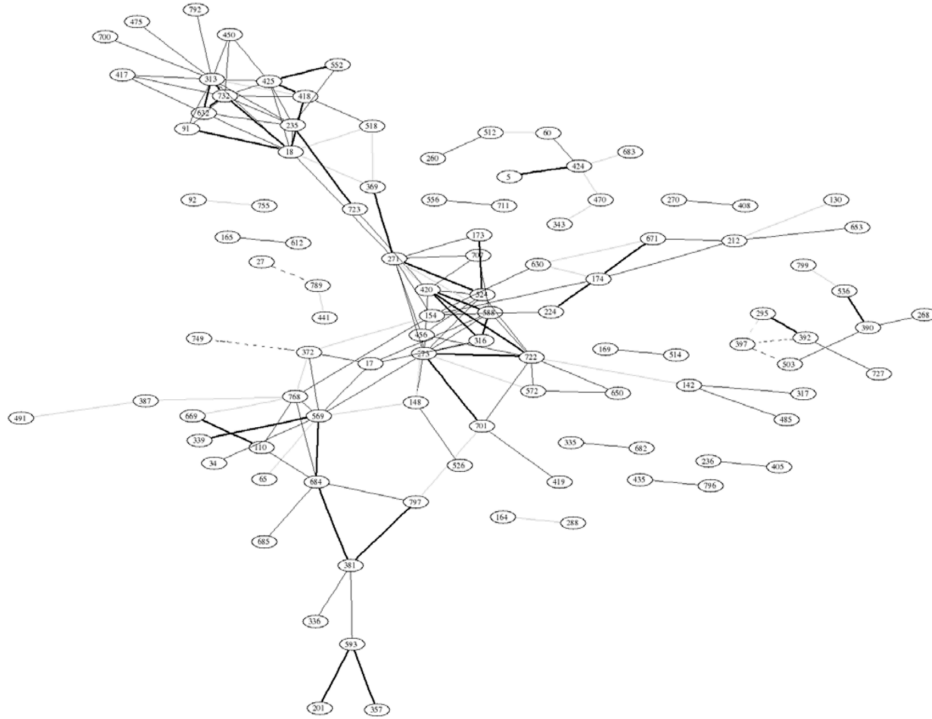
FIG 3. *This graph is reproduced from [49, Figure 1]. Although it has vertices of degree greater than four, the graph can be reduced to a graph with no vertices by iteratively finding a vertex of degree at most four and removing it and any incident edges. For example, the degree of vertex 174 is five, but once its neighbor 671, which has degree three, is removed, the degree of 174 decreases, allowing for its removal. Continuing in this manner eventually removes every vertex.*

2.2. *Examples from the applied literature.* We now consider several examples of Gaussian graphical models that appear in the applied literature on gene networks, a setting where the models are typically high-dimensional. Our rigidity theoretic methods make it easy to compute the MLT for these instances, to the extent that the work is simple enough to do by hand in many cases.

The approach is to show, first, that a graph G is d-independent by iteratively removing vertices of degree at most d until there are no vertices left. Such a graph cannot have a general position framework with a non-zero equilibrium stress in dimension d, so it must be d-independent. For the matching lower bound, we find a globally $(d-1)$-rigid subgraph with at least $d+1$ vertices. In cases where the globally rigid subgraph is complete, one could get the lower bound from the existing Theorem 1.3, but our approach gives a unified treatment of both bounds. We now turn to specific examples.

In [49, Figure 1] there is a graph G on 150 edges inferred from Arabidopsis thaliana data. By iteratively deleting vertices of degree at most 4 every vertex of the graph is deleted, implying that G is 4-independent. Theorems 1.11 and 1.5 therefore imply $\mathrm{mlt}(G) \leq 5$. Since G contains a K_5 subgraph on the vertices labeled 313,425,732,235,18, Theorem 1.3 implies moreover that $\mathrm{mlt}(G) \geq 5$, and so we have equality.

Similarly, [54, Figure 1] presents a simulated gene association network G with 100 vertices. By iteratively deleting vertices of degree at most 2 every vertex of the graph is deleted, so G is 2-independent. Since G is not cycle-free, G is not 1-independent so Theorem 1.11 implies that $\mathrm{gcr}(G) = 3$. Theorem 1.19 therefore implies $\mathrm{mlt}(G) = 3$.

Finally, [55, Figure 5(a)] gives another gene network G inferred from E. coli data, this time by a so-called shrinkage Gaussian graphical model approach. Iteratively deleting vertices of degree until there are no vertices left certifies that that G is 3-independent. Again, Theorems 1.11 and 1.5 imply $\mathrm{mlt}(G) \leq 4$, and since G has a K_4 (on b1583,lacA,yaeM,lacZ), Theorem 1.3 implies $\mathrm{mlt}(G) \geq 4$, giving equality.

While these illustrative examples are handled by our bounds and a simple heuristic algorithm, we stress that many graphs which are d-independent do not have empty $(d+1)$-core and that there is no a priori reason to expect a globally $(d-1)$-rigid subgraph of a large sparse graph to be small or easy to spot. We investigate random graph families that have this behavior next.

2.3. *Random graphs near the 2-dimensional rigidity transition.* A very sparse Erdös-Rényi random graph $G(n, c/n)$ has each of the $\binom{n}{2}$ possible edges independently with probability c/n. Hence the vertex degrees have (dependent) binomial distributions with parameters $n-1$ and c/n. As $n \to \infty$ the vertex degrees approach Poisson random variables with parameter c. Hence, when discussing such graphs we will refer to c as the "expected average degree".

A central result in the theory of random graphs describes the emergence and growth of the k-core of an Erdös-Rényi random graph $G(n, p)$. We state a simplified version (using the letter d instead of k, because it makes more sense for our application).

THEOREM 2.1 ([50]). *For each $d \geq 2$ there are constants $c_d < c'_d$, so that*

- *If $c < c_d$, then, whp, $G(n, c/n)$ has an empty $(d+1)$-core. If $c > c_d$, then, whp, $G(n, c/n)$ has a non-empty $(d+1)$-core spanning $\Omega(n)$ vertices.*
- *If $c_d < c < c'_d$, then, whp, the $(d+1)$-core of $G(n, c/n)$ has average degree lower than $2d$. If $c > c'_d$, then, whp, the $(d+1)$-core has average degree at least $2d$.*

Moreover, $c'_d < c_{d+1}$.

For reference, the value of $c_2 \approx 3.35$ and $c'_2 \approx 3.59$. As d becomes large, c'_d approaches $2d$.

What is most important for is that c'_2/n is the threshold function for an Erdös-Rényi random graph to be 2-independent.

THEOREM 2.2 ([37]). *In the notation of Theorem 2.1, if $c < c'_2$, then, whp, $G(n, c/n)$ is 2-independent[6]. If $c > c'_2$, then, whp, $G(n, c/n)$ contains a globally 2-rigid[7] subgraph spanning a $(1 - o(1))$-fraction of the 3-core (hence spanning $\Omega(n)$ vertices).*

Using Theorem 1.19, we get immediately:

THEOREM 2.3. *In the notation of Theorem 2.1,*

- *If $c_2 < c < c'_2$, then, whp, $\mathrm{mlt}(G(n, c/n)) = 3$.*
- *If $c'_2 < c < c_3$, then, whp $\mathrm{mlt}(G(n, c/n)) = 4$.*

In particular, in this range, $\mathrm{mlt}(G(n, c/n)) = \mathrm{gcr}(G(n, c/n))$, whp.

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[6]And, in fact, has no rigid component spanning more than 3 vertices.

[7]That we have global rigidity was first noted by Bill Jackson.

As a comparison, since Erdős-Rényi graphs with $p = c/n$, for any $c > 0$, are well-known to have clique number 3, whp, Theorem 1.3 would give a lower bound of 3 across this entire range. Combining Theorem 1.5 and [7, Corollary 4.5], we would get an upper bound of 4 on the MLT in the range $c_2 < c < c_2'$. Using our methods, we get an exact result in the whole range.

2.4. *Conjectures and a question.* The following conjecture generalizes Theorem 2.3.

CONJECTURE 2.4. *Let $c > 0$ be fixed. Then,* $\mathrm{mlt}(G(n, c/n)) = \mathrm{gcr}(n, c/n)$, *whp.*

In other words, we conjecture that for *all* very sparse Erdős-Rényi random graphs, the MLT and GCR coincide whp.

We will go further and make a structural conjecture that goes beyond Theorem 2.3 even for GCR 4. We need some terminology. A edge ij of a graph G is *d-redundant* if there is some generic d-dimensional framework (G, p) that has an equilibrium stress with a non-zero coefficient on the edge ij. The *d-redundant subgraph of G* is the subgraph comprising all the d-redundant edges.

CONJECTURE 2.5. *Let $c > 0$ be fixed. Then, whp, if $G = G(n, c/n)$ has GCR $d + 1 \geq 4$, the $(d - 1)$-redundant subgraph of G is generically globally $(d - 1)$-rigid.*

This conjecture implies that the MLT and GCR are equal, whp, using Theorem 1.18, but with the added precision of identifying the globally rigid subgraph that certifies the lower bound. The conjecture does not include GCR 3 and lower, where we do not expect it to be true. The reason is that, for $c < c_2'$, the 1-redundant subgraph is not 2-connected whp, and 2-connectivity is necessary for generic global rigidity [32]. We provide theoretical and experimental evidence for the conjecture in the next section.

We conclude with a question, which does not seem empirically resolved by our experiments.

QUESTION 2.6. Let $c > 0$ be fixed. Is it true that, whp, if $G = G(n, c/n)$ has MLT $d + 1 \geq 4$ that the $(d - 1)$-redundant subgraph of G is exactly the d-core?

A positive answer to the question would imply that the MLT (and also global rigidity) of sparse Erdős-Rényi random graphs has the same evolution as the d-cores.

2.5. *Theoretical evidence for the conjectures.* Aside from it being true for $d = 2$, a weaker result for $d \geq 3$ provides evidence for the conjecture. In Appendix D we will show the following result[8].

THEOREM 2.7. *Let $d \geq 1$. There is a $C_d > 0$ and a $k \in \mathbb{N}$ such that, whp, $G(n, C_d (\log n)^k / n)$ is d-rigid.*

We now get, by combining Theorem 2.7 and Theorem 4.4 below:

THEOREM 2.8. *Let $d \geq 1$. There is a $M_d > 0$ and $k \in \mathbb{N}$, such that, whp, $\mathrm{mlt}(G(n, M_d (\log n)^k / n)) \geq d$.*

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[8]While this paper was in preparation, Lew, Nevo, Peled and Raz [43] found the sharp threshold for rigidity. However, their improved result does not lead to a qualitatively better bound on the MLT than we state here.

Since, whp, the graphs in Theorem 2.8 have clique number 3, the lower bound from Theorem 1.3 becomes increasingly ineffective. On the other side, we do not have a good upper bound on $\mathrm{gcr}(G(n, M_d(\log n)^k/n))$. A simple bound on the GCR is based on maximum degree. If a graph G has maximum degree Δ then it is Δ-independent, so we can use the fact that, whp, the maximum degree of $G(n, M_d(\log n)^k/n)$ is $O(\mathrm{polylog}(n))$ to deduce:

COROLLARY 2.9. *In the notation of Theorem 2.8, then, there are $k, \ell \in \mathbb{N}$ such that, whp, for every $d \geq 1$,*

$$d \leq \mathrm{mlt}(G(n, M_d(\log n)^k/n)) \leq (\log n)^\ell$$

where M_d and k are from Theorem 2.8.

We have not tried to optimize the upper bound, since we believe that the maximum degree is not a good estimate of the GCR for these random graphs.

2.6. *Experimental evidence for the conjectures.* To test Conjectures 2.4 and 2.5, we ran experiments on Erdös-Rényi random graphs. For fixed numbers of vertices $n = 30, 500$ and expected average degree c from 3 to 30 with step size 0.1 we generated 20 samples from $G(n, c/n)$ and computed: the GCR $d + 1$; whether the $(d-1)$-redundant subgraph is generically globally rigid in dimension d; and whether the $(d-1)$-redundant subgraph is equal to the d-core.

Computing the GCR, identifying the redundant subgraph, and checking global rigidity are done using a randomized algorithm based on linear algebra over finite fields [27].

These computations rely on the results of this paper in an essential way. In particular, we need the lower bound from Theorem 1.18 to certify that the MLT is equal to the GCR. Without Theorem 1.18, we would have to rely on heuristic numerical experiments instead.

In all of our runs, when the GCR is at least 4, the $(d-1)$-redundant subgraph was generically globally rigid. This is consistent with Conjectures 2.5 and 2.4. (Recall that for GCR of 3, Conjecture 2.4 is true from Theorem 2.3, and that Conjecture 2.5 is not expected to hold.)

The charts in Figure 4 show the evolution of the GCR (which is equal to the MLT) as the expected average degree increases in our experiments for two values of n over a common range of c. The transition from 3 to 4 happens quickly between $c = 3.4$ and $c = 3.6$ as predicted by Theorem 2.3.

In all of our runs, when the GCR is in the range $3 - 6$, the equality of the $(d-1)$-redundant subgraph and the d-core held at least 97% of the time, but this fraction did not seem to go up as we increased the number of vertices from 30 to 1000. In all of our runs, when the GCR is at least 7, the equality of the $(d-1)$-redundant subgraph and the d-core held all of the time.

The code is available from the author's web site[9] and the data used to generate the charts, including the random graphs, is available upon request.

3. **Stress geometry of the MLT.** In this section we develop a detailed geometric understanding of the MLT. Our main tool for doing this is the theory of PSD equilibrium stresses of frameworks. The importance of PSD equilibrium stresses has long been known in rigidity [14] and graph theory [59, 44]. Uhler [60] has pointed out the semi-algebraic nature of the MLT problem. Here we make the connection precise enough to exactly describe the MLT in terms of equilibrium stresses.
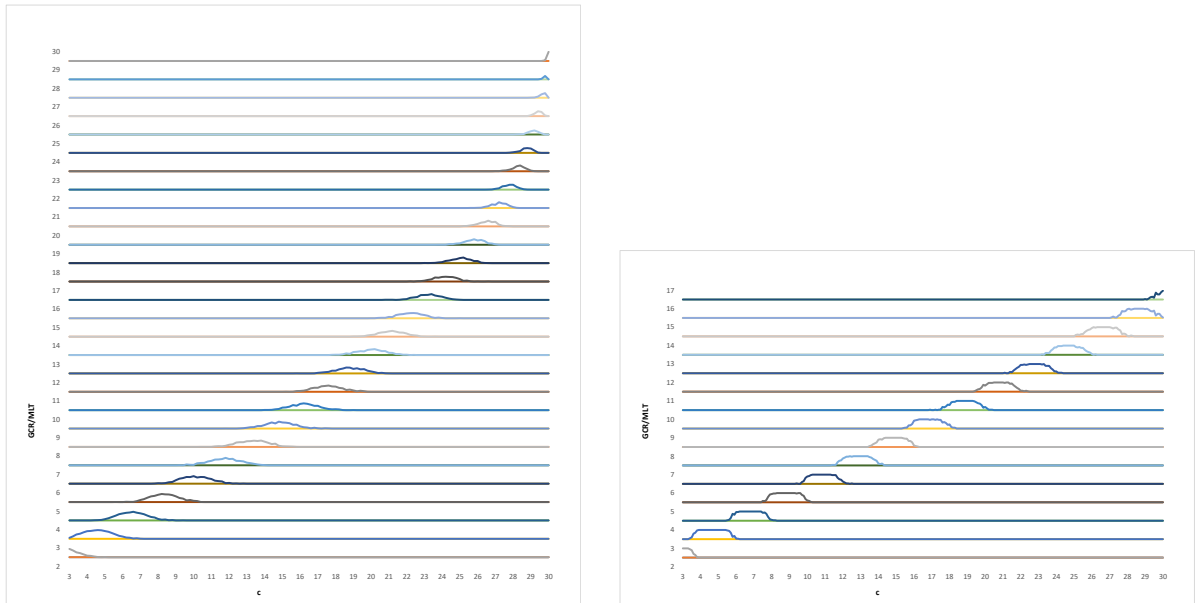
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[9]https://gist.github.com/theran/994b4d355e56529f5e6642fec4aead98

FIG 4. *Evolution of the GCR in $G(n, c/n)$ as the expected average degree, c increases. The x-axis shows the c values. The y-axis is labeled by the integer valued, possible GCR values. As stated in the text, all of the given GCR values were equal to the MLT. Each curve in horizontal slice shows the proportion of samples that produced the associated GCR. The left chart is $n = 30$ and the right is $n = 500$. Already, in the left chart, we see that each GCR values is only observed for a small range of c values. The left chart spans the range from sparse to dense graphs on 30 vertices. In the right chart, we see that, for most c under consideration, we only observe a single GCR value, and that these phase transition regions, especially in the lowest c ranges, have tightened. This is consistent with our conjectures. The right chart only looks at relatively sparse graphs on 500 vertices. Also note, in this sparse region, that the trend of the peaks of the bumps is linear, with GCR approximately equal to $3/2 + c/2 = 3/2 + m/n$, where m is the expected number of edges.*

3.1. *Linear equilibrium stresses.* To connect to the optimization problem underlying the MLT, we introduce the notion of a linear equilibrium stress, which is implicit in a number of works around rigidity in geometries with projective models (see [48] and the references therein). We start with some notation relating to vector configurations.

DEFINITION 3.1. Let q be a configuration of n vectors in $\mathbb{R}^{d+1}$. Denote by t_i the last coordinate of $q(i)$ and by Q the $(d + 1) \times n$ matrix with the $q(i)$ as its columns. We say that q is *flat* if all the t_i are one, and that q is *flattenable* if all the t_i are non-zero.

Generic configurations are clearly flattenable. There is a unique flat configuration associated with a flattenable configuration q arising from scaling $q(i)$ by $1/t_i$. Flat vector configurations in $\mathbb{R}^{d+1}$ are naturally associated with affine point configurations in $\mathbb{R}^d$.

DEFINITION 3.2. Let p be a configuration of n points in $\mathbb{R}^d$. We denote by $\hat{p}$, the vector configuration in $\mathbb{R}^{d+1}$ defined by the standard homogeneous coordinates for p, i.e.

$$\hat{p}(i) = \begin{pmatrix} p \\ 1 \end{pmatrix}.$$

The matrix $\hat{P}$ is $(d + 1) \times n$ with the vectors $\hat{p}(i)$ as its columns.

DEFINITION 3.3.   Let d be a dimension. Let G be a graph with n vertices and let q be a vector configuration of n vectors in $\mathbb{R}^{d+1}$. An assignment ω of weights ω_{ij} to the edges ij of G and ω_{ii} to the vertices of G is a *linear equilibrium stress* for q if

$$(2) \qquad\qquad \sum_{j\sim i}\omega_{ij}q(j) = \omega_{ii}q(i) \qquad (\text{all } i \in V(G)).$$

For a fixed ω, we say that q *satisfies* ω if (2) holds. A *linear equilibrium stress matrix* Ω for q is a symmetric n-by-n matrix with $\Omega_{ij} = 0$ for non-edges of G such that

$$\Omega Q^T = 0,$$

where Q is the $(d+1) \times n$ matrix with the $q(i)$ as its columns. Given a linear equilibrium stress ω for q, we can make a linear equilibrium stress matrix for it by setting $\Omega_{ij} = \Omega_{ji} = -\omega_{ij}$ on the edges and setting the diagonals $\Omega_{ii} = \omega_{ii}$. Hence the vector configurations satisfying a given set of weights arise from the kernel of the associated linear equilibrium stress matrix.

The following lemma is immediate. It gives the precise relationship between equilibrium stresses and *linear* equilibrium stresses.

LEMMA 3.4.   *Let G be a graph with n vertices and let (G,p) be a d-dimensional framework. Then for any equilibrium stress ω of (G,p), the associated stress matrix gives a linear equilibrium stress of $\hat{p}$. Any linear equilibrium stress matrix Ω for $\hat{p}$ is also an equilibrium stress matrix for (G,p).*

Linear equilibrium stresses are well-behaved under scaling. Results similar to the following can be found in e.g. [17, 20].

LEMMA 3.5.   *Let G be a graph with n vertices and let q be a vector configuration in $\mathbb{R}^{d+1}$. If Ω is a linear equilibrium stress matrix for q and $s_1,\ldots,s_n$ are any non-zero real numbers, then the configuration $\tilde{q}$, defined by*

$$\tilde{q}(i) = \frac{1}{s_i}q(i)$$

has a linear equilibrium stress matrix with the same signature as Ω.

PROOF.   Take q and the s_i as in the statement, and let ω be the linear equilibrium stress for q from the statement. For each vertex i and edge ij, define

$$\tilde{\omega}_{ij} = s_i s_j \omega_{ij} \qquad \text{and} \qquad \tilde{\omega}_{ii} = s_i^2 \omega_{ii}.$$

Then $\tilde{\omega}$ is a linear equilibrium stress for $\tilde{q}$ because for each vertex i we have

$$\sum_{j\sim i}\tilde{\omega}_{ij}\tilde{q}(j) = s_i\sum_{j\sim i}\omega_{ij}q(j) = s_i\omega_{ii}q(i) = s_i^2\omega_{ii}\tilde{q}(i) = \tilde{\omega}_{ii}q(i).$$

Let $\tilde{\Omega}$ be the stress matrix associated to ω and let S be the diagonal matrix whose diagonal entries are $s_1,\ldots,s_n$. Then $\tilde{\Omega} = S\Omega S$ and thus Ω and $\tilde{\Omega}$ have the same signature. $\square$

We get an important special case when s_i is the last coordinate of $q(i)$ for each i.

LEMMA 3.6.   *Let G be a graph with n vertices, let q be a flattenable configuration of n vectors in $\mathbb{R}^{d+1}$, and let (G,p) be the framework in $\mathbb{R}^d$ that arises from flattening q and deleting the all-ones coordinate. If there is a linear equilibrium stress matrix Ω for q, then p has an equilibrium stress matrix of the same signature as Ω.*

PROOF. If we denote by $\hat{p}$ the flattening of q, then by Lemma 3.5 there is a linear equilibrium stress for $\hat{p}$ with the same signature as Ω. This stress is an equilibrium stress of (G, p) by Lemma 3.4. □

3.2. *The optimization problem.* We now describe the MLT optimization problem. For convenience, we denote the inner product $\mathrm{Trace}(AB)$ on the set of symmetric $n \times n$ matrices by $\langle A, B \rangle$.

DEFINITION 3.7. Let G be a graph with n vertices. Let D be an $n \times (d+1)$ matrix with columns representing $(d+1)$ samples from an n-variate probability distribution. Let $S = \frac{1}{d} DD^T$ be the sample covariance matrix. *The MLT optimization problem for (G, D) is to find an $n \times n$ positive definite matrix K minimizing $f(K) = \langle S, K \rangle - \log \det K$, subject to $K_{ij} = 0$ for all $ij \notin E(G)$.*

The rigidity-theoretic viewpoint requires us to transpose our view of the data matrix. In particular, instead of thinking about S as the sample covariance obtained from $(d+1)$ samples of an n-variate distribution, we will think about S as the Gram matrix of a configuration of n points in $(d+1)$-dimensional space. This allows us to recast the MLT optimization problem in the following equivalent way.

DEFINITION 3.8. Let G be a graph with n vertices and let q be a configuration of n vectors in dimension $d+1$. Let $S = Q^T Q$ be the Gram matrix of q. *The Gram MLT optimization problem for (G, q) is to find an $n \times n$ positive definite matrix K, minimizing $g(K) = \langle S, K \rangle - \log \det K$, subject to $K_{ij} = 0$ if $ij \notin E(G)$.*

LEMMA 3.9. *Let G be a graph with n vertices and let q be a configuration of n vectors. Then the Gram MLT optimization problem (objective function g) is unbounded if and only if there is a nonzero PSD linear equilibrium stress for q.*

PROOF. Let S be the Gram matrix of q. Suppose that Ω is the PSD stress matrix of a non-zero linear equilibrium stress for q. For any $t > 0$, the matrix $I + t\Omega$ is positive definite and

$$g(I + t\Omega) = \langle S, I + t\Omega \rangle - \log \det(I + t\Omega).$$

Since

$$\langle S, I + t\Omega \rangle = \langle S, I \rangle + t \langle S, \Omega \rangle = \langle S, I \rangle + t \, \mathrm{Trace}\, Q^T Q\Omega = \langle S, I \rangle + t \, \mathrm{Trace}\, Q^T 0 = \langle S, I \rangle$$

we conclude that

$$g(I + t\Omega) = \mathrm{Trace}\, S - \log \det(I + t\Omega) \to -\infty \qquad (\text{as } t \to \infty).$$

So the optimization problem is unbounded.

For the other direction we prove the contrapositive. Suppose that there is no non-zero PSD linear equilibrium stress matrix for q. We show that the gram MLT optimization problem has a global minimum. Let $\mathbb{S}$ be be the set of symmetric $n \times n$ matrices Ω with zeros on the non-edges of G satisfying $\langle \Omega, \Omega \rangle = 1$. For any $\Omega \in \mathbb{S}$, there is a $t_0 > 0$ so that $K = I + t_0\Omega$ is a feasible point of the Gram MLT optimization problem. Define $t^* \geq t_0 > 0$ to be the supremum over values such that $I + t\Omega$ is positive definite. We will show that, for any $\Omega \in \mathbb{S}$,

$$g(I + t\Omega) \to \infty \qquad (\text{as } t \to t^*).$$

It then follows that, outside of a compact neighborhood of I, $g(I+t\Omega) > g(I)$, which implies that g has a global minimum. There are two cases. If $\Omega \in \mathbb{S}$ is not PSD, then t^* is finite, and, as $t \to t^*$,

$$g(I + t\Omega) = \operatorname{Trace} S + t\langle S, \Omega \rangle - \log \det(I + t\Omega) \to \infty,$$

since the last term grows without bound and the linear terms have bounded magnitude. If Ω is PSD, then $I + t\Omega$ is positive definite for any $t > 0$, and so $t^* = \infty$. We then have, as $t \to \infty$,

$$g(I + t\Omega) = \operatorname{Trace} S + t\langle S, \Omega \rangle - \log \det(I + t\Omega) = \operatorname{Trace} S + t\langle S, \Omega \rangle - O(\log t)$$

because the determinant is a polynomial of degree n in t. Finally, since S and Ω are PSD and Ω is not a linear equilibrium stress matrix, $\langle S, \Omega \rangle > 0$, so

$$g(I + t\Omega) \to \infty \qquad (\text{as } t \to \infty). \qquad \square$$

3.3. *Proof of Theorem 1.15.* We are now ready to prove Theorem 1.15. Lemmas 3.10 and 3.11 below each give one direction. What is left is to rigorously establish the relationship between "almost all" and generic. The most technical statements are handled in Appendix A. Recall that two measures are *mutually absolutely continuous* if they have the same null sets.

LEMMA 3.10. *Let G be a graph with $\operatorname{mlt}(G) = d + 1$. Then:*

(a) *there is a generic framework (G, p) in $\mathbb{R}^{d-1}$ with a nonzero PSD equilibrium stress, and*
(b) *no generic framework (G, p) in $\mathbb{R}^d$ has a nonzero PSD equilibrium stress.*

PROOF. Let n be the number of vertices of G. Let D be an $n \times d$ data matrix whose columns are i.i.d. samples from a distribution whose probability measure μ is mutually absolutely continuous with respect to the Lebesgue measure. Let q denote the configuration of n points in $\mathbb{R}^d$ given by the rows of D. Since $\operatorname{mlt}(G) = d + 1$, the Gram MLT optimization problem for (G, q) is unbounded with positive probability. Let X denote the set of vector configurations of n points in $\mathbb{R}^d$ for which the Gram MLT optimization problem is unbounded. Then X is semi-algebraic and not μ-null, so Lemma A.4 implies that X contains a generic vector configuration, which we continue to call q. By Lemma 3.9, there is a non-zero PSD linear equilibrium stress matrix Ω for q. Since q is generic, it is flattenable. By Lemma 3.6, the $(d-1)$-dimensional framework (G, p) arising from flattening q has an equilibrium stress matrix with the same signature as Ω, so this matrix must also be PSD and non-zero. Finally, Lemma A.5 implies that p is generic. Hence we have constructed a generic $d-1$-dimensional framework (G, p) with a non-zero PSD equilibrium stress.

Let W denote the set of configurations w of n points in $\mathbb{R}^{d+1}$ for which the Gram MLT optimization problem (G, w) is bounded. Since $\operatorname{mlt}(G) = d + 1$, the complement of W is μ-null. Let (G, p) be a generic framework in $\mathbb{R}^d$. Scaling the vectors of $\hat{p}$ by generic weights gives, via Lemma A.5, a generic configuration w in $\mathbb{R}^{d+1}$. Since W is semi-algebraic and w is generic, Lemma A.4 implies $w \in W$. By Lemma 3.9, there is no non-zero PSD linear equilibrium stress for (G, w). By Lemma 3.6, there is no non-zero PSD equilibrium stress for (G, p). $\qquad \square$

LEMMA 3.11. *Let G be a graph with n vertices and suppose that d is the smallest dimension so that no generic d-dimensional framework (G, p) has a non-zero PSD equilibrium stress. Then the MLT of G is $d + 1$.*

PROOF. By assumption, there must be a generic $(d-1)$-dimensional framework with a non-zero PSD equilibrium stress, which we will call (G, p). By scaling the vectors of $\hat{p}$ by generic numbers s_i, we obtain, by Lemma A.5, a generic vector configuration q in dimension d. By Lemma 3.6, there must be a non-zero PSD linear equilibrium stress for q. Hence, by Lemmas 3.9 and A.4, the set of configurations for which the Gram MLT optimization problem is unbounded must be non-null. We conclude that $\mathrm{mlt}(G) > d$.

Now we take a generic vector configuration q in dimension $d+1$. As noted above, by genericity, q is flattenable, and the flattened d-dimensional point configuration p is also generic by Lemma A.5. By Lemma 3.6, since (G, p) does not have a non-zero PSD equilibrium stress, there is not a non-zero PSD linear equilibrium stress for q. Hence, for every generic vector configuration q in dimension $d+1$, the Gram MLT optimization problem is bounded by Lemma 3.9. Since the set of all such vector configurations is semi-algebraic and contains all the generic points, it must have full measure by Lemma A.4. This implies that $\mathrm{mlt}(G) \leq d+1$. $\qquad\square$

The existence of a generic framework in dimension $d-1$ with a non-zero PSD equilibrium stress implies that the Gram MLT optimization problem is unbounded with positive probability, for any way of sampling data points that is continuous with respect to Lebesgue measure. However, we don't have a lower bound on this failure probability. Any general lower bound will be quite bad, since Buhl [12] showed that, for an n cycle, if the datapoints are sampled uniformly from the unit interval, the MLE exists after 2 sample points with probability $1 - 2n/n!$, even though the MLT of a cycle is 3.

3.4. *The geometric picture: lifting.* Theorem 1.15, while precise, and as we will see, convenient for deriving bounds on the MLT of a graph, is quite technical. There is an underlying geometric idea, that we now explain.

DEFINITION 3.12. Let G have n vertices. Let (G, p) be a d-dimensional framework. We say that (G, p) is *liftable* if there is an equivalent $n-1$ dimensional framework $(G, \tilde{p})$ with full affine span.

The following Lemma is due to Alfakih [2]. For completeness, we provide a proof in Appendix B that uses convex geometry ideas from [28].

LEMMA 3.13 ([2]). *A d-dimensional framework (G, p) is liftable if and only if it does not have a non-zero PSD equilibrium stress.*

Theorem 1.16 is immediate from Theorem 1.15 and Lemma 3.13.

3.5. *Remarks.* To close a circle of ideas, we note that much of the literature on the MLT, including [5, 8, 30, 60] does not work directly with the MLT optimization problem. Instead, the starting point is the following matrix completion problem.

DEFINITION 3.14. Let G be a graph with n vertices and S an $n \times n$ PSD matrix of rank $d+1$. The *MLT matrix completion problem for (G, S)* is to find an $n \times n$ positive definite matrix A that has the same diagonal entries as A and the same off diagonal entries corresponding to edges of G.

Dempster [22] showed that the MLT optmization problem is bounded if and only if the MLT matrix completion problem is feasible.[10] A less direct path to our results is to relate

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[10]One can also derive Dempster's result via duality in convex programming, see e.g. [5].

the MLT matrix completion for G problem to liftability of "coned" frameworks $(v_0 * G, p)$ [20, 62] in one dimension higher that have a new vertex v_0 connected to all the others. The proof of Theorem 2.7 in Appendix D uses this technique.

Finally, we note that we could have allowed the vector configurations in our optimization problems to satisfy a condition strictly weaker than flattenability. In particular, it would have been enough to only require that the Gram matrix $Q^T Q$ have *some* factorization that is flattenable, which happens so long as none of the vectors in q are zero. At the level of frameworks, changing factorizations corresponds to projective transformations. We elected to use the stronger condition to keep the proofs simpler, and in particular, to avoid having to define and work with generic low-rank PSD matrices.

4. MLT bounds from global rigidity. We can use the results of the previous section along with some facts about global rigidity to get improved bounds for the MLT and compute it exactly for some interesting families. The main technical tool of this section relates generic global rigidity to PSD equilibrium stresses.

THEOREM 4.1 ([18]). *Let G be a graph with $n \geq d + 2$ vertices and d a dimension. If G globally d-rigid, then there is a generic framework (G, p) with a PSD equilibrium stress of rank $n - d - 1$.*

We also need a straightforward lemma.

LEMMA 4.2. *Let G be a graph and H a subgraph of G. Then $\mathrm{mlt}(H) \leq \mathrm{mlt}(G)$.*

4.1. *Lower bounds.* The main results of this section are new lower bounds on the MLT of a graph arising from global rigidity in terms of the global rigidity number (Def. 1.17.)

PROOF OF THEOREM 1.18. Suppose that G is globally d-rigid. By Theorem 4.1, there is a generic framework (G, p) with a non-zero PSD equilibrium stress. By Theorem 1.15, $\mathrm{mlt}(G) > d + 1$. Taking d as large as possible for G to remain globally d-rigid we get $\mathrm{mlt}(G) > \mathrm{grn}(G) + 1$. The same argument works for any subgraph H of G, so Lemma 4.2 implies that $\mathrm{mlt}(G) \geq \mathrm{mlt}(H) > \mathrm{grn}(H) + 1$. Maximizing the right-hand side over H we get $\mathrm{mlt}(G) > \mathrm{grn}^*(G) + 1$. Since G is a subgraph of itself, plainly $\mathrm{grn}^*(G) \geq \mathrm{grn}(G)$. □

We can efficiently compute $\mathrm{grn}(G)$ [27], but we do not know the complexity of computing $\mathrm{grn}^*(G)$. A related graph parameter, which may be more computationally tractable is the local rigidity analogue.

DEFINITION 4.3. Let G be a graph with n vertices. The *rigidity number* $\mathrm{lrn}(G)$ is the largest d so that G is d-rigid and has at least $d + 1$ vertices. The *subgraph rigidity number* $\mathrm{lrn}^*(G)$ is the largest d so that G has a subgraph H on at least $d + 1$ vertices that is d-rigid.

THEOREM 4.4. *Let G be a graph. Then $\mathrm{lrn}(G) + 1 \leq \mathrm{lrn}^*(G) + 1 \leq \mathrm{mlt}(G)$.*

The proof needs a result of Jordán.

LEMMA 4.5 ([35]). *Let G be a graph that is $(d + 1)$-rigid. Then G is globally d-rigid.*

PROOF OF THEOREM 4.4. By Lemma 4.5 one has $\mathrm{lrn}^*(G) \leq \mathrm{grn}^*(G) + 1$. Theorem 1.18 then implies that $\mathrm{lrn}^*(G) + 1 \leq \mathrm{mlt}(G)$. Plainly $\mathrm{lrn}(G) \leq \mathrm{lrn}^*(G)$, giving the last inequality. □

Theorem 4.4 is strictly weaker than Theorem 1.18. For example, for every $n \geq 4$ there are globally rigid graphs in dimension 2 that have $2n - 2$ edges [6, 15], but if $n > 4$ then $2n - 2 < 3n - 6$, so these graphs cannot be 3-rigid.

The rigidity number of a graph is also easy to compute [4]. We do not know the complexity of computing $\mathrm{lrn}^*(G)$, but, since local rigidity is matroidal in nature, tools from submodular optimization may apply.

4.2. *Combined bounds and examples.* Combining what we know so far gives the following.

THEOREM 4.6. *For any graph G, the following inequalities hold*

(a) $\omega(G) \leq \mathrm{lrn}^*(G) + 1 \leq \mathrm{grn}^*(G) + 2 \leq \mathrm{mlt}(G) \leq \mathrm{gcr}(G) \leq \tau(G) + 1$, *and*
(b) $\mathrm{lrn}(G) + 1 \leq \mathrm{grn}(G) + 2 \leq \mathrm{grn}^*(G) + 2$.

COROLLARY 4.7. *If G is both globally d-rigid and $(d+1)$-independent, then $\mathrm{mlt}(G) = d + 2$.*

We now exhibit two new infinite families of graphs G for which the inequalities $\mathrm{grn}(G) + 2 \leq \mathrm{grn}^*(G) + 2 \leq \mathrm{mlt}(G) \leq \mathrm{gcr}(G)$ are tight. By applying Lemma 4.5 and Corollary 4.7, we obtain our first infinite family of graphs, which are the higher dimensional analogue of trees.

COROLLARY 4.8. *If G is minimally d-rigid, then $\mathrm{mlt}(G) = \mathrm{gcr}(G) = d + 1$.*

Our next example is, in essence, an extension of the cycle graphs to higher dimensions.

DEFINITION 4.9. *G is a d-circuit if it is not d-independent, but every proper subgraph is.*

COROLLARY 4.10. *Let G be a d-circuit. Then $\mathrm{gcr}(G) = d + 2$. If, furthermore, G is globally d-rigid, then $\mathrm{mlt}(G) = d + 2$ also.*

PROOF. Whiteley [62] proved that G is d-independent if and only if the coned graph $v_0 * G$, that adds a new vertex v_0 connected to every other vertex, is $(d+1)$-independent. Since G is a d-circuit, for any vertex v, $G - v$ must be d-independent. Hence, $v_0 * (G - v)$ is $(d+1)$-independent by Whiteley's result. Since G is isomorphic to a subgraph of $v_0 * (G-v)$, it is also $(d+1)$-independent. The claim now follows from Corollary 4.7. □

5. Complete bipartite graphs. To test the upper bound from Theorem 1.5 and the lower bound from Theorem 1.3, Blekherman and Sinn [8] considered the case of complete bipartite graphs. They were able to compute the MLT and generic completion ranks exactly, obtaining a number of strong results, including the first examples of graphs G with $\mathrm{mlt}(G) < \mathrm{gcr}(G)$.

Since, equilibrium stresses of complete bipartite graphs are very well understood [10, 16], we have an alternative path to the results from [8]. We require the two following results on the rigidity theory of bipartite graphs.

LEMMA 5.1 ([19]). *Fix a $d \in \mathbb{N}$ and let $m, n \geq d + 1$. If $m + n \geq \binom{d+2}{2} + 1$ then $K_{m,n}$ is globally rigid in dimension d.*

Given a finite subset S of a vector space, let $D(S)$ denote the linear space of affine dependencies among S and let S^2 be the image of S under the *Veronese map* $x \mapsto xx^T$. The following theorem collects what we need from Bolker and Roth's classic paper [10].

THEOREM 5.2. *Let $m, n, d \in \mathbb{N}$ and let $(K_{m,n}, p)$ be a d-dimensional framework. Let $A, B \subseteq \mathbb{R}^d$ denote the images under p of the partite sets of $K_{m,n}$. Then the linear space of equilibrium stresses of $(K_{m,n}, p)$ has dimension*

$$\dim(D(A)) \dim(D(B)) + \dim(D((A \cup B)^2)).$$

Moreover, if p is generic and $m + n \leq \binom{d+2}{2}$, then every equilibrium stress matrix has zeros along its diagonal.

PROOF. The first claim follows from [10, Theorem 1]. The second claim follows from [10, Lemma 5] and the fact that any set of $\binom{d+2}{2}$ generic symmetric matrices of rank 1 is a basis of the space of symmetric $(d+1) \times (d+1)$ matrices. $\qquad\square$

THEOREM 5.3 ([8]). *Let $d, m, n \in \mathbb{N}$ with $m, n \geq d + 2$. If $m + n \leq \binom{d+2}{2}$, then* $\operatorname{mlt}(K_{m,n}) \leq d + 1$ *and* $\operatorname{gcr}(K_{m,n}) \geq d + 2$.

PROOF. Let $(K_{m,n}, p)$ be a generic d-dimensional framework. Since $m + n \leq \binom{d+2}{2}$, Theorem 5.2 implies that every stress matrix has zeros along its diagonal and is therefore indefinite. Theorem 1.15 then implies that $\operatorname{mlt}(K_{m,n}) \leq d + 1$. On the other hand, Theorem 5.2 implies that the space of stresses has dimension at least $\dim(D(A)) \dim(D(B))$, which is positive as $m, n \geq d + 2$. The existence of an equilibrium stress implies that $\operatorname{gcr}(K_{m,n}) \geq d + 2$. $\qquad\square$

At this point Theorem 1.6 follows quickly.

PROOF OF THEOREM 1.6. Theorem 5.2 implies that $K_{m,m}$, for $m > 2$, is $(m - 1)$-independent but not $(m-2)$-independent. Hence $\operatorname{gcr}(K_{m,m}) = m$. By Lemma 5.1, for $n > 2$, the global rigidity number of $K_{m,m}$ is the maximum d so that $2m \geq \binom{d+2}{2} + 1$. For this d, Theorem 1.18 implies that $\operatorname{mlt}(K_{m,m}) \geq d + 2$. For any larger d', we have $2m \leq \binom{d'+2}{2}$. Theorem 5.3 then tells us that $\operatorname{mlt}(K_{m,m}) \leq (d + 1) + 1 = d + 2$. Combining both bounds, we conclude that the MLT of $K_{m,m}$ is the largest D so that $2m > \binom{D+1}{2}$ as desired. $\qquad\square$

6. A gluing construction. In this section we prove some specialized results about giving lower bounds on MLT of graphs. We do this by constructing PSD equilibrium stresses on generic frameworks of a graph obtained by gluing together smaller frameworks that each have a PSD equilibrium stress. We will need the following construction from rigidity theory.

DEFINITION 6.1. Let G be a graph with n vertices and m edges. The *rigidity matrix* $R(G, p)$ of a d-dimensional framework (G, p) is the $m \times dn$ matrix whose rows are indexed by the edges of G, columns indexed by the coordinates of $p(1), \ldots, p(n)$, where the entry corresponding to edge e and $p(v)_i$ is $p(v)_i - p(u)_i$ if $e = vu$, and 0 if v is not incident to e.

Given a d-dimensional framework (G, p) on a graph G with n vertices, $R(G, p)$ is the Jacobian of the map sending n points in $\mathbb{R}^d$ to the pairwise squared distances corresponding to the edges of G, evaluated at p. Equilibrium stresses of $R(G, p)$ are the elements of the left kernel of $R(G, p)$.

DEFINITION 6.2. A graph G is a k-*sum* of two induced subgraphs G_1 and G_2 each with at least $k+1$ vertices if G is the union of G_1 and G_2 and $G_1 \cap G_2$ is isomorphic to K_k.

The following result on equilibrium stresses of frameworks on k-sums is standard.

LEMMA 6.3. *Let $1 \le k \le d+1$ be integers and G a k-sum of subgraphs G_1 and G_2. Let (G,p) be a d-dimensional framework with the vertices of $G_1 \cap G_2$ affinely independent. Let S be the space of equilibrium stresses of (G,p) and S_i the space of equilibrium stresses of (G,p) supported on the edges of G_i. Then $S = S_1 \oplus S_2$.*

PROOF. Let $K = G_1 \cap G_2$. First observe that any equilibrium stress $\omega \in S_1 \cap S_2$ must be supported by the edges of K and so is an equilibrium stress of $(K, p|_K)$. Since K has at most $d+1$ vertices and is in general affine position, $(K, p|_K)$ supports only the zero equilibrium stress. Hence $S_1 + S_2 = S_1 \oplus S_2$.

Denote $R_i = R(G_i, p|_{G_i})$. The row spans of R_1 and R_2 are naturally included in the row span of $R(G,p)$. Both of these spans include $R(K, p|_K)$. By general position of the vertices corresponding to K, this latter space has dimension $\binom{k}{2}$. So by the interpretation of S and the S_i as cokernels of the rigidity matrix and rank-nullity, we have

$$\dim(S) = m - \operatorname{rank} R(p)$$

$$= m_1 + m_2 - \binom{k}{2} - \operatorname{rank} R(p)$$

$$= m_1 + m_2 - \binom{k}{2} - \operatorname{rank} R_1 - \operatorname{rank} R_2 + \binom{k}{2}$$

$$= m_1 - \operatorname{rank} R_1 + m_2 - \operatorname{rank} R_2$$

$$= \dim(S_1 \oplus S_2)$$

and so we can conclude $S = S_1 + S_2 = S_1 \oplus S_2$. $\qquad\square$

A framework (G,p) is *regular* if its rigidity matrix has maximum rank over all frameworks (G,q). Regularity is preserved under non-singular projective transforms applied to p. The converse of the following corollary also true, but we do not need it.

COROLLARY 6.4. *Let $1 \le k \le d+1$ and G be a k-sum of G_1 and G_2. Let (G,p) be a d-dimensional framework. If $(G_1, p|_{V(G_1)})$ and $(G_2, p|_{V(G_2)})$ are regular then (G,p) is regular.*

PROOF. Let G_i have n_i vertices and m_i edges. Assume that p is such that $(G_1, p|_{V(G_1)})$ and $(G_2, p|_{V(G_2)})$ are both regular. Let r_i be the rank of the rigidity matrix of each of these frameworks and s_i the dimension of the space of equilibrium stresses. Observing that G has $n_1 + n_2 - k$ vertices and $m_1 + m_2 - \binom{k}{2}$ edges, we can see that the maximum possible rank of the rigidity matrix for (G,p) is $r_1 + r_2 - \binom{k}{2}$. Since $K = G_1 \cap G_2$ is complete and has at most $d+1$ vertices in dimension d, regularity of $(G_1, p|_{V(G_1)})$ and $(G_2, p|_{V(G_2)})$ implies that the vertices of K are affinely independent. Otherwise there is an equilibrium stress supported only on K that is not present in all frameworks. Hence, we may apply Lemma 6.3 to (G,p) to conclude that its space of equilibrium stresses is the direct sum of equilibrium stresses supported on G_1 and G_2 respectively. The dimension of the space of equilibrium stresses of (G,p) is then $s_1 + s_2$. Then (G,p) is regular since the rank of its rigidity matrix is

$$m_1 + m_2 - \binom{k}{2} - s_1 - s_2 = m_1 + m_2 - \binom{k}{2} - (m_1 - r_1) - (m_2 - r_2) = r_1 + r_2 - \binom{k}{2}. \quad\square$$

We also have some control of the signs of stress coefficients in PSD equilibrium stresses. The following is from [17, Lemma 4.9] and the discussion around it.

LEMMA 6.5. *Let (G, p) be a d-dimensional framework and ω a PSD equilibrium stress of (G, p) and ij and edge of G so that $\omega_{ij} > 0$. Then there is a non-singular projective transformation T on $\mathbb{R}^d$ so that $(G, T(p))$ has a PSD equilibrium stress ψ so that $\psi_{ij} < 0$.*

We have things in place for the main result of this section. Given a graph G with edge ij, we let $G - ij$ denote the graph obtained from G by deleting the edge ij.

LEMMA 6.6. *Let $1 \leq k \leq d$ and G be a k-sum of subgraphs G_1 and G_2 and ij and edge of $G_1 \cap G_2$. Suppose that there are generic d-dimensional frameworks (G_1, p^1) and (G_2, p^2) that, respectively, support non-zero PSD equilibrium stresses ω^1 and ω^2, such that $\omega_{ij}^k \neq 0$ for $k = 1, 2$. Let $G' = G - ij$. Then there is a generic d-dimensional framework (G', p) that supports a non-zero PSD equilibrium stress.*

PROOF. First assume that $\omega_{ij}^1 < 0$ and $\omega_{ij}^2 > 0$. Since $G_1 \cap G_2$ has at most d vertices, any affinely independent framework on $G_1 \cap G_2$ cannot support an equilibrium stress. Hence, both ω^1 and ω^2 have some support outside of $G_1 \cap G_2$. We create a framework (G, p^0) from the frameworks (G_1, p^1) and (G_2, p^2) as follows. Pick a non-singular affine map T sending the vertices of $G_1 \cap G_2$ in (G_1, p^1) to the corresponding vertices in (G_2, p^2) and apply it to p^1. The defines a framework (G, p^0).

By Corollary 6.4 and the genericity of (G_i, p^i), the framework (G, p^0) is regular. Since equilibrium stresses are preserved under affine maps, ω^1 and ω^2 are both equilibrium stresses of (G, p^0). Our assumptions about the signs imply that some positive linear combination ω of ω^1 and ω^2 has vanishing coefficient on the edge ij. Because the ω^i have some necessarily disjoint support, ω is non-zero. Since a positive combination of PSD equilibrium stresses is PSD, we conclude that ω is. Since ω is not supported on ij, it is also an equilibrium stress of (G', p^0). Potentially, (G', p^0) is not generic, but since it is regular, a small perturbation (G, p) that is generic will have an equilibrium stress close to ω that is also PSD.

If $\omega_{ij}^1 > 0$, we reduce to the previous case by applying a projective transformation, as in Lemma 6.5. The argument is then the same as before, since we only used that the (G_i, p^i) are generic to make them regular. Regularity is preserved by projective transformations. $\square$

6.1. *Remarks.* A natural question is whether the lower bound in Theorem 1.18 is tight. The results of this section show that it is not. By Lemma 6.6, if we let G be the 2-sum of two copies of K_{d+2} over an edge ij, and G' the graph $G - ij$, there is a generic framework (G', p) in dimension d with a non-zero PSD equilibrium stress. Theorem 1.15, then implies that $\operatorname{mlt}(G') \geq d + 2$. On the other hand, since every induced subgraph of G' is independent in dimension d, $\operatorname{grn}^*(G') \leq d - 1$. Hence, $\operatorname{grn}^*(G') + 2 < \operatorname{mlt}(G')$.

If we ask, in addition that G is $(d + 1)$-connected, we do not know an example where the lower bound in Theorem 1.18 is not tight.

7. Equality of small MLT and GCR.

In this section, we prove Theorem 1.19, which rests on the rich combinatorial theory of 2-rigidity of graphs (see e.g. [42] for an overview). The cornerstone of this theory is Theorem 7.2, the Laman–Pollaczek-Geiringer theorem. We begin with the necessary definitions.

DEFINITION 7.1. A graph G with n vertices is $(2, 3)$-sparse if, for all subgraphs with n' vertices and $m' > 0$ edges, $m' \leq 2n' - 3$. If G is $(2, 3)$-sparse and, in addition has $2n - 3$ edges, it is called a *Laman graph*. A graph that is not $(2, 3)$-sparse, but becomes so after removing any edge is called a *Laman circuit*.

THEOREM 7.2 ([41, 51]). *A graph G is 2-independent if and only if G is $(2,3)$-sparse.*

Via Theorem 1.11, Theorem 7.2 immediately gives us a combinatorial characterization of the graphs with $\mathrm{gcr}(G) = 3$; these are the $(2,3)$-sparse graphs that contain a cycle. As we will see in Proposition 7.4, this also characterizes graphs with $\mathrm{mlt}(G) = 3$. In order to prove this, we need the following lemma which makes crucial use of Berg and Jordan's [6] combinatorial characterization of global rigidity in two dimensions.

LEMMA 7.3. *Let G be a Laman circuit. Then there are generic 2-dimensional frameworks (G, p) satisfying a non-zero PSD equilibrium stress.*

PROOF. If G is 3-connected, a result of Berg and Jordán [6] implies that G is globally rigid. The desired statement then follows from Theorem 4.1. If G is not 3-connected, we can find a 2-separation $\{x, y\} \subseteq V(G)$ in G. A counting argument [6, Lemma 2.4, inter alia] implies that xy is not an edge of G and that $G \cup \{xy\}$ is a 2-sum of smaller Laman circuits G_1 and G_2. By induction, we may assume that there are generic 2-dimensional frameworks (G_1, p^1) and (G_2, p^2) that each support a PSD equilibrium stress ω^1 and ω^2. Since G_1 and G_2 are circuits, the supports of ω^1 and ω^2 include the edge xy. By Lemma 6.6, there is then a generic framework (G, p) with a non-zero PSD equilibrium stress. □

PROPOSITION 7.4. *Given a graph G, the following are equivalent:*

(a) *G is $(2,3)$-sparse and contains a cycle,*
(b) *$\mathrm{gcr}(G) = 3$, and*
(c) *$\mathrm{mlt}(G) = 3$.*

PROOF. Theorems 7.2 and 1.11 imply that $\mathrm{gcr}(G) = 3$ if and only if G is $(2,3)$-sparse and contains a cycle. Now assume $\mathrm{gcr}(G) = 3$. Since cycles are globally 1-rigid, any graph G with a cycle has $\mathrm{grn}^*(G) \geq 1$, so $\mathrm{mlt}(G) \geq 3$ by Theorem 1.18. On other other hand, if a graph G is $(2,3)$-sparse then $\mathrm{gcr}(G) \leq 3$ and so $\mathrm{mlt}(G) \leq 3$ follows from Theorem 1.5.

If $\mathrm{gcr}(G) \leq 2$ or $\mathrm{mlt}(G) \leq 2$, then G cannot have a cycle. So assume $\mathrm{gcr}(G) \geq 4$. Theorems 7.2 and 1.11 now imply that G contains a Laman circuit H as a subgraph. By Lemma 7.3, H has a generic 2-dimensional framework (H, p) with non-zero PSD equilibrium stress. Theorem 1.15 implies $\mathrm{mlt}(H) \geq 4$ and therefore Lemma 4.2 implies $\mathrm{mlt}(G) \geq 4$. □

We are now ready to prove the main result of this section.

PROOF OF THEOREM 1.19. As noted in [30], $\mathrm{mlt}(G) = 1$ if and only if G has no edges and $\mathrm{mlt}(G) = 2$ if and only if G has no cycles. In both cases, it is easy to see that $\mathrm{gcr}(G) = \mathrm{mlt}(G)$. If $\mathrm{mlt}(G) = 3$ or $\mathrm{gcr}(G) = 3$, then $\mathrm{mlt}(G) = \mathrm{gcr}(G)$ follows from Proposition 7.4. If $\mathrm{gcr}(G) = 4$, then Theorem 7.2 and Lemma 7.3 together imply that $\mathrm{mlt}(G) \geq 4$ and equality follows from Theorem 1.5. □

REMARK 7.5. Theorem 1.19 is best possible in the sense that if $a \geq 4$ and $b \geq 5$, then there exist graphs G, H such that $\mathrm{mlt}(G) = a < \mathrm{gcr}(G)$ and $\mathrm{mlt}(H) < b = \mathrm{gcr}(H)$. In particular, let $n = \lfloor \frac{1}{2}\binom{a+1}{2} \rfloor$, and let D be the smallest k such that $\binom{k+1}{2} \geq 2b$. Then, Theorem 1.6 implies $\mathrm{mlt}(K_{n,n}) = a < n = \mathrm{gcr}(K_{n,n})$ and that $\mathrm{gcr}(K_{b,b}) = b > D = \mathrm{mlt}(K_{b,b})$.

8. Weak maximum likelihood threshold. This section includes connections between the *weak* maximum likelihood threshold of a graph, and two areas of classical combinatorics: partially ordered sets, and graph dimension (i.e. the minimum dimension in which a graph can be realized as a unit-distance graph).

DEFINITION 8.1. The *weak maximum likelihood threshold* of a graph G, denoted $\mathrm{wmlt}(G)$ is the smallest number of samples[11] required for the MLE of the Gaussian graphical model associated with G to exist with positive probability.

The definition of $\mathrm{wmlt}(G)$ is the same as that of $\mathrm{mlt}(G)$, but with the phrase "almost surely" swapped out for "with positive probability." Arguments along the lines of Section 3 then yield the analogue of Theorem 1.16. Since the proof is very similar, we skip it.

PROPOSITION 8.2. *Let G be a graph with n vertices. The WMLT of G is $d + 1$ if and only if d is the smallest dimension such that some generic d-dimensional framework (G, p) is liftable.*

The following implies that we can ignore genericity of our witness (cf. [30, Definition 5.1]).

PROPOSITION 8.3. *Let $d \in \mathbb{N}$ be a dimension and G be a graph with $n \geq d + 1$ vertices. If there is any liftable d-dimensional framework (G, p) then there is a generic liftable d-dimensional framework. In particular, $\mathrm{wmlt}(G) \leq d + 1$.*

PROOF. Let (G, p) be a liftable d-dimensional framework. By Lemma 3.13, (G, p) does not have a non-zero PSD equilibrium stress. By lower semi-coniuity of the rank of the rigidity matrix, there is a nbd U of p so that if $q \in U$, the space of equilibrium stresses of (G, q) has dimension at most that of (G, p). Hence any equilibrium stress of (G, q) is a small perturbation of a stress of (G, p). For sufficiently small perturbations, signature is preserved, so some neighborhood of p consists of only frameworks without non-zero PSD equilibrium stresses. This neighborhood contains a generic framework. The second statement follows from Proposition 8.2. $\square$

8.1. *Existing bounds on the WMLT.* The weak maximum likelihood threshold of a graph is one if and only if it has no edges. Examples of graphs for which MLT = WMLT = $d + 1$ are the d-*laterations*; i.e., graphs formed from K_{d+1} by a sequence of d-dimensional 0-extensions. Other than this, very little is known. Gross and Sullivant [30] showed that $\mathrm{wmlt}(G)$ is at most the chromatic number of G. Buhl [12] characterized the weak maximum likelihood thresholds of cycles, showing that $\mathrm{wmlt}(G) = 3$ if G is a three-cycle, and $\mathrm{wmlt}(G) = 2$ when G is a cycle of length four or greater.

DEFINITION 8.4. Given a directed graph, a cycle in the underlying undirected graph is *stretched* if it is of the form $v_1 \to v_2 \to \cdots \to v_k \leftarrow v_1$. Given (G, p) is a framework in $\mathbb{R}^1$ with no edges of length zero, a cycle in G is *stretched* if the corresponding cycle is stretched in the orientation of G obtained by directing each edge $i\ j$ towards j if $p(j) > p(i)$ and otherwise toward i.

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[11]Again, we are assuming that the samples are i.i.d. from a distribution whose probability measure is mutually absolutely continuous with respect to Lebesgue measure.

The following proposition can be seen as the rigidity-theoretic version of [12, Theorem 4.3].

PROPOSITION 8.5. *Let G be a cycle and let (G,p) be a generic framework in $\mathbb{R}^1$. Then (G,p) has a non-zero PSD equilibrium stress if and only if it is a stretched cycle.*

The proposition is a special case of a more general statement due Kapovich and Millson [36] which we discuss in Appendix C.

Proposition 8.5 immediately implies the following result of Gross and Sullivant.

COROLLARY 8.6 ([30, Corollary 5.4]). *If $\mathrm{wmlt}(G) = 2$, then G has an acyclic orientation with no stretched cycles.*

In [30], the property of having an acyclic orientation with no stretched cycles is called *Buhl's cycle condition*.

8.2. *A conjecture and a connection.* Based on experimental evidence, we believe that the converse to Corollary 8.6 is true.

CONJECTURE 8.7. *If G has at least one edge and an acyclic orientation with no stretched cycles, then $\mathrm{wmlt}(G) = 2$.*

Directed acyclic graphs with no stretched cycles are well-studied objects in combinatorics: they are diagrams of partially ordered sets. It is NP-hard to determine whether a given undirected graph has an acyclic orientation with no stretched cycles [11]. Thus Conjecture 8.7 would imply that the decision problem of whether a given graph has $\mathrm{wmlt}(G) = 2$ is NP-hard. Via the coning construction [62], this would imply that determining weak MLT is NP-hard in general.

The following definition is due to Erdös, Harary, and Tutte.

DEFINITION 8.8 ([25]). The *dimension* of a graph G, denoted $\dim(G)$, is the minimum d such that there exists a framework (G,p) in $\mathbb{R}^d$ such that $\|p(i) - p(j)\| = 1$ for all edges $ij \in E(G)$.

The *Hadwiger-Nelson problem* is a longstanding open problem in combinatorics which asks for the maximum chromatic number of a graph G with $\dim(G) = 2$. See [21] for the most recent progress and a brief account of the history. The connection to weak maximum likelihood thresholds is given by the following.

PROPOSITION 8.9. *Let G be a graph. Then $\mathrm{wmlt}(G) \le \dim(G) + 1$.*

PROOF. Let (G,p) be a framework in $\mathbb{R}^{\dim(G)}$ so that every edge of G has length 1. Then (G,p) is liftable. A suitable witness is the framework (G,q) in $\mathbb{R}^{n-1}$ where the $q(i)$s are the vertices of a suitably scaled unit simplex. The result now follows from Proposition 8.2. $\square$

It is well-known that $\dim(G) + 1 \le \chi(G)$. Indeed, if G has chromatic number $d + 1$, then there is a unit-distance embedding of G in dimension d by putting each of the $d + 1$ color classes on a distinct vertex of a regular simplex in dimension d. Hence, this result improves the inequality $\mathrm{wmlt}(G) \le \chi(G)$ from [30].

APPENDIX A: "ALMOST ALL" VS "GENERIC"

In this appendix, we prove some technical results needed for Theorem 1.15. We begin with a precise definition of genericity.

DEFINITION A.1.   A point $x \in \mathbb{R}^n$ is *generic* if its coordinates are algebraically independent over $\mathbb{Q}$. If $S \subseteq \mathbb{R}^n$ is an irreducible semi-algebraic set, then a point $x \in S$ is *generic in S* if whenever a polynomial f with rational coefficients satisfies $f(x) = 0$, then $f(y) = 0$ for all $y \in S$.

We record some facts about semi-algebraic sets (see, e.g. [9, 58]). Recall that a *finite boolean combination* of sets $\{S_\alpha\}_{\alpha \in J}$ is a set obtained using finitely many unions and intersections of sets in $\{S_\alpha\}_{\alpha \in J}$.

LEMMA A.2.   *Let S be an irreducible semi-algebraic set and $X \subseteq S$ semi-algebraic. Then:*

(a) *X is a finite boolean combination of open and closed (standard topology) subsets of S,*
(b) *X contains an open subset of S if and only if it has the same dimension as S,*
(c) *if X is of lower dimension than S, then each $x \in X$ satisfies some polynomial with coefficients in the field of definition for X that is not satisfied by some points in S, and*
(d) *X has finitely many irreducible components.*

LEMMA A.3.   *Let $S \subseteq \mathbb{R}^N$ be an irreducible semi-algebraic set, $X \subseteq S$ semi-algebraic, and suppose that μ is a Borel measure on S that is mutually absolutely continuous with respect to Lebesgue measure on S. Then X is μ-null if and only if every irreducible component of X is of lower dimension than S.*

PROOF. From Lemma A.2 and the fact that μ is a Borel measure, we know that each irreducible component Y of X is measurable. Since μ is a Borel measure and Lebesgue measure on S is absolutely continuous with respect to μ, if Y contains an open subset of S, $\mu(Y) > 0$. Hence, if Y has the same dimension as S, we must have $\mu(Y) > 0$. On the other hand, if Y is of lower dimension then the (standard topology) closure $\overline{Y}$ is closed and nowhere dense. Absolute continuity of μ with respect to Lebesgue measure then implies that $\mu(Y) \leq \mu(\overline{Y}) = 0$. Repeating this argument for each irreducible component of X completes the proof. $\square$

To translate between generic statements and measure theoretic ones, we use the following.

LEMMA A.4.   *Let S be an irreducible semi-algebraic subset of $\mathbb{R}^N$ and let X be a semi-algebraic subset of S, with both S and X defined over $\mathbb{Q}$. Let μ be a Borel measure on S mutually absolutely continuous with respect to Lebesgue measure. Then:*

(a) *if X is μ-null, then no generic points of S are in X,*
(b) *if X has full μ-measure, then every generic point of S is in X, and*
(c) *if neither X nor its complement are μ-null, then some generic points of S are in X and some are not.*

PROOF. Suppose, for the moment, that X is irreducible. By Lemma A.3 if X is μ-null it is of lower dimension than S. By Lemma A.2, no point of X can be generic. In general, we repeat the argument for each irreducible component, which gives (a). Part (b) follows from (a) via complementation.

For (c), Lemma A.3 implies that a μ-non-null semi-algebraic set contains an open set. Any non-generic point must lie in a nowhere dense algebraic subset of S, so if both X and its complement are μ-non-null both contain a generic point. $\qquad\square$

LEMMA A.5. *Let v be a generic configuration of n vectors in $\mathbb{R}^{d+1}$. Then v is flatten-able, and the flattened configuration p in $\mathbb{R}^d$ is also generic. Conversely, if p is a generic configuration of n points in $\mathbb{R}^d$, then there is a generic vector configuration v in $\mathbb{R}^{d+1}$ so that p is the flattening of v.*

PROOF. First suppose that v is a generic configuration of n vectors in $\mathbb{R}^{d+1}$. Letting t_i be the last coordinate of $v(i)$, we notice that if $t_i = 0$ for any i, then v satisfies a non-trivial polynomial equation and so is non-generic. Hence, v is flattenable. The map sending a flat-tenable vector configuration v to its flattening p is rational and surjective onto configurations of n points in $\mathbb{R}^d$. The result now follows from [27, Lemmas 2.7 and 2.8]. $\qquad\square$

APPENDIX B: EQUILIBRIUM STRESSES AND CONVEXITY

The goal of this appendix is to give a self-contained proof of Lemma 3.13, which originally appeared in [2]. We will denote the interior of a set S by $\text{int}(S)$.

LEMMA B.1. *Let K be a convex n-dimensional set in $\mathbb{R}^n$, let π be a rank-m linear projection from $\mathbb{R}^n$ to $\mathbb{R}^m$, and let $k := \pi(K)$ be the m-dimensional image. The following are equivalent:*

(a) $\pi^{-1}(x) \cap \text{int}(K)$ *is nonempty,*
(b) $x \in \text{int}(k)$, *and*
(c) x *does not lie on a supporting hyperplane for k.*

PROOF. Equivalence of the latter two conditions follows from the supporting hyperplane theorem [56, Ch. 8].

If $\pi^{-1}(x) \cap \text{int}(K)$ is nonempty, then there is a point $X \in \pi^{-1}(x)$ with open neighborhood N satisfying $N \subseteq \text{int}(K)$. Since π is a linear map, it is open onto its image, which is $\mathbb{R}^m$, so $\pi(N)$ is open. Since $x \in \pi(N) \subseteq k$, it is interior in k (here we used that K is full-dimensional, so that $\text{int}(K)$ is open and nonempty in $\mathbb{R}^n$).

Now assume $\pi^{-1}(x) \cap \text{int}(K) = \emptyset$. Since $\pi^{-1}(x)$ is convex, there must be an affine hyper-plane H in $\mathbb{R}^n$ weakly separating $\pi^{-1}(x)$ from $\text{int}(K)$. Since $\pi^{-1}(x)$ is an affine subspace, we have $\pi^{-1}(x) \subseteq H$. Let ℓ be the linear functional and α the real number so that

$$H = \{y \in \mathbb{R}^N : \ell(y) = \alpha\}.$$

Since $\pi^{-1}(x)$ is parallel to the kernel of π, we have that $\ker \ell \supseteq \ker \pi$. Hence, we have a well-defined linear map $\bar{\ell} : \mathbb{R}^m \cong \mathbb{R}^n / \ker \pi \to \mathbb{R}$ given by

$$\bar{\ell}(x') = \ell(y') \qquad (\text{any } y' \in \pi^{-1}(x')).$$

Hence, since H supports K, the hyperplane $\{y \in \mathbb{R}^m : \bar{\ell}(y) = \alpha\}$ supports k at x. $\qquad\square$

PROOF OF LEMMA 3.13. Let K be the PSD cone. Points in K are Gram matrices of n-point configurations in $\mathbb{R}^n$; points in the interior correspond to configurations with n-dimensional linear span. Such configurations will have an $n-1$-dimensional affine span. Fixing a graph G with m edges, π will be the map to $\mathbb{R}^m$, which measures the squared lengths of the corresponding framework; i.e. indexing $\mathbb{R}^m$ by the edges of G, for each edge ij of G, we have

$$\pi(X)_{ij} = X_{ii} + X_{jj} - 2X_{ij}.$$

The image $\pi(K)$ is an m-dimensional convex cone $k \subseteq \mathbb{R}^m$. Using Lemma B.1, it now suffices to show that $\pi(p)$ lies on the boundary of k if and only if (G, p) has a PSD equilibrium stress.

Given a configuration q of n points in $\mathbb{R}^n$, let q^l denote the vector $q(1)_l, \ldots, q(n)_l$ consisting of the lth coordinate of each point in q. Now assume $\pi(p)$ lies on the boundary of k, let ω be the normal vector of the hyperplane tangent to k at $\pi(p)$, and let Ω be the matrix obtained by setting $\Omega_{ij} = \Omega_{ji} = -\omega_{ij}$ for all edges ij of G, $\Omega_{ii} = \sum_j \omega_{ij}$ for $i = 1, \ldots, n$, and all other entries zero. This means that for any configuration q of n points in $\mathbb{R}^n$, the following inequality holds, and is moreover an equality when $q = p$:

$$(3) \qquad 0 \le \sum_{ij \text{ edge of } G} \omega_{ij} ||(q_i - q_j)||^2 = \sum_l (q^l)^T \Omega q^l.$$

This implies that Ω is PSD and that $(p^l)^T \Omega p^l = 0$ for all l. Together, these imply that $\Omega p^l = 0$ for each l, which is exactly the condition for ω to be an equilibrium stress of p. Thus ω is a PSD equilibrium stress for p.

Finally, note that if Ω is a PSD equilibrium stress of p, then the above arguments can be reversed to show that Ω defines a supporting hyperplane of k at p. $\qquad \square$

APPENDIX C: THE SIGNATURE OF A CYCLE STRESS

DEFINITION C.1. Let C_n denote the directed cycle on vertex set $\{1, \ldots, n\}$ with edges $1 \to 2, 2 \to 3, \ldots, (n-1) \to n, n \to 1$. A *framework* (C_n, p) on C_n refers to a framework on the undirected graph underlying C_n. In a general position framework (C_n, p) in $\mathbb{R}^1$, an edge $i \to (i+1)$ is *forwards* if $p(i) < p(i+1)$ and *backwards* otherwise.

Note that every general position framework (C_n, p) in $\mathbb{R}^1$ has at least one forwards edge and at least one backwards edge. Proposition 8.5 is a corollary of the following, which classifies the signatures of the stresses of cycles in $\mathbb{R}^1$.

THEOREM C.2 ([36]). *Let (C_n, p) be a generic framework in $\mathbb{R}^1$ and let f be the number of forwards edges and b the number of backwards edges. Then, for every nonzero equilibrium stress matrix Ω of (C_n, p), the signature of Ω is either $(f-1, 2, b-1)$ or $(b-1, 2, f-1)$.*

It is easy to see from Theorem C.2 that any cycle framework (C_n, p) with exactly one backwards edge, or exactly one forwards edge, must be stretched and vice versa. Since the proof in [36] uses Hodge theory, we provide a linear-algebraic argument.

LEMMA C.3. *Let (G, p) be a general position framework in $\mathbb{R}^1$ with the edge $\{1, n\}$, wlog (after cyclic relabeling) backwards and $p(n)$ the rightmost vertex. Then (G, p) has a unique, up to nonzero scaling, equilibrium stress ω, and this scaling can be chosen such that every forward edge has positive coefficient and every negative edge has negative coefficient.*

PROOF. Without loss of generality, set the coefficient $\omega_{1,n}$ on the edge $\{1, n\}$ to -1. Now walk, in cyclic order, starting from vertex 1, solving the equilibrium condition locally, by setting $\omega_{i,1+1}$ to solve (indices taken cyclically):

$$\omega_{i,i+1}(p(i+1) - p(i)) = \omega_{i-1,i}(p(i) - p(i-1)).$$

Notice that the sign changes whenever we switch from forwards to backwards edges, so we have the desired sign pattern. General position implies that we do not get any zero coefficients. We have, automatically, equilibrium at every vertex except, possibly $p(n)$. To check that we have equilibrium, notice that, by induction, all the vectors $\omega_{i,i+1}(p(i+1) - p(i))$ have magnitude $|p(n) - p(1)|$ and that $\omega_{n-1,n}$ is positive. $\qquad \square$

Our next lemma is a general fact that can be verified by direct computation.

LEMMA C.4. *Let H be any graph and ω an equilibrium stress with associated matrix Ω. For any subset S of vertices of H, let $x(S)$ be the characteristic vector of S. Then*

$$x(S)^T \Omega x(S) = \sum_{\text{edges } ij: \, i \in S, \, j \notin S} \omega_{ij}.$$

In particular, if S is the set of vertices on one side of a cut consisting of edges with positive (resp negative) stress coefficients, then $x(S)$ has positive (resp negative) Rayleigh quotient.

PROOF OF THEOREM C.2. Let (G,p) be as in the statement and Ω scaled as in Lemma C.3. Uniqueness, up to nonzero scale, of the equilibrium stress on (G,p) implies this is possible. Now recall that removing any two edges from a cycle determines a cut. If we have b backwards edges, $e_1, \ldots, e_b$, each of the cuts $\{e_1, e_j\}$ for $2 \leq j \leq b$ gives rise to a collection of $b-1$ independent incidence vectors with negative Rayleigh quotient, from Lemma C.4. Hence Ω has at least $b-1$ negative eigenvalues. Similarly, the f edges $e'_1, \ldots, e'_f$ with positive stress coefficients give $f-1$ independent incidence vectors with positive Rayleigh quotient. Since Ω has a nullity of at least 2, as an equilibrium stress matrix, the proof is complete. $\square$

APPENDIX D: RANDOM GRAPHS

In this section, we prove Theorem 2.7. We will use results of Candès and Tao [13] as a "black box" along with some ideas from Saliola and Whiteley [53].

The results in this appendix are based on those from [38], but slightly weaker. We include the proofs here to keep this paper self-contained and because they are somewhat simpler.

D.1. Low-rank matrix completion. Matrix completion is the problem of imputing an unknown $n \times n$ matrix M from a subset of its entries. A fundamental result of Candès and Tao (specialized to the case where r is a fixed constant is):

THEOREM D.1 ([13]). *If M is an $n \times n$ symmetric rank r matrix with the strong incoherence property with constant $\mu > 0$, there is a constant $C_{r,\mu} > 0$ such that, if*

$$m \geq C_r \mu^2 n (\log n)^2$$

entries of M are sampled uniformly at random, then with probability at least $1 - n^{-3}$, a nuclear norm minimization algorithm recovers all of M from the observed entries.

For our purposes we do not need to know the exact definition of strong incoherence. What we do need is that Candès and Tao also prove:

THEOREM D.2 ([13, Sec. 1.5.1]). *For any rank r and some $\mu = O(\sqrt{\log n})$, there are open sets of matrices satisfying the strong incoherence property.*

In particular, there is a generic symmetric rank r matrix satisfying the strong incoherence property.

THEOREM D.3. *Let r be a fixed rank. There is a constant $C'_r > 0$ so that, if*

$$m \geq C'_r n (\log n)^3$$

entries of a generic symmetric matrix M of rank r are sampled uniformly at random, there are finitely many symmetric matrices M' of rank r agreeing with M on the observed entries with probability at least $1 - n^{-3}$.

For a set of observed entries, if, for a generic rank r symmetric matrix M, there are finitely many rank r matrices agreeing with M on the observed entries, the pattern is said to be *finitely completable*.

PROOF. Let M be a generic, rank r symmetric matrix with the strong incoherence property where $\mu = O(\sqrt{\log n})$ from Theorem D.2. Take C'_r to be C_r from the statement of Theorem D.1. Such an M exists by Theorem D.2. By Theorem D.1, if m (from the statement) entries are sampled uniformly at random, the nuclear norm minimization algorithm finds M with probability $1 - n^{-3}$.

If there were any other way to complete the observed entries to get a different rank r matrix, this would be impossible as soon as the success probability rises above $1/2$. Hence, for $n \geq r + 1$, all but a n^{-3}-fraction of observation patterns with m observed entries yield a matrix completion problem that is generic and uniquely completable, which is the matrix completion analogue of global rigidity (see [34, 57]).

We do not know that unique completability is a generic property, even for symmetric matrices, but the existence of a generic rank r matrix M with a uniquely completable matrix completion problem for some pattern does imply that the observation pattern is finitely completable, which is a generic property [34, 57]. □

We notice that, although we are not dealing with symmetrically chosen observation patterns, since we assume the underlying matrix is symmetric, the probability of finite completability is not changed when making this assumption. Similarly, the probability of a pattern being generically finitely completable can only go up if we deterministically observe the diagonal and then uniformly sample m other entries.

D.2. Rigidity in pseduo-Euclidean spaces. A d-dimensional *pseudo-Euclidean* space $\mathbb{M}_s^d$ is $\mathbb{R}^d$ equipped with a bilinear form

$$\beta_s(x, y) = -\gamma_1 \delta_1 - \cdots - \gamma_s \delta_s + \gamma_{s+1} \delta_{s+1} + \cdots + \gamma_d \delta_d$$

where $x = (\gamma_i)$ and $y = (\delta_i)$. We can use β_s to measure length and define local and global rigidity similarly to the Euclidean case (see [29, 53] for details).

THEOREM D.4 ([53]). *For every dimension $d \geq 1$ and d-dimensional pseudo-Euclidean space $\mathbb{M}_s^d$, local d-rigidity is a generic property. Moreover, a graph G is locally d-rigid in $\mathbb{M}_s^d$ if and only if G is locally d-rigid in Euclidean space.*

Meanwhile, as observed by Gortler and Thurston in [29], any symmetric $n \times n$ matrix of rank $d+1$ and signature $(d+1-s, n-d-1, s)$ arises as the Gram matrix of a configuration of n vectors in a pseudo-Euclidean space with β_s as the bilinear form. Translating language slightly, we have:

THEOREM D.5 ([29]). *Let G be a graph on n vertices. Then G is locally d-rigid in a pseudo-Euclidean space $\mathbb{M}_s^d$ if and only if the associated symmetric set of observed entries, plus the diagonal, is generically finitely completable for rank $d+1$.*

D.3. Putting things together. We are now in a position to prove Theorem 2.7. From Theorem D.3, for each rank $d+1$, there is a constant C_{d+1} so that, if the diagonals and $m \geq C_{d+1} n \operatorname{polylog}(n)$ uniformly selected entries of a generic symmetric matrix are observed, the resulting pattern is, whp, finitely completable. Theorem D.5 then implies that the graph arising from symmetrising the observed entries is generically d-rigid in a pseudo-Euclidean space, and then, by Theorem D.4 generically d-rigid.

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