BOX DIMENSIONS OF $(\times m, \times n)$-INVARIANT SETS

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Abstract. We study the box dimensions of sets invariant under the toral endomorphism $(x, y) \mapsto (\mathbf{m}x \bmod 1, \mathbf{n}y \bmod 1)$ for integers $\mathbf{n} > \mathbf{m} \geq 2$. The basic examples of such sets are Bedford-McMullen carpets and, more generally, invariant sets are modelled by subshifts on the associated symbolic space. When this subshift is topologically mixing and sofic the situation is well-understood by results of Kenyon and Peres. Moreover, other work of Kenyon and Peres shows that the Hausdorff dimension is generally given by a variational principle. Therefore, our work is focused on the box dimensions in the case where the underlying shift is not topologically mixing and sofic. We establish straightforward upper and lower bounds for the box dimensions in terms of entropy which hold for all subshifts and show that the upper bound is the correct value for coded subshifts whose entropy can be realised by words which can be freely concatenated, which includes many well-known families such as β-shifts, (generalised) S-gap shifts, and topologically transitive sofic shifts. We also provide examples of topologically mixing coded subshifts where the general upper bound fails and the box dimension is actually given by the general lower bound. In the non-transitive sofic setting, we provide a formula for the box dimensions which is often intermediate between the general lower and upper bounds.

1. Introduction

1.1. **Background and motivation.** The dimension theory of smooth expanding dynamical systems studies the complexity of invariant sets from a dimension theoretic perspective. When the dynamical system is conformal, the Hausdorff and lower and upper box dimensions coincide for the invariant sets and can be written in terms of the root of a pressure equation [1]. The non-conformal case is more subtle and significantly less well-understood. In this case, restricting further to piecewise affine dynamical systems, generic invariant sets enjoy analogous properties, namely the three dimensions mentioned above coincide and equal the root of a "sub-additive pressure" equation [8,15]. However there is also a large class of sets invariant under non-conformal expanding dynamical systems which are exceptions to this generic rule. Certain aspects of this exceptional behaviour have been recognised since the 1980s, such as the dimensions failing to satisfy the generic formulae and failing to coincide with each other, but others are more recent phenomena, for example in [6] these sets were used to disprove a folklore conjecture concerning the existence of invariant measures of maximal Hausdorff dimension for expanding conformal repellers. All of these behaviours have been discovered in the simplest models of exceptional invariant sets, namely sets invariant under expanding maps generated by diagonal matrices, and this motivates our further investigation into the dimension theory of this model.

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Our specific focus in this paper is to study how the dynamics of an expanding map on an exceptional invariant set influences its dimension theory. This is a line of inquiry which has largely been unexplored, since most studies of exceptional invariant sets have concerned sets on which the dynamics can be modelled by a full shift. Therefore the case where the dynamics is "very far" from being modelled by a full shift is of particular interest.

1.2. **Our setting and previous work.** We study compact sets invariant under the toral endomorphism

$$T(x,y) = (\mathbf{m}x \bmod 1,\, \mathbf{n}y \bmod 1)$$

for integers $\mathbf{n} > \mathbf{m} \geq 2$, which is a basic and fundamental example of an expanding non-conformal dynamical system. The simplest examples of such invariant sets are the self-affine carpets introduced by Bedford and McMullen in 1984 [2,17]. In particular, these are modelled by a full shift. More generally, compact $(\times\mathbf{m}, \times\mathbf{n})$-invariant sets are modelled by subshifts on the associated symbolic space. Kenyon and Peres [13] studied the more general case when this subshift is topologically mixing and sofic and in [14] they resolved the Hausdorff dimension case in general by proving a variational principle. These papers provide the starting point for our investigation, which is focused on the box dimensions in the case where the underlying shift is not topologically mixing and sofic. We expand the theory in several directions.

Let $\Delta_{\mathbf{m},\mathbf{n}} = \{(a,b) \,:\, 1 \leq a \leq \mathbf{m},\ 1 \leq b \leq \mathbf{n},\ a,b \in \mathbb{N}\}$. For any $(a,b) \in \Delta_{(\mathbf{m},\mathbf{n})}$ define the contraction $S_{(a,b)} : [0,1]^2 \to [0,1]^2$ as

$$S_{(a,b)}(x,y) = \begin{pmatrix} \frac{1}{\mathbf{m}} & 0 \\ 0 & \frac{1}{\mathbf{n}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{a-1}{\mathbf{m}} \\ \frac{b-1}{\mathbf{n}} \end{pmatrix}.$$

Define the coding map $\Pi : \Delta_{\mathbf{m},\mathbf{n}}^{\mathbb{N}} \to [0,1]^2$ as

$$\Pi\left((a_1,b_1)(a_2,b_2)\ldots\right) := \lim_{n\to\infty} S_{(a_1,b_1)} \circ \cdots \circ S_{(a_n,b_n)}(0).$$

Consider any compact $(\times\mathbf{m}, \times\mathbf{n})$-invariant set F, meaning that $T(F) \subseteq F$. Then there exists a digit set $\mathcal{I} \subseteq \Delta_{\mathbf{m},\mathbf{n}}$ and a subshift Σ on the digit set $\mathcal{I}$ (meaning a compact σ-invariant subset $\Sigma \subseteq \mathcal{I}^{\mathbb{N}}$, i.e. $\sigma(\Sigma) \subseteq \Sigma$ where $\sigma : \Sigma \to \Sigma$ denotes the left shift map) such that $F = \Pi(\Sigma)$. For example, if Σ is the full shift on $\mathcal{I}$ then $\Pi(\Sigma)$ is a Bedford-McMullen carpet [2,17]. For brevity, rather than writing sequences in Σ as $(a_1,b_1)(a_2,b_2)\ldots$ and finite words which appear in sequences of Σ as $(a_1,b_2)\ldots(a_n,b_n)$ we will for the most part denote both infinite sequences and finite words by variables such as $\mathtt{i}, \mathtt{j}$, and $\mathtt{k}$.

Given a subshift Σ, let Σ^* denote the language of Σ, meaning the collection of finite words which appear in sequences $\mathtt{i} \in \Sigma$. For $n \in \mathbb{N}$ let Σ_n denote words in Σ^* which have length n. We say Σ is *topologically transitive* if for all $\mathtt{i}, \mathtt{j} \in \Sigma^*$ there exists $\mathtt{k} \in \Sigma^*$ such that $\mathtt{ikj} \in \Sigma^*$. We say Σ is *topologically mixing* if for all $\mathtt{i}, \mathtt{j} \in \Sigma^*$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists $\mathtt{k} \in \Sigma_n$ such that $\mathtt{ikj} \in \Sigma^*$. Recall that the *topological entropy* of Σ is defined as $h(\Sigma) := \lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_n$, where the limit exists by submultiplicativity arguments.

The $(\times\mathbf{m}, \times\mathbf{n})$-invariant sets are typically fractal and a key question of interest is in computing their dimensions, especially Hausdorff and box dimensions, see [2,7,13,14,17]. For more background on Hausdorff and box dimensions, see [9]. We write $\dim_{\mathrm{H}}$, $\underline{\dim}_{\mathrm{B}}$, and $\overline{\dim}_{\mathrm{B}}$ for the Hausdorff, lower and upper box dimensions, respectively. The lower and upper box dimensions are defined by

$$\underline{\dim}_{\mathrm{B}} E = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \qquad \text{and} \qquad \overline{\dim}_{\mathrm{B}} E = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta},$$

respectively, where $N_\delta(E)$ denotes the smallest number of sets of diameter $\delta > 0$ required to cover E. It is useful to keep in mind that, for all bounded sets E in Euclidean space,

$$\dim_{\mathrm{H}} E \le \underline{\dim}_{\mathrm{B}} E \le \overline{\dim}_{\mathrm{B}} E.$$

Moreover, if the upper and lower box dimensions coincide we simply refer to the box dimension, written $\dim_{\mathrm{B}}$. In the case where Σ is a full shift (over a restricted alphabet $\mathcal{I} \subseteq \Delta_{\mathbf{m},\mathbf{n}}$), the box and Hausdorff dimensions were computed independently by Bedford [2] and McMullen [17]. If Σ is a topologically mixing sofic subshift, then the box and Hausdorff dimensions were given by Kenyon and Peres [13]. We say that a subshift is *sofic* if there exists some finite directed labelled graph G such that the subshift coincides with the set of infinite sequences which label an infinite path in G (see Section 4 for more details). If Σ is a topologically transitive subshift of finite type, then the box dimension was computed by Deliu *et al* [7]. The only progress beyond the sofic setting is provided by Kenyon and Peres [14] where they show that for any compact $(\times\mathbf{m}, \times\mathbf{n})$-invariant set the Hausdorff dimension is given by a variational principle, that is, as the supremum of the Hausdorff dimensions of $(\times\mathbf{m}, \times\mathbf{n})$-invariant measures supported on the set. It is also shown that there exists a maximising (ergodic) measure, which achieves the Hausdorff dimension of the set. Moreover, it is shown in [14] that the Hausdorff dimension of an ergodic $(\times\mathbf{m}, \times\mathbf{n})$-invariant measure is given by a Ledrappier-Young formula. In some sense, this settles the question of Hausdorff dimension. The box dimensions of $(\times\mathbf{m}, \times\mathbf{n})$-invariant sets remains an interesting open problem. We recall the box dimension result of Kenyon and Peres which till now was the state of the art. Let $\pi : \Sigma \to \pi\Sigma$ denote the projection mapping $\pi\left((a_1, b_1)(a_2, b_2)\ldots\right) = a_1 a_2 \ldots$. In particular, $\pi\Sigma$ is itself a subshift.

Theorem 1.1 ((Proposition 3.5, [13]))**.** *Suppose Σ is a topologically mixing sofic subshift. Then*

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}. \tag{1}$$

It is straightforward to construct an example (see Figure 1) where (1) does not hold for a general sofic subshift Σ. For example fix $\mathbf{m} = 2$, $\mathbf{n} = 4$ and $\mathcal{I} = \{(1,1),(1,2),(1,3),(1,4),(2,1)\}$ and denote $\Sigma_2 = \{(1,1),(2,1)\}^{\mathbb{N}}$, $\Sigma_3 = \{(1,2),(1,3),(1,4)\}^{\mathbb{N}}$. Consider the subshift of finite type $\Sigma = \Sigma_2 \cup \Sigma_3$. Then,

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \max\{\dim_{\mathrm{B}} \Pi(\Sigma_2), \dim_{\mathrm{B}} \Pi(\Sigma_3)\} = 1 < 1 + \frac{\log 3 - \log 2}{\log 4} = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}$$

where in the second equality we apply (1) to $\dim_{\mathrm{B}} \Pi(\Sigma_2)$ and $\dim_{\mathrm{B}} \Pi(\Sigma_3)$. This example heavily relies on a lack of transitivity.

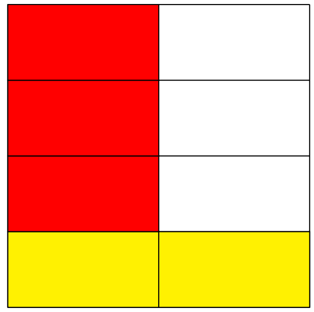

FIGURE 1. An easy non-transitive example where (1) fails. We consider the invariant set coded by the full shift on the digits associated to the red rectangles and the invariant set coded by the full shift on the digits associated to the yellow rectangles, and take the union of these two invariant sets. Note that this will simply be a union of the line segment $\{(x,0) : x \in [0,1]\}$ with a self-similar set contained in the y axis.

2. Main results

2.1. The sofic case. We fully resolve the sofic case by finding a formula that holds for any sofic subshift (which is not just the maximum over irreducible parts as above) and which simplifies to (1) in the transitive case, thus generalising Theorem 1.1 from topologically mixing to topologically transitive.

We say a graph G is *irreducible* if given any pair of vertices $v, w \in G$ there is a path in G from v to w. Given a finite directed labelled graph G which presents Σ, let $\{G_i\}_{i=1}^k$ denote the irreducible components of G, meaning the maximal irreducible subgraphs of G. Each subgraph G_i therefore presents a subshift $\Sigma_{G_i} \subseteq \Sigma$. Given $1 \le i \le k$ we let $\{i\}^+$ denote the set of all indices $1 \le j \le k$ such that there is a path in G from a vertex in G_i to a vertex in G_j, noting that $\{i\}^+$ is necessarily non-empty since we always have $i \in \{i\}^+$.

Theorem 2.1. *Let Σ be a sofic subshift which is presented by a graph G. Let $\{G_1, \dots, G_k\}$ be the irreducible components of G. Then*

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \max_{1 \le i \le k} \left\{ \frac{h(\Sigma_{G_i})}{\log \mathbf{n}} + \max_{j \in \{i\}^+} h(\pi \Sigma_{G_j}) \left(\frac{1}{\log \mathbf{m}} - \frac{1}{\log \mathbf{n}} \right) \right\}. \tag{2}$$

As in [13, Proposition 3.5], each entropy $h(\Sigma_{G_i})$ and $h(\pi\Sigma_{G_i})$ can be expressed in terms of the spectral radius of the adjacency matrix of an appropriate right-resolving presentation (of Σ_{G_i} and $\pi\Sigma_{G_i}$ respectively). When Σ is topologically transitive and sofic, Σ can be presented by an irreducible labelled graph, therefore (2) simplifies to (1). Additionally, we can also recover (1) for some sofic subshifts which are not topologically transitive, under some assumptions on the "position" of the entropy maximising irreducible components, see Corollary 4.1. Moreover, in Corollary 4.2 we obtain a characterisation of the systems which yield equal Hausdorff and box dimensions, in terms of the "position" of the entropy maximising irreducible components.

2.2. More general subshifts. Next, we turn to more general subshifts. By bounding $\underline{\dim}_{\mathrm{B}} \Pi(\Sigma)$ (and $\dim_{\mathrm{H}} \Pi(\Sigma)$) below by the box dimension of its projection and by a crude estimate involving entropy and the larger Lyapunov exponent, we show (see Proposition 3.1) that any invariant set satisfies a trivial lower bound of $\underline{\dim}_{\mathrm{B}}\Pi(\Sigma) \geq \max\left\{\frac{h(\pi\Sigma)}{\log \mathbf{m}}, \frac{h(\Sigma)}{\log \mathbf{n}}\right\}$. On the other hand, we also show (see Proposition 3.1) that the right hand side of (1) is a trivial upper bound on $\overline{\dim}_{\mathrm{B}}\Pi(\Sigma)$ in general. While Theorem 2.1 demonstrates that the box dimension can drop from this trivial upper bound if Σ is not topologically transitive, it is interesting to ask whether transitivity is sufficient for (1) to hold for general subshifts. We answer this in the negative:

Theorem 2.2. *There exists a topologically mixing subshift Σ with $0 < h(\pi\Sigma) < h(\Sigma)$ and*

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \max\left\{\frac{h(\Sigma)}{\log \mathbf{n}}, \frac{h(\pi\Sigma)}{\log \mathbf{m}}\right\}.$$

In particular, in the above example the trivial lower bound is in fact the exact value of the box dimension. Moreover this box dimension is clearly strictly smaller than the trivial upper bound and we can modify our example such that either of the trivial lower bounds equals the box dimension. The subshift Σ that we construct towards the proof of Theorem 2.2 falls into the class of *coded subshifts.*

Coded subshifts, which were first introduced in [4] and include the well-known subclasses of S-gap shifts, β-shifts and Dyck shifts, are subshifts which can be presented by an irreducible (but not necessarily finite), directed labelled graph (see Section 5). In particular, they greatly extend the class of transitive sofic subshifts and provide a natural and interesting class to investigate which, unlike subshifts of finite type and sofic subshifts in general, cannot be handled by techniques that depend on finiteness of the presentation. Unfortunately, this makes it impossible to present a universal dimension theory for invariant sets modelled by coded subshifts, as was possible in the sofic case (but after all there are only countably many possible sofic subshifts as opposed to uncountably many coded subshifts). Despite the somewhat unsatisfying lack of potential for a universal dimension theory in the coded setting, invariant sets modelled by coded subshifts have interesting properties which make their study enticing. For example, while every $(\times\mathbf{m}, \times\mathbf{n})$-invariant set modelled by a transitive sofic subshift has a unique invariant measure of maximal Hausdorff dimension [10], the same is not true of $(\times\mathbf{m}, \times\mathbf{n})$-invariant sets modelled by coded subshifts. This can be seen by noting that there are examples of coded subshifts with distinct measures of maximal entropy [12], thus if $\mathcal{I} \subset \Delta_{\mathbf{m},\mathbf{n}}$ are chosen to be indices of maps in a common column and a coded subshift $\Sigma \subset \mathcal{I}^{\mathbb{N}}$ with distinct measures of maximal entropy is fixed, then this directly implies that the corresponding invariant set $\Pi(\Sigma)$ has distinct measures of maximal Hausdorff dimension.

A useful equivalent characterisation of coded subshifts is that a subshift Σ is coded if there exists a countable collection of finite words $\mathcal{C}$, which we call generators, such that Σ is the closure of the set of sequences obtained by freely concatenating the generators. In particular, $\pi\Sigma$ is also a coded subshift which is generated by $\pi\mathcal{C}$. We say that a coded subshift Σ has *unique decomposition with respect to* $\mathcal{C}$ if no finite word can be written as a concatenation of generators in $\mathcal{C}$ in distinct ways.

We will show that if the entropy of a coded subshift Σ and $\pi\Sigma$ can be realised by counting words which can be obtained by concatenating their (respective) generators, then the box dimension $\dim_{\mathrm{B}} \Pi(\Sigma)$ equals the trivial upper bound given in Proposition 3.1. In particular let $\mathcal{G}_n$ denote all words of length n in Σ^* which can be written by concatenating generators from

$\mathcal{C}$. Analogously, $\pi\mathcal{G}_n$ are all words of length n in $(\pi\Sigma)^*$ which can be written by concatenating generators from $\pi\mathcal{C}$. We denote

$$h := \limsup_{n\to\infty} \frac{1}{n}\log \#\mathcal{G}_n \qquad \text{and} \qquad h_\pi := \limsup_{n\to\infty} \frac{1}{n}\log \#\pi\mathcal{G}_n.$$

Theorem 2.3. *Let Σ be a coded subshift and suppose $h = h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$. Then*

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}. \tag{3}$$

Note that the example constructed in Theorem 2.2 satisfies $h < h(\Sigma)$. A drawback of Theorem 2.3 is that in general it may not be straightforward to verify the equalities $h = h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$. However, under the assumption of unique decomposition of Σ and $\pi\Sigma$ we provide a more practical way of checking that the conclusion of Theorem 2.3 holds. This is based on the fact that under the assumption of unique decomposition of Σ and $\pi\Sigma$ (with respect to $\mathcal{C}$ and $\pi\mathcal{C}$), h and h_π can be understood as the Gurevic entropies of countable graphs associated with the coded subshifts Σ and $\pi\Sigma$ (see Section 5). This allows us to employ classical tools from the theory of countable Markov shifts which yields checkable criteria for Theorem 2.3 to hold, see Theorem 2.4 below, whose statement requires the introduction of some further notation.

Let $\mathcal{L}_n$ denote all words of length n in Σ^* which appear at the beginning or end of some generator in $\mathcal{C}$, analogously $\pi\mathcal{L}_n$ are all words of length n which appear at the beginning or end of some generator in $\pi\mathcal{C}$. We denote

$$\ell := \limsup_{n\to\infty} \frac{1}{n}\log \#\mathcal{L}_n \qquad \text{and} \qquad \ell_\pi := \limsup_{n\to\infty} \frac{1}{n}\log \#\pi\mathcal{L}_n.$$

Let $\mathcal{C}_n$ denote words in $\mathcal{C}$ of length $n \in \mathbb{N}$, analogously $\pi\mathcal{C}_n$ denotes words in $\pi\mathcal{C}$ of length n. Finally, define functions $f, f_\pi : [0,\infty) \to (0,\infty]$ by

$$f(x) = \sum_{n=1}^{\infty} \#\mathcal{C}_n e^{-nx} \qquad \text{and} \qquad f_\pi(x) = \sum_{n=1}^{\infty} \#\pi\mathcal{C}_n e^{-nx}. \tag{4}$$

Theorem 2.4. *Suppose Σ is a coded subshift such that Σ and $\pi\Sigma$ have unique decomposition with respect to $\mathcal{C}$ and $\pi\mathcal{C}$ respectively. Additionally, assume $f(\ell) > 1$ and $f_\pi(\ell_\pi) > 1$. Then*

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}.$$

The usefulness of Theorem 2.4 lies in the fact that $\#\mathcal{C}_n$, $\#\pi\mathcal{C}_n$, ℓ and ℓ_π are often easy to compute, which we demonstrate by applying it to generalised S-gap shifts in §5.2.2. We also note that Theorem 2.4 can easily be adapted to allow $\pi\Sigma$ to be uniquely decomposing with respect to an arbitrary generating set $\mathcal{C}^\pi$ rather than $\pi\mathcal{C}$. In particular if $\#\pi\mathcal{C}_n$ is replaced by $\#\mathcal{C}^\pi$ (words of length n in $\mathcal{C}^\pi$) in the definition of f_π, then Theorem 2.4 remains true under the assumption that $\pi\Sigma$ satisfies unique decomposition with respect to $\mathcal{C}^\pi$.

2.3. **Overview of proof strategy.** Finally, we briefly discuss the general ideas behind the proofs of the main theorems. Any invariant set can be covered by using "approximate squares". In order to obtain an estimate on the number of approximate squares required to cover the set, one needs to count the possible number of pairs of sequences $\mathtt{i} \in \Sigma^*$ and

$\mathtt{j} \in \pi\Sigma^*$ (of fixed lengths which depend on the scale) such that $\mathtt{j}$ can be "lifted" through π to a sequence in Σ^* which can be concatenated with $\mathtt{i}$. Each theorem can now be recast as a problem of counting paths on an appropriate graph. Although this is a standard approach in symbolic dynamics, what is unconventional about our setting is that we are not counting paths in a fixed graph but instead a transition occurs in the graph itself once the paths being counted reach a certain length (where this length is dependent on the scale), which adds extra subtlety and complexity to the counting problem. The proof of each result then investigates how this nuanced counting problem depends on the properties of the graph (i.e. the dynamical properties of T).

3. Preliminaries

We write $a \lesssim b$ to mean there exists a constant $C > 0$ such that $a \leq Cb$. The implicit constant C may depend on parameters which are fixed in the hypotheses, such as $\mathbf{m}, \mathbf{n}$ and Σ, but crucially do not depend on variables in the proofs, such as the covering scale δ. If we wish to emphasise that the C depends on something else, not fixed in the hypothesis such as ε, then we write $a \lesssim_\varepsilon b$. Similarly, we write $a \gtrsim b$ to mean $b \lesssim a$ and $a \approx b$ to mean $a \lesssim b$ and $a \gtrsim b$ both hold (analogously $a \gtrsim_\varepsilon b$ and $a \approx_\varepsilon b$). For $\mathtt{i} \in \Sigma_k$, we write $[\mathtt{i}]$ for the cylinder consisting of elements of Σ with prefix $\mathtt{i}$. We also refer to $\Pi([\mathtt{i}])$ as cylinders, although these are subsets of the fractal, rather than the symbolic space. Given $\mathtt{i} \in \Sigma$ or $\mathtt{i} \in \Sigma^*$ of length at least $n+1 \geq 2$ we let $\mathtt{i}|_n$ denote the truncation of $\mathtt{i}$ to its first n digits. We also write $\#A$ to denote the cardinality of a (usually finite) set A.

Let $0 < \delta < 1$, which will denote the size of the covering sets in proofs. Throughout the paper we will let $n(\delta)$ denote the unique positive integer satisfying $\mathbf{n}^{-n(\delta)} \leq \delta < \mathbf{n}^{1-n(\delta)}$ and $m(\delta)$ denote the unique positive integer satisfying $\mathbf{m}^{-m(\delta)} \leq \delta < \mathbf{m}^{1-m(\delta)}$, noting that $n(\delta) < m(\delta)$ for sufficiently small δ. Observe that by definition $m(\delta) \approx \frac{-\log \delta}{\log \mathbf{m}}$ and $n(\delta) \approx \frac{-\log\delta}{\log \mathbf{n}}$ for sufficiently small δ.

Here we prove the trivial lower and upper bounds that we alluded to in the introduction. The general strategy of relating covers to allowed words in Σ and $\pi\Sigma$ will underpin all of our subsequent proofs, therefore we take care to include all of the details here.

Proposition 3.1. *For all subshifts $\Sigma \subset \Delta_{\mathbf{m},\mathbf{n}}^{\mathbb{N}}$,*

$$\max\left\{\frac{h(\pi\Sigma)}{\log \mathbf{m}}, \frac{h(\Sigma)}{\log \mathbf{n}}\right\} \leq \dim_{\mathrm{H}} \Pi(\Sigma) \leq \underline{\dim}_{\mathrm{B}} \Pi(\Sigma) \leq \overline{\dim}_{\mathrm{B}} \Pi(\Sigma) \leq \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}.$$

Proof. Fix $\varepsilon > 0$ and $\delta > 0$.

We begin with the upper bound. Consider $\Sigma_{n(\delta)}$ and consider covers of the level $n(\delta)$ cylinders, $\Pi([\mathtt{i}])$, independently. For $\mathtt{i} \in \Sigma_k$, write

$$M(\mathtt{i}, l) = \#\pi(\mathtt{j} \in \Sigma_l : \mathtt{j}|_{n(\delta)} = \mathtt{i})$$

for the number of children of $\mathtt{i}$ at level $l > n(\delta)$ which lie in distinct columns. Then

$$\begin{aligned} N_\delta(\Pi(\Sigma)) &\approx \sum_{\mathtt{i}\in\Sigma_{n(\delta)}} N_\delta(\Pi([\mathtt{i}])) \\ &\approx \sum_{\mathtt{i}\in\Sigma_{n(\delta)}} M(\mathtt{i}, m(\delta)) \\ &\leq \sum_{\mathtt{i}\in\Sigma_{n(\delta)}} \#\pi\Sigma_{m(\delta)-n(\delta)} \qquad \text{(using shift invariance)} \\ &= \#\Sigma_{n(\delta)}\#\pi\Sigma_{m(\delta)-n(\delta)} \\ &\lesssim_\varepsilon \exp((h(\Sigma)+\varepsilon)n(\delta))\exp((h(\pi\Sigma)+\varepsilon)(m(\delta)-n(\delta))). \end{aligned}$$

In particular since $m(\delta) \approx \frac{-\log\delta}{\log \mathbf{m}}$ and $n(\delta) \approx \frac{-\log\delta}{\log \mathbf{n}}$ we have

$$\frac{\log N_\delta(\Pi(\Sigma))}{-\log\delta} \lesssim_\varepsilon \frac{(h(\Sigma)+\varepsilon)\frac{-\log\delta}{\log\mathbf{n}}}{-\log\delta} + \frac{(h(\pi\Sigma)+\varepsilon)(\frac{-\log\delta}{\log\mathbf{m}} - \frac{-\log\delta}{\log\mathbf{n}})}{-\log\delta},$$

therefore letting $\delta \to 0$ yields the desired upper bound since $\varepsilon > 0$ was chosen arbitrarily.

For the lower bounds, first observe that $\underline{\dim}_{\mathrm{B}}\Pi(\Sigma) \geq \dim_{\mathrm{H}}\Pi(\Sigma) \geq \dim_{\mathrm{H}}\pi\Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log\mathbf{m}}$, where the second inequality follows since the projection $\pi : [0,1]^2 \to [0,1]$ to the first coordinate is Lipschitz, and the final equality follows from Furstenberg's result expressing the Hausdorff dimension of a subshift in terms of entropy [11]. To see the second lower bound, let μ be a measure of maximal entropy for Σ projected onto $\Pi(\Sigma)$. Let $\varepsilon > 0$ be fixed and let $\delta > 0$. A ball of radius $\delta > 0$ centred in $\Pi(\Sigma)$ intersects at most $\lesssim 1$ many level $n(\delta)$ cylinders each with mass

$$\lesssim_\varepsilon \exp(-n(\delta)h(\Sigma)(1-\varepsilon)).$$

Therefore since $n(\delta) \approx \frac{-\log\delta}{\log\mathbf{n}}$ we deduce that $\underline{\dim}_{\mathrm{B}}\Pi(\Sigma) \geq \dim_{\mathrm{H}}\Pi(\Sigma) \geq \frac{h(\Sigma)}{\log\mathbf{n}}$ by the mass distribution principle, upon letting $\varepsilon \to 0$. □

4. Sofic $(\times\mathbf{m}, \times\mathbf{n})$-invariant sets

Fix $\mathbf{n} > \mathbf{m} \geq 2$ and $\mathcal{I} \subseteq \Delta_{\mathbf{m},\mathbf{n}}$. We say that a subshift Σ of the full shift on $\mathcal{I}$ is *sofic* if there exists a labelled directed graph G with a finite set of vertices V and edges E, where each edge $e \in E$ has a label $\ell(e) \in \mathcal{I}$, such that for each $\mathtt{i} \in \Sigma$, there exists an infinite path $e_1e_2\ldots$ $(e_i \in E)$ such that $\mathtt{i} = \ell(e_1)\ell(e_2)\ldots$. In this case we say that G *presents* Σ.

Given a presentation G of a sofic subshift Σ, there is a unique set of maximal irreducible subgraphs $\{G_1, \ldots, G_k\}$ of G, where by maximal we mean that no neighbouring vertices can be added to the subgraph while maintaining irreducibility. We call these the irreducible components of G. For each $1 \leq i \leq k$, define the subshift $\Sigma_{G_i} \subseteq \Sigma$ by

$$\Sigma_{G_i} = \{\ell(e_1)\ell(e_2)\ldots : e \in E_i\}$$

where E_i denotes the set of edges in G_i. Note that $\pi\Sigma$ is a subshift which is presented by the labelled, directed graph πG, which is constructed from G by projecting each label to its first coordinate. Its subgraphs πG_i are irreducible components of πG.

Construct a labelled directed graph H whose set of vertices is $\{1,\dots,k\}$ and where there is an edge labelled a from i to j if there is an edge labelled a in G from some vertex in G_i to some vertex in G_j. Note that H contains no cycles by definition of irreducible components. Then for each $1 \leq i \leq k$ we can define $\{i\}^+, \{i\}^- \subset \{1,\dots,k\}$ by

$$\begin{aligned} \{i\}^+ &:= \{1 \leq j \leq k \ : \ \text{there is a path in } H \text{ from } i \text{ to } j\} \\ \{i\}^- &:= \{1 \leq j \leq k \ : \ \text{there is a path in } H \text{ from } j \text{ to } i\} \end{aligned}$$

noting that the definition of $\{i\}^+$ is equivalent to that provided in the introduction. We say that an irreducible component G_i is a *source* if $\{i\}^+ = \{1,\dots,k\}$ and we say that G_i is a *sink* if $\{i\}^- = \{1,\dots,k\}$.

Before proving Theorem 2.1 we provide a couple of corollaries which follow from it. First, by exploiting the fact that $h(\Sigma) = \max\{h(\Sigma_{G_i})\}_{i=1}^k$ and $h(\pi\Sigma) = \max\{h(\pi\Sigma_{G_i})\}_{i=1}^k$, we can recover a simpler formula for the box dimension in the case that a source or sink has certain entropy maximising properties.

Corollary 4.1. *Let G be a presentation of Σ with irreducible components $\{G_1,\dots,G_k\}$. Suppose that either:*

(a) *for some $1 \leq i \leq k$, G_i is a source and $h(\Sigma_{G_i}) = h(\Sigma)$ or*
(b) *for some $1 \leq i \leq k$, G_i is a sink and $h(\pi\Sigma_{G_i}) = h(\pi\Sigma)$.*

Then

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}.$$

Secondly, by [14] we can describe which conditions guarantee (or preclude) equality of the Hausdorff and box dimensions.

Corollary 4.2. *The equality* $\dim_{\mathrm{H}} \Pi(\Sigma) = \dim_{\mathrm{B}} \Pi(\Sigma)$ *holds if and only if*

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \max_{1 \leq p \leq k} \left\{ \frac{h(\pi\Sigma_{G_p})}{\log \mathbf{m}} + \frac{h(\Sigma_{G_p}) - h(\pi\Sigma_{G_p})}{\log \mathbf{n}} \right\} \tag{5}$$

and *the measure of maximal entropy on Σ_{G_p} (for some p which maximises the expression on the right hand side of* (5)*) projects to the measure of maximal entropy on $\pi\Sigma_{G_p}$.*

In particular, if the maximum in (2) *is not obtained for a pair $i = j$, that is,*

$$\max_{1 \leq i \leq k} \left\{ \frac{h(\Sigma_{G_i})}{\log \mathbf{n}} + \max_{j \in \{i\}^+} h(\pi\Sigma_{G_j}) \left(\frac{1}{\log \mathbf{m}} - \frac{1}{\log \mathbf{n}} \right) \right\} > \max_{1 \leq p \leq k} \left\{ \frac{h(\pi\Sigma_{G_p})}{\log \mathbf{m}} + \frac{h(\Sigma_{G_p}) - h(\pi\Sigma_{G_p})}{\log \mathbf{n}} \right\}, \tag{6}$$

then $\dim_{\mathrm{H}} \Pi(\Sigma) < \dim_{\mathrm{B}} \Pi(\Sigma)$.

We will prove Corollaries 4.1 and 4.2 following the proof of Theorem 2.1 in Section 4.2.

4.1. **Example.** Before providing the proofs of the results of this section, we illustrate Theorem 2.1 with an example. Put $\mathbf{n} = 5$ and $\mathbf{m} = 3$. Let Σ be the subshift of finite type presented by the graph G in Figure 2.

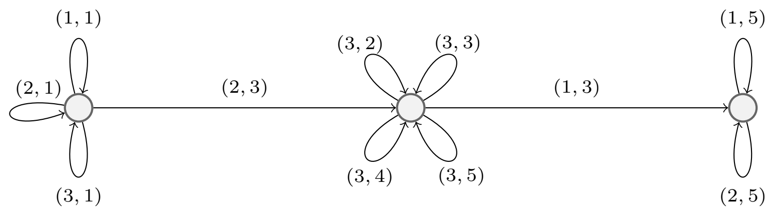

FIGURE 2. The graph G

G has three irreducible components G_1, G_2, G_3. Σ_{G_1} is the full shift on $\{(1,1),(2,1),(3,1)\}$ and $h(\Sigma_{G_1}) = h(\pi\Sigma_{G_1}) = \log 3$. Σ_{G_2} is the full shift on $\{(3,2),(3,3),(3,4),(3,5)\}$ and $h(\Sigma_{G_2}) = \log 4$, $h(\pi\Sigma_{G_2}) = 0$. Σ_{G_3} is the full shift on $\{(1,5),(2,5)\}$ and $h(\Sigma_{G_3}) = h(\pi\Sigma_{G_3}) = \log 2$.

By Theorem 2.1,

$$\begin{aligned}\dim_{\mathrm{B}} \Pi(\Sigma) &= \max\left\{\frac{\log 3}{\log 5} + \log 3\left(\frac{1}{\log 3} - \frac{1}{\log 5}\right), \frac{\log 4}{\log 5} + \log 2\left(\frac{1}{\log 3} - \frac{1}{\log 5}\right),\right. \\ &\qquad \left.\frac{\log 2}{\log 5} + \log 2\left(\frac{1}{\log 3} - \frac{1}{\log 5}\right)\right\} \\ &= \max\left\{1, \log 2\left(\frac{1}{\log 5} + \frac{1}{\log 3}\right)\right\} = \log 2\left(\frac{1}{\log 5} + \frac{1}{\log 3}\right).\end{aligned}$$

Note that

$$\dim_{\mathrm{B}} \Pi(\Sigma) < 1 + \frac{\log 4 - \log 3}{\log 5} = \frac{h(\pi\Sigma)}{\log 3} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log 5}.$$

Also note that

$$\begin{aligned}\dim_{\mathrm{B}} \Pi(\Sigma) &> \max\left\{\frac{\log 2}{\log 3} + \frac{\log 3 - \log 2}{\log 5}, \frac{\log 4}{\log 5}, \frac{\log 2}{\log 3}\right\} \\ &= \max_{1 \leq p \leq k}\left\{\frac{h(\pi\Sigma_{G_p})}{\log 3} + \frac{h(\Sigma_{G_p}) - h(\pi\Sigma_{G_p})}{\log 5}\right\}.\end{aligned}$$

4.2. **Proofs.** We begin by proving Theorem 2.1. Fix a presentation G of Σ and let $1 \leq i \leq k$ and $j \in \{i\}^+$ be parameters which achieve the maximum in (2). Roughly speaking, we show that the box dimension is exhausted by covering all regions $\Pi([\mathtt{i}])$ where $\mathtt{i} \in \Sigma_{m(\delta)}$ labels a path in G which stays in the irreducible component G_i for roughly $n(\delta)$ time steps before travelling to the irreducible component G_j and staying inside it until time $m(\delta)$.

Given a vertex v in G, let Σ_n^{v+} denote all strings in Σ_n which label a path beginning at v and Σ_n^{v-} denote all strings in Σ_n which label a path ending at v. For the lower bound we will require the following standard result which relates the entropy of an irreducible sofic subshift to paths in G. In the following $\Sigma_n^{v\pm}$ denotes either Σ_n^{v+} or Σ_n^{v-}.

Lemma 4.3. *Let Σ be an irreducible sofic subshift with irreducible presentation G and $v \in G$ be an arbitrary vertex. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log \#\Sigma_n^{v\pm} = h(\Sigma). \tag{7}$$

Proof. Since $\#\Sigma_n^{v\pm} \leq \#\Sigma_n \leq \sum_{v\in V} \#\Sigma_n^{v\pm}$ we have that $\lim_{n\to\infty} \frac{1}{n}\log\left(\max_{v\in V}\#\Sigma_n^{v\pm}\right)$ exists and equals the entropy $h(\Sigma)$. By irreducibility of G, there exists $M\in\mathbb{N}$ such that for $n>M$, $\#\Sigma_n^{v\pm} \geq \max_{w\in V}\#\Sigma_{n-M}^{w\pm}$. Hence

$$\liminf_{n\to\infty}\frac{1}{n}\log\#\Sigma_n^{v\pm} \geq \lim_{n\to\infty}\frac{1}{n}\log\left(\max_{v\in V}\#\Sigma_n^{v\pm}\right) = h(\Sigma),$$

completing the proof of (7). □

of lower bound in Theorem 2.1. Fix $\varepsilon>0$, $\delta>0$. Let $1\leq i\leq k$ and $j\in\{i\}^+$ be the indices that maximise the expression in (2). Let v be a vertex in G_i and w be a vertex in G_j. Since $j\in\{i\}^+$, there exists a path of some length N in G from v to w which is labelled by $\mathtt{k}\in\Sigma_N$. We may assume δ is small enough to ensure $n(\delta)>N$. Given $\mathtt{i}\in\Sigma_{G_i,n(\delta)-N}^{v-}$,

$$\bigcup_{\mathtt{j}\in\Sigma_{G_j,m(\delta)-n(\delta)}^{w+}}\Pi([\mathtt{ikj}])\subseteq\Pi([\mathtt{i}]).$$

Therefore,

$$N_\delta(\Pi[\mathtt{i}]) \gtrsim N_\delta\left(\bigcup_{\mathtt{j}\in\Sigma_{G_j,m(\delta)-n(\delta)}^{w+}}\Pi([\mathtt{ikj}])\right) \approx \#\pi\Sigma_{G_j,m(\delta)-n(\delta)}^{w+}.$$

Hence

$$N_\delta(\Pi(\Sigma)) \gtrsim \sum_{\mathtt{i}\in\Sigma_{G_i,n(\delta)-N}^{v-}} N_\delta([\mathtt{i}]) \gtrsim \#\Sigma_{G_i,n(\delta)-N}^{v-}\cdot\#\pi\Sigma_{G_j,m(\delta)-n(\delta)}^{w+}.$$

Therefore by (7),

$$N_\delta(\Pi(\Sigma)) \gtrsim_\varepsilon \exp\left((h(\Sigma_{G_i})-\varepsilon)(n(\delta)-N)\right)\exp\left((h(\pi\Sigma_{G_j})-\varepsilon)(m(\delta)-n(\delta))\right)$$

and by letting $\delta\to 0$ we obtain the desired lower bound since $\varepsilon>0$ was arbitrary. □

For the upper bound we will require the standard result that the entropy of a sofic subshift equals the maximum entropy of its irreducible subshifts.

Lemma 4.4. *Let Σ be a sofic subshift presented by a graph G which has irreducible components $G_1,\ldots,G_k$. Then $h(\Sigma)=\max_{1\leq i\leq k}h(\Sigma_{G_i})$.*

Proof. It is only necessary to prove the upper bound which follows by bounding $\#\Sigma_n$ above by

$$\sum_{1\leq m_1,\ldots,m_l\leq k}\ \sum_{n_{m_1}+\cdots+n_{m_l}=n}\#\Sigma_{G_{m_1},n_{m_1}}\cdot\#\Sigma_{G_{m_2},n_{m_2}}\cdots\#\Sigma_{G_{m_l},n_{m_l}},$$

where $\Sigma_{G_i,n}$ denotes all distinct strings of length n that appear in the subgraph G_i, and bounding $\#\Sigma_{G_{m_i},n_{m_i}}$ in terms of $h(\Sigma_{G_{m_i}})$. □

Let $\Sigma_n^{G_i+}$ denote all words in Σ_n which label paths in G that start at any vertex in G_i, and $\Sigma_n^{G_i-}$ denote all words in Σ_n which label paths in G that end at any vertex in G_i.

of upper bound in Theorem 2.1. Fix $\varepsilon > 0$, $\delta > 0$. Fix any $1 \leq i \leq k$ and $\mathtt{i} \in \Sigma_n^{G_i-}$. Writing $M(\mathtt{i}, l) = \#\pi(\mathtt{j} \in \Sigma_l : \mathtt{j}|_{n(\delta)} = \mathtt{i})$ and noting that by shift invariance we have $M(\mathtt{i}, m(\delta)) \leq \#\pi\Sigma_{m(\delta)-n(\delta)}^{G_i+}$ it follows that

$$N_\delta(\Pi(\Sigma)) \lesssim \sum_{i=1}^{k} \sum_{\mathtt{i} \in \Sigma_{n(\delta)}^{G_i-}} N_\delta(\Pi([\mathtt{i}])) \approx \sum_{i=1}^{k} \sum_{\mathtt{i} \in \Sigma_{n(\delta)}^{G_i-}} M(\mathtt{i}, m(\delta)) \lesssim \#\Sigma_{n(\delta)}^{G_i-} \#\pi\Sigma_{m(\delta)-n(\delta)}^{G_i+}.$$

Note that any path that ends at a vertex in G_i is contained in the minimal subgraph E of G which contains the irreducible components $\{G_j\}_{j\in\{i\}^-}$ and all edges between these components. Similarly, any path in G that begins at a vertex in G_i is contained in the minimal subgraph F of G where F contains the irreducible components $\{G_j\}_{j\in\{i\}^+}$, and all edges between these components. By Lemma 4.4,

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\Sigma_n^{G_i-} \leq h(\Sigma_E) = \max_{j\in\{i\}^-} h(\Sigma_{G_j})$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\pi\Sigma_n^{G_i+} \leq h(\pi\Sigma_F) = \max_{j\in\{i\}^+} h(\pi\Sigma_{G_j}).$$

Therefore,

$$N_\delta(\Pi(\Sigma)) \lesssim_\varepsilon \exp\left(n(\delta)(\max_{j\in\{i\}^-} h(\Sigma_{G_j}) + \varepsilon)\right) \exp\left((m(\delta) - n(\delta))(\max_{j\in\{i\}^+} h(\pi\Sigma_{G_j}) + \varepsilon)\right),$$

and by letting $\delta \to 0$ we obtain the desired upper bound since $\varepsilon > 0$ was arbitrary. □

of Corollary 4.1. First, to see (a), let G_i be the source. By assumption $h(\Sigma_{G_i}) = h(\Sigma)$. By Lemma 4.4 there exists $1 \leq j \leq k$ such that $h(\pi\Sigma_{G_j}) = h(\pi\Sigma)$. Moreover $j \in \{i\}^+$ by definition of a source. Hence by (2),

$$\dim_{\mathrm{B}} \Pi(\Sigma) \geq \frac{h(\Sigma_{G_i})}{\log \mathbf{n}} + h(\pi\Sigma_{G_j}) \left(\frac{1}{\log \mathbf{m}} - \frac{1}{\log \mathbf{n}}\right) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}.$$

On the other hand, the upper bound follows from Proposition 3.1, completing the proof of (a).

Similarly, to see (b), let G_j be the sink. By assumption $h(\pi\Sigma_{G_j}) = h(\pi\Sigma)$. Also, by Lemma 4.4 there exists $1 \leq i \leq k$ such that $h(\Sigma_{G_i}) = h(\Sigma)$. Moreover $i \in \{j\}^-$ by definition of a sink. Hence by (2),

$$\dim_{\mathrm{B}} \Pi(\Sigma) \geq \frac{h(\Sigma_{G_i})}{\log \mathbf{n}} + h(\pi\Sigma_{G_j}) \left(\frac{1}{\log \mathbf{m}} - \frac{1}{\log \mathbf{n}}\right) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}.$$

The upper bound follows from Proposition 3.1, completing the proof of (b). □

of Corollary 4.2 . First we recall that by [14], any ergodic invariant measure μ on Σ satisfies the Ledrappier-Young formula:

$$\dim_{\mathrm{H}} \mu = \frac{h(\pi\mu)}{\log \mathbf{m}} + \frac{h(\mu) - h(\pi\mu)}{\log \mathbf{n}}, \tag{8}$$

where $\dim_{\mathrm{H}} \mu$ denotes the Hausdorff dimension of μ, $h(\mu)$ denotes the measure-theoretic entropy of μ with respect to the left shift map on Σ and $h(\pi\mu)$ denotes the measure-theoretic entropy of the pushforward measure $\pi\mu$ with respect to the left shift on $\pi\Sigma$.

First, suppose the equality (5) holds and let $1 \leq p \leq k$ be an index that maximises the right hand side of (5). Let μ be the ergodic invariant measure which maximises entropy on Σ_{G_p}, and let us assume for the time being that μ projects to the measure which maximises entropy on $\pi\Sigma_{G_p}$. Then by (8),

$$\dim_{\mathrm{H}} \Pi(\Sigma) \geq \dim_{\mathrm{H}} \mu = \frac{h(\pi\mu)}{\log \mathbf{m}} + \frac{h(\mu) - h(\pi\mu)}{\log \mathbf{n}} = \dim_{\mathrm{B}} \Pi(\Sigma)$$

by (5).

For the converse, we assume that $\dim_{\mathrm{B}} \Pi(\Sigma) = \dim_{\mathrm{H}} \Pi(\Sigma)$. By [14] there exists an ergodic invariant measure μ of maximal Hausdorff dimension. [1] Since μ is ergodic, its support must be contained in Σ_{G_i} for some irreducible component G_i of G. Therefore, using (8) we obtain

$$\begin{aligned} \dim_{\mathrm{H}} \Pi(\Sigma) = \dim_{\mathrm{H}} \mu &= \frac{h(\pi\mu)}{\log \mathbf{m}} + \frac{h(\mu) - h(\pi\mu)}{\log \mathbf{n}} \\ &\leq \max_{1 \leq p \leq k} \left\{ \frac{h(\pi\Sigma_{G_p})}{\log \mathbf{m}} + \frac{h(\Sigma_{G_p}) - h(\pi\Sigma_{G_p})}{\log \mathbf{n}} \right\} \\ &\leq \dim_{\mathrm{B}} \Pi(\Sigma). \end{aligned}$$

Now, if (5) does not hold, then the second inequality above is strict and thus we get a contradiction. On the other hand, if (5) holds but the measure of maximal entropy μ_p on Σ_{G_p} does *not* project to the measure of maximal entropy on $\pi\Sigma_{G_p}$ for any $1 \leq p \leq k$ that maximises the right hand side of (5), then the first inequality above is strict yielding a contradiction and completing the proof. □

5. Coded subshifts

Fix $\mathbf{n} > \mathbf{m} \geq 2$ and $\mathcal{I} \subseteq \Delta_{\mathbf{m},\mathbf{n}}$. Let $\mathcal{C} = \{c_i\}_{i=1}^{\infty}$ be a countable family of words on the alphabet $\mathcal{I}$. We call $\mathcal{C}$ the generators. Let $\mathcal{C}_n := \mathcal{C} \cap \mathcal{I}^n$. Define

$$B := \{s c_{i_1} c_{i_2} \ldots \; : \; c_{i_j} \in \mathcal{C}, \; s \text{ is a suffix of a word in } \mathcal{C}\}.$$

Note that B is σ-invariant but may not be compact. We define $\Sigma = \overline{B}$ and say that Σ is a coded subshift. Note that $\pi\Sigma$ is also a coded subshift which is generated by $\pi\mathcal{C}$. Recall that we say that the coded subshift Σ satisfies *unique decomposition with respect to* $\mathcal{C}$ if no finite word in Σ^* can be written by concatenating generators in $\mathcal{C}$ in distinct ways. Note that if Σ satisfies unique decomposition with respect to $\mathcal{C}$, this does not necessarily mean that $\pi\Sigma$ satisfies unique decomposition with respect to $\pi\mathcal{C}$, although it may satisfy unique decomposition with respect to a different generating set (for instance if Σ satisfies unique decomposition with respect to $\mathcal{C}$ and $\{(1,2),(1,3),(1,4)(1,4)\} \subset \mathcal{C}$ then since $\{1, 11\} \subset \pi\mathcal{C}$, $\pi\Sigma$ does not satisfy unique decomposition with respect to $\pi\mathcal{C}$).

Construct a directed labelled graph by fixing a vertex v and, for each $i \in \mathbb{N}$, adding a path which begins and ends at v which is labelled by the generator c_i, such that the paths do not intersect each other apart from at the start and end points. We call these generating loops.

[1] The statement of [14, Theorem 1.1] does not make explicit that a measure of maximal Hausdorff dimension can be taken to be ergodic, however this is clear from its proof.

We say that G presents the coded subshift Σ.[2] Similarly, construct the graph πG from G by projecting each label to its first coordinate and removing any generating loop which bears the same sequence of labels as another generating loop (so that each generating loop is labelled uniquely by a generator in $\pi\mathcal{C}$). Then πG presents the coded subshift $\pi\Sigma$.

Let $\mathcal{G}_n$ denote all words in Σ_n which label a path in G that begins and ends at the vertex v, and $\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}_n$. In particular, $\mathcal{G}$ consists of concatenations of generators. Analogously, $\pi\mathcal{G}$ are all words in $\pi\Sigma$ which label a path in πG that begins and ends at the vertex v. We denote

$$h := \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{G}_n \qquad \text{and} \qquad h_\pi := \limsup_{n\to\infty} \frac{1}{n} \log \#\pi\mathcal{G}_n.$$

Note that the lim sup is necessary in the definitions above, for instance consider a coded subshift generated by a set of generators which all have even length. Also, note that these definitions are equivalent to those recorded in the introduction.

We begin by proving Theorem 2.3, namely that if $h = h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$ then $\dim_{\mathrm{B}} \Pi(\Sigma)$ equals its trivial upper bound.

of Theorem 2.3. By Proposition 3.1 it suffices to prove the lower bound. Fix $\varepsilon > 0$. Since $h = h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$ we can choose $m_\varepsilon, n_\varepsilon \in \mathbb{N}$ such that

$$\#\mathcal{G}_{m_\varepsilon} \geq e^{m_\varepsilon(h(\Sigma)-\varepsilon)} \quad \text{and} \quad \#\pi\mathcal{G}_{n_\varepsilon} \geq e^{n_\varepsilon(h(\pi\Sigma)-\varepsilon)}.$$

In particular, for all $k \in \mathbb{N}$,

$$\#\mathcal{G}_{km_\varepsilon} \geq e^{km_\varepsilon(h(\Sigma)-\varepsilon)} \quad \text{and} \quad \#\pi\mathcal{G}_{kn_\varepsilon} \geq e^{kn_\varepsilon(h(\pi\Sigma)-\varepsilon)}$$

since $\#\mathcal{G}_{kn} \geq (\#\mathcal{G}_n)^k$ and $\#\pi\mathcal{G}_{kn} \geq (\#\pi\mathcal{G}_n)^k$. Let $\delta > 0$ be sufficiently small that $n(\delta) \geq 2m_\varepsilon$ and $m(\delta) - n(\delta) \geq 2n_\varepsilon$. Hence we can find $n(\delta) - m_\varepsilon < k'(\delta) \leq n(\delta)$ which is a multiple of m_ε, that is,

$$\#\mathcal{G}_{k'(\delta)} \geq e^{k'(\delta)(h(\Sigma)-\varepsilon)}.$$

Similarly we can find $m(\delta) - n_\varepsilon < l'(\delta) \leq m(\delta)$ such that $l'(\delta) - k'(\delta)$ is a multiple of n_ε, that is,

$$\#\pi\mathcal{G}_{l'(\delta)-k'(\delta)} \geq e^{(l'(\delta)-k'(\delta))(h(\pi\Sigma)-\varepsilon)}.$$

Denoting $M(\mathtt{i}, l) = \#\pi(\mathtt{j} \in \Sigma_l : \mathtt{j}|_{k'(\delta)} = \mathtt{i})$, we have

$$\begin{aligned} N_\delta(\Pi(\Sigma)) \gtrsim \sum_{\mathtt{i}\in\mathcal{G}_{k'(\delta)}} N_\delta(\Pi([\mathtt{i}])) \quad &\gtrsim \quad \sum_{\mathtt{i}\in\mathcal{G}_{k'(\delta)}} M(\mathtt{i}, l'(\delta)) \\ &\geq \quad \#\mathcal{G}_{k'(\delta)} \#\pi\mathcal{G}_{l'(\delta)-k'(\delta)} \\ &\geq \quad e^{k'(\delta)(h(\Sigma)-\varepsilon)} e^{(l'(\delta)-k'(\delta))(h(\pi\Sigma)-\varepsilon)} \\ &\gtrsim_\varepsilon \quad e^{n(\delta)(h(\Sigma)-\varepsilon)} e^{(m(\delta)-n(\delta))(h(\pi\Sigma)-\varepsilon)}. \end{aligned}$$

The lower bound follows since ε was chosen arbitrarily. □

[2]Note that this notion of the presentation of a coded subshift differs from the notion of the presentation of a sofic subshift. If Σ is sofic then all infinite sequences in Σ label an infinite path in its presentation, whereas if Σ is coded then this need not be the case (i.e. if $\Sigma \setminus B \neq \emptyset$).

Conversely, examples can be constructed where either $h < h(\Sigma)$ or $h_\pi < h(\pi\Sigma)$ and the conclusion of Theorem 2.3 does *not* hold, that is, the dimension $\dim_B \Pi(\Sigma)$ drops from the trivial upper bound. In particular, in §5.2.3 we will construct an example where $h < h(\Sigma)$ and $\dim_B \Pi(\Sigma)$ equals the trivial lower bound $\max\left\{\frac{h(\Sigma)}{\log \mathbf{n}}, \frac{h(\pi\Sigma)}{\log \mathbf{m}}\right\}$ thereby settling Theorem 2.2.

The drawback of Theorem 2.3 is that generally it is not straightforward to verify the equalities $h = h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$. However, under the assumption of unique decomposition of Σ and $\pi\Sigma$ we can provide more checkable conditions that guarantee the box dimension $\dim_B \Pi(\Sigma)$ to equal its trivial upper bound (Theorem 2.4).

5.1. **Coded subshifts with unique decomposition.** Throughout this short section we will assume that Σ is a coded subshift with unique decomposition with respect to $\mathcal{C}$ and that the coded subshift $\pi\Sigma$ satisfies unique decomposition with respect to $\pi\mathcal{C}$. Let G and πG be the presentations of Σ and $\pi\Sigma$ as detailed in the previous section. Let $p_G(v, n)$ denote the number of paths of length n in G which begin and end at v and $p_{\pi G}(v, n)$ denote the number of paths of length n in πG which begin and end at v and write

$$h_G := \limsup_{n\to\infty} \frac{1}{n} \log p_G(v, n) \qquad \text{and} \qquad h_{\pi G} := \limsup_{n\to\infty} \frac{1}{n} \log \# p_{\pi G}(v, n).$$

In particular, h_G is the Gurevic entropy of G and $h_{\pi G}$ is the Gurevic entropy of πG, noting that the limsups are actually independent of the choice of vertex. Since Σ and $\pi\Sigma$ satisfy unique decomposition with respect to $\mathcal{C}$ and $\pi\mathcal{C}$ respectively, we have $h = h_G$ and $h_\pi = h_{\pi G}$. This will enable us to apply techniques from the theory of countable Markov shifts.

Recall from the introduction the functions $f, f_\pi : [0, \infty) \to (0, \infty]$ which we defined by

$$f(x) = \sum_{n=1}^{\infty} \#\mathcal{C}_n e^{-nx} \qquad \text{and} \qquad f_\pi(x) = \sum_{n=1}^{\infty} \#\pi\mathcal{C}_n e^{-nx}. \tag{9}$$

We can apply the classical work of Vere-Jones [19] to deduce behaviour of f and f_π at h and h_π.

Lemma 5.1. *Let Σ and $\pi\Sigma$ be coded subshifts with unique decomposition with respect to generating sets $\mathcal{C}$ and $\pi\mathcal{C}$ respectively. Then*

$$f(h) \le 1 \qquad \text{and} \qquad f_\pi(h_\pi) \le 1. \tag{10}$$

Proof. Let $q_G(v, n)$ denote the number of generating loops of length n in G. Let $q_{\pi G}(v, n)$ denote the number of generating loops of length n in πG. In particular, $q_G(v, n) = \#\mathcal{C}_n$ and $q_{\pi G}(v, n) = \#\pi\mathcal{C}_n$, so $f(x) = \sum_{n=1}^{\infty} q_G(v, n)e^{-nx}$ and $f_\pi(x) = \sum_{n=1}^{\infty} q_{\pi G}(v, n)e^{-nx}$. By using the recurrence relation $p(v, n) = \sum_{k=1}^{n} p_G(v, n-k)q_G(v, k)$ and an application of a renewal theorem, Vere-Jones [19, Lemma 2] showed that $\sum_{n=1}^{\infty} q_G(v, n)e^{-nh_G} \le 1$, and analogously $\sum_{n=1}^{\infty} q_{\pi G}(v, n)e^{-nh_{\pi G}} \le 1$. This implies the result since $h = h_G$ and $h_\pi = h_{\pi G}$ by unique decomposition. □

Next recall the definitions from the introduction

$$\ell := \limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{L}_n \qquad \text{and} \qquad \ell_\pi := \limsup_{n\to\infty} \frac{1}{n} \log \#\pi\mathcal{L}_n$$

where $\mathcal{L}_n$ and $\pi\mathcal{L}_n$ denote words of length n which appear at the beginning or end of generators in $\mathcal{C}$ and $\pi\mathcal{C}$ respectively. In [5] it was shown that $\ell < h$ implies existence of a measure of maximal entropy for the coded subshift Σ. The behaviour of f at a quantity related to ℓ was used in [18] to characterise coded subshifts in terms of the properties of their measures of maximal entropy.

To prove Theorem 2.4 we will show that $f(\ell) > 1$ implies $\ell < h$ by using (10) and the fact that f is strictly decreasing, and then by naturally decomposing words in Σ into concatenations of generators and subwords of generators we will deduce that this implies $h = h(\Sigma)$ (respectively $h_\pi = h(\pi\Sigma)$).

Lemma 5.2. *Suppose Σ is a coded subshift which satisfies unique decomposition with respect to a generating set $\mathcal{C}$ and $f(\ell) > 1$. Then*

$$\limsup_{n\to\infty} \frac{1}{n} \log \#\mathcal{G}_n = h(\Sigma).$$

Proof. Assume that $f(\ell) > 1$. Since $h_G = h$ by unique decomposition it follows that $f(h) = f(h_G) \leq 1$ by (10) and therefore since f is strictly decreasing we have $\ell < h_G = h \leq h(\Sigma)$ (the second inequality follows trivially from the definition of $\mathcal{G}$). We will show that $\ell < h(\Sigma)$ implies that $h = h(\Sigma)$, using arguments similar to those contained in [5, §5.1].

Let $\varepsilon > 0$ be sufficiently small that $\ell - h(\Sigma) + 2\varepsilon < 0$.

Then

$$\begin{aligned} \#\Sigma_n &\leq \sum_{i+j+k=n} \#\mathcal{L}_i \#\mathcal{G}_j \#\mathcal{L}_k \\ &\lesssim_\varepsilon \sum_{i+j+k=n} e^{(i+k)(\ell+\varepsilon)} \#\mathcal{G}_j \\ &= \sum_{j=0}^{n} (n-j) e^{(n-j)(\ell+\varepsilon)} \#\mathcal{G}_j. \end{aligned} \tag{11}$$

Hence

$$\sum_{j=0}^{n} e^{(n-j)(\ell-h(\Sigma)+2\varepsilon)}(n-j)\frac{\#\mathcal{G}_j}{\#\Sigma_j} \gtrsim_\varepsilon \sum_{j=0}^{n} e^{(n-j)(\ell+\varepsilon)}(n-j)\frac{\#\mathcal{G}_j}{\#\Sigma_j}\frac{\#\Sigma_j}{\#\Sigma_n} \gtrsim_\varepsilon 1.$$

In particular, there exists $c_\varepsilon > 0$ such that

$$\sum_{j=0}^{n} e^{(n-j)(\ell-h(\Sigma)+2\varepsilon)}(n-j)\frac{\#\mathcal{G}_j}{\#\Sigma_j} \geq c_\varepsilon. \tag{12}$$

Since $\ell - h(\Sigma) + 2\varepsilon < 0$ we can choose $N \in \mathbb{N}$ sufficiently large that

$$\begin{aligned} \sum_{j=0}^{n-N} e^{(n-j)(\ell-h(\Sigma)+2\varepsilon)}(n-j)\frac{\#\mathcal{G}_j}{\#\Sigma_j} &\leq \sum_{j=0}^{n-N} e^{(n-j)(\ell-h(\Sigma)+2\varepsilon)}(n-j) \\ &\leq \sum_{m\geq N} e^{m(\ell-h(\Sigma)+2\varepsilon)} m \leq \frac{c_\varepsilon}{2}. \end{aligned}$$

Hence by (12)

$$\sum_{j=n-N+1}^{n} e^{(n-j)(\ell-h(\Sigma)+2\varepsilon)}(n-j)\frac{\#\mathcal{G}_j}{\#\Sigma_j} \geq \frac{c_\varepsilon}{2}. \tag{13}$$

If we let C be a uniform upper bound on $e^{m(\ell-h(\Sigma)+2\varepsilon)}m$ (for $m \geq 0$), we can deduce from (13) that

$$\sum_{j=n-N+1}^{n} \frac{\#\mathcal{G}_j}{\#\Sigma_j} \geq \frac{c_\varepsilon}{2C}$$

hence for all $n \geq N+1$ we have $\#\mathcal{G}_j \geq \frac{c_\varepsilon}{2CN}\#\Sigma_j$ for some $n-N+1 \leq j \leq n$. This implies that $h = \limsup_{n\to\infty} \frac{1}{n}\log\#\mathcal{G}_n = h(\Sigma)$. □

Clearly by combining Lemma 5.2 with Theorem 2.3 we establish Theorem 2.4: that if Σ and $\pi\Sigma$ satisfy unique decomposition with respect to $\mathcal{C}$ and $\pi\mathcal{C}$ and we have that $f(\ell) > 1$ and $f_\pi(\ell_\pi) > 1$ then

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}. \tag{14}$$

Hence to establish (14) for uniquely decomposing coded subshifts Σ and $\pi\Sigma$, it is sufficient to calculate $f(\ell)$ and $f_\pi(\ell_\pi)$, which solely depend on $\#\mathcal{C}_n, \#\pi\mathcal{C}_n, \#\mathcal{L}_n$ and $\#\pi\mathcal{L}_n$ which are often easy to compute. We demonstrate this with some examples in the next section.

5.2. **Examples.** In this section, we illustrate Theorems 2.2, 2.3 and 2.4 with some examples. First, in §5.2.1 we describe how Theorem 2.3 can be applied to β-shifts. In §5.2.2 we apply Theorem 2.4 to (generalised) S-gap shifts. Finally in §5.2.3 we construct an example of a coded subshift Σ where $h < h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$ and

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \max\left\{\frac{h(\Sigma)}{\log \mathbf{n}}, \frac{h(\pi\Sigma)}{\log \mathbf{m}}\right\} < \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}$$

thereby proving Theorem 2.2.

5.2.1. *β-shifts.* Fix $\mathbf{n} > \mathbf{m}$ and $\mathcal{I} \subset \Delta_{\mathbf{m},\mathbf{n}}$. We begin by describing a subshift on the set of digits $\mathcal{I}$ which is conjugate to the β-shift on the usual digit set $\{0, \ldots, \lfloor\beta\rfloor\}$, for more details see [3] or [5] and references therein.

Fix a bijection $\mathcal{O} : \{0, \ldots, |\mathcal{I}|-1\} \to \mathcal{I}$ which will determine an ordering on the elements in $\mathcal{I}$. We extend $\mathcal{O}$ to finite and infinite words with digits in $\{0, \ldots, |\mathcal{I}|-1\}$ by $\mathcal{O}(i_1 i_1 \ldots) = \mathcal{O}(i_1)\mathcal{O}(i_2)\ldots$. Fix $|\mathcal{I}| < \beta < |\mathcal{I}|+1$ and let $(b_n)_{n\in\mathbb{N}}$ be the greedy β-expansion of 1, meaning the lexicographically maximal solution to

$$\sum_{n=1}^{\infty} b_n \beta^{-n} = 1.$$

We define

$$\Sigma = \left\{\mathcal{O}((x_n)_{n\in\mathbb{N}}) \;:\; (x_n)_{n\in\mathbb{N}} \in \{0, \ldots, |\mathcal{I}|-1\}^{\mathbb{N}} \text{ s.t. } \sigma^k((x_n)_{n\in\mathbb{N}}) \preceq (b_n)_{n\in\mathbb{N}}\ \forall k \in \mathbb{N}\right\}$$

where $\preceq$ stands for the lexicographic order. In particular, Σ is conjugated by $\mathcal{O}$ to the β-shift on the set of digits $\{0, \dots, \lfloor \beta \rfloor\}$. Therefore it is known [3] that Σ is a coded subshift where the set of generators is given by

$$\mathcal{C} = \bigcup_{n \geq 1 : b_n > 0} \{\mathcal{O}(b_1 \dots b_{n-1} 0), \dots, \mathcal{O}(b_1 \dots b_{n-1}(b_n - 1))\}.$$

Note that any word in $\mathtt{i} \in \Sigma^*$ can be written $\mathtt{i} = c_1 \dots c_k w$ where $c_i \in \mathcal{C}$ and w is a word that appears at the beginning of a generator in $\mathcal{C}$. Hence

$$\#\Sigma_n \leq \sum_{k=1}^{n} \#\mathcal{G}_{n-k} \tag{15}$$

since for each $k \in \mathbb{N}$, $\mathcal{O}(b_1 \dots b_k)$ is the unique word of length k that appears at the beginning of a generator in $\mathcal{C}$. Similarly, we have

$$\#\pi\Sigma_n \leq \sum_{k=1}^{n} \#\pi\mathcal{G}_{n-k}. \tag{16}$$

Using (15) and (16) it is easy to adapt the set of inequalities (11) and the estimates that follow it to deduce that $h = h(\Sigma)$ and $h_\pi = h(\pi\Sigma)$. In particular $\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}$ by Theorem 2.3.

5.2.2. *Generalised S-gap shifts.* We begin by considering the following natural generalisation of the S-gap shifts [5, 16]. Fix any $\mathbf{n} > \mathbf{m}$, $\mathcal{I} \subset \Delta_{\mathbf{m},\mathbf{n}}$ and $(i_0, j_0) \in \mathcal{I}$ such that $\pi(\mathcal{I} \setminus \{(i_0, j_0)\}) \cap \{i_0\} = \emptyset$. Fix a countable set $S \subset \mathbb{N}$. Put

$$\mathcal{C} = \{w\,(i_0, j_0) \;:\; w \in (\mathcal{I} \setminus \{(i_0, j_0)\})^s,\ s \in S\}.$$

We consider the coded subshift Σ generated by $\mathcal{C}$. Under the assumptions on $\mathcal{I}$, both Σ and $\pi\Sigma$ satisfy unique decomposition with respect to $\mathcal{C}$ and $\pi\mathcal{C}$ respectively. The classical S-gap shifts correspond to the case that $\#\mathcal{I} = 2$, however since analysis of the box dimension of $\Pi(\Sigma)$ is trivial for subshifts on 2 symbols we are primarily interested in the case that $\#\mathcal{I} \geq 3$.

Observe that

$$\#\mathcal{C}_n = \begin{cases} (\#\mathcal{I} - 1)^{n-1} & n \in S \\ 0 & n \notin S. \end{cases}$$

Also clearly $\ell = \log(\#\mathcal{I} - 1)$. Therefore,

$$f(\ell) = \frac{(\#\mathcal{I} - 1)^{n-1}}{(\#\mathcal{I} - 1)^n} = \sum_{n \in S} \frac{1}{\#\mathcal{I} - 1} = \infty > 1.$$

Similarly we can calculate that

$$\#\pi\mathcal{C}_n = \begin{cases} (\#\pi\mathcal{I} - 1)^{n-1} & n \in S \\ 0 & n \notin S. \end{cases}$$

and $\ell_\pi = \log(\#\pi\mathcal{I} - 1)$, so $f_\pi(\ell_\pi) = \infty$. In particular, Theorem 2.4 is applicable and we deduce that $\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{h(\pi\Sigma)}{\log \mathbf{m}} + \frac{h(\Sigma) - h(\pi\Sigma)}{\log \mathbf{n}}$.

5.2.3. *Example whose box dimension equals the trivial lower bound.* Fix $\mathbf{m} \geq 2$ and $\mathbf{n} \geq \max\{\mathbf{m}+1, 5\}$. Let

$$\mathcal{I} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1)\}$$

and $\Omega = \{(1,3),(1,4),(1,5)\}$. Put

$$\mathcal{C} = \{(1,1)\} \cup \{(2,1)\} \cup \{w\,(1,2)^m \;:\; w \in \Omega^*,\ m \geq 2^{|w|}\}$$

where $(1,2)^m$ denotes the concatenation of m instances of the digit $(1,2)$, and let Σ be the coded subshift generated by $\mathcal{C}$. Note that $\pi\Sigma = \{1,2\}^{\mathbb{N}}$. It is easy to see that $h_\pi = h(\pi\Sigma) = \log 2$, and we will show that $h \leq \log 2 < \log 3 = h(\Sigma)$, see Lemma 5.4. The graph G (see Figure 3) presents Σ. We will be interested in words which label a path that begins and ends at the vertex v.

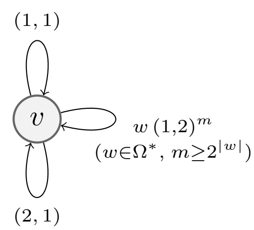

FIGURE 3. The graph G

Definition 5.3. *For each $n \in \mathbb{N}$ let I_n denote all strings in Σ_n which can be presented by a path on G ending at v. Let $I = \bigcup_{n=1}^{\infty} I_n$.*

Lemma 5.4. *We have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \# I_n \leq \log 2.$$

Proof. Suppose a word in I_n has c digits from Ω and a digits from $\{(1,1),(2,1)\}$.

By definition of the code words $\mathcal{C}$, we must have $a + c + 2^c \leq n$ therefore $c \leq \log_2(n-a)$. Now, assuming $c > 0$, for each $1 \leq k \leq c$ there are $\binom{c-1}{k-1}$ ways to divide the c digits into k groups.

Following each of the k blocks of digits from Ω there must be a string of $(1,2)$'s whose length is equal to the exponential of the length of that block. That leaves $n - c - 2^c - a$ extra $(1,2)$'s to be distributed. These can be placed after any of the k blocks of $(1,2)$'s, or directly before the first block of digits from Ω. This gives $\binom{n-c-2^c-a+k}{k}$ different ways in which we can distribute the excess $(1,2)$'s.

Finally, we can distribute the a digits from $\{(1,1),(2,1)\}$ directly preceding any of the k blocks of $(1,2)$'s or at the end of the word. This gives $\binom{a+k}{k}$ possibilities for distributing the a digits from $\{(1,1),(2,1)\}$.

Note that since $k \leq c \leq \log_2(n-a)$ we have

$$\binom{a+k}{k} \leq \binom{a+\log_2(n-a)}{\log_2(n-a)} \leq (e+en)^{\log_2 n}$$

where we have used that $\binom{N}{k} \leq (\frac{eN}{k})^k$. Similarly

$$\binom{n-c-2^c-a+k}{k} \leq \binom{n-c-2^c-a+\log_2(n-a)}{\log_2(n-a)} \leq (e+en)^{\log_2 n}.$$

Also, since $c \leq \log_2(n-a)$,

$$\binom{c-1}{k-1} \leq 2^{\log_2(n-a)} \leq 2^{\log_2 n}$$

where we have first bounded $\binom{c-1}{k-1}$ by the central binomial term and used that $\binom{2N}{N} \leq 4^N$.

Therefore

$$\begin{aligned} \#I_n &\leq \sum_{a=0}^{n} \sum_{c=0}^{\log_2(n-a)} \sum_{k=1}^{c} \binom{c-1}{k-1} 3^c \binom{n-c-2^c-a+k}{k} \binom{a+k}{k} 2^a \\ &\leq (e+en)^{2\log_2 n} 2^{\log n} \sum_{a=0}^{n} \sum_{c=0}^{\log_2(n-a)} \sum_{k=1}^{c} 3^c 2^a \\ &\leq (e+en)^{2\log_2 n} 2^{\log_2 n} 3^{\log_2 n} \sum_{a=0}^{n} \sum_{c=0}^{\log_2(n-a)} \sum_{k=1}^{c} 2^a \end{aligned}$$

from which the result follows. $\square$

Using the above estimate for $\#I_n$, it is now easy to compute the entropy of Σ.

Lemma 5.5. *$h(\Sigma) = \log 3$.*

Proof. The lower bound $h(\Sigma) \geq \log 3$ follows from the fact that $\Omega^{\mathbb{N}} \subset \Sigma$. So it is sufficient to prove the upper bound. Fix any $\varepsilon > 0$. Suppose $\mathtt{i} \in \Sigma_n$. Then $\mathtt{i}$ falls into one of the following mutually exclusive categories:

(i) $\mathtt{i} \in I_n$.
(ii) $\mathtt{i} = \mathtt{jk}$ for $\mathtt{j} \in I$ and $\mathtt{k} = w\,(1,2)^m$ where $w \in \Omega^*$, $m \geq 0$.
(iii) $\mathtt{i} = w\,(1,2)^m$ for $w \in \Omega^*$ and $m \geq 0$.

By Lemma 5.4 the number of strings in category (i) is $\lesssim_\varepsilon 2^{(1+\varepsilon)n}$. The number of strings in category (iii) is given by

$$\sum_{m=0}^{n} 3^{n-m} \lesssim_\varepsilon 3^{(1+\varepsilon)n}.$$

Finally, the number of strings in category (ii) is given by

$$\sum_{j=1}^{n-1} \sum_{m=0}^{n-j-1} \#I_j 3^{n-m-j} \leq \sum_{j=1}^{n-1} \sum_{m=0}^{n-j-1} 2^j 3^{n-m-j} \lesssim_\varepsilon 3^{(1+\varepsilon)n}.$$

Hence $\frac{1}{n} \log \#\Sigma_n \lesssim_\varepsilon (1+\varepsilon) \log 3$ which concludes the proof of the upper bound since $\varepsilon > 0$ was chosen arbitrarily. $\square$

We will now prove Theorem 2.2 by showing that

$$\dim_{\mathrm{B}} \Pi(\Sigma) = \max\left\{\frac{\log 3}{\log \mathbf{n}}, \frac{\log 2}{\log \mathbf{m}}\right\} = \max\left\{\frac{h(\Sigma)}{\log \mathbf{n}}, \frac{h(\pi\Sigma)}{\log \mathbf{m}}\right\}.$$

Note that $\dim_{\mathrm{B}} \Pi(\Sigma)$ can attain either $\frac{h(\Sigma)}{\log \mathbf{n}}$ or $\frac{h(\pi\Sigma)}{\log \mathbf{m}}$. For instance if $\mathbf{n} = 5$, $\mathbf{m} = 2$ then $\dim_{\mathrm{B}} \Pi(\Sigma) = 1 = \frac{h(\pi\Sigma)}{\log \mathbf{m}}$. Whereas if $\mathbf{n} = 6$, $\mathbf{m} = 5$ then $\dim_{\mathrm{B}} \Pi(\Sigma) = \frac{\log 3}{\log 6} = \frac{h(\Sigma)}{\log \mathbf{n}}$.

of Theorem 2.2. The lower bound corresponds to the trivial lower bound from Proposition 3.1. So we just need to prove the upper bound. Fix $\varepsilon > 0$, $\delta > 0$. Let $k = n(\delta)$ and $l = m(\delta)$ and $\mathtt{i} \in \Sigma_l$. Then $\mathtt{i}$ falls into one of the following mutually exclusive categories.

(1) $\mathtt{i} = \mathtt{jk}$ where $\mathtt{j} \in \Sigma_k$ and $\pi(\mathtt{k}) = 1^{l-k}$.
(2) $\mathtt{i} = \mathtt{jk}\,(2,1)\,\mathtt{l}$ where: for some $1 \le m \le l-k$, $\pi(\mathtt{l}) \in \{1,2\}^{m-1}$; $\pi(\mathtt{k}) = 1^{l-k-m}$; $\mathtt{j} \in \Sigma_k$ has the form $\mathtt{j} = uw$ for $u \in I$ and $w \in \Omega^*$ with length $1 \le |w| \le \log_2(l-k-m)$.
(3) $\mathtt{i} = \mathtt{jk}\,(2,1)\,\mathtt{l}$ where: for some $1 \le m \le l-k$, $\pi(\mathtt{l}) \in \{1,2\}^{m-1}$; $\pi(\mathtt{k}) = 1^{l-k-m}$; $\mathtt{j} = uw\,(1,2)^z$ where $u \in I$, $1 \le z \le k$, $w \in \Omega^*$ with length $0 \le |w| \le \log_2(l-k+z-m)$.

For each $j = 1, 2, 3$ we define

$$A_j := \bigcup_{\mathtt{i} \in \Sigma_l \text{ in category } (j)} \Pi([\mathtt{i}]).$$

Then

$$N_\delta(\Pi(\Sigma)) \le \sum_{j=1}^{3} N_\delta(A_j). \tag{17}$$

Firstly,

$$N_\delta(A_1) = \#\Sigma_k \lesssim_\varepsilon 3^{(1+\varepsilon)k}$$

by Lemma 5.5. Secondly,

$$\begin{aligned} N_\delta(A_2) = \sum_{m=1}^{l-k} \sum_{|w|=1}^{\log_2(l-k-m)} \#I_{k-|w|} 3^{|w|} 2^{m-1} \quad &\lesssim_\varepsilon \quad \sum_{m=1}^{l-k} \sum_{|w|=1}^{\log_2(l-k-m)} 2^{(1+\varepsilon)(k-|w|)} 3^{|w|} 2^{m-1} \\ &\lesssim_\varepsilon \quad 2^{(1+\varepsilon)k} \left(\frac{3}{2^{1+\varepsilon}}\right)^{\log_2(l-k)} 2^{(1+\varepsilon)(l-k)} \lesssim_\varepsilon 2^{(1+2\varepsilon)l}. \end{aligned}$$

Finally,

$$\begin{aligned} N_\delta(A_3) &= \sum_{m=1}^{l-k}\sum_{z=1}^{k}\sum_{|w|=0}^{\log_2(l-k+z-m)} \#I_{k-|w|-z}3^{|w|}2^{m-1} \\ &\lesssim_\varepsilon \sum_{m=1}^{l-k}\sum_{z=1}^{k}\sum_{|w|=0}^{\log_2(l-k+z-m)} 2^{(1+\varepsilon)(k-|w|-z)}3^{|w|}2^{m-1} \\ &\lesssim_\varepsilon \sum_{m=1}^{l-k}\sum_{z=1}^{k} 2^{(1+\varepsilon)(k-z)}\left(\frac{3}{2^{1+\varepsilon}}\right)^{\log_2(l-k+z-m)} 2^{m-1} \\ &\lesssim_\varepsilon 2^{(1+\varepsilon)k}\left(\frac{3}{2^{1+\varepsilon}}\right)^{\log_2 l} 2^{(1+\varepsilon)(l-k)} \lesssim_\varepsilon 2^{(1+2\varepsilon)l}. \end{aligned}$$

By (17) we deduce that

$$\dim_B \Pi(\Sigma) \le \max\left\{\frac{\log 3}{\log \mathbf{n}}, \frac{\log 2}{\log \mathbf{m}}\right\},$$

as required. □

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