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A survey on conjugacy class graphs of groups

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Abstract

There are several graphs defined on groups. Among them we consider graphs whose vertex set consists conjugacy classes of a group G and adjacency is defined by properties of the elements of conjugacy classes. In particular, we consider commuting/nilpotent/solvable conjugacy class graph of G where two distinct conjugacy classes a^G and b^G are adjacent if there exist some elements $x \in a^G$ and $y \in b^G$ such that $\langle x, y\rangle$ is abelian/nilpotent/solvable. After a section of introductory results and examples, we discuss all the available results on connectedness, graph realization, genus, various spectra and energies of certain induced subgraphs of these graphs. Proofs of the results are not included. However, many open problems for further investigation are stated.

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1. Introduction

Characterizing finite groups using graphs defined over groups has gained traction as a research topic in recent times. A number of graphs have been defined on groups (see [23]), among which the commuting graph has been studied widely. Let G be a finite

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non-abelian group. The *commuting graph* of G is a simple undirected graph whose vertex set is G, in which two vertices x and y are adjacent if they commute. The complement of this graph is the *non-commuting graph* of G. The concept of commuting graph appeared in an important work of Brauer and Fowler [21], in the year 1955, a step towards the Classification of Finite Simple Groups. After the work of Erdős and Neumann [62] on its complement in the year 1976, it was studied in its own right.

The property that x and y commute is equivalent to saying that $\langle x, y\rangle$ is abelian. Using other group types such as cyclic, nilpotent, solvable, ... graphs have been defined on groups. Given a group type $\mathcal{P}$ (for instance, cyclic, abelian, nilpotent, solvable etc.), we define a graph on a group G, called the $\mathcal{P}$ graph of G, whose vertex set is G and two distinct vertices x and y are adjacent if $\langle x, y\rangle$ is a $\mathcal{P}$ group. In this nomenclature 'abelian graph' is nothing but the commuting graph. These graphs forms the following hierarchy (where $A \subseteq B$ denotes that A is a spanning subgraph of B):

$$\text{Cyclic graph} \subseteq \text{Commuting graph} \subseteq \text{Nilpotent graph} \subseteq \text{Solvable graph}. \tag{1}$$

It is worth mentioning that there are other graphs in the above hierarchy (for details one can see [23]).

A *dominant vertex* of a graph is a vertex that is adjacent to all other vertices. Let $\mathcal{P}(G) = \{g \in G : \langle g, h\rangle \text{ is a } \mathcal{P} \text{ group for all } h \in G\}$. Then $\mathcal{P}(G)$ is the set of all dominant vertices of $\mathcal{P}$ graph of G.

In the four cases just described, $\mathcal{P}(G)$ is a subgroup of G, the cyclicizer, centre, hypercentre, and solvable radical of G respectively. The question of connectedness of the subgraphs of $\mathcal{P}$ graph induced by $G \setminus \mathcal{P}(G)$ is an interesting problem (see [8,19,22,37,49,58,64]). Of course, this problem is trivial unless $\mathcal{P}(G)$ is removed (otherwise the graph has diameter at most 2). However for other studies such as independence number or clique number, it makes either no difference or just a trivial difference.

Graphs are also defined from (finite) groups by considering the vertex set as the set of conjugacy classes (or class sizes), with adjacency defined by certain properties of the elements of conjugacy classes or the class sizes. A survey on graphs whose vertex set consists of class sizes of a finite group can be found in [54]. Graphs whose vertex set consists conjugacy classes of a group and adjacency is defined by properties of their sizes were first considered in [13].

In this survey, we shall consider graphs whose vertex set consists conjugacy classes of a group G, with adjacency defined by properties of the elements of these classes. We call such a graph the $\mathcal{P}$ *conjugacy class graph* of G, or for short the $\mathcal{P}$CC-graph. The $\mathcal{P}$ conjugacy class graph of G is a simple undirected graph whose vertex set is the set of all the conjugacy classes of G and two vertices (conjugacy classes) a^G and b^G are adjacent if there exist some elements $x \in a^G$ and $y \in b^G$ such that $\langle x, y\rangle$ is a $\mathcal{P}$ group. We have the following hierarchy in case of $\mathcal{P}$ conjugacy class graph:

$$\text{Cyclic CC-graph} \subseteq \text{CCC-graph} \subseteq \text{NCC-graph} \subseteq \text{SCC-graph}, \tag{2}$$

where here and subsequently we use CCC-graph, NCC-graph, SCC-graph to denote commuting, nilpotent and solvable conjugacy class graph. (Commuting conjugacy class graph is synonymous with 'abelian conjugacy class graph'.) Clearly, 1^G (the conjugacy class of the identity element) is a dominant vertex if $\mathcal{P}(G)$ is a subgroup. To make the question of connectedness interesting, we should consider the induced subgraph on the set of non-dominant vertices. However, as we will see, it is not always known what this set is. Sometimes we just remove the identity class from the vertex set.

Note that the cyclic conjugacy class graph of a group has not yet been studied. In what follows, we shall consider commuting/nilpotent/solvable conjugacy class graph.

An outline of the paper follows. In the next section we give some general results about conjugacy class graphs, including a discussion of when they are complete (Theorem 2.3), when it happens that the $\mathcal{P}$ graph is a "blow-up" of the $\mathcal{P}CC$-graph (Theorem 2.6), and some discussion of the dominant vertices (a characterization is known only for the CCC-graph, see Proposition 2.5 and Problem 2.2). The following three sections survey the three graph types, discussing connectedness, detailed structure for special groups, and properties such as genus, spectrum and energy. (These results are taken from the literature and proofs are not given.) The final section includes some open problems.

2. General remarks and examples

We begin with a general observation about conjugacy class graphs which is often useful. Let $\mathcal{P}$ be any group-theoretic property, and let Γ be the $\mathcal{P}CC$-graph of G; that is, the vertices are conjugacy classes, and there is an edge $\{C_1, C_2\}$ if and only if there exist $g_i \in C_i$ for $i = 1, 2$ such that the group $\langle g_1, g_2\rangle$ has property $\mathcal{P}$. In fact a stronger condition holds: if $\{C_1, C_2\}$ is an edge, then for any $h_1 \in C_1$ there exists $h_2 \in C_2$ such that $\langle h_1, h_2\rangle$ has property $\mathcal{P}$. For let g_1, g_2 be as in the definition. There exists $x \in G$ such that $g_1^x = h_1$; then, letting $h_2 = g_2^x$, we see that

$$\langle g_1, g_2\rangle^x = \langle h_1, h_2\rangle,$$

and since $\mathcal{P}$ is a group-theoretic property it is preserved by conjugation.

The first result relevant to conjugacy class graphs is the theorem of Landau [53] from 1903: given the number of conjugacy classes of a finite group G, there is an upper bound on the order of G. This implies the following result.

Proposition 2.1. *Given a graph Γ, there are only finitely many finite groups G whose commuting, nilpotent or solvable conjugacy class graph is isomorphic to Γ.*

The number of such groups is not usually 1. For example, for any abelian group of order n, the CCC-graph is the complete graph on n vertices.

Problem 2.1. Which finite groups are uniquely determined by their commuting, nilpotent, or solvable conjugacy class graph?

Fig. 1 shows the conjugacy classes of the symmetric group S_4, with the CCC- and NCC-graphs. (The conjugacy classes of S_n are defined by cycle types, and are labelled with partitions of n; so their number is $p(n)$, where p is the partition function, sequence A000041 in the On-line Encyclopedia of Integer Sequences [1].) Solid lines show edges in the CCC-graph, while the dotted line is the additional edge in the NCC-graph. Note that the partitions 1111, 112, 22 and 4 form a clique in the NCC-graph. This observation leads to our first general result.

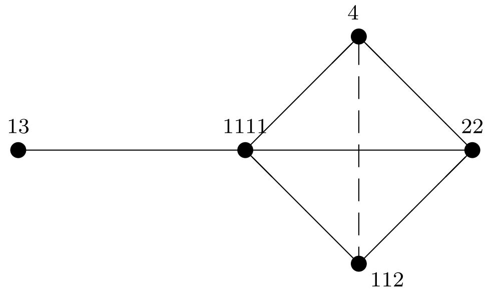

Fig. 1. Conjugacy class graphs of S_4.

Proposition 2.2. *Let G be a finite group and p a prime. Then the conjugacy classes of elements of p-power order form a clique in the NCC-graph.*

Proof. Let P be a Sylow p-subgroup of G. By Sylow's theorem, every element of p-power order is conjugate to an element of P; so P meets every conjugacy class of p-elements. Let C_1 and C_2 be two such classes, and take $g_i \in C_i \cap P$. Then $\langle g_1, g_2 \rangle \leq P$, so this group is a p-group, hence nilpotent. □

Thus the NCC-graph of S_n has a clique whose size is the number of partitions of n into powers of 2 (sequence A018819 in the OES).

The SCC-graph of S_4 is complete, because S_4 is a solvable group (and the class of solvable groups is subgroup-closed). This is also a special case of the following general result.

Theorem 2.3. *Let G be a finite group. Then the CCC-graph (resp., the NCC-graph, the SCC-graph) of G is complete if and only if G is abelian (resp., nilpotent, solvable).*

The "if" statements are clear. For the converses, we use the following result. Its roots lie in the work of Jordan.

Proposition 2.4. *Let H be a proper subgroup of the finite group G. Then G has a conjugacy class disjoint from H.*

This holds because G acts transitively on the set of right cosets of H by right multiplication; by Jordan's theorem, G contains an element x fixing no coset, that is, no conjugate of x lies in H.

Thus, if we choose one element from each conjugacy class of G, these elements generate G.

Now we prove Theorem 2.3 for the CCC-graph. Suppose that the CCC-graph is complete. Choose any element h, say $h \in C_1$. By our general remark about conjugacy class graphs, there exist $h_i \in C_i$ for all i such that h_i commutes with $h = h_1$. Then $\langle h_1, \ldots, h_r \rangle = G$, and so $h \in Z(G)$. Since h was arbitrary, G is abelian.

For the SCC-graph, this is immediate from the main theorem in [25], according to which G is solvable if and only if, for all $g, h \in G$, there exists $x \in G$ such that $\langle g, h^x \rangle$ is solvable.

For the NCC-graph, we use [25, Corollary E], which states that a finite group G is nilpotent if the following condition holds: for distinct primes p, q, and for $g, h \in G$ where g is a p-element and h a q-element, there exists $x \in G$ such that g commutes with h^x. Now suppose that G has complete NCC-graph, and let p, q, g, h be as stated. Then there exists $x \in G$ such that $\langle g, h^x \rangle$ is nilpotent. But in a nilpotent group, a p-element and a q-element necessarily commute.

We would like to have a strengthening of this describing the dominant vertices of one of our graphs (those joined to all others). The dominant vertices of the commuting, nilpotent and solvable graphs are known; they are respectively the centre, hypercentre, and solvable radical of the group [23, Theorem 11.2]. One might expect that the analogous result would hold for conjugacy class graphs, since each of these sets is the union of conjugacy classes. But this is not the case. The groups $\mathrm{PSL}(2, 2^a)$ for $a \geq 2$ have a single conjugacy class of involutions, and every element is conjugate to its inverse by some involution. This means that for any element g there is an involution h such that $\langle g, h \rangle$ is dihedral. So the class of involutions is dominant in the SCC-graph, even though the solvable radical is trivial. It is, however, true for the CCC-graph:

Proposition 2.5. *The set of dominant vertices in the CCC-graph of a finite group G is the set of central conjugacy classes of G.*

Proof. Clearly the central classes are dominant. Suppose that the class of the element g is dominant; that is, for all $h \in G$, there exists $x \in G$ such that g commutes with h^x. Then the centralizer of g meets every conjugacy class; by Jordan's result, it is the whole of G, so $g \in Z(G)$ as required. $\square$

Problem 2.2. Describe the dominant vertices of the NCC- or SCC-graph of a finite group.

Finally in this section, we compare our conjugacy class graphs with the conjugacy supergraphs as defined, for example, in [10]. In these graphs, the vertex set is the group G, and two vertices g and h are joined if and only if there exist conjugates g' and h' of g and h respectively so that $\langle g', h' \rangle$ has the appropriate property. These graphs are obtained from the conjugacy class graphs defined here by "inflating" each vertex to the number of vertices in its conjugacy class. In the other direction, we shrink a conjugacy class to a single vertex.

It is clear that many properties of the two graphs (such as connectedness and dominant vertices) will be unaltered by these transformations, while others such as spectrum and clique number will change. We will not discuss this further.

The question of when these graphs are equal is answered by the next theorem.

Theorem 2.6. *Let $\mathcal{P}$ be one of the properties "commutative", "nilpotent", or "solvable". A necessary and sufficient condition for the $\mathcal{P}$ graph and the conjugacy super $\mathcal{P}$ graph to be equal is as follows.*

(a) *For $\mathcal{P}$ = "commutative": G is a 2-Engel group (one satisfying the identity $[x, y, y] = 1$).*

(b) *For* $\mathcal{P}$ = *"nilpotent": G is nilpotent.*
(c) *For* $\mathcal{P}$ = *"solvable": G is solvable.*

Proof. (a) This is [10, Theorem 2].

(b) It is clear that, if G is nilpotent, then both graphs are complete. So suppose that they are equal. Suppose that G is not nilpotent. Then G contains a Schmidt group (a minimal non-nilpotent group). These groups were classified by Schmidt [69]; a convenient reference is [12].

By inspection, any such group contains a p-element x acting non-trivially on a q-group Q, where p and q are distinct primes. If $y \in Q$ with $y^x \neq y$, then

$$(x^{-1})^y x = [y, x] = y^{-1} y^x$$

is a non-identity q-element. But $\langle x, x\rangle$ is nilpotent, so by assumption $\langle x^y, x\rangle$ is nilpotent. This is a contradiction since all p-elements of a nilpotent group are contained in a single Sylow p-subgroup.

(c) The key ingredient is the fact that any finite simple group can be generated by two conjugate elements. In fact, by [39], if G is a finite simple group, then there exists $s \in G$ such that for all nontrivial $x \in G$ there exists $g \in G$ such that $\langle x, s^g\rangle = G$ (we can take $x = s$ to get the previous claim).

It is clear that, if G is solvable, then both graphs are complete. So suppose that they are equal. Any two conjugates of an element g are joined in the $\mathcal{P}$ supergraph, and therefore are joined in the $\mathcal{P}$ graph. Suppose that G is not solvable. Let $N < M < G$ be a subnormal series such that M/N is a non-abelian simple group. As noted above, there exists $g, y \in M$ such that $M/N = \langle Ng, Ng^y\rangle$. In particular, $\langle Ng, Ng^y\rangle$ is non-solvable and hence $\langle g, g^y\rangle$ is non-solvable. However, $\langle g, g\rangle = \langle g\rangle$ is solvable, which is a contradiction. Therefore, G is solvable, completing the proof. □

The simplicity of the conditions in (b) and (c) compared to (a) is striking.

In the next three sections, we collate and survey some properties of the CCC-, NCC- and SCC-graphs of finite groups.

3. Commuting conjugacy class graph

We write $\mathcal{CCC}(G)$ to denote the CCC-graph of a group G. The CCC-graph of G is a graph whose vertex set is $\mathrm{Cl}(G) := \{x^G : x \in G\}$, where x^G denotes the conjugacy class of x in G, and two distinct vertices a^G and b^G are adjacent if there exist some elements $x \in a^G$ and $y \in b^G$ such that $\langle x, y\rangle$ is an abelian group. In this section, we discuss results on various induced subgraphs of $\mathcal{CCC}(G)$. Herzog et al. [48] considered three induced graphs of $\mathcal{CCC}(G)$ induced by $\mathrm{Cl}(G \setminus 1)$, $\mathrm{Cl}(G \setminus Z(G))$ and $\mathrm{Cl}(G \setminus FC(G))$, where $\mathrm{Cl}(S) = \{x^G : x \in S\}$ for any subset S of G, and $FC(G) = \{x \in G : x^G \text{ is finite}\}$ is the *FC-centre* of G.

3.1. Connectivity of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$

Let X be the class of groups which cannot be written as a union of conjugates of a proper subgroup. The following theorem gives a characterization of residually X-group G such that $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is complete.

Theorem 3.1 (*[48, Proposition 1]*). *Let G be a residually X-group. Then the graph* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is complete if and only if G is abelian. In particular, the claim holds if G is residually (finite or solvable)-group.*

Connectedness of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is discussed in the next two theorems.

Theorem 3.2 (*[48, Theorem 10 and 12]*). *Let G be a finite solvable group or a periodic solvable group. Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *has at most two connected components, each of diameter* ≤ 9.

Theorem 3.3 (*[48, Theorems 13–14]*). *Let G be a finite group or a locally finite group. Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *has at most six connected components, each of diameter* ≤ 19.

The following theorems give characterization of supersolvable/solvable groups such that $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is disconnected.

Theorem 3.4 (*[48, Proposition 7]*). *Let G be a supersolvable group. Then the graph* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is disconnected if and only if G is either of the groups given in the following two types:*

(a) $G = A \rtimes \langle x \rangle$, *where* $x \in G$, $|x| = 2$ *and A is a subgroup of G on which x acts fixed-point-freely.*
(b) *G is finite and* $G = A \rtimes B$, *where A, B are non-trivial subgroups of G, A is nilpotent and B is cyclic, and B acts on A fixed-point-freely (in particular, G is a Frobenius group with kernel A and a cyclic complement B).*

Theorem 3.5 (*[48, Theorem 16]*). *Let G be a finite solvable group such that the graph* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is disconnected. Then there exists a nilpotent normal subgroup H of G such that one of the following holds:*

(a) $G = H \rtimes T$ *is a Frobenius group with the kernel H and a complement T.*
(b) $G = (H \rtimes S) \rtimes \langle x \rangle$, *where S is a non-trivial cyclic subgroup of G of odd order which acts fixed-point-freely on H,* $x \in N_G(S)$ *is such that* $\langle x \rangle$ *acts fixed-point-freely on S, and there exist* $h_1 \in H \setminus \{1\}$ *and* $i \in \mathbb{N}$ *such that* $x^i \neq 1$, *which satisfy* $[x^i, h_1] = 1$.

Conversely, if either (a) *or* (b) *holds, then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is disconnected.*

3.2. Properties of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$

In the following theorem Herzog et al. [48] determined all periodic groups G such that $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is empty.

Theorem 3.6 (*[48, Theorem 19]*). *Let G be a periodic non-abelian group. Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is empty if and only if G is isomorphic to* D_8, Q_8 *or* S_3.

In 2016, Mohammadian et al. [57] classified all finite groups G such that the graph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is triangle-free and obtained the following results.

Theorem 3.7 (*[57, Theorem 2.3]*). *If G is a finite group of odd order and the graph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is triangle-free, then $|G| = 21$ or 27.*

Theorem 3.8 (*[57, Theorem 3.4]*). *Suppose G is a finite group of even order which is not a 2-group and $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is triangle-free. If $|Z(G)| \neq 1$, then G is isomorphic to D_{12} or $T_{12} = \langle a, b : a^4 = b^3 = 1, aba^{-1} = b^{-1}\rangle$.*

Theorem 3.9 (*[57, Theorem 3.5]*). *If G is a centreless non-solvable finite group and $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is triangle-free, then G is isomorphic to one of the groups $PSL(2, q)$ $(q \in \{4, 7, 9\})$, $PSL(3, 4)$ or* SmallGroup(960, 11357).

Theorem 3.10 (*[57, Theorem 3.6]*). *If G is a centreless non-abelian solvable finite group and $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is triangle-free, then G is isomorphic to one of the following groups: S_3, D_{10}, A_4, S_4,* SmallGroup(72, 41), SmallGroup(192, 1023) *or* SmallGroup(192, 1025).

Theorem 3.11 (*[57, Theorem 3.7]*). *If G is a finite non-abelian 2-group such that $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is triangle-free, then $\Phi(G) \leq Z(G)$ and $C_G(x) = \langle x, Z(G)\rangle$ whenever $x \in G \setminus Z(G)$. Furthermore, either $G \cong D_8$, $G \cong Q_8$ or $|G : Z(G)| = |Z(G)|$.*

3.3. Structure of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$

In [66–68] structures of commuting conjugacy class graphs of certain finite non-abelian groups were determined. In this section, we shall discuss the structures of CCC-graphs of dihedral group, generalized quaternion group, semi-dihedral group, the groups $U_{(n,m)}$, V_{8n} and $G(p, m, n)$ along with some other groups such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2n}, where p is a prime and $D_{2n} = \langle x, y : \ x^n = y^2 = 1, \ yxy^{-1} = x^{-1}\rangle$.

Theorem 3.12 (*[66, Theorem 1.2]*). *Let G be a finite group with centre $Z(G)$ and $\frac{G}{Z(G)}$ is isomorphic to the dihedral group D_{2n}. Then*

$$\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))] = \begin{cases} K_{\frac{(n-1)|Z(G)|}{2}} \cup 2K_{\frac{|Z(G)|}{2}}, & \text{for } 2 \mid n \\ K_{\frac{(n-1)|Z(G)|}{2}} \cup K_{|Z(G)|}, & \text{for } 2 \nmid n. \end{cases}$$

As a corollary to Theorem 3.12, we get the structure of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ when G is the dihedral group D_{2n}, the generalized quaternion group $Q_{4m} = \langle x, y : \ x^{2m} = 1, x^m = y^2, y^{-1}xy = x^{-1}\rangle$, the semi-dihedral group $SD_{8n} = \langle x, y : \ x^{4n} = y^2 = 1, yxy = x^{2n-1}\rangle$ the group $U_{(n,m)} = \langle x, y : \ x^{2n} = y^m = 1, \ x^{-1}yx = y^{-1}\rangle$ and the group $U_{6n} = \langle x, y : \ x^{2n} = y^3 = 1, \ x^{-1}yx = y^{-1}\rangle$. Further, Salahshour and Ashrafi [67, Proposition 2.4 and 2.6] determined the structures of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ when G is the group $V_{8n} = \langle x, y : \ x^{2n} = y^4 = 1, \ yx = x^{-1}y^{-1}, \ y^{-1}x = x^{-1}y\rangle$ and the group $G(p, m, n) = \langle x, y : \ x^{p^m} = y^{p^n} = [x, y]^p = 1, [x, [x, y]] = [y, [x, y]] = 1\rangle$ as given below:

$$\mathcal{CCC}(V_{8n})[\mathrm{Cl}(V_{8n} \setminus Z(V_{8n}))] = \begin{cases} K_{2n-2} \cup 2K_2, & \text{for } 2 \mid n \\ K_{2n-1} \cup 2K_1, & \text{for } 2 \nmid n \end{cases}$$

and

$$\mathcal{CCC}(G(p,m,n))[\mathrm{Cl}(G(p,m,n) \setminus Z(G(p,m,n)))] \\ = 2K_{p^{m+n-1}-p^{m+n-2}} \cup (p^n - p^{n-1})K_{p^{m-n}(p^n-p^{n-1})}.$$

Note that all the groups considered above are AC-groups (non-abelian groups whose centralizers of non-central elements are abelian). Salahshour and Ashrafi [67] also obtained the structure of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ if G is a finite AC-group. Let $\mathrm{Cent}(G) = \{C_G(a) : a \in G\}$, where $C_G(a)$ is the centralizer of $a \in G$. Consider the equivalence relation $\sim$ on $\mathrm{Cent}(G) \setminus \{G\}$ given by $C_G(a) \sim C_G(b)$ if and only if $C_G(a)$ and $C_G(b)$ are conjugate in G. Then we have the following result.

Theorem 3.13 (*[67, Theorem 3.3]*)**.** *Let G be a finite AC-group with centre $Z(G)$. Then*

$$\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))] = \bigcup_{\frac{C_G(a)}{\sim} \in EC(G)} K_{n_{\frac{C_G(a)}{\sim}}}$$

where $EC(G) = \frac{Cent(G)\setminus\{G\}}{\sim}$ is the set of all equivalence classes of $\sim$ and $n_{\frac{C_G(a)}{\sim}} = \frac{|C_G(a)|-|Z(G)|}{[N_G(C_G(a)):C_G(a)]}$.

Salahshour and Ashrafi [68] determined the structures of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ when G is a finite non-abelian group such that $\frac{G}{Z(G)}$ has order p^2 or p^3 as given in the following theorems.

Theorem 3.14 (*[68, Theorem 3.1]*)**.** *Let G be a finite non-abelian group with centre $Z(G)$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is prime. Then $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))] = (p+1)K_n$, where $n = \frac{(p-1)|Z(G)|}{p}$.*

Theorem 3.15 (*[68, Theorem 3.3]*)**.** *Let G be a finite non-abelian group with centre $Z(G)$ and $|\frac{G}{Z(G)}| = p^3$, where p is a prime. Then one of the following is satisfied:*

(a) *If $\frac{G}{Z(G)}$ is abelian then $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))] = K_m \cup p^2K_n$ or $(p^2+p+1)K_n$, where $m = \frac{(p^2-1)|Z(G)|}{p}$ and $n = \frac{(p-1)|Z(G)|}{p^2}$.*

(b) *If $\frac{G}{Z(G)}$ is non-abelian then $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))] = K_m \cup kpK_{n_1} \cup (p-k)K_{n_2}$, $(kp+1)K_{n_1} \cup (p+1-k)K_{n_2}$, $K_m \cup pK_{n_2}$, $(p^2+p+1)K_{n_1}$ or $K_{n_1} \cup (p+1)K_{n_2}$, where $m = \frac{(p^2-1)|Z(G)|}{p}$, $n_1 = \frac{(p-1)|Z(G)|}{p^2}$, $n_2 = \frac{(p-1)|Z(G)|}{p}$, $1 \le k \le p$.*

As a corollary, it follows that $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))] = (p+1)K_{p(p-1)}$ or $K_{(p^2-1)} \cup pK_{p-1}$ if G is a non-abelian p-group of order p^4. Ashrafi and Salahshour [11, Theorem 1.2] also obtained the structure of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ when $\frac{G}{Z(G)}$ is isomorphic to $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_{p^2}$, where p is a prime. In a recent work, Rezaei and Foruzanfar [65] have determined the structure of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ when $\frac{G}{Z(G)}$ is isomorphic to a Frobenius group of order pq or p^2q, where p, q are primes.

3.4. Genus of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$

For any graph Γ, we write $\gamma(\Gamma)$ to denote its genus. The genus of Γ is the smallest integer $k \geq 0$ such that Γ can be embedded on the surface obtained by attaching k handles to a sphere. If $\gamma(\Gamma)$ is equal to $0, 1, 2$, or 3, then Γ is called planar, toroidal, double-toroidal, or triple-toroidal, respectively. Clearly, $\gamma(K_1) = \gamma(K_2) = 0$. For $n \geq 3$, by [75, Theorem 6-38], we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil,$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a for any real number a. It is worth mentioning that finite non-abelian groups G for which $\mathcal{C}(G)[G \setminus Z(G)]$ (the induced subgraph of commuting graph of G induced by $G \setminus Z(G)$) is planar have been characterized (see [7, Theorem 2.2]), toroidal (see [7, Theorem 2.2] and [30, Theorem 3.3]), double-toroidal (see [63, Theorem 3.3]) and triple-toroidal (see [63, Theorem 3.7]). In this regard, we have the following problem.

Problem 3.1. Characterize all finite non-abelian groups G such that the induced subgraph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ of $\mathcal{CCC}(G)$ is planar, toroidal, double-toroidal or triple-toroidal.

This problem was considered by Bhowal and Nath [17] and they characterized the dihedral groups, generalized quaternion groups and semidihedral groups such that $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is planar, toroidal, double-toroidal or triple-toroidal. We have the following theorems for instance.

Theorem 3.16 (*[18, Theorem 2.2]*). *Let G be the dihedral group D_{2n}. Then*

(a) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is planar if and only if* $3 \leq n \leq 10$.
(b) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is toroidal if and only if* $11 \leq n \leq 16$.
(c) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is double-toroidal if and only if* $n = 17, 18$.
(d) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is triple-toroidal if and only if* $n = 19, 20$.
(e) $\gamma(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]) = \begin{cases} \left\lceil \frac{(n-7)(n-9)}{48} \right\rceil, & \text{for } 2 \nmid n \text{ and } n \geq 21 \\ \left\lceil \frac{(n-8)(n-10)}{48} \right\rceil, & \text{for } 2 \mid n \text{ and } n \geq 22. \end{cases}$

Theorem 3.17 (*[18, Theorem 2.4]*). *Let G be the generalized quaternion group Q_{4m}. Then*

(a) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is planar if and only if* $m = 2, 3, 4$ *or* 5.
(b) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is toroidal if and only if* $m = 6, 7$ *or* 8.
(c) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is double-toroidal if and only if* $m = 9$.
(d) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is triple-toroidal if and only if* $m = 10$.
(e) $\gamma(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]) = \left\lceil \frac{(m-4)(m-5)}{12} \right\rceil$ *for* $m \geq 11$.

Theorem 3.18 (*[18, Theorem 2.3]*). *Let G be the semidihedral group SD_{8n}. Then*

(a) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is planar if and only if* $n = 2$ *or* 3.
(b) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is toroidal if and only if* $n = 4$.
(c) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is double-toroidal if and only if* $n = 5$.
(d) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is not triple-toroidal.*
(e) $\gamma(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]) = \begin{cases} \left\lceil \frac{(n-3)(2n-5)}{6} \right\rceil, & \text{for } 2 \nmid n \text{ and } n \geq 7 \\ \left\lceil \frac{(n-2)(2n-5)}{6} \right\rceil, & \text{for } 2 \mid n \text{ and } n \geq 6. \end{cases}$

Bhowal and Nath [18] also considered the groups V_{8n}, $U_{(n,m)}$ and $G(p, m, n)$ in their study and obtained the following result.

Theorem 3.19 (*[18, Corollary 2.8]*). *Let G be a group isomorphic to D_{2n}, SD_{8n}, Q_{4m}, V_{8n}, $U_{(n,m)}$ or $G(p, m, n)$. Then*

(a) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is planar if and only if* $G = D_6, D_8, D_{10}, D_{12}, D_{14}, D_{16}, D_{18}, D_{20}, SD_{16}, SD_{24}, Q_8, Q_{12}, Q_{16}, Q_{20}, V_{16}, U_{(2,2)}, U_{(2,3)}, U_{(2,4)}, U_{(2,5)}, U_{(2,6)}, U_{(3,2)}, U_{(3,3)}, U_{(3,4)}, U_{(4,2)}, U_{(4,3)}, U_{(4,4)}, G(2, 1, 1), G(3, 1, 1), G(5, 1, 1), G(2, 2, 1), G(2, 3, 1), G(2, 1, 2), G(2, 2, 2)$ *or* $G(2, 1, 3)$.
(b) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is toroidal if and only if* $G = D_{22}, D_{24}, D_{26}, D_{28}, D_{30}, D_{32}, SD_{32}, Q_{24}, Q_{28}, Q_{32}, V_{24}, V_{32}, U_{(2,7)}, U_{(2,8)}, U_{(3,5)}$ *or* $U_{(3,6)}$.
(c) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is double-toroidal if and only if* $G = D_{34}, D_{36}, SD_{40}, Q_{36}, U_{(2,9)}, U_{(2,10)}, U_{(4,5)}, U_{(4,6)}, U_{(5,2)}, U_{(5,3)}, U_{(6,2)}, U_{(6,3)}, U_{(7,2)}, U_{(7,3)}$ *or* $G(3, 1, 2)$.
(d) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is triple-toroidal if and only if* $G = D_{38}, D_{40}, Q_{40}, V_{40}, U_{(3,7)}, U_{(3,8)}, U_{(5,4)}, U_{(6,4)}$ *or* $U_{(7,4)}$.

It may be interesting to continue similar study for the groups with known/unknown structures of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ and answer Problem 3.1.

3.5. Various spectra and energies of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$

The spectrum of a finite graph Γ with vertex set $V(\Gamma)$, denoted by $\mathrm{Spec}(\Gamma)$, is the set of eigenvalues of its adjacency matrix with multiplicities. If $\mathrm{Spec}(\Gamma) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \ldots, \alpha_k^{a_k}\}$ for some Γ then we mean that $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the eigenvalues of the adjacency matrix of Γ with multiplicities $a_1, a_2, \ldots, a_k$ respectively. Similarly, L-spec(Γ) and Q-spec(Γ) denote the Laplacian spectrum (L-spectrum) and signless Laplacian spectrum (Q-spectrum) of Γ i.e., the set of eigenvalues of the Laplacian and signless Laplacian matrices of Γ respectively. A graph Γ is called integral/L-integral/Q-integral if Spec(Γ)/L-spec(Γ)/Q-spec(Γ) contains only integers. To determine all the integral/L-integral/Q-integral graphs is a general problem in graph theory. Various spectra of $\mathcal{C}(G)[G \setminus Z(G)]$ were computed in [29–31,59] and obtained various groups such that $\mathcal{C}(G)[G \setminus Z(G)]$ is integral/L-integral/Q-integral. Note that the following problem is still open.

Problem 3.2. Determine all the finite non-abelian groups G such that $\mathcal{C}(G)[G \setminus Z(G)]$ is integral/L-integral/Q-integral.

The energy, Laplacian energy (L-energy) and signless Laplacian energy (Q-energy) of Γ denoted by $E(\Gamma)$, $LE(\Gamma)$ and $LE^+(\Gamma)$ respectively are given by

$$E(\Gamma) := \sum_{\alpha \in \mathrm{Spec}(\Gamma)} |\alpha|, \qquad LE(\Gamma) := \sum_{\beta \in \text{L-spec}(\Gamma)} \left| \beta - \frac{tr(\mathcal{D}(\Gamma))}{|V(\Gamma)|} \right|$$
$$\text{and } LE^+(\Gamma) := \sum_{\gamma \in \text{Q-spec}(\Gamma)} \left| \gamma - \frac{tr(\mathcal{D}(\Gamma))}{|V(\Gamma)|} \right|,$$

where $\mathcal{D}(\Gamma)$ is the degree matrix of Γ and $tr(\mathcal{D}(\Gamma))$ is the trace of $\mathcal{D}(\Gamma)$. In 1978, Gutman [41] introduced the notion of energy of a graph. In Huckel theory, π-electron energy of a conjugated carbon molecule is approximated by $E(\mathcal{G})$. Subsequently, Gutman and Zhou [46] in 2006 and Abreua et al. [6] in 2008 introduced L-energy and Q-energy of a graph. Applications of these energies can be found in crystallography, theory of macromolecules, analysis and comparison of protein sequences, network analysis, satellite communication, image analysis and processing etc. (see [45] and the references therein). In 2009, Gutman et al. [44] conjectured (E-LE conjecture) that

$$E(\Gamma) \leq LE(\Gamma). \tag{3}$$

Though (3) was disproved in [55,71] people wanted to know whether this conjecture is true for various graphs defined on groups. The following problem for $\mathcal{C}(G)[G \setminus Z(G)]$ is considered in [27,32].

Problem 3.3. Determine all the finite non-abelian groups G such that $\mathcal{C}(G)[G \setminus Z(G)]$ satisfy the following inequalities:

(a) $E(\mathcal{C}(G)[G \setminus Z(G)]) \leq LE(\mathcal{C}(G)[G \setminus Z(G)])$.
(b) $LE^+(\mathcal{C}(G)[G \setminus Z(G)]) \leq LE(\mathcal{C}(G)[G \setminus Z(G)])$.

A graph Γ is called hyperenergetic, L-hyperenergetic and Q-hyperenergetic if $E(\Gamma) > E(K_{|V(\Gamma)|})$, $LE(\Gamma) > LE(K_{|V(\Gamma)|})$ and $LE^+(\Gamma) > LE^+(K_{|V(\Gamma)|})$ respectively. The concept of hyperenergetic graph was given by Walikar et al. [74] and Gutman [42], independently in 1999. The concept of L-hyperenergetic and Q-hyperenergetic graph can be found in [34]. Again, Γ is called borderenergetic (introduced by Gong et al. [38]), L-borderenergetic (introduced by Tura [73]) and Q-borderenergetic (introduced by Tao et al. [72]) if $E(\Gamma) = E(K_{|V(\Gamma)|})$, $LE(\Gamma) = LE(K_{|V(\Gamma)|})$ and $LE^+(\Gamma) = LE^+(K_{|V(\mathcal{G})|})$ respectively. The following conjecture was posed by Gutman [41] in 1978.

Conjecture 3.20. *Any finite graph $\Gamma \ncong K_{|v(\mathcal{G})|}$ is non-hyperenergetic.*

This conjecture was also disproved by different mathematicians providing counter examples (see [43]). However, the search for counter examples to Conjecture 3.20 continued. In [70], the following problem for $\mathcal{C}(G)[G \setminus Z(G)]$ was considered.

Problem 3.4. Determine all the finite non-abelian groups G such that $\mathcal{C}(G)[G\setminus Z(G)]$ is hyperenergetic/borderenergetic/L-hyperenergetic/L-borderenergetic/ Q-hyper-energetic/Q-borderenergetic.

In [16,17], Bhowal and Nath considered problems corresponding to the Problems 3.2–3.4 for $\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]$. They considered the dihedral groups, generalized quaternion groups and semidihedral groups and obtained the following results.

Theorem 3.21 (*[17, Theorem 3.1]*)**.** *If G is the dihedral group D_{2n}, then*

(a) $\mathrm{Spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))])$

$$= \begin{cases} \left\{(0)^1, (-1)^{\frac{n-3}{2}}, \left(\frac{n-3}{2}\right)^1\right\}, & \text{for } 2\nmid n \\ \left\{(0)^2, (-1)^{\frac{n}{2}-2}, \left(\frac{n}{2}-2\right)^1\right\}, & \text{for } 2\mid n \text{ and } 2\mid \frac{n}{2} \\ \left\{(-1)^{\frac{n}{2}-1}, (1)^1, \left(\frac{n}{2}-2\right)^1\right\}, & \text{for } 2\mid n \text{ and } 2\nmid \frac{n}{2} \end{cases}$$

and $E(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} n-3, & \text{for } 2\nmid n \\ n-4, & \text{for } 2\mid n \text{ and } 2\mid \frac{n}{2} \\ n-2, & \text{for } 2\mid n \text{ and } 2\nmid \frac{n}{2}. \end{cases}$

(b) $\text{L-spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))])$

$$= \begin{cases} \left\{(0)^2, \left(\frac{n-1}{2}\right)^{\frac{n-3}{2}}\right\}, & \text{for } 2\nmid n \\ \left\{(0)^3, \left(\frac{n}{2}-1\right)^{\frac{n}{2}-2}\right\}, & \text{for } 2\mid n \text{ and } 2\mid \frac{n}{2} \\ \left\{(0)^2, 2^1, \left(\frac{n}{2}-1\right)^{\frac{n}{2}-2}\right\}, & \text{for } 2\mid n \text{ and } 2\nmid \frac{n}{2} \end{cases}$$

and $LE(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \frac{2(n-1)(n-3)}{n+1}, & \text{for } 2\nmid n \\ \frac{3(n-2)(n-4)}{n+2}, & \text{for } 2\mid n \text{ and } 2\mid \frac{n}{2} \\ 4, & \text{for } n=6 \\ \frac{(n-4)(3n-10)}{n+2}, & \text{for } 2\mid n,\ n\geq 10 \text{ and } 2\nmid \frac{n}{2}. \end{cases}$

(c) $\text{Q-spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))])$

$$= \begin{cases} \left\{(0)^1, (n-3)^1, \left(\frac{n-5}{2}\right)^{\frac{n-3}{2}}\right\}, & \text{for } 2\nmid n \\ \left\{(0)^2, (n-4)^1, \left(\frac{n}{2}-3\right)^{\frac{n}{2}-2}\right\}, & \text{for } 2\mid n \text{ and } 2\mid \frac{n}{2} \\ \left\{(0)^1, (2)^1, (n-4)^1, \left(\frac{n}{2}-3\right)^{\frac{n}{2}-2}\right\}, & \text{for } 2\mid n \text{ and } 2\nmid \frac{n}{2} \end{cases}$$

$$\text{and } LE^{+}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \frac{(n-3)(n+3)}{n+1}, & \text{for } 2\nmid n \\ \frac{(n-4)(n+6)}{n+2}, & \text{for } n=4,8 \\ \frac{2(n-2)(n-4)}{n+2}, & \text{for } 2\mid n,\ \frac{n}{2} \text{ and } n\geq 12 \\ 4, & \text{for } n=6 \\ \frac{22}{3}, & \text{for } n=10 \\ \frac{2(n-2)(n-6)}{n+2}, & \text{for } 2\mid n,\ 2\nmid \frac{n}{2} \\ & \quad \text{and } n\geq 14. \end{cases}$$

Theorem 3.22 (*[17, Theorem 3.2]*)**.** *If G is the generalized quaternion group Q_{4m}, then*

(a) $$\mathrm{Spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \left\{(-1)^{m-1},(1)^1,(m-2)^1\right\}, & \text{for } 2\nmid m \\ \left\{(-1)^{m-2},(0)^2,(m-2)^1\right\}, & \text{for } 2\mid m \end{cases}$$

$$\text{and } E(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} 2\ m-2, & \text{for } 2\nmid m \\ 2\ m-4, & \text{for } 2\mid m. \end{cases}$$

(b) $$\text{L-spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \left\{(0)^2,(2)^1,(m-1)^{m-2}\right\}, & \text{for } 2\nmid m \\ \left\{(0)^3,(m-1)^{m-2}\right\}, & \text{for } 2\mid m \end{cases}$$

$$\text{and } LE(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} 4, & \text{for } m=3 \\ \frac{2(m-2)(3\ m-5)}{m+1}, & \text{for } 2\nmid m \text{ and } m\geq 5 \\ \frac{6(m-1)(m-2)}{m+1}, & \text{for } 2\mid m. \end{cases}$$

(c) $$\text{Q-spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \left\{(2)^1,(0)^1,(2\ m-4)^1,(m-3)^{m-2}\right\}, & \text{for } 2\nmid m \\ \left\{(0)^2,(2\ m-4)^1,(m-3)^{m-2}\right\}, & \text{for } 2\mid m \end{cases}$$

$$\text{and } LE^{+}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} 4, & \text{for } m=3 \\ \frac{22}{3}, & \text{for } m=5 \\ \frac{4(m-1)(m-3)}{m+1}, & \text{for } 2\nmid m \text{ and } m\geq 7 \\ \frac{2(m-2)(m+3)}{m+1}, & \text{for } m=2,4 \\ \frac{4(m-1)(m-2)}{m+1}, & \text{for } 2\mid m \text{ and } m\geq 6. \end{cases}$$

Theorem 3.23 (*[17, Theorem 3.5]*)**.** *If G is the semidihedral group SD_{8n}, then*

(a) $$\mathrm{Spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \left\{(-1)^{2n},(3)^1,(2n-3)^1\right\}, & \text{for } 2\nmid n \\ \left\{(-1)^{2n-2},(0)^2,(2n-2)^1\right\}, & \text{for } 2\mid n \end{cases}$$

$$\text{and } E(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} 4n, & \text{for } 2\nmid n \\ 4n-4, & \text{for } 2\mid n. \end{cases}$$

(b) $$\text{L-spec}(\mathcal{CCC}(G)[\mathrm{Cl}(G\setminus Z(G))]) = \begin{cases} \left\{(0)^2,(4)^3,(2n-2)^{2n-3}\right\}, & \text{for } 2\nmid n \\ \left\{(0)^3,(2n-1)^{2n-2}\right\}, & \text{for } 2\mid n \end{cases}$$

$$\text{and } LE(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]) = \begin{cases} 12, & \text{for } n = 3 \\ \frac{2(2n-3)(5n-11)}{n+1}, & \text{for } 2 \nmid n \text{ and } n \geq 5 \\ \frac{6(2n-1)(2n-2)}{2n+1}, & \text{for } 2 \mid n. \end{cases}$$

(c) Q-spec$(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))])$

$$= \begin{cases} \left\{(6)^1, (2)^3, (4n-6)^1, (2n-4)^{2n-3}\right\}, & \text{for } 2 \nmid n \\ \left\{(0)^2, (4n-4)^1, (2n-3)^{2n-2}\right\}, & \text{for } 2 \mid n \end{cases}$$

$$\text{and } LE^+(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]) = \begin{cases} 12, & \text{for } n = 3 \\ 22, & \text{for } n = 5 \\ \frac{16(n-1)(n-3)}{n+1}, & \text{for } 2 \nmid n \text{ and } n \geq 7 \\ \frac{28}{5}, & \text{for } n = 2 \\ \frac{4(2n-1)(2n-2)}{2n+1}, & \text{for } 2 \mid n \text{ and } n \geq 4. \end{cases}$$

Bhowal and Nath [16,17] also considered the groups V_{8n}, $U_{(n,m)}$ and $G(p, m, n)$ in their study and obtained the following results.

Theorem 3.24 ([17, Corollary 3.6]). *If G is isomorphic to D_{2n}, Q_{4m}, SD_{8n}, V_{8n}, $U_{(n,m)}$ and $G(p, m, n)$ then $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is integral, L-integral and Q-integral.*

Comparing various energies of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ they also obtained the following results.

Theorem 3.25 ([17, Theorem 4.6]). *Let G be a finite non-abelian group and $\Gamma = \mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$. Then*

(a) *$E(\Gamma) = LE^+(\Gamma) = LE(\Gamma)$ if G is isomorphic to $D_6, D_8, D_{12}, Q_8, Q_{12}, U_{(n,2)}, U_{(n,3)}, U_{(n,4)}$ $(n \geq 2)$, V_{16}, SD_{24} or $G(p, m, 1)$ $(p \geq 2, m \geq 1)$.*
(b) *$LE^+(\Gamma) < E(\Gamma) < LE(\Gamma)$ if G is isomorphic to D_{20}, Q_{20}, $U_{(2,5)}, U_{(3,5)}, U_{(2,6)}$ or $G(2, 2, 2)$.*
(c) *$E(\Gamma) < LE^+(\Gamma) < LE(\Gamma)$ if G is isomorphic to $D_{14}, D_{16}, D_{18}, D_{2n}$ $(n \geq 11)$, Q_{16}, Q_{24}, Q_{4m} $(m \geq 8)$, $U_{(n,5)}$ $(n \geq 4)$, $U_{(n,m)}$ $(m \geq 6$ and $n \geq 3)$, $U_{(n,m)}$ $(m \geq 8$ and $n \geq 2)$, V_{8n} $(n \geq 3)$, SD_{16}, SD_{8n} $(n \geq 4)$, $G(2, m, 2)$ $(m \geq 3)$, $G(p, m, 2)$ $(p \geq 3, m \geq 1)$ or $G(p, m, n)$ $(n \geq 3, p \geq 2, m \geq 1)$.*
(d) *$E(\Gamma) = LE^+(\Gamma) < LE(\Gamma)$ if G is isomorphic to Q_{28} or $U_{(2,7)}$*
(e) *$E(\Gamma) < LE^+(\Gamma) = LE(\Gamma)$ if G is isomorphic to D_{10} and $G(2, 1, 2)$.*

Theorem 3.26 ([17, Theorem 5.6]). *Let G be a finite non-abelian group. Then*

(a) *$\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is neither hyperenergetic, borderenergetic, L-hyperenergetic, L- borderenergetic, Q-hyperenergetic nor Q-borderenergetic if G is isomorphic to D_8, D_{12}, D_{2n} (n is odd), Q_8, Q_{12}, Q_{16}, $U_{(2,6)}$, $U_{(n,3)}$, $U_{(n,4)}$ $(n \geq 2)$, V_{16}, SD_{16}, SD_{24}, $G(p, m, 1)$ $(p \geq 2$ and $m \geq 1)$, $G(2, 1, 2)$ or $G(2, 2, 2)$.*
(b) *$\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ is L-borderenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic, Q-hyperenergetic nor Q-borderenergetic if G is isomorphic to Q_{20} or $U_{(2,5)}$.*

(c) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is L-hyperenergetic but neither hyperenergetic, borderenergetic, L- borderenergetic, Q-hyperenergetic nor Q-borderenergetic if G is isomorphic to* D_{16}, D_{20}, D_{24}, D_{28}, Q_{24}, Q_{28}, $U_{(3,5)}$, $U_{(3,6)}$, $U_{(2,7)}$, V_{24}, V_{32}, $G(2,3,2)$ *or* $G(2,1,3)$.

(d) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is L-hyperenergetic and Q-borderenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-hyperenergetic if G is isomorphic to* SD_{40}.

(e) $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is L-hyperenergetic and Q-hyperenergetic but neither hyperenergetic, borderenergetic, L-borderenergetic nor Q-borderenergetic if G is isomorphic to* D_{2n} *(n is even,* $n \geq 16$*),* Q_{4m} ($m \geq 8$), $U_{(n,5)}$ ($n \geq 4$), $U_{(n,6)}$ ($n \geq 4$), $U_{(n,7)}$ ($n \geq 3$), $U_{(n,m)}$ ($n \geq 2$ *and* $m \geq 8$), V_{8n} ($n \geq 5$), SD_{32}, SD_{8n} ($n \geq 6$), $G(2,m,2)$ ($m \geq 4$), $G(p,m,2)$ ($p \geq 3$ *and* $m \geq 1$), $G(2,m,3)$ ($m \geq 2$) *or* $G(p,m,n)$ ($n \geq 4$, $p \geq 2$ *and* $m \geq 1$).

Theorem 3.27 (*[17, Theorem 5.7]*)**.** *Let G be a finite non-abelian group.*

(a) *Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is neither hyperenergetic nor borderenergetic, for G is isomorphic to* D_{2n}, Q_{4m}, $U_{(n,m)}$, V_{8n}, SD_{8n} *or* $G(p,m,n)$.

(b) *Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is L-hyperenergetic, for G is isomorphic to* D_{2n} *(n is even,* $n \geq 8$*),* Q_{4m} ($m \geq 6$), $U_{(n,5)}$ ($n \geq 3$), $U_{(n,6)}$ ($n \geq 3$), $U_{(n,m)}$ ($n \geq 2$ *and* $m \geq 7$), V_{8n} ($n \geq 3$), SD_{8n} ($n \geq 4$), $G(2,m,2)$ ($m \geq 3$), $G(p,m,2)$ ($p \geq 3$ *and* $m \geq 1$), $G(2,m,3)$ ($m \geq 1$) *or* $G(p,m,n)$ ($n \geq 4$, $p \geq 2$ *and* $m \geq 1$).

(c) *Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is L-borderenergetic, for G is isomorphic to* Q_{20} *or* $U_{(2,5)}$.

(d) *Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is Q-hyperenergetic, for G is isomorphic to* D_{2n} *n is even,* ($n \geq 16$), Q_{4m} ($m \geq 8$), $U_{(n,5)}$ ($n \geq 4$), $U_{(n,6)}$ ($n \geq 4$), $U_{(n,7)}$ ($n \geq 3$), $U_{(n,m)}$ ($n \geq 2$ *and* $m \geq 8$), V_{8n} ($n \geq 5$), SD_{32}, SD_{8n} ($n \geq 6$), $G(2,m,2)$ ($m \geq 4$), $G(p,m,2)$ ($p \geq 3$ *and* $m \geq 1$), $G(2,m,3)$ ($m \geq 2$) *or* $G(p,m,n)$ ($n \geq 4$, $p \geq 2$ *and* $m \geq 1$).

(e) *Then* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ *is Q-borderenergetic, for G is isomorphic to* SD_{40}.

We conclude this section noting that problems analogous to Problems 3.2–3.4 for various common neighbourhood spectrum and energies of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ were considered in [50,51]. Common neighbourhood spectrum/energy, common neighbourhood Laplacian spectrum/energy and common neighbourhood signless Laplacian spectrum/energy of graphs were introduced in [9,52]. Various common neighbourhood spectrum and energies of $\mathcal{C}(G)[G \setminus Z(G)]$ were considered in [34,35,61].

3.6. Properties of $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$

The FC-centre of a group G is the set of elements $x \in G$ such that x^G is finite. Herzog et al. [48] obtained the following results for the induced subgraph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$ of $\mathcal{CCC}(G)$ induced by $\mathrm{Cl}(G) \setminus \{g^G : g \in FC(G)\}$ when G is a periodic group.

Theorem 3.28 (*[48, Theorem 22]*)**.** *Let G be a periodic group such that the graph* $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$ *is empty. Write* $F = FC(G)$ *and suppose that there exists*

$xF \in G/F$ such that $|xF| = 3$. Then G has the following structure:

$$G = F \rtimes \langle a, b \rangle,$$

where $|a| = 3$, $|b| = 2$, $a^b = a^{-1}$ (i.e.$\langle a, b \rangle \cong S_3$), F is an elementary abelian 2-group, $F = D_1 \times D_2$, where $D_1 = \{[d, b] : d \in F\}$, $D_2 = \{[d, ab] : d \in F\}$, D_1 and D_2 are infinite subgroups of F and a acts fixed-point-freely on F.

Conversely, if G has the above structure, then $F = FC(G)$ and the graph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$ is empty.

Theorem 3.29 (*[48, Theorem 23]*)**.** *Let G be a periodic group such that the graph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$ is empty. Write $F = FC(G)$. Then G is locally finite and either G is a hypercentral 2-group with G/F abelian of exponent 2 or G/F is finite. In the latter case, either G/F is a finite elementary abelian 2-group or $G/F \cong S_3$ and G has the structure described in Theorem 3.28.*

4. Nilpotent conjugacy class graph

We write $\mathcal{NCC}(G)$ to denote the NCC-graph of a group G. The NCC-graph of G is a graph whose vertex set is $\mathrm{Cl}(G)$ and two distinct vertices a^G and b^G are adjacent if there exist some elements $x \in a^G$ and $y \in b^G$ such that $\langle x, y \rangle$ is a nilpotent group. In this section, we discuss various results on induced subgraphs of $\mathcal{NCC}(G)$. Mohammadian and Erfanian [56] considered two induced subgraphs of $\mathcal{NCC}(G)$ induced by $\mathrm{Cl}(G \setminus 1)$ and $\mathrm{Cl}(G \setminus \mathrm{Nil}(G))$, where $\mathrm{Nil}(G) := \{g \in G : \langle g, x \rangle$ is nilpotent for all $x \in G\}$ is the hypercentre of G.

4.1. Connectivity of $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$

Mohammadian and Erfanian [56] obtained the following results analogous to Theorems 3.2 and 3.3.

Theorem 4.1. *Let G be any group.*

(a) *[56, Theorem 2.6-2.7] If G is finite solvable or periodic solvable then the graph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ has at most two connected components whose diameters are at most 7.*

(b) *[56, Theorem 2.10-2.11] If G is finite or locally finite then $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ has at most six connected components whose diameters are at most 10.*

Mohammadian and Erfanian [56] also obtained the following characterizations of supersolvable and solvable groups G such that $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is disconnected.

Theorem 4.2 (*[56, Theorem 3.2]*)**.** *Let G be a supersolvable group. Then the graph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is disconnected if and only if one of the following holds:*

(a) *If G is an infinite group, then $G = H \rtimes \langle a \rangle$, where $a \in G$, $|a| = 2$ and H is a subgroup of G on which a acts fixed-point-freely.*

(b) *If G is a finite group, then $G = H \rtimes K$ is a Frobenius group with kernel H and a cyclic complement K.*

Theorem 4.3 (*[56, Theorem 3.4]*)**.** *Let G be a finite solvable group with disconnected graph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$. Then there exists a nilpotent normal subgroup N of G such that one of the following holds:*

(a) *$G = N \rtimes H$ is a Frobenius group with the kernel N and a complement H.*
(b) *$G = (N \rtimes L) \rtimes \langle x \rangle$, where L is a non-trivial cyclic subgroup of G of odd order which acts fixed-point-freely on N, $x \in N_G(L)$ is such that $\langle x \rangle$ acts fixed-point-freely on L, and there exist $a \in N \setminus \{1\}$ and $i \in \mathbb{N}$ such that $x^i \neq 1$ and $[a, x^i] = 1$.*

Conversely, if either (a) *or* (b) *holds, then $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is disconnected.*

Notice that the two situations in Theorems 3.5 and 4.3, where we get disconnected $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ and $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$, are identical. Therefore, the following problem arises naturally.

Problem 4.1. Determine whether $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)] = \mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ if and only if one of the cases in Theorem 4.3 holds.

4.2. Properties of $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$

Mohammadian and Erfanian [56] also considered the subgraph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ of $\mathcal{NCC}(G)$ induced by the set $\mathrm{Cl}(G \setminus \mathrm{Nil}(G))$ in their study. They obtain the following characterizations of finite non-nilpotent groups G such that $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ is empty/triangle-free.

Theorem 4.4 (*[56, Theorem 4.3]*)**.** *Let G be a finite non-nilpotent group. Then the graph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ is an empty graph if and only if $G \cong S_3$.*

Theorem 4.5. *Let G be a finite non-nilpotent group.*

(a) *[56, Theorem 4.8] If $|G|$ is odd then $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ is triangle-free if and only if $|G| = 21$.*
(b) *[56, Theorem 4.9] If $|G|$ is even then $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ is triangle-free if and only if G is isomorphic to one of the groups S_3, D_{10}, D_{12}, A_4, T_{12} or $\mathrm{PSL}(2, q)$ where $q \in \{4, 7, 9\}$.*

Note that the structure of graph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ is not realized much. If the structures of $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ for various families of finite groups are known then one can consider problems similar to Problems 3.2–3.4 for the graph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$. Therefore, the following problem is worth mentioning.

Problem 4.2. Determine the structure of $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ for various families of finite non-nilpotent groups.

We conclude this section with the following problem analogous to Problem 3.1.

Problem 4.3. Characterize all finite non-nilpotent groups G such that the induced subgraph $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Nil}(G))]$ of $\mathcal{NCC}(G)$ is planar, toroidal, double-toroidal or triple-toroidal.

5. Solvable conjugacy class graph

We write $\mathcal{SCC}(G)$ to denote the SCC-graph of a group G. The SCC-graph of G is a graph whose vertex set is $\mathrm{Cl}(G)$ and two distinct vertices a^G and b^G are adjacent if there exist some elements $x \in a^G$ and $y \in b^G$ such that $\langle x, y\rangle$ is a solvable group. In this section, we discuss results on an induced subgraph of $\mathcal{SCC}(G)$. Bhowal et al. [14,15] considered the subgraph of $\mathcal{SCC}(G)$ induced by $\mathrm{Cl}(G \setminus 1)$.

5.1. Connectivity of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$

Not much is known about the connectivity of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$. We have the following problem whose answer is not known.

Problem 5.1. If G is a finite non-solvable group then determine the number of components of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ and an upper bound for diameters of its components.

The answers to Problem 5.1 for $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ and $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ are given in Theorems 3.3 and 4.1 respectively. The following results are known regarding the connectivity of the graph $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$.

Theorem 5.1 (*[14, Theorem 2.1]*)**.** *Let G be a finite group. Then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is complete if and only if G is solvable.*

Theorem 5.2 (*[14, Theorem 2.9]*)**.** *If G, H are arbitrary non-trivial groups then the graph* $\mathcal{SCC}(G \times H)[\mathrm{Cl}(G \times H \setminus (1_G, 1_H))]$ *is connected with diameter* ≤ 3. *In particular,* $\mathcal{SCC}(G \times G)[\mathrm{Cl}(G \times G \setminus (1_G, 1_G))]$ *is connected with diameter* ≤ 3. *Further,* $\mathrm{diam}(\mathcal{SCC}(G \times G)[\mathrm{Cl}(G \times G \setminus (1_G, 1_G))]) = 3$ *if and only if* $\mathrm{diam}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) \ge 3$ *(possibly infinite).*

Theorem 5.3 (*[14, Theorem 4.4]*)**.** *Let G be a finite group. Let H and K be two subgroups of G such that H is normal in G,* $G = HK$ *and* $\mathcal{SCC}(H)[\mathrm{Cl}(H \setminus 1)]$, $\mathcal{SCC}(K)[\mathrm{Cl}(K \setminus 1)]$ *are connected. If there exist two elements* $h \in H \setminus \{1\}$ *and* $x \in G \setminus H$ *such that* h^G *and* x^G *are connected in* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is connected.*

Theorem 5.4 (*[14, Theorem 3.3]*)**.** *Let G be a finite group. If G has an element of order* $n = \Pi_{i=1}^m p_i^{k_i}$, *where* p_i*'s are distinct primes. Then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *has a clique of size* $\Pi_{i=1}^m (k_i + 1) - 1$.

Theorem 5.5 (*[14, Theorem 3.5]*)**.** *For any positive integer d, there are only finitely many finite groups G such that the clique number of $\mathcal{SCC}(G)$ is d.*

Problem 5.2. Do the analogues of Theorem 5.5 hold for the graphs $\mathcal{CCC}(G)$ and $\mathcal{NCC}(G)$?

We conclude this section with the following result which shows that the graph $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is triangle-free when $G \cong \{1\}, \mathbb{Z}_2, \mathbb{Z}_3$ or S_3, the symmetric group of degree 3.

Theorem 5.6 (*[14, Theorem 3.4]*)**.** *With the exception of the cyclic groups of orders* 1, 2 *and* 3 *and the symmetric group of degree* 3, *every finite group G has the property that $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ contains a triangle (that is, has girth* 3*).*

5.2. Genus of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$

We have seen various results on genus of the graph $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$ in Section 3.4. The genus of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus \mathrm{Sol}(G))]$, where $\mathrm{Sol}(G) := \{g \in G : \langle g, x\rangle$ is solvable for all $x \in G\}$, the solvable radical of G, is not studied so far. However, the following problem is considered in [15].

Problem 5.3. Characterize all finite groups G such that the graph $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ is planar, toroidal, double-toroidal, triple-toroidal or projective.

Let N_k be the connected sum of k projective planes. A simple graph Γ which can be embedded in N_k but not in N_{k-1}, is called a graph with crosscap k. We write $\bar{\gamma}(\Gamma)$ to denote the crosscap of Γ. A graph Γ is called projective if $\bar{\gamma}(\Gamma) = 1$. Bhowal et al. [15] obtained the following results related to Problem 5.3.

Theorem 5.7 (*[15, Theorem 3.1]*)**.** *Let $G = D_{2n}$. Then*

(a) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is planar if and only if $n = 2, 3, 4, 5$ and 7.*
(b) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is toroidal if and only if $n = 6, 8, 9, 10, 11$ and 13.*
(c) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is double-toroidal if and only if $n = 12$ and 15.*
(d) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is triple-toroidal if and only if $n = 14$ and 17.*
(e) $$\gamma(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{48} \right\rceil, & \text{when } n \geq 19 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{48} \right\rceil, & \text{when } n \geq 16 \text{ and } n \text{ is even.} \end{cases}$$
(f) $\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = 0$ *if and only if $n = 2, 3, 4, 5$ and 7.*
(g) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is projective if and only if $n = 6, 8, 9$ and 11.*
(h) $$\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{24} \right\rceil, & \text{when } n \geq 13 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{24} \right\rceil, & \text{when } n \geq 10 \text{ and } n \text{ is even.} \end{cases}$$

Theorem 5.8 (*[15, Theorem 3.2]*)**.** *Let $G = Q_{4m}$. Then*

(a) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is planar if and only if* $m = 1$ *and* 2.
(b) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is toroidal if and only if* $m = 3, 4$ *and* 5.
(c) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is double-toroidal if and only if* $m = 6$.
(d) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is triple-toroidal if and only if* $m = 7$.
(e) $\gamma(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \left\lceil \frac{(m-1)(m-2)}{12} \right\rceil$ *for* $m \geq 8$.
(f) $\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = 0$ *if and only if* $m = 1$ *and* 2.
(g) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is projective if and only if* $m = 3$ *and* 4.
(h) $\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{24} \right\rceil, & \text{when } n \geq 13 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{24} \right\rceil, & \text{when } n \geq 10 \text{ and } n \text{ is even.} \end{cases}$

Theorem 5.9 (*[15, Theorem 3.3]*)**.** *Let* G *be a finite solvable group. Then*

(a) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is planar if and only if* $|\mathrm{Cl}(G)| = 1, 2, 3, 4$ *and* 5.
(b) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is toroidal if and only if* $|\mathrm{Cl}(G)| = 6, 7$ *and* 8.
(c) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is double-toroidal if and only if* $|\mathrm{Cl}(G)| = 9$.
(d) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is triple-toroidal if and only if* $|\mathrm{Cl}(G)| = 10$.
(e) $\gamma(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \left\lceil \frac{(|\mathrm{Cl}(G)|-4)(|\mathrm{Cl}(G)|-5)}{12} \right\rceil$ *for* $|\mathrm{Cl}(G)| \geq 11$.
(f) $\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = 0$ *if and only if* $|\mathrm{Cl}(G)| = 1, 2, 3, 4$ *and* 5.
(g) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is projective if and only if* $|\mathrm{Cl}(G)| = 6$ *and* 7.
(h) $\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \begin{cases} 3, & \text{when } m = 5 \\ \left\lceil \frac{(m-1)(m-2)}{6} \right\rceil, & \text{when } m \geq 6. \end{cases}$

Theorem 5.10 (*[15, Theorem 3.4]*)**.** *Let* $G = S_n$. *Then*

(a) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is planar if and only if* $n \leq 5$.
(b) *If* $n \geq 7$ *then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is neither planar, toroidal, double-toroidal nor triple-toroidal.*
(c) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is not toroidal if* $n = 6$.
(d) *If* $n \geq 6$ *then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is not projective.*

Theorem 5.11 (*[15, Theorem 3.5]*)**.** *Let* $G = A_n$, *the alternating group of degree* n. *Then*

(a) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is planar if and only if* $n \leq 6$.
(b) *If* $n \geq 9$ *then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is neither planar, toroidal, double-toroidal nor triple-toroidal.*
(c) $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is toroidal if and only if* $n = 7$.
(d) *If* $n \geq 8$ *then* $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ *is not projective.*

Bhowal et al. [15] also obtained the genus of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ when G is certain projective special linear group.

Theorem 5.12 (*[15, Theorem 3.6]*)**.** *Let* $G = \mathrm{PSL}(2, q)$, *where* $q = 2^d$ *with* $d \geq 3$. *Then*

$$\gamma(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = \gamma(K_{q/2}) + \gamma(K_{q/2+1}) \text{ or } \gamma(K_{q/2}) + \gamma(K_{q/2+1}) - 1.$$

Regarding bounds for genus and crosscap of $\mathcal{SCC}(G)[\mathrm{Cl}(G\setminus 1)]$ we have the following results.

Theorem 5.13 (*[15, Theorem 2.4]*)**.** *Given $n \geq 10$, let $k = p(\lfloor n/2 \rfloor) - 1$. If $G = S_n$, the symmetric group of degree n, and $\Gamma = \mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$ then*

$$\gamma(\Gamma) \geq \left\lceil \frac{(k-3)(k-4)}{12} \right\rceil \quad \textit{and} \quad \bar{\gamma}(\Gamma) \geq \left\lceil \frac{(k-3)(k-4)}{6} \right\rceil.$$

Theorem 5.14 (*[15, Theorem 2.6]*)**.** *Let G be a finite non-solvable group with non-trivial centre $Z(G)$. Then*

$$4\gamma(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) \geq (|Z(G)| - 3)(|\mathrm{Cl}(G)| - |Z(G)| - 2).$$

As an application of Theorem 5.14, we have the following bound for $\Pr(G)$ which is the probability that a randomly chosen pair of elements of G commute; also known as commutativity degree of G.

Theorem 5.15 (*[15, Corollary 2.7]*)**.** *Let G be a finite non-solvable group and $|Z(G)| > 3$. Then*

$$\Pr(G) \leq \frac{4\gamma(\Gamma_{sc}(G)) + (|Z(G)| - 3)(|Z(G)| + 2)}{|G|(|Z(G)| - 3)}.$$

Many other bounds for $\Pr(G)$ using various group theoretic notions can be found in [24,40,60]. It is worth noting that similar bounds for nilpotency degree [26] and solvability degree [36] can be obtained by obtaining bounds for $\gamma(\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)])$ and $\gamma(\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)])$ similar to Theorem 5.14.

It is intuitive that the order of a finite group G is bounded (above) by a function of the genus of $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]$. In this regard, we have the following problem.

Problem 5.4 (*[15, Problem 2.2]*)**.** Find an explicit bound for the order of a finite group G for which

(a) $\gamma(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = k$.
(b) $\bar{\gamma}(\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus 1)]) = k$.

We conclude this section noting that problems similar to Problem 5.4 for the graphs $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus 1)]$ and $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus 1)]$ are worth considering.

6. Concluding remarks

In [23], conditions for holding equalities in the hierarchy (1) were discussed for various $\mathcal{P}$ graphs of finite groups. For instance, the cyclic graph of G is equal to the commuting graph of G if and only if G contains no subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ (where p is prime); the commuting graph of G is equal to the nilpotent graph of G if and only if the Sylow subgroups of G are abelian; the nilpotent graph of G is equal to the solvable graph of G if and only if G is nilpotent. It may be interesting to obtain conditions for holding equalities in the hierarchy (2) for various $\mathcal{PCC}$-graphs.

Let $DV(\mathcal{PCC}(G))$ be the set of all dominant vertices of $\mathcal{PCC}(G)$. In view of Proposition 2.5, $DV(\mathcal{CCC}(G)) = \mathrm{Cl}(Z(G))$. It is easy to see that $\mathrm{Cl}(\mathrm{Nil}(G)) \subseteq DV(\mathcal{NCC}(G))$ and $\mathrm{Cl}(\mathrm{Sol}(G)) \subseteq DV(\mathcal{SCC}(G))$. The following problem along with similar problems for $\mathcal{CCC}(G)$ and $\mathcal{NCC}(G)$ are worth mentioning here.

Problem 6.1 (*[15, Problem 3.8]*)**.** Which non-solvable finite groups G have the property that $|DV(\mathcal{SCC}(G))| = 2$?

To make the question of connectedness of $\mathcal{P}$ conjugacy class graphs interesting it is necessary to determine $DV(\mathcal{PCC}(G))$ (also see Problem 2.2). Now consider the following problem.

Problem 6.2. Determine whether the induced subgraphs $\mathcal{CCC}(G)[\mathrm{Cl}(G \setminus Z(G))]$, $\mathcal{NCC}(G)[\mathrm{Cl}(G) \setminus DV(\mathcal{NCC}(G))]$ and $\mathcal{SCC}(G)[\mathrm{Cl}(G) \setminus DV(\mathcal{SCC}(G))]$ of $\mathcal{CCC}(G)$, $\mathcal{NCC}(G)$ and $\mathcal{SCC}(G)$ respectively are connected. Also, find upper bounds for the diameters of their components.

Note that the induced subgraphs $\mathcal{NCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$ and $\mathcal{SCC}(G)[\mathrm{Cl}(G \setminus FC(G))]$ of $\mathcal{NCC}(G)$ and $\mathcal{SCC}(G)$ respectively are not studied yet. (Recall that $FC(G)$ is the FC-centre of G, see Section 3.6.) Therefore, researchers working in this area may consider these graphs in their study.

The complements of cyclic/commuting/nilpotent/solvable graphs (in other words non-cyclic/non-commuting/non-nilpotent/non-solvable graphs) of finite groups were well-studied over the years (see [2–5,20,28,33,47,76]). However, the complements of $\mathcal{PCC}$-graphs of G are not studied. Note that $DV(\mathcal{PCC}(G))$ is the set of isolated vertices of the complement of $\mathcal{PCC}(G)$. Thus, by Proposition 2.5, it follows that the set of isolated vertices of the complement of $\mathcal{CCC}(G)$ is $\mathrm{Cl}(Z(G))$. The following problem is equivalent to Problem 2.2.

Problem 6.3. Describe the set of isolated vertices of the complements of NCC- or SCC-graph of a finite group.

Similarly, we have equivalent problems corresponding to Problems 2.1, 4.1, 4.2 and 6.1 for the complement of $\mathcal{PCC}(G)$. We conclude this paper noting that problems analogous to Problems 3.1–3.4, 4.3, 5.1–5.4 and 6.2 for complements of $\mathcal{PCC}$-graphs of G are worth considering.

CRediT authorship contribution statement

Peter J. Cameron: Writing – review & editing, Writing – original draft, Methodology, Investigation. **Firdous Ee Jannat:** Writing – review & editing, Writing – original draft, Methodology, Investigation. **Rajat Kanti Nath:** Writing – review & editing, Writing – original draft, Methodology, Investigation. **Reza Sharafdini:** Writing – review & editing, Writing – original draft, Methodology, Investigation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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