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A dichotomy on the self-similarity of graph-directed attractors

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Abstract. This paper seeks conditions that ensure that the attractor of a graph directed iterated function system (GD-IFS) cannot be realised as the attractor of a standard iterated function system (IFS). For a strongly connected directed graph, it is known that, if all directed circuits go through a particular vertex, then for any GD-IFS of similarities on $\mathbb{R}$ based on the graph and satisfying the convex open set condition (COSC), its attractor associated with that vertex is also the attractor of a (COSC) standard IFS. In this paper we show the following complementary result. If there exists a directed circuit which does not go through a certain vertex, then there exists a GD-IFS based on the graph such that the attractor associated with that vertex is not the attractor of any standard IFS of similarities. Indeed, we give algebraic conditions for such GD-IFS attractors not to be attractors of standard IFSs, and thus show that 'almost-all' COSC GD-IFSs based on the graph have attractors associated with this vertex that are not the attractors of any COSC standard IFS.

1. Introduction

An *iterated function system* (IFS) $\{S_i\}_i$ is a finite set of distinct contracting maps on a complete metric space which we will assume here to be $\mathbb{R}^n$ [11]. The *attractor* of the IFS is the unique non-empty compact set $K \subset \mathbb{R}^n$ such that

$$K = \bigcup_{i=1}^{m} S_i(K). \tag{1.1}$$

If these maps are all contracting similarities, we say that this IFS is a *standard IFS*, and call K a *self-similar set*. A contracting similarity $S(x)$ on $\mathbb{R}$ can be written as $S(x) = \rho x + b$, where $\rho \in (-1, 1) \setminus \{0\}$ is the *contraction ratio*.

Separation conditions for IFSs are often required to ensure 'not too much overlapping' in the union (1.1). A frequent condition is the *open set condition* (OSC), meaning that there exists a non-empty open set $U \subseteq \mathbb{R}^n$ such that $\bigcup_{i=1}^{m} S_i(U) \subseteq U$ with this union disjoint. We say that the IFS satisfies the *convex open set condition* (COSC) if U can be chosen to be convex, or we can (equivalently) take $U =$

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int(conv K) where 'conv' denotes the convex hull, 'int' denotes the interior of a set. We say that the IFS satisfies the *convex strong separation condition* (CSSC) if we can take $U = \text{int}(\text{conv } K)$ such that $S_i(\text{conv } K) \cap S_j(\text{conv } K) = \emptyset$ for any $i \neq j$.

We also consider graph-directed IFSs [12] based on a given digraph. A *directed graph* (or *a digraph* for brevity), $G := (V, E)$, consists of a finite set of vertices V and a finite set of directed edges E (for brevity, we often omit 'directed') with loops and multiple edges allowed. Let $E_{uv} \subset E$ be the set of edges from the *initial* vertex u to the *terminal* vertex v. A *graph-directed iterated function system* (GD-IFS) on $\mathbb{R}^n$ consists of a finite collection of contracting similarities $\{S_e : e \in E_{uv}\}$ from $\mathbb{R}^n_v$ to $\mathbb{R}^n_u$ for $u, v \in V$, where $\mathbb{R}^n_u$ is a copy of $\mathbb{R}^n$ associated with vertex u. We write $\rho_e \in (-1, 1) \setminus \{0\}$ for the contraction ratio of the similarity S_e in $\mathbb{R}$. We always require the digraph satisfies that $d_u \geq 1$ for every $u \in V$ ([12], [4, Section 4.3]), where d_u is the *out-degree* of u (the number of directed edges leaving u). For a GD-IFS $(V, E, (S_e)_{e\in E})$ based on such a digraph, there exists a unique list of non-empty compact sets $(F_u \subset \mathbb{R}^n_u)_{u\in V}$ such that, for all $u \in V$,

$$F_u = \bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(F_v), \tag{1.2}$$

see [12] or [4, Theorem 4.3.5 on p.128]. We call the above $(F_u)_{u\in V}$ the *(list of) attractors* of the GD-IFS, and each F_u is called a *GD-attractor*. A (finite) *directed path* $e_1e_2\cdots e_k$ is a consecutive sequence of directed edges $e_i \in E$ $(i = 1, \ldots, k)$ for which the terminal vertex of e_i is the initial vertex of e_{i+1} $(i = 1, \ldots, k-1)$. For a directed path $\mathbf{e} = e_1e_2\cdots e_k$ with edges e_i $(1 \leq i \leq k)$, the corresponding contractive mapping is given by $S_{\mathbf{e}} = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$, and its contraction ratio along $\mathbf{e}$ is $\rho_{\mathbf{e}} = \rho_{e_1}\rho_{e_2}\cdots\rho_{e_k}$.

For a GD-IFS there are analogous separation conditions. The *open set condition* (OSC) is satisfied if there exist non-empty bounded open sets $(U_u \subset \mathbb{R}^n_u)_{u\in V}$, with

$$\bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(U_v) \subset U_u$$

and the union is disjoint for each $u \in V$. The *convex open set condition* (COSC) means that these $(U_u)_{u\in V}$ can all be chosen to be convex. In one-dimensional case, one can take

$$(U_u)_{u\in V} = (\text{int}(\text{conv } F_u))_{u\in V}, \tag{1.3}$$

since $\text{conv } F_u \subset \overline{U_u}$ for each $u \in V$ (see Proposition 5.2 in the Appendix). We say that a GD-IFS satisfies the CSSC (*convex strong separation condition*), if the union

$$\bigcup_{v\in V} \bigcup_{e\in E_{uv}} S_e(\text{conv } F_v) \qquad \text{(which belongs to conv } F_u)$$

is disjoint for each $u \in V$.

GD-attractors and GD-IFSs appear naturally in dynamical systems and fractal geometry. For example, certain complex dynamical systems can be regarded as conformal GD-IFSs using a Markov partition, see [7, Section 5.5]. For another occurrence, the orthogonal projection of certain self-similar sets may be GD-attractors [8, Theorem 1.1]. We will work with COSC (including CSSC) GD-IFSs defined on $\mathbb{R}$ based on digraphs with $d_u \geq 2$ for every vertex u in V throughout this paper.

We say that a digraph is strongly connected if, for all vertices $u, v \in V$, there is a *directed path* from u to v (we allow $u = v$). For brevity, we will assume throughout that a strongly connected digraph always satisfies $d_u \geq 2$ for all $u \in V$. This is because, if $d_v = 1$ ($v \in V$), then F_v is just a scaled copy of another GD-attractor F_w ($w \in V \setminus \{v\}$). Then F_v is self-similar (with the COSC) if and only if F_w is self-similar (with the COSC), since if K is the attractor of the IFS $\{\rho_i x + b_i\}_i$, then $\eta K + l$ is the attractor of the IFS $\{\rho_i x + \eta b_i + (1 - \rho_i)l\}_i$ ($\eta, l \in \mathbb{R}$). We can do a reduction as in [5, pp.607] on any strongly connected digraph and associated GD-IFS, to obtain a subgraph and new GD-IFS with $d_u \geq 2$ for all $u \in V$ such that each attractor is similar to one of the original ones.

A natural question arises, “When does a GD-IFS of similarity mappings have attractors which cannot be realised as attractors of any standard IFS?”. In particular, we seek algebraic conditions involving the parameters underlying the GD-IFS similarities that ensure this is so. Some cases were examined in an earlier paper [3] which showed that, for a class of strongly connected digraphs, it is possible to construct CSSC GD-IFSs on $\mathbb{R}$ with attractors that cannot be obtained from a standard IFS, with or without the CSSC. Another paper [2] uses a different argument to construct CSSC GD-IFSs on $\mathbb{R}$ with attractors that cannot be obtained from a standard IFS. This paper further investigates this issue for all *strongly connected* digraphs (or even wider classes of digraphs).

For a strongly connected digraph G, it is known from [2, Lemma 5.1] (see also Theorem 5.4 in the Appendix) that, if all directed circuits in G go through a vertex $u \in V$, then for any (COSC) GD-IFS based on G, its attractor F_u is also the attractor of a (COSC) standard IFS. By way of contrast, we will show that if, for some vertex $u \in V$, not all directed circuits in G go through u, then it is possible to define GD-IFSs of similarities satisfying the COSC so that the corresponding attractor F_u is not the attractor of a standard IFS of similarities satisfying the COSC (Lemma 4.4). Moreover, this is true for ‘almost all’ choices of similarities in a natural sense (Theorem 4.8). The proof basically relies on identifying a characteristic of the ‘gap length set’, where we use a shorter systematical algebraic argument ‘ratio analysis’ rather than the categorising method of [3, Section 6] which only works for certain classes of digraphs. In fact, we can relax the strong connectivity of G in this construction (Lemma 4.1) and the ‘ratio analysis’ method may have further applications to other related problems. We finally apply [2, Theorem 1.4] (see also Theorem 5.6 in the

Appendix) to show immediately that there exists GD-IFSs of similarities with the CSSC so that the corresponding attractor F_u is not the attractor of a standard IFS.

GD-IFSs considered in this paper are *inhomogeneous*, by which we mean GD-IFSs of contracting similarities with not all contraction ratios equal. We will require the COSC condition, which is easy to verify from the parameters of a GD-IFS by solving simultaneous linear inequalities. There are difficulties in relaxing this condition to OSC (even in $\mathbb{R}$) where many problems still remain open even for standard IFSs, such as the affine-embedding problem [10, Conjecture 1.1] or the inverse fractal problem (determining the *generating* IFSs of a standard IFS attractor) [9]. The question considered here can be viewed as an inverse-type problem, where we show certain GD-attractors have no generating standard IFS (with or without the COSC). Previous results on inhomogeneous self-similar sets also require this condition [9, Section 4] or stronger conditions such as SSC and restrictions on Hausdorff dimension [1,6,10]. Thus, one might expect similar difficulties for inhomogeneous GD-attractors.

This paper is organised as follows. In Section 2, we first introduce and obtain an expression for the gap length set of COSC GD-attractors, and we then introduce our algebraic method 'ratio analysis', and derive a key lemma (Lemma 2.9) relating the ratio sets of GD-IFSs and standard IFSs with the COSC. In Section 3 we introduce natural vector sets and construct GD-IFSs satisfying the COSC or the CSSC. In Section 4 we use the GD-IFSs constructed in Section 3 to show that the corresponding GD-attractors are not the attractors of COSC standard IFSs using both the 'ratio analysis' lemmas and the tool developed in [2]. We provide some examples to illustrate our assertions.

2. Gap length sets and ratio analysis

2.1. Gap length sets

For a compact set $K \subset \mathbb{R}$ with $(\mathrm{conv}\, K) \setminus K \neq \emptyset$, let

$$(\mathrm{conv}\, K) \setminus K = \bigcup_i U_i$$

be the unique decomposition of the disjoint non-empty bounded complementary intervals $\{U_i = (a_i, b_i)\}_i$ (see for example [13, Chapter 2, Theorem 9]), which will be called the *gaps* of K numbered by decreasing length (and left to right for equal length intervals).

Definition 2.1 (Gap length set). Define the *gap length set* of a compact set $K \subset \mathbb{R}$ to be

$$\mathrm{GL}(K) := \{b_i - a_i\}_i$$

that is, the set of lengths of all the gaps of K. If $(\text{conv}\, K) \setminus K = \emptyset$, that is, if K is an interval (or a singleton), we define $\text{GL}(K) := \emptyset$.

For each vertex $u \in V$, we arrange the edges leaving u, denoted by $e_u^{(k)}$ ($k = 1, \ldots, d_u$) in the following way. Denote by $\omega(e)$ the terminal vertex of an edge $e \in E$, then the interiors of the intervals $S_e(\text{conv}\, F_{\omega(e)})$ are disjoint due to the COSC. We rank these intervals in order from left to right, and denote the kth interval by

$$S_u^{(k)}\big(\text{conv}\, F_{\omega(e_u^{(k)})}\big) \qquad (1 \le k \le d_u)$$

with the edges (and also the GD-IFS $\{S_e\}_{e \in E}$) arranged according to this order.

Definition 2.2 (Basic gaps). With the above notation, for each $u \in V$ and $1 \le k \le d_u - 1$ ($d_u \ge 2$), let $\lambda_u^{(k)}$ be the length of the complementary open interval between $S_u^{(k)}(\text{conv}\, F_{\omega(e_u^{(k)})})$ and $S_u^{(k+1)}(\text{conv}\, F_{\omega(e_u^{(k+1)})})$ (possibly $\lambda_u^{(k)} = 0$). All such complementary intervals (possibly empty) are called the *basic gaps* of this ordered COSC GD-IFS $\{S_u^{(k)}\}$ sitting at vertex u. Let

$$\Lambda_u := \{\lambda_u^{(k)} \colon \lambda_u^{(k)} > 0, 1 \le k \le d_u - 1\} \tag{2.1}$$

be the set of strictly positive lengths of the basic gaps associated with vertex $u \in V$, see Figure 1.

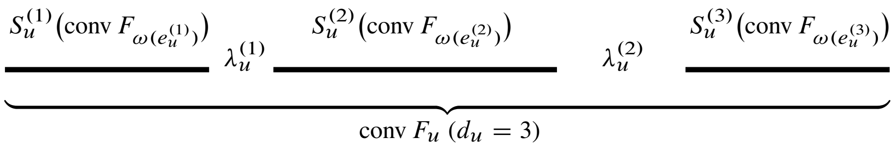

Figure 1. Basic gaps of F_u.

As standard IFSs are one-vertex GD-IFSs, this definition is also applicable to standard IFSs when we will omit the single vertex.

The GD-attractors $(F_u)_{u \in V}$ of any GD-IFS can be determined in the following way, see [12, Equation (15)]. For any list of compact sets $(I_u)_{u \in V}$, we define

$$I_u^m := \bigcup_{\mathbf{e} \in E_u^m} S_{\mathbf{e}}(I_{\omega(\mathbf{e})}) \quad \text{for any } m \ge 1, \tag{2.2}$$

where E_u^m denotes the set of paths of length m leaving u and $\omega(\mathbf{e})$ denotes the terminal vertex of path $\mathbf{e}$. Note that if

$$I_u^1 \subset I_u \text{ for each } u \in V, \tag{2.3}$$

then the sequence I_u^m decreases in m in the sense that $I_u^{m+1} \subseteq I_u^m$ for every $m \geq 1$, since

$$\begin{aligned} I_u^{m+1} &= \bigcup_{\tilde{\mathbf{e}} \in E_u^{m+1}} S_{\tilde{\mathbf{e}}}\big(I_{\omega(\tilde{\mathbf{e}})}\big) = \bigcup_{\mathbf{e} \in E_u^m} \bigcup_{e \in E^1_{\omega(\mathbf{e})}} S_{\mathbf{e}} \circ S_e\big(I_{\omega(e)}\big) \\ &= \bigcup_{\mathbf{e} \in E_u^m} S_{\mathbf{e}}\Big(\bigcup_{e \in E^1_{\omega(\mathbf{e})}} S_e\big(I_{\omega(e)}\big)\Big) = \bigcup_{\mathbf{e} \in E_u^m} S_{\mathbf{e}}\big(I^1_{\omega(\mathbf{e})}\big) \\ &\subseteq \bigcup_{\mathbf{e} \in E_u^m} S_{\mathbf{e}}\big(I_{\omega(\mathbf{e})}\big) = I_u^m. \end{aligned} \tag{2.4}$$

From this, it is known that for each $u \in V$,

$$F_u = \bigcap_{m=1}^{\infty} I_u^m, \tag{2.5}$$

provided that (2.3) is satisfied.

In particular, taking $I_u = \operatorname{conv} F_u$ for each $u \in V$, we see that (2.3) is satisfied, since by (1.2)

$$F_u \subseteq \bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(\operatorname{conv} F_v) = I_u^1 \subseteq \operatorname{conv}\Big(\bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(F_v)\Big) = \operatorname{conv} F_u = I_u. \tag{2.6}$$

In this case, the (2.5) is true. Moreover, by taking convex hulls in (2.6), we know that

$$\operatorname{conv} F_u \subseteq \operatorname{conv} I_u^1 \subseteq \operatorname{conv}\operatorname{conv} F_u = \operatorname{conv} F_u,$$

which gives that

$$\operatorname{conv} I_u^1 = \operatorname{conv} F_u = I_u, \tag{2.7}$$

meaning that the two endpoints of the interval $\operatorname{conv} I_u^1$ coincide with those of the interval $\operatorname{conv} F_u = I_u$. This fact will be used shortly.

Throughout this paper, the product AB of sets $A, B \subset \mathbb{R}$ is defined to be $AB = \{ab : a \in A, b \in B\}$, and when we encounter the product of a set in $\mathbb{R}$ and a constant, regard the constant as a set in $\mathbb{R}$. If A is an empty set, then AB is also empty.

The following proposition gives a characterization for the gap length set of an attractor F_u of any COSC GD-IFS, which slightly extends a result in [3, below equation (5.2) in Section 5] to the case when a GD-IFS satisfies the COSC.

Proposition 2.3. *Let (V, E) be a digraph with $d_u \geq 2$ for all $u \in V$, and let F_u be a GD-attractor of a GD-IFS in $\mathbb{R}$ with the COSC based on (V, E). With the above* *notation, the gap length set* $\mathrm{GL}(F_u)$ *of the attractor* F_u *is given by*

$$\mathrm{GL}(F_u) = \Lambda_u \bigcup \Big(\bigcup_{m=1}^{\infty} \bigcup_{v \in V} \Lambda_v \{ |\rho_{\mathbf{e}}| : \mathbf{e} \textit{ is a directed path from } u \textit{ to } v \textit{ with length } m \} \Big). \tag{2.8}$$

When there is no directed path from u *to* v*, the set* $\{|\rho_{\mathbf{e}}|\}$ *is understood to be empty.*

Proof. When $\mathrm{GL}(F_u) = \emptyset$, that is, $F_u = \operatorname{conv} F_u$, we have for all $m \geq 1$

$$\operatorname{conv} F_u \supseteq \bigcup_{v \in V} \bigcup_{\mathbf{e} \in E_{uv}^m} S_{\mathbf{e}}(\operatorname{conv} F_v) \supseteq \bigcup_{v \in V} \bigcup_{\mathbf{e} \in E_{uv}^m} S_{\mathbf{e}}(F_v) = F_u = \operatorname{conv} F_u,$$

where E_{uv}^m is a collection of paths from vertex u to vertex v with length m. From this and using the COSC, we see that

$$S_{\mathbf{e}}(\operatorname{conv} F_v) = S_{\mathbf{e}}(F_v)$$

for every $v \in V$ and every directed path $\mathbf{e}$ of length m from u to v, showing that $\Lambda_v = \emptyset$ for all $v \in V$ to which a directed path from u exists. Thus, (2.8) is trivial in this case.

In the sequel, we assume that $\mathrm{GL}(F_u) \neq \emptyset$. Let $u \in V$ be a vertex. Set $I_u := \operatorname{conv} F_u$ for each $u \in V$, and (2.5) holds true by virtue of (2.6). So the gaps of F_u will be given by

$$(\operatorname{conv} F_u) \setminus F_u = I_u \setminus \Big(\bigcap_{m=1}^{\infty} I_u^m \Big) = \Big(I_u \setminus I_u^1 \Big) \bigcup \Big(\bigcup_{m=1}^{\infty} I_u^m \setminus I_u^{m+1} \Big), \tag{2.9}$$

which consists of the complementary open intervals in $I_u \setminus I_u^1$ and $I_u^m \setminus I_u^{m+1}$ ($1 \leq m < \infty$). We need to calculate the lengths of these open intervals.

Indeed, for the open set $I_u \setminus I_u^1$, we know by definition (2.2) that

$$I_u \setminus I_u^1 = \operatorname{conv} F_u \setminus \bigcup_{\mathbf{e} \in E_u^1} S_{\mathbf{e}}\big(I_{\omega(\mathbf{e})}\big) = \bigcup_{i=1}^{d_u - 1} G_u^{(r)}, \tag{2.10}$$

where $G_u^{(r)}$ for $1 \leq r \leq d_u - 1$ form the basic gaps of F_u, whose lengths form the set Λ_u by using (2.7) with the property that two intervals I_u and I_u^1 have the same endpoints, see Figure 1.

On the other hand, for any $m \geq 1$, due to the COSC, the interiors of the level-m intervals $\{S_{\mathbf{e}}(I_{\omega(\mathbf{e})})\}_{\mathbf{e} \in E_u^m}$ are disjoint for any m. We know by (2.4) that

$$I_u^m \setminus I_u^{m+1} = \bigcup_{\mathbf{e} \in E_u^m} S_{\mathbf{e}}\big(I_{\omega(\mathbf{e})} \setminus I_{\omega(\mathbf{e})}^1\big) = \bigcup_{\mathbf{e} \in E_u^m} S_{\mathbf{e}}\Bigg(\bigcup_{r=1}^{d_{\omega(\mathbf{e})} - 1} G_{\omega(\mathbf{e})}^{(r)} \Bigg) \qquad \text{(using (2.10))}.$$

The above union consists of disjoint complementary open intervals $S_{\mathbf{e}}(G^{(r)}_{\omega(\mathbf{e})})$, whose lengths are given by $|\rho_{\mathbf{e}}| \cdot \lambda^{(r)}_{\omega(\mathbf{e})}$, which form the gap length sets at the mth-level for any $m \geq 1$. Summing up over m will give the double union in the right-hand side of (2.8), and so (2.8) follows from (2.9) and the definition of $\mathrm{GL}(F_u)$. ∎

2.2. Ratio analysis

We will use "ratio analysis" to analyse sets Θ of positive real numbers in $(0, \infty)$, in terms of strictly decreasing geometric sequences $\{\theta' r^k\}_{k=0}^{\infty}$ that are contained in Θ.

Definition 2.4. Let $\Theta \subset (0, \infty)$. For $\theta \in \Theta$, let

$$R_{\Theta}(\theta) = \left\{r \in (0,1)\colon \text{there exists some } \theta' \in \Theta \text{ such that } \theta \in \{\theta' r^k\}_{k=0}^{\infty} \subset \Theta\right\} \quad (2.11)$$

be the set of common ratios of strictly decreasing geometric sequences in Θ that contains θ (the set $R_{\Theta}(\theta)$ may be empty).

This concept arises quite naturally as the characteristic set $\mathrm{GL}(F_u)$ contains many geometric sequences. The following definition will be used in studying $R_{\mathrm{GL}(F_u)}(\theta)$ later on.

Definition 2.5. For a finite set $A = \{a_i\}_{i=1}^n \subset (0, \infty)$, define $A^{\mathbb{Z}_+^*}$ (resp. $A^{\mathbb{Q}_+^*}$, $A^{\mathbb{Q}^*}$) to be the union of all products $\prod_{i=1}^n a_i^{m_i}$, where $(m_i)_{i=1}^n$ are non-zero vectors whose entries are nonnegative integers (resp. nonnegative rationals, rationals). Let $A^{\mathbb{Z}_+} = \{1\} \cup A^{\mathbb{Z}_+^*}$, that is, the union of all products $\prod_{i=1}^n a_i^{m_i}$, where $(m_i)_i$ are nonnegative integer vectors (including the zero vector). Similarly, $A^{\mathbb{Q}} = \{1\} \cup A^{\mathbb{Q}^*}$ and $A^{\mathbb{Q}_+} = \{1\} \cup A^{\mathbb{Q}_+^*}$.

We will analyse $\mathrm{GL}(F_u)$ given by (2.8) with the following Lemma.

Lemma 2.6. *Let $A = \{a_i\}_{i=1}^n \subset (0,1)$ for $n \in \mathbb{Z}_+^* := \{1, 2, \ldots\}$, and λ_j ($j = 1, \ldots, m$) be positive real numbers (not necessarily distinct). Let $\Theta = \bigcup_{j=1}^m \lambda_j A_j$, where $A_j \subset A^{\mathbb{Z}_+}$ for $1 \leq j \leq m$.*

(i) *Then $R_{\Theta}(\theta) \subset A^{\mathbb{Q}_+^*}$ for all $\theta \in \Theta$.*

(ii) *If $\lambda_p/\lambda_q \notin A^{\mathbb{Q}}$ for all distinct $p, q \in \{1, \ldots, m\}$ when $m \geq 2$, then for every strictly decreasing geometric sequence $\{\theta' r^k\}_{k=0}^{\infty} \subset \Theta$, there exists a unique $l \in \{1, \ldots, m\}$ such that $\{\theta' r^k\}_{k=0}^{\infty} \subset \lambda_l A_l$, and*

$$\theta' r^k \notin \lambda_j A_j \quad \text{for all } j \neq l \text{ and all } k \geq 0. \quad (2.12)$$

Condition (ii) in Lemma 2.6 means that the sets $\{\lambda_j A_j\}_{j=1}^m$ are disjoint.

Proof. (i) Let $\theta \in \Theta$. Assume that $R_{\Theta}(\theta) \neq \emptyset$. Let $r \in R_{\Theta}(\theta)$. By (2.11), there exists $\theta' \in \Theta$ such that $\theta \in \{\theta' r^k\}_{k=0}^{\infty} \subset \Theta$, so by the pigeonhole principle we can find some

λ_l such that $\{\theta' r^k\}_{k=0}^{\infty} \bigcap \lambda_l A_l$ is infinite. Write this infinite subsequence as

$$\theta' r^{k_t} = \lambda_l \prod_{i=1}^{n} a_i^{m_{i,t}}, \tag{2.13}$$

where $(m_{i,t})_{i=1}^n \in \mathbb{Z}_+^n$ and $\{k_t\} \subset \mathbb{Z}_+ := \{0, 1, 2, \dots\}$ with $k_t < k_{t+1}$, for $t \in \mathbb{Z}_+$. Applying Proposition 5.1 in the Appendix with $B = \{(m_{i,t})_{i=1}^n\}_{t \in \mathbb{Z}_+}$, there exist two distinct vectors $(m_{i,p})_{i=1}^n$ and $(m_{i,q})_{i=1}^n$ in $\mathbb{Z}_+^n$ for some two indices $p < q$ in $\mathbb{Z}_+$, such that

$$(m_{i,p})_{i=1}^n \le (m_{i,q})_{i=1}^n \tag{2.14}$$

under the partial order defined by inequality of all coordinates. Therefore, we have by (2.13)

$$r^{k_q - k_p} = \frac{\theta' r^{k_q}}{\theta' r^{k_p}} = \frac{\lambda_l \prod_{i=1}^n a_i^{m_{i,q}}}{\lambda_l \prod_{i=1}^n a_i^{m_{i,p}}} = \prod_{i=1}^{n} a_i^{m_{i,q} - m_{i,p}} \, (\text{or } r = \prod_{i=1}^{n} a_i^{(m_{i,q} - m_{i,p})/(k_q - k_p)}).$$

Since

$$\Big(\frac{m_{i,q} - m_{i,p}}{k_q - k_p}\Big)_{i=1}^n \in (\mathbb{Q}_+^n)^*$$

using (2.14), it follows that $r \in A^{\mathbb{Q}_+^*}$ by definition. Therefore,

$$R_\Theta(\theta) \subset A^{\mathbb{Q}_+^*}$$

for all $\theta \in \Theta$, thus proving our assertion (i).

(ii) For $m \ge 2$, suppose that there exist distinct $p, q \in \{1, \dots, m\}$ such that $\theta' r^k \in \lambda_p A_p$ and $\theta' r^j \in \lambda_q A_q$ for some $k, j \in \mathbb{Z}_+$. Write

$$\theta' r^k = \lambda_p \prod_{i=1}^{n} a_i^{p_{i,k}} \quad \text{and} \quad \theta' r^j = \lambda_q \prod_{i=1}^{n} a_i^{q_{i,j}} \quad (p_{i,k}, q_{i,j} \in \mathbb{Z}_+). \tag{2.15}$$

By (i), $r \in R_\Theta(\theta') \subset A^{\mathbb{Q}_+^*}$ since $\theta' \in \Theta$, and so $r^{k-j} \in A^{\mathbb{Q}}$. It follows that

$$\frac{\lambda_p}{\lambda_q} = r^{k-j} \prod_{i=1}^{n} a_i^{q_{i,j} - p_{i,k}} \in A^{\mathbb{Q}} A^{\mathbb{Q}} = A^{\mathbb{Q}},$$

leading to a contradiction to our assumption. Thus, there exists a unique integer $l \in \{1, \dots, m\}$ such that $\{\theta' r^k\}_{k=0}^{\infty} \subset \lambda_l A_l$.

It remains to show (2.12). In fact, if (2.12) were not true, then $\theta' r^k \in \lambda_t A_t$ for some integer $k \ge 0$ and some $t \ne l$. Taking $p = l$, $j = k$, $q = t$ in (2.15), we would have

$$\frac{\lambda_l}{\lambda_t} = \prod_{i=1}^{n} a_i^{q_{i,k} - p_{i,k}} \in A^{\mathbb{Q}},$$

leading to a contradiction. The assertion (2.12) follows. ∎

The following corollary will be used to describe a certain 'homogeneity' property of (the gap length sets of) attractors of COSC standard IFSs.

Corollary 2.7. *Let $X \subset (0,1)$ and $\Lambda \subset (0,\infty)$ be two finite sets. Then*

$$X^{\mathbb{Z}_+^*} \subset R_{\Lambda X^{\mathbb{Z}_+}}(\theta) \subset X^{\mathbb{Q}_+^*}$$

for every $\theta \in \Lambda X^{\mathbb{Z}_+}$.

Proof. Let $\theta \in \Lambda X^{\mathbb{Z}_+}$. Since $X^{\mathbb{Z}_+^*} \subset X^{\mathbb{Z}_+}$,

$$\theta X^{\mathbb{Z}_+^*} \subset (\Lambda X^{\mathbb{Z}_+})X^{\mathbb{Z}_+} = \Lambda(X^{\mathbb{Z}_+}X^{\mathbb{Z}_+}) = \Lambda X^{\mathbb{Z}_+}.$$

For any $r \in X^{\mathbb{Z}_+^*}$ and $k \in \mathbb{Z}_+$, we have $r^k \in X^{\mathbb{Z}_+}$ and so

$$\theta r^k \in \big(\Lambda X^{\mathbb{Z}_+}\big)X^{\mathbb{Z}_+} = \Lambda X^{\mathbb{Z}_+},$$

thus showing that $r \in R_{\Lambda X^{\mathbb{Z}_+}}(\theta)$ by definition (2.11) with $\Theta = \Lambda X^{\mathbb{Z}_+}$, so the first inclusion follows.

The second inclusion also follows by taking $A = X$, $\lambda_j \in \Lambda$ and each $A_j = X^{\mathbb{Z}_+}$ in Lemma 2.6 (i) (so that $\Theta = \Lambda X^{\mathbb{Z}_+}$). ■

As an application of Lemma 2.6 and Corollary 2.7, we derive a key lemma that will be used to distinguish the attractor of a COSC GD-IFS from that of a COSC standard IFS.

Definition 2.8 (Absolute contraction ratio set). The *absolute contraction ratio set* of a GD-IFS is defined to be the set of the absolute values of the contraction ratios of the similarities, that is $\{|\rho_e| : e \in E\}$.

Lemma 2.9. *Let $(F_v)_{v\in V}$ be the attractors of a COSC GD-IFS based on a digraph with $d_v \geq 2$ for all $v \in V$, with absolute contraction ratio set A. Assume that for some u, the set F_u is not an interval (or a singleton) and is the attractor of some COSC standard IFS with absolute contraction ratio set X.*

(i) *Then for all $\theta \in \mathrm{GL}(F_u)$*

$$X \subset X^{\mathbb{Z}_+^*} \subset R_{\mathrm{GL}(F_u)}(\theta) \subset A^{\mathbb{Q}_+^*} \cap X^{\mathbb{Q}_+^*}. \tag{2.16}$$

(ii) *If $A_1 \cup A_2 = A$ and $A_1^{\mathbb{Q}^*} \cap A_2^{\mathbb{Q}_+^*} = \emptyset$, then the following dichotomy is true: either*

$$R_{\mathrm{GL}(F_u)}(\theta) \cap A_1^{\mathbb{Q}_+^*} \neq \emptyset \quad \textit{for all } \theta \in \mathrm{GL}(F_u), \tag{2.17}$$

or

$$R_{\mathrm{GL}(F_u)}(\theta) \cap A_1^{\mathbb{Q}_+^*} = \emptyset \quad \textit{for all } \theta \in \mathrm{GL}(F_u). \tag{2.18}$$

Assertion (ii) of Lemma 2.9 gives a necessary condition that a COSC GD-attractor F_u is also the attractor of some COSC standard IFS in the following way: if there exist two elements $\theta_1, \theta_2 \in \mathrm{GL}(F_u)$ such that (2.17) holds for θ_1 whilst (2.18) holds for θ_2, then F_u is not the attractor of any COSC standard IFS. This assertion will be used in Lemma 4.1 below.

Proof. (i) Let Λ be the set of non-zero basic gap lengths of some COSC standard IFS with the attractor F_u, and let X be the absolute contraction ratio set. Regard this standard IFS as a GD-IFS based on $(\{v\}, \{e_j\}_{j=1}^m)$, where e_j are loops of the single vertex v, all directed paths of length $k \geq 1$ are now $e_{i_1} e_{i_2} \cdots e_{i_k}$, where $i_l = 1, 2, \ldots, m$ for all $l = 1, 2, \ldots, k$. By (2.8),

$$\begin{aligned}\mathrm{GL}(F_u) &= \Lambda \bigcup \Big(\bigcup_{m=1}^{\infty} \Lambda \big\{ |\rho_{\mathbf{e}}| \colon \mathbf{e} \text{ is a directed path from } v \text{ to } v \text{ with length } m \big\} \Big) \\ &= \Lambda \cup \Lambda X^{\mathbb{Z}_+^*} = \Lambda X^{\mathbb{Z}_+}.\end{aligned}$$

Note that $\mathrm{GL}(F_u)$ is non-empty by using our assumption that F_u is not an interval or a singleton.

On the other hand, Corollary 2.7 implies that

$$X^{\mathbb{Z}_+^*} \subset R_{\mathrm{GL}(F_u)}(\theta) = R_{\Lambda X^{\mathbb{Z}_+}}(\theta) \subset X^{\mathbb{Q}_+^*} \tag{2.19}$$

for all $\theta \in \mathrm{GL}(F_u)$. Recall that a directed circuit containing u is a directed path from u to u. We write the union given by (2.8) as

$$\begin{aligned}\Theta := \mathrm{GL}(F_u) = \bigg(\bigcup_{\lambda \in \Lambda_u} \lambda \Big(\{1\} \bigcup \big\{ |\rho_{\mathbf{e}}| \colon \mathbf{e} \text{ is a directed circuit containing } u \big\} \Big) \bigg) \\ \bigcup \Big(\bigcup_{v \in V \setminus \{u\}} \bigcup_{\lambda \in \Lambda_v} \lambda \big\{ |\rho_{\mathbf{e}}| \colon \mathbf{e} \text{ is a directed path from } u \text{ to } v \big\} \Big).\end{aligned} \tag{2.20}$$

Since the absolute contraction ratios are all in A (so that $|\rho_{\mathbf{e}}| \in A^{\mathbb{Z}_+}$), it follows from Lemma 2.6 (i) that $R_{\mathrm{GL}(F_u)}(\theta) \subset A^{\mathbb{Q}_+^*}$ for all $\theta \in \mathrm{GL}(F_u)$, which combines with (2.19) to give that

$$R_{\mathrm{GL}(F_u)}(\theta) \in A^{\mathbb{Q}_+^*} \cap X^{\mathbb{Q}_+^*},$$

leading to the inclusions in (2.16), as desired.

(ii) If $X \cap A_1^{\mathbb{Q}_+^*} \neq \emptyset$, it follows from (2.16) that

$$X \cap A_1^{\mathbb{Q}_+^*} \subset R_{\mathrm{GL}(F_u)}(\theta) \cap A_1^{\mathbb{Q}_+^*}$$

for all $\theta \in \mathrm{GL}(F_u)$, thus showing that (2.17) is true.

Now assume that $X \cap A_1^{\mathbb{Q}_+^*} = \emptyset$. We will show that (2.18) is true.

We first claim that $A^{\mathbb{Q}_+^*}$ is the union of two disjoint sets $A_1^{\mathbb{Q}_+^*}$ and $A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}$. To see this, as $A_1^{\mathbb{Q}^*} \cap A_2^{\mathbb{Q}_+^*} = \emptyset$ by assumption, it follows that

$$A_1^{\mathbb{Q}_+^*} \cap A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*} = \emptyset. \tag{2.21}$$

In fact, if (2.21) were not true, there would exist three elements

$$a \in A_1^{\mathbb{Q}_+^*}, \quad b \in A_1^{\mathbb{Q}_+}, \quad c \in A_2^{\mathbb{Q}_+^*}$$

with $a = bc$, from which $\frac{a}{b} \in A_1^{\mathbb{Q}}$ and $\frac{a}{b} = c \in A_2^{\mathbb{Q}_+^*}$. As $A_1^{\mathbb{Q}} = \{1\} \cup A_1^{\mathbb{Q}^*}$ by definition and $\{1\} \cap A_2^{\mathbb{Q}_+^*} = \emptyset$ due to $A_2^{\mathbb{Q}_+^*} \subset (0, 1)$, we see that

$$\frac{a}{b} \in A_1^{\mathbb{Q}} \cap A_2^{\mathbb{Q}_+^*} = \left(\{1\} \cup A_1^{\mathbb{Q}^*}\right) \cap A_2^{\mathbb{Q}_+^*} = \left(\{1\} \cap A_2^{\mathbb{Q}_+^*}\right) \cup \left(A_1^{\mathbb{Q}^*} \cap A_2^{\mathbb{Q}_+^*}\right) = \emptyset,$$

a contradiction.

We need to show

$$A^{\mathbb{Q}_+^*} = A_1^{\mathbb{Q}_+^*} \cup A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}. \tag{2.22}$$

In fact, let

$$A_1 = \{b_i\}_{i=1}^m, \qquad A_2 = \{c_j\}_{j=1}^n.$$

As $A_1 \cap A_2 \subset A_1^{\mathbb{Q}^*} \cap A_2^{\mathbb{Q}_+^*} = \emptyset$, any element $a \in A^{\mathbb{Q}_+^*} = (A_1 \cup A_2)^{\mathbb{Q}_+^*}$ can be written as

$$a = \prod_{i=1}^m b_i^{p_i} \prod_{j=1}^n c_j^{q_j} \quad \text{for some } (p_i)_{i=1}^m \in \mathbb{Q}_+^m \text{ and } (q_j)_{j=1}^n \in \mathbb{Q}_+^n,$$

where not all p_i, q_j are zero. Thus, if all q_j are zero, then $a = \prod_{i=1}^m b_i^{p_i} \in A_1^{\mathbb{Q}_+^*}$; otherwise $a \in A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}$. This proves (2.22) by using (2.21).

As $X \cap A_1^{\mathbb{Q}_+^*} = \emptyset$ and

$$X \subset A^{\mathbb{Q}_+^*} = A_1^{\mathbb{Q}_+^*} \cup A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}$$

by using (2.16) and (2.22), we have

$$X \subset A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}. \tag{2.23}$$

We will show the following inclusion

$$X^{\mathbb{Q}_+^*} \subset A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}. \tag{2.24}$$

Since X is finite, let

$$X = \{x_l\}_{l=1}^k, \quad A_1 = \{b_i\}_{i=1}^m, \quad A_2 = \{c_j\}_{j=1}^n.$$

By (2.23), we write for each $l = 1, 2, \dots, k$

$$x_l = \prod_{i=1}^m b_i^{p_{i,l}} \prod_{j=1}^n c_j^{q_{j,l}} \quad \text{for some } (p_{i,l})_{i=1}^m \in \mathbb{Q}_+^m \text{ and } (q_{j,l})_{j=1}^n \in (\mathbb{Q}_+^n)^*.$$

Then any element $x \in X^{\mathbb{Q}_+^*}$ can be written as

$$\begin{aligned} x = \prod_{l=1}^k x_l^{r_l} &= \prod_{l=1}^k \Big(\prod_{i=1}^m b_i^{p_{i,l}} \prod_{j=1}^n c_j^{q_{j,l}} \Big)^{r_l} \\ &= \prod_{i=1}^m b_i^{\sum_{l=1}^k p_{i,l} r_l} \prod_{j=1}^n c_j^{\sum_{l=1}^k q_{j,l} r_l} \qquad \text{for some } (r_l)_{l=1}^k \in \big(\mathbb{Q}_+^k\big)^*. \end{aligned}$$

Note that the numbers $\sum_{l=1}^k p_{i,l} r_l$ and $\sum_{l=1}^k q_{j,l} r_l$ all belong to $\mathbb{Q}_+$. Since $r_{l'} > 0$ for some l' while $q_{j',l'} > 0$ for this l' and some j', we have $\sum_{l=1}^k q_{j',l} r_l > 0$ for this j'. Therefore, we obtain (2.24).

Finally, by (2.16) and (2.24), we have for all $\theta \in \mathrm{GL}(F_u)$,

$$R_{\mathrm{GL}(F_u)}(\theta) \subset X^{\mathbb{Q}_+^*} \subset A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*},$$

from which we easily conclude that (2.18) holds by using (2.21). ∎

3. Construction of GD-IFSs

We will construct COSC (CSSC) GD-IFSs in terms of vector sets in Euclidean spaces, to analyse the existence and extent of *non-trivial* GD-IFSs whose attractors are not attractors of any (COSC) standard IFS.

For a digraph $G = (V, E)$ with $d_i \geq 2$ for $i \in V = \{1, 2, \dots, N\}$, we set

$$n := 2\#E - \#V = 2(d_1 + d_2 + \cdots + d_N) - N \tag{3.1}$$

so that $n \geq N$ (recall that d_i denotes the number of the edges leaving vertex i). Define the subset P_0 in the Euclidean space $\mathbb{R}^n$, with n given in (3.1), by

$$\begin{aligned} P_0 := \Big\{ x = \big(& x_1^{(1)}, \dots, x_1^{(d_1)}, x_2^{(1)}, \dots, x_2^{(d_2)}, \dots, x_N^{(1)}, \dots, x_N^{(d_N)}, \\ & \xi_1^{(1)}, \dots, \xi_1^{(d_1-1)}, \dots, \xi_N^{(1)}, \dots, \xi_N^{(d_N-1)} \big) \\ & \text{where } x_i^{(k)}, x_i^{(d_i)} \in (-1, 1) \setminus \{0\} \text{ and } \xi_i^{(k)} \geq 0 \\ & \text{for each vertex } i \in V, 1 \leq k \leq d_i - 1 \Big\}. \end{aligned} \tag{3.2}$$

Each vector x in P_0 consists of two kinds of entries: the entries $\{x_i^{(k)}\}_{i\in V,1\le k\le d_i}$ all lie in the set $(-1,1)\setminus\{0\}$, and will specify the contraction ratios of GD-IFSs to be constructed, whilst the other entries $\{\xi_i^{(k)}\}_{i\in V,1\le k\le d_i-1}$ are all nonnegative, and will specify the basic gap lengths.

For vertex $i\in V$, let $\{e_i(k)\colon 1\le k\le d_i\}$ be the set of edges leaving i, which are arranged in some order which will henceforth remain fixed. For a point x in P_0, we look at its entries $\{x_i^{(k)}\}_{i\in V,1\le k\le d_i}$ and define an $N\times N$ matrix $M_x(s)$ for any $s>0$ by

$$M_x(s)=\big(M_{ij}(s)\big)_{1\le i,j\le N}, \tag{3.3}$$

where

$$M_{ij}(s)=\sum_{e_i(k)\in E_{ij}}|x_i^{(k)}|^s \tag{3.4}$$

if $E_{ij}\neq\emptyset$, and $M_{ij}(s)=0$ if $E_{ij}=\emptyset$ (recall that E_{ij} is the set of (multiple) edges from vertex i to vertex j).

Let $\mathscr{b}$, ℓ, be two vectors defined by

$$\mathscr{b}:=(b_i^{(1)})_{i\in V},\qquad \text{where } b_i^{(1)}\in\mathbb{R}, \tag{3.5}$$

$$\ell:=(l_i)_{i\in V},\qquad \text{where } l_i\ge 0. \tag{3.6}$$

For each edge $e_i(k)$ $(1\le k\le d_i)$ leaving vertex $i\in V$, we define the mappings associated with a point x in P_0 by

$$S_{e_i(k)}(t):=x_i^{(k)}(t-b^{(1)}_{\omega(e_i(k))})+b_i^{(k)}-x_i^{(k)}l_{\omega(e_i(k))}\mathbf{1}_{\{x_i^{(k)}<0\}}\quad\text{for a variable } t\in\mathbb{R}, \tag{3.7}$$

where $\mathbf{1}_{\{x_i^{(k)}<0\}}=1$ if $x_i^{(k)}<0$, and $\mathbf{1}_{\{x_i^{(k)}<0\}}=0$ otherwise, and

$$b_i^{(k+1)}:=b_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))}+\xi_i^{(k)}\quad\text{for } i\in V \text{ and } 1\le k\le d_i-1, \tag{3.8}$$

and $\omega(e_i(k))$ denotes the terminal vertex of the edge $e_i(k)$ as before.

Note that for any point $x\in P_0$, the mapping $S_{e_i(k)}$ defined as in (3.7) has the contraction ratio $x_i^{(k)}\in(-1,1)\setminus\{0\}$, therefore it is a contracting similarity, and

$$F(x,\mathscr{b},\ell):=\{S_{e_i(k)}\colon i\in V, 1\le k\le d_i\} \tag{3.9}$$

forms a GD-IFS on the digraph $(V,\{e_i(k)\})$, thus having a unique list of GD-attractors $\{F_i\}_{i\in V}$.

For any two vectors $\mathscr{b}$, ℓ as in (3.5), (3.6) and any point x in P_0, we define the closed intervals (which may be singletons) for each vertex $i\in V$ by

$$\begin{aligned} I_i&=[b_i^{(1)},b_i^{(1)}+l_i],\\ I_i(k)&=\big[b_i^{(k)},b_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))}\big]\quad\text{for } 1\le k\le d_i,\end{aligned} \tag{3.10}$$

where $b_i^{(k+1)}$ for $1\le k\le d_i-1$ are given by (3.8).

We will work with a subset P of P_0 defined by

$$P := \left\{x \in P_0 \colon r_\sigma(M_x(1)) < 1, \sum_{k=1}^{d_i-1} \xi_i^{(k)} > 0 \quad \text{for all } 1 \le i \le N\right\}, \tag{3.11}$$

where the matrix $M_x(1)$ is defined by (3.3) with $s = 1$, and $r_\sigma(M)$ denotes the *spectral radius* of a matrix M, which is the largest absolute value (complex modulus) of the eigenvalues of M.

We show that any point in P will give arise to at least one COSC GD-IFS on G, in form of (3.7), whose contraction ratios are $\{x_i^{(k)}\}_{i\in V, 1\le k\le d_i}$ and whose attractor F_i at each vertex i has the convex hull I_i given by (3.10), having the basic gap lengths $\{\xi_i^{(k)}\}_{1\le k\le d_i-1}$, provided that l_i satisfies (3.12) below.

Lemma 3.1 (Construction of GD-IFSs). *Let $G = (V, E)$ be a digraph with $d_i \ge 2$ for $i \in V$. With the same notation above, let x be any point in P as in (3.11) and $\mathscr{b}$ be any vector as in (3.5). Let $(l_i)_{i\in V}$ be a vector of real numbers given by*

$$(l_i)_{i\in V}^T := (\mathrm{id} - M_x(1))^{-1} \left(\sum_{k=1}^{d_i-1} \xi_i^{(k)}\right)_{i\in V}^T, \tag{3.12}$$

where M^T denotes the transpose of a matrix M. Then any GD-IFS $F(x, \mathscr{b})$, given by (3.7), (3.9) and (3.12) and having attractors $\{F_i\}_{i\in V}$, satisfies the following properties.

(i) *For each vertex $i \in V$, we have $l_i > 0$ and*

$$\mathrm{conv}\, F_i = I_i = [b_i^{(1)}, b_i^{(1)} + l_i]. \tag{3.13}$$

(ii) *The GD-IFS $F(x, \mathscr{b})$ satisfies the COSC. The basic gaps of attractor F_i for $i \in V$ are given by the following open intervals in $\mathbb{R}$*

$$\left\{\left(b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}, b_i^{(k+1)}\right)\right\}_{1\le k\le d_i-1}, \tag{3.14}$$

which are arranged in order from left to right. The corresponding basic gap lengths are

$$\left\{b_i^{(k+1)} - \left(b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}\right) = \xi_i^{(k)}\right\}_{1\le k\le d_i-1}. \tag{3.15}$$

If further all $\xi_i^{(k)} > 0$ for $i \in V$ and $1 \le k \le d_i - 1$, then $F(x, \mathscr{b})$ satisfies the CSSC.

Proof. Note that

$$b_i^{(d_i)} = b_i^{(1)} + \sum_{k=1}^{d_i-1} (\xi_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}), \tag{3.16}$$

since, by repeatedly using the defining expression (3.8) of $b_i^{(k+1)}$,

$$\begin{aligned} b_i^{(d_i)} &= b_i^{(d_i-1)} + |x_i^{(d_i-1)}| l_{\omega(e_i(d_i-1))} + \xi_i^{(d_i-1)} \\ &= \big(b_i^{(d_i-2)} + |x_i^{(d_i-2)}| l_{\omega(e_i(d_i-2))} + \xi_i^{(d_i-2)}\big) + |x_i^{(d_i-1)}| l_{\omega(e_i(d_i-1))} + \xi_i^{(d_i-1)} \\ &= \cdots \\ &= b_i^{(1)} + \sum_{k=1}^{d_i-1} (\xi_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}). \end{aligned}$$

Also note that

$$l_i > 0 \quad \text{for each } i \in V, \tag{3.17}$$

since, by using the defining expression (3.11) of P, the matrix $(\mathrm{id} - M_x(1))$ is invertible and can be written as

$$(\mathrm{id} - M_x(1))^{-1} = \mathrm{id} + M_x(1) + M_x^2(1) + \cdots ,$$

(see for example [14, Lemma **B**.1, Appendix B]), from which it follows by definition (3.12) that

$$\begin{aligned} l_i &= \sum_{j \in V} \big((\mathrm{id} - M_x(1))^{-1}\big)_{ij} \left(\sum_{k=1}^{d_j-1} \xi_j^{(k)} \right) \\ &= \sum_{j \in V} \big(\mathrm{id} + M_x(1) + M_x^2(1) + \cdots\big)_{ij} \left(\sum_{k=1}^{d_j-1} \xi_j^{(k)} \right) \\ &\geq \sum_{k=1}^{d_i-1} \xi_i^{(k)} > 0 \end{aligned}$$

by using the fact that $M_x(1)$ is a nonnegative matrix and that $\sum_{k=1}^{d_i-1} \xi_i^{(k)} > 0$ by (3.11).

We claim that

$$b_i^{(d_i)} + |x_i^{(d_i)}| l_{\omega(e_i(d_i))} = b_i^{(1)} + l_i \quad \text{for each vertex } i \in V. \tag{3.18}$$

Indeed, we know by definition (3.12) that

$$(\mathrm{id} - M_x(1))(l_i)_{i \in V}^T = \left(\sum_{k=1}^{d_i-1} \xi_i^{(k)} \right)_{i \in V}^T ,$$

from which, by definition (3.3) and (3.4),

$$\sum_{k=1}^{d_i-1}\xi_i^{(k)}=l_i-\sum_{j=1}^{N}M_{ij}(1)l_j=l_i-\sum_{j=1}^{N}\Big(\sum_{e_i(k)\in E_{ij}}|x_i^{(k)}|\Big)l_j=l_i-\sum_{k=1}^{d_i}|x_i^{(k)}|l_{\omega(e_i(k))},$$

so that

$$l_i=\sum_{k=1}^{d_i-1}\xi_i^{(k)}+\sum_{k=1}^{d_i}|x_i^{(k)}|l_{\omega(e_i(k))}\quad\text{for each vertex } i\in V.$$

Combining this with (3.16),

$$\begin{aligned}l_i&=\sum_{k=1}^{d_i-1}\xi_i^{(k)}+\sum_{k=1}^{d_i}|x_i^{(k)}|l_{\omega(e_i(k))}=\sum_{k=1}^{d_i-1}(\xi_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))})+|x_i^{(d_i)}|l_{\omega(e_i(d_i))}\\&=b_i^{(d_i)}-b_i^{(1)}+|x_i^{(d_i)}|l_{\omega(e_i(d_i))},\end{aligned}$$

thus showing (3.18). This proves our claim.

We next show that the contracting similarity $S_{e_i(k)}$ associated with the edge $e_i(k)$ satisfies

$$S_{e_i(k)}(I_{\omega(e_i(k))})=[b_i^{(k)},b_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))}]=I_i(k)\tag{3.19}$$

for each vertex $i\in V$ and each $1\le k\le d_i$. This is easily seen by looking at the two endpoints of interval $I_{\omega(e_i(k))}$, depending on whether $x_i^{(k)}>0$ or not. Indeed, by definition (3.10) with vertex i being replaced by vertex $\omega(e_i(k))$,

$$I_{\omega(e_i(k))}=[b^{(1)}_{\omega(e_i(k))},b^{(1)}_{\omega(e_i(k))}+l_{\omega(e_i(k))}].$$

If $x_i^{(k)}>0$, we have by definition (3.7) that $S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))})=b_i^{(k)}$ and

$$S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))}+l_{\omega(e_i(k))})=x_i^{(k)}l_{\omega(e_i(k))}+b_i^{(k)}=b_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))},$$

from which

$$\begin{aligned}S_{e_i(k)}(I_{\omega(e_i(k))})&=[S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))}),S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))}+l_{\omega(e_i(k))})]\\&=[b_i^{(k)},b_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))}],\end{aligned}$$

thus showing (3.19). On the other hand, if $x_i^{(k)}<0$, we similarly have that

$$S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))}+l_{\omega(e_i(k))})=b_i^{(k)}$$

and

$$S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))})=b_i^{(k)}-x_i^{(k)}l_{\omega(e_i(k))}=b_i^{(k)}+|x_i^{(k)}|l_{\omega(e_i(k))},$$

so

$$S_{e_i(k)}(I_{\omega(e_i(k))}) = [S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))} + l_{\omega(e_i(k))}), S_{e_i(k)}(b^{(1)}_{\omega(e_i(k))})]$$
$$= [b_i^{(k)}, b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}],$$

thus showing (3.19) again. Thus (3.19) is always true.

Since by definition (3.8)

$$b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))} = b_i^{(k+1)} - \xi_i^{(k)} \le b_i^{(k+1)},$$

we know by (3.19) that the closed intervals $\{I_i(k)\colon 1 \le k \le d_i\}$ are arranged in order from left to right, which together with (3.18) implies that

$$\bigcup_{k=1}^{d_i} \mathrm{int}(I_i(k)) = \bigcup_{k=1}^{d_i} \big(b_i^{(k)}, b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}\big)$$
$$\subset \big(b_i^{(1)}, b_i^{(d_i)} + |x_i^{(d_i)}| l_{\omega(e_i(d_i))}\big) = \big(b_i^{(1)}, b_i^{(1)} + l_i\big) \tag{3.20}$$

with the disjoint union.

We are now in a position to prove the assertions (i), (ii).

(i) We will use (3.19) and (3.12) to derive (3.13). Indeed, recall that the intervals I_i are defined in (3.10). Note that $l_i > 0$ for each $i \in V$ by (3.17). As in (2.2), for each vertex $i \in V$ we let

$$I_i^m := \bigcup_{\mathbf{e} \in E_i^m} S_{\mathbf{e}}\big(I_{\omega(\mathbf{e})}\big) \quad \text{for } m = 1, 2, \ldots, \tag{3.21}$$

where E_i^m is the set of edges of length m leaving vertex i, and $\omega(\mathbf{e})$ is the terminal of path $\mathbf{e}$ as before. We show that for each vertex $i \in V$

$$\min I_i^m = b_i^{(1)} = \min I_i, \quad \max I_i^m = b_i^{(1)} + l_i = \max I_i \qquad \text{for } m = 1, 2, \ldots, \tag{3.22}$$

so that the left and right endpoints, respectively, of all the intervals I_i^m are the same.

Indeed, we know by definition (3.21) that for each vertex $i \in V$

$$I_i^1 = \bigcup_{\mathbf{e} \in E_i^1} S_{\mathbf{e}}\big(I_{\omega(\mathbf{e})}\big) = \bigcup_{k=1}^{d_i} S_{e_i(k)}\big(I_{\omega(e_i(k))}\big)$$
$$= \bigcup_{k=1}^{d_i} [b_i^{(k)}, b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}] \quad \text{(using (3.19))},$$

from which, using the fact that $b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))} \le b_i^{(k+1)}$ by (3.8), it follows that $\min I_i^1 = b_i^{(1)}$, and

$$\max I_i^1 = b_i^{(d_i)} + |x_i^{(d_i)}| l_{\omega(e_i(d_i))} = b_i^{(1)} + l_i$$

by using (3.18). Hence, (3.22) is true when $m = 1$ by the defining expression (3.10) of I_i.

Assume inductively that (3.22) holds for some $m \ge 1$. Since for each vertex $i \in V$

$$I_i^{m+1} = \bigcup_{\mathbf{e}' \in E_i^{m+1}} S_{\mathbf{e}'}\big(I_{\omega(\mathbf{e}')}\big) = \bigcup_{\mathbf{e} \in E_i^m} S_{\mathbf{e}}\big(I^1_{\omega(\mathbf{e})}\big)$$

by using (2.4), it follows that

$$\begin{aligned}\min I_i^{m+1} &= \min\big\{S_{\mathbf{e}}(I^1_{\omega(\mathbf{e})}) \colon \mathbf{e} \in E_i^m\big\}\\ &= \min\big\{S_{\mathbf{e}}(I_{\omega(\mathbf{e})}) \colon \mathbf{e} \in E_i^m\big\}\\ &= \min I_i^m = b_i^{(1)}.\end{aligned}$$

Similarly,

$$\max I_i^{m+1} = \max I_i^m = b_i^{(1)} + l_i.$$

Therefore, the (3.22) holds for all $m \ge 1$ by induction.

Since condition (2.3) holds using that $I_i^1 \subset I_i = [b_i^{(1)}, b_i^{(1)} + l_i]$, and we know by (2.5) that $F_i = \bigcap_{m=1}^{\infty} I_i^m$, (3.22) gives that,

$$\operatorname{conv} F_i = \operatorname{conv} \bigcap_{m=1}^{\infty} I_i^m = [b_i^{(1)}, b_i^{(1)} + l_i],$$

showing that (3.13) holds true.

(ii) Applying (3.13) with i replaced by vertex $\omega(e_i(k))$, the terminal of the edge $e_i(k)$,

$$\operatorname{conv} F_{\omega(e_i(k))} = I_{\omega(e_i(k))} = [b^{(1)}_{\omega(e_i(k))}, b^{(1)}_{\omega(e_i(k))} + l_{\omega(e_i(k))}],$$

from which it follows by (3.19) that

$$S_{e_i(k)}(\operatorname{conv} F_{\omega(e_i(k))}) = S_{e_i(k)}(I_{\omega(e_i(k))}) = [b_i^{(k)}, b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}] = I_i(k) \tag{3.23}$$

for $1 \le k \le d_i$.

We show that $F(x, \mathscr{b})$ satisfies the COSC. Taking $U_i = \operatorname{int}(\operatorname{conv} F_i)$, from (3.13)

$$U_i = \operatorname{int}(\operatorname{conv} F_i) = (b_i^{(1)}, b_i^{(1)} + l_i) = \operatorname{int}(I_i),$$

so that each open set U_i is not empty as $l_i > 0$. It follows that

$$\begin{aligned}
\bigcup_{j\in V}\bigcup_{e\in E_{ij}} S_e(U_j) &= \bigcup_{k=1}^{d_i} S_{e_i(k)}(U_{\omega(e_i(k))}) = \bigcup_{k=1}^{d_i} S_{e_i(k)}\big(\mathrm{int}(I_{\omega(e_i(k))})\big)\\
&= \bigcup_{k=1}^{d_i} \mathrm{int}\big(S_{e_i(k)}(I_{\omega(e_i(k))})\big) = \bigcup_{k=1}^{d_i} \mathrm{int}\big(I_i(k)\big) \quad \text{(using (3.19))}\\
&\subseteq \big(b_i^{(1)}, b_i^{(1)} + l_i\big) \quad \text{(using (3.20))}\\
&= U_i
\end{aligned}$$

with the union disjoint. Thus $F(x, \boldsymbol{b})$ satisfies the COSC.

For each vertex $i \in V$, the basic gaps of the attractor F_i are the complementary open intervals between the closed interval

$$S_{e_i(k)}(\mathrm{conv}\, F_{\omega(e_i(k))}) = [b_i^{(k)}, b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}] = I_i(k) \quad \text{(using (3.23))}$$

and its neighbour

$$S_{e_i(k+1)}(\mathrm{conv}\, F_{\omega(e_i(k+1))}) = [b_i^{(k+1)}, b_i^{(k+1)} + |x_i^{(k+1)}| l_{\omega(e_i(k+1))}] = I_i(k+1)$$

for $1 \le k \le d_i - 1$. Specifically, they are the following open intervals

$$\Big\{\big(b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}, b_i^{(k+1)}\big)\Big\}_{1\le k\le d_i-1}$$

that are arranged in order from left to right, thus showing (3.14) for each vertex $i \in V$.

The basic gap lengths of the attractor F_i are the lengths of the open intervals in (3.14), which are equal to

$$b_i^{(k+1)} - \big(b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}\big) = \xi_i^{(k)} \quad (1 \le k \le d_i - 1)$$

by using the definition (3.8), thus showing (3.15).

Finally, if all $\xi_i^{(k)} > 0$, then $F(x, \boldsymbol{b})$ satisfies the CSSC, since

$$\bigcup_{v\in V}\bigcup_{e\in E_{uv}} S_e\big(\mathrm{conv}\, F_v\big) = \bigcup_{k=1}^{d_i} S_{e_i(k)}\big(\mathrm{conv}\, F_{\omega(e_i(k))}\big) = \bigcup_{k=1}^{d_i} I_i(k)$$

with the disjoint union, as the intervals $I_i(k)$ and $I_i(k+1)$ are separated by distance $\xi_i^{(k)}$, which are strictly positive. ∎

Remark 3.2. Note that any point x belongs to P if

$$\max_{i\in V}\left\{\sum_{k=1}^{d_i}|x_i^{(k)}|\right\}<1. \tag{3.24}$$

This is because

$$r_\sigma(M_x(1))\le\max_{i\in V}\left\{\sum_{j=1}^{N}\sum_{e_i(k)\in E_{ij}}|x_i^{(k)}|\right\}=\max_{i\in V}\left\{\sum_{k=1}^{d_i}|x_i^{(k)}|\right\}<1,$$

using the elementary fact that the spectral radius of a nonnegative matrix is no greater than any row sum, see for example [14, Equation (1.9)]. Therefore, every $x\in P_0$ satisfying (3.24) belongs to P, and all the assertions (i), (ii) in Lemma 3.1 hold true, provided that $(l_i)_{i\in V}$ are chosen as in (3.12).

We now look at subsets P, depending on a number $\delta>0$, which will give rise to a special class of GD-IFSs, satisfying the CSSC, having attractors $\{F_i\}_{i\in V}$ with the property that $\operatorname{conv} F_i=[0,1]$, and all the basic gaps of F_i have the same length δ.

Definition 3.3. Let δ be a small number such that

$$0<\delta<\min_{i\in V}\left\{\frac{1}{d_i-1}\right\} \tag{3.25}$$

(recall our assumption that the out-degree d_i at vertex i satisfies $d_i\ge 2$ for all i). We define a set $\mathcal{A}(\delta)$ by

$$\begin{aligned}\mathcal{A}(\delta):=\big\{&(x_1^{(1)},\dots,x_1^{(d_1)},x_2^{(1)},\dots,x_2^{(d_2)},\dots,x_N^{(1)},\dots,x_N^{(d_N)},\delta,\dots,\delta)\in\mathbb{R}^n:\\ &|x_i^{(k)}|>0 \text{ and } |x_i^{(1)}|+\cdots+|x_i^{(d_i)}|=1-(d_i-1)\delta \text{ for all } i\in V, 1\le k\le d_i\big\}.\end{aligned} \tag{3.26}$$

Let $M_x(1)$ be an $N\times N$ matrix associated with point x as in (3.3) for $s=1$. For each $x\in\mathcal{A}(\delta)$, the spectral radius of matrix $M_x(1)$ is less than 1, since

$$\max_{i\in V}\left\{\sum_{k=1}^{d_i}|x_i^{(k)}|\right\}=\max_{i\in V}\{1-(d_i-1)\delta\}<1\quad\text{(using (3.25))}$$

and hence,

$$\mathcal{A}(\delta)\subset P, \tag{3.27}$$

where the set P is as in (3.11). Moreover,

$$\big(\mathrm{id}-M_x(1)\big)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}:=\begin{pmatrix}(d_1-1)\delta\\(d_2-1)\delta\\\vdots\\(d_N-1)\delta\end{pmatrix} \tag{3.28}$$

so that (3.12) is satisfied with

$$l_i = 1 \quad \text{and} \quad \xi_i^{(k)} = \delta \quad \text{for } i \in V; 1 \le k \le d_i - 1,$$

this is because for each $i \in V$, by definitions (3.26) and (3.4),

$$\begin{aligned}(d_i - 1)\delta &= 1 - \left(|x_i^{(1)}| + \cdots + |x_i^{(d_i)}|\right) \\ &= 1 - \sum_{j=1}^{N}\Bigg(\sum_{e_i(k)\in E_{ij}} |x_i^{(k)}|\Bigg) = \sum_{j=1}^{N}\big(\mathrm{id} - M_x(1)\big)_{ij}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}.\end{aligned} \tag{3.29}$$

Let $\{b_i^{(k)}\}_{i\in V, 1\le k\le d_i}$ be a family of real numbers given by

$$b_i^{(1)} = 0 \quad \text{and} \quad b_i^{(k+1)} = b_i^{(k)} + |x_i^{(k)}| + \delta \quad \text{for } i \in V, 1 \le k \le d_i - 1,$$

so that

$$b_i^{(k+1)} = |x_i^{(1)}| + |x_i^{(2)}| + \cdots + |x_i^{(k)}| + k\delta \quad (i \in V, 1 \le k \le d_i - 1). \tag{3.30}$$

Clearly, each $b_i^{(k+1)} \in (0,1)$ for $1 \le k \le d_i - 1$ by using (3.29).

Let $\mathscr{b}_0$, ℓ_1 be two vectors defined by

$$\begin{aligned}\mathscr{b}_0 &:= (b_i^{(1)}, b_2^{(1)}, \ldots, b_N^{(1)}) = (0, 0, \ldots, 0), \\ \ell_1 &:= (l_1, l_2, \ldots, l_N) = (1, 1, \ldots, 1).\end{aligned}$$

In this situation, for $x \in \mathcal{A}(\delta)$, the contracting similarities defined in (3.7) read

$$S_{e_i(k)}(t) = x_i^{(k)} t + b_i^{(k)} - x_i^{(k)} \mathbf{1}_{\{x_i^{(k)}<0\}} \quad \text{for a variable } t \in \mathbb{R} \tag{3.31}$$

for $i \in V$, $1 \le k \le d_i$, which will give arise to a GD-IFS satisfying the CSSC. This will be used in Theorem 4.10 below.

Corollary 3.4. *Let $G = (V, E)$ be a digraph with $d_i \ge 2$ for $i \in V = \{1, 2, \ldots, N\}$. Let δ satisfy (3.25). For $x \in \mathcal{A}(\delta)$, let*

$$F(x) := \{S_{e_i(k)} : i \in V, 1 \le k \le d_i\}$$

be a GD-IFS given as in (3.31), with attractors $(F_i)_{i\in V}$. Then the following statements hold.

(i) *For each vertex $i \in V$,* $\operatorname{conv} F_i = [0, 1]$.

(ii) *For each vertex* $i \in V$,

$$S_{e_i(1)}([0,1]) = \big[0, |x_i^{(1)}|\big] \tag{3.32}$$

so that $|x_i^{(1)}| \in F_i$. *The basic gaps of the attractor* F_i *are given by*

$$\big(|x_i^{(1)}| + \cdots + |x_i^{(k)}| + (k-1)\delta, |x_i^{(1)}| + \cdots + |x_i^{(k)}| + k\delta\big) \tag{3.33}$$

for every $1 \le k \le d_i - 1$, *so that the basic gap lengths are all equal to the same number,* δ *say. Moreover, the GD-IFS* $F(x)$ *satisfies the CSSC.*

Proof. Let $x \in \mathcal{A}(\delta)$. Then $x \in P$ by using (3.27), and condition (3.12) is also satisfied by (3.28). Thus all the assumptions in Lemma 3.1 are satisfied. Applying Lemma 3.1 (i) and using (3.10) with $b_i^{(1)} = 0$ and $l_i = 1$,

$$\operatorname{conv} F_i = [b_i^{(1)}, b_i^{(1)} + l_i] = [0,1],$$

thus showing (i).

To show (ii), noting that $I_{\omega(e_i(k))} = [0,1]$ and $l_{\omega(e_i(k))} = 1$, $b_i^{(1)} = 0$, we know by (3.19) that

$$S_{e_i(1)}([0,1]) = S_{e_i(1)}(I_{\omega(e_i(k))}) = [b_i^{(1)}, b_i^{(1)} + |x_i^{(1)}| l_{\omega(e_i(1))}] = [0, |x_i^{(1)}|]$$

thus showing (3.32).

By (3.14) and (3.30), the basic gaps of the attractor F_i are given by

$$\begin{aligned} &\big(b_i^{(k)} + |x_i^{(k)}| l_{\omega(e_i(k))}, b_i^{(k+1)}\big) = \big(b_i^{(k)} + |x_i^{(k)}|, b_i^{(k+1)}\big) \\ &= \big(|x_i^{(1)}| + \cdots + |x_i^{(k)}| + (k-1)\delta, |x_i^{(1)}| + \cdots + |x_i^{(k)}| + k\delta\big) \end{aligned}$$

for every $1 \le k \le d_i - 1$, thus showing (3.33). From this, it is clear that the basic gap lengths all are equal to the same number which we call δ. Finally, $F(x)$ satisfies the CSSC by Lemma 3.1 (i) since all $\xi_i^{(k)} = \delta > 0$. ■

4. Criteria for graph directed attractors not to be self-similar sets

In this section we give some sufficient conditions under which GD-attractors cannot be realised as attractors of any standard IFSs with or without the COSC.

For a directed path L, let $A(L)$ (resp. $A(L^c)$) be the set of the absolute values of the contraction ratios of the similarities associated with the edges in L (resp. not in L). Recall the definition of Λ_u from (2.1).

Lemma 4.1. *Assume that (V, E) is a digraph with $d_w \geq 2$ for all $w \in V$ and L is a directed circuit that does not go through every vertex in V. Let u be a vertex outside L and v a vertex in L, assume that there exists a directed path from u to v. Consider a COSC GD-IFS based on this digraph. With the notation above, suppose that the following three conditions hold:*

(i) $(A(L))^{\mathbb{Q}^*} \cap (A(L^c))^{\mathbb{Q}^*_+} = \emptyset$.

(ii) $\Lambda_u \neq \emptyset$ *and* $\Lambda_v \neq \emptyset$.

(iii) *For all pairs* $(w, k) \neq (z, m)$ *with* $\lambda_z^{(m)} \neq 0$, *where* $w, z \in V$ *and* $1 \leq k \leq d_w - 1, 1 \leq m \leq d_z - 1$,

$$\lambda_w^{(k)}/\lambda_z^{(m)} \notin (A(L) \cup A(L^c))^{\mathbb{Q}}.$$

Then the GD-IFS attractor F_u is not the attractor of any COSC standard IFS.

Basically, condition (i) means that linear combinations of numbers $\{\log|\rho_e| \colon e \in A(L)\}$ over $\mathbb{Q}^*$, that is,

$$\sum_{e \in A(L)} q_e \log|\rho_e| \quad \text{for} \quad (q_e)_{e \in A(L)} \in \left(\mathbb{Q}^{\#A(L)}\right)^*,$$

where $(\mathbb{Q}^{\#A(L)})^*$ is the set of non-zero vectors in $\mathbb{Q}^{\#A(L)}$ as before, are different from those of numbers $\{\log|\rho_e| \colon e \in A(L^c)\}$ over $\mathbb{Q}^*_+$, while condition (ii) means that not all basic gaps associated with u and v are empty, and condition (iii) means that $\log(\lambda_w^{(k)}/\lambda_z^{(m)})$ for all distinct basic gaps of positive lengths are different from linear combinations of numbers $\{\log|\rho_e| \colon e \in E\}$ over $\mathbb{Q}$. Note that condition (i) requires a certain homogeneity, on the ratios of the gap length set of a COSC self-similar GD-attractor, which does not necessarily hold when (ii) and (iii) are satisfied. Note that among the three conditions (i), (ii), (iii), no two of them imply the third.

Proof. We show that the strict dichotomy required by Lemma 2.9 (ii) for a GD attractor fails for F_u satisfying the conditions of this theorem.

Let u be a vertex outside L and v a vertex in L. For any $w \neq u$ in V, let

$$R(uw) = \{|\rho_{\mathbf{e}}| \colon \mathbf{e} \text{ is a directed path from } u \text{ to } w\},$$

and let

$$R(uu) = \{1\} \cup \{|\rho_{\mathbf{e}}| \colon \mathbf{e} \text{ is a directed circuit containing } u\}.$$

With the above notation, the union (2.20) becomes

$$\Theta := \mathrm{GL}(F_u) = \bigcup_{w \in V} \Lambda_w R(uw) = \bigcup_{w \in V} \bigcup_{\lambda \in \Lambda_w} \lambda R(uw). \tag{4.1}$$

By condition (ii), we can choose two non-zero basic gap lengths $\lambda_u \in \Lambda_u$, $\lambda_v \in \Lambda_v$. Since there exists a directed path $\mathbf{e}$ from u to v, we can choose a number

$$\theta := \lambda_v |\rho_{\mathbf{e}}| \in \lambda_v R(uv) \subset \mathrm{GL}(F_u).$$

Recall that ρ_L denotes the product of the contraction ratios on the edges of L. For each integer $k \geq 0$, we define $\mathbf{e}L^k$ by $\mathbf{e}L^0 := \mathbf{e}$ and

$$\mathbf{e}L^k := \mathbf{e}\,\underbrace{L \cdots L}_{k \text{ times}} \quad \text{for } k \geq 1,$$

all of which are directed paths from u to v, so that $|\rho_{\mathbf{e}L^k}| \in R(uv)$. Note that

$$|\rho_L| \in R_{\mathrm{GL}(F_u)}(\theta), \tag{4.2}$$

since for every $k \geq 0$,

$$\theta |\rho_L|^k = \lambda_v |\rho_{\mathbf{e}}||\rho_L|^k = \lambda_v |\rho_{\mathbf{e}L^k}| \in \lambda_v R(uv) \subset \mathrm{GL}(F_u)$$

by using (4.1), which implies (4.2) using (2.11) with θ' being replaced by $\theta \in \mathrm{GL}(F_u)$.

Set $A_1 := A(L)$, $A_2 := A(L^c)$. Since $|\rho_L| \in (A(L))^{\mathbb{Z}_+^*} = A_1^{\mathbb{Z}_+^*} \subset A_1^{\mathbb{Q}_+^*}$, we obtain by (4.2)

$$R_{\mathrm{GL}(F_u)}(\theta) \cap A_1^{\mathbb{Q}_+^*} \neq \emptyset. \tag{4.3}$$

Let

$$r \in R_{\mathrm{GL}(F_u)}(\lambda_u).$$

By definition (2.11), there exists a geometric sequence $\{\theta' r^k\}_{k=0}^{\infty} \subset \mathrm{GL}(F_u)$ containing λ_u with $\theta' \in \mathrm{GL}(F_u)$. Note that $\lambda_u \in \lambda_u R(uu) \subset \mathrm{GL}(F_u)$ by (4.1).

We claim that

$$\theta' = \lambda_u. \tag{4.4}$$

To see this, taking the decomposition of $\Theta = \mathrm{GL}(F_u)$ given by (4.1), the requirements for Lemma 2.6 (ii), with λ_j varying in $\{\lambda \in \Lambda_w : w \in V\}$, A_j varying in $\{R_{uw} : w \in V\}$ and with $A = A(L) \bigcup A(L^c)$, are satisfied by assumption (iii). Thus, there is a unique $w \in V$ and a unique $\lambda \in \Lambda_w$ such that

$$\{\theta' r^k\}_{k=0}^{\infty} \subset \lambda R(uw), \tag{4.5}$$

and

$$\theta' r^k \notin \lambda' R(uz) \quad \text{for all } (\lambda', z) \neq (\lambda, w) \text{ and all } k \geq 0 \tag{4.6}$$

by (2.12). Thus, $\lambda_u \in \{\theta' r^k\}_{k=0}^{\infty} \subset \lambda R(uw)$. On the other hand, noting that $1 \in R(uu)$ so that

$$\lambda_u \in \lambda_u R(uu),$$

we conclude that $\lambda = \lambda_u$, $w = u$ by (4.6).

Since $r < 1$, we have $\lambda_u = \theta' r^k \leq \theta'$ for some k. As $R(uu) \subset (0, 1]$, we know by (4.5) that

$$\theta' r^k \in \lambda R(uw) = \lambda_u R(uu) \tag{4.7}$$

for every $k \geq 0$, which gives that $\theta' \leq \lambda_u$ on taking $k = 0$, and so $\lambda_u = \theta'$, thus proving our claim (4.4).

By (4.4) and (4.7) with $k = 1$,

$$\lambda_u r = \theta' r \in \lambda_u R(uu),$$

from which we see that $r \in R(uu)$, thus showing that

$$R_{\mathrm{GL}(F_u)}(\lambda_u) \subset R(uu), \tag{4.8}$$

since r is any number in $R_{\mathrm{GL}(F_u)}(\lambda_u)$.

On the other hand, since u is not in L, any directed circuit L' containing u must also visit some edge outside L as well, implying that $|\rho_{L'}| \in (A(L))^{\mathbb{Z}_+}(A(L^c))^{\mathbb{Z}_+^*} = A_1^{\mathbb{Z}_+} A_2^{\mathbb{Z}_+^*}$ and

$$\begin{aligned} R(uu) &= \{1\} \cup \{|\rho_{L'}| \colon L' \text{ is a directed circuit containing } u\} \\ &\subset \{1\} \cup A_1^{\mathbb{Z}_+} A_2^{\mathbb{Z}_+^*} \subset \{1\} \cup A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}. \end{aligned} \tag{4.9}$$

Noting that by assumption (i)

$$A_1^{\mathbb{Q}^*} \cap A_2^{\mathbb{Q}_+^*} = (A(L))^{\mathbb{Q}^*} \cap (A(L^c))^{\mathbb{Q}_+^*} = \emptyset$$

so that $A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*} \cap A_1^{\mathbb{Q}_+^*} = \emptyset$ by (2.21), it follows that

$$\begin{aligned} R_{\mathrm{GL}(F_u)}(\lambda_u) \cap A_1^{\mathbb{Q}_+^*} &\subset R(uu) \cap A_1^{\mathbb{Q}_+^*} \quad \text{(using (4.8))} \\ &\subset \big(\{1\} \cup A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*}\big) \cap A_1^{\mathbb{Q}_+^*} \quad \text{(using (4.9))} \\ &= \big(\{1\} \cap A_1^{\mathbb{Q}_+^*}\big) \cup \big(A_1^{\mathbb{Q}_+} A_2^{\mathbb{Q}_+^*} \cap A_1^{\mathbb{Q}_+^*}\big) = \emptyset \end{aligned} \tag{4.10}$$

using that $\{1\} \cap A_1^{\mathbb{Q}_+^*} = \emptyset$, since all numbers in $A_1^{\mathbb{Q}_+^*}$ are strictly less than 1.

Finally, since (4.3) and (4.10) hold simultaneously, Lemma 2.9 (ii) implies that F_u cannot be the attractor of any COSC standard IFS. ■

Note that the assumption 'there exists a directed path from u to v' in Lemma 4.1 is necessary. The following example shows that without this assumption, the GD-attractor may be an attractor of some standard IFS (with or without the COSC).

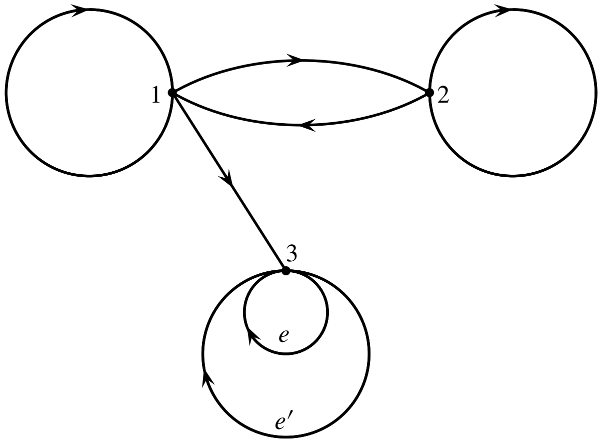

Figure 2. F_3 is an attractor of a standard IFS.

Example 4.2. Let $G = (V, E)$ be the digraph (not strongly connected) in Figure 2 with $V = \{1, 2, 3\}$ and E consisting of seven edges, three of which leave vertex 1 (including one loop). Let $\{S_e\}_{e\in E}$ be any COSC GD-IFS, having GD-attractors F_1, F_2, F_3 associated with vertices $1, 2, 3$, respectively. By (1.2), the set F_3 satisfies

$$F_3 = S_e(F_3) \cup S_{e'}(F_3),$$

which is an attractor of the standard IFS $\{S_e, S_{e'}\}$. Note that there is no directed path from vertex 3 to other two vertices 1, 2.

We give an example to illustrate Lemma 4.1. Our example is a digraph that has three vertices and is not strongly connected.

Example 4.3 (Three-vertex digraph). Let $G = (V, E)$ be the (not strongly) connected digraph in Figure 3 with $V = \{1, 2, 3\}$ and E consisting of seven edges. Note that the out-degrees of the vertices are respectively

$$d_1 = d_2 = 2, \quad d_3 = 3.$$

Let $L = e_1(1)$ be a loop (circuit) so that vertex $u = 3$ is outside L whilst vertex $v = 1$ is inside L. A directed path from u to v is labelled by $e_3(2)$.

Let x' be a point given by

$$x' = (p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}, \lambda, p_5\lambda, 0, \pi\lambda),$$

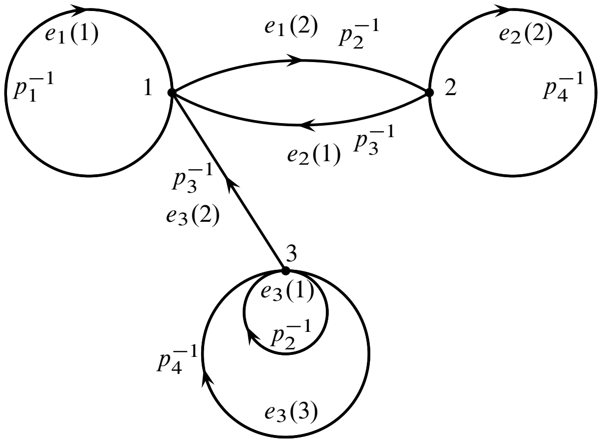

Figure 3. F_3 is not an attractor of any standard COSC IFS.

where $\{p_j\}_{1\le j\le 5}$ are five distinct positive prime numbers. The matrix $M_{x'}(1)$ defined in (3.3) is given by

$$M := M_{x'}(1) = \begin{pmatrix} p_1^{-1} & p_2^{-1} & 0 \\ p_3^{-1} & p_4^{-1} & 0 \\ p_3^{-1} & 0 & p_2^{-1}+p_4^{-1} \end{pmatrix}.$$

The point x' belongs to the set P in (3.11) by using (3.24) as the sum of each row of matrix M is bounded by 1, that is,

$$\max\{p_1^{-1}+p_2^{-1}+0,\, p_3^{-1}+p_4^{-1}+0,\, p_3^{-1}+0+p_2^{-1}+p_4^{-1}\} < 1.$$

Let $\ell = (l_1, l_2, l_3)$ be determined by (3.12), that is,

$$\ell^T = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix} = (\mathrm{id} - M)^{-1} \begin{pmatrix} \lambda \\ p_5\lambda \\ 0+\pi\lambda \end{pmatrix}. \tag{4.11}$$

Let $\mathscr{b} = (0,0,0)$ and let $F(x') := F(x', \mathscr{b})$ be a GD-IFS constructed as in Lemma 3.1, which is given by

$$\begin{aligned} &S_{e_1(1)}(t) = p_1^{-1}t, \quad S_{e_1(2)}(t) = p_2^{-1}t + b_1^{(2)}, \\ &S_{e_2(1)}(t) = p_3^{-1}t, \quad S_{e_2(2)}(t) = p_4^{-1}t + b_2^{(2)}, \\ &S_{e_3(1)}(t) = p_2^{-1}t, \quad S_{e_3(2)}(t) = p_3^{-1}t + b_3^{(2)}, \quad S_{e_3(3)}(t) = p_4^{-1}t + b_3^{(3)} \ \text{ for } t \in \mathbb{R}, \end{aligned}$$

where $\{b_1^{(2)}, b_2^{(2)}, b_3^{(2)}, b_3^{(3)}\}$ are determined by (3.8), with $\ell = (l_1, l_2, l_3)$ determined by (4.11).

By Lemma 3.1, such a GD-IFS, $F(x')$, satisfies the COSC, and the basic gap length sets at three vertices are respectively

$$\begin{aligned} &\lambda_1^{(1)} = \lambda && \text{(at vertex 1)}, \\ &\lambda_2^{(1)} = p_5\lambda && \text{(at vertex 2)}, \\ &\lambda_3^{(1)} = 0, \quad \lambda_3^{(2)} = \pi\lambda && \text{(at vertex 3)}, \end{aligned} \tag{4.12}$$

so that the sets of positive gap lengths at the vertices are given by

$$\begin{aligned} &\Lambda_1 = \{\lambda\} && \text{(at vertex 1)}, \\ &\Lambda_2 = \{p_5\lambda\} && \text{(at vertex 2)}, \\ &\Lambda_3 = \{\pi\lambda\} && \text{(at vertex 3)}. \end{aligned} \tag{4.13}$$

Since $L = e_1(1)$ is a loop, we see that

$$A(L) = \{p_1^{-1}\} \quad \text{and} \quad A(L^c) = \{p_2^{-1}, p_3^{-1}, p_4^{-1}\}, \tag{4.14}$$

so that the contraction ratio set A is given by

$$A = A(L) \cup A(L^c) = \{p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}\}.$$

We show that conditions (i), (ii), (iii) in Lemma 4.1 are all satisfied when $L = e_1(1)$, $u = 3$ and $v = 1$. Thus, the attractor F_3 of the GD-IFS, $F(x')$ above, is not the attractor of any COSC standard IFS.

To verify condition (i), we need to show that

$$\big(A(L)\big)^{\mathbb{Q}^*} \cap (A(L^c))^{\mathbb{Q}^*_+} = \emptyset,$$

where $A(L)$, $A(L^c)$ are given as in (4.14). Otherwise, there would exist some non-zero rational number q such that

$$\big(p_1^{-1}\big)^q \in \{p_2^{-1}, p_3^{-1}, p_4^{-1}\}^{\mathbb{Q}^*_+},$$

which would imply

$$1 \in \{p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}\}^{\mathbb{Q}^*_+},$$

a contradiction by using Proposition 5.5 in the Appendix.

Condition (ii) holds by directly using (4.13).

Finally, for condition (iii), we know from (4.12) that all the ratios $\lambda_w^{(k)}/\lambda_z^{(m)}$ of basic gap lengths for $(w,k) \neq (z,m)$ lie in the following set

$$\left\{\frac{1}{p_5}, \frac{1}{\pi}, p_5, \frac{p_5}{\pi}, 0, \pi, \frac{\pi}{p_5}\right\},$$

each number in which does not belong to $A^{\mathbb{Q}} = \{p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}\}^{\mathbb{Q}}$ by using Proposition 5.5 in Appendix and the fact that π is transcendental. Thus, condition (iii) is satisfied.

We mention in passing that one can also construct a GD-IFS with the *CSSC*, whose GD-attractor is not attractor of any standard IFS. For example, let p_6 be a prime different from other p_j $(1 \le j \le 5)$, and let

$$x'' = (p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}, \lambda, p_5\lambda, p_6\lambda, \pi\lambda).$$

Such a point x'' also belongs to the set P, and the corresponding GD-IFS, $F(x'')$ associated with x'' in a way of Lemma 3.1, satisfies the CSSC. When $L = e_1(1)$, $u = 3$ and $v = 1$, the attractor F_3 of this $F(x'')$ is not an attractor of any standard IFS. We omit the details.

Lemma 4.4. *For a strongly connected digraph $G = (V, E)$ with $d_w \ge 2$ for all $w \in V$, let A be the absolute contraction ratio set of a COSC GD-IFS based on G, having $(F_u)_{u\in V}$ as its attractors. Suppose that the following conditions hold.*

(i′) *All the contraction ratios have different absolute values, and $1 \notin A^{\mathbb{Q}^*}$.*

(ii′) *$\Lambda_w \ne \emptyset$ for all $w \in V$.*

(iii) *For all pairs $(w, k) \ne (z, m)$ with $\lambda_z^{(m)} \ne 0$, where $w, z \in V$ and $1 \le k \le d_w - 1$, $1 \le m \le d_z - 1$,*

$$\lambda_w^{(k)}/\lambda_z^{(m)} \notin (A(L) \cup A(L^c))^{\mathbb{Q}}.$$

If G contains a directed circuit not passing through a vertex u, then F_u is not the attractor of any COSC standard IFS.

Proof. Let L be a directed circuit that does not go through u. By condition (ii′) we know that condition (ii) in Lemma 4.1 holds upon taking any vertex v in L. Since the digraph G is strongly connected, there exists a directed path from u to v. Since condition (iii) remains the same, we only need to verify condition (i) in Lemma 4.1 under the stronger condition (i′). For, suppose that there exists some $\theta \in (A(L))^{\mathbb{Q}} \cap (A(L^c))^{\mathbb{Q}^*_+}$. Setting

$$A(L) = \{b_i\}_{i=1}^m, \quad A(L^c) = \{c_j\}_{j=1}^n$$

so that $A = A(L) \cup A(L^c)$, we write

$$\theta = \prod_{i=1}^m b_i^{p_i} = \prod_{j=1}^n c_j^{q_j}$$

for some two vectors $(p_i)_{i=1}^m \in \mathbb{Q}^m$ and $(q_j)_{j=1}^n \in (\mathbb{Q}_+^n)^*$. Then

$$1 = \prod_{i=1}^m b_i^{p_i} \prod_{j=1}^n c_j^{-q_j} \in A^{\mathbb{Q}^*},$$

where we have used that $A(L) \cap A(L^c) = \emptyset$ since all the contraction ratios have different absolute values by condition (i′). However, this contradicts our assumption $1 \notin A^{\mathbb{Q}^*}$. Therefore, all conditions in Lemma 4.1 are satisfied, thus the conclusion of the lemma follows. ■

Remark 4.5. Lemma 4.4 is an extension of [3, Theorem 6.3]. The assertion of Lemma 4.4 is optimal in the sense that the restriction on the graph 'there is a circuit not passing through u' cannot be relaxed (see Theorem 5.4 in the Appendix).

The following example, with a digraph that has two vertices with two loops and is strongly connected, illustrates Lemma 4.4.

Example 4.6 (Two-vertex digraph). Let $G = (V, E)$ be a strongly connected digraph, where $V = \{1, 2\}$, $E = \{e_1(1), e_1(2), e_2(1), e_2(2)\}$, so that $d_1 = 2$, $d_2 = 2$, see Figure 4.

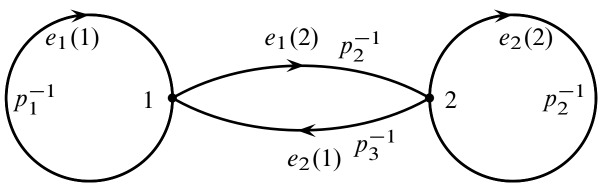

Figure 4. F_1 and F_2 are not the attractors of any standard COSC IFS.

Let $\{p_j\}_{1 \le j \le 4}$ be four distinct primes arranged in ascending order so that $2 \le p_j < p_{j+1}$, and let p_5 be a positive number such that $\log p_5$ is not a rational linear combination of $\{\log p_j\}_{1 \le j \le 4}$. Let $\lambda > 0$ be any real number, and let x be a vector given by

$$x = (p_1^{-1}, p_2^{-1}, p_3^{-1}, p_4^{-1}, \lambda, p_5\lambda) =: (x_1^{(1)}, x_1^{(2)}, x_2^{(1)}, x_2^{(2)}, \xi_1^{(1)}, \xi_2^{(1)}). \tag{4.15}$$

The matrix $M_x(1)$ in (3.3), (3.4) associated with point x is given by

$$M := M_x(1) = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix} = \begin{pmatrix} p_1^{-1} & p_2^{-1} \\ p_3^{-1} & p_4^{-1} \end{pmatrix}.$$

Note that $x \in P$ in (3.11) by using (3.24), since

$$\max\{p_1^{-1} + p_2^{-1}, p_3^{-1} + p_4^{-1}\} < 1.$$

Let $\ell = (l_1, l_2)^T$ be given by (3.12), that is,

$$\ell = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = (\mathrm{id} - M_x(1))^{-1} \begin{pmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 - p_1^{-1} & -p_2^{-1} \\ -p_3^{-1} & 1 - p_4^{-1} \end{pmatrix}^{-1} \begin{pmatrix} \lambda \\ p_5\lambda \end{pmatrix}.$$

Let $\boldsymbol{b} := (b_1^{(1)}, b_2^{(1)})$ for $b_1^{(1)}, b_2^{(1)} \in \mathbb{R}$, and let

$$b_1^{(2)} := b_1^{(1)} + p_1^{-1} l_1 + \lambda,$$
$$b_2^{(2)} := b_2^{(1)} + p_3^{-1} l_2 + \lambda p_5.$$

We define four similarities $F(x, \boldsymbol{b}) := \{S_{e_1(1)}, S_{e_1(2)}, S_{e_2(1)}, S_{e_2(2)}\}$, depending on x, $\boldsymbol{b}$, by

$$3S_{e_1(1)}(t) = p_1^{-1} t + b_1^{(1)}, \qquad S_{e_1(2)}(t) = p_2^{-1} t + b_1^{(2)},$$
$$S_{e_2(1)}(t) = p_3^{-1} t + b_2^{(1)}, \qquad S_{e_2(2)}(t) = p_4^{-1} t + b_2^{(2)} \qquad \text{for } t \in \mathbb{R}.$$

Clearly, such a GD-IFS $F(x, \boldsymbol{b})$ has absolute contraction ratio set given by $A := \{p_i^{-1}\}_{i=1}^4$. Applying Lemma 3.1, $F(x, \boldsymbol{b})$ satisfies the CSSC, whose basic gap lengths sets are $\Lambda_1 = \{\xi_1^{(1)}\} = \{\lambda\}$ (at vertex 1) and $\Lambda_2 = \{\xi_2^{(1)}\} = \{\lambda p_5\}$ (at vertex 2). Let F_1, F_2 be the attractors of $F(x, \boldsymbol{b})$ at vertices 1 and 2.

We will use Lemma 4.4 to show that F_1 (or F_2) is not the attractor of any COSC standard IFS, noting that (V, E) contains a directed circuit (loop) not passing through vertex 1 (or through vertex 2).

Condition (i′) is clear since the contraction ratios $A = \{p_i^{-1}\}_{i=1}^4$ are distinct, and $1 \notin A^{\mathbb{Q}^*}$ by using Proposition 5.5 in the Appendix. Condition (ii′) is trivial since the basic gap lengths are $\lambda, \lambda p_5$ that are strictly positive.

It remains to verify condition (iii) or equivalently to check that

$$\frac{\lambda_2^{(1)}}{\lambda_1^{(1)}} = \frac{\xi_2^{(1)}}{\xi_1^{(1)}} = \frac{\lambda p_5}{\lambda} = p_5 \notin A^{\mathbb{Q}},$$
$$\frac{\lambda_1^{(1)}}{\lambda_2^{(1)}} = \frac{1}{p_5} \notin A^{\mathbb{Q}}.$$

However, this is trivial by noting that

$$p_5 \neq \left(p_1^{-1}\right)^{s_1} \left(p_2^{-1}\right)^{s_2} \left(p_3^{-1}\right)^{s_3} \left(p_4^{-1}\right)^{s_4} \quad \text{(the same is true for } p_5^{-1}\text{)}$$

for any rationals $(s_i)_{i=1}^4$, since $\log p_5$ is not a rational linear combination of $\{\log p_j\}_{1 \leq j \leq 4}$.

Therefore, all the assumptions (i′), (ii′), (iii) in Lemma 4.4 are satisfied, so the GD-attractor F_1 (or F_2) is not the attractor of any standard IFS with the COSC.

We next show that for n-dimensional Lebesgue almost all vectors in P, all the conditions in Lemma 4.4 hold for their corresponding GD-IFSs. Let P_1 be a subset of

P given by

$$P_1 := \{x \in P_0 : r_\sigma(M_x(1)) < 1, \xi_i^{(k)} > 0 \quad \text{for each vertex } i \in V, 1 \le k \le d_i - 1\}. \tag{4.16}$$

Clearly, $P_1 \subset P$ since each $\sum_{k=1}^{d_i-1} \xi_i^{(k)} > 0$.

Definition 4.7 (Admissible set). With the notation as above, we say that a point $x = (x_1, x_2, \ldots, x_n)$ in the set P_1 is *admissible* if

$$\prod_{i=1}^{n} |x_i|^{p_i} \neq \prod_{i=1}^{n} |x_i|^{q_i} \tag{4.17}$$

for any two distinct vectors $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ of nonnegative rationals. The set of all admissible points is denoted by $\mathcal{A}$.

Note that the admissible set $\mathcal{A}$ depends only on the numbers of vertices and their out-degrees, but is independent of any vertex itself and the order of edges. If $(x_1, x_2, \ldots, x_n) \in \mathcal{A}$, then for any two distinct indices i, j, taking $p_i = 1$, $p_k = 0$ for all $k \neq i$ and $q_j = 1$, $q_k = 0$ for all $k \neq j$ in (4.17),

$$|x_i| \neq |x_j|, \tag{4.18}$$

and so the entries of any vector in $\mathcal{A}$ all have distinct absolute values.

By Lemma 3.1, we know that each admissible point x leads to a COSC GD-IFS

$$F(x, \mathscr{b}) \tag{4.19}$$

as in (3.7), (3.12), for any $\mathscr{b}$ in (3.5).

The following says that the size of the admissible set $\mathcal{A}$ is very large.

Theorem 4.8. *Let $G = (V, E)$ be a strongly connected digraph with $d_w \ge 2$ for all $w \in V$, containing a vertex $u \in V$ outside a directed circuit. With the notation as above, if $x \in \mathcal{A}$, then the attractor F_u of the corresponding GD-IFS, $F(x, \mathscr{b})$, defined as in* (4.19) *for any $\mathscr{b}$, is not the attractor of any COSC standard IFS. Moreover, with n given as in* (3.1),

$$\mathcal{L}^n(P \setminus \mathcal{A}) = 0,$$

that is, the complement of the set $\mathcal{A}$ in P has n-dimensional Lebesgue measure zero.

Proof. Let $\mathscr{b} = (b_i^{(1)})_{i \in V}$ for $b_i^{(1)} \in \mathbb{R}$ and let $x = (x_1, x_2, \ldots, x_n)$ be an admissible point. By Lemma 3.1, the corresponding GD-IFS, $F(x, \mathscr{b})$ associated with the vectors $x, \mathscr{b}$, satisfies the CSSC. We will show that such a GD-IFS $F(x, \mathscr{b})$ also satisfies all three conditions (i′), (ii′), (iii) in Lemma 4.4.

Clearly, the GD-IFS $F(x, \mathscr{b})$ satisfies condition (ii′) by noting that $\Lambda_i \neq \emptyset$ for each vertex $i \in V$, since all the basic gap lengths sitting at vertex i are $\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(d_i-1)}$ by Lemma 3.1 (ii), which are strictly positive since the vector x belongs to P_1.

We show condition (i′). Let

$$X := \{|x_i|\}_{i=1}^n \subset (0, \infty).$$

We need to prove

$$1 \notin X^{\mathbb{Q}^*}. \tag{4.20}$$

For, suppose that $1 = \prod_{i=1}^n |x_i|^{s_i}$ for some $(s_i)_{i=1}^n \in (\mathbb{Q}^n)^*$, then

$$\prod_{i=1}^n |x_i|^{s_i^-} = \prod_{i=1}^n |x_i|^{s_i^+},$$

where $s_i^+ = \max\{s_i, 0\}$, $s_i^- = \max\{-s_i, 0\}$ so that $s_i = s_i^+ - s_i^-$. As not all s_i are zero, we see that $(s_i^+)_{i=1}^n \neq (s_i^-)_{i=1}^n$ are two distinct nonnegative rational vectors. This contradicts the admissibility of x as defined in (4.17), thus (4.20) is true.

By using (3.7) and (4.18), all the contraction ratios of the COSC GD-IFS $F(x, \mathscr{b})$ have different absolute values. Since $1 \notin A^{\mathbb{Q}^*}$ as $A^{\mathbb{Q}^*} \subset X^{\mathbb{Q}^*}$, where A is the absolute contraction ratio set of $F(x, \mathscr{b})$, condition (i′) is satisfied.

For condition (iii), suppose that there exists some $a \in A^{\mathbb{Q}}$ such that $a = \lambda_w^{(k)}/\lambda_z^{(m)}$, where $\lambda_w^{(k)} \in \Lambda_w$, $\lambda_z^{(m)} \in \Lambda_z$. Then

$$1 = \lambda_w^{(k)}\big(\lambda_z^{(m)}\big)^{-1} a^{-1} \in \{|x_i|\}^{\mathbb{Q}^*} = X^{\mathbb{Q}^*}$$

by noting that $\lambda_w^{(k)} = x_i > 0$, $\lambda_z^{(m)} = x_j > 0$ for some two indices $i \neq j$ in virtue of definition (3.2), contradicting (4.20). Thus, condition (iii) is also satisfied.

Therefore, by applying Lemma 4.4, the attractor F_u of the GD-IFS $F(x, \mathscr{b})$ is not the attractor of any COSC standard IFS.

We finally show that $\mathcal{L}^n(P \setminus \mathcal{A}) = 0$. For this, note that

$$\mathcal{L}^n(P \setminus P_1) = 0,$$

where P_1 is defined as in (4.16), since $P \setminus P_1$ lies in the union of hyperplanes $\xi_i^{(k)} = 0$. We just need to show $\mathcal{L}^n(P_1 \setminus \mathcal{A}) = 0$. Let

$$x = (x_1, x_2, \ldots, x_n) \in P_1 \setminus \mathcal{A},$$

that is, for some two distinct vectors $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ of nonnegative rationals,

$$\prod_{i=1}^n |x_i|^{p_i} = \prod_{i=1}^n |x_i|^{q_i}.$$

As $p_i \neq q_i$ for some i, say without loss of generality for $i = 1$, then

$$|x_1| = \prod_{i=2}^{n} |x_i|^{(q_i - p_i)/(p_1 - q_1)},$$

from which, it follows that any vector in $P_1 \setminus \mathcal{A}$ lies in an at most $(n-1)$-dimensional manifold. Since there are countably many such equations, the union of countably many such manifolds has n-dimensional Lebesgue measure zero in $\mathbb{R}^n$. ∎

There are plenty of examples of admissible points so that the assertions of Theorem 4.8 hold. However, there are also some other interesting examples such that the first assertion in Theorem 4.8 still holds, but points are not admissible.

Example 4.9. The point x given by (4.15) in Example 4.6 is not admissible in the sense of Definition 4.7 for a certain class of λ. To see this, we need to show that (4.17) fails for suitable λ. In fact, if (4.17) fails, then by definition (4.15)

$$\begin{aligned}(p_1^{-1})^{s_1}(p_2^{-1})^{s_2}(p_3^{-1})^{s_3}(p_4^{-1})^{s_4}\lambda^{s_5}(p_5\lambda)^{s_6} &= \prod_{i=1}^{6}|x_i|^{s_i} = \prod_{i=1}^{6}|x_i|^{t_i}\\ &= (p_1^{-1})^{t_1}(p_2^{-1})^{t_2}(p_3^{-1})^{t_3}(p_4^{-1})^{t_4}\lambda^{t_5}(p_5\lambda)^{t_6}\end{aligned}$$

for some two distinct vectors $(s_i)_{i=1}^{6}$ and $(t_i)_{i=1}^{6}$ of nonnegative rationals. From this, we know that

$$\lambda^{(s_5-t_5)+(s_6-t_6)} = p_1^{s_1-t_1} p_2^{s_2-t_2} p_3^{s_3-t_3} p_4^{s_4-t_4} p_5^{-(s_6-t_6)}. \tag{4.21}$$

Thus, condition (4.17) fails if λ is chosen as in (4.21). In particular, condition (4.17) fails if $\lambda = 1/\sqrt{p_5}$ on taking $s_i = t_i$ for $i = 1, 2, 3, 4$ whilst $s_i = t_i + 1$ for $i = 5, 6$.

However, the GD-attractor F_u, associated with such a non-admissible point x, is not the attractor of any COSC standard IFS by Example 4.6.

We further consider the situation by removing the 'COSC'. We will apply Corollary 3.4 and Theorem 5.6 in the Appendix.

Theorem 4.10. *Let $G = (V, E)$ be a strongly connected digraph with $d_j \geq 2$ for every vertex $j \in V$, containing a vertex $i \in V$ outside a directed circuit. Let $x \in \mathcal{A}(\delta)$ (see definition (3.26)) satisfying that, for every vertex $j \neq i$ in V,*

$$|x_i^{(1)}| \in \big(|x_j^{(1)}| + \cdots + |x_j^{(m_j)}| + (m_j - 1)\delta,\, |x_j^{(1)}| + \cdots + |x_j^{(m_j)}| + m_j\delta\big), \tag{4.22}$$

$$1 - |x_i^{(1)}| \in \big(|x_j^{(1)}| + \cdots + |x_j^{(n_j)}| + (n_j - 1)\delta,\, |x_j^{(1)}| + \cdots + |x_j^{(n_j)}| + n_j\delta\big), \tag{4.23}$$

where $m_j, n_j \in [1, d_j - 1]$ are integers. Let $F(x)$ be corresponding CSSC GD-IFS constructed as in Corollary 3.4, with GD-attractors $(F_j)_{j\in V}$. Then F_i is not the attractor of any standard IFS.

Proof. Let $x \in \mathcal{A}(\delta)$. Recall that the corresponding GD-IFS, $F(x) = \{S_{e_i(k)}\}_{i \in V, 1 \le k \le d_i}$ associated with point x, is given by (3.31), where $\{b_i^{(k+1)}\}_{i \in V, 1 \le k \le d_i - 1}$ are real numbers in $(0, 1)$ defined as in (3.30) ($b_i^{(1)} = 0$ for every $i \in V$).

We apply Theorem 5.6 in the Appendix to prove this theorem. Clearly, conditions (1), (2) in Theorem 5.6 are satisfied. In order to verify condition (3), we need to show that for every vertex $j \neq i$,

$$F_i \not\subseteq F_j \tag{4.24}$$

and

$$1 - F_i \not\subseteq F_j. \tag{4.25}$$

We first show (4.24). Indeed, note that the point $|x_i^{(1)}|$ belongs to the attractor F_i by Corollary 3.4 (ii). However, this point does not belong to any attractor F_j $(j \neq i)$, since it falls in some basic gap (see formula (3.33)) of F_j by using assumption (4.22).

Similarly, the point $1 - |x_i^{(1)}|$ belongs to the set $1 - F_i$ but does not belong to any attractor F_j $(j \neq i)$, since it also falls in some basic gap of F_j by using assumption (4.23), and thus (4.25) is also true, as required. ∎

To illustrate Theorem 4.10, we give an example. Let i be a fixed vertex in $V = \{1, 2, \ldots, N\}$. Let $x \in \mathcal{A}(\delta)$ satisfy

$$|x_i^{(1)}| \in \big(|x_j^{(1)}|, |x_j^{(1)}| + \delta\big) \quad \text{and} \quad 1 - |x_i^{(1)}| \in \big(|x_j^{(1)}|, |x_j^{(1)}| + \delta\big) \quad \text{for any } j \neq i, \tag{4.26}$$

so that both conditions (4.22), (4.23) are satisfied with $m_j = n_j = 1$. To secure (4.26), we let

$$0 < \delta < \min_{j \in V} \left\{ \frac{1}{2(d_j - 1)} \right\}.$$

By the definition of $\mathcal{A}(\delta)$, any vector $x \in \mathcal{A}(\delta)$ satisfies that for all $j \in V$,

$$|x_j^{(1)}| + \cdots + |x_j^{(d_j)}| = 1 - (d_j - 1)\delta > \frac{1}{2}. \tag{4.27}$$

Now we first choose $\{x_j^{(1)}\}_{j \in V}$ by

$$|x_i^{(1)}| = \frac{1}{2} \quad \text{and} \quad |x_j^{(1)}| \in \left(\frac{1}{2} - \delta, \frac{1}{2}\right) \quad \text{for any } j \neq i,$$

and then we choose $\big\{x_j^{(2)}, x_j^{(3)}, \ldots, x_j^{(d_j)}\big\}_{j \in V}$ to be any numbers such that (4.27) is satisfied. Such a class of points satisfy condition (4.26), which implies that conditions (4.22), (4.23) are both satisfied.

5. Appendix

In this appendix we derive some general properties and secondary results that are used in the main part of the paper.

The following proposition on ordering integer lattice points is used in Lemma 2.6. Recall that $\mathbb{Z}_+$ denotes the set of all nonnegative integers.

Proposition 5.1. *Let $B \subset \mathbb{Z}_+^n$ be an infinite set. Then B contains two distinct vectors $\overrightarrow{x} \leq \overrightarrow{y}$ under the partial order defined by inequality of all coordinates.*

Proof. We write $\overrightarrow{x} := (x_i)_{i=1}^n \in \mathbb{Z}_+^n$. Consider the set of integers:

$$S := \left\{ \min\{x_i\}_{i=1}^n : \overrightarrow{x} \in B \right\}.$$

If S is unbounded, then we are done by fixing some vector $\overrightarrow{x}$ and taking $\overrightarrow{y} = (y_i)_{i=1}^n \in B \subset \mathbb{Z}_+^n$ with $\min\{y_i\}_{i=1}^n > \max\{x_i\}_{i=1}^n$ so that

$$\overrightarrow{x} < \overrightarrow{y}.$$

Otherwise, S is bounded by an integer N in which case we prove the proposition by induction on n. When $n = 1$ it is trivial. Assume that the proposition holds for $n - 1$. For each $1 \leq j \leq n$ and each $\alpha \in \{0, 1, \dots, N\}$, define

$$B(\alpha, j) := \{\overrightarrow{x} \in B : x_j = \min\{x_i\}_{i=1}^n = \alpha\},$$

a (possibly empty) collection of all vectors in B whose j-th entries equal to the same number α and take the smallest value. Since

$$\bigcup_{j=1}^{n} \bigcup_{\alpha=0}^{N} B(\alpha, j) = B,$$

we can assume that some $B(\alpha, j)$, say $B(\beta, m)$, contains infinitely many elements. Deleting the mth coordinate $x_m = \beta$ of all the vectors in such a set $B(\beta, m)$, we obtain an infinite set $B'(\beta, m) \subset \mathbb{Z}_+^{n-1}$, and by induction assumption, $B'(\beta, m) \subset \mathbb{Z}_+^{n-1}$ has two distinct vectors $\overrightarrow{x'} \leq \overrightarrow{y'}$. Inserting the mth coordinate $x_m = \beta$ into $\overrightarrow{x'}$, $\overrightarrow{y'}$ to get $\overrightarrow{x}$, $\overrightarrow{y}$, respectively, we obtain two distinct vectors $\overrightarrow{x} \leq \overrightarrow{y}$ in $B(\beta, m) \subset B$, showing the assertion for $\mathbb{Z}_+^n$. ∎

The next proposition generalises a well-known result for standard IFSs to GD-IFSs.

Proposition 5.2. *Let $G = (V, E)$ be a digraph and $(F_u)_{u\in V}$ be the GD-attractors of a GD-IFS $F = (V, E, (S_e)_{e\in E})$ based on it. If there exist non-empty sets $(U_u)_{u\in V}$ such that*

$$\bigcup_{v\in V}\bigcup_{e\in E_{uv}} S_e(U_v) \subset U_u \quad \textit{for each } u \in V, \tag{5.1}$$

then $F_u \subset \overline{U_u}$, the closure of the set U_u, for each $u \in V$.

Proof. Set $I_u := \overline{U_u}$ for each $u \in V$. Let I_u^m be defined by (2.2) for $m \geq 1$. Then the inclusion (2.3) is satisfied, since

$$\begin{aligned} I_u^1 &= \bigcup_{e\in E_u^1} S_e(I_{\omega(e)}) = \bigcup_{e\in E_u^1} S_e(\overline{U_{\omega(e)}}) = \bigcup_{e\in E_u^1} \overline{S_e(U_{\omega(e)})} \\ &\subset \overline{\bigcup_{e\in E_u^1} S_e(U_{\omega(e)})} \subset \overline{U_u} = I_u \quad (\text{using (5.1)}), \end{aligned}$$

thus showing that $F_u \subset I_u^1 \subset \overline{U_u}$ by virtue of (2.5). The proof is complete. ∎

The directed paths in GD-IFSs play the same role as the finite-length words in standard IFSs, as the following proposition suggests. We will frequently use the following fact that, for any $u \in V$ and $m \geq 1$,

$$F_u = \bigcup_{\mathbf{e}\in E_u^m} S_{\mathbf{e}}(F_{\omega(\mathbf{e})}), \tag{5.2}$$

by repeatedly using definition (1.2) (recall that E_u^m is the totality of all paths of length m leaving u). The following proposition concerns the disjointness of images of components under mappings corresponding to different words.

Proposition 5.3. *Let $G = (V, E)$ be a digraph and $(F_u)_{u\in V}$ be the GD-attractors of a GD-IFS $F = (V, E, (S_e)_{e\in E})$ based on it. Assume that each F_u is not a singleton. Let $\mathbf{e}'$, $\mathbf{e}''$ be two directed paths with $\mathbf{e}'' \neq \mathbf{e}'\mathbf{e}$ if $|\mathbf{e}'| \leq |\mathbf{e}''|$ (where $\mathbf{e}$ is a directed path which may be empty). If F satisfies the COSC on $\mathbb{R}$, then the interiors of $S_{\mathbf{e}'}(\mathrm{conv}\, F_{\omega(\mathbf{e}')})$ and $S_{\mathbf{e}''}(\mathrm{conv}\, F_{\omega(\mathbf{e}'')})$ are disjoint. Similarly, if F satisfies the CSSC, then $S_{\mathbf{e}'}(F_{\omega(\mathbf{e}')})$ and $S_{\mathbf{e}''}(F_{\omega(\mathbf{e}'')})$ are disjoint.*

Proof. By (5.2), for any path $\mathbf{e}$, we have $S_{\mathbf{e}}(F_{\omega(\mathbf{e})}) \subset F_{\alpha(\mathbf{e})}$, where $\alpha(\mathbf{e})$ denotes the initial vertex of path $\mathbf{e}$. As $S_{\mathbf{e}}$ is a similarity on $\mathbb{R}$, taking the convex hulls gives that

$$S_{\mathbf{e}}(\mathrm{conv}\, F_{\omega(\mathbf{e})}) = \mathrm{conv}\, S_{\mathbf{e}}(F_{\omega(\mathbf{e})}) \subset \mathrm{conv}\, F_{\alpha(\mathbf{e})},$$

from which, we see that, for any path $\mathbf{e}_1\mathbf{e}_2$ (meaning that $\omega(\mathbf{e}_1) = \alpha(\mathbf{e}_2)$, the terminal of $\mathbf{e}_1$ is the initial of $\mathbf{e}_2$)

$$S_{\mathbf{e}_1\mathbf{e}_2}(\mathrm{conv}\, F_{\omega(\mathbf{e}_2)}) = S_{\mathbf{e}_1}(S_{\mathbf{e}_2}(\mathrm{conv}\, F_{\omega(\mathbf{e}_2)})) \subset S_{\mathbf{e}_1}(\mathrm{conv}\, F_{\alpha(\mathbf{e}_2)}) = S_{\mathbf{e}_1}(\mathrm{conv}\, F_{\omega(\mathbf{e}_1)}). \tag{5.3}$$

Assume now that F satisfies the COSC. By (1.3), one can take

$$U_u = \operatorname{int}(\operatorname{conv} F_u) \quad \text{for each } u \in V,$$

which is non-empty by our assumption that F_u is not a singleton.

For any two paths $\mathbf{e}e_1$, $\mathbf{e}e_2$ with common path $\mathbf{e}$ and distinct edges e_1, e_2, the interiors of two intervals

$$\operatorname{int}\big(S_{\mathbf{e}e_1}(\operatorname{conv} F_{\omega(e_1)})\big) \cap \operatorname{int}\big(S_{\mathbf{e}e_2}(\operatorname{conv} F_{\omega(e_2)})\big) = \emptyset \tag{5.4}$$

by using the COSC, since

$$S_{\mathbf{e}e_1}(\operatorname{conv} F_{\omega(e_1)}) = S_{\mathbf{e}}(S_{e_1}(\operatorname{conv} F_{\omega(e_1)}))$$

and

$$S_{\mathbf{e}e_2}(\operatorname{conv} F_{\omega(e_2)}) = S_{\mathbf{e}}\big(S_{e_2}(\operatorname{conv} F_{\omega(e_2)})\big)$$

and the interiors of $S_{e_1}(\operatorname{conv} F_{\omega(e_1)})$ and $S_{e_2}(\operatorname{conv} F_{\omega(e_2)})$ are disjoint as the edges e_1, e_2 have the same initial vertex, namely the terminal of path $\mathbf{e}$.

Let $\mathbf{e}$ be the longest common path of $\mathbf{e}''$ and $\mathbf{e}'$ (which may be empty). Write $\mathbf{e}' = \mathbf{e}e_1\mathbf{p}_1$ and $\mathbf{e}'' = \mathbf{e}e_2\mathbf{p}_2$, where $e_1 \neq e_2$ are two distinct edges and $\mathbf{p}_1, \mathbf{p}_2$ are some paths (possibly empty). By (5.3),

$$\begin{aligned} S_{\mathbf{e}'}(\operatorname{conv} F_{\omega(\mathbf{e}')}) &= S_{\mathbf{e}e_1\mathbf{p}_1}(\operatorname{conv} F_{\omega(\mathbf{p}_1)}) \subset S_{\mathbf{e}e_1}(\operatorname{conv} F_{\omega(e_1)}), \\ S_{\mathbf{e}''}(\operatorname{conv} F_{\omega(\mathbf{e}'')}) &= S_{\mathbf{e}e_2\mathbf{p}_2}(\operatorname{conv} F_{\omega(\mathbf{p}_2)}) \subset S_{\mathbf{e}e_2}(\operatorname{conv} F_{\omega(e_2)}), \end{aligned}$$

thus, the interiors of $S_{\mathbf{e}'}(\operatorname{conv} F_{\omega(\mathbf{e}')})$ and $S_{\mathbf{e}''}(\operatorname{conv} F_{\omega(\mathbf{e}'')})$ are disjoint by using (5.4).

The assertion for the CSSC is similar. The proof is complete. ∎

The following was essentially proved in [2, Lemma 5.1], except that we also consider the COSC case.

Theorem 5.4. *Let $G = (V, E)$ be a strongly connected digraph with $d_v \geq 2$ for all $v \in V$. If every directed circuit goes through a vertex $u \in V$, then for any (resp. COSC) GD-IFS based on G, its attractor F_u is also the attractor of a (resp. COSC) standard IFS.*

Proof. Set $N := \#V$, the number of vertices in V. Let $L(u)$ be the set of all circuits having u as their initial and terminal, and which do not contain another shorter circuits, that is,

$$L(u) := \{\mathbf{e} = e_{uv_1v_2\cdots v_ku} \colon \text{each } v_i \neq u \in V, |\mathbf{e}| \leq N\},$$

where the symbol $e_{uv_1v_2\cdots v_ku} = e_{uv_1}e_{v_1v_2}\cdots e_{v_ku}$ is understood to be a path consisting of consecutive edges.

We claim that

$$F_u = \bigcup_{\mathbf{e}' \in L(u)} S_{\mathbf{e}'}(F_u), \tag{5.5}$$

by using the fact that every circuit goes through vertex u.

To see this, we have by (5.2) that

$$F_u = \bigcup_{\mathbf{e} \in E_u^N} S_{\mathbf{e}}(F_{\omega(\mathbf{e})}).$$

Since any directed path $\mathbf{e}$ in E_u^N can be written as

$$\mathbf{e} = e_{uv_1v_2\cdots v_N},$$

we see that at least one of vertices $v_1, v_2, \ldots, v_N$ must be u, otherwise, one of them would appear twice, thus producing a circuit, contradicting the assumption that every directed circuit goes through vertex u. There exists some index k such that $v_k = u$ and the path visits u the second time (besides the initial time), and

$$\mathbf{e} = e_{uv_1v_2\cdots v_{k-1}uv_{k+1}\cdots v_N} = e_{uv_1v_2\cdots v_{k-1}u}e_{uv_{k+1}\cdots v_N} = \mathbf{e}'\mathbf{e}'',$$

where $\mathbf{e}' = e_{uv_1v_2\cdots v_{k-1}u} \in L(u)$ and $\mathbf{e}''$ is a path with initial u if it exists (possibly $\mathbf{e}''$ is empty and the following argument will become easier). From this, we know that

$$S_{\mathbf{e}}(F_{\omega(\mathbf{e})}) = S_{\mathbf{e}'}(S_{\mathbf{e}''}(F_{\omega(\mathbf{e}'')})) \subset S_{\mathbf{e}'}(F_u),$$

since $S_{\mathbf{e}''}(F_{\omega(\mathbf{e}'')}) \subset F_u$ by (5.2). It follows that

$$F_u = \bigcup_{\mathbf{e} \in E_u^N} S_{\mathbf{e}}(F_{\omega(\mathbf{e})}) \subset \bigcup_{\mathbf{e}' \in L(u)} S_{\mathbf{e}'}(F_u).$$

The opposite inclusion is also clear since, by (5.2),

$$S_{\mathbf{e}'}(F_u) \subset F_u,$$

thus showing that our claim (5.5) holds true. Therefore, F_u is the attractor of the IFS

$$\Phi := \{S_{\mathbf{e}'} : \mathbf{e}' \in L(u)\}. \tag{5.6}$$

If the GD-IFS F further satisfies the COSC, we claim that the IFS Φ given by (5.6) also satisfies the COSC. Indeed, by definition of the COSC and the fact that Φ has attractor F_u, we need only to show that the interiors of two intervals $S_{\mathbf{e}'}(\mathrm{conv}\, F_u)$ and $S_{\mathbf{e}''}(\mathrm{conv}\, F_u)$ are disjoint, where $\mathbf{e}'$, $\mathbf{e}''$ are in $L(u)$. But this assertion immediately follows from Proposition 5.3. ∎

The following easy property of powers of primes is used in the examples in Section 4.

Proposition 5.5. *Let $\{a_i\}_{i=1}^n$ be distinct positive prime numbers. Then*

$$1 \notin A^{\mathbb{Q}^*} \quad \text{for} \quad A = \{a_i^{-1}\}_{i=1}^n.$$

Proof. Suppose to the contrary, that $1 \in A^{\mathbb{Q}^*}$. Then $1 = \prod_{i=1}^n (a_i^{-1})^{s_i}$ for some non-zero vector $(s_i)_{i=1}^n$ of rationals. Let q be the least common denominator of the rationals s_i. Taking the qth power, it follows that

$$m := \prod_{i=1}^n a_i^{qs_i^-} = \prod_{i=1}^n a_i^{qs_i^+},$$

where $s_i^+ = \max\{s_i, 0\}$, $s_i^- = \max\{-s_i, 0\}$ so that $s_i = s_i^+ - s_i^-$. As the s_i are not all zero, the vectors of integers $(qs_i^+)_{i=1}^n$ and $(qs_i^-)_{i=1}^n$ are distinct. By the uniqueness of the prime factorisation of the integer m, we see that $(qs_i^+)_{i=1}^n = (qs_i^-)_{i=1}^n$, a contradiction. ∎

The following assertion was essentially obtained in [2, Theorem 1.4 and the end of Section 1]. Here we give a simpler proof under stronger assumptions with conditions (2), (3) in the next theorem.

Theorem 5.6. *Let $G = (V, E)$ be a strongly connected digraph with $d_w \geq 2$ for each $w \in V$. Suppose that a given GD-IFS of similarities based on G satisfies the CSSC, and* $\operatorname{conv} F_w = [0, 1]$ *for each $w \in V$. For some vertex $u \in V$, suppose the following conditions hold.*

(1) *There is a directed circuit that does not pass through u.*

(2) *All basic gaps have the same length $\delta > 0$.*

(3) *For each vertex $v \neq u$, we have $F_u \not\subseteq F_v$ and $1 - F_u \not\subseteq F_v$.*

Then F_u is not the attractor of any standard IFS defined on $\mathbb{R}$.

Proof. The proof is divided into two steps.

Step 1. We claim that, for any $v \in V$ and any contracting similarity f with $f(F_u) \subset F_v$, there exists some path $\mathbf{e}$ leaving v with terminal $\omega(\mathbf{e}) = u$ such that

$$f(F_u) \subset S_{\mathbf{e}}(F_u). \tag{5.7}$$

Indeed, as F_v consists of the level-1 cells $S_e(F_{\omega(e)})$ for edges e leaving v by using (1.2), the $f(F_u)$ must belong to only one of those cells, say

$$f(F_u) \subset S_e(F_{\omega(e)}) \quad \text{for some edge } e \text{ leaving } v. \tag{5.8}$$

Otherwise, there are two points in $f(F_u)$ lying in two distinct level-1 cells, and as $f(F_u) \subset F_v$, we know that $f(F_u)$ spans a basic gap of F_v, implying that $f(F_u)$ has a gap, containing a basic gap of F_v, whose length is clearly greater than or equal to δ. However, this is impossible, because all gap lengths of F_u do not exceed δ by assumption (2) and (2.8), so that all the gap lengths of $f(F_u)$ are strictly smaller than δ by using the contractivity of f.

By (5.2), it follows that

$$f(F_u) \subset F_v = \bigcup_{\mathbf{e}' \in E_v^m} S_{\mathbf{e}'}(F_{\omega(\mathbf{e}')}) \quad \text{for any } m \geq 1,$$

where E_v^m is the set of all paths leaving v with the same length m as before. As $f(F_u)$ has fixed diameter and cells $S_{\mathbf{e}'}(F_{\omega(\mathbf{e}')})$ have arbitrarily small diameters by taking m large, we can choose a longest directed path $\mathbf{e}_1$ leaving v, which exists by using (5.8) and the fact that distinct cells of the same length are disjoint (see Proposition 5.3), such that

$$f(F_u) \subset S_{\mathbf{e}_1}(F_{\omega(\mathbf{e}_1)}). \tag{5.9}$$

We show that the contraction ratio ρ of the mapping $S_{\mathbf{e}_1}^{-1} \circ f$ satisfies $\rho = \pm 1$.

The diameter of each F_w equals 1, since $\operatorname{conv} F_w = [0, 1]$ for each $w \in V$ by our assumption. By (5.9)

$$S_{\mathbf{e}_1}^{-1} \circ f(F_u) \subset F_{\omega(\mathbf{e}_1)}, \tag{5.10}$$

implying that $|\rho| \leq 1$ by comparing the diameters of F_u and $F_{\omega(\mathbf{e}_1)}$ and noting that $S_{\mathbf{e}_1}^{-1} \circ f$ is a similarity.

If $|\rho| < 1$, we will derive a contradiction. Indeed, by (5.10), we can apply (5.8) with f being replaced by $S_{\mathbf{e}_1}^{-1} \circ f$ and v replaced by $\omega(\mathbf{e}_1)$, and obtain

$$S_{\mathbf{e}_1}^{-1} \circ f(F_u) \subset S_e(F_{\omega(e)})$$

for some edge e leaving $\omega(\mathbf{e}_1)$. From this and that $\omega(e) = \omega(\mathbf{e}_1 e)$,

$$f(F_u) \subset S_{\mathbf{e}_1}(S_e(F_{\omega(e)})) = S_{\mathbf{e}_1 e}(F_{\omega(\mathbf{e}_1 e)}),$$

which contradicts the fact that $\mathbf{e}_1$ is the longest path by virtue of (5.9). Thus, $|\rho| = 1$.

Therefore, if $\rho = 1$, then $F_u + c \subset F_{\omega(\mathbf{e}_1)}$ for some translation $c \in \mathbb{R}$ using (5.10), which implies that

$$[0, 1] + c = (\operatorname{conv} F_u) + c = \operatorname{conv}(F_u + c) \subset \operatorname{conv} F_{\omega(\mathbf{e}_1)} = [0, 1]$$

using our assumption that $\operatorname{conv} F_w = [0, 1]$ for each $w \in V$. Then $c = 0$, and so

$$F_u \subset F_{\omega(\mathbf{e}_1)},$$

showing that $\omega(\mathbf{e}_1) = u$ by assumption (3) that $F_u \not\subseteq F_v$ if $v \neq u$.

Similarly, if $\rho = -1$, then $-F_u + c \subset F_{\omega(\mathbf{e}_1)}$ for some translation $c \in \mathbb{R}$ by (5.10), which implies that

$$[-1, 0] + c = \mathrm{conv}(-F_u) + c = \mathrm{conv}(-F_u + c) \subset \mathrm{conv}(F_{\omega(\mathbf{e}_1)}) = [0, 1]$$

using our assumption that $\mathrm{conv}\, F_w = [0, 1]$ for each $w \in V$. We must have $c = 1$, and so

$$1 - F_u \subset F_{\omega(\mathbf{e}_1)},$$

showing that $\omega(\mathbf{e}_1) = u$ again by assumption (3) that $1 - F_u \not\subseteq F_v$ if $v \neq u$.

Therefore, noting that $\omega(\mathbf{e}_1) = u$ in (5.9), we obtain (5.7) with $\mathbf{e} = \mathbf{e}_1$, proving our claim.

Step 2. We show that F_u is not the attractor of any standard IFS.

Assume to the contrary that there exists a standard IFS $\{f_i\}$ such that

$$F_u = \bigcup_i f_i(F_u).$$

As $f_i(F_u) \subset F_u$, using (5.7) with $v = u$, we know that $f_i(F_u) \subset S_{\mathbf{e}_i}(F_u)$, and so

$$F_u = \bigcup_i f_i(F_u) \subset \bigcup_i S_{\mathbf{e}_i}(F_u), \tag{5.11}$$

where each $\mathbf{e}_i$ is a directed circuit from initial u to terminal u. By condition (1), there is a vertex $w \neq u$ contained in a circuit L that does not pass through u. By the strong connectivity, we can find a simple path L_1 (i.e., a path visits any vertex at most for once) from u to w.

Note that the path $L_1 L^m$ from u to w visits u only once. We can pick an integer m so large that the path length is greater than $\max\{|\mathbf{e}_i|\}_i$. By (5.2) and (5.11),

$$S_{L_1 L^m}(F_w) \subset F_u \subset \bigcup_i S_{\mathbf{e}_i}(F_u). \tag{5.12}$$

Note that $\{S_e\}$ satisfies the CSSC by assumption (2), and so $S_{L_1 L^m}(F_w)$ is disjoint with any set $S_{\mathbf{e}_i}(F_u)$ in (5.12) using Proposition 5.3, since the path $L_1 L^m$ does not start with any of these paths $\mathbf{e}_i$, otherwise $L_1 L^m$ would visit u twice. This contradicts (5.12), thus showing that F_u is not self-similar. ∎

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