Separability properties of semigroups and algebras

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A thesis submitted for the degree of PhD
at the
University of St Andrews

2021

Identifier to use to cite or link to this thesis:

DOI: https://doi.org/10.17630/sta/931

Abstract

Separability properties can be seen as generalisations of residual finiteness. In this thesis we investigate four such properties: monogenic subalgebra separability, weak subalgebra separability, strong subalgebra separability and complete separability.

In Chapter 1 we outline the necessary preliminary definitions and results. We define separability properties in terms of universal algebra, in order to be able to study these properties in a range of different settings. We also provide a topological interpretation of these properties. The chapter concludes with the necessary preliminary information to be able to study these properties in semigroups.

In Chapter 2 we investigate the separability properties of free objects in different semigroup varieties. This builds upon work by Hall which shows that the free group is weakly subgroup separable. The varieties considered are groups, semigroups, completely simple semigroups, Clifford semigroups and completely regular semigroups. We also define a new variety, known as α-groups, to aid in our investigation of the free completely simple semigroup.

We begin Chapter 3 by investigating which separability properties are inherited by the Schützenberger groups of a semigroup. We use the theory developed to classify precisely when a finitely generated commutative semigroup has each of four separability properties considered. We conclude the chapter by studying when separability properties of Schützenbeger groups pass to semigroups with finitely many $\mathcal{H}$-classes.

In the final chapter, we consider the preservation of separability properties under various semigroup-theoretic constructions. The constructions considered are the 0-direct union, the direct product, the free product, as well as an investigation into large subsemigroups. We classify precisely when a finite semigroup preserves both monogeinc subsemigroup separability and strong subsemigroup separability in the direct product. We conclude the work by indicating some directions that future research may take.

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I, Gerard Aidan O'Reilly, do hereby certify that this thesis, submitted for the degree of PhD, which is approximately 42 000 words in length, has been written by me, and that it is the record of work carried out by me, or principally by myself in collaboration with others as acknowledged, and that it has not been submitted in any previous application for any degree. I confirm that any appendices included in my thesis contain only material permitted by the 'Assessment of Postgraduate Research Students' policy.

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Acknowledgements

General Acknowledgements

I would like to thank my supervisors, Professor Nik Ruškuc and Dr Martyn Quick, for their thoughtful, patient and invaluable guidance throughout my PhD. I would also like to thank the staff of the School of Mathematics and Statistics of the University of St Andrews for all their work in supporting me in this project.

I wish to extend my gratitude to the following people for the support (virtual or otherwise) during the last four years: Ashley Clayton, Craig Miller, Finn Smith, Fanny Empacher and Tao Chen.

Finally, I wish to thank Tom Howson and Hope Thompson for all the quizzes, boardgames and sea swims that made lockdown and writing more bearable.

Funding

This work was supported by the University of St Andrews (School of Mathematics and Statistics).

Contents

Chapter 1

Introduction and Preliminaries

Separability properties have proven to be an important tool in the study of algebras. The notion of separability involves separating an element of an algebra from a subset of its complement in a finite homomorphic image. The most extensively studied separability property is residual finiteness. One reason for this is the connection of residual finiteness to the word problem. It was shown that finitely presented, residually finite groups have solvable word problem by Dyson in [14], and also by Mostowski in [41], who made use of methods developed by McKisney in [39]. In fact this result can be generalised to say that any finitely presented, residually finite algebra has solvable word problem. This was shown by Evans in [15], although Evans acknowledges that Mal'cev was the first to note this result in [37]. Moving away from the word problem, Mal'cev showed in [36] that finitely presented, residually finite groups are Hopfian. More recently residual finiteness has arisen in relation to Zelmanov's solution to the restricted Burnside problem. This result can be interpreted as saying that a finitely generated, residually finite group of finite exponent is necessarily finite, see [54] and [55]. These results demonstrate the importance of residual finiteness in understanding the structure and behaviour of groups and algebras in general.

Within the class of semigroups, residual finiteness has also been a major focus of study. Mal'cev showed that finitely generated commutative semigroups are residually finite in [37], a result reproved by Lallement in [31]. Many papers focus on residual finiteness in certain classes of semigroups, such as the work of Lesohin and Golubov on residual finiteness in commutative semigroups [33], or Golubov's work on completely 0-simple semigroups in [18]. Another theme of research has been investigating how the property of residual finiteness is preserved under certain constructions. For example, in [19] Golubov shows that an arbitrary free product of residually finite semigroups is itself residually finite but an analogous result does not hold for an arbitrary wreath product of semigroups. More recently, in [22] Gray and Ruškuc considered how residual finiteness interacts with the direct product for algebras in general and in the same paper showed that the direct product of two semigroups is residually finite if and only if both factors are residually finite. Residual finiteness of algebras continues to be an active area, with de Witt characterising when the direct product of two monounary algebras is residually finite in [53].

In this thesis, we investigate four generalisations of residual finiteness. One of the earliest examples of these in the literature comes from a 1949 paper by Hall, [27]. Hall showed that in a free group an element can always be separated from a finitely generated subgroup contained within its complement. We refer to this property in groups as weak subgroup separability and as weak subalgebra separability for algebras in general, although many different names exist in the literature. Just as there is a connection between residual finiteness and the word problem, there is a link between weak subalgebra separability and the generalised word problem. The generalised word problem for a finitely presented, weakly subalgebra separable algebra is solvable, as shown by Evans in [16]. Many classes of groups been have shown to be weakly subgroup separable. Scott showed that surface groups are weakly subgroup separable in [51] and Agol showed that fundamental groups of geometric 3-manifolds are also weakly subgroup separable in [2]. The relation between group-theoretic constructions and weak subgroup separability has also been investigated. In [7] Burns showed that the free product of two weakly subgroup separable groups is itself weakly subgroup separable. However, in [3] Allenby and Gregorac give an example of the direct product of two weakly subgroup separable groups which is not weakly subgroup separable. The semigroup version of this property has not received as much attention as the group version, but does appear in the literature, for example in [20]. In this paper, Golubov considers many different separability properties for semigroups and investigates the intersection and containments of the classes of semigroups satisfying these properties.

A weaker property than that of weak subalgebra separability is that of monogenic subalgebra separability. This is where it is possible to separate an element from a monogenic subalgebra contained within its complement. Within group theory, this property was first introduced by Stebe in 1968 in [52]. The purpose of its introduction was to use this property a tool in the investigation into the residual finiteness of Knot groups. In the same paper, Stebe was able to show that both the direct product of two monogenic subgroup separable groups and the free product of two monogenic subgroup separable groups are themselves monogenic subgroup separable. Within group theory, this property has continued to receive significant attention. For example in [6], Burillo and Martino consider how this property interacts with the property of quasi-potency. As with weak subsemigroup separability, monogenic subsemigroup separability has only received limited attention, but it is among the properties considered by Golubov in [20].

It is also possible to consider a strengthening of weak subalgebra separability. Strong subalgebra separability concerns separating an element from any subalgebra contained within its complement. Recent work includes [48], where Robinson et al. investigate this property for classes of groups including nilpotent groups, soluble groups and locally finite groups. It is folklore that strong subgroup separability is not closed under free products. The fact the direct product preserves strong subgroup separability has been attributed to Mal'cev in [37]. For semigroups, this property has received considerable attention, especially in the 1960s and 1970s. For example, in [17] Golubov studied this property in commutative semigroups, semigroups without idempotents, and weakly cancellative semigroups, amongst others. In

[30], Kublanovskiĭ and Lesohin give a characterisation of when a finitely generated commutative semigroup is strongly subsemigroup separable. Golubov also showed that there exist two strongly subsemigroup separable semigroups whose direct product is not strongly subsemigroup separable in [21].

The final property we consider is that of complete separability. In this case we are able to separate an element of an algebra from its complement in a finite homomorphic image. This means complete separability is the strongest property we consider. In the case of groups, this property turns out to be equivalent to being finite. However, this is not the case for all algebras. For example, there exist infinite completely separable semigroups. In [17], Golubov was able to characterise when a semigroup is completely separable. The preservation of complete separability (also weak subsemigroup separability and strong subsemigroup separability) for the direct product of two monounary algebras was investigated by de Witt in [53].

The purpose of this thesis is to investigate the properties of monogenic subalgebra separability, weak subalgebra separability, strong subalgebra separability and complete separability for algebras in general with a specific focus on semigroups. The strands of research undertaken are motivated by the previous research outlined above. In the remainder of Chapter 1, we outline the content of this thesis. We will then introduce the necessary preliminary definitions and results. We formalise the definitions of separability properties using the machinery of universal algebra. This allows us to to apply these notions in a general context, and consider many different types of algebras. Therefore we begin the preliminaries with some basic notions from universal algebra. Concepts defined include algebras, subalgebras, homomorphisms and congruences. We also introduce varieties. The guaranteed existence of free objects within varieties makes them the natural setting to search for results analogous to Hall's result for free groups. After this we formally define separability and give the first preliminary results. At this point, we also explain the naming convention as well as discuss the various different names used for these properties in the literature. The remainder of Chapter 1 is devoted to giving the necessary semigroup preliminaries, including partial orders and Green's relations.

In Chapter 2 we turn our attention to free objects. Hall's result that the free group is weakly subgroup separable motivates an investigation into the separability properties of other free objects. We show that free semigroups and free monoids are completely separable, a result previously known to Golubov (see [17]). Given the stark difference between the separability properties of the free group and the free semigroup, we investigate free objects in other semigroup varieties to establish how this difference arises. The varieties considered are inverse semigroups, completely simple semigroups, Clifford semigroups and completely regular semigroups. The complex nature of free completely regular semigroups motivates the definition of a variety of semigroups, which we call α-groups. The definition of α-groups is designed so that free α-groups capture some of the structure and behaviour of free completely regular semigroups whilst being less complex and easier to work with. Although some success comes from this strategy, the separability properties of both free completely regular semigroups and free α-groups are yet to be fully determined.

The material for Chapter 3 is centred around finitely generated commutative semigroups. Motivation for this is provided by the fact that finitely generated abelian groups are strongly subgroup separable. We therefore set out to establish the separability properties of finitely generated commutative semigroups. To do so we make use of the theory of Schützenberger groups. We investigate which of our separability properties are inherited by Schützenberger groups. Through these investigations we are able to show that for finitely generated commutative semigroups, the properties of complete separability, strong subsemigroup separability and weak subsemigroup separability coincide and are equivalent to every $\mathcal{H}$-class being finite. We then show that for the classes of finitely generated semigroups and commutative semigroups the three separability properties mentioned above are in fact distinct. We conclude Chapter 3 by returning to Schützenberger groups. For a semigroup with finitely many $\mathcal{H}$-classes, we ask if every Schützenberger group having a separability property is sufficient for the semigroup to have the same separability property itself.

In Chapter 4, the final chapter, we consider how certain semigroup constructions interact with these separability properties. The constructions under consideration are the direct product and the free product, as well as an investigation into how these properties pass from large subsemigroups to their oversemigroups. For all our separability properties except complete separability, we find that they are not preserved by the direct product, even when one of the factors is finite. This leads us to ask which semigroups preserve the separability properties of the other factor in a direct product. We are able to characterise precisely when a finite semigroup preserves strong subsemigroup separability and monogenic subsemigroup separability. The free product preserves both complete separability and monogenic separability. However, there exists a strongly subsemigroup separable semigroup whose free product with any semigroup fails to be strongly subsemigroup separable. The situation is also similar for weak subsemigroup separability. Although it is not necessarily true that monogenic subsemigroup separability passes from a large subsemigroup to its oversemigroup, we conclude Chapter 4 by showing that for our other separability properties, the property passes from large subsemigroups to oversemigroups. We conclude the thesis by outlining some possible directions of future work.

1.1 Algebras and Varieties

The reader is assumed to have an undergraduate level of knowledge in general algebra, although we reintroduce many familiar notions in terms of universal algebra. In this section we begin by introducing fundamental concepts within the area of universal algebra. First we define what we mean by an algebra, providing examples. We then discuss the notions of subalgebras, homomorphisms, congruences and quotient algebras, culminating with the First Isomorphism Theorem (Theorem 1.1.22), which links homomorphic images and quotient algebras.

Finding subalgebras and quotients of an algebra can be thought of as a way of constructing new algebras from old. Another way of building new algebras is through the process of taking direct products, which we will define. This brings us to varieties, which are classes of algebras closed under taking subalgebras, quotients, and direct products. After providing examples of varieties, we establish an equivalent definition of varieties as equational classes. The final notion we introduce in this section is that of free algebras, which are guaranteed to exist within varieties. The definitions and notations of this section are based upon [8, Chapter 2].

1.1.1 Algebras

Definition 1.1.1. Let A be a non-empty set. For a non-negative integer n, define

$$A^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in A \text{ for } 1 \leq i \leq n\}.$$

Note that we adopt the convention that $A^0 = \{\emptyset\}$. An *n-ary operation on A* is any function $f : A^n \to A$. We say the *arity* of such a function f is n. The image of $(a_1, a_2, \dots, a_n)$ under f is denoted by $f(a_1, a_2, \dots, a_n)$. A *finitary* operation is an n-ary operation for some n. An operation of arity 0, also called a *nullary* operation, can be thought as an element of A.

Definition 1.1.2. A *type* of algebras is a set $\mathcal{F}$ of *function symbols* such that for each $f \in \mathcal{F}$ there is a non-negative integer n associated to f. This integer is called the *arity* of f and f is said to be a *n-ary function symbol.* Let $\mathbf{ar} : \mathcal{F} \to \mathbb{Z}$ be the function which returns the arity of a function symbol. If $\mathcal{F} = \{f_1, f_2, \dots, f_k\}$ is finite, then we adopt the convention that

$$\mathbf{ar}(f_1) \geq \mathbf{ar}(f_2) \geq \cdots \geq \mathbf{ar}(f_k).$$

Definition 1.1.3. For a type of algebras $\mathcal{F}$, an *algebra A* of *type $\mathcal{F}$* is an ordered pair $\mathbf{A} = (A, F)$, where A is a non-empty set and F is a family of finitary operations on A in correspondence with $\mathcal{F}$. That is, for every n-ary function symbol $f \in \mathcal{F}$ there exists an n-ary operation $f^A \in F$. The set A is known as the *underlying set* or *universe* of $\mathbf{A}$. The elements of F are known as the *fundamental operations* of $\mathbf{A}$. When the context is clear, we shall write

f instead of f^A. When $\mathcal{F} = \{f_1, f_2, \cdots, f_k\}$ is finite, we often denote

$$\mathbf{A} = (A, f_1, f_2, \ldots, f_k).$$

In this case we say that the *signature* of $\mathbf{A}$ is the (ordered) tuple of non-negative integers

$$(\mathbf{ar}(f_1), \mathbf{ar}(f_2), \ldots, \mathbf{ar}(f_k)).$$

When it is clear, for brevity will refer to an algebra $\mathbf{A} = (A, F)$ by the underlying set A.

Example 1.1.4. A *semigroup* $(S, \cdot)$ is an algebra of signature (2). For the binary operation $\cdot$, we write $x \cdot y$ to mean $\cdot(x, y)$. Semigroups satisfy the identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

That is, the binary operation $\cdot$ is *associative.* When it is clear, we write xy to mean $x \cdot y$ and x^n to mean $\underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ times}}$.

For a finite example, let $L = \{x, y\}$. We define the binary operation $\cdot$ by the following table.

$\cdot$	x	y
x	x	x
y	y	y

An exhaustive check confirms that $\cdot$ is associative and so $(L, \cdot)$ is a semigroup, which we now will refer to as L.

A semigroup $(S, \cdot)$ is called *commutative* if it satisfies the additional property that

$$x \cdot y = y \cdot x.$$

In this case we also say that $\cdot$ is *commutative.* For example consider $(\mathbb{N}, +)$, where $+$ is the usual addition on $\mathbb{N} = \{1, 2, 3, \ldots\}$. Then, as $+$ is both associative and commutative, we have that $(\mathbb{N}, +)$ is a commutative semigroup. From now on we will refer to $(\mathbb{N}, +)$ simply as $\mathbb{N}$. The semigroup L given above is not commutative as $x \cdot y = x$ but $y \cdot x = y$.

Example 1.1.5. A *monoid* $(M, \cdot, 1)$ is an algebra of signature $(2, 0)$. The binary operation $\cdot$ is associative. The nullary operation 1 is interpreted as an element of the underlying set M. Apart from the associativity of $\cdot$, monoids also satisfy the following identity

$$1 \cdot x = x \cdot 1 = x.$$

That is, the element 1 is an *identity* of M. An example of a monoid is $(\mathbb{N}_0, +, 0)$, where $+$ is the usual addition on non-negative integers and 0 is the number zero.

Every monoid can be viewed as a semigroup by only considering the underlying set and the binary operation. For every semigroup $(S, \cdot)$ we can define an associated monoid. Let $S^1 = S \cup \{1\}$, where S and $\{1\}$ are assumed to be disjoint. Extend the binary operation $\cdot$ on S to a binary operation $\bar{\cdot}$ on S^1 by defining

$$x \mathbin{\bar{\cdot}} y = x \cdot y, \quad z \mathbin{\bar{\cdot}} 1 = 1 \mathbin{\bar{\cdot}} z = z$$

for all $x, y \in S$ and $z \in S^1$. It easy to check that $\bar{\cdot}$ is associative. Then $(S^1, \bar{\cdot}, 1)$ is a monoid. We refer to S^1 as S *with an identity adjoined.* For example, it is easy to see that $(\mathbb{N}_0, +, 0)$ is actually $\mathbb{N}^1$.

For the semigroup L from Example 1.1.4, we have that L^1 is the monoid with multiplication table given below.

$\bar{\cdot}$	x	y	1
x	x	x	x
y	y	y	y
1	x	y	1

Again, a monoid $(M, \cdot, 1)$ is called *commutative* if $\cdot$ is commutative. We have that $\mathbb{N}^1$ is a commutative monoid but L^1 is not a commutative monoid. In general we have that for a semigroup S, the monoid S^1 is commutative if and only if the semigroup S is commutative.

Example 1.1.6. A *group* $(G, \cdot, ^{-1}, 1)$ is an algebra of signature (2, 1, 0). The binary operation of $\cdot$ is associative and 1 gives an identity element. Hence groups can be thought of as both semigroups and monoids. For the unary operation ${}^{-1}$, we write x^{-1} to mean ${}^{-1}(x)$. Groups satisfy the additional identity that

$$x \cdot x^{-1} = x^{-1} \cdot x = 1.$$

We say that x^{-1} is the *inverse* of x and that ${}^{-1}$ is the *inversion map*. When it is clear we will write x^{-n} to mean $\underbrace{x^{-1} \cdot x^{-1} \cdot \ldots \cdot x^{-1}}_{n \text{ times}}$ and we adopt the convention that $x^0 = 1$ for all $x \in G$. We have that $(x^n)^{-1} = x^{-n}$.

Consider $(\mathbb{Z}, +, -, 0)$ where $+$ is the usual addition of integers, $-$ represents the negation of an integer and 0 is the number zero. Then $(\mathbb{Z}, +, -, 0)$ is a group with identity 0 and for $z \in \mathbb{Z}$, the integer $-z$ is the (additive) inverse of z. From now on will refer to $(\mathbb{Z}, +, -, 0)$ simply as $\mathbb{Z}$.

If the binary operation of a group is commutative, then the group is called *abelian*. The group $\mathbb{Z}$ is an example of an abelian group.

Definition 1.1.7. Let $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, F)$ be two algebras of the same type. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$ if $B \subseteq A$ and for every function symbol f, the map f^B is the restriction of f^A to the set B. In this case we write $\mathbf{B} \leq \mathbf{A}$.

Remark 1.1.8. For an algebra A, any non-empty set $B \subseteq A$ that is closed under the fundamental operations of A can be viewed as a subalgebra of A by taking the fundamental operations of B to be the restrictions to B of the fundamental operations of A. In this manner, we can view finding all the subsets of A which are closed under the fundamental operations of A as a method of finding new algebras.

Example 1.1.9. Consider $\mathbb{Z}$ as a semigroup. Then it is clear to see that $\mathbb{N} \leq \mathbb{Z}$, i.e. $\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$.

If we consider $\mathbb{Z}$ as a monoid, then we have that $\mathbb{N}^1 \leq \mathbb{Z}$. That is, $\mathbb{N}^1$ is a submonoid of $\mathbb{Z}$. The algebra $\mathbb{N}$ cannot be considered as a submonoid of $\mathbb{Z}$, as $\mathbb{N}$ is not equipped with an identity operation.

Not every subset of an algebra need be closed under the fundamental operations. Therefore it is natural to ask for any subset, what is the smallest subalgebra it is contained within. The following definition provides the answer.

Definition 1.1.10. Let $\mathbf{A} = (A, F)$ be an algebra and let X be a non-empty subset of A. Define

$$\langle X \rangle = \bigcap \{B \mid X \subseteq B \text{ and } (B, F) \text{ is a subalgebra of } \mathbf{A}\}.$$

As $X \subseteq A$, this is an intersection of a non-empty set. It is also clear that $X \subseteq \langle X \rangle$, and so $\langle X \rangle$ is non-empty. For each function symbol f, define an n-ary operation $f^{\langle X \rangle}$ on $\langle X \rangle$ by setting $f^{\langle X \rangle}$ to be the restriction of f^A to the set $\langle X \rangle$. This is well-defined, as if $a_1, a_2, \cdots, a_n \in \langle X \rangle$, then $f^A(a_1, a_2, \cdots, a_n) \in \langle X \rangle$ because $\langle X \rangle$ is an intersection of subalgebras. Then $\langle X \rangle$ can be viewed as a subalgebra of $\mathbf{A}$. We refer to this subalgebra as the *subalgebra generated by* X. When $X = \{x_1, x_2, \ldots, x_k\}$ is finite, we often write $\langle x_1, x_2, \ldots, x_k \rangle$ to denote the subalgebra generated by X. In this case we say that $\langle X \rangle$ is *finitely generated*. When an algebra can be generated by a set of size 1, then we say that it is *monogenic*.

Example 1.1.11. We have already seen that $\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$. It is clear that we have $\mathbb{N} = \langle \mathbb{N} \rangle$ but we also have that $\mathbb{N} = \langle 1 \rangle$. So $\mathbb{N}$ is a finitely generated subsemigroup of $\mathbb{Z}$ and $\mathbb{N}$ is monogenic.

Remark 1.1.12. If at least one of the fundamental operations of an algebra A is a nullary operation, then we can extend Definition 1.1.10 to include the subalgebra generated by the empty set. Let $X \subseteq A$ be the non-empty set of constants that corresponds to the set of nullary operations. Then X must be contained in every subalgebra of A. So we define $\langle \emptyset \rangle = \langle X \rangle$. For example, when we have a group G, the subgroup generated by the empty set is $\{1\}$, the subgroup consisting of just the identity element. Even though every group G can be considered as a semigroup, we cannot talk of the subsemigroup generated by the empty set because none of the fundamental operations of a semigroup is nullary.

When dealing with functions between algebras of the same type, it is natural to restrict our attention to those which respect the fundamental operations and hence preserve, to varying extents, algebraic structure. This is formalised below.

Definition 1.1.13. Suppose $\mathbf{A} = (A, F)$ and $\mathbf{B} = (B, F)$ are algebras of the same type. A function $\phi : A \to B$ is called a *homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ if

$$\phi(f^A(a_1, a_2, \ldots, a_n)) = f^B(\phi(a_1), \phi(a_2), \ldots, \phi(a_n)),$$

for each n-ary $f \in F$ and $a_1, a_2, \ldots, a_n \in A$. If ϕ is surjective it is known as an *epimorphism*. An injective epimorphism is known as an *isomorphism*. An isomorphism can be considered as a relabelling of elements, without altering any of the algebraic structure. Hence if there exists an isomorphism between $\mathbf{A}$ and $\mathbf{B}$, we consider $\mathbf{A}$ and $\mathbf{B}$ to represent the same algebraic object. In this case we say that $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic*. We write $\mathbf{A} \cong \mathbf{B}$.

Example 1.1.14. The inclusion map $\iota : \mathbb{N} \to \mathbb{Z}$ is a semigroup homomorphism. It is not epimorphism. For example there does not exist $n \in \mathbb{N}$ such that $\iota(n) = 0$.

Recalling the semigroup L from Example 1.1.4, the constant map $\phi : \mathbb{Z} \to L$ given by $z \mapsto x$ for all $z \in \mathbb{Z}$ is a semigroup homomorphism. Note that ϕ is not a group homomorphism as L is not a group.

Let $\psi : L \to \mathbb{N}$ be any function. Let $n = \psi(x)$. Then $\psi(xx) = \psi(x) = n$ but $\psi(x) + \psi(n) = 2n \neq n$. Hence ψ is not a homomorphism. This argument has shown that there are no semigroup homomorphisms from L to $\mathbb{N}$.

Consider $\sigma : L \to L$ given by $x \mapsto y$ and $y \mapsto x$. This is clearly a bijection and it is easy to check that it is a semigroup homomorphism. Hence σ is an isomorphism. An isomorphism from an algebra to itself is known as an *automorphism*.

The following lemma shows that homomorphisms are preserved under composition.

Lemma 1.1.15. *Let A, B, C be algebras of type F and let $\phi : A \to B$ and*

$\psi : B \to C$ be homomorphisms. Then $\psi \circ \phi : A \to C$ is also a homomorphism.

Proof. This follows since for an n-ary operation $f \in F$, we have that

$$\begin{aligned}\psi \circ \phi \left(f^A(a_1, \ldots, a_n)\right) &= \psi \left(f^B(\phi(a_1), \ldots, \phi(a_n))\right) \\ &= f^C \left(\psi \circ \phi(a_1), \ldots, \psi \circ \phi(a_n)\right). \qquad \square\end{aligned}$$

Example 1.1.16. For a singleton set $A = \{a\}$ and type of algebra $\mathcal{F}$, A can be viewed as an algebra of type $\mathcal{F}$. This is done by setting $f^A(\underbrace{a, a, \ldots, a}_{n \text{ times}}) = a$, where $f \in \mathcal{F}$ is of arity n. The algebra A is said to be *trivial*. If $A = \{a\}$ and $B = \{b\}$ are both trivial algebras of type $\mathcal{F}$, then the only function $\phi : A \to B$ is a isomorphism. Therefore as any two trivial algebras are isomorphic, we will refer to *the trivial algebra* non-ambiguously.

Just as functions that are homomorphisms interest us, for partitions of an algebra we will focus our attention to those which respect the fundamental operations, as defined below.

Definition 1.1.17. For a set A, a set $\rho \subseteq A \times A$ is known as an *equivalence relation* on A if for $a, b, c \in A$ we have

1. **reflexivity:** $(a, a) \in \rho$;
2. **symmetry:** if $(a, b) \in \rho$ then $(b, a) \in \rho$;
3. **transitivity:** if $(a, b), (b, c) \in \rho$ then $(a, c) \in \rho$.

We will often write $a \,\rho\, b$ to mean $(a, b) \in \rho$. The *equivalence class* of an element $a \in A$ is the set $[a]_\rho = \{b \in A \mid a \,\rho\, b\}$. The set of all equivalence classes forms a partition of A. Additionally, for an algebra $\mathbf{A} = (A, F)$ and equivalence ρ on A, if for each n-ary operation $f \in F$ we have

4. **compatibility:** if $(a_i, b_i) \in \rho$ for $i \in \{1, 2, \ldots, n\}$, then

$$(f^A(a_1, a_2, \ldots, a_n), f^A(b_1, b_2, \ldots, b_n)) \in \rho,$$

then ρ is said to be a *congruence* on $\mathbf{A}$. In this case we refer to the sets $[a]_\rho$ as congruences classes.

For a congruence ρ, we denote the set of all congruence classes as A/ρ. The set A/ρ can be viewed as an algebra of type F in the following way. For an n-ary operation $f \in F$ and $[a_1]_\rho, [a_2]_\rho, \ldots, [a_n]_\rho \in A/\rho$, define

$$f^{A/\rho}([a_1]_\rho, [a_2]_\rho, \ldots, [a_n]_\rho) = [f^A(a_1, a_2, \ldots, a_n)]_\rho.$$

The fact this operation is well-defined follows from the compatibility of ρ. The algebra $(A/\rho, F)$ is known as the *quotient of* $\mathbf{A}$ *by* ρ. The map $\phi_\rho : A \to A/\rho$ given by $a \mapsto [a]_\rho$ is a homomorphism known as the *canonical map*. In the case that a congruence ρ has only finitely many congruence classes, we call it a *finite index congruence*.

Example 1.1.18. Any non-trivial algebra A comes equipped with two distinct congruences. The *universal congruence* on A is the set $A \times A$, and is denoted by ∇_A. The quotient A/∇_A is isomorphic to the trivial algebra. The *diagonal congruence* on A is the set $\{(a, a) \mid a \in A\}$ and is denoted by Δ_A. The quotient A/Δ_A is isomorphic to A. For the trivial algebra, the universal congruence and the diagonal congruence coincide.

Consider the group $\mathbb{Z}$. For $n \in \mathbb{N}$, consider $\rho_n \subseteq \mathbb{Z} \times \mathbb{Z}$ given by $(y, z) \in \rho$ if $y \equiv z \pmod{n}$. It is easy to check that ρ_n satisfies all four conditions from Definition 1.1.17, and therefore ρ_n is a congruence on $\mathbb{Z}$. Note that ρ_n is a group congruence, but also a semigroup congruence and a monoid congruence.

Now consider the quotient group $\mathbb{Z}/\rho_n$. As ρ_n has n congruence classes, $\mathbb{Z}/\rho_n$ is a finite group. We also have that $\{[1]_{\rho_n}\}$ generates $\mathbb{Z}/\rho_n$. When a group can be generated by a set of size 1 it is known as *cyclic*. As two cyclic groups of the same size are isomorphic, we have that $\mathbb{Z}/\rho_n$ is the cyclic group of order n. We will normally denote $\mathbb{Z}/\rho_n$ by C_n.

Remark 1.1.19. Just as subalgebras provide us with a way of finding new algebras, congruences also provide means for constructing new algebras, namely the quotient algebras.

Remark 1.1.20. For a group G, a subgroup $N \leq G$ is known as *normal* if $g^{-1}Ng = N$ for all $g \in N$. We denote that N is a normal subgroup of G by $N \trianglelefteq G$. If ρ is a congruence on G, then the congruence class of the identity element $[1]_\rho$ is a normal subgroup and $g \,\rho\, h$ if and only if $gh^{-1} \in [1]_\rho$. Equally, if N is a normal subgroup of G, then the binary relation ρ_N defined on G by

$$g \,\rho_N\, h \text{ if and only if } gh^{-1} \in N$$

is a congruence on G. Hence, congruences on G are in correspondence with normal subgroups and a congruence on a group is completely determined by the congruence class of the identity element. We will often write G/N instead of G/ρ_N and Ng instead of $[g]_{\rho_N}$. We can do this because $[g]_{\rho_N} = \{ng \mid n \in N\}$. A congruence class Ng is known as a *coset* of N.

We now establish a link between quotient algebras and homomorphic images. Before we can state the result, we will need some definitions.

Definition 1.1.21. Let A and B be algebras of the same type and let $\phi : A \to B$ be a homomorphism. Define the *kernel of* ϕ to be the set

$$\ker(\phi) = \{(a_1, a_2) \mid \phi(a_1) = \phi(a_2)\} \subseteq A \times A.$$

The *image of* ϕ is the set

$$\mathrm{im}(\phi) = \{\phi(a) \mid a \in A\} \subseteq B.$$

Theorem 1.1.22 (First Isomorphism Theorem)**.** *Let A and B be algebras of the same type and let $\phi : A \to B$ be a homomorphism. Then*

(i) *ker(ϕ) is a congruence on A;*
(ii) *im(ϕ) is a subalgebra of B;*
(iii) *im(ϕ) $\cong A/\rho$.*

For a proof of Theorem 1.1.22, see [8, Section 2.6]

We have seen two important ways of constructing new algebras from old ones; finding subalgebras and finding homomorphic images. We conclude this subsection by presenting another important method for constructing algebras.

Definition 1.1.23. Let $(A_i)_{i\in I}$ be an indexed family of algebras of type $\mathcal{F}$. The *direct product* $A = \Pi_{i\in I} A_i$ is an algebra of type $\mathcal{F}$. The underlying set is the Cartesian product $\Pi_{i\in I} A_i$. For $a \in \Pi_{i\in I} A_i$ we denote the ith coordinate of a by $a(i)$. For an n-ary operation $f \in \mathcal{F}$ and $a_1, a_2, \ldots, a_n \in A$, we define

$$f^A(a_1, a_2, \ldots, a_n)(i) = f^{A_i}\left(a_1(i), a_2(i), \ldots, a_n(i)\right).$$

That is, f^A is defined coordinate wise. The empty direct product $\Pi\emptyset$ is defined to be the trivial algebra.

For $j \in I$ we define the *projection map* $\pi_j : \Pi_{i\in I} A_i \to A_j$ by $a \mapsto a(j)$. For each $j \in I$, the map π_j is a surjective homomorphism.

In the case that $I = \{1, 2, \ldots, n\}$ is a finite set, we often write $A_1 \times A_2 \times \cdots \times A_n$ instead of $\Pi_{i\in I} A_i$.

Example 1.1.24. Recall the semigroup L from Example 1.1.4. The underlying set of $L \times L$ is the set $\{(x, x), (x, y), (y, x), (y, y)\}$. The binary operation is given by the following multiplication table.

$\cdot$	(x,x)	(x,y)	(y,x)	(y,y)
(x,x)	(x,x)	(x,x)	(x,x)	(x,x)
(x,y)	(x,y)	(x,y)	(x,y)	(x,y)
(y,x)	(y,x)	(y,x)	(y,x)	(y,x)
(y,y)	(y,y)	(y,y)	(y,y)	(y,y)

The value of $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Note that when the argument of a function is a tuple, we will omit one set of brackets for ease of reading.

1.1.2 Varieties

Now we have met subalgebras, homomorphic images and direct products, we can can define varieties.

Definition 1.1.25. A non-empty class of algebras K of type $\mathcal{F}$ is a *variety* if K is closed under taking subalgebras, homomorphic images and direct products. We often identity $\mathcal{F}$ by giving the signature of the algebras of type $\mathcal{F}$.

Example 1.1.26. The class of all semigroups forms a variety of signature (2). It is easy to see that the associativity of the binary operation is inherited by subsemigroups, homomorphic images and direct products. Perhaps more interestingly, the class of all commutative semigroups forms a variety.

The class of all groups forms a variety of signature (2, 1, 0). However the class of all groups does not form a variety of signature (2). Not every subsemigroup of a group need be a group. For example, $\mathbb{N}$ is a subsemigroup of $\mathbb{Z}$, but $\mathbb{N}$ is not a group.

Previously we have described some varieties given in Example 1.1.26 by giving their type and a set of identities that they satisfy. It turns out that this is another way of thinking about varieties. Before we can state this in proper, we need to formalise the notion of identity which itself relies on the notion of terms, which we now define.

Definition 1.1.27. Let X be a set. The elements of X will be known as *variables*. Let $\mathcal{F}$ be a type of algebras. Let $\mathcal{F}_0 \subseteq \mathcal{F}$ be the set of function symbols with arity 0. The set $T_{\mathcal{F}}(X)$ of *terms of type* $\mathcal{F}$ *over* X is the smallest set such that

(i) $X \cup \mathcal{F}_0 \subseteq T_{\mathcal{F}}(X)$;
(ii) if $p_1, \ldots, p_n \in T_{\mathcal{F}}(X)$ and $f \in \mathcal{F}$ is an n-ary operation, then the string $f(p_1, \ldots, p_n) \in T_{\mathcal{F}}(X)$.

Example 1.1.28. Suppose that $\mathcal{F} = \{\cdot, ^{-1}\}$ is of signature (2, 1) and $X = \{x, y\}$. Then the following are terms:

$$x, \quad x \cdot y, \quad (y \cdot x) \cdot y, \quad y \cdot (x \cdot y), \quad y^{-1}, \quad ((x \cdot x^{-1}) \cdot y^{-1})^{-1}.$$

Now suppose that $\mathcal{F} = \{1\}$ is of signature (0). Then $T_{\mathcal{F}}(\emptyset) = \{1\}$.

Definition 1.1.29. An *identity of type* $\mathcal{F}$ over X is an expression of the form

$$p = q,$$

where $p, q \in T_{\mathcal{F}}(X)$. Given an algebra A of type $\mathcal{F}$ and a function $\phi : X \to A$ we can replace each variable $x \in X$ appearing in p with $\phi(x)$ to obtain a string p_ϕ, and replace each variable $x \in X$ appearing in q with $\phi(x)$ to obtain a string q_ϕ. An algebra A of type $\mathcal{F}$ *satisfies* an identity $p = q$ if for any function $\phi : X \to A$ the strings p_ϕ and q_ϕ represent the same element of A.

Example 1.1.30. We have already seen that semigroups by definition satisfy the identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

Consider the semigroup $\mathbb{N}$. It does not satisfy the identity

$$x \cdot x = x.$$

For example, if we replace x by the number 1, the left hand side of this identity becomes $1 + 1 = 2$ which is not equal to the right hand side which is 1.

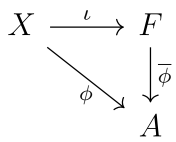
Definition 1.1.31. Let Σ be a set of identities of type $\mathcal{F}$ and define $M(\Sigma)$ to be the class of algebras of type $\mathcal{F}$ satisfying Σ. Note that $M(\Sigma)$ is non-empty as the trivial algebra satisfies any identity. A class K of algebras is an *equational class* if there exists a set of identities such that $K = M(\Sigma)$.

Theorem 1.1.32 (Birkhoff)**.** *A class of algebras K is an equational class if and only if K is variety.*

Birkhoff's Theorem (a proof of which can be found in [8, Theorem 11.9]) is powerful as it allows us to describe a variety by the set of identities that its members satisfy. In many instances this will be easier than showing a variety is closed under subalgebras, homomorphic images and direct products. The final concept that we introduce in this section is that of free algebras.

Definition 1.1.33. Let K be a class of algebras of type $\mathcal{F}$. For an algebra $F \in K$ and a subset $X \subseteq F$, we say that F is *free on X* if for any algebra $A \in$

K and any function $\phi : X \to A$ we can uniquely extend ϕ to a homomorphism $\overline{\phi} : F \to A$. That is, the following diagram commutes



where ι is the inclusion map. An algebra which is free is said to have the *universal property*. The set X is called a *basis*.

Example 1.1.34. The semigroup $\mathbb{N}$ is a free semigroup on the set $\{1\}$. To see this, suppose that S is any semigroup and $\phi : \{1\} \to S$ is given by $\phi(1) = s$, for some $s \in S$. As $\{1\}$ is a generating set for $\mathbb{N}$, if ϕ can be extended to homomorphism then this extension is unique. The map $\overline{\phi} : \mathbb{N} \to S$ where for $n \in \mathbb{N}$ we have $\phi(n) = s^n$ is a homomorphic extension of ϕ and hence $\mathbb{N}$ is free on $\{1\}$ as desired.

The group $\mathbb{Z}$ is a free group on the set $\{1\}$. To see this, suppose that G is any group with identity element e and $\phi : \{1\} \to G$ is given by $\phi(1) = g$, for some $g \in G$. As $\{1\}$ is a (group) generating set for $\mathbb{Z}$, if ϕ can be extended to a homomorphism then this extension is unique. The map $\overline{\phi} : \mathbb{Z} \to G$, where for $z \in \mathbb{Z}$ we have $\phi(z) = g^z$, is a homomorphic extension of ϕ and hence $\mathbb{Z}$ is a free group on $\{1\}$.

However, there does not exist a set $X \subseteq \mathbb{Z}$ such that $\mathbb{Z}$ is a free as a semigroup on X. To show this, it is sufficient to show that there cannot exist a homomorphism from $\mathbb{Z}$ to $\mathbb{N}$. Suppose that $\phi : \mathbb{Z} \to \mathbb{N}$ is any function. Then $\phi(0) = n$ for some $n \in \mathbb{N}$. Then $\phi(0 + 0) = \phi(0) = n$ and $\phi(0) + \phi(0) = n + n = 2n$. But $2n > n$. Hence ϕ is not a homomorphism and $\mathbb{Z}$ is not a free semigroup.

The next lemma shows that free algebras with a basis of a given cardinality are unique up to isomorphism.

Lemma 1.1.35. *Let A and B be algebras of type $\mathcal{F}$ such that A is free on the set X and B is free on the set Y. If $|X| = |Y|$ then A and B are isomorphic.*

Proof. [8, Theorem 10.7] □

The last result of this section establishes the existence of free algebras within varieties.

Theorem 1.1.36. *For a variety K and a non-empty set X there exists an algebra $A \in K$ such that A is free on X.*

Proof. [8, Theorem 10.12] □

Remark 1.1.37. The trivial group is a free group on the empty set. However, even within varieties, it need not be that there exists an algebra which is free on empty set.

Of course free objects can exist in classes of algebras which are not varieties. But if we are given a variety, we are guaranteed that there exist free algebras. Given a variety K and an algebra A, we have that A is the homomorphic image of some free algebra in K. To see this, let X be a non-empty generating set for A. The there exists $F_{\overline{X}} \in K$ which is free on $\overline{X}$, where $|\overline{X}| = |X|$. Then there exists a bijection $\phi : \overline{X} \to X$. As $F_{\overline{X}}$ is free, there is a unique extension of ϕ to a homomorphism $\overline{\phi} : F_{\overline{X}} \to A$. As X is a generating set for A, it must be that $\overline{\phi}$ is an epimorphism and hence A is a homomorphic image of a free algebra.

1.2 Separability

We now come to separability properties, which are the main topic of this thesis.

Definition 1.2.1. For a class of algebras K of type $\mathcal{F}$, an algebra $A \in K$ and a collection $\mathcal{C}$ of subsets of A, we say that A has the *separability property with respect to* $\mathcal{C}$ if for any $a \in A$ and any subset $X \subseteq A \setminus \{a\}$ belonging to the collection $\mathcal{C}$, there exists a finite algebra $B \in K$ and homomorphism $\phi : A \to B$ such that $\phi(a) \notin \phi(X)$. In this case we say that a can be *separated* from X and that ϕ *separates* a from X. Equivalently, A has the separability property with respect to $\mathcal{C}$ if for any $a \in A$ and any $\mathcal{C}$-subset $X \subseteq A \setminus \{a\}$ there exists a finite index congruence ρ on A such that $[a]_\rho \neq [x]_\rho$ for all $x \in X$. In this case we say that ρ *separates* a from S.

For an algebra A we say that:

- A is *monogenic subalgebra separable* if A has the separability property with respect to the collection of all monogenic subalgebras;
- A is *weakly subalgebra separable* if A has the separability property with respect to the collection of all finitely generated subalgebras;
- A is *strongly subalgebra separable* if A has the separability property with respect to the collection of all subalgebras;
- A is *completely separable* (CS) if A has the separability property with respect to the collection of all subsets.

Notation 1.2.2. When our class K is the variety of all groups we shall say *monogenic subgroup separable*, *weakly subgroup separable* and *strongly subgroup separable*. Similarly when we are working within the variety of semigroups we shall say *monogenic subsemigroup separable* (MSS), *weakly subsemigroup separable* (WSS) and *strongly subsemigroup separable* (SSS). Note that we are reserving the use of the MSS, WSS and SSS for semigroup separability properties only.

Remark 1.2.3. The four separability properties listed in Definition 1.2.1 have been studied for many classes of algebras under various different names. Monogenic subgroup separable groups are known both as *cyclic subgroup separable* (for example see [6]) and Π_C groups ([52]). Weakly subgroup separable groups are known both as *locally extended residually finite* groups (LERF) ([3]) or simply as *subgroup separable* groups ([24]). Weak subalgebra separability is also known as *finite divisibility* [16]. Strongly subgroup separable groups are also know as *extended residually finite groups* (ERF) ([3]). SSS semigroups have been called *finitely divisible* ([17]) and *finitely separable* ([30]). CS semigroups are also known as *semigroups with finitely divisible subsets* ([17]). Due to the many different and sometimes inconsistent names used for these properties, we have decided to introduce our own terms, i.e., monogenic subalgebra separability, weak subalgebra separability, strong subalgebra separability and complete separability. In our nomenclature, the names are designed to describe the properties and highlight the relationship between the different properties.

Below we define what is perhaps the best known separability property, although it is not normally thought of in such terms.

Definition 1.2.4. An algebra A of type $\mathcal{F}$ is called *residually finite* if for every pair of distinct elements $a, b \in A$ there exists a finite algebra C of type $\mathcal{F}$ and a homomorphism $\phi : A \to C$ such that $\phi(a) \neq \phi(b)$.

Residual finiteness therefore can be viewed as the separability property with respect to the collection of all singleton subsets. It turns out that knowing an algebra is residually finite provides huge insight into the structure of the algebra. The structural characterisation of when an algebra is residually finite will be discussed in the commentary around Theorem 3.4.4. For now we present the first example of separability in a semigroup.

Example 1.2.5. The semigroup $\mathbb{Z}$ is residually finite but not MSS. To see that it is residually finite, let $y, z \in \mathbb{Z}$ be such that $y \neq z$. Let $n = |y - z| + 1$. Note that $n \in \mathbb{N}$ and that $y \not\equiv z \pmod{n}$, as the absolute difference between y and z is $n - 1 \neq 0$. Hence the congruence ρ_n defined in Example 1.1.18 separates y from z.

Now we show that $\mathbb{Z}$ is not MSS. Consider the monogenic subsemigroup $\mathbb{N}$ and the element $0 \in \mathbb{Z} \setminus \mathbb{N}$. Let $\sim$ be an arbitrary finite index congruence on $\mathbb{Z}$. Then, as $\mathbb{N}$ is an infinite set, there exist $i, j \in \mathbb{N}$ with $i < j$ such that $i \sim j$. As $\sim$ is a congruence we have

$$0 = i + (-i) \sim j + (-i) \in \mathbb{N}.$$

Therefore 0 cannot be separated from the monogenic subsemigroup $\mathbb{N}$ and hence $\mathbb{Z}$ is not MSS.

Remark 1.2.6. It is true that $\mathbb{Z}$ is monogenic subgroup separable. In fact it is strongly subgroup separable. This is shown in Theorem 3.1.4.

Separability properties are examples of finiteness conditions, which are defined below.

Definition 1.2.7. For a class of algebras K, a *finiteness condition* is a property which every finite member of K satisfies.

Lemma 1.2.8. *Every separability property is a finiteness condition.*

Proof. Let $\mathscr{P}$ be a separability property, let A be a finite algebra, let $\mathcal{C}$ be the collection of subsets corresponding to $\mathscr{P}$, let $X \in \mathcal{C}$ and let $a \in A \setminus X$. Then the diagonal congruence Δ_A separates a from X. Hence A has property $\mathscr{P}$. $\square$

Finiteness conditions are studied because they allow a way of understanding the structure and behaviour of infinite algebras by some finite description. As already alluded to, if we know that an algebra is residually finite then we have significant insight into its structure. Throughout this thesis, we will see that we can classify algebras that satisfy certain separability properties. One of the earliest (and most straightforward) examples is Theorem 1.2.19, which states that a group is CS if and only if it finite. Before this, we establish some basic but useful results linking different separability properties.

Proposition 1.2.9. *For an algebra A the following hold.*

(i) *If A is completely separable then it is strongly subalgebra separable.*
(ii) *If A is strongly subalgebra separable then it is weakly subalgebra separable.*
(iii) *If A is weakly subalgebra separable then it is monogenic subalgebra separable.*

Proof. Each claim follows immediately from the definitions. $\square$

The next two lemmas show that within the varieties of groups and semigroups, monogenic subalgebra separability also implies residual finiteness.

Lemma 1.2.10. *If G is a monogenic subgroup separable group then G is residually finite.*

Proof. For a group, the property of being residually finite is equivalent to being able to separate every non-identity element from the identity 1. As the trivial subgroup $\{1\}$ is monogenic, the result follows. $\square$

Lemma 1.2.11. *If S is a monogenic subsemigroup separable semigroup then S is residually finite.*

Proof. Let $s, t \in S$ such that $s \neq t$. If $s \notin \langle t \rangle$, then we can separate s from $\langle t \rangle$ using the monogenic subsemigroup separability of S. In particular we can separate s from t. Similarly if $t \notin \langle s \rangle$ then we can separate s and t.

Now assume that $s \in \langle t \rangle$ and $t \in \langle t \rangle$. Then $s = t^i$ for some $i \in \mathbb{N}$ and $t = s^j$ for some $j \in \mathbb{N}$. Then we have that $s = s^{ij}$. Let $k \in \mathbb{N} \setminus \{1\}$ be minimal such that $s = s^k$. Note that $k \neq 2$, as in that case $s^n = s$ for all $n \in \mathbb{N}$ and in particular $s = t$. Then

$$s^{k-1} \cdot s^{k-1} = s^{k-1} \cdot s \cdot s^{k-2} = s^k \cdot s^{k-2} = s \cdot s^{k-2} = s^{k-1}.$$

That is $\langle s^{k-1} \rangle = \{s^{k-1}\}$. Then $s^{j-1} \neq s^{k-1}$. If we had equality then $t = s^j = s^k = s$. So, using the monogenic subsemigroup separability of S, we can separate s^{j-1} from s^{k-1}. That is, there exists a finite semigroup P and homomorphism $\phi : S \to P$ such that $\phi(s^{j-1}) \neq \phi(s^{k-1})$. Now suppose that $\phi(s) = \phi(t) = \phi(s^j)$. Then as ϕ is a homomorphism we would have that

$$\phi(s^{k-1}) = \phi(s)\phi(s^{k-2}) = \phi(s^j)\phi(s^{k-2}) = \phi(s^{j-1})$$

which is a contradiction. Hence ϕ separates s and t and S is residually finite. $\square$

It is not true that in every class of algebras we have that monogenic subalgebra separability implies residual finiteness, as the following example shows.

Example 1.2.12. A group G is called *simple* if its only normal subgroups are the trivial subgroup $\{1\}$ and G itself. An infinite simple group cannot be residually finite as it only has one finite quotient, the trivial group, which does not separate any points. There exist infinite simple groups which are generated (as groups) by two elements, for example see [38].

We will consider a variety K of signature (2, 1, 0, 0, 0). Elements of this variety can be viewed as groups with an additional two nullary operations and no additional identities. Consider an infinite simple group G generated by the set $\{g, h\}$. We can consider G as a member of K by setting the first extra nullary operation to be g and the second to be h. Then every subalgebra contains g and h and, as every subalgebra is closed under the fundamental group operations, the only subalgebra is G itself. Hence G is trivially monogenic subalgebra separable. But we have already observed that G is not residually finite.

We have already seen in Example 1.2.5 that there exist semigroups which are residually finite but not MSS. Within this thesis we will show that our four stronger separability properties do not coincide in the class of semigroups. An MSS semigroup which is not WSS is given in Example 3.3.24. A WSS semigroup which is not SSS is given in Example 3.4.6. Finally, an SSS semigroup which is not CS is given in Example 3.4.7.

Now we show that separability properties are inherited by subalgebras.

Proposition 1.2.13. *Let A be an algebra and let B be a subalgebra of A. Let $\mathscr{P}$ be any of the following properties: complete separability, strong subalgebra separability, weak subalgebra separability, monogenic subalgebra separability and residual finiteness. If A has property $\mathscr{P}$ then B also has property $\mathscr{P}$.*

Proof. As A has property $\mathscr{P}$, it has the separability property with respect to $\mathcal{C}$, where $\mathcal{C}$ is the collection of subsets of the type associated with $\mathscr{P}$. Let $X \subseteq B$ be a subset of the relevant type and let $b \in B \setminus X$. Then X is also a subset of the relevant type in A and $b \in A \setminus X$. Then as A has property $\mathscr{P}$, there exists a finite algebra U and homomorphism $\phi : A \to U$ such that $\phi(b) \notin \phi(A \setminus X)$. Let $\phi|_B : B \to U$ be the restriction of ϕ to U. Then $\phi|_B(b) \notin \phi|_B(B \setminus X)$ and B has property $\mathscr{P}$. □

In Chapter 3 we will be interested in the semigroup separability properties of groups. Example 1.2.5 and Proposition 1.2.13 allows to give conditions for a group to be MSS, WSS or SSS, but first we need the following definition.

Definition 1.2.14. A group G is known as *torsion* or *periodic* if for every $g \in G$, there exists $n \in \mathbb{N}$ such that $g^n = 1$, the identity element. The smallest such n is known as the *order of* g and this is denoted by $o(g)$.

Lemma 1.2.15. *If a group G is not torsion then there exists a subgroup of G isomorphic to $\mathbb{Z}$.*

Proof. As G is not torsion there exists $g \in G$ such that $g^n \neq 1$ for all $n \in \mathbb{N}$. We claim that if $g^i = g^j$ for $i, j \in \mathbb{Z}$, then $i = j$. For a contradiction assume that $g^i = g^j$ but that $i \neq j$. Without loss of generality, assume $i < j$. Then $g^{j-i} = 1$. But this contradicts that g is a non-torsion element. Hence $\mathrm{Gp}\langle g \rangle$ is an infinite monogenic group and therefore $\mathrm{Gp}\langle g \rangle$ is isomorphic to $\mathbb{Z}$. □

Lemma 1.2.16. *Every subsemigroup of a torsion group is a subgroup.*

Proof. Let G be a torsion group and let S be a subsemigroup of G. Let $g \in S$. Then as G is torsion there exists some $n \in \mathbb{N}$ such that $g^n = 1$. As S is closed under multiplication, we have that $1 \in S$ and S is closed under the identity operation. Furthermore, as $g^{n-1} \cdot g = g \cdot g^{n-1} = 1$ we have that $g^{-1} = g^{n-1}$. Hence S is closed under the inversion map. As S is closed under all the fundamental group operations, we have that S is a subgroup of G. □

Proposition 1.2.17. *For a group G we have:*

(i) *G is monogenic subsemigroup separable if and only if G is torsion and monogenic subgroup separable;*

(ii) *G is weakly subsemigroup separable if and only if G is torsion and weakly subgroup separable;*

(iii) *G is strongly subsemigroup separable if and only if G is torsion and strongly subgroup separable.*

Proof. Let $\mathscr{P}$ be one of the following properties: monogenic subsemigroup separability, weak subsemigroup separability and strong subsemigroup separability. First we show that if G is not torsion, that G does not have property $\mathscr{P}$. If G is not torsion, then by Lemma 1.2.15 there exists a subgroup H of G isomorphic to $\mathbb{Z}$. Therefore H is not MSS by Example 1.2.5. As subsemigroups of MSS semigroups are themselves MSS by Proposition 1.2.13, it cannot be the case that G is MSS. As all SSS and WSS semigroups are MSS by Proposition 1.2.9, it cannot be the case that G has property $\mathscr{P}$.

Now suppose that G is torsion and monogenic subgroup separable. As G is torsion, every monogenic subsemigroup is actually a monogenic subsemigroup by Lemma 1.2.16. Hence G is MSS. The cases for weak subsemigroup separability and strong subsemigroup separability are similar. □

We are also able to classify when a group is CS, which relies on the following theorem.

Theorem 1.2.18 (Lagrange)**.** *Let G be a group and H be a subgroup of G. Then any two cosets Hg and Hk have the same cardinality.*

Theorem 1.2.19. *A group is completely separable if and only if it is finite.*

Proof. As separability properties are finiteness conditions by Lemma 1.2.8, we have that every finite group is CS. Now suppose that G is completely separable. Then there exists a finite index congruence on G such that $\{1\}$ is a congruence class. Then every congruence class is a singleton by Theorem 1.2.18. As there are only finitely many singleton congruence classes, it must be that G is finite. □

The next result will be utilised many times. It says that if we can separate an element from a finite number of sets then we can separate it from their union.

Proposition 1.2.20. *Let A be an algebra of type $\mathcal{F}$, let X_1, X_2, $\ldots$, X_n be subsets of A and let $a \in A$ be such that $a \notin X_i$ for $i \in \{1, 2, \ldots, n\}$. If a can* *be separated from X_i for all $i \in \{1, 2, \ldots, n\}$, then a can be separated from $X = \bigcup_{i=1}^{n} X_i$.*

Proof. For each i, as a can be separated from X_i, there exists a finite index congruence ρ_i on A such that $[a]_{\rho_i} \neq [x]_{\rho_i}$ for all $x \in X_i$. Consider $\rho = \bigcap_{i=1}^{n} \rho_i \subseteq A \times A$. It is known that an intersection of congruences is also a congruence (see [8, Chapter 1]). Furthermore, as each ρ_i is a finite index congruence, then ρ will have finitely many congruences classes. Indeed, the congruence classes of ρ are the non-empty intersections of the form

$$[x_1]_{\rho_1} \cap [x_2]_{\rho_2} \cap \cdots \cap [x_n]_{\rho_n}.$$

Finally, let $y \in X$. Then there exists $i \in \{1, \ldots, n\}$ such that $y \in X_i$. Then $(a, y) \notin \rho_i$, and so $(a, y) \notin \rho$. Hence ρ separates a from X. □

An immediate consequence of this is that residual finiteness can be regarded as the separability property with respect to finite subsets. The following result utilises Proposition 1.2.20 to show that in a weakly subgroup separable group, we can extend separation from finitely generated subgroups to finite unions of cosets of such subgroups.

Corollary 1.2.21. *Let G be a weakly subgroup separable group and let $H \leq G$ be finitely generated. If for cosets $Hg_1, Hg_2, \ldots, Hg_n$ we have that $g \notin \bigcup_{i=1}^{n} Hg_i$ then g can be separated from $\bigcup_{i=1}^{n} Hg_i$.*

Proof. By Proposition 1.2.20 it is sufficient to show that for $1 \leq i \leq n$, g can be separated from Hg_i. As $g \notin Hg_i$ then $gg_i^{-1} \notin H$. As G is weakly subgroup separable then there exists as finite group K and homomorphism $\phi : G \to K$ such that $\phi(gg_i^{-1}) \notin \phi(H)$. If $\phi(g) \in \phi(Hg_i)$, then $\phi(gg_i^{-1}) \in \phi(H)$, which is a contradiction and so the result holds. □

1.2.1 A Topological Viewpoint

Separability properties can be viewed topologically. Although for the purpose of this thesis we consider them as algebraic properties, for completeness we briefly outline the topological viewpoint. Basic knowledge of topology is assumed. We begin with the definition of a topological algebra.

Definition 1.2.22. A *topological algebra* is a pair (A, τ), where A is an algebra of type $\mathcal{F}$ and τ is a topology on A such that for each $f \in \mathcal{F}$, the fundamental operation f^A is continuous in the product topology.

The requirement for each of the fundamental operations to be a continuous map is designed to ensure that the topology interacts with the algebraic operations. The topology linked to separability properties is known as the profinite topology. We introduce this topology via a basis.

Lemma 1.2.23. *Let A be an algebra. Then the set $\mathcal{B}_A$, consisting of all congruences classes of all finite index congruences, is a basis for a topology.*

Proof. To show that $\mathcal{B}_A$ is a basis it is sufficient to show:

(i) for all $a \in A$, there exists $B \in \mathcal{B}_A$ such that $a \in B$; and
(ii) for $B_1, B_2 \in \mathcal{B}_A$, there exists $B_3 \in \mathcal{B}_A$ such that $B_1 \cap B_2 \subseteq B_3$.

As ∇_A is a finite index congruence on A, we have that $A \in \mathcal{B}_A$ and so (i) holds. Suppose now that $B_1, B_2 \in \mathcal{B}_A$. If $B_1 \cap B_2 = \emptyset$, then condition (ii) holds trivially. Now suppose that $B_1 \cap B_2 \neq \emptyset$. As $B_1, B_2 \in \mathcal{B}_A$, there exist finite index congruence ρ_1, ρ_2 on A such that B_1 is a congruence class of ρ_1 and B_2 is a congruence class of ρ_2. Furthermore, $\rho_1 \cap \rho_2$ is also a finite index congruence on A and $B_1 \cap B_2$ is a congruence class of $\rho_1 \cap \rho_2$. Hence $B_1 \cap B_2 \in \mathcal{B}_A$ and condition (ii) holds, completing the proof that $\mathcal{B}_A$ is a basis. □

Definition 1.2.24. The *profinite topology* τ_A on an algebra A is the topology with a basis $\mathcal{B}_A$.

Proposition 1.2.25. *For an algebra A, (A, τ_A) is a topological algebra.*

Proof. The fact that each fundamental operation is continuous follows from the fact that every (finite index) congruence is compatible with each fundamental operation. □

We now express our separability properties in topological terms.

Proposition 1.2.26. *For an algebra A we have:*

(i) *A is residually finite if and only if the profinite topology is Hausdorff;*

(ii) *A is monogenic subalgebra separable if and only if every monogenic subalgebra is closed in the profinite topology;*

(iii) *A is weakly subalgebra separable if and only if every finitely generated subalgebra is closed in the profinite topology;*

(iv) *A is strongly subalgebra separable if and only if every subalgebra is closed in the profinite topology;*

(v) *A is completely separable if and only if the profinite topology is discrete.*

Proof. (i) ($\Rightarrow$) If A is residually finite, for every distinct pair $a, b \in A$, there exists a finite index congruence ρ such that $[a]_\rho \neq [b]_\rho$. Then $[a]_\rho, [b]_\rho \in \mathcal{B}_A$ and $[a]_\rho$ is an open neighbourhood of a and $[b]_\rho$ is an open neighbourhood of b. As congruence classes of ρ partition A, we conclude that $[a]_\rho \cap [b]_\rho = \emptyset$ and hence τ_A is Hausdorff.

($\Leftarrow$) Assume that τ_A is Hausdorff and let $a, b \in A$ be distinct. Then there exists $B_1, B_1 \in \mathcal{B}_A$ such that $a \in B_1$, $b \in B_2$ and $B_1 \cap B_1 = \emptyset$. That is, there is exist finite index congruences ρ on A such that B_1 is a congruence class of ρ. As $b \notin B_1$, we conclude that $B_1 = [a]_\rho \neq [b]_\rho$ and hence A is residually finite.

(ii) ($\Rightarrow$) As A is monogenic subsemigroup separable, for every monogenic subalgebra C and every point $x \in A \backslash C$, there exists a finite index congruence ρ_x on A such that $[x]_{\rho_x} \neq [c]_{\rho_x}$ for all $c \in C$. That is, $C \subseteq A \setminus [x]_{\rho_x}$. Note that $A \setminus [x]_{\rho_x}$ is a closed set. Then $C = \bigcap_{x \in A \backslash C} A \setminus [x]_{\rho_x}$ and we conclude that C is closed in the profinite topology.

($\Leftarrow$) Let C be a monogenic subalgebra of A and let $x \in A \setminus C$. As we are assuming that C is closed in the profinite topology, we have that $A \setminus C$ is an open set. That is, there exists $B \in \mathcal{B}_A$ such that $x \in B$ and $B \subseteq A \setminus C$. Then B is a congruence class for some finite index congruence ρ on A and $B = [x]_\rho \neq [c]_\rho$ for all $c \in C$. Hence ρ separates x from C and A is monogenic subalgebra separable.

The proofs for (iii) and (iv) are similar to the proof for (ii).

(v) ($\Rightarrow$) As A is completely separable, for each $a \in A$ there exists a finite congruence ρ_a such that $[a]_{\rho_a} = \{a\}$. Hence $\{a\} \in \mathcal{B}_A$ for all $a \in A$ and we conclude that the profinite topology is discrete.

($\Leftarrow$) Assume that the profinite topology is discrete. Then for each $a \in A$ the set $\{a\} \in \mathcal{B}_A$. That is, $\{a\}$ is a congruence class of some finite index congruence ρ_a. From this we conclude that A is completely separable. □

1.2.2 The Word Problem and the Generalised Word Problem

In this subsection, we provide more motivation for the study of separability properties via two decision problems; the word problem and the generalised word problem. We state these problems in terms of presentations, although they do not have to be couched in this language.

Definition 1.2.27. Let A be an algebra and let $R \subseteq A \times A$. Define

$$R^\sharp = \bigcap \{\rho \mid R \subseteq \rho \text{ and } \rho \text{ is a congruence on } A\}.$$

Note that as $A \times A$ is a congruence on A, this intersection is non-empty. As the intersection of congruences is a congruence, we have that $R^\sharp$ is a congruence. We call $R^\sharp$ the *congruence generated by* R.

Definition 1.2.28. Let K be a variety of algebras. For a non-empty set Y, denote the free algebra on Y by F_Y. A *presentation* is a pair (X, R), where X is a non-empty set of *generators* and $R \subseteq F_X \times F_X$ is a set of *relations*. For $(u, w) \in R$, we often write $u = w$. We write the presentation (X, R) as $\langle X \mid R \rangle$. An algebra A is said to be *given by the presentation* $\langle X \mid R \rangle$ if $A \cong F_X/R^\sharp$. In this case we write $A \cong \langle X \mid R \rangle$. The algebra A is said to be *finitely presented* if there exists a finite set of generators X and a finite set of relations R such that $A \cong \langle X \mid R \rangle$.

Example 1.2.29. We have already seen that $\mathbb{N}$ is a free semigroup which is monogenic. Hence $\mathbb{N} \cong \langle \{x\} \mid \emptyset \rangle$. When both set of generators and the set of relations are finite, we often dispense with the set brackets. So here we write $\mathbb{N} \cong \langle x \mid \rangle$.

Consider the finite cyclic group C_n. From Example 1.1.18 we know that $C_n = \mathbb{Z}/\rho_n$. We have observed that $\mathbb{Z}$ is the free group which is monogenic. It can be shown that ρ_n is generated as a congruence by the set $\{(0, n)\}$. Hence C_n is given by the group presentation $\langle x \mid x^n = 1 \rangle$.

Remark 1.2.30. We have already observed that within a variety, every algebra is a homomorphic image of a free algebra. That is, for every algebra A, there exists a non-empty set X and epimorphism $\phi : F_X \to A$. Then $A \cong F_X/\ker(\phi)$. In other words, A is given by the presentation $\langle X \mid \ker(\phi) \rangle$. This shows that every element of a variety can be given by a presentation. However, there is no need for this presentation to be unique.

Definition 1.2.31. The following question is known as the *word problem*. Given a finitely presented algebra $\langle X \mid R \rangle$ of type $\mathcal{F}$, does there exist an algorithm that on input of $u, v \in T_{\mathcal{F}}(X)$ decides if u and v represent the same element in $\langle X \mid R \rangle$?

The following variation is known as the *generalised word problem*. Given a finitely presented algebra $\langle X \mid R \rangle$ of type $\mathcal{F}$, does there exist an algorithm that on input of $u, v_1, v_2, \ldots, v_n \in T_{\mathcal{F}}(X)$ decides if u represents an element of the subalgebra $\langle v_1, v_2, \ldots, v_n \rangle$ of the algebra $\langle X \mid R \rangle$?

Theorem 1.2.32. *If a finitely presented algebra is residually finite then its word problem is decidable. Equally, if a finitely presented algebra is weakly subalgebra separable then its generalised word problem is decidable.*

Proof. [16, Theorem 2.1 and 2.2] □

1.3 Semigroup Preliminaries

In this section we introduce some basic notions in the field of semigroup theory. The notation used is based on [28]. For the remainder of this work the adopt the following notation.

Notation 1.3.1. We reserve $\langle X \rangle$ to denote the subsemigroup generated by the set X. In the case we wish to consider an algebra of another type generated the set X, we will use some appropriate additional notation. For example, if we wish to denote the subgroup generated by X, we will write $\mathrm{Gp}\langle X \rangle$. Similarly, we reserve $A \leq B$ to mean that A is a subsemigroup of B. Again we will use additional notation where appropriate. For example, we will use $A \leq_{\mathrm{Gp}} B$ to show that A is a subgroup of B. Finally, we reserve $\langle\, X \mid R \,\rangle$ for semigroup presentations. When we wish to consider presentations of other types of algebras, we will use additional notation. For example, we will use $\mathrm{Gp}\langle\, X \mid R \,\rangle$ for group presentations.

The results we present are needed to understand the structure and substructures of semigroups. We introduce Green's relations $\mathcal{J}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$. The relation $\mathcal{J}$ partitions a semigroup based upon principle ideals, and the parts come equipped with a partial ordering revealing the structure of a semigroup. Green's relations $\mathcal{L}$ and $\mathcal{R}$ provide additional structural information. The relation $\mathcal{H}$ allows us to identity the maximal subgroups of a semigroup. The separability properties of these subgroups often prove vital in understanding the separability properties of the semigroup as a whole. But first we begin with partially ordered sets and semilattices.

1.3.1 Partially ordered sets and semilattices

In this subsection we introduce the notion of partially ordered sets. These are interesting mathematical objects in their own right and also often appear in theorems concerning the structure of semigroups. There is also one class of partially ordered sets, semilattices, which can be viewed as a variety of semigroups. We present both viewpoints of semilattices and justify their equivalence. Following this we describe free semilattices. We begin with the definition of a partially ordered set.

Definition 1.3.2. For a set X, a subset $\omega \subseteq X \times X$ is known as a *partial order on* X if for $x, y, z \in X$ we have

1. **reflexivity:** $(x, x) \in \omega$;
2. **anti-symmetry:** if $(x, y), (y, x) \in \omega$ then $x = y$; and
3. **transitivity:** if $(x, y), (y, z) \in \omega$ then $(x, z) \in \omega$.

We often write $x \leq y$ to mean $(x, y) \in \omega$. A *partially ordered set* is a pair $(X, \leq)$, where X is a set and $\leq$ is a partial order on X.

Example 1.3.3. The normal ordering $\leq$ on $\mathbb{Z}$, given by $x \leq y$ if $x - y$ is negative or zero, is a partial order.

For a set X, there is a partial order on the power set $\mathcal{P}(X)$ given by $A \leq B$ if $A \subseteq B$.

Definition 1.3.4. In a partially ordered set $(X, \leq)$, an element $x \in X$ is *minimal* if there does not exist $y \in X$ such that $y \leq x$. An element $x \in X$ is called *the minimum* element if $x \leq z$ for all $z \in X$.

Remark 1.3.5. It follows from the definition that if a minimum element exists, it must be unique. This is why we call it *the* minimum element. It is also clear that the minimum element is a minimal element, and in the case that the minimum element exists, it is the only minimal element.

Example 1.3.6. In $(\mathbb{Z}, \leq)$ there are no minimal elements. This follows as for any $z \in \mathbb{Z}$, we have that $z - 1 \leq z$. In $(\mathcal{P}(X), \subseteq)$ it is clear that the empty set $\emptyset$ is the minimum element. An example of a partially ordered set with minimal elements but not minimum element is given in Example 1.3.8.

Definition 1.3.7. For a partially ordered set $(X, \leq)$, an element z is called a *lower bound* of x and y if $z \leq x$ and $z \leq y$. A lower bound z of x and y is called the *greatest lower bound* of x and y if for any lower bound t of x and y we have $t \leq z$.

Example 1.3.8. In $(\mathcal{P}(X), \subseteq)$ the least lower bound of A and B is $A \cap B$.

Given any set X, the diagonal congruence Δ_X is also a partial order on X. Given any two distinct elements of x and y, there are no lower bounds of x and y, so certainly there is no greatest lower bound.

Consider the set $X = \{x, y, z, t\}$ with partial order

$$\{(x, x), (y, y), (z, z), (t, t), (z, x), (t, x), (z, y), (t, y)\}.$$

Then both z and t are lower bounds of x and y. However, as z and t are incomparable, there is not greatest lower bound for x and y. Note that both z and t are both minimal elements, but there does not exists a minimum element.

Definition 1.3.9. A *semilattice* is a partially ordered set $(X, \leq)$ such that for each pair of elements $x, y \in X$, the greatest lowest bound of x and y exists. We denote the greatest lowest bound by $x \wedge y$. Note by definition we have that $x \wedge y = y \wedge x$.

Example 1.3.10. For a non-empty set X, let $\mathcal{P}_\omega(X)$ denote the set of all finite subsets of X. Then $(\mathcal{P}_\omega(X) \setminus \{\emptyset\}, \supseteq)$ is a partially ordered set. Furthermore $(\mathcal{P}_\omega(X) \setminus \{\emptyset\}, \supseteq)$ is a semilattice with $A \wedge B = A \cup B$.

There is an equivalent definition of semilattices as semigroups.

Definition 1.3.11. A *semilattice* is a semigroup S which satisfies the following identities:

$$x^2 = x, \quad xy = yx.$$

The following proposition establishes the equivalence of the two definitions for semilattices.

Proposition 1.3.12. *Given a partially order set $(X, \leq)$ which is a semilattice in the order-theoretic sense, we can define a binary operation $\cdot$ on X by defining $x \cdot y = x \wedge y$. Under this binary operation, X is a semilattice in the semigroup-theoretic sense.*

Equivalently, given a semigroup S which is a semilattice in the semigroup-theoretic sense, we can define a partial order on S by $s \leq t$ if $st = s$. Under this partial ordering, S is a semilattice in the order-theoretic sense.

Proof. [28, Proposition 1.3.2] □

Definition 1.3.11 defines semilattices in terms of identities, and hence the class of all semilattices forms a variety of signature (2). For a non-empty set X, the free semilattice on X is precisely the object $\mathcal{P}_\omega(X) \setminus \{\emptyset\}$ given in Example 1.3.10. The underlying set is the collection of all non-empty finite subsets of X and the binary operation is set union.

Theorem 1.3.13. *The semigroup $\mathcal{P}_\omega(X) \setminus \{\emptyset\}$ is the free semilattice on X.*

Proof. We show that $\mathcal{P}_\omega(X) \setminus \{\emptyset\}$ is free on the $\overline{X} = \{\{x\} \mid x \in X\}$, which is clearly in bijection with X. Note that as every element of $\mathcal{P}_\omega(X) \setminus \{\emptyset\}$ is a finite union of elements of $\overline{X}$, we have that $\overline{X}$ is a generating set for $\mathcal{P}_\omega(X) \setminus \{\emptyset\}$.

Let S be a semilattice and $\phi : \overline{X} \to S$ given by $\{x\} \mapsto s_x$, for some $s_x \in S$. Then the only way to extend ϕ to a homomorphism $\overline{\phi} : \mathcal{P}_\omega(X) \setminus \{\emptyset\} \to S$ is by $\{x_1, x_2, \ldots, x_n\} \mapsto s_{x_1} s_{x_2} \ldots s_{x_n}$, where $x_i \in X$ for $1 \leq i \leq n$. Note that as S is commutative, it does not matter what order the elements of the subsets occur in. Furthermore, $\overline{\phi}$ is a homomorphism as

$$\begin{aligned}\overline{\phi}(\{x_1, \ldots, x_m\} \cup \{x_{m+1}, \ldots, x_n\}) &= \overline{\phi}(\{x_1, \ldots, x_m, x_{m+1}, \ldots, x_n\} \\ &= s_{x_1} \ldots_{x_m} s_{x_{m+1}} \ldots s_{x_n} \\ &= \overline{\phi}(\{x_1, \ldots, x_m\})\overline{\phi}(\{x_{m+1}, \ldots, x_n\}),\end{aligned}$$

where $x_i \in X$ for $1 \leq i \leq n$. $\square$

Not only is it intriguing that this class of partially ordered sets can be viewed algebraically as semigroups, it is also true that semilattices provide the underlying structure for some classes of semigroups. Examples of this will be seen throughout this thesis; one such instance is Proposition 3.3.6. In the next subsection we see that each semigroup comes equipped with a partial ordering via Green's relation $\mathcal{J}$.

1.3.2 Ideals and Green's Relations

Green's relations are arguably the most important tools for understanding the structure of semigroups. These can be defined in terms of ideals, so we begin with ideals.

Definition 1.3.14. An subset I of a semigroup S is:

- a *left ideal* if $IS \subseteq S$;
- a *right ideal* if $SI \subseteq S$;
- an *ideal* if I is both a left ideal and a right ideal.

Remark 1.3.15. For an ideal $I \subseteq S$, we have $I^2 \subseteq I$. Therefore I is closed under multiplication and is a subsemigroup. However, not every subsemigroup is an ideal as the following example shows.

Example 1.3.16. Consider $I = \{n \in \mathbb{N} \mid n \geq 4\} \subseteq \mathbb{N}$. Then I is an ideal. To see this let $i \in I$ and $n \in \mathbb{N}$. As $i + n = n + i > i \geq 4$ we have that $n + i, i + n \in I$ as required.

Now consider $T = \{2\} \cup I \subseteq \mathbb{N}$. We have already observed I is an ideal, so to check that T is closed under multiplication we only need check that $2 + 2 \in T$, which it is. Hence T is a subsemigroup. But T is not an ideal as $2 + 1 = 3 \notin T$.

Definition 1.3.17. For a semigroup S and an element $a \in S$, the *principal ideal generated by a* is the set $S^1 a S^1$. This can be viewed as the smallest ideal (by containment) that contains the element a.

Example 1.3.18. The ideal I from Example 1.3.16 is the principal ideal generated by the element 4.

For every ideal of semigroup there is an associated congruence, which we now define.

Definition 1.3.19. For an ideal $I \leq S$, we can define a congruence ρ_I on S by:

$$s \,\rho_I\, t \text{ if and only if } s, t \in I \text{ or } s = t.$$

This congruence is known as the *Rees congruence* on S by I. We denote the quotient of this congruence by S/I and the congruence class of $s \in S$ by $[s]_I$.

Note that if $s \in S \setminus I$, then $[s]_I = \{s\}$. Furthermore, I itself forms the remaining congruence class. We have observed that ρ_I has partitioned S, so it is an equivalence relation. To confirm that ρ_I is a congruence we need to check that it is compatible with multiplication. Let $[x]_I, [y]_I$ be two ρ_I classes. Now either both $[x]_I$ and $[y]_I$ are singletons, in which case there is nothing to check, or at least one of $[x]_I$ or $[y]_I$ is the ideal I. Then no matter what representatives we pick, $xy \in I$ and hence ρ_I is an ideal.

Example 1.3.20. Consider the ideal I from Example 1.3.16. Then the corresponding Rees congruence has four classes: $[1]_I, [2]_I, [3]_I$ and I. The multiplication table of the quotient $\mathbb{N}/I$ is given below.

$\cdot$	$[1]_I$	$[2]_I$	$[3]_I$	I
$[1]_I$	$[2]_I$	$[3]_I$	I	I
$[2]_I$	$[3]_I$	I	I	I
$[3]_I$	I	I	I	I
I	I	I	I	I

Definition 1.3.21. An element 0 of a semigroup S is said to be a *zero* if $0s = s0 = 0$ for all $s \in S$.

Example 1.3.22. In a Rees quotient S/I, the congruence class I is a zero element. This follows from the fact I is an ideal. Given a semigroup S with zero element 0, the set $\{0\}$ is an ideal of S.

Definition 1.3.23. For a semigroup $(S, \cdot)$, we can define an associated semigroup which has a zero. Let $S^0 = S \cup \{0\}$, where S and $\{0\}$ are assumed to be disjoint, and define multiplication $\bar{\cdot}$ on S^0 by

$$x \mathbin{\bar{\cdot}} y = x \cdot y, \quad z \mathbin{\bar{\cdot}} 0 = 0 \mathbin{\bar{\cdot}} z = 0,$$

where $x, y \in S$ and $z \in S^0$. It is easy to check that $\bar{\cdot}$ is associative and therefore $(S^0, \bar{\cdot})$ is a semigroup. We refer to S^0 as S *with a zero adjoined.*

Example 1.3.24. Recall the semigroup L from Example 1.1.4. The L^0 is the semigroup with the following multiplication table.

$\cdot$	x	y	0
x	x	x	0
y	y	y	0
0	0	0	0

We now turn our attention to Green's relation $\mathcal{J}$ and its related partial ordering.

Definition 1.3.25. *Green's relation* $\mathcal{J}$ on a semigroup is an equivalence relation on S given by

$$s \,\mathcal{J}\, t \text{ if and only if } S^1 s S^1 = S^1 t S^1.$$

That is, two elements are $\mathcal{J}$-related if and only if they generate the same principal ideal. We denote the $\mathcal{J}$-class of an element x by J_x.

Example 1.3.26. For $k \in \mathbb{N}$, the set $\mathbb{N}^1 k \mathbb{N}^1 = \{n \in \mathbb{N} : n \geq k\}$. Therefore $k \mathcal{J} m$ if and only if $k = m$. That is $J_k = \{k\}$. Hence, for $\mathbb{N}$, Green's relation $\mathcal{J}$ coincides with the diagonal equivalence $\Delta_{\mathbb{N}}$.

Note that for any semigroup with a zero element 0, we have that $J_0 = \{0\}$.

Definition 1.3.27. A partial order can be put on the $\mathcal{J}$-classes of a semigroup S in the following way:

$$J_a \leq J_b \text{ if } S^1 a S^1 \subseteq S^1 b S^1.$$

Remark 1.3.28. It follows immediately that for $x, y \in S^1$ and $a \in S$ we have

$$J_{xay} \leq J_a.$$

Example 1.3.29. For $k \in \mathbb{N}$ we have already observed that $\mathbb{N}^1 k \mathbb{N}^1 = \{n \in \mathbb{N} : n \geq k\}$. Therefore $J_k \leq J_m$ if and only $k \geq m$.

Using the above partial ordering we can define the following ideal.

Definition 1.3.30. For a non-minimal $\mathcal{J}$-class J of a semigroup S, define

$$I(J) = \bigcup \{J_s \mid s \in S, J \nleq J_s\}.$$

Then $I(J)$ is an ideal in S. To see this first note that as J is non-minimal, there must exist a $\mathcal{J}$-class J_x such that $J_x \leq J$ and $J_x \neq J$. Therefore $I(J)$ is non-empty. Now let $a \in I(J)$ and $s \in S$. From Remark 1.3.28, we have that both $J_{as} \leq J_a$ and $J_{sa} \leq J_a$. If $as \notin I(J)$, then $J \leq J_{as}$. But, as the ordering on the $\mathcal{J}$-classes is transitive, this would imply that $J \leq J_a$, which is a contradiction. Hence $as \in I$ and similarly $sa \in I$, completing the proof that $I(J)$ is an ideal. Note that $J \subseteq S \setminus I(J)$.

Example 1.3.31. For $k \in \mathbb{N}$, we have that $I(J_k) = \{n \in \mathbb{N} \mid n > k\}$.

We now define consider Green's relations $\mathcal{L}$ and $\mathcal{R}$.

Definition 1.3.32. *Green's relation* $\mathcal{L}$ on a semigroup is an equivalence relation on S given by

$$s \,\mathcal{L}\, t \text{ if and only if } S^1 s = S^1 t.$$

That is, two elements are $\mathcal{L}$-related if and only if they generate the same principal left ideal. We denote the $\mathcal{L}$-class of an element x by L_x. *Green's relation* $\mathcal{R}$ on a semigroup is an equivalence relation on S given by

$$s \,\mathcal{R}\, t \text{ if and only if } sS^1 = tS^1.$$

That is, two elements are $\mathcal{R}$-related if and only if they generate the same principal right ideal. We denote the $\mathcal{R}$-class of an element x by R_x.

Example 1.3.33. Consider the semigroup L from Example 1.1.4. As $xL^1 = \{x\}$ and $yL^1 = \{y\}$, we conclude that L is $\mathcal{L}$-trivial, i.e. its $\mathcal{L}$ classes are singletons. But $L^1x = L^1y = \{x, y\}$ are so have that L has just one $\mathcal{R}$-class.

From the definitions, it follows that $\mathcal{L}, \mathcal{R} \subseteq \mathcal{J}$. Finally, we consider Green's relation $\mathcal{H}$.

Definition 1.3.34. *Green's relation* $\mathcal{H}$ on a semigroup is an equivalence relation on S given by

$$s\,\mathcal{H}\,t \text{ if and only if } s\,\mathcal{L}\,t \text{ and } s\,\mathcal{R}\,t.$$

We denote the $\mathcal{H}$-class of an element x by H_x.

It follows from the definitions that $\mathcal{H} \subseteq \mathcal{L}, \mathcal{R}$. Green also defined one further relation, $\mathcal{D}$, but as we do not utilise it during this thesis we refrain from defining it.

Example 1.3.35. As $\mathcal{H} \subseteq \mathcal{J}$ and from Example 1.3.26 we know that $\mathcal{J}$ and the diagonal congruence $\Delta_{\mathbb{N}}$ coincide for $\mathbb{N}$, we must have that $\mathcal{H}$ also coincides with $\Delta_{\mathbb{N}}$.

It turns out that the $\mathcal{H}$-classes that contain idempotents are precisely the maximal subgroups of a semigroup, as the following proposition formalises.

Proposition 1.3.36. *For an $\mathcal{H}$-class H of semigroup S, the following are equivalent:*

(i) *H is a maximal subgroup of S;*
(ii) *H contains an idempotent;*
(iii) *the intersection $H \cap H^2$ is non-empty.*

Proof. [28, Corollary 2.2.6]. □

The following extension of Proposition 1.3.36 is considered folklore, but we provide a proof for completeness.

Lemma 1.3.37. *Let H be a $\mathcal{H}$-class of a semigroup S. If $H \cap H^n \neq \emptyset$ for any $n \geq 2$, then H is a maximal subgroup.*

Proof. Suppose $h_1, h_2, \ldots, h_n, h \in H$ such that $h_1 h_2 \ldots h_n = h$. Since h_1, h_2, h_{n-1}, h_n and h are pairwise $\mathcal{H}$-related there exist $s, t, u, v \in S^1$ such that

$$hs = h_1,\ h_n t = h_2,\ uh = h_n,\ vh_1 = h_{n-1}.$$

Then

$$h \cdot sh_n = h_1h_n, \; h_1h_n \cdot th_3h_4 \dots h_n = h,$$

$$h_1u \cdot h = h_1h_n, \; h_1h_2 \dots h_{n-2}v \cdot h_1h_n = h.$$

Hence $(h, h_1h_n) \in \mathcal{H}$. Then $H \cap H^2 \neq \emptyset$ and it follows that H is a group by Proposition 1.3.36. □

To complete this section, we consider monogenic semigroups. Not only is understanding the structure of monogenic subsemigroups important for understanding monogenic subsemigroup separability, we also use finite monogenic subsemigroups to show examples of Green's relation $\mathcal{H}$.

Example 1.3.38. A monogenic semigroup $A = \langle a \rangle$ is either infinite, in which case $A \cong \mathbb{N}$, or it is finite. In the case where A is finite, there exists positive integers m and r such that $a^m = a^{m+r}$. The smallest such value of m is called the *index* of A and the smallest such value of r is called the *period* of A. To simplify statements, we will say that $\mathbb{N}$ has index ∞. Every finite monogenic semigroup has a presentation of the form $\langle\, a \mid a^m = a^{m+r} \,\rangle$, where m is the index and r is the period.

We have already seen Green's relation $\mathcal{H}$ for $\mathbb{N}$ in Example 1.3.35. For a finite monogenic semigroup A, given by the presentation $\langle\, a \mid a^m = a^{m+r} \,\rangle$, the $\mathcal{H}$ relation is as follows. For $1 \leq i \leq r-1$, the $\mathcal{H}$-class of a^i is the set $\{a_i\}$. None of these $\mathcal{H}$-classes are groups. The set $\{a^m, a^{m+1}, \dots, a^{m+r-1}\}$ forms a group $\mathcal{H}$-class. This $\mathcal{H}$-class will always be a cyclic group. Hence a monogenic semigroup is a group if and only if its index is 1. For proof of the above claims see [28, Theorem 1.2.2].

1.3.3 Actions

The actions of semigroups and groups will feature throughout this thesis. They often appear in constructions of specific semigroups, such as in Definition 2.3.2, or in describing how a subalgebra acts on some subset, for example Schützenberger groups which feature in Chapter 3. Here we will define both semigroup and group actions and provide some basic examples and properties.

Definition 1.3.39. Let S be a semigroup with binary operation $*$. A *right semigroup action* of S is a set X along with an operation $\cdot : X \times S \to X$ which is compatible with the multiplication of S. That is,

$$x \cdot (s * t) = (x \cdot s) \cdot t,$$

for all $x \in X$ and $s, t \in S$. We can define left semigroup actions in an analogous way.

Example 1.3.40. Every semigroup acts on itself by right multiplication. This follows from the fact that multiplication is associative. More generally, given a homomorphism $\phi : S \to T$, we can define an action $\cdot : T \times S \to T$ by $t \cdot s = t * \phi(s)$, where $*$ is the binary operation of T. The compatibility of ϕ guarantees that this is an action.

Definition 1.3.41. Let G be a group with binary operation $*$ and identity 1. A *right group action* of G is a set X along with an operation $\cdot : X \times G \to G$ which is a right semigroup action along with the additional property that

$$x \cdot 1 = x$$

for all $x \in X$. Left group actions are defined in an analogous way.

Example 1.3.42. Every group acts on itself by conjugation. That is, for a group G with multiplication $*$, the map $\cdot : G \times G \to G$ given by $x \cdot g = g^{-1} * x * g$ is an action.

Although actions are not the main area of study of this thesis, it will be useful to know some properties of group actions, especially to understand Schützenbeger groups. We conclude this chapter by listing the properties of group actions that are of interest to us and providing examples.

Definition 1.3.43. The action of a group G on a set X is:

- *transitive* if for all $x, y \in X$ there exists $g \in G$ such that $x \cdot g = y$;
- *free* if $x \cdot g = x$ implies that $g = 1$;

- *regular* if it is both free and transitive.

Example 1.3.44. Consider a non-trivial abelian group A with multiplication $*$ acting upon itself by conjugation. Then $x \cdot a = a^{-1} * x * a = x$ for all $x \in X$ and $a \in A$. Hence, this action is neither transitive nor free.

Consider the action of $\mathbb{Z}$ on the set $\{0, 1\}$ by $x \cdot z = (x + z) \pmod 2$. Then as $0 \cdot 0 = 0$, $1 \cdot 0 = 1$, $0 \cdot 1 = 1$ and $1 \cdot 1 = 0$, the action is transitive. However, this action is not free. For example $0 \cdot 2 = 0$ but 2 is not the identity of $\mathbb{Z}$.

On the other hand, the action of any group G on itself by right multiplication is regular.

Chapter 2

Separability Properties of Free Objects

In this chapter we consider the separability properties of free objects in different semigroup varieties. By semigroup varieties we mean varieties where there is at least one associative binary operation. This work is motivated by a result of Hall. In [27, Theorem 5.1], Hall shows that free groups are weakly subgroup separable. We also show that the free group is not strongly subsemigroup separable, except in two special cases. Our aim is to understand the separability properties of free objects in other semigroup varieties. The obvious variety to start with is that of semigroups itself. It turns out that the free semigroup exhibits very different behaviour to that of the free group; the free semigroup is completely separable.

Having established this difference, we further endeavour to understand for which semigroup varieties the separability properties of the free objects behave like those of the free group and which behave like those of the free semigroup. Somewhat surprisingly it turns out that free inverse monoids are also completely separable. To try and find examples of semigroup varieties which behave like the variety of groups, we consider semigroups which are unions of groups and hence turn our attention to the variety of completely regular semigroups. We are able to deal with the sub-varieties of completely simple semigroups and Clifford semigroups, whose free objects indeed do mirror the behaviour of free groups. However, free completely regular semigroups prove to be hard to handle. In order to simplify matters, we define a new variety of semigroups, known as α-groups. The variety is designed to encapsulate the behaviour of completely regular semigroups whilst removing some of the structural complexity. We are able to show that free α-groups are monogenic subalgebra separable but it still remains an open question if they are weakly subsemigroup separable. Whether free completely regular semigroups are weakly subsemigroup separable also remains an open question.

2.1 Free Groups

We begin by defining the basic concepts of alphabets, words and then free semigroups and free monoids. This is necessary as it allows us to describe the underlying set not only of free groups, but the underlying set of several of the algebraic objects we meet in this chapter. Although we define free semigroups and free monoids here, we leave the discussion of their separability properties to the next subsection. This ordering reflects that separability properties of the free group were studied earlier than those of free semigroups and free monoids.

Definition 2.1.1. An *alphabet* A is a set. A *word* over A is a finite sequence $a_1 a_2 \dots a_n$, where $a_i \in A$ for $1 \leq i \leq n$. For a word w, the *length* of w is the length of the sequence that makes up w and is denoted by $|w|$. The unique word of length zero is called *the empty word* and is denoted by ϵ. The set of all non-empty words over an alphabet A is denoted by F_A. The set of all words over an alphabet A is denoted by FM_A. For words $w = a_1 \dots a_m$ and $u = b_1 \dots b_n$ the sequence $a_1 \dots a_m b_1 \dots b_m$ is the *concatenation* of w by u and is denoted by wu. It is clear that concatenation of words is associative and therefore both F_A and FM_A are semigroups under the operation of concatenation. Furthermore FM_A is a monoid with identity ϵ. The semigroup F_A is the free semigroup on A and the monoid FM_A is the free monoid on A.

More information on free monoids and free semigroups, including a proof that FM_A is a monoid and free on A and that F_A is a semigroup and free on A, can be found in [28, Section 1.6 and Chapter 7].

Example 2.1.2. Let $A = \{a, b\}$. Then $w = ab$ is a word of length 2 over A and $u = bba = b^2a$ is a word of length 3 over A. We have that $wu = ab^3a$ and $uw = b^2a^2b$.

We can now define free groups.

Definition 2.1.3. Let X be a set and let X^{-1} be a set disjoint from X such that there exists a bijection $\phi : X \to X^{-1}$. For each $x \in X$, set $x^{-1} = \phi(x)$. Furthermore, for $x^{-1} \in X^{-1}$, set $(x^{-1})^{-1} = x$. At this stage x^{-1} does not yet represent the inverse of x. Let $X^{\pm} = X \cup X^{-1}$. We call a word over $X^{\pm}$ *reduced* if it contains no occurrences of contiguous subwords of the form xx^{-1} or $x^{-1}x$, where $x \in X$. Such pairs are known as *cancelling pairs*. Let FG_X be the set of all reduced words over $X^{\pm}$.

We now define a binary operation $*$ on FG_X. Let $v = x_1 \dots x_m$, $w = y_1 \dots y_n$ be elements of FG_X, where $x_i, y_j \in X^{\pm}$ for $1 \le i \le m$ and $1 \le j \le n$. Define

$$v * w = x_1 \dots x_{m-r} y_{r+1} \dots y_n,$$

where r is the largest value of $0 \le k \le \min\{m, n\}$ for which none of $x_m y_1$, $x_{m-1}y_2$, $\dots$, $x_{m-k+1}y_k$ is reduced. When it is clear from the context, we will write uv for $u * v$. If $r > 0$ then we say that *cancellation* has occurred between v and w. Then under this operation FG_X is a group with identity ϵ. The inverse of v is $v^{-1} = x_m^{-1} x_{m-1}^{-1} \dots x_1^{-1}$. Furthermore FG_X is free on the set X.

For more on FG_X, including a proof that it is a group and that it is free on X, see [29, Chapter 1].

Several of the free objects we meet in this chapter can be defined in terms of free groups. The following theory of Nielsen will prove useful in understanding the separability properties of these objects. In [42], Nielsen proved that every finitely generated subgroup of FG_X is free. In order to do this, Nielsen defined *Nielsen reduced* sets and showed that if $H \le_{\mathrm{Gp}} \mathrm{FG}_X$ is finitely generated, then there exists a finite generating set for H which is Nielsen reduced. For more on Nielsen's method, see [29, Chapter 3].

Before we give the definition of Nielsen reduced sets, we establish some notation.

Notation 2.1.4. For a set $W \subseteq \mathrm{FG}_X$, define $W^{-1} = \{w^{-1} \mid w \in W\}$ and $W^{\pm} = W \cup W^{-1}$.

Definition 2.1.5. A set $W \subseteq \mathrm{FG}_X$ is called *Nielsen reduced* if for all $u, v, w \in W^{\pm}$, the following conditions hold:

(i) $|u| \neq 0$;
(ii) $u \neq v^{-1} \implies |uv| \geq |u|, |v|$;
(iii) $u \neq v^{-1}$ and $v^{-1} \neq w \implies |uvw| > |u| - |v| + |w|$.

Example 2.1.6. Consider the set $X = \{a, b, c\}$. Then it is easy to check that $W = \{ab^{-1}c, c^{-1}ab\}$ is Nielsen reduced. However the set $U = \{abca, c^{-1}c^{-1}bc, a^{-1}c^{-1}bc\}$ is not Nielsen reduced. It is true that U satisfies the first two conditions of Definition 2.1.5. However, by setting $u = abca$, $v = a^{-1}c^{-1}bc$ and $w = c^{-1}b^{-1}cc$ we have that $uvw = abcc$. Hence $|uvw| = 4$ and $|u| - |v| + |w| = 4$ and so the third condition fails to be satisfied.

The following lemma gives some sufficient conditions for a subset of the free group to be Nielsen reduced.

Lemma 2.1.7. *Let $W \subseteq FG_X$ such that for all $u, v \in W^{\pm}$, the following hold:*

(i) $|u| \neq 0$;
(ii) $u \neq v^{-1} \implies |uv| > |u|, |v|$.

Then W is Nielsen reduced.

Proof. By assumption, W satisfies conditions (i) and (ii) from Definition 2.1.5. Therefore we only need show that W satisfies condition (iii) from Definition 2.1.5. Let $u, v, w \in W^{\pm}$ such that $u \neq v^{-1}$ and $v^{-1} \neq w$. Now, by condition (ii) of the assertion of the lemma, we can write $u = u_1u_2$ and $v^{-1} = u_2^{-1}v'$, so that $uv^{-1} = u_1v'$ where there is no cancellation between u_1 and v', $|u_2| < \frac{|u|}{2}$ and $|u_2| < \frac{|v^{-1}|}{2}$. Indeed, if $|u_2| \geq \frac{|u|}{2}$ then

$$|uv^{-1}| = |u_1v'| = |u| + |v^{-1}| - 2|u_2| < |v^{-1}|,$$

which contradicts assumption (ii) of this lemma. A similar contradiction occurs if $|u_2| \geq \frac{|v^{-1}|}{2}$. In a similar manner we write $w = w_1w_2$ and $v^{-1} = v''w_1^{-1}$, so that $v^{-1}w = v''w_2$ where there is no cancellation between v'' and w_2, $|w_1| < \frac{|w|}{2}$ and $|w_1| < \frac{|v^{-1}|}{2}$. Putting both of these observations together we can write $v = u_2^{-1}v_1w_1^{-1}$, where $|v_1| \geq 1$. Then

$$|uv^{-1}w| = |u_1v_1w_2| = |u| + |v| + |w| - 2|u_2| - 2|w_1| > |u| - |v| + |w|,$$

as $2|u_2| + 2|w_1| < 2|v|$. Hence condition (iii) of Definition 2.1.5 is satisfied and W is Nielsen reduced. □

For a finite set $W \subseteq \mathrm{FG}_X$, Nielsen gave an algorithm that would output a finite set $\overline{W} \subseteq \mathrm{FG}_X$, such that $\overline{W}$ is Nielsen reduced and $\mathrm{Gp}\langle W\rangle = \mathrm{Gp}\langle \overline{W}\rangle$. This is summarised below.

Theorem 2.1.8. [29, Lemma 3.1 and Theorem 3.1] *If $H \leq_{\mathrm{Gp}} \mathrm{FG}_X$ is finitely generated, then there exists a finite set $W \subseteq \mathrm{FG}_X$ such that W is Nielsen reduced and $\mathrm{Gp}\langle W\rangle = H$.*

We now state the motivating theorem for this chapter.

Theorem 2.1.9. [27, Theorem 5.1] *The free group FG_X is weakly subgroup separable.*

We turn our attention to the other separability properties of free groups. As free groups are weakly subsemigroup separable, we immediately have that they are monogenic subgroup separable and residually finite by Proposition 1.2.9 and Lemma 1.2.10. As we have seen in Theorem 1.2.19, a group is completely separable if and only if it is finite. It is well known that the group FG_X is finite if and only if $X = \emptyset$. In this case FG_X is isomorphic to the trivial group.

When $|X| = 1$, we have that $\mathrm{FG}_X \cong \mathbb{Z}$. As every subgroup of a cyclic group is cyclic, every subgroup of $\mathbb{Z}$ is finitely generated. Then by Theorem 2.1.9, $\mathbb{Z}$ is strongly subgroup separable.

For $|X| \geq 2$, the free group FG_X is not strongly subsemigroup separable as the following example shows.

Example 2.1.10. Let X be a set such that $|X| \geq 2$. Let $a, b \in X$ be distinct and consider the subgroup $H \leq_{\mathrm{Gp}} \mathrm{FG}_X$ with generating set $W = \{a^i b a^{-i} \mid i \in \mathbb{N}\}$. Let w be a reduced word over $W^{\pm}$, whose length over $W^{\pm}$ is n. Then, as a reduced word over X, w contains precisely n occurrences of b or b^{-1}. Furthermore, w must begin and end with a or a^{-1}. From these observations we conclude that $b \in \mathrm{FG}_X \setminus H$. Let $\phi : \mathrm{FG}_X \to G$, where G is a finite group. Let $m = o(\phi(a))$. Then

$$\phi(b) = \phi(a^m b a^{-m}) \in \phi(H).$$

Hence FG_X is not strongly subgroup separable.

The above observations are summarised below.

Lemma 2.1.11. *The free group* FG_X *is:*

(i) *completely separable if and only is* $X = \emptyset$*;*
(ii) *strongly subgroup separable if and only if* $|X| \leq 1$*.*

2.2 Free Monoids and Free Semigroups

Given Hall's result that the free group is weakly subgroup separable, it is natural to consider the separability properties of free objects in other semigroup varieties. The obvious variety to begin with is that of semigroups itself. This question was considered by Golubov who showed that free semigroups and free commutative semigroups are completely separable [17, Corollaries 1 and 2]. Golubov's proof relied on upon a characterisation of completely separable semigroups which we will consider in Chapter 4. This characterisation allowed Golubov to construct homomorphisms to show that these semigroups are completely separable. We present a proof that it is completely separable. This proof makes use of an ideal and its Rees quotient, though it can be easily shown that the resulting congruences are the same as those Golubov gives.

Theorem 2.2.1. *The free monoid* FM_X *is completely separable.*

Proof. Let $w \in \mathrm{FM}_X$. Let $X_w \subseteq X$ be the set of letters that appear in the word w. As w has finite length it must be that X_w is finite. Let $|w| = n$. Define the set $I(w)$ as follows:

$$I(w) = \{u \in \mathrm{FM}_X : |u| > n \text{ or } u \text{ contains a letter in } X \setminus X_w\}.$$

Then $I(w)$ is an ideal. Indeed, if $u \in I(w)$ and $v, z \in \mathrm{FM}_X$, then either u contains a letter in $X \setminus X_w$ and therefore so does vuz, or $|u| > n$ and therefore $|vuz| > n$. In either case $vuz \in I(w)$, and hence $I(w)$ is an ideal.

Furthermore, from the definition of $I(w)$ it is clear that $w \in \mathrm{FM}_X \setminus I(w)$. If a word v is in $\mathrm{FM}_X \setminus I(w)$, then $|v| \leq n$ and v is a word over the finite set X_w. Hence the set $\mathrm{FM}_X \setminus I(w)$ is finite. Therefore the Rees quotient $\mathrm{FM}_X / I(w)$ is finite. Furthermore, as $w \in \mathrm{FM}_X \setminus I(w)$, we have $[w]_{I(w)} = \{w\}$. Hence, FM_X is completely separable. □

As F_X is a subsemigroup of FM_X, any separability property of FM_X is inherited by F_X by Proposition 1.2.13.

Corollary 2.2.2. *The free semigroup* F_X *is completely separable.*

The above result can also be found in [45, Proposition 2.4]. In these lectures notes, Pin uses the language of topology. Indeed, he gives a metric on the free semigroup and uses this metric to define a topology, which turns out to be the same as the profinite topology on the free semigroup. Pin uses this theory to give results concerning recognisable languages, a well studied application of semigroup theory, but one which is not considered in this thesis.

2.3 Free Inverse Semigroups

Having observed the stark difference between the separability properties of the free semigroup and the free group, it could be conjectured that the varieties closer to the variety of semigroups have completely separable free objects, whilst the free objects in those varieties which are closer to the variety of groups have weaker separability properties.

This conjecture is backed up by the evidence that the free commutative semigroup, the free monoid and free commutative monoid are completely separable. Each of these three varieties can be considered closer to the variety of semigroups than to the variety of groups, in particular as there is no notion of inversion within these varieties.

To test this conjecture, it seems natural to investigate the separability properties of the free objects in a variety close to the variety of groups. The obvious choice of variety is that of inverse semigroups. Indeed, when inverse semigroups where first introduced by Vagner, they were referred to as "generalised groups".

Inverse semigroups form a variety of signature (2,1), with the binary operation of multiplication which will be represented by concatenation of elements, and the unary operation of inversion which will be represented by $^{-1}$. The class of inverse semigroups satisfy the following identities:

$$(xy)z = x(yz), \quad (x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Where groups algebraically represent symmetries, realised as permutations, inverse semigroups capture the notion of partial symmetries represented by partial bijections of a set. More on inverse semigroups can be found in [28, Chapter 5].

We will consider free inverse monoids, as free inverse semigroups are subsemigroups of free inverse monoids. There exists different ways of representing the free inverse monoids. Here we present them as *McAlister triples.* This method is chosen as it easily allows the Green's relations of free inverse monoids to be described. In particular, Green's relation $\mathcal{J}$ is used in the construction of ideals that show that the free inverse monoid is completely separable.

Definition 2.3.1. Let $(\mathcal{X}, \leq)$ be a partially ordered set and let $\mathcal{Y} \subseteq \mathcal{X}$ such that

(i) $\mathcal{Y}$ is a meet semilattice with respect to $\leq$,
(ii) $\mathcal{Y}$ is downward closed.

Let G be a group that acts on $\mathcal{X}$ from the left such that for $g, h \in G$ and $A, B \in \mathcal{X}$ we have the following three conditions:

$$(\forall B \in \mathcal{X})(\forall g \in G)(\exists A \in \mathcal{X})(gA = B),$$
$$gA = gB \iff A = B,$$
$$A \leq B \iff gA \leq gB.$$

Suppose that the triple has $(G, \mathcal{X}, \mathcal{Y})$ has the properties

(iii) $G\mathcal{Y} = \mathcal{X}$,
(iv) for all $g \in G$, $g\mathcal{Y} \cap \mathcal{Y} \neq \emptyset$.

A triple $(G, \mathcal{X}, \mathcal{Y})$ satisfying conditions (i)-(iv) is a *McAlister triple.* Given such a triple, let

$$\mathcal{M}(G, \mathcal{X}, \mathcal{Y}) = \{(A, g) \in \mathcal{Y} \times G \mid g^{-1}A \in \mathcal{Y}\}$$

with multiplication

$$(A, g)(B, h) = (A \wedge gB, gh),$$

and inversion

$$(A, g)^{-1} = (g^{-1}A, g^{-1}).$$

Then $\mathcal{M}(G, \mathcal{X}, \mathcal{Y})$ is an inverse semigroup, see [28, Theorem 5.9.2].

In fact, McAlister triples form a special class of inverse monoids known as E-unitary inverse semigroups. The definition of this class is not important for our purposes except that free inverse monoids can be represented as McAlister triples, as is shown in the following definition.

Definition 2.3.2. Let X be a non-empty set and let $G = \mathrm{FG}_X$. For a word $w = w_1w_2\dots w_n \in \mathrm{FG}_X$, let $w^{\downarrow} = \{\epsilon, w_1, w_1w_2, \dots, w_1w_2\dots w_n\}$. That is $w^{\downarrow}$ is the set of all prefixes of w. A non-empty subset $A \subseteq \mathrm{FG}_X$ is called *saturated* if for every $w \in A$ we have that $w^{\downarrow} \subseteq A$. Let $\mathcal{Y} = \{A \subseteq \mathrm{FG}_X \mid A \text{ is finite and saturated}\}$. Define an order $\leq$ on $\mathcal{Y}$ by

$$A \leq B \text{ if and only if } A \supseteq B.$$

Then $\mathcal{Y}$ is a meet semilattice (where $A \wedge B = A \cup B$). The group FG_X acts on $\mathcal{Y}$ by

$$gA = \{gw \mid w \in A\}.$$

Let $\mathcal{X} = \{gA \mid g \in \mathrm{FG}_X, A \in \mathcal{Y}\}$. Then $\mathcal{X}$ is a partially ordered set under the ordering

$$A \leq B \text{ if and only if } A \supseteq B.$$

The action of FG_X on $\mathcal{Y}$ can be extended naturally to an action on $\mathcal{X}$. Then the triple $(\mathrm{FG}_X, \mathcal{X}, \mathcal{Y})$ is a McAlister triple. Furthermore, $\mathrm{FIM}_X \cong \mathcal{M}(\mathrm{FG}_X, \mathcal{X}, \mathcal{Y})$, where FIM_X is the *free inverse monoid* on the set X.

For a proof that FIM_X is isomorphic to $\mathcal{M}(\mathrm{FG}_X, \mathcal{X}, \mathcal{Y})$, see [28, Theorem 5.10.2]. From now on, we will identify FIM_X with $(\mathrm{FG}_X, \mathcal{X}, \mathcal{Y})$. In order to show that FIM_X is completely separable, we will use Green's relation $\mathcal{J}$. We make use of the following lemma.

Lemma 2.3.3. *Let $(A, g) \in \mathrm{FIM}_X$. Then there only exist finitely many $(B, h) \in \mathrm{FIM}_X$ such that $(A, g) \leq_{\mathcal{J}} (B, h)$.*

Proof. If $(A, g) \leq_{\mathcal{J}} (B, h)$, then there exist $(C, t), (D, z) \in \mathrm{FIM}_X$ such that $(C, t)(B, h)(D, z) = (A, g)$. That is $C \cup tB \cup thD = A$ and $thz = g$. As B is a saturated set, $\epsilon \in B$. Hence $t \in tB \subseteq A$. Since A is finite, there are only finitely many choices for t. Similarly, as D is a saturated set, $\epsilon \in D$. Hence $th \in thD \subseteq A$. Given that A is finite and we have already established there are only finitely many choices for t, it must be that there are only finitely many choices for h.

Now, as $tB \subseteq A$, it must be that $B = t^{-1}Z$, where $Z \subseteq A$. Given that A only has finitely many subsets and that there are only finitely many choices for t, we conclude that there can only be finitely many possibilities for B. Putting this together, we get that there are only finitely many possibilities for (B, h), as desired. □

Theorem 2.3.4. *The free inverse monoid* FIM_X *is completely separable.*

Proof. Let $(A, g) \in \mathrm{FIM}_X$. Define

$$I(A, g) = \{(B, h) \in \mathrm{FIM}_X \mid (B, h) \not\geq_{\mathcal{J}} (A, g).\}.$$

Then by Definition 1.3.30, $I(A, g)$ is an ideal and $(A, g) \in \mathrm{FIM}_X \setminus I(A, g)$. Furthermore, by Lemma 2.3.3, $\mathrm{FIM}_X \setminus I(A, g)$ is finite. Therefore the Rees quotient $\mathrm{FIM}_X / I(A, g)$ is finite and $[(A, g)]_{I(A,g)} = \{(A, g)\}$. Hence FIM_X is completely separable. □

2.4 Free Completely Simple Semigroups

The fact that FIM_X is completely separable is somewhat surprising. The idea that the extra structure associated with inversion would mean that FIM_X would behave like FG_X has shown to be wrong. On closer inspection, we can see this expectation was perhaps a little naive. From [28, Proposition 5.9.4, (2) and (3)], we can see that for $(A, g), (B, h) \in \mathrm{FIM}_X$ we have that $(A, g)\mathcal{H}(B, h)$ if and only if $A = B$ and $g^{-1}A = h^{-1}B$. In other words, FIM_X is $\mathcal{H}$-trivial and in particular every subgroup of FIM_X is trivial. Therefore even though FG_X plays an important role in the description of FIM_X, none of the group structure of FG_X is inherited by FIM_X.

In order to identify the point where free objects in semigroup varieties switch from being completely separable to having separability properties similar to that of the free group, we need to take one step closer to the variety of groups. The obvious variety to choose would be that of completely regular semigroups, indeed a completely regular semigroup is a union of groups.

However, the completely regular case proves to be difficult. Therefore we start with two examples of semigroups varieties which are contained within the class of completely regular semigroups and whose separability properties prove relatively easy to determine. The first of these is the variety of completely simple semigroups.

A semigroup S is *simple* if the only ideal of S is S itself. A simple semigroup is called *completely simple* if it is both simple and a union of groups. For equivalent definitions of completely simple semigroups, see [28, Theorems 3.3.2 and Theorems 3.3.3]. Completely simple semigroups form a variety of signature (2,1), with the binary operation of multiplication which will be represented by concatenation of elements, and the unary operation of inversion which will be represented by $^{-1}$. The class of completely simple semigroups satisfies the following identities:

$$(xy)z = x(yz), \quad (x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad xx^{-1} = x^{-1}x,$$
$$xyx(xyx)^{-1} = xx^{-1}.$$

Rees showed that the class of completely simple semigroups coincides with the class of *Rees matrix semigroups over groups* [28, Theorem 3.3.1]. Let G be a group. Let I, Λ be non empty sets and $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries from G. The Rees matrix semigroup over the group G is $S = M[G; I, \Lambda; P] = (I \times G \times \Lambda)$ with multiplication

$$(i, a, \lambda)(j, b, \mu) = (i, ap_{\lambda j}b, \mu),$$

and inversion

$$(i, g, \lambda)^{-1} = (i, p_{\lambda i}^{-1} g^{-1} p_{\lambda i}^{-1}, \lambda).$$

For $i \in I$ and $\lambda \in \Lambda$ the set $\{i\} \times G \times \{\lambda\}$ is a maximal subgroup of $M[G : I, \Lambda; P]$ isomorphic to G.

In [10, Theorem 7.4] Clifford gave the following description of the free completely simple semigroup on a set X.

Definition 2.4.1. Let X be a non-empty set. Let 1 be a distinguished element of X. Let $Q = \{q_x \mid x \in X\} \cup \{p_{yz} \mid y, z \in X \setminus \{1\}\}$. The set Q is a set of formal symbols consisting of two disjoint subsets, one indexed by X and the other indexed by $(X \setminus \{1\}) \times (X \setminus \{1\})$. Let FG_Q be the free group on Q. Define $p_{y1} = p_{1z} = \epsilon$ for all $y, z \in X$ and let P be the $X \times X$ matrix (p_{yz}) over FG_Q. Then $\mathrm{FCS}_X \cong \mathcal{M}(\mathrm{FG}_Q; X, X; P)$, where FCS_X is the *free completely simple semigroup* on the set X. The set $\overline{X} = \{(x, q_x, x) \mid x \in X\}$ forms a free generating set.

We use this description to show that FCS_X is weakly subalgebra separable. Recall that subalgebras are non-empty subsets that are closed under multiplication and inversion and hence are completely simple semigroups. In the proof, we will make use of the following lemma.

Lemma 2.4.2. *A completely simple semigroup $M[G; I, \Lambda, P]$ is finitely generated if and only if G is finitely generated and the sets I and Λ are finite.*

Proof. This follows from [4, Main Theorem]. □

Theorem 2.4.3. *The free completely simple semigroup FCS_X is weakly subalgebra separable.*

Proof. Let $\mathrm{FCS}_X = M[\mathrm{FG}_Q; X, X; P]$. Let U be a finitely generated subalgebra of FCS_X and let $v = (y, w, z) \in \mathrm{FCS}_X \setminus U$. As U is finitely generated, then by Lemma 2.4.2 there exists a finitely generated group G, finite sets I and Λ and a $\Lambda \times I$ matrix M with entries from G such that $U \cong M[G; I, \Lambda; M]$. Then there exists an embedding $\iota : M[G; I, \Lambda; M] \to M[\mathrm{FG}_Q; X, X; P]$. Therefore, we identify $I, \Lambda \subseteq X$, $G \leq_{\mathrm{Gp}} \mathrm{FG}_Q$ and M as the induced submatrix of P obtained by considering only the rows indexed by the set Λ and columns indexed by the set I.

To show that FCS_X is weakly subalgebra separable we will find a finite group H, a finite set Ω, an $\Omega \times \Omega$ matrix A over H and a homomorphism $\phi : M[\mathrm{FG}_Q; X, X; P] \to M[H; \Omega, \Omega; A]$ such that $\phi(v) \notin \phi(U)$.

As U is finitely generated, there exists a finite subset $\overline{\Omega} \subseteq X$ such that $U \cup \{v\} \subseteq \mathrm{FCS}_{\overline{\Omega}}$. Let $\Omega = \overline{\Omega} \cup \{0\}$, where $\{0\}$ is disjoint from $\overline{\Omega}$.

If $U \cap (\{y\} \times \mathrm{FG}_Q \times \{z\}) = \emptyset$, set $H = \{1\}$, the trivial group, and set $\overline{\phi} : \mathrm{FG}_Q \to H$ to be the trivial homomorphism. Otherwise $U \cap (\{y\} \times \mathrm{FG}_Q \times \{z\})$ is a maximal subgroup of U. In this case $U \cap (\{y\} \times \mathrm{FG}_Q \times \{z\})$ is isomorphic to the finitely generated subgroup G of FG_Q.

Let $\psi : \{y\} \times \mathrm{FG}_Q \times \{z\} \to \mathrm{FG}_Q$ be given by $(y, u, z) \mapsto up_{zy}$. We show ψ is an isomorphism. As

$$\psi((y, u, z)(y, v, z)) = \psi(y, up_{zy}v, z) = up_{zy}vp_{zy} = \psi(y, u, z)\psi(y, v, z),$$

we have that ψ is an homomorphism. If $\psi(y, u, z) = \psi(y, v, z)$ then $up_{zy} = vp_{zy}$ and therefore $u = v$, showing that ψ is injective. Finally, for $u \in \mathrm{FG}_Q$ we have $\psi(y, up_{zy}^{-1}, z) = u$, showing that ψ is surjective and indeed a homomorphism.

Therefore $U \cap (\{y\} \times \mathrm{FG}_Q \times \{z\}) = \{y\} \times Gp_{zy}^{-1} \times \{z\}$. Then saying that $v \in \mathrm{FCS}_X \setminus U$ is equivalent to saying that $w \in \mathrm{FG}_Q \setminus Gp_{zy}^{-1}$. As FG_Q is weakly subgroup separable by Theorem 2.1.9 and Corollary 1.2.21, there exists a finite semigroup H and a homomorphism $\overline{\phi} : \mathrm{FG}_Q \to H$ such that $\overline{\phi}(w) \notin \overline{\phi}(Gp_{zy}^{-1})$.

Let A be the $\Omega \times \Omega$ matrix $(a_{\omega\mu})$ with entries from H where

$$a_{\omega\mu} = \begin{cases} \overline{\phi}(p_{\omega\mu}) & \text{if } \omega, \mu \in \overline{\Omega}, \\ 1_H & \text{otherwise.} \end{cases}$$

Let $\phi : M[\mathrm{FG}_Q; I, I; P] \to M[H; \Omega, \Omega; A]$ be the unique extension to a homomorphism of the function given by

$$(x, q_x, x) \mapsto \begin{cases} (x, \overline{\phi}(q_x), x) & \text{if } x \in \overline{\Omega}, \\ (0, 1_H, 0) & \text{otherwise.} \end{cases}$$

By construction, $\phi(\mu, u, \nu) = (\mu, \overline{\phi}(u), \nu)$ for (μ, u, ν) in the subalgebra generated by the set $\{(\omega, q_\omega, \omega) \mid \omega \in \overline{\Omega}\}$. As v is an element of this subalgebra and U is contained in this subalgebra, we can conclude that $\phi(v) \notin \phi(U)$ and that FCS_X is weakly subalgebra separable. □

When $|X| \geq 2$, every maximal subgroup of FCS_X is isomorphic to FG_Y, where $|Y| = |X| + |X - 1|^2$. In particular, FG_Y is not strongly subgroup separable by Lemma 2.1.11. As subgroups are example of subalgebras in the variety of completely simple semigroups, it cannot be the case that FCS_X is strongly subalgebra separable, as otherwise FG_Y would have inherited strong subgroup separability by Proposition 1.2.13.

When $|X| = 1$, it is easy to see that $\mathrm{FCS}_X \cong \mathbb{Z}$. In this case the only subalgebras are subgroups, and therefore FCS_X is strongly subalgebra separable but not completely separable by Lemma 2.1.11. These observations are summarised below. Note that FCS_X is not defined for $X = \emptyset$.

Lemma 2.4.4. *The free completely simple semigroup* FCS_X *is not completely separable and is strongly subalgebra separable if and only if* $|X| = 1$.

Not only do the separability properties of FCS_X mirror those of free groups, they rely heavily upon the separability properties of free groups. In showing that FCS_X is weakly subalgebra separable we utilised the fact the free group is subgroup separable. When FCS_X is not strongly subalgebra separable, it is because it has a subgroup which is isomorphic to a non-strongly subgroup separable free group. This seems to demonstrate that the separability properties of unions of groups are very closely tied to the separability properties of those groups.

2.5 Free Clifford Semigroups

Another variety that lives within the class of completely regular semigroups is that of Clifford semigroups. Whereas all maximal subgroups of a completely simple semigroup are isomorphic, this does not have to be the case with Clifford semigroups. This gives rise to the potential of a more nuanced relationship between the separability properties of a Clifford semigroup and the separability properties of the underlying groups. However, in the case of the free Clifford semigroup, we obtain a similar result to that of the free completely simple semigroup.

Clifford semigroups form a variety of signature (2,1), with the binary operation of multiplication which will be represented by concatenation and the unary map of inversion which will be represented by $^{-1}$. The variety of Clifford semigroups satisfies the following identities:

$$(xy)z = x(yz), \quad xx^{-1}x = x, \quad (x^{-1})^{-1} = x, \quad xx^{-1} = x^{-1}x,$$
$$xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

A semigroup S is a Clifford semigroup if and only if it is a strong semilattice of groups, which we define below. For equivalent definitions of Clifford semigroups see [28, Theorem 4.2.1].

Definition 2.5.1. We say that a semigroup S is a *semilattice of semigroups* if for some semilattice Y, we can write S as a disjoint union of subsemigroups $S = \bigcup_{\alpha \in Y} S_\alpha$, such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for all $\alpha, \beta \in Y$. In this case we write $S = \mathcal{S}[Y; \{S_\alpha\}]$.

Additionally suppose that for all $\alpha \geq \beta \in Y$, there exists a homomorphism $\phi_{\alpha,\beta} : S_\alpha \to S_\beta$ such that

(i) $\phi_{\alpha,\alpha} = 1_{S_\alpha}$ for all $\alpha \in Y$;
(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for all $\alpha \geq \beta \geq \gamma \in Y$.

We define a multiplication on $S = \bigcup_{\alpha \in Y} S_\alpha$. For $x \in S_\alpha$ and $y \in S_\beta$, define

$$xy = \phi_{\alpha,\alpha\beta}(x)\phi_{\beta,\alpha\beta}(y).$$

Under this multiplication S is a semigroup known as a *strong semilattice of semigroups.* In this case we write $S = \mathcal{S}[Y; \{S_\alpha\}; \{\phi_{\alpha,\beta}\}]$.

The following is a description of the free Clifford semigroup on a set X. This is taken from [28, Exercise 5.40]. In the notes at the end of Chapter 5 of [28], Howie attributes this description to Liber in [34].

Definition 2.5.2. Let X be a non-empty set. Let Y be the semilattice of non-empty finite subsets of X under the operation of union. Then Y is the free semilattice on X. For $\alpha \in Y$, let FG_α denote a copy of the free group on α. We will write elements of FG_α as ordered pairs (w, α), where w is an element of the free group on α. A free basis for FG_α is the set $\{(a, \alpha) \mid a \in \alpha\}$. For $\alpha, \beta \in Y$, such that $\alpha \subseteq \beta$ define the group homomorphism $\phi_{\alpha,\beta} : \mathrm{FG}_\alpha \to \mathrm{FG}_\beta$ to be the unique homomorphic extension of the map given by $(a, \alpha) \mapsto (a, \beta)$ for all $a \in \alpha$. Then $\mathrm{FCliff}_X \cong \mathcal{S}[Y; \{\mathrm{FG}_\alpha\}; \{\phi_{\alpha,\beta}\}]$, where FCliff_X is the free Clifford semigroup on X. With this notation, we can see that for $(w, \alpha), (v, \beta) \in \mathrm{FCliff}_X$ we have

$$(w, \alpha)(v, \beta) = (wv, \alpha \cup \beta),$$

and

$$(w, \alpha)^{-1} = (w^{-1}, \alpha).$$

The set $\{(x, \{x\}) \mid x \in X\}$ is a free generating set for FCliff_X.

Theorem 2.5.3. *The free Clifford semigroup* FCliff_X *is weakly subalgebra separable.*

Proof. Let $\mathrm{FCliff}_X = \mathcal{S}[Y; \mathrm{FG}_\alpha; \phi_{\alpha,\beta}]$. Let U be a finitely generated subalgebra of FCliff_X and let $(w, \alpha) \in \mathrm{FCliff}_X \setminus U$. To show that FCliff_X is weakly subalgebra separable we will find: a finite semilattice Z; a family of finite groups $\{H_\mu\}_{\mu \in Z}$; for $\mu \geq \nu \in Z$, a family of homomorphisms $\psi_{\mu,\nu} : H_\mu \to H_\nu$; and a homomorphism $\xi : \mathcal{S}[Y; \{\mathrm{FG}_\alpha\}; \{\phi_{\alpha,\beta}\}] \to \mathcal{S}[Z; \{H_\mu\}; \{\psi_{\mu,\nu}\}]$ such that $\xi(w, \alpha) \notin \xi(U)$.

As U is finitely generated, there exists a non-empty finite set $\overline{\Omega} \subseteq Y$ such that $U \cup \{(w, \alpha)\} \subseteq \mathrm{FCliff}_{\overline{\Omega}}$. Let $\Omega = \overline{\Omega} \cup \{0\}$, where $\{0\}$ is disjoint from $\overline{\Omega}$. Let Z be the free semilattice on Ω. That is, let Z be the set of all non-empty (finite) subsets of Ω under union. Note that $|Z| = 2^{|\Omega|} - 1$.

If $G = U \cap \mathrm{FG}_\alpha = \emptyset$, then for $\mu \in Z$ set $H_\mu = \{1\}$, the trivial group. Then for $\mu \geq \nu \in Z$ set $\psi_{\mu,\nu}$ to be the identity map. This family of homomorphisms certainly satisfies conditions (i) and (ii) from definition Definition 2.5.1.

Otherwise G is non-empty. In this case G is a finitely generated subgroup of FG_α. Indeed, if V is a finite generating set for U, then G has generating set

$$\overline{V} = \{(v,\alpha) \mid \text{ there exists } \beta \geq \alpha \in Y \text{ such that } (v,\beta) \in V\}.$$

To show that $\overline{V}$ is generating set for G first assume that $(u,\alpha) \in G$. Then

$$(u,\alpha) = (u_1,\beta_1)^{\delta_1} \dots (u_n,\beta_n)^{\delta_n},$$

where for $1 \leq i \leq n$ we have $(u_i,\beta_i) \in V$ and $\delta_i \in \{\pm 1\}$. Furthermore, it must be the case that in the semilattice Y, $\alpha = \beta_1 \dots \beta_n$. In particular, $\beta_i \geq \alpha$ for $1 \leq i \leq n$. Hence

$$(u,\alpha) = (u_1,\alpha)^{\delta_1} \dots (u_n,\alpha)^{\delta_n}$$

and $(u,\alpha) \in \mathrm{Cliff}\langle\overline{V}\rangle$, the subalgebra generated by $\overline{V}$. That is, $G \subseteq \mathrm{Cliff}\langle\overline{V}\rangle$.

Now assume that $(v,\alpha) \in \mathrm{Cliff}\langle\overline{V}\rangle$. Then there exist $(v_1,\gamma_1),\dots,(v_m,\gamma_m) \in V$ such that $v = v_1^{\eta_1} \dots v_m^{\eta_m}$, where for $1 \leq i \leq m$ we have $\eta_i \in \{\pm 1\}$ and $\gamma_i \geq \alpha$. As G is non-empty, there exists some $(v',\alpha) \in G$. Then

$$(v,\alpha) = (v_1,\gamma_1)^{\eta_1} \dots (v_m,\gamma_m)^{\eta_m}(v',\alpha)(v',\alpha)^{-1},$$

and $(v,\alpha) \in G$. Hence $\mathrm{Cliff}\langle\overline{V}\rangle \subseteq G$, completing the proof that G is a finitely generated group.

As $\alpha \subseteq \Omega$, we can realise G as a finitely generated subsemigroup of FG_Ω such that $w \in \mathrm{FG}_\Omega \setminus G$. Then as FG_Ω is weakly subgroup separable by Theorem 2.1.9, there exists a finite group H and a homomorphism $\sigma : \mathrm{FG}_\Omega \to H$ such that $\sigma(w) \notin \sigma(G)$. Let $Z = \{0,1\}$ be the two element semilattice with identity element 1 and zero element 0. Let $H_1 = H$, let $H_0 = \{e\}$ be the trivial group and let $\psi_{1,0} : H_1 \to H_0$ be the trivial homomorphism. Then it is clear that $\mathcal{S}[Z;\{H_\mu\};\phi_{\mu,\nu}]$ is isomorphic to H with a zero element adjoined. Define $\xi : \mathcal{S}[Y;\{\mathrm{FG}_\alpha\};\{\phi_{\alpha,\beta}\}] \to \mathcal{S}[Z;\{H_\mu\};\{\psi_{\mu,\nu}\}]$ by

$$(z,\delta) \to \begin{cases} (\sigma(z),1) & \text{if } \delta \geq \alpha, \\ (e,0) & \text{otherwise.} \end{cases}$$

Now if $(z,\delta) \in U$ and $\delta \geq \omega$, then $(z,\alpha) \in U$. This follows as $U \cap \mathrm{FG}_\alpha \neq \emptyset$, and so for any $(v',\alpha) \in U$, we have that $(z,\alpha) = (z,\delta)(v',\alpha)(v',\alpha)^{-1}$. Using this observation, we can see that ξ is a homomorphism which separates (w,α) from U as desired. □

When $|X| \geq 2$, FCliff_X contains a subgroup isomorphic to FG_X. In particular, FG_X is not strongly subgroup separable by Lemma 2.1.11. As subgroups are example of subalgebras in the variety of Clifford semigroups, it cannot be the case that FCliff_X is strongly subalgebra separable, as otherwise FG_X would have inherited strong subgroup separability by Proposition 1.2.13.

When $|X| = 1$, it is easy to see that $\mathrm{FCliff}_X \cong \mathbb{Z}$. In this case the only subalgebras are subgroups, and therefore FCliff_X is strongly subalgebra separable but not completely separable by Lemma 2.1.11. These observations are summarised below. Note that FCliff_X is not defined for $X = \emptyset$.

Lemma 2.5.4. *The free Clifford semigroup* FCliff_X *is not completely separable and is strongly subalgebra separable if and only if* $|X| = 1$.

2.6 Free Completely Regular Semigroups

We now come to the variety of completely regular semigroups. A completely regular semigroup is a union of groups and so it seems most likely that the behaviour of free objects in this variety will closely match that of free groups. However, even giving a description of the free completely regular semigroup proves to be difficult. We present a partial description of the free completely regular semigroup on a set of size two to demonstrate its complexity. In order to understand the separability properties of free completely regular semigroups, we define a new variety of semigroups, which we call α-groups.

The definition was chosen so that the free objects in this new variety mimic some of the behaviour of free completely regular semigroups. We are able to show that the free α-groups are monogenic subalgebra separable. However, it remains an open question if the free objects are weakly subalgebra separable. Even with this result, it remains an open question if free completely regular semigroups are monogenic subalgebra separable. The title of this section is somewhat of a misnomer, as most of the section is dedicated to α-groups and not free completely regular semigroups. However, it is hoped that α-groups will provide a means to understanding free completely regular semigroups in the future.

Completely regular semigroups form a variety of signature $(2, 1)$ with the binary operation of multiplication which will be represented by concatenation of elements, and the unary operation of inversion which will be represented by $^{-1}$. The class of of completely regular semigroups satisfies the following identities:

$$(xy)z = x(yz), \quad (x^{-1})^{-1} = x, \quad xx^{-1}x = x, \quad xx^{-1} = x^{-1}x.$$

Observe that both completely simple semigroups and Clifford semigroups satisfy these identities and so these classes are contained within the class of completely regular semigroups. For more on completely regular semigroups, including a proof that they are a union of groups, see [28, Chapter 4].

For a non-empty set X, let FCR_X denote the free completely regular semigroup on X. When $|X| = 1$, it is the case that $\mathrm{FCR}_X \cong \mathbb{Z}$. However, when $|X| \geq 2$ the situation becomes increasingly more complex. Indeed, Clifford dedicates the majority of [10] to giving a description of FCR_X for $|X| = 2$. Here, we provide a partial summary of that description, highlighting the underlying structure without specifying the multiplication.

For $|X| = 2$, the semigroup FCR_X is a semilattice of completely simple semigroups. The underlying semilattice is the free semilattice on a set of size two. The completely simple semigroups in the two maximal positions of this semilattice are copies of the integers, which we denote by $\mathbb{Z}$ and $\overline{\mathbb{Z}}$. The remaining completely simple semigroup is $M[\mathrm{FG}_Y; \mathbb{Z} \cup \overline{\mathbb{Z}}, \mathbb{Z} \cup \overline{\mathbb{Z}}; P]$, where Y is a countably infinite set. Clifford gives a formula for the entries of the matrix P, and then ten equations to describe the multiplication between the three completely simple subsemigroups.

From the underlying semilattice structure, we can see that the two copies of the integers are acting on the semigroup $M[\mathrm{FG}_Y; \mathbb{Z} \cup \overline{\mathbb{Z}}, \mathbb{Z} \cup \overline{\mathbb{Z}}; P]$. The semigroup $M[\mathrm{FG}_Y; \mathbb{Z} \cup \overline{\mathbb{Z}}, \mathbb{Z} \cup \overline{\mathbb{Z}}; P]$ is difficult to get a handle on, but it is built from copies of FG_Y, the free group on a countably infinite set. Therefore the variety of α-groups will be designed so that the free objects will contain a copy of FG_Y, with the free cyclic group acting upon it.

It is already known that FCR_X is residually finite [44, Theorem 3.12]. From the limited description above we can establish precisely when FCR_X is strongly subalgebra separable and completely separable. When $|X| \geq 2$, FCR_X contains a copy of FG_Y, the free group on a countable set. As subgroups are examples of subalgebras (i. e. subsets which are closed under multiplication and inversion), and FG_Y is not strongly subgroup separable by Lemma 2.1.11, it cannot be that FCR_X is strongly subalgebra separable.

When $|X| = 1$, we have that $\mathrm{FCR}_X \cong \mathbb{Z}$. In this case the only subalgebras are subgroups, and therefore FCR_X is strongly subalgebra separable but not completely separable by Lemma 2.1.11. These observations are summarised below.

Lemma 2.6.1. *For any non-empty X, the free completely regular semigroup* FCR_X *is not completely separable. Furthermore,* FCR_X *is strongly subalgebra separable if and only if $|X| = 1$.*

So the separability properties of FCR_X left to determine are those of monogenic-subalgebra separability and weak subalgebra separability.

2.6.1 Free α-groups

The class of α-groups is defined as a variety of signature $(2, 1, 1, 1, 0)$. The binary operation is that of multiplication which will be represented by concatenation of elements. The first unary operation is inversion which will be represented by $^{-1}$. The next unary operation will be represented by σ, with $\sigma(x)$ meaning σ applied to the element x. The final unary operation will be represented by σ^{-1}, again with $\sigma^{-1}(x)$ meaning σ^{-1} applied to x. Finally, the nullary operation is identity, represented by the element 1.

An α-group satisfies the identities for a group, i.e.

$$(xy)z = x(yz), \quad (x^{-1})^{-1} = x, \quad xx^{-1} = x^{-1}x = 1, \quad 1x = x1 = x.$$

Additionally it also satisfies the following identities:

$$\sigma(\sigma^{-1}(x)) = \sigma^{-1}(\sigma(x)) = x, \quad \sigma(xy) = \sigma(x)\sigma(y).$$

These identities guarantee that σ and σ^{-1} are mutually inverse automorphisms of the underlying group. To see this, first assume that $\sigma(x) = \sigma(y)$. Then

$$x = \sigma^{-1}(\sigma(x)) = \sigma^{-1}(\sigma(y)) = y.$$

This shows that σ is injective. Similarly we have that σ^{-1} is injective. For any x, we have that $\sigma(\sigma^{-1}(x)) = x$, which shows that σ is surjective. Similarly we have that σ^{-1} is surjective. As $\sigma(\sigma^{-1}(x)) = \sigma^{-1}(\sigma(x)) = x$, it must be that σ and σ^{-1} are mutually inverse bijections. Finally, as $\sigma(xy) = \sigma(x)\sigma(y)$, we have that σ, and therefore σ^{-1}, are automorphisms. With this observation, any α-group can be given by a pair (G, σ), where G is a group and $\sigma : G \to G$ is an automorphism of G. This explains the choice of the name α-group, with α chosen to represent the fact that each of these algebras is a group with an associated automorphism.

Our aim is to understand the separability properties of free α-groups. Before we can do this, we must first understand α-group homomorphisms. Let (G, σ) and (H, ψ) be α-groups and let $\phi : G \to H$. For ϕ to be an α-group homomorphism, it it necessary that ϕ is a group homomorphism. Additionally, we need that $\phi \circ \sigma = \psi \circ \phi$. That is, the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{\sigma} & G \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ H & \xrightarrow{\psi} & H \end{array}$$

We now give a description of free α-groups.

Definition 2.6.2. For some non-empty set $X = \{x_i \mid i \in I\}$, where I is an index set, let $\overline{X} = \{x_{(i,z)} \mid i \in I, z \in \mathbb{Z}\}$. Let $\mathrm{FG}_{\overline{X}}$ be the free group on $\overline{X}$ and let $\sigma : \mathrm{FG}_{\overline{X}} \to \mathrm{FG}_{\overline{X}}$ be the unique extension to a homomorphism of the map given by $x_{(i,z)} \mapsto x_{(i,z+1)}$. As the restriction $\sigma|_{\overline{X}} : \overline{X} \to \overline{X}$ of σ to the basis set $\overline{X}$ is both injective and surjective, then σ itself is an automorphism of $\mathrm{FG}_{\overline{X}}$. Hence $(\mathrm{FG}_{\overline{X}}, \sigma)$ is an α-group. Furthermore, $(\mathrm{FG}_{\overline{X}}, \sigma)$ is free on the set $\{x_{(i,0)} \mid i \in I\}$. Note that $|\{x_{(i,0)} \mid i \in I\}| = |X|$.

Theorem 2.6.3. *The α-group $(\mathrm{FG}_{\overline{X}}, \sigma)$ is free on the set $\{x_{(i,0)} \mid i \in I\}$.*

Proof. Let $Y = \{x_{(i,0)} \mid i \in I\}$. For Y to be a free basis for $(\mathrm{FG}_{\overline{X}}, \sigma)$, it must be that for any α-group (G, ψ) and any function $\phi : Y \to (G, \psi)$, ϕ can be extended uniquely to an homomorphism $\overline{\phi} : (\mathrm{FG}_{\overline{X}}, \sigma) \to (G, \psi)$.

Let (G, ψ) be an α-group and let $\phi : Y \to (G, \psi)$ be a function. For ϕ to be extended to a homomorphism, it must be the case that $\phi(\sigma(x)) = \psi(\phi(x))$ for $x \in X$. That is, any extension of ϕ to a homomorphism must also be an extension to the map $x_{(i,z)} \mapsto \psi^z(\phi(x_{(i,0)}))$. This map is defined on the group basis $\overline{X}$, so there is a unique extension to a group homomorphism $\overline{\phi} : \mathrm{FG}_{\overline{X}} \to G$. This group homomorphism has the property that $\overline{\phi}(x_{(i,z+1)}) = \psi(\overline{\phi}(x_{(i,z)}))$. To check that $\overline{\phi}$ is also an α-group homomorphism, we need to check that for all $w \in \mathrm{FG}_{\overline{X}}$ we have $\overline{\phi}(\sigma(w)) = \psi(\overline{\phi}(w))$.

Let $w = x^{\delta_1}_{(i_1,z_1)} x^{\delta_2}_{(i_2,z_2)} \dots x^{\delta_n}_{(i_n,z_n)}$, where for $1 \leq j \leq n$ we have $i_j \in I$, $z_j \in \mathbb{Z}$

and $\delta_j \in \{\pm 1\}$. Then $\sigma(w) = x^{\delta_1}_{(i_1,z_1+1)} x^{\delta_2}_{(i_2,z_2+1)} \dots x^{\delta_n}_{(i_n,z_n+1)}$. So

$$\begin{aligned}
\overline{\phi}(\sigma(w)) &= \overline{\phi}(x_{(i_1,z_1+1)})^{\delta_1} \overline{\phi}(x_{(i_2,z_2+1)})^{\delta_2} \dots \overline{\phi}(x_{(i_n,z_n+1)})^{\delta_n} \\
&= \psi(\overline{\phi}(x_{(i_1,z_1)})^{\delta_1}) \psi(\overline{\phi}(x_{(i_2,z_2)})^{\delta_2}) \dots \psi(\overline{\phi}(x_{(i_n,z_n)})^{\delta_n}) \\
&= \psi((\overline{\phi}(x_{(i_1,z_1)})^{\delta_1})(\overline{\phi}(x_{(i_2,z_2)})^{\delta_2}) \dots (\overline{\phi}(x_{(i_n,z_n)})^{\delta_n})) \\
&= \psi((\overline{\phi}(x^{\delta_1}_{(i_1,z_1+1)} x^{\delta_2}_{(i_2,z_2+1)} \dots x^{\delta_n}_{(i_n,z_n+1)}))) \\
&= \psi(\overline{\phi}(w)),
\end{aligned}$$

as desired. □

Therefore we call $(\mathrm{FG}_{\overline{X}}, \sigma)$ the *free α-group on the set X*, and denote it by $\mathrm{F}\alpha\mathrm{G}_X$. Note that the underlying group of $\mathrm{F}\alpha\mathrm{G}_X$ is the free group $\mathrm{FG}_{\overline{X}}$. When X is countable, this is the free group on an countably infinite set. Furthermore, the automorphism σ can be realised as an action of $\mathbb{Z}$ on $\mathrm{FG}_{\overline{X}}$, being the extension of the mapping $x_{(i,z)} \cdot 1 = x_{(i,z+1)}$. Hence $\mathrm{F}\alpha\mathrm{G}_X$ captures the structure of FCR_X that we wished to simulate.

We now turn to the combinatorics of $\mathrm{F}\alpha\mathrm{G}_X$, which proves to be an important tool in understanding the separability properties of $\mathrm{F}\alpha\mathrm{G}_X$.

Lemma 2.6.4. *Let $w \in \mathrm{F}\alpha\mathrm{G}_X$, let $\eta_1, \eta_2 \in \{\pm 1\}$ and let $k \in \mathbb{Z} \setminus \{0\}$. Then*

$$|w^{\eta_1} \sigma^k (w^{\eta_2})| \geq |w|.$$

Proof. Let $w = x^{\delta_1}_{(i_1,z_1)} x^{\delta_2}_{(i_2,z_2)} \dots x^{\delta_n}_{(i_n,z_n)}$ be a reduced word in $\mathrm{FG}_{\overline{X}}$, where for $1 \leq j \leq n$ we have $i_j \in I$, $z_j \in \mathbb{Z}$ and $\delta_j \in \{\pm 1\}$. Then $\sigma^k(w) = x^{\delta_1}_{(i_1,z_1+k)} x^{\delta_2}_{(i_2,z_2+k)} \dots x^{\delta_n}_{(i_n,z_n+k)}$.

Case 1: First we assume that $n = 2m+1$ is odd and that $\eta_1 = \eta_2 = 1$. Then

$$w^{\eta_1} \sigma^k (w^{\eta_2}) = x^{\delta_1}_{(i_1,z_1)} x^{\delta_2}_{(i_2,z_2)} \dots x^{\delta_n}_{(i_n,z_n)} x^{\delta_1}_{(i_1,z_1+k)} x^{\delta_2}_{(i_2,z_2+k)} \dots x^{\delta_n}_{(i_n,z_n+k)}.$$

We argue that $x^{\delta_{m+1}}_{(i_{m+1},z_{m+1})}$ is not cancelled. Indeed, if it were then for $1 \leq j \leq m+1$ we would have that $\delta_j = -\delta_{n-(j-1)}$, $i_j = i_{n-(j-1)}$ and $z_1 + k = z_n$.

But this implies that $\delta_{m+1} = -\delta_{m+1}$, which is not possible. Hence less than half of either w^{η_1} and $\sigma(w^{\eta_2})$ cancel, and the result holds.

Case 2: The case when n is odd and $\eta_1 = \eta_2 = -1$ works in a similar manner to Case 1.

Case 3: Now assume that n is odd and that $\eta_1 = 1$ and $\eta_2 = -1$. Then

$$w^{\eta_1}\sigma^k(w^{\eta_2}) = x^{\delta_1}_{(i_1,z_1)}x^{\delta_2}_{(i_2,z_2)}\dots x^{\delta_n}_{(i_n,z_n)}x^{-\delta_n}_{(i_n,z_n+k)}x^{-\delta_{n-1}}_{(i_{n-1},z_{n-1}+k)}\dots x^{-\delta_1}_{(i_1,z_1+k)}.$$

Cancellation can only occur if $z_n = z_n + k$. But as $k \neq 0$ this is not possible and so no cancellation occurs. Hence the result holds.

Case 4: The case when n is odd and $\eta_1 = -1$ and $\eta_2 = 1$ works in a similar manner to Case 2.

Case 5: Now assume that $n = 2m$ is even and that $\eta_1 = \eta_2 = 1$. Then

$$w^{\eta_1}\sigma^k(w^{\eta_2}) = x^{\delta_1}_{(i_1,z_1)}x^{\delta_2}_{(i_2,z_2)}\dots x^{\delta_n}_{(i_n,z_n)}x^{\delta_1}_{(i_1,z_1+k)}x^{\delta_2}_{(i_2,z_2+k)}\dots x^{\delta_n}_{(i_{n-1},z_n+k)}.$$

We argue that $x^{\delta_m}_{(i_m,z_m)}$ is not cancelled. Indeed, if it were cancelled then for $1 \leq j \leq m+1$ we have $\delta_j = -\delta_{n-(j-1)}$, $i_j = i_{n-(j-1)}$ and $z_j + k = z_{n-(j-1)}$. But then $z_m + k = z_{m+1}$ and $z_{m+1} + k = z_m$. Hence $k = 0$, which is a contradiction. Hence at most half of w_1^η and $\sigma^k(w_2^\eta)$ cancel and the result holds.

Case 6: The case when n is even and $\eta_1 = \eta_2 = -1$ works in a similar manner to Case 5.

Case 7 and 8: The case when n is even and $\eta_1 = -\eta_2$ works in a similar manner to Case 3. □

Corollary 2.6.5. *Let $w \in \mathrm{F}\alpha\mathrm{G}_X \setminus \{\epsilon\}$. Then there exists a decomposition of $w = vu\sigma^k(v^{-1})$, where $v, u \in \mathrm{F}\alpha\mathrm{G}_X$, $k \in \mathbb{Z}$ and there is no cancellation between v and u nor between u and $\sigma^k(v^{-1})$.*

Proof. Let $w = x^{\delta_1}_{(i_1,z_1)}x^{\delta_2}_{(i_2,z_2)}\dots x^{\delta_n}_{(i_n,z_n)}$ be a reduced word in $\mathrm{FG}_{\overline{X}}$, where for $1 \leq j \leq n$ we have $i_j \in I$, $z_i \in \mathbb{Z}$ and $\delta_i \in \{\pm 1\}$. If $\delta_n = \delta_1$ then the only option for v is $v = \epsilon$, and so we take $u = w$.

Otherwise, consider the case when $\delta_n = -\delta_1$. If $i_1 \neq i_n$, then again the only option for v is $v = \epsilon$, and so we take $u = w$. Then we are left with the case that $i_1 = i_n$. We first deal with the case that $z_n = z_1$. In this case $k = 0$. Let j be maximal index such that $(x^{\delta_1}_{(i_1,z_1)} \dots x^{\delta_j}_{(i_j,z_j)})^{-1} = x^{\delta_{n-(j-1)}}_{(i_{n-(j-1)},z_{n-(j-1)})} \dots x^{\delta_n}_{(i_n,z_n)}$. We know that $j \geq 1$ and as $w \neq \epsilon$, we have that $j < \frac{n}{2}$. Let $v = x^{\delta_1}_{(i_1,z_1)} \dots x^{\delta_j}_{(i_j,z_j)}$. Then we take u to be $x^{\delta_{j+1}}_{(i_{j+1},z_{j+1})} \dots x^{\delta_{n-j}}_{(i_{n-j},z_{n-j})}$. In this case we have that $u \neq \epsilon$. Hence we have u and v satisfying the statement and we have chosen v to be as long as possible.

Finally, let $z_n - z_1 = k \neq 0$. Then there is cancellation between w and $\sigma^k(w^{-1})$. Lemma 2.6.4 tells us that the number of cancelling pairs is at most $\frac{n}{2}$. Suppose there are j cancelling pairs. Set v to be the prefix of w of length j and u to be the remaining middle section as we did above. Note that in this case, we can have that $u = \epsilon$. Again we have u and v satisfying the statement and we have chosen v to be as long as possible. □

Our aim is to show that $\mathrm{F}\alpha\mathrm{G}_X$ is monogenic subalgebra separable. Separation involves mapping into some finite α-group (G, τ). As τ is an automorphism of a finite group, it has finite order. With this in mind, we define a family of infinite α-groups, where the associated automorphisms have finite order. It turns out that any α-group homomorphism from $\mathrm{F}\alpha\mathrm{G}_X$ into a finite α-group will have to factor through one of these infinite α-groups.

Definition 2.6.6. For a non-empty set $X = \{x_i \mid i \in I\}$, where I is some index set, let $X_n = \{a_{(i,k)} \mid i \in I, k \in \{0, 1, \dots, n-1\}\}$. Let FG_{X_n} be the free group on X_n. Let ψ_n be the unique extension to a homomorphism of the map $a_{(i,k)} \mapsto a_{(i,(k+1) \bmod n)}$. Then as ψ_n induces a permutation of the basis of FG_{X_n}, ψ_n is an automorphism. Hence $(\mathrm{FG}_{X_n}, \psi_n)$ is an α-group.

The following lemma asserts that any α-group homomorphism from $\mathrm{F}\alpha\mathrm{G}_X$ to a finite α-group has to factor through $(\mathrm{FG}_{X_n}, \psi_n)$ for some $n \in \mathbb{N}$.

Lemma 2.6.7. *Let $\xi : \mathrm{F}\alpha\mathrm{G}_X \to (G, \tau)$ be an α-group homomorphism into a finite α-group where the order of τ is n. Let $\phi_n : \mathrm{F}\alpha\mathrm{G}_X \to (\mathrm{FG}_{X_n}, \psi_n)$*

be the unique extension to an α-group homomorphism of the map given by $x_{(i,0)} \mapsto a_{(i,0)}$ where $i \in I$. Then there exists an α-group homomorphism $\overline{\xi} : (\mathrm{FG}_{X_n}, \psi_n) \to (G, \tau)$ such that $\xi = \overline{\xi} \circ \phi_n$.

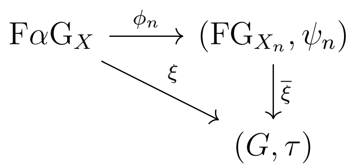
Proof. The homomorphism ξ is completely determined by where it sends the set $\{x_{(i,0)} \mid i \in I\}$. For $i \in I$, let $g_i = \xi(x_{(i,0)})$. Then define $\overline{\xi} : (\mathrm{FG}_{X_n}, \psi_n) \to (G, \tau)$ to be the unique extension to a group homomorphism of the map $a_{(i,j)} \mapsto \tau^j(g_i)$, where $i \in I$ and $0 \leq j \leq n-1$. A unique extension exists as $\{a_{(i,j)} \mid i \in I, 1 \leq j \leq n-1\}$ is a basis for the free group FG_{X_n}. To show that $\overline{\xi}$ is an α-group homomorphism we need to show that $\overline{\xi} \circ \psi_n = \tau \circ \overline{\xi}$. It is sufficient to show this for the group generating set $\{a_{(i,j)} \mid i \in I, 0 \leq j \leq n-1\}$. Let $i \in I$ and $0 \leq j \leq n-1$. Then

$$\begin{aligned}
\overline{\xi}(\psi_n(a_{(i,j)})) &= \overline{\xi}(a_{(i,(j+1) \bmod n)}) \\
&= \tau^{j+1}(g_i) \\
&= \tau(\tau^j(g_i)) \\
&= \tau(\overline{\xi}(a_{(i,j)})),
\end{aligned}$$

as desired. Finally, we note that for $i \in I$ we have

$$\xi(x_{(i,0)}) = g_i = \overline{\xi}(a_{(i,0)}) = \overline{\xi}(\phi_n(x_{(i,0)})).$$

Then $\xi = \overline{\xi} \circ \phi_n$. That is, the following diagram commutes.



□

We make use of this factorisation when showing that $\mathrm{F}\alpha\mathrm{G}_X$ is monogenic subalgebra separable. For this to work, we need $(\mathrm{FG}_{X_n}, \psi_n)$ to be monogenic subalgebra separable. We are able to go one better, as the following theorem demonstrates.

Theorem 2.6.8. *The α-group (FG_{X_n}, ψ_n) is weakly subalgebra separable.*

Proof. Let H be a subalgebra of $(\mathrm{FG}_{X_n}, \psi_n)$ finitely generated by the set $W = \{w_1, w_2, \cdots, w_k\}$. Note that H is finitely generated as a subgroup of FG_{X_n}. Indeed, as H is the closure of W under multiplication, inversion and the unary operations ψ_n and ψ_n^{-1}, and ψ_n is an automorphism of finite order, it follows that $W \cup \psi_n(W) \cup \psi_n^2(W) \ldots \psi_n^{n-1}(W)$ is a group generating set for H. Let $u \in (\mathrm{FG}_{X_n}, \psi_n) \setminus H$. As FG_{X_n} is weakly subgroup separable by Theorem 2.1.9, there exists a finite group G and a group homomorphism $\phi : \mathrm{FG}_{X_n} \to G$ such that $\phi(u) \notin \phi(H)$. We will use this to build a finite α-group in which u can be separated from H.

Consider the finite group $G^n = \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}$. The mapping $\sigma_n : G^n \to G^n$ given by $(g_1, g_2, \ldots, g_n) \mapsto (g_2, g_3, \ldots, g_n, g_1)$ is an automorphism of G^n. To show this, let $g = (g_1, g_2, \ldots, g_n)$ and $h = (h_1, h_2, \ldots, h_n) \in G^n$. If $\sigma_n(g) = \sigma_n(h)$, then for $1 \leq j \leq n$ we have $g_i = h_i$ and therefore $g = h$. Hence σ_n is injective. Note that $\sigma_n(g_n, g_1, \ldots, g_{n-1}) = g$ and so σ_n is surjective. Finally,

$$\begin{aligned}\sigma_n(gh) &= \sigma_n(g_1h_1, g_2h_2, \ldots, g_nh_n)\\ &= (g_2h_2, g_3h_3, \ldots, g_1h_1)\\ &= (g_2, g_2, \ldots, g_1)(h_2, h_3, \ldots, h_1)\\ &= \sigma_n(g)\sigma_n(h)\end{aligned}$$

and therefore σ_n is a homomorphism and so also an automorphism. Hence (G^n, σ_n) is a finite α-group.

Let $\phi(a_{(i,k)}) = b_{(i,k)} \in G$. Define

$$c_{(i,k)} = (b_{(i,k \bmod n)}, b_{(i,(k+1) \bmod n)}, \ldots, b_{(i,(k+n-1) \bmod n)}) \in G^n.$$

Note that $\sigma_n(c_{(i,k)}) = c_{(i,(k+1) \bmod n)}$. Define $\overline{\phi} : (\mathrm{FG}_{X_n}, \psi_n) \to (G^n, \sigma_n)$ by the unique extension to a group homomorphism of the map $a_{(i,k)} \mapsto c_{(i,k)}$. A unique extension exists as FG_{X_n} is a free group. To confirm that that $\overline{\phi}$ is an α-group homomorphism we need to check that for all $w \in (\mathrm{FG}_{X_n}, \psi_n)$ we have $\overline{\phi}(\psi_n(w)) = \sigma_n(\overline{\phi}(w))$. It is sufficient to check that this holds for some generating set of $(\mathrm{FG}_{X_n}, \psi_n)$. The set $\{a_{(i,0)} \mid i \in I\}$ generates $(\mathrm{FG}_{X_n}, \psi_n)$. Then

$$\begin{aligned}\overline{\phi}(\psi_n(a_{(i,0)})) &= \overline{\phi}(a_{(i,1 \bmod n)}) \\ &= c_{(i,1 \bmod n)} \\ &= \sigma_n(c_{(i,0)}) \\ &= \sigma_n(\overline{\phi}(a_{(i,0)}))\end{aligned}$$

as desired. Hence $\overline{\phi}$ is an α-group homomorphism.

Finally, to see that $\overline{\phi}(u) \notin \overline{\phi}(H)$ we consider the projection $\pi_1 : G^n \to G$ given by $(g_1, g_2, \ldots, g_n) \mapsto g_1$. By construction we have $\phi = \pi_1 \circ \overline{\phi}$. So it cannot be the case that $\overline{\phi}(u) \in \overline{\phi}(H)$, else $\phi(u) \in \phi(H)$ which is a contradiction. Hence $(\mathrm{FG}_{X_n}, \psi_n)$ is weakly subalgebra separable. □

We are now ready for the main result of this subsection.

Theorem 2.6.9. *The free α-group $\mathrm{F}\alpha\mathrm{G}_X$ is monogenic subalgebra separable.*

Proof. Let H be a monogenic subalgebra of $\mathrm{F}\alpha\mathrm{G}_X$ generated by the set $\{w\}$ and let $y \in \mathrm{F}\alpha\mathrm{G}_X \setminus H$. Let $\phi_n : \mathrm{F}\alpha\mathrm{G}_X \to (\mathrm{FG}_{X_n}, \psi_n)$ be the unique extension to a α-group homomorphism given by $x_{(i,0)} \mapsto a_{(i,0)}$, where $i \in I$. By Lemma 2.6.7 and Theorem 2.6.8 it is sufficient to find an $N \in \mathbb{N}$ such that $\phi_N(y) \notin \phi_N(H)$. If we can find such an N we can factor through $(\mathrm{FG}_{X_N}, \psi_N)$ and use the weak subalgebra separability of $(\mathrm{FG}_{X_N}, \psi_N)$.

Case 1: The first case is when $w = \epsilon$. In this case we need to find $N \in \mathbb{N}$ such that $\phi_N(y) \neq \epsilon$. Let

$$y = x^{\delta_1}_{(b_1,s_1)} x^{\delta_2}_{(b_2,s_2)} \ldots x^{\delta_p}_{(b_p,s_p)}$$

be a reduced word in $\mathrm{FG}_{\overline{X}}$, where for $1 \leq j \leq p$ we have $b_j \in I$, $s_j \in \mathbb{Z}$ and $\delta_j \in \{\pm 1\}$. Furthermore, we may assume that $s_j \geq 0$ for $1 \leq j \leq p$. If not, there exists some $n \in \mathbb{N}$ such that $s_j + n \geq 0$ for $1 \leq j \leq p$. We can separate

y from H if and only if we can separate $\sigma^n(y)$ from H. To show this assume that we cannot separate $\sigma^n(y)$ from H. Then for every finite α-group (G, ψ) and α-group homomorphism $\phi : \mathrm{F}\alpha\mathrm{G}_X \to (G, \psi)$, there exists $h \in H$ such that $\phi(h) = \phi(\sigma^n(y)) = \psi^n(\phi(y))$. But as H is invariant under σ and σ^{-1} we have that $\sigma^{-n}(h) \in H$. Then

$$\phi(\sigma^{-n}(h)) = \psi^{-n}(\phi(h)) = \psi^{-n}(\psi^n(\phi(y)) = \phi(y)$$

and y cannot be separated from H. A similar argument shows that if y cannot be separated from H then neither can $\sigma^n(H)$.

Let $m_1 = \max\{s_j \mid 1 \leq j \leq p\}$. Let $N > m_1$. Then

$$\phi_N(y) = a_{(b_1,s_1)}^{\delta_1} a_{(b_2,s_2)}^{\delta_2} \dots a_{(b_p,s_p)}^{\delta_p}.$$

In particular, $|\phi_N(y)| = p > 0$ and so $\phi_N(y) \neq \epsilon$ as desired. Note that this shows that $\mathrm{F}\alpha\mathrm{G}_X$ is residually finite. This follows as $\mathrm{F}\alpha\mathrm{G}_X$ can be viewed as a group, and for a group to be residually finite it is sufficient that every non-identity element can be separated from the identity. We have shown that for $\mathrm{F}\alpha\mathrm{G}_X$, this separation can also occur in a finite α-group.

Now we will deal with the case when $w \neq \epsilon$. By Corollary 2.6.5 $w = vu\sigma^k(v^{-1})$, and here we choose v to be as long as possible.

Case 2: We will deal with the case when $|u| \geq 1$. The proof for this case is organised as follows.

(1) Establish a constant m.
(2) For $n > m$, give a group basis B_n for $\phi_n(H)$.
(3) Show that there exists $N > m$ such that $\phi_N(y) \notin \phi_N(H)$.

(1) Let

$$v = x_{(c_1,t_1)}^{\eta_1} x_{(c_2,t_2)}^{\eta_2} \dots x_{(c_q,t_q)}^{\eta_q},$$

be a reduced word in $\mathrm{FG}_{\overline{X}}$, where for $1 \leq j \leq q$ we have $c_j \in I$, $t_j \in \mathbb{Z}$ and $\eta_j \in \{\pm 1\}$. Note that v could be empty. Let

$$u = x_{(d_1,z_1)}^{\kappa_1} x_{(d_2,z_2)}^{\kappa_2} \dots x_{(d_r,z_r)}^{\kappa_r},$$

be a reduced word in $\mathrm{FG}_{\overline{X}}$, where for $1 \leq j \leq r$ we have $d_j \in I$, $z_j \in \mathbb{Z}$ and $\kappa_j \in \{\pm 1\}$. If $v \neq \epsilon$, let $k \in \mathbb{Z}$ be such that $w = vu\sigma^k(v^{-1})$. We may assume that for $1 \leq j \leq q$ we have $t_j \geq 0$ and $t_j + k \geq 0$ and for $1 \leq j \ \leq r$ we have $z_j \geq 0$. If not, there exists some $n \in \mathbb{N}$ such that $t_j + n \geq 0$ and $t_j + k + n$ for $1 \leq j \leq q$ and $z_j + n \geq 0$ for $1 \leq j \leq r$. As H is invariant under σ and σ^{-1}, we have that $\{\sigma^n(w)\}$ is also a generating set for H.

Let y be as in Case 1 and let m_1 be as in Case 1. We may assume that $\min\{s_1, s_2, \ldots, s_p\} = \min\{z_1, z_2, \ldots, z_r\}$ by applying σ or σ^{-1} an appropriate number of times to y. Let

$$m_2 = \max\{\{t_j \mid 1 \leq j \leq q\} \cup \{t_j + k \mid 1 \leq j \leq q\}\},$$

and let

$$m_3 = \max\{\{z_j \mid 1 \leq j \leq r\} \cup \{z_j + k \mid 1 \leq j \leq r\}\}.$$

Let $m = \max\{m_1, m_2, m_3\}$.

(2) Consider $n > m$. We claim that the set

$$B_n = \{\phi_n(w), \psi_n(\phi_n(w)), \psi_n^2(\phi_n(w)), \ldots, \psi_n^{n-1}(\phi_n(w))\}$$

is a group basis for $\phi_n(H)$. To see this, first note that $B' = \{\sigma^z(w) \mid z \in \mathbb{Z}\}$ is a group generating set for H. Then $\phi_n(B') = B_n$ is a group generating set for $\phi_n(H)$. To show it is a basis for $\phi_n(H)$ it is sufficient to show that it is Nielsen reduced (see Definition 2.1.5).

Firstly, let $\alpha = \psi_n^{j_1}(\phi_n(w)) \in B_n$, were $0 \leq j_1 \leq n-1$. Then

$$\begin{aligned}\alpha =& a^{\eta_1}_{(c_1,(t_1+j_1) \bmod n)} \cdots a^{\eta_q}_{(c_q,(t_q+j_1) \bmod n)} a^{\kappa_1}_{(d_1,(z_1+j_1) \bmod n)} \cdots a^{\kappa_r}_{(d_r,(z_r+j_1) \bmod n)} \\ & a^{-\eta_q}_{(c_q,(t_q+j_1+k) \bmod n)} \cdots a^{-\eta_1}_{(c_1,(t_1+j_1+k) \bmod n)}.\end{aligned}$$

But as $n > m_2, m_3$, this is a reduced word. In particular $|\alpha| = 2q + r > 0$. Hence, condition (i) of Definition 2.1.5 is satisfied.

For $0 \leq j_2 \leq n-1$ and $\mu_1, \mu_2 \in \{\pm 1\}$, let $\alpha^{\mu_1} = \psi_n^{j_1}(\phi_n(w))^{\mu_1}$, $\beta^{\mu_2} = \psi_n^{j_2}(\phi_n(w))^{\mu_2} \in B_n^{\pm}$ be such that if $j_1 = j_2$ then $\mu_1 \neq -\mu_2$. Cancellation in $\alpha^{\mu_1}\beta^{\mu_2}$ only occurs if $\mu_1 = \mu_2 = 1$ and $(t_1 + j_1 + k) \equiv (t_1 + j_2) \pmod{n}$, or $\mu_1 = \mu_2 = -1$ and $(t_1 + j_1) \equiv (t_1 + j_2 + k) \pmod{n}$. In the first case we have that $j_2 \equiv (j_1 + k) \pmod{n}$. We claim that the pair

$$a^{\kappa_r}_{(d_r,(z_r+j_1) \bmod n)} a^{\kappa_1}_{(d_1,(z_1+j_2) \bmod n)}$$

does not cancel in the product $\alpha\beta$. Indeed, if it were to cancel then $d_r = d_1$, $\kappa_r = -\kappa_1$ and $(z_r + j_1) \equiv (z_1 + j_2) \pmod{n}$. If this were the case it cannot be that $z_r = z_1 + k$, as we have chosen v to be as long as possible. But

$$\begin{aligned}(z_r + j_1) \equiv (z_1 + j_2) \pmod{n} &\implies (z_r + j_1) \equiv (z_1 + j_1 + k) \pmod{n} \\ &\implies z_r \equiv (z_1 + k) \pmod{n},\end{aligned}$$

since we have already observed that $j_2 \equiv (j_1 + k) \pmod{n}$. As we have chosen $n > m_3$, this would imply that $z_r = z_1 + k$, which is a contradiction. Hence the pair does not cancel. Similarly in the second case the pair

$$a^{-\kappa_1}_{(d_1,(z_1+j_1) \bmod n)} a^{-\kappa_r}_{(d_r,(z_r+j_2) \bmod n)}$$

does not cancel. Hence

$$|\alpha^{\mu_1}\beta^{\mu_2}| \geq 2q + 2r > 2q + r = |\alpha^{\mu_1}|, |\beta^{\mu_2}|.$$

So B_n satisfies condition (ii) of Definition 2.1.5. As in condition (ii) we actually have a strict inequality, it follows that condition (iii) must also be satisfied by Lemma 2.1.7. Hence B_n is a group basis for $\phi_n(H)$.

(3) As $n > m_1$, we have that

$$\phi_n(y) = a^{\delta_1}_{(b_1,s_1)} a^{\delta_2}_{(b_2,s_2)} \dots a^{\delta_p}_{(b_p,s_p)}.$$

Now suppose that that $\phi_n(y) \in \phi_n(H)$. Then $\phi_n(y)$ can be expressed uniquely as a reduced word over B_n. For $0 \leq \ell \leq n-1$, if the element

$$\alpha = \psi^\ell(\phi_n(w)) = \psi^\ell(\phi_n(vu\sigma^k(v^{-1})) \in B_n$$

appears in the decomposition of $\phi_n(y)$, then

$$\{(z_j + \ell) \pmod{n} \mid 1 \leq j \leq r\} \subseteq \{s_1, s_2, \ldots, s_p\}.$$

This is because the $\psi_n^{\ell}(\phi_n(u))$ will not be cancelled by our analysis in part (2).

Let $\ell_0 = \max\{|s_i - s_j| \mid 1 \leq i, j \leq p\} + 1$. Then there exists $n_0 \in \mathbb{N}$ and such that for all $n \geq n_0$ and $\ell_0 \leq \hat{\ell} \leq n-1$ there exists $1 \leq j \leq r$ such that

$$(z_j + \hat{\ell}) \pmod{n} \notin \{s_1, s_2, \ldots, s_p\}.$$

Hence, for $n \geq n_0$, if $\phi_n(y) \in \phi_n(H)$, then $\phi_n(y)$ is in the subgroup generated by the set

$$\overline{B_n} = \{\phi_n(w), \psi_n(\phi_n(w)), \psi_n^2(\phi_n(w)), \ldots, \psi_n^{\ell_0-1}(\phi_n(w))\}.$$

There exists $N \geq n_0$ such that

$$\begin{aligned}\psi_N^{\ell_0}(\phi_N(w)) =& a_{(c_1,t_1+\ell_0)}^{\eta_1} \cdots a_{(c_q,t_q+\ell_0)}^{\eta_q} a_{(d_1,z_1+\ell_0)}^{\kappa_1} \cdots a_{(d_r,z_r+\ell_0)}^{\kappa_q} \\ & a_{(c_q,t_q+\ell_0)}^{-\eta_q} \cdots a_{(c_1,t_1+\ell_0)}^{-\eta_1}.\end{aligned}$$

That is, the map ϕ_N, which replaces an occurrence of x with an a and also reduces the second coordinate in the subscript, does not change any of the subscripts for words in the subgroup generated by the set

$$\overline{B}' = \{w, \sigma(w), \sigma^2(w), \ldots, \sigma^{\ell_0-1}(w)\}.$$

Hence, if $\phi_N(y) \in \phi_N(H)$, then y is in the subgroup generated by $\overline{B}'$ and hence $y \in H$. This is a contradiction and so $\phi_N(y) \notin \phi_N(H)$.

Case 3: Finally we consider the case when $u = \epsilon$ and $|v| \geq 1$. As previously, the proof for this case is organised as follows.

(1) Establish a constant m.
(2) For $n > m$, give a group basis B_n for $\phi_n(H)$.
(3) Show that there exists $N > m$ such that $\phi_N(y) \notin \phi_N(H)$.

(1) Let y and m_1 be as in Case 1. Let v be as in Case 2 and let m_2 be as in Case 2. Again assume $\min\{s_1, s_2, \ldots, s_p\} = \min\{z_1, z_2, \ldots, z_r\}$. Now let $m = \max\{m_1, m_2\}$.

(2) Consider $n > m$ such that k divides n. Then just as in Case 2, we have that

$$Y_n = \{\psi_n^j(\phi_n(w)) \mid 0 \leq j \leq n-1\}$$

is a group generating set for $\phi_n(H)$. However, it is not obvious that Y_n is a group basis for $\phi_n(H)$. We claim that

$$B_n = \{\psi_n^j(\phi_n(v)) \cdot \psi_n^{j+dk}(\phi_n(v^{-1})) \mid 0 \leq j \leq k-1, 1 \leq d \leq \frac{n}{k} - 1\}$$

is a group basis for $\phi_n(H)$. Note that we use $\cdot$ to show multiplication in (FG_{X_n}, ϕ_n) here for clarity.

First we will show that B_n is a generating set for $\phi_n(H)$. To do this we will show that $Y_n \subseteq \mathrm{Gp}\langle B_n\rangle$ and that $B_n \subseteq \mathrm{Gp}\langle Y_n\rangle$. Firstly, for some $0 \leq j \leq n-1$ let $z = \psi_n^j(\phi_n(w)) \in Y_n$. Then

$$z = \psi_n^j(\phi_n(v\sigma^k(v^{-1})) = \psi_n^j(\phi_n(v)) \cdot \psi_n^{j+k}(\phi_n(v^{-1})).$$

Now $j = \lambda k + \rho$ for some $\lambda \in \{0, 1, \ldots, \frac{n}{k} - 1\}$ and $\rho \in \{0, 1, \ldots, k-1\}$. If $\lambda = 0$ then

$$z = \psi_n^\rho(\phi_n(v)) \cdot \psi_n^{\rho+k}(\phi_n(v^{-1})) \in B.$$

If $\lambda = \frac{n}{k} - 1$ then

$$\begin{aligned} z^{-1} &= \psi_n^{\lambda k+\rho+k}(\phi_n(v)) \cdot \psi_n^{\lambda k+\rho}(\phi_n(v^{-1})) \\ &= \psi_n^\rho(\phi_n(v)) \cdot \psi_n^{\rho+\lambda k}(\phi_n(v^{-1})) \in B_n. \end{aligned}$$

Otherwise $0 < \lambda < \frac{n}{k} - 1$. Then $b_1 = \psi_n^\rho(\phi_n(v)) \cdot \psi_n^{\rho+\lambda k}(\phi_n(v^{-1}))$,

$b_2 = \psi_n^{\rho}(\phi_n(v)) \cdot \psi_n^{\rho+(\lambda+1)k}(\phi_n(v^{-1})) \in B_n$ and

$$\begin{aligned} b_1^{-1} \cdot b_2 &= \left(\psi_n^{\rho+\lambda k}(\phi_n(v)) \cdot \phi_n^{\rho}(\phi_n(v^{-1}))\right) \cdot \left(\psi_n^{\rho}(\phi_n(v)) \cdot \psi_n^{\rho+(\lambda+1)k}(\phi_n(v^{-1}))\right) \\ &= \psi_n^{\rho+\lambda k}(\phi_n(v)) \cdot \psi_n^{\rho+(\lambda+1)k}(\phi_n(v^{-1})) \\ &= \psi_n^{j}(\phi_n(v)) \cdot \psi_n^{j+k}(\phi_n(v^{-1})) \\ &= z. \end{aligned}$$

Hence $Y_n \subseteq \mathrm{Gp}\langle B_n \rangle$.

Now for some $0 \leq j \leq k-1$ and $1 \leq d \leq \frac{n}{k} - 1$ let $b = \psi_n^{j}(\phi_n(v)) \cdot \psi_n^{j+dk}(\phi_n(v^{-1})) \in B_n$. Then

$$\begin{aligned} z_1 &= \psi_n^{j}(\phi_n(v)) \cdot \psi_n^{j+k}(\phi_n(v^{-1})), \\ z_2 &= \psi_n^{j+k}(\phi_n(v)) \cdot \psi_n^{j+2k}(\phi_n(v^{-1})), \\ &\vdots \\ z_d &= \psi_n^{j+(d-1)k}(\phi_n(v)) \cdot \psi_n^{j+dk}(\phi_n(v^{-1})) \in Y_n. \end{aligned}$$

Furthermore, we have that

$$b = z_1 \cdot z_2 \cdot \ldots \cdot z_d.$$

Hence $B_n \subseteq \mathrm{Gp}\langle Y_n \rangle$ and therefore we conclude that B_n is a group generating set for $\phi_n(H)$.

To show that B_n is a basis for $\phi_n(H)$ it is sufficient that any non-empty reduced product over B_n is not equal to ϵ. To do this we will make use of facts about the length over $\overline{X}$ of reduced products of one or two elements of B_n. Let $\alpha = \psi_n^{j_1}(\phi_n(v)) \cdot \psi_n^{j_1+d_1k}(\phi_n(v^{-1})) \in B_n$, where $0 \leq j_1 \leq k-1$ and $1 \leq d_1 \leq \frac{n}{k} - 1$. Then

$$\begin{aligned} \alpha =& a^{\eta_1}_{(c_1,(t_1+j_1) \bmod n)} a^{\eta_2}_{(c_2,(t_2+j_1) \bmod n)} \cdots a^{\eta_q,}_{(c_q,(t_q+j_1) \bmod n)} \\ & a^{-\eta_q}_{(c_q,(t_q+j_1+d_1k) \bmod n)} a^{-\eta_{q-1}}_{(c_{q-1},(t_{q-1}+j_1+d_1k) \bmod n)} \cdots a^{-\eta_1}_{(c_1,(t_1+j_1+d_1k) \bmod n)}. \end{aligned}$$

As $n > m_2$, cancellation could only occur if $(t_q+j_1) \equiv (t_q+j_1+d_1k) \pmod{n}$. However, $d_1k < n$. Hence no cancellation occurs and so $|\alpha| = 2q > 0$.

Now let $\beta = \psi_n^{j_2}(\phi_n(v)) \cdot \psi_n^{j_2+d_2k}(\phi_n(v^{-1})) \in B_n$, where $0 \leq j_2 \leq k-1$ and $1 \leq d_2 \leq \frac{n}{k} - 1$. Also let $\mu_1, \mu_2 \in \{\pm 1\}$ be such that if $(j_1, d_1) = (j_2, d_2)$ then $\mu_1 \neq -\mu_2$. Consider $\alpha^{\mu_1}\beta^{\mu_2}$. If $\mu_1 = \mu_2 = 1$ then no cancellation occurs as $(t_1+j_1+d_1k) \not\equiv (t_1+j_2) \bmod \pmod{n}$. This follows as $0 \leq j_1, j_2, \leq k-1$, and $j_1+d_1k < n$ but $d_1 \geq 1$. Similarly, if $\mu_1 = \mu_2 = -1$ there is no cancellation as $(t_1+j_1) \not\equiv (t_1+j_2+d_2k) \pmod{n}$. If $\mu_1 = 1$ and $\mu_2 = -1$ then no cancellation occurs as this would imply that $(j_1, d_1) = (j_2, d_2)$. Finally, if $\mu_1 = -1$ and $\mu_2 = 1$ then cancellation will occur if $j_1 = j_2$. However, only q pairs of letters will cancel as the pair

$$a^{\eta_q}_{(c_q,(t_1+j_1+d_1k) \bmod n)} a^{-\eta_q}_{(c_q,(t_1+j_2+d_2k) \bmod n)}$$

will not cancel as $(j_1, d_1) \neq (j_2, d_2)$. Hence $|\alpha^{\mu_1}\beta^{\mu_2}| \geq 4q - 2q = |\alpha|, |\beta|$.

Now consider a reduced product over B_n. That is, let

$$g = b_1^{\mu_1} b_2^{\mu_2} \dots b_r^{\mu_r},$$

where for $1 \leq j \leq r$ we have $b_j \in B_n$, $\mu_j \in \{\pm 1\}$. The fact that this product is reduced means that for $1 \leq j \leq r-1$ if $b_j = b_{j+1}$ then $\mu_j = \mu_{j+1}$. From the analysis above, for $1 \leq j \leq r-1$ we can see that cancellation occurs in $b_j^{\mu_j} b_{j+1}^{\mu_{j+1}}$ if and only if $(\mu_j, \mu_{j+1}) = (-1, 1)$, in which case precisely half of $b_j^{\mu_j}$ and $b_{j+1}^{\mu_{j+1}}$ cancel. Also there is no cancellation between $b_{j-1}^{\mu_{j-1}}$ and $b_j^{\mu_j}$, nor between $b_{j+1}^{\mu_{j+1}}$ and $b_{j+2}^{\mu_{j+2}}$ as the ordered indices cannot be $(-1, 1)$. Hence, for $1 \leq j \leq r$, at most half of $b_j^{\mu_j}$ cancels and so $g \neq \epsilon$. Therefore B_n is a group basis for $\phi_n(H)$.

We note that $|B_n| = |Y_n|$, and as $\mathrm{Gp}\langle B_n \rangle = \mathrm{Gp}\langle Y_n \rangle$ we also have that Y_n is a group basis for $\phi_n(H)$. However, it will be more convenient for us to work with B_n than Y_n. Furthermore, we note that if cancellation occurs in a reduced word over B_n, the only possibility is that $\psi_n^j(\phi_n(v^{-1}))$ cancels with $\psi_n^j(\phi_n(v^{-1}))$ where $1 \leq j \leq k-1$.

(3) As $n > m_1$, we have that

$$\phi_n(y) = a^{\delta_1}_{(b_1,s_1)} a^{\delta_2}_{(b_2,s_2)} \dots a^{\delta_p}_{(b_p,s_p)}.$$

Now suppose that that $\phi_n(y) \in \phi_n(H)$. Then $\phi_n(y)$ can be expressed uniquely as a reduced word over B_n. For $0 \leq j' \leq k-1$ and $1 \leq d \leq \frac{n}{k} - 1$, if the element $\alpha = \psi_n^{j'}(\phi_n(v)) \cdot \psi_n^{j'+dk}(\phi_n(v^{-1})) \in B$ appears in the decomposition of $\phi_n(y)$, then

$$\{(t_j + j' + dk) \pmod n \mid 1 \leq j \leq q\} \subseteq \{s_1, s_2, \ldots, s_p\}.$$

This is because the $\psi_n^{j+dk}(\phi_n(v^{-1}))$ will not be cancelled by our analysis in part (2).

Let ℓ_0 be the same as in Case 2. Then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ such that k divides n and for all $\ell_0 < \hat{\ell} \leq n-1$, there exists $1 \leq j \leq r$ such that

$$(z_j + j' + \hat{\ell}k) \pmod n \notin \{s_1, s_2, \ldots, s_p\}.$$

Hence, for such an n, if $\phi_n(y) \in \phi_n(H)$, then $\phi_n(y)$ is in the subgroup generated by the set

$$\overline{B_n} = \{\psi_n^j(\phi_n(v)) \cdot \psi_n^{j+dk}(\phi_n(v^{-1})) \mid 0 \leq j \leq k-1, 1 \leq d \leq \ell_0\}.$$

There exists $N \geq n_0$ such that

$$\psi_N^{k-1+\ell_0 d}(\phi_N(v^{-1})) = a^{-\eta_q}_{(c_1, t_q + k - 1\ell_0 d)} \cdots a^{-\eta_1}_{(c_q, t_1 + k - 1 + \ell_0 d)}.$$

That is, the map ϕ_N, which rewrites an occurrence of x with an a and also reduces the second coordinate in the subscript, does not change any of the subscripts for words in the subgroup generated by the set

$$\overline{B}' = \{\sigma^j(v) \cdot \sigma^{j+dk}(v^{-1}) \mid 0 \leq j \leq k-1, 1 \leq d \leq \ell_0\}.$$

Hence, if $\phi_N(y) \in \phi_N(H)$, then y is in the subgroup generated by $\overline{B}'$ and hence $y \in H$. This is a contradiction and so $\phi_N(y) \notin \phi_N(H)$.

Hence, in all three cases we have been able to separate y from H an we conclude that $\mathrm{F}\,\alpha G_X$ is monogenic subalgebra separable. □

Remark 2.6.10. The above argument works because for every $n \in \mathbb{N}$ we are able to find a group basis B_n for $\phi_n(H)$ that has useful properties. In the case that H is finitely generated, it is not clear if we would be able to find a basis with such nice properties. Thus whether $\mathrm{F}\alpha\mathrm{G}_X$ is weakly subalgebra separable remains unknown. However, $\mathrm{F}\alpha\mathrm{G}_X$ is not strongly subalgebra separable by the following example.

Example 2.6.11. Let $X = \{x_i \mid i \in I\}$ be a non-empty set and fix $i \in I$. Let H be the subalgebra of $\mathrm{F}\alpha\mathrm{G}_X$ generated by the set

$$Y = \{x_{(i,0)}x_{(i,1)}x_{(i,0)}^{-1}\} \cup \{x_{(i,0)}x_{(i,n)}x_{(i,0)} \mid n \geq 2\}.$$

First we show that $x_{(i,0)}x_{(i,0)}$ is not an element of H.

To see this observe that

$$Z = \{x_{(i,z)}x_{(i,z+1)}x_{(i,z)}^{-1} \mid z \in \mathbb{Z}\} \cup \{x_{(i,z)}x_{(i,z+n)}x_{(i,z)} \mid z \in \mathbb{Z}, n \geq 2\}$$

is a group generating set for H. We show that Z is Nielsen reduced. For $u \in Z^{\pm}$ we have that $|z| = 3 > 0$ and so condition (i) of Definition 2.1.5 holds.

Let $u, v \in Z^{\pm}$ such that $u \neq v^{-1}$. Cancellation in uv can only occur in the following cases:

- $u = x_{(i,z)}x_{(i,z+1)}x_{(i,z)}^{-1}$ and $v = x_{(i,z)}x_{(i,z+n)}x_{(i,z)}$,
- $u = x_{(i,z)}x_{(i,z+1)}^{-1}x_{(i,z)}^{-1}$ and $v = x_{(i,z)}x_{(i,z+n)}x_{(i,z)}$,
- $u = x_{(i,z)}^{-1}x_{(i,z+n)}^{-1}x_{(i,z)}^{-1}$ and $v = x_{(i,z)}x_{(i,z+1)}x_{(i,z)}^{-1}$,
- $u = x_{(i,z)}^{-1}x_{(i,z+n)}^{-1}x_{(i,z)}^{-1}$ and $v = x_{(i,z)}x_{(i,z+1)}^{-1}x_{(i,z)}^{-1}$.

In all cases we have that $|uv| \geq 4 > |u|, |v|$ as $n \geq 2$ and hence condition (ii) of Definition 2.1.5 holds. Because we achieved a strict inequality in regards to condition (ii), condition (iii) will automatically hold by Lemma 2.1.7, and so Z is Nielsen reduced. By the above analysis, for any $u \in Z^{\pm}$, the middle letter of u will not cancel when u appears as part of a reduced product over Z. Then, by induction on the length of an element of a product over $Z^{\pm}$, any element of H has length at least 3 and therefore $x_{(i,0)}x_{(i,0)}$ is not an element of H.

Now we show that $x_{(i,0)}x_{(i,0)}$ cannot be separated from H. Let (G,ψ) be a finite α-group and let $\phi : \mathrm{F}\alpha\mathrm{G}_X \to (G,\psi)$ be an α-group homomorphism. Let n_0 be the order of the group G and n_1 be the order of the automorphism ϕ. Then

$$\begin{aligned}\phi(H) \ni &\phi((x_{(i,0)}x_{(i,1)}x_{(i,0)}^{-1})^{n_0-1}\cdot x_{(i,0)}x_{(i,n_1+1)}x_{(i,0)})\\ =&\phi(x_{(i,0)}x_{(i,1)}^{n_0-1}x_{(i,n_1+1)}x_{(i,0)})\\ =&\phi(x_{(i,0)})\psi(\phi(x_{(i,0)}))^{n_0-1}\psi^{n_1+1}(\phi(x_{(i,0)}))\phi(x_{(i,0)})\\ =&\phi(x_{(i,0)})\psi(\phi(x_{(i,0)}))^{n_0-1}\psi(\phi(x_{(i,0)}))\phi(x_{(i,0)})\\ =&\phi(x_{(i,0)})\phi(x_{(i,0)})\\ =&\phi(x_{(i,0)}x_{(i,0)}).\end{aligned}$$

Hence $x_{(i,0)}x_{(i,0)}$ cannot be separated from H and so $\mathrm{F}\alpha\mathrm{G}_X$ is not strongly subalgebra separable.

The above example provides more evidence that the behaviour of $\mathrm{F}\alpha\mathrm{G}_X$ mimics that of FCR_X; see Lemma 2.6.1.

In this chapter we have investigated the separability properties of the free objects in different varieties of semigroups. We have seen that the free monoid, free semigroup and free inverse monoid are all completely separable. However, the free group, the free completely simple semigroup and free Clifford semigroup are all weakly subalgebra separable and only strongly subalgebra separable in a limited number of instances. This naturally led us to consider the separability properties of the free completely regular semigroup. Given the complexity of the structure of the free completely regular semigroup, we defined a new variety of semigroups, α-groups, in such a way that the structure of the free objects of this variety capture some of the behaviour of free completely regular semigroups. We were able to show that free α-groups are monogenic subsemigroup separable. We conclude this chapter with some open problems.

Open Problem 2.6.12. Is the free α-group $\mathrm{F}\alpha\mathrm{G}_X$ weakly subalgebra separable?

Open Problem 2.6.13. For $|X| \geq 2$, is the free completely regular semigroup FCR_X weakly subalgebra separable or monogenic subalgebra separable?

Chapter 3

Schützenberger Groups and Finitely Generated Commutative Semigroups

In this chapter we consider the separability properties of finitely generated commutative semigroups. This line of enquiry arises from the fact that finitely generated abelian groups are strongly subgroup separable. In Section 3.1 we see why this is true, by way of the classification of finitely generated abelian groups. However, an equivalent classification for finitely generated commutative semigroups is not known and so we must approach their separability properties by a different route.

We have already seen that there exists finitely generated commutative semigroups which are not even MSS in Example 1.2.5. This example will prove key in understanding the separability properties of finitely generated commutative semigroups. In order to understand these properties we make use of two pieces of machinery. The first is the theory of Schützenberger groups. A Schützenberger group is a group associated with an $\mathcal{H}$-class of a semigroup. In Section 3.2 we show that Schützenberger groups inherit complete separability and strong subsemigroup separability. The Schützenberger groups of commutative semigroups also inherit weak subsemigroup separability. Given the additional fact that any Schützenberger group of a finitely generated commutative semigroup is a finitely generated abelian group, we end up with a strong understanding of separability properties of Schützenberger groups of finitely generated commutative semigroups.

The second piece of machinery we use is a structural theorem for finitely generated commutative semigroups. This states that a finitely generated commutative semigroup is a semilattice of archimedean semigroups. Armed with these tools, in Section 3.3 we are able to show that a finitely generated commutative semigroup is MSS if and only if every maximal subgroup is finite. Additionally, we show that in the class of finitely generated commutative semigroups, the properties of complete separability, strong subsemigroup separability and weak subsemigroup separability coincide, and are equivalent to every $\mathcal{H}$-class being finite. The fact that the properties of complete separability and strong subsemigroup separability coincide for finitely generated commutative semigroups was already known to Kublanovskiĭ and Lesohin, see [30, Corollary 1]. We present their methods in contrast to those developed within this chapter.

Given that the properties of complete separability, strong subsemigroup separability and weak subsemigroup separability coincide for finitely generated commutative semigroups, it is natural to ask whether they coincide in some larger class of semigroups. In Section 3.4 we show that this is not the case for the class of commutative semigroups nor the class of finitely generated semigroups. We also consider the semigroup separability properties of abelian groups.

In the final section of this chapter, we turn our attention once again to Schützenberger groups. This time we ask if all Schützenberger groups of a semigroup have a separability property, then does the semigroup itself have this property. The answer to this question in general is no. However, by restricting our attention to semigroups with only finitely many $\mathcal{H}$-classes, and therefore only finitely many Schützenberger groups, we are able to show in this case the answer will be yes for the properties of complete separability and monogenic subsemigroup separability. For the property of strong subsemigroup separability the answer is still no even when there are only finitely many $\mathcal{H}$-classes, whilst for weak subsemigroup separability the problem remains open.

This chapter is largely based on the paper [40], co-written by the author. This material is included with the permission of all co-authors.

3.1 Finitely Generated Abelian Groups

This chapter arises from the desire to investigate the separability properties of finitely generated commutative semigroups. This is motivated by the fact that finitely generated abelian groups are strongly subgroup separable. This result is folklore, but in this section we provide a proof. This proof is based upon a characterisation of finitely generated abelian groups, which we give below.

Theorem 3.1.1. *For a finitely generated abelian group A, we have that*

$$A \cong \underbrace{\mathbb{Z} \times \mathbb{Z} \cdots \times \mathbb{Z}}_{n \text{ times}} \times C_{d_1} \times C_{d_2} \times \cdots \times C_{d_k},$$

where n is finite and, for $1 \leq i \leq k$, C_{d_i} is a finite cyclic group.

More on finitely generated abelian groups, including a proof of Theorem 3.1.1, can be found in [29, Chapter 6]. This characterisation allows us to show that finitely generated abelian groups are strongly subgroup separable, making use of the following corollary.

Corollary 3.1.2. *Every finitely generated abelian group is residually finite.*

Proof. This follows as any finitely generated abelian group can be realised as the direct product of finitely many residually finite groups. We have already observed that all finite groups are residually finite and Example 1.2.5 establishes that $\mathbb{Z}$ is residually finite. In Lemma 4.1.1, we see that the direct product preserves residual finiteness. □

We also make the following observation, which proves important throughout this chapter.

Corollary 3.1.3. *A finitely generated abelian group is infinite if and only if it contains a subgroup which is isomorphic to* $\mathbb{Z}$.

We are now able to state the main theorem of this section. This result is folklore, but we provide a proof for completeness.

Theorem 3.1.4. *Every finitely generated abelian group is strongly subgroup separable.*

Proof. Let A be a finitely generated abelian group, let H be a subgroup of A and let $a \in A \setminus H$. As A is abelian, we have that H is a normal subgroup of A. Let $\phi : A \to A/H$ be the canonical homomorphism. Then we have that $\phi(h) = H1_A$ for all $h \in H$ but $\phi(a) = Ha \neq H1_A$. As A/H is the quotient of a finitely generated abelian group, it is also finitely generated. By Corollary 3.1.2, we have that A/H is residually finite. Hence there exists a finite group G and a group homomorphism $\psi : A/H \to G$ such that $\psi(H1_a) \neq \psi(Ha)$. Then the homomorphism $\psi \circ \phi : A \to G$ separates a from H. Hence, A is strongly subgroup separable. $\square$

This result relies upon the characterisation given in Theorem 3.1.1. Whilst there is a characterisation of finitely generated commutative semigroups (Proposition 3.3.6), it does not readily lead to results concerning separability properties in the same way that Theorem 3.1.1 does for finitely generated abelian groups. One would hope we could gain a result similar to Theorem 3.1.4. However this is not the case. Indeed we have already seen in Example 1.2.5 that $\mathbb{Z}$, which is finitely generated as a commutative semigroup by the set $\{1, -1\}$, is not even MSS.

We investigate the separability properties of finitely generated commutative semigroups via the separability properties of their Schützenberger groups. There proves to be a strong link between the two.

3.2 Schützenberger Groups

In this section we investigate which semigroup separability properties are inherited by Schützenberger groups. We first define Schützenberger groups.

For an $\mathcal{H}$-class H of a semigroup S, we are able to associate a group called the Schützenberger group of H. Schützenberger groups capture how a certain submonoid of S^1 acts of H. We being by defining this submonoid.

Definition 3.2.1. For an $\mathcal{H}$-class H of a semigroup S, the *right stabiliser* of H in S is

$$\mathrm{Stab}(H) = \{s \in S^1 \mid Hs = H\},$$

where $Hs = \{hs \mid h \in H\}$.

Lemma 3.2.2. *For an $\mathcal{H}$-class H of a semigroup S, we have that* $\mathrm{Stab}(H)$ *is a submonoid if S^1.*

Proof. First note that as $s1 = s$ for all $s \in S$, it is certainly true that $H1 = H$, and therefore $1 \in \mathrm{Stab}(H)$. Now let $s, t \in \mathrm{Stab}(H)$. Then

$$H(st) = (Hs)t = Ht = H,$$

and so $st \in \mathrm{Stab}(H)$. Hence $\mathrm{Stab}(H)$ is a submonoid of S^1. □

The following lemma gives a useful criterion for an element of a semigroup to belong to the stabiliser of an $\mathcal{H}$-class.

Lemma 3.2.3. *For an $\mathcal{H}$-class H of a semigroup S and $s \in S^1$, if there exists some $h \in H$ such that $hs \in H$, then $s \in \mathrm{Stab}(H)$.*

The above result result follows directly from Green's Lemma, which can be found in [28, Lemma 2.2.1].

The following lemma is utilised several times throughout this chapter.

Lemma 3.2.4. *For a non-group $\mathcal{H}$-class H of a semigroup S, we have $H \cap \mathrm{Stab}(H) = \emptyset$.*

Proof. For a contradiction suppose that there exists $x \in H \cap \text{Stab}(H)$. Then, as $x \in \text{Stab}(H)$, we would have $x^2 \in H$. So $x^2 \in H \cap H^2$. Then H is a group by Lemma 1.3.37, which is a contradiction. □

We now define Schützenberger groups via the Schützenberger congruence.

Definition 3.2.5. For an $\mathcal{H}$-class H of a semigroup S, define a congruence σ_H on $\text{Stab}(H)$, called the *Schützenberger congruence of* H, by

$$(x, y) \in \sigma_H \iff hx = hy \text{ for all } h \in H.$$

Then $\Gamma(H) = \text{Stab}(H)/\sigma_H$ is a group, known as the *Schützenberger group* of H.

Schützenberger groups are known to have many useful properties, some of which are summarised in the following proposition.

Proposition 3.2.6. *For $\Gamma(H)$, the Schützenberger group of an $\mathcal{H}$-class H, we have that:*

(i) *The map $\cdot : H \times \Gamma(H) \to H$ given by $h \cdot [x]_{\sigma_H} = hx$ is a group action.*
(ii) *This action is regular, that is both transitive and free.*
(iii) $|\Gamma(H)| = |H|$.
(iv) *If H is a group then $\Gamma(H) \cong H$.*

It should noted that one could similarly define a group $\Gamma_l(H)$ in an analogous way by considering the left stabiliser of H. However, it turns out that $\Gamma_l(H) \cong \Gamma(H)$. For more on Schützenberger groups and proofs of the above claims, see [32, Section 2.3].

We now consider which of our separability properties are inherited by Schützenberger groups. This is motivated by the following result of Gray and Ruškuc.

Theorem 3.2.7. [23, Theorem 3.1] *Every Schützenberger group of a residually finite semigroup is residually finite.*

We show that the semigroup properties of CS and SSS are inherited by

Schützenberger groups. By way of contrast, it is not true that every Schützenberger group of a WSS semigroup is WSS, as will be seen in Example 3.4.13. However, we are able to give a sufficient condition on the stabiliser of an $\mathcal{H}$-class of a WSS semigroup to ensure that the corresponding Schützenberger group is also WSS. We begin with the following result.

Proposition 3.2.8. *If a semigroup S has an infinite non-group $\mathcal{H}$-class H, then S is not strongly subsemigroup separable.*

Proof. Fix some $h \in H$ and let $T = \langle H \setminus \{h\} \rangle$. If $h \in T$, then $h \in H^n$ for some $n \geq 2$. But this contradicts that H is not a group by Lemma 1.3.37. Therefore $h \notin T$.

Let $\sim$ be a finite index congruence on S. Then there exist distinct elements $x, y \in H \setminus \{h\}$ such that $x \sim y$. As $x \,\mathcal{H}\, h$, there exists some $s \in S^1$ such that $xs = h$. So

$$h = xs \sim ys.$$

By Green's Lemma [28, Lemma 2.2.4], multiplication on the right by s permutes H. Therefore $ys \neq xs$ and $ys \in H \setminus \{h\} \subseteq T$. Hence, h cannot be separated from T in a finite quotient and S is not SSS. □

With the above proposition, we are able to show that the properties of strong subsemigroup separability and complete separability are inherited by Schützenberger groups.

Corollary 3.2.9. *Every Schützenberger group of a strongly subsemigroup separable semigroup is itself strongly subsemigroup separable.*

Proof. Let S be an SSS semigroup. First we consider the Schützenberger groups of non-group $\mathcal{H}$-classes. Let H be an $\mathcal{H}$-class which is not a group. Then, by Proposition 3.2.8, H is finite. Also by Proposition 3.2.6, we have that $|\Gamma(H)| = |H|$, where $\Gamma(H)$ is the Schützenberger group of H. So $\Gamma(H)$ is finite and certainly SSS.

Now consider a group $\mathcal{H}$-class G. By Proposition 3.2.6 we have that $\Gamma(G) \cong$

G. As G is a subsemigroup of an SSS semigroup, it is itself SSS by Proposition 1.2.13. Hence $\Gamma(G)$ is SSS, as desired. □

Corollary 3.2.10. *Every $\mathcal{H}$-class of a completely separable semigroup is finite (and hence every Schützenberger group is completely separable).*

Proof. Let S be a CS semigroup. First we consider the Schützenberger groups of non-group $\mathcal{H}$-classes. Let H be an $\mathcal{H}$-class which is not a group. Then by Proposition 3.2.8 H is finite. Also by Proposition 3.2.6, we have that $|\Gamma(H)| = |H|$, where $\Gamma(H)$ is the Schützenbeger group if H. So $\Gamma(H)$ is finite.

Now consider a group $\mathcal{H}$-class G. By Proposition 3.2.6 we have that $\Gamma(G) \cong G$. As G is a subsemigroup of a CS semigroup, it is itself CS by Proposition 1.2.13. Hence G is finite by Theorem 1.2.19, as desired. □

It is not true that every Schützenberger group of a WSS semigroup is itself WSS, as will be demonstrated in Section 3.4. Indeed, Example 3.4.13 shows that a Schützenberger group of a WSS semigroup need not even be MSS. Therefore it also true that Schützenberger groups of MSS semigroups need not be MSS. However, it is true that Schützenberger groups of WSS commutative semigroups are also WSS. We deduce this from the following result.

Lemma 3.2.11. *Let S be a weakly subsemigroup separable semigroup and let H be an $\mathcal{H}$-class of S. If there exists an element $h \in H$ such that $ah = ha$ for all $a \in \mathrm{Stab}(H)$, then the Schützenberger group $\Gamma(H)$ is weakly subsemigroup separable.*

Proof. If H is a group then $\Gamma(H) \cong H$ by Proposition 3.2.6. As H is a subsemigroup of a WSS semigroup, it is itself WSS by Proposition 1.2.13. Hence $\Gamma(H)$ is WSS.

Now assume that H is not a group. For $x \in \mathrm{Stab}(H)$, we will denote $[x]_{\sigma_H}$ by $[x]$, where σ_H is the Schützenberger congruence. Let $T = \langle t_1, t_2, \dots, t_n \rangle \leq$

$\Gamma(H)$ and let $z \in \Gamma(H) \backslash T$. For $i \in \{1, \ldots, n\}$ choose $x_i \in \text{Stab}(H)$ such that $[x_i] = t_i$, and also choose $u \in \text{Stab}(H)$ such that $[u] = z$. Let $\overline{T} = \langle h, x_1, x_2, \ldots, x_n \rangle \leq S$.

First we will show that $hu \notin \overline{T}$. For a contradiction assume that $hu \in \overline{T}$. Then as $hx_j = x_j h$ for $1 \leq j \leq n$, we have $hu = h^i t$ for some $t \in \langle x_1, x_2, \ldots, x_n \rangle^1$ and $i \geq 0$. We split into three cases: $i = 0$, $i = 1$, and $i > 1$.

(i) If $i = 0$ we have $hu = t \in \text{Stab}(H)$. But then $h \cdot hu = ht \in H$. But as $h, hu \in H$ we have that $h^2 u \in H \cap H^2$. Then by Lemma 1.3.37, H is a group. This is a contradiction, so $i \neq 0$.

(ii) If $i = 1$ we have $hu = ht$. As $[u] \notin T$, it must be that $[u] \neq [t]$. But this contradicts that the action of $\Gamma(H)$ on H is free; see Proposition 3.2.6. Hence $i \neq 1$.

(iii) Finally, assume $i > 1$. Then as $hu \in H$, we have that $h^i t \in H$. But as $h, ht \in H$, we have that $h^i t \in H^i$. So $h^i t \in H \cap H^i$. Then, by Lemma 1.3.37, H is a group. This is a contradiction, so $i \not> 1$.

As all possibilities lead to a contradiction, it must be the case that $hu \notin \overline{T}$. As S is WSS, there exists a finite semigroup U and homomorphism $\phi : S \to U$ such that $\phi(hu) \notin \phi(\overline{T})$. Let $H_{\phi(h)} \subseteq U$ be the $\mathcal{H}$-class of $\phi(h)$.

Now $\phi(\text{Stab}(H)) \subseteq \text{Stab}(H_{\phi(h)})$. To see this let $b \in \text{Stab}(H)$. Then $hb \in H$. So there exist $w, v, x \in S^1$ such that $wh = hb$, $vhb = h$ and $hbx = h$. Then as ϕ is a homomorphism, we have that

$$\begin{aligned} \phi(h)\phi(b) = \phi(hb), \quad \phi(hb)\phi(x) = \phi(h), \\ \phi(w)\phi(h) = \phi(hb), \quad \phi(v)\phi(hb) = \phi(h). \end{aligned}$$

That is, $\phi(h)\phi(b) \,\mathcal{H}\, \phi(h)$. Then, by Lemma 3.2.3, $\phi(b) \in \text{Stab}(H_{\phi(h)})$.

Consider the Schützenberger group $\Gamma(H_{\phi(h)})$. Again, for $x \in \text{Stab}(H_{\phi(h)})$, we denote $[x]_{\sigma_{H_{\phi(h)}}}$ by $[x]$. Then the map $\theta : \Gamma(H) \to \Gamma(H_{\phi(h)})$ given by $[x] \mapsto [\phi(x)]$ is a homomorphism. First we show that θ is well-defined. That is, if we have $x, y \in \text{Stab}(H)$ such that $[x] = [y]$, then $\theta([x]) = \theta([y])$. This follows as if $[x] = [y]$, then $h'x = h'y$ for all $h' \in H$. Then, as ϕ is a homomorphism, we have

$$\phi(h')\phi(x) = \phi(h')\phi(y). \tag{3.1}$$

for all $h' \in H$. We want to show that $h''\phi(x) = h''\phi(y)$ for all $h'' \in H_{\phi(h)}$. For a contradiction, assume that there exists $h'' \in H_{\phi(h)}$ such that $h''\phi(x) \neq h''\phi(y)$. As $h'' \mathcal{H} \phi(h)$, there exists $\ell \in \phi(S)^1$ such that $\ell h'' = \phi(h)$. Furthermore left multiplication by ℓ permutes $H_{\phi(h)}$ by Green's Lemma ([28, Lemma 2.2.4]). Hence we conclude that $\phi(h)\phi(x) = \ell h''\phi(x) \neq \ell h''\phi(y) = \phi(h)\phi(y)$. But this contradicts equation 3.1, and so $h''\phi(x) = h''\phi(y)$ for all $h'' \in H_{\phi(h)}$. That is $\phi(x)\, \sigma_{H_{\phi(h)}}\, \phi(y)$, completing the claim that θ is well-defined.

To see θ is a homomorphism, let $x, y \in \mathrm{Stab}(H)$. Then

$$\theta([x][y]) = \theta([xy]) = [\phi(xy)] = [\phi(x)\phi(y)] = [\phi(x)][\phi(y)] = \theta([x])\theta([y]),$$

and so θ is a homomorphism.

Finally, we show that $\theta([u]) \notin \theta(T)$. Indeed, if $\theta([u]) \in \theta(T)$ then $\theta([u]) = \theta([t])$ for some $[t] \in T$. In this case $\phi(h)\phi(u) = \phi(h)\phi(t)$, where $t \in \langle x_1, x_2, \ldots, x_n\rangle^1$. This contradicts $\phi(hu) \notin \phi(\overline{T})$. Hence $\Gamma(H)$ is WSS. $\square$

We immediately obtain the following lemma concerning commutative semigroups.

Corollary 3.2.12. *Every Schützenberger group of a weakly subsemigroup separable commutative semigroup is itself weakly subsemigroup separable.*

Although Schützenberger groups of weakly subsemigroup separable semigroups need not in general be weakly subsemigroup separable, it remains an open question if they are necessarily weakly *subgroup* separable.

Open Problem 3.2.13. Let S be a semigroup and let H be an $\mathcal{H}$-class of S. If S is weakly subsemigroup separable, is $\Gamma(H)$ weakly subgroup separable?

In the final part of this section we provide some partial solutions to this problem, one of which is utilized in the proof of Proposition 3.4.11.

Proposition 3.2.14. *Let S be a semigroup and let H be an $\mathcal{H}$-class of S. If S is weakly subsemigroup separable, then $\Gamma(H)$ satisfies the separability property with respect to the collection of all finitely generated abelian subgroups.*

Proof. First we consider when H is a group. By Proposition 3.2.6 we have that $\Gamma(H) \cong H$. As H is a subsemigroup of an WSS semigroup, it is itself WSS by Proposition 1.2.13. Then as every finitely generated abelian subgroup is also a finitely generated commutative subsemigroup, $\Gamma(H)$ satisfies the separability property with respect to the collection of all finitely generated abelian subgroups.

Suppose that H is not a group. Let G be a finitely generated abelian subgroup of $\Gamma(H)$ and let $b \in \Gamma(H) \setminus G$. Now, G is generated (as a group) by some set $\{a_1, \ldots, a_n\} \cup G_0$, where each a_i is non-torsion and G_0 is the finite torsion subgroup of G, see Theorem 3.1.1. As in the proof of Lemma 3.2.11, we shall just write $[s]$ for $[s]_{\sigma_H}$. Let U denote the subsemigroup

$$\{u \in \mathrm{Stab}(H) : [u] \in G\}$$

of $\mathrm{Stab}(H)$. For each $i \in \{1, \ldots, n\}$, select $x_i, y_i \in U$ such that $[x_i] = a_i$ and $[y_i] = a_i^{-1}$. Also, fix an element $h \in H$.

The remainder of this proof is organised as follows.

(1) We show that for each $i \in \{1, \ldots, n\}$, there exists $\alpha(i), \beta(i) \in \mathbb{N}$ such that $x_i^{\alpha(i)} h = h y_i^{\beta(i)}$.
(2) We build a finitely generated subsemigroup T of S such that $T \cap H = \{hu \mid u \in U\}$.
(3) We find a finite group K and a homomorphism $\theta : \Gamma(H) \to K$ such that $\theta(b) \notin \theta(G)$.

(1) Let $i \in \{1, \ldots, n\}$, and write $x = x_i$, $y = y_i$. We first show that $hy \in \langle h, x \rangle$. Since S is weakly subsemigroup separable, it suffices to show that hy cannot be separated from $\langle h, x \rangle$ by a finite index congruence. Indeed, if $\sim$ is a finite index congruence on S, there exist $k, \ell \in \mathbb{N}$ with $k < \ell$ such that

$hx^k \sim hx^\ell$. Then

$$hy = hx^k y^{k+1} \sim hx^\ell y^{k+1} = hx^{\ell-k-1} \in \langle h, x\rangle.$$

Since $\langle x\rangle \subseteq \mathrm{Stab}(H)$ and H is not a group, it must be the case that $\langle x\rangle \cap H = \emptyset$ by Lemma 3.2.4. This means that $hy \notin \langle x\rangle$. If $hy \in \langle h\rangle$, then $hy = h^n$ for some $n \geq 2$. In this case $H \cap H^n \neq \emptyset$ and so H is a group by Lemma 1.3.37. This is a contradiction and so $hy \notin \langle h\rangle$. Therefore, as $hy \in \langle h, x\rangle$, every way of expressing hy as a product over $\{h, x\}$ contains at least one occurrence of h and at least one occurrence of x. Fix one such product. Post-multiplying hy by an appropriate power of y, we deduce that $hy^j = uh$ for some $j \in \mathbb{N}$ and $u \in \langle h, x\rangle$. Note that for $z \in \mathbb{N}$ we have

$$hy^{jz} = uhy^{j(z-1)} = u^2hy^{j(z-2)} = \cdots = u^z h.$$

We now show that $hx^j \in \langle h, u\rangle$ using a similar argument. Indeed, if $\sim$ is a finite index congruence on S, there exist $k, \ell \in \mathbb{N}$ with $k < \ell$ such that $hy^{jk} \sim hy^{j\ell}$, and hence

$$hx^j = hy^{jk}x^{j(k+1)} \sim hy^{j\ell}x^{j(k+1)} = hy^{j(\ell-k-1)} = u^{\ell-k-1}h \in \langle h, u\rangle.$$

Noting that u is in the left stabiliser of H, by a similar argument as for $\langle x\rangle$, we deduce that $\langle u\rangle \cap H = \emptyset$. Therefore $hx^j \notin \langle u\rangle$. It is also true that $hx^j \notin \langle h\rangle$, by an similar argument to that which shows that $hy \notin \langle h\rangle$. Therefore, as $hx^j \in \langle h, u\rangle$, every way of expressing hx^j as a product over $\{h, u\}$ contains at least one occurrence of h and at least one occurrence of u.

So $hx^j = w_1hw_2$ for some $w_1 \in \langle u\rangle^1$ and $w_2 \in \langle h, u\rangle^1 \subseteq \langle h, x\rangle^1$, with $(w_1, w_2) \neq (1, 1)$. Then $w_1 = u^k$ for some $k \in \mathbb{N}_0$. Hence $w_1h = u^kh = hy^{jk}$. It cannot be the case that $hx^j = w_1h$. If it were, then $hx^j = hy^{jk}$, which would imply that

$$hx^{j(k+1)} = hx^jx^{jk} = hy^{jk}x^{jk} = h.$$

In this case, $[x]$ is a torsion element of $\Gamma(H)$, which is a contradiction. Therefore $hx^j \neq w_1h$ and it must be that $w_2 \neq 1$.

We cannot write w_2 as sht for some $s \in \langle x \rangle^1$ and $t \in \langle h, x \rangle^1$. Indeed, if we could, then since $w_1 h \in H$ and $x \in \mathrm{Stab}(H)$, we would have $hx^j \in H \cap H^n$ for some $n \geq 2$. Then by Lemma 1.3.37 H is a group, which is a contradiction. We must then have $w_2 \in \langle u \rangle$. But $u \in \langle h, x \rangle$, so we conclude that $u = x^m$ for some $m \in \mathbb{N}$. Then, as $hy^j = uh$, we have $hy^j = x^m h$, as desired. Now set $\alpha(i) = m$ and $\beta(i) = j$. We fix $\alpha(i)$ and $\beta(i)$ for the remainder of this proof.

(2) Now we build a finitely generated subsemigroup T of S such that $T \cap H = \{hu \mid u \in U\}$. For each $i \in \{1, \ldots, n\}$, let $m_i = \max\{\alpha(i), \beta(i)\}$. For each $g \in G_0$, select $u_g \in U$ such that $[u_g] = g$. We define a finite set

$$W = \{x_1^{j_1} \ldots x_n^{j_n} \mid 0 \leq j_i \leq m_i - 1 \text{ for } 1 \leq i \leq n\} \cup \{u_g \mid g \in G_0\} \subseteq U.$$

Let $X = \{x_i^{\alpha(i)} \mid 1 \leq i \leq n\}$, and let T be the subsemigroup of S generated by

$$Z = X \cup \{hw \mid w \in W\}.$$

Note, as $U \subseteq \mathrm{Stab}(H)$, by Lemma 3.2.4 we have that $U \cap Z = X$ and $H \cap Z = \{hw \mid w \in W\}$. We also have Z is finite as both X and W are finite. We prove that $T \cap H = \{hu \mid u \in U\}$.

First, let $h' \in T \cap H$. Then $h' = z_1 \ldots z_k$ for some $z_j \in Z$. If every $z_j \in X$, then $h' \in \mathrm{Stab}(H)$. Then by Lemma 3.2.4 we have that H is a group, which is a contradiction. Therefore, there exists a minimal j such that $z_j = hw$ for some $w \in W$. Then for each $i < j$, we have $z_i \in X$ and hence $z_i h \in hU$ from (1). So we deduce that $h' = hw' z_{j+1} \ldots z_k$ for some $w' \in U$. Let $u = w' z_{j+1} \ldots z_k$. We shall show that $u \in U$.

Let $i_1, i_2, \ldots, i_m$ be the indices such that $z_{i_1}, z_{i_2}, \ldots, z_{i_m} \in H$, where

$$j + 1 \leq i_1 < \cdots < i_m \leq k.$$

Let $h_\ell = z_{i_\ell} z_{i_\ell + 1} \ldots z_{i_{\ell+1} - 1}$ for $\ell \in \{1, \ldots, m-1\}$, and let $h_m = z_{i_m} \ldots z_k$. Then $h_\ell \in H$ for each $\ell \in \{1, \ldots, m\}$. But then

$$h' = (hw' z_{j+1} \ldots z_{i_1 - 1}) h_1 \ldots h_m \in H \cap H^{m+1}.$$

Then, by Lemma 1.3.37, H is a group, which is a contradiction. We conclude that $z_i \in X$ for every $i \in \{j+1, \dots, k\}$. It follows that $u \in U$ and hence $h' \in \{hu \mid u \in U\}$.

For the reverse containment, let $u \in U$. Since G is abelian, we have $[u] = a_1^{k_1} \dots a_n^{k_n} c$, where for $i \in \{1, \dots, n\}$ we have $k_i \in \mathbb{Z}$ and $c \in G_0$. Consider $i \in \{1, \dots, n\}$. If $k_i \geq 0$, let $p_i \in \mathbb{N} \cup \{0\}$ and $r_i \in \{0, \dots, \alpha(i) - 1\}$ be such that $k_i = p_i \alpha(i) + r_i$, and let $q_i = s_i = 0$. If $k_i < 0$, let $q_i \in \mathbb{N}$ and $s_i \in \{0, \dots, \beta(i) - 1\}$ be such that $k_i = -q_i \beta(i) + s_i$, and let $p_i = r_i = 0$. Now let $t_i = \max\{r_i, s_i\}$. Recalling that we have chosen $u_c \in \mathrm{Stab}(H)$ such that $[u_c] = c$, it follows that

$$hu = \left(x_1^{\alpha(1)}\right)^{q_1} \dots \left(x_n^{\alpha(n)}\right)^{q_n} \left(hx_1^{t_1} \dots x_n^{t_n} u_c\right) \left(x_1^{\alpha(1)}\right)^{p_1} \dots \left(x_n^{\alpha(n)}\right)^{p_n},$$

so $hu \in T \cap H$, and hence $T \cap H = \{hu \mid u \in U\}$.

(3) Finally, we find a finite group K and a homomorphism $\theta : \Gamma(H) \to K$ such that $\theta(b) \notin \theta(G)$. Choose $v \in \mathrm{Stab}(H)$ such that $[v] = b$. Then $hv \notin T$. This is because $hv \notin \{hu \mid u \in U\}$, since $v \notin U$ and the action of $\Gamma(H)$ is regular. Since S is WSS, there exists a finite semigroup P and a homomorphism $\phi : S \to P$ such that $\phi(hv) \notin \phi(T)$. Let $H_{\phi(h)}$ denote the $\mathcal{H}$-class of $\phi(h)$, and let K be the finite group $\Gamma(H_{\phi(h)})$. As in the proof of Lemma 3.2.11, the map $\theta : \Gamma(H) \to K$, given by $[t] \mapsto [\phi(t)]$, is a homomorphism such that $\theta(b) \notin \theta(G)$, as required. □

The fact that the weak subsemigroup separability of a semigroup guarantees that its Schützenberger groups have the separability property with respect to finitely generated abelian subgroups immediately gives us the following.

Corollary 3.2.15. *Let S be a semigroup and let H be an $\mathcal{H}$-class of S. If S is weakly subsemigroup separable and $\Gamma(H)$ is abelian, then $\Gamma(H)$ is weakly subgroup separable.*

As every cyclic group is a finitely generated abelian group, we also have the following corollary.

Corollary 3.2.16. *Let S be a semigroup and let H be an $\mathcal{H}$-class of S. If S is weakly subsemigroup separable, then $\Gamma(H)$ is monogenic subgroup separable.*

However the following remains an open question.

Open Problem 3.2.17. Let H be an $\mathcal{H}$-class of a semigroup S. If S is monogenic subsemigroup separable, is $\Gamma(H)$ monogenic subgroup separable?

In Section 3.5 we return to Schützenberger groups and consider the following question: if all the Schützenberger groups of a semigroup have a separability property, does the semigroup itself have that property?

3.3 Finitely Generated Commutative Semigroups

In this section, we consider when a finitely generated commutative semigroup has each of our separability properties. The following is already known.

Theorem 3.3.1. [31, Theorem 3] *Every finitely generated commutative semigroup is residually finite.*

Lallement attributes the above theorem to Mal'cev in [37]. To discern other separability properties of finitely generated commutative semigroups we make use of the theory concerning separability properties of Schützenberger groups which was developed in the previous section. We also make use of the structural theory of finitely generated commutative semigroups, which we now present.

In commutative semigroups, Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$ and $\mathcal{H}$ coincide. Therefore, instead of talking of the partial ordering of $\mathcal{J}$-classes that was described in Remark 1.3.28, we will instead speak of a partial ordering on $\mathcal{H}$-classes. This ordering, in terms of $\mathcal{H}$-classes, is restated below.

Definition 3.3.2. For a commutative semigroup S, inclusion among principal ideals induces a partial ordering on $\mathcal{H}$-classes:

$$H_x \leq H_y \text{ if } S^1x \subseteq S^1y.$$

Commutative semigroups are built from archimedean semigroups, which we now define.

Definition 3.3.3. An *archimedean* semigroup is a commutative semigroup S such that for each $a, b \in S$ there exists $n > 0$ such that $H_{a^n} \leq H_b$.

It is worth noting that some authors do not include commutativity as part of the definition for archimedean semigroups. However, here we follow the definition given by Grillet in [25, Chapter 4].

Example 3.3.4. The semigroup $\mathbb{N}$ is archimedean. As we have seen in Example 1.3.35, $\mathbb{N}$ is $\mathcal{H}$-trivial. This, combined with Example 1.3.29, gives us that for $m, n \in \mathbb{N}$ we have $H_m \leq H_n$ if and only if $m \geq n$. As for all $m, n \in \mathbb{N}$, there exists a $k \in \mathbb{N}$ such that $km \geq n$, we have that $H_{km} \leq H_n$ and so $\mathbb{N}$ is archimedean.

The following lemma will be useful in ascertaining the number of idempotents in a finitely generated commutative semigroup.

Lemma 3.3.5. *An archimedean semigroup has at most one idempotent.*

Proof. Let A be an archimedean semigroup and suppose that $e, f \in A$ are idempotents. Then by definition there exist $m, n > 0$ such that $H_{e^m} \leq H_f$ and $H_{f^n} \leq H_e$. But as $e^m = e$ and $f^n = f$, we have that $H_e \leq H_f$ and $H_f \leq H_e$. Hence $H_e = H_f$. As a consequence of Proposition 1.3.36, any $\mathcal{H}$-class can have at most one idempotent, and so we conclude that $e = f$. □

We immediately see that any (commutative) semigroup with more than one idempotent is not archimedean. Recalling Definition 2.5.1, we have the following structural theorem for commutative semigroups.

Proposition 3.3.6. [25, Theorem 4.2.2] *A commutative semigroup S is a semilattice of archimedean semigroups $\mathcal{S}(Y, \{S_\alpha\}_{\alpha \in Y})$. Furthermore, if S is finitely generated, then Y is finite.*

We refer to the subsemigroups that make up the semilattice as the *archimedean components* of S.

Corollary 3.3.7. *A finitely generated commutative semigroup contains only finitely many idempotents.*

Proof. This follows as Proposition 3.3.6 tells us that a finitely generated commutative semigroup is a finite semilattice of archimedean semigroups and Lemma 3.3.5 tells us an archimedean semigroup has at most one idempotent. □

In the remainder of this section it will be important to understand how archimedean semigroups with idempotents behave. To achieve this we need two definitions.

Definition 3.3.8. We call a semigroup with a zero *nilpotent* if every element has a power equal to 0.

Example 3.3.9. Any null semigroup is also a nilpotent semigroup.

The following lemma helps us understand finitely generated nilpotent semigroups.

Lemma 3.3.10. *A finitely generated commutative semigroup in which every element has finite index is finite.*

Proof. Let S be a finitely generated commutative semigroup. Let $X = \{x_1, x_2, \ldots, x_k\}$ be a generating set for S. For $i \in \{1, 2, \ldots, k\}$, define m_i to be the index of x_i and define r_i to the period of x_i. Define $n = \max\{m_i + r_i \mid 1 \leq i \leq k\}$. Let $s \in S$ be such that s can be written as a product over X of length ℓ such that $\ell \geq kn$. Then by the pigeonhole principle, there exists $x_i \in X$ such that there are at least n occurrences of x_i in this product. Then, as $n \geq m_i + r_i$, these n occurrences of x_i can be replaced by $n - r_i$ occurrences of x_i. We have found a new expression for s of length $\ell - r_i$. If $\ell - r_i \geq kn$, we may iterate the process until we have an expression for s of length strictly less that kn. Therefore any element of S can be written as a product over X of length at most $kn - 1$, and hence S is finite. □

Corollary 3.3.11. *A finitely generated commutative nilpotent semigroup is finite.*

Proof. It follows from the definition of nilpotency that every element in a nilpotent semigroup has finite index. □

We now give the second definition needed to characterise archimedean semigroups with idempotent.

Definition 3.3.12. An *ideal extension* of a semigroup S by a semigroup Q is a semigroup E such that S is an ideal of E and the Rees quotient E/S is isomorphic to Q.

Example 3.3.13. Let $\overline{\mathbb{N}} = \{\overline{n} \mid n \in \mathbb{N}\}$ be a copy of $\mathbb{N}$. Let $E = \overline{\mathbb{N}} \cup \mathbb{N}$ with multiplication (written additively) inherited from $\overline{\mathbb{N}}$ and $\mathbb{N}$, and for $\overline{m} \in \overline{\mathbb{N}}$ and $n \in \mathbb{N}$ define

$$\overline{m} + n = n + \overline{m} = m + n.$$

This multiplication is associative and therefore E is a semigroup. Furthermore, $\mathbb{N}$ is an ideal of E. From the observation of Example 1.3.16, we have that $E/\mathbb{N} = \overline{\mathbb{N}}^0$. As $\overline{\mathbb{N}}^0 \cong \mathbb{N}^0$, we have that E is an ideal extension of $\mathbb{N}$ by $\mathbb{N}^0$.

The following result provides a characterisation of archimedean semigroups with idempotents.

Proposition 3.3.14. [25, Proposition 4.2.3] *A commutative semigroup S is archimedean with an idempotent if and only if S is either a group or an ideal extension of a group by a nilpotent semigroup.*

In general, the structure of archimedean semigroups is complex. For more about the decomposition of commutative semigroups into archimedean subsemigroups, see [25, Chapter 4].

As with the ordering of $\mathcal{J}$-classes, for all $a, x \in S$ we have that

$$H_{xa} \leq H_a.$$

With this observation is mind, we are able to give the following well-known result regarding the Schützenberger groups of finitely generated commutative semigroups.

Lemma 3.3.15. *Every Schützenberger group of a finitely generated commutative semigroup is a finitely generated abelian group.*

Proof. Let S be a finitely generated commutative semigroup and let H be an $\mathcal{H}$-class of S. Let X be a finite generating set for S. Then the semigroup S^1 is generated by the set $X \cup \{1\}$. Therefore S^1 is a finitely generated commutative monoid. We show that $\mathrm{Stab}(H)$ is also finitely generated and commutative.

As $\mathrm{Stab}(H)$ is a submonoid of S^1, it is commutative. We claim that $Y = (X \cup \{1\}) \cap \mathrm{Stab}(H)$ is a generating set for $\mathrm{Stab}(H)$. As $\mathrm{Stab}(H)$ is a submonoid of S^1, we certainly have that $\langle Y \rangle \subseteq \mathrm{Stab}(H)$. We need to show that $\mathrm{Stab}(H) \subseteq \langle Y \rangle$. For a contradiction, assume that there exists $s \in \mathrm{Stab}(H)$ such that $s \notin \langle Y \rangle$. Therefore, any possible way as expressing s as a product of elements of $X \cup \{1\}$ must contain at least one occurrence of an element in $X \setminus Y$. Consider $x \in X \setminus Y$. Then, by the definition of Y, it must be the case that $x \notin \mathrm{Stab}(H)$. Then for $h \in H$ we have $H_{hx} < H_h = H$, as $Hx \neq H$. Given that any decomposition of s must contain such an x, we conclude that $H_{hs} \leq H_{hx} < H$. Hence $Hs \neq H$. This is a contradiction. Therefore $\mathrm{Stab}(H) \subseteq \langle Y \rangle$.

As $\mathrm{Stab}(H)$ is a finitely generated commutative monoid, any quotient of $\mathrm{Stab}(H)$ is also finitely generated and commutative. Hence, we conclude that $\Gamma(H)$, the Schützenberger group of H, is a finitely generated abelian group. □

The ordering of $\mathcal{H}$-classes leads to a family of ideals. Before we define these ideals, we give the following result about the ordering of $\mathcal{H}$-classes.

Lemma 3.3.16. *For a commutative semigroup S, if there exists a minimal $\mathcal{H}$-class H, then H is the minimum $\mathcal{H}$-class.*

Proof. Suppose that H is minimal. Then if $H_s \leq H$ it must be that $H_s = H$. We must show that $H \leq H_t$ for all $t \in S$. Let $t \in S$ and fix $h \in H$. Then $H_{ht} \leq H$ and $H_{ht} \leq H_t$. But as H is minimal, it must be the case that $H_{ht} = H$. Then $H \leq H_t$, as required. $\square$

Recall that for each non-minimum $\mathcal{J}$-class of a semigroup S, there is an associated ideal $I(J)$, see Definition 1.3.30. Since Green's relations $\mathcal{J}$ and $\mathcal{H}$ coincide for commutative semigroups, we restate the definition of this ideal in terms of $\mathcal{H}$-classes.

Definition 3.3.17. For an non-minimal $\mathcal{H}$-class H of a commutative semigroup S, define

$$I(H) = \bigcup \{H_s \mid s \in S, H_s \not\geq H\}.$$

In order to determine the separability properties of finitely generated commutative semigroups, when dealing with a non-minimal $\mathcal{H}$-class H we shall often pass to the Rees quotient $S/I(H)$. The following proposition is needed to justify this strategy.

Proposition 3.3.18. *Let H be a non-minimal $\mathcal{H}$-class of a commutative semigroup S, and denote $I(H)$ by I. For an $\mathcal{H}$-class H' of S, let $\phi : S \to S/I$ to the canonical homomorphism. Let $x \in S$. Then the following hold:*

(i) *$\phi(H')$ is an $\mathcal{H}$-class of S/I;*
(ii) *$H' \geq H$ if and only if $\phi(H') \geq \phi(H)$;*
(iii) *$\phi(H)$ is the unique minimal non-zero $\mathcal{H}$-class in S/I;*
(iv) *$x \in \mathrm{Stab}(H)$ if and only if $\phi(x) \in \mathrm{Stab}(\phi(H))$.*

Proof. Recall that S/I consists of a zero element, namely I, and singleton sets $\{s\}$, where $s \in S \setminus I$. Denote the zero element of S/I by 0.

(i) Let $h' \in H'$ and let V denote that the $\mathcal{H}$-class of $\phi(h')$. We show that $V = \phi(H')$. If $H' \not\geq H$, then $\phi(H') = \{0\}$, and in particular $\phi(h') = 0$. As the $\mathcal{H}$-class of 0 is $\{0\}$, we have that $V = \phi(H')$.

Now suppose that $H' \geq H$. In this case $\phi(h') \neq 0$ and hence $V \neq \{0\}$. Let

$\phi(s) \in \phi(H')$. Then as $h', s \in H'$ there exist $x, y \in S^1$ such that $h'x = s$ and $sy = h'$. Then $\phi(h')\phi(x) = \phi(h'x) = \phi(s)$ and $\phi(s)\phi(y) = \phi(sy) = \phi(h')$. This shows that $\phi(h')\,\mathcal{H}\,\phi(s)$ and so $\phi(H') \subseteq V$.

Now let $\phi(v) \in V$. Then there exist $a, b \in (S/I)^1$ such that $\phi(h')a = \phi(v)$ and $\phi(v)b = \phi(h')$. If a is not the identity element of $(S/I)^1$, let $a' \in S$ such that $\phi(a') = a$. Otherwise let a' be the identity of S^1, in which case we say $\phi(a')$ is the identity of $(S/I)^1$. Similarly, if b is not the identity element of $(S/I)^1$, let $b' \in S$ such that $\phi(b') = b$. Otherwise let b' be the identity of S^1. As $V \neq \{0\}$, we have $\phi^{-1}(\phi(h')) = \{h'\}$ and $\phi^{-1}(\phi(v)) = \{v\}$. Then

$$\phi(h'a') = \phi(h')\phi(a') = \phi(h')a = \phi(v)$$

and so $h'a' = v$. Similarly $vb' = h'$. Therefore

$$h'\,\mathcal{H}\,v \text{ and } V \subseteq \phi(H').$$

We conclude that $\phi(H')$ is an $\mathcal{H}$-class of S/I.

(ii) First suppose that $H' \geq H$. Then for $h' \in H'$ and $h \in H$, we have that $S^1h \subseteq S^1h'$. Let $x \in (S/I)^1\phi(h)$. Then there exists $y \in (S/I)^1$ such that $y\phi(h) = x$. If y is not the identity in $(S/I)^1$, pick $y' \in S$ such that $\phi(y') = y$. Otherwise let y' be the identity of S^1. Then $y'h \in S^1h'$. So there exists $z \in S^1$ such that $y'h = zh'$. If z is the identity of S^1 set $\phi(z)$ to the identity of $(S/I)^1$. Then

$$x = y\phi(h) = \phi(y')\phi(h) = \phi(y'h) = \phi(zh) = \phi(z)\phi(h')$$

and $x \in (S/I)^1\phi(h')$. Hence $\phi(H') \geq \phi(H)$.

Now suppose that $\phi(H') \geq \phi(H)$. For a contradiction assume that $H' \not\geq H$. Then by the definition of I, we have that $\phi(H') = \{0\}$. As $\{0\}$ is the $\mathcal{H}$-class of the zero element, we have $\phi(H') = \{0\} < \phi(H)$. This is a contradiction, and so $H' \geq H$.

(iii) By the definition of I, we have that $\phi(H)$ is a non-zero $\mathcal{H}$-class in S/I. Let V be a non-zero $\mathcal{H}$-class in S/I. We need to show that $V \geq \phi(H)$. There exists $s \in S$ such that $\phi(s) \in V$. Let H' denote the $\mathcal{H}$-class of s. Then by part (i) we have that $V = \phi(H')$. As $V = \phi(H')$ is non-zero we have that $H' \geq H$ and so by part (ii), $V = \phi(H') \geq \phi(H)$, as desired.

(iv) Let $x \in \text{Stab}(H)$. Then $Hx = H$. Then $\phi(Hx) = \phi(H)\phi(x) = \phi(H)$ and so $\phi(x) \in \text{Stab}(\phi(H))$.

Now suppose that $\phi(x) \in \text{Stab}(H_I)$. Let $h \in H$. Then by part (i), $\phi(h)\phi(x) = \phi(\overline{h})$ for some $\overline{h} \in H$. By the definition of I we have that $\phi^{-1}(\phi(\overline{h})) = \{\overline{h}\}$. Then $hx = \overline{h}$. By Lemma 3.2.3, this is sufficient to show that $x \in \text{Stab}(H)$, as desired. □

In order to make use of the strategy of using the Rees quotient $S/I(H)$, we need to understand the structure of $S/I(H)$. In particular, we want to know the relationship between the archimedean components of $S/I(H)$ and the stabiliser of H. We first need the following lemma.

Lemma 3.3.19. *Let S be a finitely generated commutative semigroup and let H be an $\mathcal{H}$-class. Fix $s \in S$ with $H_s \geq H$. Then*

$$\text{Stab}(H) = \{x \in S^1 \mid H_{sx^n} \geq H \textit{ for all } n \in \mathbb{N}\}.$$

Proof. First we assume that H is the minimum $\mathcal{H}$-class in S. In this case $S^1 = \text{Stab}(H)$. To see this fix $h \in H$. Then for any $x \in S^1$ we have $H_{hx} \leq H_h = H$. But as H is the minimum $\mathcal{H}$ class we have that $H_{hx} = H$. That is, $hx \in H$ and so $x \in \text{Stab}(H)$ by Lemma 3.2.3. Furthermore, as H is the minimum $\mathcal{H}$-class we have $H_{sx^n} \geq H$ for all $s \in S$, $x \in S^1$ and $n \in \mathbb{N}$. Hence the result follows.

Now assume that H is a non-minimal $\mathcal{H}$-class. We claim that

$$I(H) \cap \{x \in S^1 \mid H_{sx^n} \geq H \text{ for all } n \in \mathbb{N}\} = \emptyset.$$

To see this let $i \in I(H)$. Then by the definition of $I(H)$, we have $H_i \ngeq H$. But $H_{si^n} \leq H_i$ for all $n \in \mathbb{N}$. In particular we have that $H_{si^n} \ngeq H$ for all $n \in \mathbb{N}$. Hence $i \notin \{x \in S^1 \mid H_{sx^n} \geq H \text{ for all } n \in \mathbb{N}\}$, completing the proof of the claim. As both $\mathrm{Stab}(H) \subseteq S^1 \setminus I$, and $\{x \in S^1 \mid H_{sx^n} \geq H \text{ for all } n \in \mathbb{N}\} \subseteq S^1 \setminus I$, we may factor out by $I(H)$. Hence we assume that H is the unique minimal non-zero $\mathcal{H}$-class.

Let $x \in \mathrm{Stab}(H)$. For a contradiction assume that there exists $n \in \mathbb{N}$ such that $H_{sx^n} \ngeq H$. Then, as H is the unique minimal non-zero $\mathcal{H}$-class, we have $sx^n = 0$. As $H_s \geq H$, there exists $t \in S^1$ such that $st \in H$. Then

$$0 = sx^n t = stx^n \in Hx^n = H,$$

which is a contradiction. Hence $\mathrm{Stab}(H) \subseteq \{x \in S^1 \mid H_{sx^n} \geq H \text{ for all } n \in \mathbb{N}\}$.

Now assume that $H_{sx^n} \geq H$ for all $n \in \mathbb{N}$. Fix $h \in H$. Assume for a contradiction that $x \notin \mathrm{Stab}(H)$. Then as $H_{hx} < H$ and as H is the minimal no-zero $\mathcal{H}$-class, we have that $hx = 0$. As S is finitely generated and commutative, by Theorem 3.3.1 it is residually finite. Let $\sim$ be an arbitrary finite index congruence on S. Then there exist $m, n \in \mathbb{N}$, with $m < n$, such that $sx^m \sim sx^n$. As $H_{sx^m} \geq H$, there exists $t \in S^1$ such that $sx^m t = h$. Then

$$h = sx^m t \sim sx^n t = sx^m t x^{n-m} = hx^{n-m} = 0.$$

As $\sim$ is arbitrary, we have shown we cannot separate h and 0 in a finite quotient. This contradicts S being residually finite and so $x \in \mathrm{Stab}(H)$. Hence the result holds. □

Corollary 3.3.20. *Let S be a finitely generated commutative semigroup and let H be a non-minimal $\mathcal{H}$-class. Let $I = I(H)$. If A is an archimedean component in S/I not containing the zero element 0, then $A \subseteq \mathrm{Stab}(H_I)$.*

Proof. Let $a \in A$. As A is a subsemigroup we have $a^n \in A$ for all $n \in \mathbb{N}$. In particular we have that $a^n \neq 0$ for all $n \in \mathbb{N}$. So $H_{a \cdot a^n} \geq H_I$ for all $n \in \mathbb{N}$ by part (iii) of Proposition 3.3.18. Hence $a \in \mathrm{Stab}(H_I)$ by Lemma 3.3.19. □

We now present a lemma that plays a crucial role in determining the separability properties of finitely generated commutative semigroups.

Lemma 3.3.21. *Let S be a finitely generated commutative semigroup and let H be a finite $\mathcal{H}$-class of S. Let $h \in H$. Then there exists a finite index congruence ρ on S such that $[h]_\rho = \{h\}$.*

Proof. We split into two cases, the first is that H is not a group, and the second is that H is a group.

Case 1. Assume that H is not a group. Then by Lemma 1.3.37, $H^2 \neq H$. In particular, H is not the minimal $\mathcal{H}$-class. Denote $I(H)$ by I. Factoring out I, we may assume that H is the unique minimal non-zero $\mathcal{H}$-class in S.

For $x \in S$, let A_x denote the archimedean component of the element x. As H is not a group, we have $H^2 = \{0\}$ and in particular $h^2 = 0$. We claim that $A_h = A_0$. To see this, recall that the S is a semilattice of archimedean components. For an element a of a semilattice, we have that $a = a^2$. As we have observed that $h^2 = 0$, we conclude that h and 0 in the same archimedean component, completing the claim. Then as 0 is an idempotent, we have that A_h is the ideal extension of a group by a nilpotent semigroup (Proposition 3.3.14). As the $\mathcal{H}$-class of a zero element is trivial, we conclude that the only subgroup of A_h is trivial. That is, A_h is a nilpotent semigroup. Now from Corollary 3.3.20, it follows that all archimedean components apart from A_h are subsets of $\mathrm{Stab}(H)$. Furthermore $\mathrm{Stab}(H) \cap A_h = \emptyset$. This follows as A_h is a nilpotent semigroup. Hence for all $a \in A_h$, there exists $n \in \mathbb{N}$ such that $a^n = 0$. Then $ha^n = 0$, showing it is not possible for a to be in $\mathrm{Stab}(H)$. Therefore, it follows that S^1 is the disjoint union of $\mathrm{Stab}(H)$ and A_h. Consider any finite generating set for S, and write it as $X \cup Y$, where $X \subseteq \mathrm{Stab}(H)$ and $Y \subseteq A_h$. Then $\langle X \rangle \subseteq \mathrm{Stab}(H)$ and, as A_h is non-empty, it must be the case that Y is non-empty. Note that $U = \langle Y \rangle$ is finite by Corollary 3.3.11 as A_h is a finitely generated commutative nilpotent semigroup. We may assume that X is non-empty, for otherwise $S = U$ is a finite semigroup and the result holds.

Let $X = \{x_1, x_2, \ldots, x_m\}$ and let $\overline{X} = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m\}$ be disjoint from X. Let $\mathrm{FC}_{\overline{X}}$ denote the free commutative monoid on $\overline{X}$. Let $\phi : \mathrm{FC}_{\overline{X}} \to \mathrm{Stab}(H)$ be the unique extension to a homomorphism of the map given by $\overline{x}_i \to x_i$.

For $u \in U$ define

$$I_u = \{w \in \mathrm{FC}_{\overline{X}} \mid u\phi(w) \in H\}.$$

If I_u is non-empty, it is an ideal of $\mathrm{FC}_{\overline{X}}$. Indeed, if $w \in I_u$ and $z \in \mathrm{FC}_{\overline{X}}$, then since $u\phi(w) \in H$ and $\phi(z) \in \mathrm{Stab}(H)$, we have that $u\phi(wz) = (u\phi(w))\phi(z) \in H$.

We claim that I_u is finitely generated as an ideal. To see this note that $\mathrm{FC}_{\overline{X}}$ is isomorphic to $\mathbb{N}_0^{|X|}$. The ideals of the latter semigroup are upward closed sets under the component-wise ordering on tuples. By Dickson's Lemma ([13, Lemma A]), $\mathbb{N}_0^{|X|}$ has no infinite antichains when considered as a partially ordered set with the aforementioned partial order. That is, in $\mathbb{N}_0^{|X|}$ there does not exist an infinite subset such that every pair of elements is pairwise incomparable. It then follows that every ideal of $\mathbb{N}_0^{|X|}$ is finitely generated as an ideal, and hence so is every ideal of $\mathrm{FC}_{\overline{X}}$, proving the claim.

Let $U' = \{u \in U \mid I_u \neq \emptyset\}$. For each $u \in U'$ let Z_u be a finite generating set for I_u, and let

$$Z = \bigcup_{u \in U'} Z_u.$$

As U is finite, we have that Z is finite. For each $z \in Z$, we have

$$z = \overline{x}_1^{\alpha_1(z)} \overline{x}_2^{\alpha_2(z)} \ldots \overline{x}_m^{\alpha_m(z)}$$

for some $\alpha_i(z) \in \mathbb{N}_0$. Define

$$n = \max\{\alpha_i(z) \mid z \in Z, 1 \leq i \leq m\}.$$

Let ρ be the congruence on S generated by $\{(x_i^n, x_i^{n+|H|}) \mid 1 \leq i \leq m\}$. First we note that S/ρ is finite. To see this let $\phi : S \to S/\rho$ be the canonical homomorphism. Then S/ρ is a finitely generated commutative semigroup generated by the set $\phi(X) \cup \phi(Y)$. As each of the generators has finite index, S/ρ is finite by Lemma 3.3.10. We show that $[h]_\rho = \{h\}$.

Let $(h, t) \in \rho$. We need to show that $t = h$. Clearly it is sufficient to assume that t is obtained from h by a single application of a pair from the generating set of ρ. So, let $h = sx_i^p$ and $t = sx_i^q$ where $1 \leq i \leq m$, $s \in S^1$ and $\{p, q\} = \{n, n + |H|\}$.

If $(p, q) = (n, n + |H|)$ then $t = hx_i^{|H|}$. Since $x_i \in \mathrm{Stab}(H)$, $[x_i]_\sigma$ is an element of the Schützenberger group $\Gamma(H)$. As $|\Gamma(H)| = |H|$, it follows that $[x_i^{|H|}]_\sigma = [x_i]_\sigma^{|H|} = [1]_\sigma$. Hence $t = hx_i^{|H|} = h$.

Now we consider the case when $(p,\ q) = (n + |H|, n)$. As $h \notin \mathrm{Stab}(H)$, we have $s \in A_h \backslash \{0\}$. Any way of decomposing s into generators must contain at least one element from Y. Therefore, we have that $s = us'$, where $u \in U'$ and $s' \in \mathrm{Stab}(H)$. Fix some

$$w = \overline{x}_1^{\beta_1} \overline{x}_2^{\beta_2} \dots \overline{x}_m^{\beta_m} \in \mathrm{FC}_{\overline{X}}$$

such that $\phi(w) = s'$. As $h = us' x_i^{n+|H|} \in H$, we have that

$$w\overline{x}_i^{n+|H|} = \overline{x}_1^{\beta_1} \overline{x}_2^{\beta_2} \dots \overline{x}_i^{\beta_i + n + |H|} \dots \overline{x}_m^{\beta_m} \in I_u.$$

Then there exist $z \in Z$ and $w' = \overline{x}_1^{\gamma_1} \dots \overline{x}_m^{\gamma_m} \in \mathrm{FC}_{\overline{X}}$ such that

$$zw' = w\overline{x}_i^{n+|H|}.$$

For $1 \leq j \leq m$, we have that

$$\alpha_j(z) + \gamma_j = \begin{cases} \beta_j & \text{if } j \neq i, \\ \beta_i + n + |H| & \text{if } j = i. \end{cases}$$

As $\alpha_i(z) \leq n$ by definition, it must be the case that $\gamma_i \geq |H|$. Then

$$w\overline{x}_i^n = z\overline{x}_1^{\gamma_1} \dots \overline{x}_i^{\delta} \dots \overline{x}_m^{\gamma_m}$$

where $\delta = \gamma_i - |H| \geq 0$. Hence $w\overline{x}_i^n \in I_u$ and so $sx_i^n = t \in H$. As $h = tx_i^{|H|}$, a similar argument as above proves that $h = t$, as required.

Case 2. Now we assume that H is a group. By Proposition 3.3.14, A_h is either a group or the ideal extension of H by a nilpotent semigroup. Hence $A_h \subseteq \mathrm{Stab}(H)$, as H is an ideal of A_h. If H is the minimal $\mathcal{H}$-class of S, then $S^1 = \mathrm{Stab}(H)$. If H is not minimal, we may assume that it is the unique minimal non-zero $\mathcal{H}$-class of S (by taking the Rees quotient by $I(H)$). We claim that for $a \in S^1 \setminus \{0\}$, we have $a \in \mathrm{Stab}(H)$. This is because we have that $H_a \geq H$ (from the definition of $I(H)$). Then there exists $b \in S^1$ such that $ab \in H$. As H is a group, we have that $H \subseteq \mathrm{Stab}(H)$ and therefore $ab \in \mathrm{Stab}(H)$. Hence $a \in \mathrm{Stab}(H)$, completing the proof of the claim. As it is clear that $0 \notin \mathrm{Stab}(H)$ we conclude $S^1 \setminus \{0\} = \mathrm{Stab}(H)$.

In either case, let X be a finite generating set for $\mathrm{Stab}(H)$. As in Case 1, let $\overline{X} = \{\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m\}$ be a set in bijection with X, let $\mathrm{FC}_{\overline{X}}$ denote the free commutative monoid on X, and let $\phi : \mathrm{FC}_{\overline{X}} \to \mathrm{Stab}(H)$ be the unique extension to a homomorphism of a bijection $\overline{X} \to X$. Define

$$J = \{w \in \mathrm{FC}_X \mid \phi(w) \in H\}.$$

Then J is an ideal of $\mathrm{FC}_{\overline{X}}$ and hence is finitely generated as an ideal. Let Z be a finite generating set for J (as an ideal). As before let

$$z = \overline{x}_1^{\alpha_1(z)} \overline{x}_2^{\alpha_2(z)} \ldots \overline{x}_m^{\alpha_m(z)}$$

for some $\alpha_i(z) \in \mathbb{N}_0$ and define

$$n = \max\{\alpha_i(z) \mid z \in Z, 1 \leq i \leq m\}.$$

Let ρ be the congruence on S with generating set $\{(x^n, x^{n+|H|}) \mid x \in X\}$. Again we have that S/ρ is finite. An argument essentially the same as that of Case 1 shows that $[h]_\rho = \{h\}$, completing the proof. □

We are now ready to give a characterisation of when a finitely generated commutative semigroup is MSS. We have already seen that $\mathbb{Z}$ is not MSS in Example 1.2.5. It turns out that the only way a finitely generated commutative semigroup can fail to be MSS is if it contains a copy of $\mathbb{Z}$.

Theorem 3.3.22. *A finitely generated commutative semigroup S is monogenic subsemigroup separable if and only if every subgroup is finite.*

Proof. ($\Leftarrow$) Assume that every subgroup of S is finite. Let $U = \langle u \rangle \leq S$ for some $u \in S$ and let $v \in S \setminus U$. Consider H, the $\mathcal{H}$-class of v. If H is a group, then it is finite. By Lemma 3.3.21, there exists a finite index congruence ρ on S such that $[v]_\rho = \{v\}$. In particular, ρ separates v from U.

Now consider the case when H is not a group. We claim that $|H \cap U| \leq 1$. To see this assume that $|H \cap U| \geq 2$. Then there exist positive integers i, j, with $i < j$, such that $u^i, u^j \in H$. Let $k = j - i$. So we have $u^k \in \mathrm{Stab}(H)$ by Lemma 3.2.3. Then $(u^i)^{k+1} = (u^i)(u^k)^i \in H$. So by Lemma 1.3.37 we have that H is a group. This is a contradiction and so $|H \cap S| \leq 1$.

Now we note that in a commutative semigroup, $\mathcal{H}$ is a congruence. To see this let $a, b \in S$ such that $a \,\mathcal{H}\, b$. Then there exist $x, y \in S^1$ such that $ax = b$ and $by = a$. We show that $at \,\mathcal{H}\, bt$ for all $t \in S$. This follows as $atx = axt = bt$ and $bty = byt = at$. Hence $\mathcal{H}$ is a congruence.

We have that $[v]_{\mathcal{H}} = H$. Note every $\mathcal{H}$-class of $S/\mathcal{H}$ is a singleton. Then by factoring through $S/\mathcal{H}$ and evoking Lemma 3.3.21, there exists a finite index congruence ρ on S such that $[v]_\rho = H$. If $U \cap H = \emptyset$, then ρ separates v from U. Otherwise $|U \cap H| = 1$. Let $u^\ell \in U \cap H$. By Theorem 3.3.1 we have that S is residually finite and so there exists a finite index congruence σ on S such that $[v]_\sigma \neq [u^\ell]_\sigma$. Then $\rho \cap \sigma$ is a finite index congruence on S that separates v from U, giving monogenic subsemigroup separability.

($\Rightarrow$) We show the contrapositive. Assume that S contains an infinite subgroup G. As the maximal subgroups of S are the group $\mathcal{H}$-classes (Proposition 1.3.36), we may assume that G is an $\mathcal{H}$-class. Then the Schützenberger group $\Gamma(G)$ is isomorphic to G by Proposition 3.2.6. Then by Lemma 3.3.15 we have that $\Gamma(G)$, and therefore G, is an infinite finitely generated abelian group. So G contains a copy of $\mathbb{Z}$ by Corollary 3.1.3. As $\mathbb{Z}$ is not MSS by Example 1.2.5, S cannot be MSS, as subsemigroups inherit the property of monogenic subsemigroup separability by Proposition 1.2.13. □

We now turn our attention to the properties of weak subsemigroup separability, strong subsemigroup separability and complete separability.

Theorem 3.3.23. *Let S be a finitely generated commutative semigroup.*

Then the following are equivalent:

(1) *S is completely separable;*

(2) *S is strongly subsemigroup separable;*

(3) *S is weakly subsemigroup separable;*

(4) *every $\mathcal{H}$-class of S is finite.*

Proof. From Proposition 1.2.9 we have that (1) implies (2), and (2) implies (3).

(3) $\implies$ (4). We will show the contrapositive. Assume that S contains an infinite $\mathcal{H}$-class H. Then, by Proposition 3.2.6, we have that the Schützenberger group $\Gamma(H)$ is infinite. By Lemma 3.3.15, we have that $\Gamma(H)$ is an infinite finitely generated abelian group. Hence $\Gamma(H)$ contains a copy of $\mathbb{Z}$ by Corollary 3.1.3. As $\mathbb{Z}$ is not WSS by Example 1.2.5, $\Gamma(H)$ cannot be WSS as subsemigroups inherit the property of weak subsemigroup separability by Proposition 1.2.13. But then S cannot be WSS as Schützenberger groups of commutative WSS semigroups are themselves WSS by Corollary 3.2.12.

Lemma 3.3.21 gives us (4) implies (1). □

Using these characterisations we are able to give an example of a finitely generated commutative semigroup which is MSS but not WSS (and therefore not SSS nor CS).

Example 3.3.24. Let $T = \mathbb{N} \times \mathbb{N}$. Let $N = \{x_z \mid z \in \mathbb{Z}\} \cup \{0\}$ be a null semigroup with zero element 0. Let $S = T \cup N$ with multiplication inherited from T and N, and for $z \in \mathbb{Z}$ and $(i, j) \in T$ define

$$x_z \cdot (i, j) = (i, j) \cdot x_z = x_{z+i-j},$$
$$0 \cdot (i, j) = (i, j) \cdot 0 = 0.$$

An exhaustive check confirms that this multiplication is associative and hence S is a semigroup. It is also clear that the multiplication is commutative so S is a commutative semigroup. Note that the set $\{(1,0),(0,1),x_0\}$ is a generating set for S and so S is a finitely generated commutative semigroup.

There is only one idempotent contained in S, the element 0. Hence any subgroup of S is contained in H_0, the $\mathcal{H}$-class of 0. As 0 is a zero element for the entirety of S we have that $H_0 = \{0\}$. Hence every subgroup of S is finite and S is MSS by Theorem 3.3.22

Now consider the set $X = \{x_z \mid z \in \mathbb{Z}\} \subseteq S$. We claim that X is contained in H_{x_0}. Let $x_z \in X$. If $z = 0$ then it is clear that $x_z \in H_{x_0}$. If $z < 0$ then $x_z \cdot (z, 0) = x_0$ and $x_0 \cdot (0, z) = x_z$. As S is commutative, this is sufficient to show that $x_z \in H_{x_0}$. A similar argument deals with the case when $z > 0$. Hence we have that H_{x_0} is infinite. So S is not WSS by Theorem 3.3.23.

3.3.1 A Comparison with Previously Known Results

In [30] Kublanovskiĭ and Lesohin show that the properties of complete separability and strong subsemigroup separability coincide for finitely generated commutative semigroups. They also give characterisations of when a finitely generated commutative semigroup has one, and hence both, of these separability properties. To understand these characterisations we briefly outline their setup and results, without giving proofs.

Let S be a finitely generated commutative semigroup with finite generating set A. For $s \in S$, let $C_s = A \cap \operatorname{Stab}(H_s)$. Then C_s is finite and can be empty. We denote $|C_s|$ by k_s. Then $\langle C_s \rangle^1 = \operatorname{Stab}(H_s)$. Consider the free commutative monoid $\mathbb{N}_0^{k_s}$ on k_s generators. There is a canonical homomorphism $\phi : \mathbb{N}_0^{k_s} \to \operatorname{Stab}(H_s)$. We note that $\operatorname{Stab}(s)$, the point stabiliser of s, is a submonoid of $\operatorname{Stab}(H_s)$. Let $W_s = \phi^{-1}(\operatorname{Stab}(s)) \leq \mathbb{N}_0^{k_s}$ be the pre-image of $\operatorname{Stab}(s)$. We can view $\mathbb{N}_0^{k_s}$ as a submonoid of the free abelian group $\mathbb{Z}^{k_s}$. Consider the subgroup $G_s \leq \mathbb{Z}^{k_s}$ generated by W_s. As G_s is a subgroup of $\mathbb{Z}^{k_s}$, we have $G_s \cong \mathbb{Z}^{m_s}$ for some $m_s \leq k_s$. Using this, the authors obtain the following characterisation for SSS finitely generated commutative semigroups

Theorem 3.3.25. [30, Theorem 1] *A finitely generated commutative semi-*

group S is strongly subsemigroup separable if and only if $m_s = k_s$ for all $s \in S$.

From the proof of this result they obtained two corollaries, summarised as follows:

Corollary 3.3.26. [30, Corollaries 2 and 3] *For a finitely generated commutative semigroup S the following are equivalent:*

(1) *S is completely separable;*
(2) *S is strongly subsemigroup separable;*
(3) *if $a, b \in S$ are such that $a \in b^n S$ for all $n \in \mathbb{N}$, then there exists $m \in \mathbb{N}$ such that $a = b^m a$.*

Theorem 3.3.23 enhances the result of Corollary 3.3.26 by showing that for a finitely generated commutative semigroup, weak subsemigroup separability is also equivalent to complete separability. The characterisation of CS finitely generated commutative semigroups in terms of $\mathcal{H}$-classes given in Theorem 3.3.23 removes the need for the parameters k_s and m_s of Theorem 3.3.25. The methods developed to give this characterisation were also utilised in characterising when a finitely generated commutative semigroup is MSS. In particular, Lemma 3.3.21 is used in both the proof Theorem 3.3.22 and Theorem 3.3.23. The proofs we provide are independent of the work of Kublanovskiĭ and Lesohin, although the reader may note parallels between the methods used.

3.4 Beyond Finitely Generated Commutative Semigroups

Given that for finitely generated commutative semigroups, the three properties of complete separability, strong subsemigroup separability and weak subsemigroup separability coincide, the following questions naturally arise.

- For commutative semigroups in general (not necessarily finitely generated), do the properties of complete separability and strong subsemigroup separability coincide?

- For commutative semigroups in general, do the properties of strong subsemigroup separability and weak subsemigroup separability coincide?
- For finitely generated semigroups in general (not necessarily commutative), do the properties of complete separability and strong subsemigroup separability coincide?
- For finitely generated semigroups in general, do the properties of strong subsemigroup separability and weak subsemigroup separability coincide?

In this section we answer all of these questions in the negative. But we begin with a brief survey of the semigroup separability properties of abelian groups.

3.4.1 Semigroup Separability Properties of Abelian Groups

Recall the following result concerning the semigroup-theoretic separability properties for groups.

Proposition 1.2.17. *For a group G we have:*

(i) *G is monogenic subsemigroup separable if and only if G is torsion and monogenic subgroup separable;*

(ii) *G is weakly subsemigroup separable if and only if G is torsion and weakly subgroup separable;*

(iii) *G is strongly subsemigroup separable if and only if G is torsion and strongly subgroup separable.*

We will now discuss the situation for abelian groups. It is known that a group is residually finite if and only if it is isomorphic to a subdirect product of finite groups, see [12, Corollary 2.7.2]. As every finite abelian group is itself a direct product of finite cyclic groups, it follows that an abelian group is residually finite if and only if it is isomorphic to a subdirect product of finite cyclic groups.

In order to discuss monogenic subsemigroup separability and weak subsemigroup separability in abelian groups, we first give the following definition and lemma.

Definition 3.4.1. A semigroup is called *locally finite* if every finitely generated subsemigroup if finite.

Lemma 3.4.2. *Let S be a semigroup which is both residually finite and locally finite. Then S is weakly subsemigroup separable.*

Proof. Let $T \leq S$ be finitely generated and $x \notin T$. Then as S is locally finite, T is finite. Then as S is residually finite, we can separate x from T. □

For an abelian group to be weakly subsemigroup separable, it is necessary for it to be residually finite. It is also necessary for it to be torsion, as noted above. Since torsion abelian groups are locally finite, being residually finite and torsion are sufficient conditions for an abelian group to be weakly subsemigroup separable by Lemma 3.4.2. It also follows that an abelian group is monogenic subsemigroup separable if and only if it is residually finite and torsion. That is, the properties of monogenic subsemigroup separability and weak subsemigroup separability coincide for abelian groups.

In [17], Golubov characterises when commutative semigroups are strongly subsemigroup separable. We can apply his result to abelian groups but first we need the following definition.

Definition 3.4.3. For an abelian group A and for a prime p, the *p-primary component* of A is the set

$$A_p = \{a \in A \mid o(a) = p^n \text{ for some } n \in \mathbb{N}\}$$

where $o(a)$ is the order of the element a. We say that A_p has *finite exponent* if there exists $n \in \mathbb{N}$ such that $o(a) \leq p^n$ for all $a \in A_p$.

Then [17, Theorem 2] tells us that an abelian group A is strongly subsemigroup separable if and only if A is torsion and for each prime p, the p-primary component has finite exponent.

Finally, Theorem 1.2.19 tells us that a group is completely separable if and only if it is finite. We summarise all these observations in the following theorem.

Theorem 3.4.4. *Let A be an abelian group.*

(1) *A is residually finite if and only if it is isomorphic to a subdirect product of finite cyclic groups.*
(2) *A is monogenic subsemigroup separable if and only if A is weakly subsemigroup separable.*
(3) *A is weakly subsemigroup separable (and hence monogenic subsemigroup separable) if and only if it is torsion and residually finite.*
(3) *A is strongly subsemigroup separable if and only if it is torsion and for each prime p, the p-primary component has finite exponent.*
(4) *A is completely separable if and only if it is finite.*

3.4.2 Non-finitely Generated Commutative Semigroups

In this subsection we show two things. Firstly, that the properties of weak subsemigroup separability and strong subsemigroup separability do not coincide for commutative semigroups. Secondly, that the properties of strong subsemigroup separability and complete separability do not coincide for commutative semigroups. We first give an example of a commutative semigroup that is weakly subsemigroup separable but not strongly subsemigroup separable. In order to do this, we first establish the following result.

Proposition 3.4.5. *If a residually finite semigroup S has $\mathbb{N}$ as a homomorphic image, then it is weakly subsemigroup separable.*

Proof. Let $T \leq S$ be finitely generated and let $x \in S \setminus T$. By assumption there exists a homomorphism $\phi : S \to \mathbb{N}$. Let $n = \phi(x)$. The set $I = \{m \mid m > n\} \subseteq \mathbb{N}$ is an ideal of $\mathbb{N}$. Let $\psi : \mathbb{N} \to \mathbb{N}/I$ be the canonical homomorphism.

Since T is finitely generated and $\phi(st) > \phi(s)$ for any $s, t \in S$, it follows that the set

$$Y = \{t \in T \mid \phi(t) = n\}$$

is finite. Since S is residually finite, there exists a finite semigroup P and homomorphism $\sigma : S \to P$ such that $\sigma(x) \notin \sigma(Y)$. Then $(\psi \circ \phi) \times \sigma : S \to \mathbb{N}/I \times P$ given by $s \mapsto (\psi \circ \phi(s), \sigma(s))$ separates x from T. □

Example 3.4.6. Consider $S = \mathbb{N} \times \mathbb{Z}$. Now, S is residually finite since it is the direct product of two residually finite semigroups, see Lemma 4.1.1. As the projection map onto the first factor gives a homomorphic image which is $\mathbb{N}$, we conclude that S is weakly subsemigroup separable by Proposition 3.4.5.

We now show that S is not strongly subsemigroup separable. Consider $\mathbb{N} \times \mathbb{N} \leq S$ and the element $(2, 0) \notin \mathbb{N} \times \mathbb{N}$. Let $\sim$ be a finite index congruence on S. Then there exist $i, j \in \mathbb{Z}$ with $i < j$ such that $(1, i) \sim (1, j)$. Then

$$(2, 0) = (1, i)(1, -i) \sim (1, j)(1, -i) = (2, j - i) \in \mathbb{N} \times \mathbb{N}.$$

Hence, S is a commutative semigroup which is weakly subsemigroup separable but not strongly subsemigroup separable.

We are left to find an example of a strongly subsemigroup separable commutative semigroup which is not completely separable. Our example is a group.

Example 3.4.7. Let C_2 denote the cyclic group of order 2. Let $G = C_2^{\mathbb{N}}$ be the Cartesian product of countably many copies of C_2. By Theorem 1.2.19, G is not completely separable. But from Theorem 3.4.4, an abelian group is strongly subsemigroup separable if and only if it is torsion and for each prime p, the p-primary component is bounded in the exponent. As every non-identity element in G has order 2, G certainly satisfies these conditions. Hence G is strongly subsemigroup separable.

3.4.3 Finitely Generated Semigroups

In the previous section we showed that being commutative is not on its own a sufficient condition for the three properties complete separability, strong subsemigroup separability and weak separability to coincide. In this section we will show that being finitely generated is also not on its own a sufficient condition. That is, we provide two examples of finitely generated semigroups, one of which is weakly subsemigroup separable but not strongly subsemigroup separable, and the other strongly subsemigroup separable but not completely separable.

First we give an example of a finitely generated semigroup which is weakly subsemigroup separable but not strongly subsemigroup separable. We do this by introducing a construction of semigroups and establishing necessary and sufficient conditions for this construction to be weakly subsemigroup separable and finitely generated. For the construction and proof, we will use the following notation.

Notation 3.4.8. For a subset $Z \subseteq G$ of an abelian group G, let $X_Z = \{x_z \mid z \in Z\}$ be a set disjoint from G.

Construction 3.4.9. Let T be a semigroup, and let G be an abelian group such that there exists a surjective homomorphism $\phi : T \to G$. Let $N = X_G \cup \{0\}$ be a null semigroup with identity element 0, such that N is disjoint from T. Let $\mathcal{S}(T, G, \phi) = T \cup N$, with multiplication inherited from T and N, and for $t \in T$ and $x_g \in X_G$ we define the following multiplication:

$$x_g \cdot t = x_{g\phi(t)},$$
$$t \cdot x_g = x_{g(\phi(t))^{-1}},$$
$$t \cdot 0 = 0 \cdot t = 0.$$

An exhaustive check confirms this multiplication is associative and therefore $\mathcal{S}(T, G, \phi)$ is a semigroup.

This construction is designed to have the following property.

Lemma 3.4.10. *In Construction 3.4.9, the set X_G forms a non-group $\mathcal{H}$-*

class and the Schützenberger group of this $\mathcal{H}$-class is isomorphic to G.

Proof. Let e denote the identity of G and let H denote the $\mathcal{H}$-class of x_e. First we show that $H = X_G$. Note that for all $s \in \mathcal{S}(T, G, \phi)$, we have that $s \cdot x_e, x_e \cdot s \in X_G \cup \{0\}$. This means that $H \subseteq X_G \cup \{0\}$. But as 0 is a zero element for the entirety of $\mathcal{S}(T, G, \phi)$, it cannot be the case that $0 \in H$. Hence $H \subseteq X_G$.

Now let $x_g \in X_G$. As ϕ is surjective there exist $u, v \in T$ such that $\phi(u) = g$ and $\phi(v) = g^{-1}$. Then

$$x_e \cdot u = x_g, \quad x_g \cdot v = x_e, \quad v \cdot x_e = x_g, \quad u \cdot x_g = x_e.$$

That is, $x_g \,\mathcal{H}\, x_e$ and so $X_G \subseteq H$. Hence we have that $H = X_G$, as desired. As X_G is idempotent free, we have that H is not a group.

Now we show that $\Gamma(H)$ is isomorphic to G. First note that $\mathrm{Stab}(H) = T^1$. Let $\overline{\phi} : T^1 \to G$ be the extension of ϕ given by $1 \mapsto e$ and $t \mapsto \phi(t)$ for all $t \in T$. Then for $u, v \in T^1$, we claim that $[u]_\sigma = [v]_\sigma$ if and only if $\overline{\phi}(u) = \overline{\phi}(v)$, where σ is the Schützenberger congruence. To see this, first assume that $[u]_\sigma = [v]_\sigma$. Then for all $g \in G$, we have that $x_g \cdot u = x_g \cdot v$. So it must be that $g\overline{\phi}(u) = g\overline{\phi}(v)$ and so $\overline{\phi}(u) = \overline{\phi}(v)$. Equally, if $\overline{\phi}(u) = \overline{\phi}(v)$ then it must be that $x_g \cdot u = x_g \cdot v$ for all $g \in G$ and hence $[u]_\sigma = [v]_\sigma$. The claim gives us $\psi : \Gamma(H) \to G$, given by $[u]_\sigma \mapsto \overline{\phi}(u)$, is well-defined and injective.

We now show that ψ is a group isomorphism. First let $[u]_\sigma, [v]_\sigma \in \Gamma(H)$. Then

$$\psi([u]_\sigma [v]_\sigma) = \psi([uv]_\sigma) = \overline{\phi}(uv) = \overline{\phi}(u)\overline{\phi}(v) = \psi([u]_\sigma)\psi([v]_\sigma).$$

Hence ψ is a homomorphism. Finally, as ϕ is surjective, we have that ψ is also surjective. Hence ψ is an isomorphism, as desired. □

We now give necessary and sufficient conditions for $\mathcal{S}(T, G, \phi)$ to be weakly subsemigroup separable.

Proposition 3.4.11. *Let T be a semigroup, let G be an abelian group such that there exists a surjective homomorphism $\phi : T \to G$, and let $S = \mathcal{S}(T, G, \phi)$. Then S is weakly subsemigroup separable if and only if T is weakly subsemigroup separable and G is weakly subgroup separable.*

Proof. ($\Rightarrow$) First assume that S is weakly subsemigroup separable. Since T is a subsemigroup of S, it must be weakly subsemigroup separable by Proposition 1.2.13. Since G is abelian and isomorphic to a Schützenberger group of S, it follows from Corollary 3.2.15 that G is weakly subgroup separable.

($\Leftarrow$) Now assume that T is weakly subsemigroup separable and G is weakly subgroup separable. Let $Y \subseteq S$ be a finite set, $U = \langle Y \rangle \leq S$ and $v \in S \backslash U$. Let $N \subseteq S$ be as in Construction 3.4.9. Let $Y_1 = Y \cap T$ and $Y_2 = Y \cap N$. We split into cases.

Case 1. Assume that $v \in T$. Note that $T \cap U = \langle Y_1 \rangle$. As T is weakly subsemigroup separable and $v \notin \langle Y_1 \rangle$, there exists a finite semigroup P and homomorphism $\psi : T \to P$ such that $\psi(v) \notin \psi(T \backslash \langle Y_1 \rangle)$. Define $\overline{\psi} : S \to P^0$ by

$$s \mapsto \begin{cases} \psi(s) & \text{if } s \in T, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\psi}$ is a homomorphism and $\overline{\psi}(v) \notin \overline{\psi}(U)$.

Case 2. Now assume that $v \in N$ and $Y_2 = \emptyset$. Then $U \subseteq T$. Let ρ be the congruence on S with classes T and N. Then $[v]_\rho \neq [u]_\rho$ for all $u \in U$ as required.

Case 3. Finally assume that $v \in N$ and $Y_2 \neq \emptyset$. Note that $0 \in Y_2^2 \subseteq N$, and hence $v \neq 0$. Let $v = x_g$ and $Y_2 \cap X_G = \{x_{g_1}, \ldots, x_{g_n}\}$. Let $H \leq_{\mathrm{Gp}} G$ be the subgroup generated (as a group) by the set $\phi(Y_1)$. Then we claim that

$$U \cap N = X_Z \cup \{0\},$$

where $Z = \bigcup_{i=1}^{n} Hg_i$. For the proof of this claim we extend the definition of ϕ to a function from T^1 to G, by setting $\phi(1) = e$, where e is the identity of G. We first show that $U \cap N \subseteq X_Z \cup \{0\}$. We have already noted that $0 \in U$. Now let $x_k \in (U \cap N) \setminus \{0\}$. Then $x_k \in \langle Y \rangle$. As $\langle Y_1 \rangle \leq T$, it must be that any way as expressing x_k as a product of elements of Y must contain at least one element of $Y_2 \setminus \{0\}$. Let x_{g_i} be that element. Then the decomposition of x_k as product of elements of Y has the form $x_k = s_1 x_{g_i} s_2$, where $s_1, s_2 \in U^1$. Now if either of s_1 or $s_2 \in N$, we we would have that $x_k = 0$, which is a contradiction. Hence $s_1, s_2 \in T^1$ and hence $s_1, s_2 \in \langle Y_1 \rangle^1$. Then $k = g_i \phi(s_1)^{-1} \phi(s_2) \in Z$, as desired. Hence $U \cap N \subseteq X_Z \cup \{0\}$.

Now we show $X_Z \cup \{0\} \subseteq U \cap N$. Again $\{0\} \in U \cap N$. Now let $x_k \in X_Z$. Then $k \in Hg_i$ for some $i \in \{1, 2, \ldots, n\}$. That is $k = hg_i$ for some $h \in \phi(\langle Y_1 \rangle)^1$. Then $h = \phi(s_1)^{-1} \phi(s_2)$, for some $s_1, s_2 \in \langle Y_1 \rangle^1$. Then $x_k = s_1 x_{g_i} s_2 \in U \cap N$, as desired. This completes the proof of the claim.

As $v \notin U$, it follows that $g \notin \bigcup_{i=1}^n Hg_i$. As G is weakly subgroup separable there exists a finite group K and homomorphism $\psi : G \to K$ such that $\psi(g) \notin \psi\left(\bigcup_{i=1}^n Hg_i\right)$ by Corollary 1.2.21. Let $P = \mathcal{S}(K, K, \mathrm{id}) = K \cup X_K \cup \{0\}$. Note that P is finite. Let $\overline{\psi} : S \to P$ be given by

$$s \mapsto \begin{cases} (\psi \circ \phi)(s) & \text{if } s \in T, \\ x_{\psi(a)} & \text{if } s = x_a \text{ for some } a \in G, \\ 0 & \text{if } s = 0. \end{cases}$$

Then it is straightforward to check that $\overline{\psi}$ is a homomorphism with $\overline{\psi}(v) \notin \overline{\psi}(U)$. □

The next lemma provides necessary and sufficient conditions for $\mathcal{S}(T, G, \phi)$ to be finitely generated.

Lemma 3.4.12. *Let T be a semigroup, let G be an abelian group such that there exists a surjective homomorphism $\phi : T \to G$, and let $S = \mathcal{S}(T, G, \phi)$. Then S is finitely generated if and only if T is finitely generated.*

Proof. If S is finitely generated, then as T is the complement of an ideal, it must also be finitely generated. Conversely, if T is generated by a finite set Y, then it is easy to see that S is generated by $Y \cup \{x_e\}$, where e is the identity of G. $\square$

We provide an example of a weakly subsemigroup separable semigroup S that has the following properties:

- S is finitely generated, non-commutative, but not strongly subsemigroup separable;
- S has a Schützenberger group which is not weakly subsemigroup separable.

Example 3.4.13. Let $\mathrm{F}_2 = \{a, b\}^+$ be the free semigroup on $\{a, b\}$. Let $\phi : \mathrm{F}_2 \to \mathbb{Z}$ be the surjective homomorphism given by $a \mapsto 1$ and $b \mapsto -1$. As F_2 is completely separable by Corollary 2.2.2 and $\mathbb{Z}$ is weakly subgroup separable by Theorem 3.1.4, it follows that $\mathcal{S}(\mathrm{F}_2, \mathbb{Z}, \phi)$ is weakly subsemigroup separable by Proposition 3.4.11. Since F_2 is finitely generated, $\mathcal{S}(\mathrm{F}_2, \mathbb{Z}, \phi)$ is finitely generated by Lemma 3.4.12. It is clear that $\mathcal{S}(\mathrm{F}_2, \mathbb{Z}, \phi)$ is not commutative.

By Lemma 3.4.10 we have that $X_{\mathbb{Z}}$ is an infinite non-group $\mathcal{H}$-class with Schützenberger group isomorphic to $\mathbb{Z}$, which is infinite. Hence $\mathcal{S}(\mathrm{F}_2, \mathbb{Z}, \phi)$ is not strongly subsemigroup separable by Proposition 3.2.8. Also note that from Example 1.2.5, we have the Schützenberger group isomorphic to $\mathbb{Z}$ is not MSS, and therefore certianly not WSS. Notice that, due to the way the right and left actions of F_2 on $X_{\mathbb{Z}}$ are defined, the $\mathcal{H}$-class $X_{\mathbb{Z}}$ does not satisfy the condition of Lemma 3.2.11.

We conclude this section by exhibiting an example of a finitely generated semigroup which is strongly subsemigroup separable but not completely separable.

Example 3.4.14. Let $F = \{a, b, c\}^+$ be the free semigroup on the set $\{a, b, c\}$. Let $I \leq F$ be the ideal generated by the set $\{x^2 \mid x \in F\}$. Let $S = F/I$ be the Rees quotient of F by I. We can view S as the set of all square-free words over the alphabet $\{a, b, c\}$ with a zero adjoined. Multiplication in S is concatenation, unless concatenation creates a word containing a contiguous subword which is a square, in which case the product is zero. Certainly S is finitely generated by $\{a, b, c\}$.

First we will show that S is not completely separable. It is known that there exists an infinite square-free sequence $w = x_1x_2x_3\dots$ over $\{a, b, c\}$, see [35, Chapter 2]. Then every finite prefix of w is a non-zero element of S. Let $w_i = x_1x_2\dots x_i \in S$. For $i < j$, let $v_{i,j} = x_{i+1}x_{i+2}\dots x_j \in S$. Let $\sim$ be a finite index congruence class on S. Then there exist $i, j \in \mathbb{N}$, with $i < j$, such that $w_i \sim w_j$. Then

$$w_j = w_iv_{i,j} \sim w_jv_{i,j} = w_iv_{i,j}v_{i,j} = 0.$$

So we have shown that it is not possible for 0 to be separated from $S \setminus \{0\}$ in a finite quotient. Hence, S is not completely separable.

Now let $T \leq S$. Then $0 \in T$. For $x \in S \setminus \{0\}$ let $|x|$ denote the length of x in terms of the generators $\{a, b, c\}$. Now let $v \notin T$ where $|v| = n$. Let

$$I = \{x \in S \mid |x| > n\} \cup \{0\}.$$

Then I is an ideal. Clearly the Rees quotient S/I is finite. Furthermore, $[v]_I = \{v\}$. Hence, S is strongly subsemigroup separable.

3.5 Semigroups with Finitely Many $\mathcal{H}$-classes

In Section 3.2 we asked which of our separability properties are inherited by Schützenberger groups. We showed in Corollary 3.2.9 and Corollary 3.2.10 that the properties of complete separability and strong subsemigroup separability are inherited by Schützenberger groups. Although it is not true that every Schützenberger group of a WSS semigroup is itself WSS or even MSS, as demonstrated in Example 3.4.13, we showed in Corollary 3.2.15 that weak subsemigroup separability is inherited by Schützenberger groups of commutative semigroups.

One may ask whether the properties are inherited in the opposite direction, that is, if every Schützenberger group of a semigroup S has a separability property must S itself satisfy the same property? This, however, turns out not to be true. Let $\mathscr{P}$ be any of the properties of complete separability, strong subsemigroup separability, weak separability, monogenic separability or residual finiteness. A semigroup whose Schützenberger groups all have property $\mathscr{P}$ may not itself have property $\mathscr{P}$. One example is the bicyclic monoid, given by the monoid presentation $\mathrm{Mon}\langle\, b, c \mid bc = 1\,\rangle$. The bicyclic monoid is $\mathcal{H}$-trivial, meaning that every $\mathcal{H}$-class is a singleton, so every Schützenberger group is the trivial group and certainly completely separable. However the bicyclic monoid is not even residually finite [11, Corollary 1.12]. In fact this direction fails comprehensively even for commutative semigroups, as the next example shows.

Example 3.5.1. Let $A = \langle a \rangle \cong \mathbb{N}$. Let $B = \{b_i \mid i \in \mathbb{N}\} \cup \{0\}$ be the countable null semigroup. Let $S = A \cup B$ with multiplication between A and B as follows:

$$a^i b_j = b_j a^i = \begin{cases} b_{j-i} & \text{for } j > i, \\ 0 & \text{otherwise,} \end{cases}$$
$$a^i 0 = 0 a^i = 0.$$

An exhaustive case analysis shows that this multiplication is associative and clearly it is commutative. We claim that S is $\mathcal{H}$-trivial. First note that 0 is a zero element for the entirety of S. We always have that the $\mathcal{H}$-class of a zero element is a singleton. Not consider an element $b^i \in B \setminus \{0\}$. Now, $b_i S = \{b_j \mid 1 \leq j < i\} \cup \{0\}$. So there does not exist $t \in b_i S$ such that $b_i \in tS$. Hence $H_{b_i} = \{b_i\}$. Finally consider an element $a^i \in A$. We have that $a^i S = \{a^j \mid j > i\} \cup B$. So there cannot exist $t \in a^i S$ such that $a^i \in tS$. So $H_{a^i} = \{a^i\}$ and so S is $\mathcal{H}$-trivial.

However, S is not residually finite. Suppose that $\sim$ is a finite index congruence on S. Then there exist $i, j \in \mathbb{N}$, with $i < j$, such that $b_i \sim b_j$. Then

$$0 = b_i a^{j-1} \sim b_j a^{j-1} = b_1.$$

So we cannot separate 0 and b_1 in a finite quotient so S is not residually finite.

Remark 3.5.2. In the semigroup S of Example 3.5.1, both the ideal B and the Rees quotient $S/B \cong \mathbb{N}_0$ are CS. That is, S is the ideal extension of a CS semigroup by a CS semigroup. However, this is not enough to guarantee that S is residually finite, let alone CS.

In the remainder of this section we restrict our attention to the class of semigroups which have only finitely many $\mathcal{H}$-classes. This is motivated by the following result.

Theorem 3.5.3. [23, Theorem 7.2] *Let S be a semigroup with finitely many $\mathcal{H}$-classes. Then S is residually finite if and only if all its Schützenberger groups are residually finite.*

We shall investigate whether there are analogous results for the properties of complete separability, strong subsemigroup separability, weak subsemigroup separability and monogenic subsemigroup separability.

For complete separability, the analogous result holds.

Proposition 3.5.4. *Let S be a semigroup with only finitely many $\mathcal{H}$-classes. Then the following are equivalent:*

(1) *S is completely separable;*
(2) *all the Schützenberger groups of S are completely separable;*
(3) *S is finite.*

Proof. (1) $\Rightarrow$ (2). If S is completely separable, then all of its Schützenberger groups are completely separable by Corollary 3.2.10.

(2) $\Rightarrow$ (3). If a Schützenberger group is completely separable it is finite by Theorem 1.2.19. As a Schützenberger group is in bijection with the corresponding $\mathcal{H}$-class, and S has only finitely many $\mathcal{H}$-classes, we conclude that S is finite.

(3) $\Rightarrow$ (1). Clear. □

From Corollary 3.2.9, we know that every Schützenberger group of an SSS semigroup is itself SSS. However, even when a semigroup has only finitely many $\mathcal{H}$-classes, every Schützenberger group being SSS does not guarantee that the semigroup is SSS, as the following example demonstrates.

Example 3.5.5. Let G be an infinite strongly subsemigroup separable abelian group. The existence of such a group is established by Theorem 3.4.4. Then, recalling Construction 3.4.9, $S = \mathcal{S}(G, G, \mathrm{id})$ has three $\mathcal{H}$-classes: G, X_G and $\{0\}$. The Schützenberger groups of the $\mathcal{H}$-classes are isomorphic to G, G and the trivial group respectively. Then certainly every Schützenberger group is SSS. However, since X_G is an infinite non-group $\mathcal{H}$-class, S is not strongly subsemigroup separable by Proposition 3.2.8.

We now turn our attention to weak subsemigroup separability. Along with monogenic subsemigroup separability, this is an example of a separability property that is not necessarily inherited by Schützenberger groups, as demonstrated by Example 3.4.13. However, when we restrict to a semigroup with only finitely many $\mathcal{H}$-classes, the following remains an open problem.

Open Problem 3.5.6. Is it true that a semigroup with only finitely many $\mathcal{H}$-classes is weakly subsemigroup separable if and only if all its Schützenberger groups are weakly subsemigroup separable?

Indeed, we do not even know if either direction of the above statement holds. We restrict our attention to locally finite semigroups with only finitely many $\mathcal{H}$-classes. By concentrating on this smaller class of semigroups we will be able to evoke Lemma 3.4.2, which says that a semigroup which is both residually finite and locally finite is weakly subsemigroup separable. We will focus our attention on semigroups where every maximal subgroup is solvable, which is defined below.

Definition 3.5.7. A group G is solvable if there exist subgroups

$$1 = G_0 <_{\mathrm{Gp}} G_1 <_{\mathrm{Gp}} \cdots <_{\mathrm{Gp}} G_k = G$$

such that G_{j-1} is normal in G_j and G_j/G_{j-1} is abelian for $j \in \{1, 2, \cdots, k\}$.

It is clear from the definition that abelian groups are solvable. This line of investigation allows us to give the following partial answer to Open Problem 3.5.6.

Theorem 3.5.8. *Let S be a semigroup with only finitely many $\mathcal{H}$-classes whose maximal subgroups are all solvable. Then S is weakly subsemigroup separable if and only if all its Schützenberger groups are weakly subsemigroup separable.*

In particular we note that a commutative semigroup with finitely many $\mathcal{H}$-classes is weakly subsemigroup separable if and only if all its Schützenberger groups are weakly subsemigroup separable. To prove Theorem 3.5.8 we make use of several lemmas.

A semigroup S is called an *epigroup* if every element of S has a power which lies in a subgroup of S.

Lemma 3.5.9. *A semigroup S with finitely many $\mathcal{H}$-classes is an epigroup.*

Proof. Let $s \in S$. As S has finitely many $\mathcal{H}$-classes there exist i, $j \in \mathbb{N}$ with $i < j$ such that $s^i \mathcal{H} s^j$. Let H be the $\mathcal{H}$-class of s^i. Then $s^{j-i} \in \mathrm{Stab}(H)$. Hence $(s^i)^{j-i} = (s^{j-i})^i \in \mathrm{Stab}(H)$, so $(s^i)^{j-i+1} = s^i(s^i)^{j-i} \in H$. Therefore $H^{j-i+1} \cap H \neq \emptyset$, and so H is a group by Lemma 1.3.37. □

Lemma 3.5.10. *Let S be a semigroup with finitely many $\mathcal{H}$-classes. If every maximal subgroup of S is torsion then S is periodic.*

Proof. By Lemma 3.5.9, S is an epigroup. Let $s \in S$. Then there is a power of s in a torsion subgroup of S; in particular, there exists $i \in \mathbb{N}$ such that $s^i = e$, where e is idempotent. Hence $s^{2i} = e^2 = e = s^i$ and S is periodic. □

Corollary 3.5.11. *Let S be a semigroup with finitely many $\mathcal{H}$-classes. If S is monogenic subsemigroup separable, then S is periodic.*

Proof. As S is MSS, then so are all of its maximal subgroups by Proposition 1.2.13. For a group to be MSS it is necessary for it to be torsion. Indeed, if a group is not torsion it contains a subgroup isomorphic to $\mathbb{Z}$ and therefore is not MSS by Example 1.2.5 and Proposition 1.2.13. Therefore the result follows by Lemma 3.5.10. □

Corollary 3.5.12. *Let S be a semigroup with finitely many $\mathcal{H}$-classes. If every Schützenberger group of S is monogenic subsemigroup separable then S is periodic.*

Proof. If every Schützenberger group is MSS, then every maximal subgroup of S is MSS and hence torsion, so S is periodic by Lemma 3.5.10. □

At this point on our journey to prove Theorem 3.5.8, we briefly pause to consider the property of monogenic subsemigroup separability. The theory developed so far allows us to give the following result.

Theorem 3.5.13. *Let S be a semigroup with finitely many $\mathcal{H}$-classes. Then S is monogenic subsemigroup separable if and only if every Schützenberger group of S is monogenic subsemigroup separable.*

Proof. ($\Rightarrow$) Assume that S is MSS. Then by Corollary 3.5.11 we have that S is periodic. Let $\Gamma(H)$ be some Schützenberger group of S. As $\Gamma(H)$ is the quotient of a periodic monoid, it must be that $\Gamma(H)$ is torsion. So any monogenic subsemigroup U of $\Gamma(H)$ is finite. Let $v \in \Gamma(H) \setminus U$. As S is MSS and therefore residually finite, we have that $\Gamma(H)$ is residually finite by Theorem 3.5.3. Therefore we can separate v from U.

($\Leftarrow$) Assume that every Schützenberger group of S is MSS. Let U be a monogenic subsemigroup of S and let $v \in S \setminus U$. By Corollary 3.5.12 we have that S is periodic. Hence U is a finite set. As every Schützenberger group of S is MSS, they are certainly residually finite. Then by Theorem 3.5.3, we have that S is residually finite. Hence we can separate v from U, as desired. □

Remark 3.5.14. We note that a semigroup with finitely many $\mathcal{H}$-classes is MSS if and only if it is residually finite and periodic. The forward direction follows as MSS semigroups are residually finite by Lemma 1.2.11 and MSS semigroups with finitely many $\mathcal{H}$-classes are periodic by Corollary 3.5.11. The backward directions follows as in a periodic semigroup every monogenic subsemigroup is finite and and residually finiteness can be thought of as the separability property with respect to the collection of finite subsets.

We now continue on our quest to prove Theorem 3.5.8. We show that an epigroup with finitely many $\mathcal{J}$-classes is locally finite if and only if all its maximal subgroups are locally finite. We first consider the case when a semigroup with a zero only has two $\mathcal{J}$-classes. For this we need some definitions.

Definition 3.5.15. A semigroup S with a zero is called *0-simple* if the only ideals of S are $\{0\}$ and S and $S^2 \neq \{0\}$. A 0-simple semigroup is called *completely 0-simple* if it is both 0-simple and an epigroup.

For equivalent definitions of completely 0-simple semigroups see [28, Theorem 3.2.11]. Rees showed that the class of completely 0-simple semigroups coincides with the class of Rees matrix semigroups over zero-groups [28, Theorem 3.2.3], which we now define.

Definition 3.5.16. For a group G, let the zero-group G^0 be G with a zero adjoined. Let I, Λ be non empty sets and $P = (p_{\lambda i})$ be a $\Lambda \times I$ matrix with entries from G^0 such that no row or column of P consists entirely of zeros. The *Rees matrix semigroup over the zero-group* G^0 is $S = M^0[G; I, \Lambda; P] = (I \times G \times \Lambda) \cup \{0\}$ with multiplication given as follows:

$$(i, a, \lambda)(j, b, \mu) = \begin{cases} (i, ap_{\lambda j}b, \mu) & \text{if } p_{\lambda j} \neq 0, \\ 0 & \text{if } p_{\lambda j} = 0, \end{cases}$$
$$(i, a, \lambda)0 = 0(i, a, \lambda) = 0 \cdot 0 = 0.$$

In a completely 0-simple semigroup $M^0[G; I, \Lambda; P]$, the $\mathcal{H}$-class of the element (i, a, λ) is $\{i\} \times G \times \{\lambda\}$. This is a maximal subgroup if and only if $p_{\lambda i} \neq 0$, in which case it is isomorphic to G.

Lemma 3.5.17. *A completely 0-simple semigroup $S = M^0[G; I, \Lambda; P]$ is locally finite if and only if G is locally finite.*

Proof. Suppose that S is locally finite. As S has a subsemigroup isomorphic to G, and local finiteness is inherited by subsemigroups, it follows that G is locally finite.

Now suppose that G is locally finite. Let $T = \langle (i_1, g_1, \lambda_1), \ldots, (i_n, g_n, \lambda_n) \rangle$. We will show that the intersection of T with a non-zero $\mathcal{H}$-class is finite. Let K be the subgroup of G generated by the set

$$\{g_1, g_2, \ldots, g_n\} \cup \{p_{\lambda_k i_l} \mid p_{\lambda_k i_l} \neq 0,\ 1 \leq k,\ l \leq n\}.$$

As K is finitely generated, it is finite. Let $H = \{i\} \times G \times \{\lambda\}$ be a non-zero $\mathcal{H}$-class. Then $H \cap T \subseteq \{i\} \times K \times \{\lambda\}$ and hence $H \cap T$ is finite. Now T can only intersect finitely many non-zero $\mathcal{H}$-classes. Indeed, if T intersects $\{j\} \times H \times \{\mu\}$, then $j \in \{i_1, i_2, \ldots, i_n\}$ and $\mu \in \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Hence it follows that T is finite. □

Lemma 3.5.18. *Let S be an epigroup with finitely many $\mathcal{J}$-classes. Then S is locally finite if and only if all its maximal subgroups are locally finite.*

Proof. ($\Rightarrow$) The forward direction follows as subsemigroups inherit local finiteness.

($\Leftarrow$) Now assume that all the maximal subgroups of S are locally finite. We assume that S has a zero element. If not we may adjoin one as S^0 is also an epigroup with finitely many $\mathcal{J}$-classes (precisely one more than S) and all the maximal subgroups of S^0 are also locally finite. Our argument shall show that S^0 is locally finite, and as $S \leq S^0$, we will have also shown that S is locally finite. We proceed by induction on the number of $\mathcal{J}$-classes.

A semigroup with a zero and only one $\mathcal{J}$-class is trivial, so the result certainly holds for the base case.

We now assume that S has more than one $\mathcal{J}$-class. Let I be the minimal non-zero ideal of S. Then I is 0-simple or a null semigroup, see [25, Proposition 2.4.9]. In the case that I is 0-simple, as S is an epigroup then so is I and therefore I is completely 0-simple. Furthermore, the maximal subgroups of I are the maximal subgroups of S contained in I, see [28, Proposition 2.4.2] Therefore we conclude that I is locally finite by Lemma 3.5.17. Note that null semigroups are also locally finite. Indeed, if X is a non-empty finite set of a null semigroup with zero element 0, then $\langle X \rangle = X \cup \{0\}$ (note this union is not necessarily disjoint).

As I is a minimal non-zero ideal, I is the union of at least two $\mathcal{J}$-classes and so the Rees quotient S/I has fewer $\mathcal{J}$-classes than S. Hence S/I is locally finite by the inductive hypothesis. Let T be a finitely generated subsemigroup of S. As S/I is locally finite, it follows that the Rees quotient $T/(T \cap I)$ is finite. Then the set $T \setminus (T \cap I)$ is finite. Hence $T \cap I$ is a subsemigroup with finite complement in T. Therefore, as T is finitely generated, $T \cap I$ is also finitely generated by [49, Theorem 1.1]. Then $T \cap I$ is finite as I is locally finite. Then $T = (T \setminus (T \cap I)) \cup (T \cap I)$ is finite and hence S is locally finite. □

Lemmas 3.5.9 and 3.5.18 together yield:

Corollary 3.5.19. *Let S be a semigroup with finitely many $\mathcal{H}$-classes. Then S is locally finite if and only if all its maximal subgroups are locally finite.*

We are now in a position to prove Theorem 3.5.8.

Proof of Theorem 3.5.8. Let S be a semigroup with finitely many $\mathcal{H}$-classes whose maximal subgroups are all solvable.

($\Rightarrow$) Firstly assume that S is WSS. Then S is residually finite and therefore its Schützenberger groups are residually finite by Theorem 3.5.3. As S is WSS and only has finitely many $\mathcal{H}$-classes, S is periodic by Corollary 3.5.11. Then all its Schützenberger groups are torsion. Torsion solvable groups are locally finite, see [47, p. 5.4.11]. Hence the Schützenberger groups are both residually finite and locally finite, so they are WSS by Lemma 3.4.2.

($\Leftarrow$) Now assume that all the Schützenberger groups of S are WSS. Then they are certainly residually finite and it follows that S is residually finite by Theorem 3.5.3. Furthermore, as S only has finitely many $\mathcal{H}$-classes and all its Schützenberger groups are weakly subsemigroup separable, S is periodic by Corollary 3.5.12. Then all its maximal subgroups are torsion and solvable so it follows that they are locally finite. Hence S is locally finite by Corollary 3.5.19 and therefore S is WSS by Lemma 3.4.2. $\square$

If there is any hope of solving Open Problem 3.5.6, we must consider cases where an infinite Schützenberger group is weakly subsemigroup separable but not solvable. One such example is the Grigorchuk group, which is a finitely generated infinite torsion group that is weakly subgroup separable (and hence WSS), see [24]. In particular, the following problem remains open.

Open Problem 3.5.20. Let G be the Grigorchuk group, let I and Λ finite sets, and let $P = (p_{\lambda i})$ a $\Lambda \times I$ matrix with entries from G^0 such that no row or column consists entirely of zeros. Is the semigroup $M^0[G; I, \Lambda; P]$ weakly subsemigroup separable?

Chapter 4

Separability Properties and Semigroup Constructions

In this chapter we consider how various semigroup constructions preserve our separability properties. The constructions we consider are natural and important ways of building new semigroups from old. This motivates the investigation into how these constructions interact with different separability properties. More motivation is provided by the fact that the group-theoretic versions of many of these constructions have been studied in relation to group separability properties. Furthermore, it is also known how residual finiteness interacts with the semigroup version of these constructions. All this information is presented when we introduce each construction.

In Section 4.1 we consider direct products. In this section we ask two questions. Firstly, for a separability property $\mathcal{P}$, if semigroups S and T both have property $\mathcal{P}$, does $S \times T$ necessarily have property $\mathcal{P}$? Secondly, if $S \times T$ has property $\mathcal{P}$, do both S and T necessarily have property $\mathcal{P}$? We work through the properties in turn. The answer to the first question turns out be no for all our separability properties except complete separability. This motivates us to ask when a semigroup preserves a separability property in the direct product. That is, what conditions do we need on a semigroup T such that $S \times T$ has property $\mathcal{P}$ for every semigroup S with property $\mathcal{P}$. We are able to characterise when a finite semigroup is SSS-preserving (Theorem 4.1.18) and when a finite semigroup is MSS-preserving (Theorem 4.1.42). In Section 4.2, we perform a similar investigation for free products of semigroups. In Section 4.3 we consider if separability properties can pass from large subsemigroups to their oversemigroups. Whilst being a large subsemigroup can be considered a semigroup property rather than a semigroup construction, the theory developed does allow us to say that if S has a property $\mathcal{P}$, then so do S^1 and S^0.

We now give a flavour of the results in this chapter by considering one of the simplest semigroup constructions: the 0-direct union.

Definition 4.0.1. Let $(S_i)_{i\in I}$ be a family of pairwise disjoint semigroups, with $|I| > 1$. The underlying set of the *0-direct union* $Z = {}^0\bigcup_{i\in I} S_i$ is the set $\left(\bigcup_{i\in I} S_i\right) \cup \{0\}$, where $\{0\}$ is assumed to be disjoint from each S_i. For each $a \in Z \setminus \{0\}$, there exists a unique $k \in I$ such that $a \in S_k$. We call k the index of a. Let $\sigma : Z \setminus \{0\} \to I$ be the map which takes an element to its index. We define a product on Z in the following way. For $a, b \in Z$ define

$$a \cdot b = \begin{cases} ab & \text{if } \sigma(a) = \sigma(b), \\ 0 & \text{otherwise,} \end{cases}$$

$$0 \cdot a = a \cdot 0 = 0.$$

It is easy to check that this multiplication is associative and hence Z is a semigroup.

Example 4.0.2. Let $S_1 = \{e_1\}$ and $S_2 = \{e_2\}$ be copies of the trivial semigroup. Then $Z = {}^0\bigcup_{i\in\{1,2\}} S_i$ is a three element semilattice. It has two maximal elements, e_1 and e_2, and one minimal element, 0.

We now show that all our separability properties except complete separability are inherited by the 0-direct union.

Proposition 4.0.3. *Let $\mathcal{P}$ be on the following separability properties: residual finiteness, monogenic subsemigroup separability, weak subsemigroup separability and strong subsemigroup separability. If a family $(S_i)_{i\in I}$ of pairwise disjoint semigroups each have property $\mathcal{P}$, then so does $Z = {}^0\bigcup_{i\in I} S_i$.*

Proof. Let $X \subseteq Z$ be a subset of the type associated with property $\mathcal{P}$ and let $y \in Z \setminus X$. We split into cases.

Case 1. First we assume that $y \in Z \setminus \{0\}$. Then $y \in S_i$ for some $i \in I$. We now split into two subcases: $S_i \cap X = \emptyset$ and $S_i \cap X \neq \emptyset$.

Case 1a. Assume that $S_i \cap X = \emptyset$. Let $\phi : S_i \to \{e\}$ be the homomorphism from S_i to the trivial semigroup. Define $\overline{\phi} : Z \to \{e\}^0$ by

$$a \mapsto \begin{cases} \phi(a) & \text{if } a \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism. Observe that $\overline{\phi}(y) = e$ but $\overline{\phi}(X) = \{0\}$. Hence $\overline{\phi}$ separates y from X.

Case 1b. Now assume that $S_i \cap X \neq \emptyset$. In this case $S_i \cap X$ is a subset of S_i of the type associated with property $\mathcal{P}$. As S_i has property $\mathcal{P}$, there exists a finite semigroup P and a homomorphism $\phi : S_i \to P$ such that $\phi(y) \notin \phi(S_i \cap X)$. Define $\overline{\phi} : Z \to P^0$ by

$$a \mapsto \begin{cases} \phi(a) & \text{if } a \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism. We have that $\overline{\phi}(y) \notin \overline{\phi}(S_i \cap X)$. Furthermore, $\overline{\phi}(y) \in P$ but $\overline{\phi}(X \setminus (S_i \cap X)) = \{0\}$. Hence $\overline{\phi}$ separates y from X.

Case 2. Now assume that $y = 0$. We claim that there exists $i \in I$ such that $X \subseteq S_i$. This is clear when property $\mathcal{P}$ is residual finiteness and X is a singleton. For the other properties X is a subsemigroup, and so if there exist $x, x' \in X$ such that $x \in S_i$ and $x' \in S_j$, where $i \neq j$, then $xx' = 0 \in X$. This contradicts that $y \notin X$ and hence the claim holds.

Let $\phi : S_i \to \{e\}$ be the homomorphism from S_i to the trivial semigroup.

Define $\overline{\phi} : Z \to \{e\}^0$ by

$$a \mapsto \begin{cases} \phi(a) & \text{if } a \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism. Observe that $\overline{\phi}(y) = 0$ but $\overline{\phi}(X) = \{e\}$. Hence $\overline{\phi}$ separates y from X. This completes the proof of this claim and of the proposition. □

The following lemma shows that the 0-direct union of an infinite family of pairwise disjoint is never complete separable.

Lemma 4.0.4. *Let $(S_i)_{i\in I}$ be a family of pairwise disjoint semigroups, where I is infinite. Then $Z = {}^0\bigcup_{i\in\mathbb{N}} S_i$ is not completely separable.*

Proof. Let $\sim$ be an arbitrary finite index congruence on Z. Then there exists $z, z' \in Z \setminus \{0\}$ such that $z \sim z'$ but $z \in S_i$ and $z' \in S_j$ with $i \neq j$. Therefore

$$0 = zz' \sim z'z' \in S_j.$$

Hence $|[0]_\sim| > 1$. As $\sim$ was arbitrary we conclude that Z is not completely separable. □

However, when we are taking the 0-direct union of a finite number of completely separable semigroups the resulting semigroup is also completely separable.

Proposition 4.0.5. *For completely separable semigroups $S_1, S_2, \ldots, S_k$ the 0-direct union $Z = {}^0\bigcup_{i\in\{1,\ldots,k\}} S_i$ is completely separable.*

Proof. Let $y \in Z$. We split into two cases: $y \in Z \setminus \{0\}$ and $y = 0$.

Case 1. Assume that $y \in Z \setminus \{0\}$. Then $y \in S_i$ for some $i \in \{1, 2, \ldots, k\}$. As S_i is completely separable there exists a finite semigroup P and homomorphism $\phi : S_i \to P$ such that $\phi(y) \notin \phi(S_i \setminus \{y\})$. Define $\overline{\phi} : Z \to P^0$

by

$$a \mapsto \begin{cases} \phi(a) & \text{if } a \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism. We have that $\overline{\phi}(y) \notin \overline{\phi}(S_i \setminus \{y\})$. Furthermore $\overline{\phi}(y) \in P$ but $\overline{\phi}(Z \setminus S_i) = \{0\}$. Hence $\overline{\phi}$ separates y from $Z \setminus \{y\}$ as desired.

Case 2. Now assume that $y = 0$. For $i \in \{1, 2, \ldots, k\}$, let $T_i = \{e_i\}$ be a copy of the trivial semigroup. Let $T = {}^0\bigcup_{i \in \{1,\ldots,k\}} T_i$. Note that $|T| = k + 1$ and so T is certainly finite. Define $\phi : Z \to T$ as

$$a \mapsto \begin{cases} e_i & \text{if } a \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then ϕ is a homomorphism the only element ϕ sends to 0 in T is 0. That is, $\phi(0) \notin \phi(Z \setminus \{0\})$. This completes the proof that Z is completely separable.

□

Combining the observation from Lemma 4.0.4 with Proposition 4.0.5, along with the fact subsemigroups inherit complete separability by Proposition 1.2.13, we obtain the following result.

Corollary 4.0.6. *A 0-direct union $Z =^0 \bigcup_{i \in I} S_i$ is completely separable if and only if I is finite and for each $i \in I$ the semigroup S_i is completely separable.*

The fact that in general the 0-direct union does not preserve complete separability indicates that separability properties do not always behave in predictable ways when it comes to semigroup constructions. We note that if a 0-direct union $Z = {}^0\bigcup_{i \in I} S_i$ has a separability property $\mathcal{P}$, then for each $i \in I$, the semigroup S_i has property $\mathcal{P}$. This is because S_i is a subsemigroup of Z and inherits property $\mathcal{P}$ by Proposition 1.2.13. However, the situation may not always be so straightforward. Indeed, in Section 4.1, we see that a direct product may have a separability property when one of the factors that make up the direct product does not have that separability property.

4.1 Direct Products

The material in this section is largely based upon [43], co-written by the author. In this section we investigate how semigroup separability properties interact with the direct product. We have already seen in Example 1.2.5 that $\mathbb{Z}$ is residually finite but not MSS. Recall that every residually finite algebra is a subdirect product of finite algebras ([12, Corollary 2.7.2]). Therefore $\mathbb{Z}$ can be realised as a subsemigroup of some direct product of finite semigroups, all of which are completely separable. But as $\mathbb{Z}$ is not even MSS, this suggests that an investigation into how arbitrary direct products preserve separability properties is unlikely to bear much fruit. Therefore, we will restrict our attention to direct products of finitely many semigroups. It is sufficient to consider a direct product containing only two factors. Hence, throughout we use direct product to refer to the direct product of two algebras, and not in its more general meaning. The starting point for this investigation are some previously known results concerning residual finiteness and the direct product. The fact that residual finiteness of the factors implies the residual finiteness of the direct product is universally true for algebraic structures. We provide a proof of this fact for completeness.

Lemma 4.1.1. *Let A and B be residually finite algebras of type $\mathcal{F}$. Then the direct product $A \times B$ is residually finite.*

Proof. Let $(a, b), (a', b')$ be distinct elements of $A \times B$. Then at least one of the following must be true: $a \neq a'$ or $b \neq b'$. Suppose that $a \neq a'$. Then there exists a finite algebra U of type $\mathcal{F}$ and homomorphism $\phi : A \to U$ such that $\phi(a) \neq \phi(a')$. Then the homomorphism $\phi \circ \pi_S : A \times B \to U$ separates (a, b) from (a', b'). A similar argument deals with the case when $b \neq b'$. $\square$

However, it is non-trivial to show that if a direct product of semigroups is residually finite then both factors are residually finite. This was done by Gray and Ruškuc in [22], giving the following theorem.

Theorem 4.1.2. [22, Theorem 2] *The direct product of two semigroups is residually finite if and only if both factors are residually finite.*

Group Property $\mathcal{P}$	**CS**	**SSS**	**WSS**	**MSS**
G, H are $\mathcal{P}$	✓	✓	✗	✓
$\implies G \times H$ is $\mathcal{P}$	Theorem 1.2.19	[37]	[3, "The Example"]	[52, Thm 4]

Table 4.1: Separability properties of direct products of groups

In a similar vein, it is also universally true that the direct product of two completely separable algebras is completely separable. This was shown by Golubov for semigroups in [21, Lemma 2]. On the other hand, the fact that complete separability is inherited by the factors of the direct product of two semigroups is non-trivial. We will fully prove both of these facts in Section 4.1.1.

Our aim is to determine for which of our three remaining separability properties an analogous result holds. In the cases where it does not, we investigate further to determine what can be said about the separability properties of direct products.

The preservation of group separability properties with respect to the direct product has been studied. In [52, Theorem 4], Stebe showed that the direct product of two monogenic subgroup separable groups is itself monogenic subgroup separable. It is not true that the direct product of two weakly subgroup separable groups is necessarily weakly subgroup separable. An example is given by Allenby and Gregorac in [3, "The Example"]. The direct product of two strongly subgroup separable groups is known to be strongly subgroup separable. In [3] Allenby and Gregorac attribute this result to Mal'cev in [37]. As we have already seen in Theorem 1.2.19, a group is completely separable if and only if it is finite. Therefore it is trivial that the direct product of two group is CS if and only if both factors are CS. The results for groups are recorded in Table 4.1.

For groups G and H, both G and H are always isomorphic to subgroups of $G \times H$. By Proposition 1.2.13, subgroups inherit all of these properties. So if $G \times H$ has one of these properties then so will both G and H. However, the situation for algebras in general, and for semigroups in particular, may not always be so straightforward. Indeed, in [53] de Witt was able to show

Semigroup Property $\mathcal{P}$	**CS**	**SSS**	**WSS**	**MSS**
S, T are $\mathcal{P}$ $\implies$ $S \times T$ is $\mathcal{P}$	✓ Theorem 4.1.7	✗ Ex 4.1.10	✗ Ex 4.1.24	✗ Ex 4.1.43
$S \times T$ is $\mathcal{P}$ $\implies$ S, T are $\mathcal{P}$	✓ Theorem 4.1.7	✓ Thm 4.1.11	✗ Ex 3.4.6	✗ Ex 3.4.6

Table 4.2: Separability properties of direct products of semigroups

that there exist monounary algebras A and B such that A is not residually finite but $A \times B$ is completely separable. So factors of a direct product of monounary algebras which has separability property $\mathcal{P}$ need not be $\mathcal{P}$ themselves. We shall observe that within the class of semigroups, a similar situation occurs with the properties of weak subsemigroup separability and monogenic subsemigroup separability.

By working through the separability properties in turn, we are able to give an analogous results table for semigroups to that of groups. We also consider which separability properties are inherited from the direct product by its factors. These results are recorded in Table 4.2.

It turns out that the properties of strong subsemigroup separability, weak subsemigroup separability and monogenic subsemigroup separability are not necessarily preserved in the direct product. This motivates the following definition.

Definition 4.1.3. Let $\mathcal{P}$ be one of the following properties: strong subsemigroup separability, weak subsemigroup separability or monogenic subsemigroup separability. We say that a semigroup T is *$\mathcal{P}$-preserving* (in direct products) if for every semigroup S which has property $\mathcal{P}$, the direct product $S \times T$ also has property $\mathcal{P}$.

We note that if a semigroup T is $\mathcal{P}$-preserving then T must have property $\mathcal{P}$. This is because the trivial semigroup has property $\mathcal{P}$ and T is isomorphic to the direct of itself with the trivial semigroup. In this section we focus on characterising when finite semigroups are $\mathcal{P}$-preserving. The remainder of this section is organised by property.

4.1.1 Complete Separability

In this subsection, we first prove the claim that the direct product of two completely separable algebras is itself completely separable. For semigroups, this was shown by Golubov in [21, Lemma 2], though the arguments used do not generalise to all algebras.

Lemma 4.1.4. *Let A and B be completely separable algebras of type $\mathcal{F}$. Then $A \times B$ is also completely separable.*

Proof. Let $(a, b) \in A \times B$. As A is CS, there exists a finite algebra U_1 of type $\mathcal{F}$ and homomorphism from $\phi_1 : A \to U_1$ such that $\phi_1(a) \notin \phi_1(A \setminus \{a\})$. Similarly, as B is CS, there exists a finite algebra U_2 of type $\mathcal{F}$ and homomorphism from $\phi_2 : B \to U_2$ such that $\phi_2(b) \notin \phi_2(B \setminus \{b\})$. Then the map $\phi : A \times B \to U_1 \times U_2$ given by $(x, y) \mapsto (\phi_1(x), \phi_2(y))$ is a homomorphism. Furthermore $\phi(a, b) \notin \phi(A \times B \setminus \{(a, b)\})$. Hence $A \times B$ is completely separable. $\square$

In order to show that complete separability is inherited by factors of the semigroup direct product, we will use a characterisation of complete separability given by Golubov in [17]. This characterisation depends upon the following definition.

Definition 4.1.5. For a semigroup S and $a, b \in S$ define

$$[a : b] = \{(u, v) \in S^1 \times S^1 \mid ubv = a\}.$$

Using this definition, we have the following characterisation of CS semigroups.

Theorem 4.1.6. *A semigroup S is completely separable if and only if for each $a \in S$ the set $\{[a : s] \mid s \in S\}$ is finite.*

Using this, we give the following result.

Theorem 4.1.7. *The direct product of semigroups S and T is completely separable if and only if S and T are completely separable.*

Proof. ($\Leftarrow$) This follows from Lemma 4.1.4.

($\Rightarrow$) This direction is the same as [21, Lemma 2]. We will prove the contrapositive, so assume that S is not CS. We show that $S \times T$ is not CS. By Theorem 4.1.6 there exists some $a \in S$ such that the set $\{[a : s] \mid s \in S\}$ is infinite. Let $\sim$ be a finite index congruence on $S \times T$. Fix $t \in T$. Then there exist $b, c \in S$ such that $[a : b] \neq [a : c]$ and $(b, t) \sim (c, t)$. Assume, without loss of generality, that there exist $u, v \in S^1$ such that $ubv = a$ but $ucv \neq a$. We split into two cases. The first case is that $u, v \in S$. Then

$$(a, t^3) = (u, t)(b, t)(v, t) \sim (u, t)(c, t)(v, t) = (ucv, t^3).$$

The second case is that either $u = 1$ or $v = 1$. We will deal with the case that $u = 1$. A similar argument deals with the case that $v = 1$. Then

$$(a, t^3) = (b, t)(v, t^2) \sim (c, t)(v, t^2) = (cv, t^3) = (ucv, t^3).$$

In either case we cannot separate (a, t^3) from its complement. Therefore $S \times T$ is not CS. $\square$

4.1.2 Strong Subsemigroup Separability

The behaviour of strong subsemigroup separability with respect to the direct product is not as predictable as that of complete separability or residual finiteness. Indeed, Golubov showed in [21] that the direct product of two SSS semigroups need not be itself SSS. The example given involved the direct product of two infinite SSS semigroups. We show that even when one of the factors is finite, the direct product of two SSS semigroups need not be SSS (Example 4.1.10). However, in creating these examples we develop sufficient theory to show that if a direct product of semigroups is SSS then both factors are SSS (Theorem 4.1.11).

Lemma 4.1.8. *Let S be a semigroup which is not completely separable and let T be a monogenic semigroup with index $m \geq 4$. Then $S \times T$ is not strongly subsemigroup separable.*

Proof. Let x be the generator of T. As S is not CS, by Theorem 4.1.6 there exists some $a \in S$ such that the set $\{[a : s] \mid s \in S\}$ is infinite. Let $\sim$ be a finite index congruence on $S \times T$. Then there exist $b, c \in S$ such that $[a : b] \neq [a : c]$ and $(b, x) \sim (c, x)$. Assume, without loss of generality, that there exist $u, v \in S^1$ such that $ubv = a$ but $ucv \neq a$. Let $U = \langle (S \setminus \{a\}) \times \{x^3\} \rangle \leq S \times T$. We claim that $(a, x^3) \notin U$. Indeed, if $(a, x^3) \in U$ then $x^3 \in \{x^{3k} \mid k > 1\}$. But this contradicts that $m \geq 4$ and hence we conclude that $(a, x^3) \notin U$.

We split into two cases. The first is that $u, v \in S$. Then

$$(a, x^3) = (u, x)(b, x)(v, x) \sim (u, x)(c, x)(v, x) = (ucv, x^3) \in U.$$

The second case is when either $u = 1$ or $v = 1$. We will deal with the case that $u = 1$. Then

$$(a, x^3) = (b, x)(v, x^2) \sim (c, x)(v, x^2) = (ucv, x^3) \in U.$$

In either case we cannot separate (a, x^3) from U. A similar argument deals with the case that $v = 1$. Therefore $S \times T$ is not SSS. □

As there exist SSS semigroups which are not CS (see Example 3.4.7 and Example 3.4.14), we can use Lemma 4.1.8 to construct an example of a direct product of a SSS semigroup with a finite semigroup which itself is not SSS. Before we give the example, we state the following definition.

Definition 4.1.9. A semigroup S with a zero element $\{0\}$ is called k-nilpotent if $S^k = \{0\}$, where $k \in \mathbb{N}$.

Example 4.1.10. As an immediate consequence of Lemma 4.1.8, if we take the direct product of an SSS semigroup S which is not CS with the finite 4-nilpotent semigroup $\langle x \mid x^4 = x^5 \rangle$, the resulting semigroup is not SSS.

Despite the negative nature of Lemma 4.1.8, it actually plays a key part in the proof of the following result.

Theorem 4.1.11. *If $S \times T$ is strongly subsemigroup separable then both S*

and T are strongly subsemigroup separable.

Proof. Assume $S \times T$ is SSS. Without loss of generality we show that S is SSS. We split into two cases.

The first case is that T contains an idempotent. Then S is isomorphic to a subsemigroup of the SSS semigroup $S \times T$ and hence S must also be SSS by Proposition 1.2.13

The second case is that T has no idempotents. For a contradiction assume that S is not SSS. Then S is certainly not CS. As T is idempotent free, it contains a copy of $\mathbb{N}$. Then $S \times \mathbb{N} \leq S \times T$. But $S \times \mathbb{N}$ is not SSS by Lemma 4.1.8. This contradicts the strong subsemigroup separability of $S \times T$, as all subsemigroups of an SSS semigroup are also SSS by Proposition 1.2.13. Therefore it must be the case that S is SSS. □

This theorem is the final step needed in classifying the separability properties of the direct product of a semigroup with $\mathbb{N}$, which we now present.

Theorem 4.1.12. *For a semigroup S, the following fully characterise the separability properties of $\mathbb{N} \times S$:*

(1) $\mathbb{N} \times S$ *is completely separable if and only if S is completely separable;*
(2) $\mathbb{N} \times S$ *is weakly subsemigroup separable but not strongly subsemigroup separable if and only if S is residually finite but not completely separable;*
(3) $\mathbb{N} \times S$ *is not residually finite if and only if S is not residually finite.*

Proof. (1) As we know that $\mathbb{N}$ is completely separable from Corollary 2.2.2, the result follows from Theorem 4.1.7.

(2) ($\Rightarrow$) WSS semigroups are residually finite (a consequence of Proposition 1.2.9 and Lemma 1.2.11). Therefore it follows that S is residually finite by Theorem 4.1.2 The fact that S cannot be CS is a consequence of Theorem 4.1.7.

(⇐) The fact that $\mathbb{N} \times S$ is WSS follows from Proposition 3.4.5, as $\mathbb{N} \times S$ must be residually finite and certainly has a homomorphic image isomorphic to $\mathbb{N}$. The fact that S cannot be SSS follows from Lemma 4.1.8.

(3) This follows from Theorem 4.1.2. □

Remark 4.1.13. In the statement of Theorem 4.1.12, the semigroup $\mathbb{N}$ can be replaced by any CS semigroup which has $\mathbb{N}$ as a homomorphic image and the result still holds. Such semigroups include all free semigroups and free commutative semigroups.

Theorem 4.1.12 highlights the intriguing behaviour of the direct product with respect to separability properties. On one hand we have $\mathbb{N}$ which has the strongest separability property we know: complete separability. On the other hand, if we have a semigroup which is just residually finite, the weakest separability property we consider, and take its direct product with $\mathbb{N}$, we end up with a separability property in between those of complete separability and residual finiteness. Furthermore, we cannot take a direct product with $\mathbb{N}$ and end up with a semigroup which is SSS but not CS. Neither can we end up with a semigroup which is MSS but not WSS.

The rest of this subsection is dedicated to establishing when a finite semigroup is SSS-preserving. To do this we show that SSS semigroups enjoy a separability property, in the sense of Definition 1.2.1, with respect to a more general class of subsets than just subsemigroups.

Proposition 4.1.14. *Let S be a strongly subsemigroup separable semigroup and let $V \subseteq S$ such that $V^{n+1} \subseteq V$ for some $n \in \mathbb{N}$ and let $s \in S \setminus V$. Then s can be separated from V.*

Proof. We proceed by induction on n. The base case $n = 1$ corresponds to V being a subsemigroup, and the result follows from S being SSS.

Now consider $n > 1$. Our inductive hypothesis is that for all $U \subseteq S$ such that $U^{k+1} \subseteq U$ for some $k \in \{1, 2, \dots, n-1\}$, and for all $s' \in S \setminus U$, we can separate s' from U. Let V and s be as in the statement of the proposition.

If $s \notin \langle V \rangle$ then as S is SSS, we can separate s from $\langle V \rangle$ and in particular from V.

So suppose that $s \in \langle V \rangle$. Note that as $V^{n+1} \subseteq V$, we have that $\langle V \rangle = V \cup V^2 \cup \cdots \cup V^n$ and that V^n is a subsemigroup of S. Let

$$L = \{\ell \in V \mid s \in \ell V^i, \text{ for some } 1 \leq i \leq n-1\},$$

which is some set of left-divisors of s. As $s \in \langle V \rangle$, we have that L is non-empty. Let $Z = V \setminus L$. Then we claim that $s \notin \langle Z \rangle$. Indeed, if $s \in \langle Z \rangle$, then $s = zw$ for some $z \in Z$ and $w \in Z^k$, where $k \in \mathbb{N} \cup \{0\}$. If $k = 0$ then $s = z \in Z \subseteq V$, which is a contradiction. Otherwise, as $V^{n+1} \subseteq V$, we have $w \in V^i$ for some $i \in \{1, 2, \ldots, n\}$. If $w \in V^n$, then $s \in V^{n+1} \subseteq V$, which is a contradiction. Then $w \in V^i$ for some $1 \leq i \leq n-1$. But then $z \in L$, which is a contradiction. So $s \notin \langle Z \rangle$. As S is SSS, s can be separated from $\langle Z \rangle$ and in particular s can be separated from Z.

For $i \in \{1, 2, \ldots, n-1\}$, define $X_i = V \cap V^{i+1}$. Then we claim that

$$X_i^{n-i+1} \subseteq X_i. \tag{4.1}$$

To see this let $x_1, x_2, \ldots, x_{n-i+1} \in X_i$. Firstly, as $X_i \subseteq V \cap V^{i+1}$, we have $x_1 \in V^{i+1}$ and $x_2, \ldots, x_{n-i+1} \in V$. Then

$$x_1 x_2 \ldots x_{n-i+1} \in V^{(i+1)+(n-i+1-1)} = V^{n+1} \subseteq V.$$

Secondly, noting that $n - i + 1 \geq 2$, we have $x_1, x_2 \in V^{i+1}$ Then

$$x_1 x_2 \ldots x_{n-i+1} \in V^{2(i+1)+(n-i+1-2)} = V^{n+i+1} \subseteq V^{i+1}.$$

Hence $x_1 x_2 \cdots x_{n-i+1} \in V \cap V^{i+1} = X_i$ and we conclude that $X_i^{n-i+1} \subseteq X_i$ and (4.1) holds. As $s \notin V$ we have that $s \notin X_i$.

Since $1 \leq n - i \leq n-1$ and $X_i^{n-i+1} \subseteq X_i$, by our inductive hypothesis we can separate s from each X_i. For $i \in \{1, \ldots, n-1\}$, define

$$L_i = \{\ell \in L \mid s \in \ell V^i\}.$$

Note that $L = \bigcup_{1 \leq i \leq n-1} L_i$. We show that s can be separated from each L_i. Suppose that s cannot be separated from some L_i. We claim that $sV^{n-1} \subseteq V^i \cap V^n$. First for a contradiction, assume that there exists $u \in V^{n-1}$ such that $su \in S \backslash V^n$. Then, as V^n is a subsemigroup and S is SSS, it must be the case that su can be separated from V^n. Let $\sim$ be a finite index congruence which separates su from V^n. As s cannot be separated from L_i which is a subset of V, there exists $v \in V$ such that $s \sim v$. But then $su \sim vu \in V^n$, which is a contradiction. Hence if s cannot be separated from L_i, we have $sV^{n-1} \subseteq V^n$. As L_i is non-empty, we have that $s \in V^{i+1}$. But as $s \in V^{i+1}$ and $V^{n+1} \subseteq V$, we have that $sV^{n-1} \subseteq V^i$. Hence $sV^{n-1} \subseteq V^n \cap V^i$ as claimed. From this we get

$$sV^n \subseteq V \cap V^{i+1} = X_i. \tag{4.2}$$

Now let $\sim$ be a finite index congruence that separates s from X_i. As s cannot be separated from L_i, there exists $\ell \in L_i$ such that $\ell \sim s$. But as $\ell \in L_i$, there exists $u \in V^i$ such that $\ell u = s$. Then, by the compatibility of $\sim$, we have

$$s = \ell u \sim su \sim su^2 \sim \cdots \sim su^{n-1} \sim su^n.$$

By the transitivity of $\sim$, we obtain that $s \sim su^n$. As $u \in V^i$, we have that $u^n \in V^{in}$. But as $V^{n+1} \subseteq V$, we conclude that $u^n \in V^n$. So by Equation (4.2) we have $su^n \in sV^n \subseteq X_i$. That is, $\sim$ does not separate s from X_i. This is a contradiction and so s can be separated from L_i.

As $V = Z \cup L_1 \cup L_2 \cup \cdots \cup L_{n-1}$, and for each of the sets in this finite union there exists a finite index congruence which separates s from said set, we can separate s from their union by Proposition 1.2.20. That is, we can separate s from V as desired. □

Example 4.1.15. Here we exhibit an example of Proposition 4.1.14 in action. Let $C_2 = \{0, 1\}$ be the cyclic group of order 2 with identity element 0. Consider $G = C_2^{\mathbb{N}}$, the Cartesian product of countably many copies of C_2. In Example 3.4.7 we showed that G is strongly subsemigroup separable. Now consider $V = \{g \in G \mid \pi_1(g) = 1\}$. That is, V is the set of all elements that have the non-identity element of C_2 in the first coordinate. Note that V is not a subsemigroup. If we take any two elements of V, their product will have the identity of C_2 is the first coordinate and so V is not closed under multiplication. However, as in C_2 we have that $1^3 = 1$, we conclude that $V^3 \subseteq V$. Now let e denote the identity of G. Then $e \notin V$. Proposition 4.1.14 tells us that we can separate e from V. Here we are truly relying on Proposition 4.1.14. Example 3.4.7 tells us that G is not CS, and we have already observed that V is not a subsemigroup and so we are unable to use the strong subsemigroup separability of G. Furthermore, the set V is infinite and so we cannot use the residual finiteness of G either. Although the proof of Proposition 4.1.14 does not give a way of constructing a homomorphism to separate e from V, in this case it is easy to find one. Indeed, we have that $\pi_1(V) = \{1\}$, but $\pi_1(e) = 0$. Then the projection map π_1 separates e from V.

The fact that SSS semigroups satisfy the additional separability property of Proposition 4.1.14 allows us to characterise finite semigroups which are SSS-preserving. Our characterisation relies on the notion of indecomposability of elements.

Definition 4.1.16. For a semigroup S and an element $s \in S$, we say that s is *decomposable* if $s \in S^2$. In this case, there exist $t, u \in S$ such that $s = tu$. Otherwise we say that s is *indecomposable.*

Example 4.1.17. Consider $\mathbb{N}$. The element 1 is indecomposable. For any $n \geq 2$, we have that $n = (n-1) + 1$ and so n is decomposable.

We are now ready to characterise when a finite semigroup is SSS-preserving.

Theorem 4.1.18. *A finite semigroup P is strong subsemigroup separability preserving if and only if every element of P is indecomposable or belongs to a subgroup.*

Proof. ($\Rightarrow$) We prove the contrapositive, so assume that $p \in P$ is not contained in a subgroup but there exist $s, t \in P$ such that $st = p$. As p is not contained in a subgroup, we have that $p^n \neq p$ for all $n \geq 2$. Let G be an infinite SSS group (one could take G to be the group from Example 3.4.7). Let $U \leq G \times P$ be generated by the set $\{(g, p) \mid g \in G \setminus \{1_G\}\}$. As $p^n \neq p$ for all $n \geq 2$, we have that $(1_G, p) \notin U$. Let $\sim$ be a finite index congruence on $G \times P$. Then there exist $g, h \in G$ with $g \neq h$ such that $(g, s) \sim (h, s)$. Then

$$(1_G, p) = (g, s)(g^{-1}, t) \sim (h, s)(g^{-1}, t) = (hg^{-1}, p) \in U.$$

Hence $G \times P$ is not SSS and therefore P is not SSS-preserving.

($\Leftarrow$) Now assume that P is a finite semigroup in which every element not contained in a subgroup is indecomposable. Let S be an SSS semigroup, let $U \leq S \times P$ and let $(s, p) \in (S \times P) \setminus U$. If $s \notin \pi_S(U) \leq S$, then we can separate (s, p) from U by factoring through S and invoking the strong subsemigroup separability of S. If $p \notin \pi_P(U)$, then π_P separates (s, p) from U.

Now assume that both $s \in \pi_S(U)$ and $p \in \pi_P(U)$. If p is not contained within a subgroup of P, then p is indecomposable. Then (s, p) is indecomposable in $S \times P$. Let $I = (S \times P) \setminus \{(s, p)\}$. Then I is an ideal of finite complement in $S \times P$ and $[(s, p)]_I = \{(s, p)\}$. In particular, (s, p) is separated from U in the Rees quotient of $S \times P$ by I.

The final case to consider is that p is contained in a subgroup of P. Then for some $n \in \mathbb{N}$, we have that $p^{n+1} = p$. Let $V = \pi_S(U \cap (S \times \{p\})$. First note that $s \notin V$ as $(s, p) \notin U$. Secondly, as $p^{n+1} = p$, we have $V^{n+1} \subseteq V$. Then, by Proposition 4.1.14, we have that s can be separated from V. So there exists a finite semigroup Q and a homomorphism $\phi : S \to Q$ such that $\phi(v) \notin \phi(V)$. Define $\overline{\phi} : S \times P \to Q \times P$ by $(a, b) \mapsto (\phi(a), b)$. Then $\overline{\phi}$ is a homomorphism which separates (s, p) from U. Hence $S \times P$ is SSS and so P is SSS-preserving. $\square$

Remark 4.1.19. As the set S^2 is an ideal of a semigroup S, Theorem 4.1.18 is equivalent to saying that a finite semigroup P is SSS-preserving if and only if P is the ideal extension of a union of groups by a null semigroup, see

Definition 3.3.12.

Corollary 4.1.20. *The following families of finite semigroups are strong subsemigroup separability preserving: groups, Clifford semigroups, completely simple semigroups, completely regular semigroups, bands, and null semigroups.*

Proof. As we have already seen groups, completely simple semigroups and completely regular groups are unions of groups and so finite semigroups from these classes certainly satisfy the criterion of Theorem 4.1.18. Bands form a variety of semigroups satisfying the additional identity $x^2 = x$. Hence every element is idempotent and certainly contained in a subgroup and so bands are also union of groups. In null semigroups, every non-zero element is indecomposable and the zero element is idempotent, so finite null semigroups also satisfy the criterion of Theorem 4.1.18. □

Corollary 4.1.21. *A finite monoid is strong subsemigroup separability preserving if and only if it is a union of groups.*

Proof. In a monoid M there are no indecomposable elements. This is because if we have an element s in a monoid M, then $s1 = s$. Hence, by Theorem 4.1.18, for a finite monoid to be SSS-preserving all of its elements must be contained in subgroups. □

We conclude this subsection with some open problems.

Open Problem 4.1.22. Is there a characterisation of when the direct product of two strongly subsemigroup separable semigroups is itself strongly subsemigroup separable?

Open Problem 4.1.23. Is it true that the direct product of two strongly subsemigroup separable semigroups is weakly subsemigroup separable?

4.1.3 Weak Subsemigroup Separability

How weak subsemigroup separability interacts with the direct product is more complex than any of the other separability properties we consider. This has already been exhibited in Open Problem 4.1.23, which states that even when we strengthen the factor semigroups to be SSS, we still do not know if this guarantees that the direct product is WSS. In Example 3.4.6, we have already seen that $\mathbb{N} \times \mathbb{Z}$ is WSS even though Example 1.2.5 tells us that $\mathbb{Z}$ is not even MSS. This provides the first example of a separability property which is not necessarily inherited by the factors of a direct product. This complexity means the results in this section are less comprehensive than those for the other separability properties considered. In a vein similar to the strong subsemigroup separability situation, we show that the direct product of a WSS semigroup with a finite semigroup is not necessarily WSS (Example 4.1.24). Although we do not characterise when a finite semigroup is WSS-preserving, we show that finite nilpotent semigroups are WSS-preserving (Theorem 4.1.25). In contrast to the Open Problem 4.1.23, here we are able to show that the direct product of two WSS semigroups is MSS (Theorem 4.1.27).

Example 4.1.24. Let $S = \mathcal{S}[\mathrm{FC}_2, \mathbb{Z}, \phi]$ be as in Example 3.4.13 and let L be a non-trivial left-zero semigroup (we could take L from Example 1.1.4). Then $S \times L$ is not WSS.

Let $y, z \in L$ be distinct. Let

$$U = \langle (a, y), (x_1, z) \rangle \leq S \times L.$$

We claim that

$$U = \{(a^i, y) \mid i \in \mathbb{N}\} \cup \{(x_i, z) \mid i \in \mathbb{N}\} \cup \{(x_i, y) \mid i \in \mathbb{Z}\} \cup \{(0, y), (0, z)\}.$$

We can see this from the following observations. Firstly recall that $x_i a^j = x_{i+j}$, $a^j x_i = x_{i-j}$ and $x_i x_j = 0$. Then, if any product over $\{(a, y), (x_1, z)\}$ contains two or more occurrences of (x_1, z), the first coordinate of this product will be zero. If this product begins with (a, y) then we obtain $(0, y)$ and otherwise we obtain $(0,z)$. If we want an element of U whose first coordinate is non-zero, then there is at most one occurrence of (x_1,z). First suppose that there is precisely one occurrence of (x_1,z) in the decomposition of our product. If the second coordinate of the product is z then our decomposition is of the form $(x_1,z)(a,y)^n$, where $n \geq 0$. This gives us the set $\{(x_i,z) \mid i \in \mathbb{N}\}$. Otherwise the second coordinate of the product is y and our decomposition is of the form $(a,y)(a,y)^m(x_1,z)(a,y)^n$, where $m,n \geq 0$. This corresponds to the set $\{(x_i,y) \mid i \in \mathbb{Z}\}$. The final case is that our product only contains occurrences of (a,y). This gives us the set $\{(a^i,y) \mid i \in \mathbb{N}\}$, completing the proof of the claim.

In particular, $(x_0,z) \notin U$. Let $\sim$ be a finite index congruence on $S \times L$. Then there exist $i,j \in \mathbb{N}$ with $i<j$ such that $(x_i,z) \sim (x_j,z)$. Then

$$(x_0,z) = (x_i,z)(b^i,z) \sim (x_j,z)(b^i,z) = (x_{j-i},z) \in U.$$

Hence $S \times L$ is not weakly subsemigroup separable. By a similar argument, it can be shown that the direct product of $S = \mathcal{S}[\mathrm{FC}_2,\mathbb{Z},\phi]$ with a non-trivial right-zero semigroup is not WSS.

Example 4.1.24 shows that (non-trivial) finite left-zero and right-zero semigroups are not weakly subsemigroup separability preserving. This is somewhat surprising as finite left-zero and finite right-zero semigroups, which are examples of bands, are strongly subsemigroup preserving by Corollary 4.1.20. Furthermore, they also turn out to be MSS-preserving by Theorem 4.1.42. Adding yet another twist to the story, the following theorem shows that finite k-nilpotent semigroups are WSS-preserving, even though this class contains semigroups which are neither SSS-preserving nor MSS-preserving. We will see an example of this in Example 4.1.26.

Theorem 4.1.25. *The direct product of a weakly subsemigroup separable semigroup with a residually finite k-nilpotent semigroup is weakly subsemigroup separable.*

Proof. Let S be a WSS semigroup and let N be a residually finite k-nilpotent semigroup with zero element 0, for some $k \in \mathbb{N}$. Let $U \leq S \times N$ be finitely generated and let $(s, n) \in (S \times N)\backslash U$. Fix a finite generating set for U with the form

$$(X_0 \times \{0\}) \cup (X_1 \times \{n_1\}) \cup \cdots \cup (X_j \times \{n_k\}),$$

where $n_1, n_2, \ldots, n_j \in N \setminus \{0\}$ and $X_0, X_1, \ldots, X_j$ are finite subsets of S. Let $X = \bigcup_{i=0}^{j} X_i$ and let $T = \langle X \rangle \leq S$.

Let $Z = \pi_N(U) = \{z_0, z_1, z_2, \cdots, z_m\}$ where $z_0 = 0$. Note Z is finite as k-nilpotent semigroups are locally finite. For $0 \leq i \leq m$ let

$$Y_i = \pi_S(U \cap (S \times \{z_i\})).$$

Then for $1 \leq i \leq m$ the set Y_i is finite. To see this first note that Y_i is a subset of T. Then for $y \in Y_i$, we can write y as a product of elements of X. The maximum length of the product is $k-1$, as N is a k-nilpotent semigroup and z_i is a non-zero element of N. As X is a finite set, there are only finitely many such y and so Y_i is finite.

Now consider Y_0. Certainly $T^k \subseteq Y_0$ as N is a k-nilpotent semigroup. We have that $Y_0 \setminus T^k$ is finite as any element of $Y_0 \setminus T^k$ can be expressed as product over X of length at most $k-1$.

If $s \notin \pi_S(U) = T$ then we can separate (s, n) from U by factoring through S and invoking the weak subsemigroup separability of S. Similarly, if $t \notin \pi_N(U) = Z$ then we can separate (s, n) from U by factoring through N and using the residual finiteness of N.

Now assume that $s \in \pi_S(U)$ and $n \in \pi_N(U)$. Then either

(i) $n = 0$ and $s \notin Y_0$, or
(ii) $n = z_i$ for some $1 \leq i \leq m$ and $s \notin Y_i$.

(i) First note that $T^k \leq S$ is a finitely generated by the set

$$X^k \cup X^{k+1} \cup \cdots \cup X^{2k-1}.$$

As $s \notin T^k$ and S is WSS, we can separate s from T^k. As $s \notin Y_0 \setminus T^k$ and S is residually finite, we can separate s from $Y_0 \backslash T^k$. Hence, by Proposition 1.2.20, we can separate s from $Y_0 = T^K \cup (Y_0 \setminus T^k)$. That is, there exists a finite semigroup P and homomorphism $\phi : S \to P$ such that $\phi(s) \notin \phi(Y_0)$.

As N is residually finite, there exists a finite semigroup Q and homomorphism $\psi : N \to Q$ such that $\psi(0) \notin \psi(Z \setminus \{0\})$. Then the homomorphism $\phi \times \psi : S \times N \to P \times Q$ given by $(a, b) \mapsto (\phi(a), \psi(b))$ separates (s, n) from U.

(ii) As S is WSS, it is residually finite by Lemma 1.2.11, then there exists a finite semigroup P and homomorphism $\phi : S \to P$ such that $\phi(s) \notin \phi(Y_j)$. As N is residually finite, there exists a finite semigroup Q and homomorphism $\psi : N \to Q$ such that $\psi(z_j) \notin \psi(Z \setminus \{z_j\})$. Then $\phi \times \psi : S \times N \to P \times Q$ given by $(a, b) \mapsto (\phi(a), \psi(b))$ separates (s, n) from U. This completes the proof that $S \times N$ is WSS. □

Example 4.1.26. Consider the semigroup S given by the presentation $\langle\, x \mid x^3 = x^4 \,\rangle$. This is a finite 3-nilpotent semigroup and hence is WSS-preserving by Theorem 4.1.25. The element x^2 is decomposable but it is not contained in a subgroup by Example 1.3.38. Hence S is not SSS-preserving by Theorem 4.1.18. Theorem 4.1.42 states that a finite semigroup is MSS-preserving if and only if it is a union of groups. Hence, as S itself is a monogenic semigroup which is not a union of group, we have that S is not MSS-preserving.

At the point of writing, we are not yet able to characterise when a finite semigroup is WSS-preserving. We leave this as one of the open problems concerning weak subsemigroup separability and direct products. However, before this we provide a positive result showing that the direct product of two WSS semigroups is MSS.

Theorem 4.1.27. *The direct product of two weakly subsemigroup separable semigroups is monogenic subsemigroup separable.*

Proof. Let S and T be WSS semigroups. Let $U = \langle (s, t) \rangle \leq S \times T$ be a monogenic subsemigroup and let $(x, y) \in (S \times T) \setminus U$. If $x \notin \pi_S(U) = \langle s \rangle$

then we can separate (x, y) from U by factoring through S and using the weak subsemigroup separability of S. By a similar argument, we can separate (x, y) from U if $y \notin \pi_T(U)$.

Now consider the case that $x \in \pi_S(U)$ and $y \in \pi_S(T)$. Then $x = s^i$ and $y = t^j$, where $i, j \in \mathbb{N}$ are such that $i \neq j$, $s^i \neq s^j$ and $t^i \neq t^j$. Firstly we consider the case when at least one of $\langle s \rangle$ and $\langle t \rangle$ is infinite. Without loss of generality, assume that $\langle s \rangle \cong \mathbb{N}$. In this instance $s^i \notin \langle s^{i+1}, s^{i+2}, \ldots, s^{2i+1} \rangle = \{s^k \mid k \geq i+1\}$. Hence we can separate (x, y) from $\{(s^k, t^k) \mid k \geq i+1\}$ by factoring through S and using the weak subsemigroup separability of S. As S and T are both WSS, they are both residually finite by Lemma 1.2.11. Hence $S \times T$ is residually finite by [22, Theorem 2]. Then we can separate (x, y) from the finite set $\{(s^k, t^k) \mid k \leq i\}$. As we can separate (x, y) from both $\{(s^k, t^k) \mid k \leq i\}$ and $\{(s^k, t^k) \mid k \geq i+1\}$, we can separate (x, y) from their union by Proposition 1.2.20. Hence we can separate (x, y) from U as required.

The final case to consider is when both $\langle s \rangle$ and $\langle t \rangle$ are finite, in which case U is finite. As $S \times T$ is residually finite we can separate (x, y) from U. Hence $S \times T$ is MSS, as desired. □

We conclude this subsection with some open problems.

Open Problem 4.1.28. Is there a characterisation of when a finite semigroup is weakly subsemigroup preserving?

Open Problem 4.1.29. Is there a characterisation of when a direct product of two weakly subsemigroup separable semigroups is itself weakly subsemigroup separable?

Open Problem 4.1.30. If the direct product of two semigroups is weakly subsemigroup separable, is at least one of the factors weakly subsemigroup separable?

4.1.4 Monogenic subsemigroup Separability

In Example 3.4.6, we have already seen that the factors of an MSS semigroup need not themselves be MSS. Also, as with properties of strong subsemigroup separability and weak subsemigroup separability, the direct product of two MSS semigroups need not be MSS itself, even when one the factors is finite (Example 4.1.43). In the process of constructing this example, we develop the necessary theory to determine when a finite semigroup is MSS-preserving (Theorem 4.1.42). To show that monogenic subsemigroup separability is not preserved by the direct product, we consider the semigroup A, given by the presentation $\langle\, a, b, c \mid ab^2c = b \,\rangle$. This is an MSS semigroup which is not WSS. The key step in showing this is embedding A into a group. To show that A can be embedded in a group, we will use a criterion given by Adian in [1].

Definition 4.1.31. Let $\langle\, X \mid R \,\rangle$ be a presentation. Consider a relation $(u, w) \in R$. The *left pair* of (u, w) is the pair (x, y), where x is the leftmost letter of u and y is the leftmost letter of w. The *right pair* of (u, w) is the pair (z, t), where z is the rightmost letter of u and t is the rightmost letter of w. The *left graph of the presentation* is the graph with vertex set X such that $\{x, y\}$ is an edge if and only if (x, y) is a left pair of some relation. Note that multiple edges and loops are allowed. The *right graph of the presentation* is the graph with vertex set X such that $\{z, t\}$ is an edge if and only if (z, t) is a right pair of some relation. The presentation $\langle\, X \mid R \,\rangle$ is said to have no *cycles* if both its left graph and its right graph have no cycles (here loops are cycles and multiple edges create cycles).

Theorem 4.1.32. ([1, Theorem 2.3]) *If a presentation has no cycles then the natural mapping from the semigroup given by the presentation to the group given by the presentation is an embedding.*

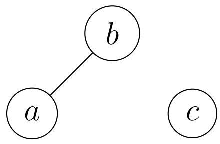
Corollary 4.1.33. *The mapping ϕ given by*

$$a \mapsto x, \quad b \mapsto y, \quad c \mapsto y^{-2}x^{-1}y,$$

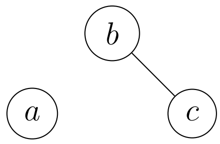
from the semigroup $A = \langle\, a, b, c \mid ab^2c = b \,\rangle$ to the free group FG_2 on the set

$\{x, y\}$ *is an embedding.*

Proof. Below we draw the left graph of the presentation $\langle\, a, b, c \mid ab^2c = b \,\rangle$.



Now we present the right graph of the same presentation.



As neither of these graphs contains cycles, the presentation $\langle\, a, b, c \mid ab^2c = b \,\rangle$ has no cycles. By Theorem 4.1.32, the natural map from A to the group G given by the group presentation $\langle\, x, y, z \mid xy^2z = y \,\rangle$ is an embedding (note that to avoid confusion, for the group presentation we have replaced a with x, b with y, and c with z). As G is a group, the single relation of the group presentation can be rewritten as $z = y^{-2}x^{-1}y$. This means that the set $\{x, y\}$ is a generating set for G. Hence, we can remove the generator z and any relation containing z from the presentation for G. This is an example of a Tietze transformation. Because we only make limited use of this theory, we do not formally define such transformations here. They are discussed in detail in [29, Section 4.4], which also justifies their use in finding new presentations for groups from existing presentations. So we have that G is given by the group presentation $\langle\, x, y \mid \emptyset \,\rangle$. This shows that G is the free group on the set $\{x, y\}$ and the proof in complete. □

Definition 4.1.34. In the semigroup $A = \langle\, a, b, c \mid ab^2c = b \,\rangle$, the strings *abbc* and b represent the same element. Therefore we can define a rewriting system on A that replaces the string ab^2c by b. As the one rule of this rewriting system is length reducing, the process is terminating. Also, as ab^2c does not overlap with itself, the process is locally confluent and hence confluent.

For more on rewriting systems, see [5, Section 1.1]. Therefore, each element of A is represented by a unique word in $\{a, b, c\}^+$, where this representative does not contain ab^2c as a contiguous subword. Such a word is said to be in *normal form*. Note that a contiguous subword of a normal form word is also a word in normal form. We shall therefore consider the underlying set of A to be the set of all words over $\{a, b, c\}^+$ in normal form. Multiplication is concatenation, except in the case where concatenation creates instances of ab^2c as contiguous subwords, in which case the rewriting rule is applied to convert the product into normal form.

Lemma 4.1.35. *Let A, FG_2 and $\phi : A \to \mathrm{FG}_2$ be as in Corollary 4.1.33. Then $\phi(w) \neq \epsilon$ for all $w \in A$.*

Proof. Let w be an element of A. We proceed by a case analysis based upon the number of occurrences of contiguous strings of the letter c appearing in the word w.

Case 1. The first case in when there are no occurrences of the letter c in w. Then $w \in \{a, b\}^+$. As ϕ rewrites an occurrence of a with an x and an occurrence of b with a y, we have that $\phi(w)$ is a non-empty word, as desired.

Case 2. Now we consider the case when w contains precisely one string of the letter c. We split into three subcases.

Case 2a. Consider $w = c^n$, where $n \geq 1$. Observe that

$$\phi(c^n) = y^{-2}(x^{-1}y^{-1})^{n-1}x^{-1}y.$$

Hence the assertion of the lemma holds. For future cases note that $\phi(c^n)$ ends with the suffix $x^{-1}y$.

Case 2b. Consider $w = uc^n$, where $u \in \{a, b\}^+$ and $n \geq 1$. We claim that when concatenating the words $\phi(u)$ and $\phi(c^n)$, there are at most two cancelling pairs of letters. Indeed, if there were three cancelling pairs of letters, then $\phi(u)$ would end with xy^2 as by Case (2a) we know that $\phi(c^n)$ begins with $y^{-2}x^{-1}$. Hence u would end with ab^2. But in this case uc^n

contains the string ab^2c, contradicting that uc^n is in normal form. It follows that $(x^{-1}y^{-1})^{n-1}x^{-1}y$ is a suffix of $\phi(uc^n)$ and hence $\phi(uc^n)$ is non-empty. Again we note that $\phi(uc^n)$ ends with $x^{-1}y$. For future cases we consider when $\phi(uc^n)$ can begin with a negative letter. In such a case, the entirety of $\phi(u)$ has been cancelled by $\phi(c^n)$. By our analysis of cancellation between $\phi(u)$ and $\phi(c)$ there are only two options: $u = b$ or $u = b^2$.

Case 2c. Consider $w = uc^n v$, where $u \in \{a, b\}^*$, $v \in \{a, b\}^+$ and $n \geq 1$. Then $\phi(v)$ consists of positive letters. By Cases (2a) and (2b), $\phi(uc^n)$ ends with a positive letter. Then when concatenating $\phi(uc^n)$ and $\phi(v)$, there can be no pairs of cancelling letters. Hence $\phi(uc^n v)$ is non-empty, as desired.

Case 3. Finally we consider the case when w contains more than one string of the letter c. We can decompose $w = w_1 w_2 \dots w_k v$ where:

- $w_1 = uc^{n_1}$, where $u \in \{a, b\}^*$ and $n_1 \geq 1$;
- $w_i = u_i c^{n_i}$, where $u_i \in \{a, b\}^+$ and $n_i \geq 1$ for $2 \leq i \leq k$;
- $v \in \{a, b\}^*$; and $k \geq 2$;

By our previous case analysis, for each i we have that $\phi(w_i)$ is a non-empty word. We now claim that for $1 \leq i \leq k$, when we concatenate $\phi(w_i)$ and $\phi(w_{i+1})$ there is at most one cancelling pair of letters. We have already observed in Cases (2a) and (2b) that $\phi(w_i)$ must end with $x^{-1}y$. Therefore, for cancellation to occur, $\phi(w_{i+1})$ must begin with y^{-1}. By the final observation of Case (2b), $\phi(w_{i+1})$ can only begin with a negative letter if $u_{i+1} = b$ or $u_{i+1} = b^2$. In the first case, $\phi(w_{i+1}) = y^{-1}(x^{-1}y^{-1})^{n_{i+1}-1}x^{-1}y$ and there is precisely one cancelling pair when we concatenate $\phi(w_i)$ and $\phi(w_{i+1})$. In the second case, $\phi(w_{i+1}) = (x^{-1}y^{-1})^{n_{i+1}-1}x^{-1}y$ and no cancellation occurs when we concatenate $\phi(w_i)$ and $\phi(w_{i+1})$ completing the proof of the claim. Note that if cancellation occurs, than the cancelling pair is yy^{-1}.

Now consider $\phi(w_1)\phi(w_2)\dots\phi(w_k)\phi(v)$. As already observed, $\phi(w_1)$ and $\phi(w_2)$ both end with $x^{-1}y$. By the claim of the previous paragraph, there is at most one cancelling pair of letters, yy^{-1}, between $\phi(w_1)$ and $\phi(w_2)$. So it must be the case that $\phi(w_1 w_2)$ also ends with $x^{-1}y$. Continuing in this manner, we conclude that $\phi(w_1 w_2 \dots w_k)$ ends with $x^{-1}y$. As $\phi(v)$ is either empty or consists of positive letters, there is no cancellation between $\phi(w_1 w_2 \dots w_k)$ and $\phi(v)$ and we conclude $\phi(w)$ is non-empty, completing the proof of this case and of the lemma. $\square$

In showing that the semigroup A is not WSS, we introduce the idea of stability.

Definition 4.1.36. A semigroup S is *stable* if for all $s, t \in S$ the following hold:

(i) $s \,\mathcal{J}\, st \implies s \,\mathcal{R}\, st$;
(ii) $s \,\mathcal{J}\, ts \implies s \,\mathcal{L}\, ts$.

Example 4.1.37. The semigroup $\mathbb{N}$ is stable. We have seen in Example 1.3.26 that in $\mathbb{N}$, Green's relation $\mathcal{J}$ coincides with the diagonal equivalence $\Delta_{\mathbb{N}}$. From the definitions of Green's relations $\mathcal{L}$ and $\mathcal{R}$ it follows that $\Delta_{\mathbb{N}} \subseteq \mathcal{L} \subseteq \mathcal{J}$ and $\Delta_{\mathbb{N}} \subseteq \mathcal{R} \subseteq \mathcal{J}$. Hence we conclude that $\mathcal{J} = \mathcal{R} = \mathcal{L}$ and therefore $\mathbb{N}$ is stable.

Lemma 4.1.38. *Finite semigroups are stable.*

Proof. [46, Theorem A.2.4]. $\square$

The reason the notion of stability will be useful is because of the following lemma.

Lemma 4.1.39. *Let S be a stable semigroup. If for $s \in S$ we have that $s \,\mathcal{J}\, s^2$, then H_s is a group.*

Proof. As $s \,\mathcal{J}\, s^2$ and S is stable, we have that $s \,\mathcal{L}\, s^2$ and $s \,\mathcal{R}\, s^2$. That is, $s \,\mathcal{H}\, s^2$. Then $H_s \cap H_s^2 \neq \emptyset$ and we have that H_s is a group by Proposition 1.3.36. $\square$

We are now able to establish the separability properties of the semigroup A.

Proposition 4.1.40. *The semigroup A given by the presentation $\langle\, a, b, c \mid ab^2c = b\,\rangle$ is monogenic subsemigroup separable but not weakly subsemigroup separable.*

Proof. Let $T = \langle u \rangle \leq A$ be a monogenic subsemigroup and let $v \in A \backslash T$. Let FG_2 and $\phi : A \to \mathrm{FG}_2$ be as in Corollary 4.1.33. As ϕ is an embedding, we have that $\phi(v) \notin \phi(T)$. Let $H \leq \mathrm{FG}_2$ be the cyclic subgroup with generator $\phi(u)$. First we show that $\phi(v) \notin H$.

For a contradiction, assume that $\phi(v) \in H$. As $\phi(v) \notin \phi(T)$, we have that $\phi(v) \notin \{\phi(u)^n \mid n \in \mathbb{N}\}$. By Lemma 4.1.35, $\phi(v)$ has positive length and therefore $\phi(v) \neq \phi(u)^0$. Therefore $\phi(v) = \phi(u)^{-n}$ for some $n \in \mathbb{N}$. But then $\phi(vu^n) = \phi(v)\phi(u^n) = \epsilon$. This contradicts Lemma 4.1.35 and we conclude $\phi(v) \notin H$.

As FG_2 is weakly subgroup separable (Theorem 2.1.9), there exists a finite group G and homomorphism $\psi : \mathrm{FG}_2 \to G$ such that $\psi(\phi(v)) \notin \psi(H)$. In particular, $\psi \circ \phi : A \to G$ is a homomorphism from A to a finite semigroup such that $(\psi \circ \phi)(u) \notin (\psi \circ \phi)(T)$. Hence A is MSS.

To show that $A = \langle\, a, b, c \mid ab^2c = b\,\rangle$ is not weakly subsemigroup separable, consider the subsemigroup $V = \langle b^2, b^3 \rangle$. As $\langle b \rangle \cong \mathbb{N}$, we have $b \in S \backslash V$. Let P be a finite semigroup and let $\sigma : A \to P$ be a homomorphism. The relation $ab^2c = b$ ensures that $b \,\mathcal{J}\, b^2$ and hence $\sigma(b) \,\mathcal{J}\, \sigma(b^2)$. As $\sigma(A)$ is a finite semigroup, we have that $\sigma(b) \,\mathcal{H}\, \sigma(b^2)$ by Lemma 4.1.38. Hence we have that $H_{\sigma(b)}$ is a group by Lemma 4.1.39. Hence $\sigma(V) = \langle \sigma(b) \rangle$ is a finite cyclic group and in particular $\sigma(b) \in \sigma(V)$. Hence A is not weakly subsemigroup separable. □

We use the semigroup A in the proof of our characterisation of finite MSS-preserving semigroups. In fact, we are able to go one step further and characterise when residually finite periodic semigroups are MSS-preserving. We note the following fact.

Lemma 4.1.41. *A periodic semigroup is monogenic subsemigroup separable* *if and only if it is residually finite.*

Proof. Let S be a periodic semigroup. First assume that S is MSS. Then by Lemma 1.2.11 we have that S is residually finite.

Now assume that S is residually finite. Let $T = \langle t \rangle$ be a monogenic subsemigroup of S and let $s \in S \setminus T$. As S is periodic we have that T is finite. Since S is residually finite, we can separate s from the finite set T and therefore we have that S is MSS. $\square$

Theorem 4.1.42. *A residually finite periodic semigroup is monogenic subsemigroup separability preserving if and only if it is a union of groups.*

Proof. ($\Leftarrow$) Let T be a residually finite periodic semigroup which is a union of groups. Let S be a MSS semigroup and let $U = \langle (s,t) \rangle \leq S \times T$. Let $(x,y) \in (S \times T) \backslash U$. We separate into cases. Note, some cases may overlap.

Case 1. Suppose $\langle s \rangle \leq S$ is finite. As T is periodic, we also have that $\langle t \rangle$ is finite. Therefore U is also finite. As S is MSS it is also residually finite and therefore $S \times T$ is residually finite. Hence we can separate (x,y) from U.

Case 2. Suppose that $x \notin \pi_S(U)$. Then $x \notin \langle s \rangle \leq S$. So we can separate (x,y) from U by factoring through S and invoking the monogenic subsemigroup separability of S.

Case 3. Suppose that $y \notin \pi_T(U)$. As $\pi_T(U) = \langle t \rangle$ and T is periodic, we have that $\pi_T(U)$ is finite. So we can separate (x,y) from U by factoring though T and invoking the residual finiteness of T.

Case 4. Now suppose that $\langle s \rangle \cong \mathbb{N}$, $x \in \pi_S(U)$ and $y \in \pi_T(U)$. Let $r \in \mathbb{N}$ be minimal such that $t^{r+1} = t$. Such an r exists as T is periodic and a union of groups. As $x \in \pi_S(U)$, we have that $x = s^i$ for some $i \in \mathbb{N}$. As $y \in \pi_T(U)$, we have that $y = t^j$ for some $j \in \{1, 2, \ldots, r\}$. Observe that

$$\pi_S(U \cap (S \times \{t^j\})) = \{s^{j+rn} \mid n \geq 0\}.$$

As $(x,y) \notin U$, it must be the case that $i \not\equiv j \pmod r$. We split into two subcases.

Case 4a. First we will deal with the case that $y = t^r$. In this case, y is an idempotent and we have

$$\pi_S(U \cap (S \times \{t^r\})) = \langle s^r \rangle.$$

Hence $x = s^i \notin \langle s^r \rangle$. As S is MSS, there exists a finite semigroup P_1 and homomorphism $\phi_1 : S \to P_1$ such that $\phi(x) \notin \phi(\langle s^r \rangle)$. As T is residually finite, there exists a finite semigroup P_2 and homomorphism $\phi_2 : T \to P_2$ such that $\phi_2(y) \neq \phi(\pi_T(U) \setminus \{y\})$. Then $\phi : S \times T \to P_1 \times P_2$ given by $(a,b) \mapsto (\phi_1(a), \phi_2(b))$ is a homomorphism that separates (x,y) from U.

Case 4b. Now assume that $j \in \{1, 2, \ldots, r-1\}$. Let k be such that $j + k = r$. Then as $i \not\equiv j \pmod r$, we have that $i + k \not\equiv j + k \pmod r$. Hence $(s^{i+k}, t^{j+k}) = (s^{i+k}, t^r) \notin U$. By Case 4a we can separate (s^{i+k}, t^r) from U.

We now show that we can separate $(x,y) = (s^i, t^j)$ from U. For a contradiction suppose it cannot be separated. Let $\sim$ be a finite index congruence which separates (s^{i+k}, t^r) from U. As (s^i, t^j) cannot be separated from U, there exists $\ell \in \mathbb{N}$ such that $(s^i, t^j) \sim (s,t)^\ell$. But then

$$(s^{i+k}, t^r) = (s^i, t^j)(s,t)^k \sim (s,t)^\ell (s,t)^k \in U.$$

This contradicts that $\sim$ separates (s^{i+k}, t^r) from U. Hence (x,y) can be separated from U, completing the proof of this case and of the backward direction of the proof.

($\Rightarrow$) Now suppose that T is a residually finite semigroup which is not a union of groups. We will show that T is not MSS-preserving. As T is not a union of groups, there exists an element $t \in T$ such that for all $n \geq 2$ we have that $t^n \neq t$. Let m be the index and let r be the period of the monogenic subsemigroup $\langle t \rangle$. We have that $m \geq 2$ and $t^m = t^{m+r}$. Let $i = m - 1$.

As before, let A be the monogenic subsemigroup separable semigroup given by the presentation $\langle\, a, b, c \mid ab^2c = b\,\rangle$. We will see in Corollary 4.3.11 that A^1 is also MSS. Let $U = \langle (b,t) \rangle \le A^1 \times T$. Observe that

$$\pi_{A^1}(U \cap (A^1 \times \{t^{i+r}\}) = \{b^{i+nr} \mid n \ge 1\}. \tag{4.3}$$

Then $(b^i, t^{i+r}) \notin U$.

Let $d \in \{m, m+1, \ldots, m+r-1\}$ be such that $d \equiv 0 \pmod r$. Then it must be the case that t^d is an idempotent. Furthermore, $t^{i+r}t^d = t^{i+r}$. Let ρ be a finite index congruence on $A^1 \times T$. Then

$$[(a,t^d)]_\rho[(b^2,t^d)]_\rho[(c,t^d)]_\rho = [(ab^2c,t^d)]_\rho = [(b,t^d)]_\rho.$$

As also $[(b,t^d)]_\rho[(b,t^d)]_\rho = [(b^2,t^d)]_\rho$ we have that $[(b,t^d)]_\rho \mathcal{J} [(b,t^d)]_\rho^2$. Since $(A^1 \times T)/\rho$ is finite, we conclude that the $\mathcal{H}$-class of $[(b,t^d)]_\rho$ is a finite group by Lemma 4.1.38 and Lemma 4.1.39. In particular there exists $k > 2$ such that $[(b,t^d)]_\rho^k = [(b,t^d)]_\rho$. From this we conclude that $[(b,t^d)]_\rho = [(b^{1+(k-1)n},t^d)]_\rho$ for all $n \in \mathbb{N}$. Adopting the convention that if $i = 1$ then b^{i-1} is the identity of A^1, we observe that

$$\begin{aligned}[(b^i,t^{i+r})]_\rho &= [(b,t^d)]_\rho[(b^{i-1},t^{i+r})]_\rho \\ &= [(b^{1+(k-1)r},t^d)]_\rho[(b^{i-1},t^{i+r})]_\rho \\ &= [(b^{i+(k-1)r},t^{i+r})]_\rho.\end{aligned}$$

From Equation (4.3), we have that $(b^{i+(k-1)r}, t^{i+r}) \in U$. So ρ does not separate (b^i, t^{i+r}) from U. As ρ was arbitrary we conclude that $A^1 \times T$ is not MSS and in particular T is not MSS-preserving. □

The following example is a consequence of the proof of Theorem 4.1.42. It shows that the direct product of two MSS semigroups need not be MSS, even when one of the factors is finite.

Example 4.1.43. The direct product of $A^1 = \langle\, a, b, c \mid ab^2c = b\,\rangle^1$ and the two element zero semigroup $N = \{x, 0\}$ (with the zero element 0) is not monogenic subsemigroup separable. This follows from the proof of Theorem 4.1.42 and as N is not a union of groups.

Although it is not true that the direct product of an MSS semigroup with a residually finite semigroup is necessarily MSS, if we strengthen the assumption of monogenic subsemigroup separability to weak subsemigroup separability, we obtain a positive result, which we present below. Proposition 4.1.44 can be seen as a successor of Theorem 4.1.42, but it can also be viewed as a variation of Theorem 4.1.27. The similarities between Proposition 4.1.44 and Theorem 4.1.27 are discussed in Remark 4.1.45.

Proposition 4.1.44. *The direct product of a weakly subsemigroup separable semigroup S with a residually finite periodic semigroup T is monogenic subsemigroup separable.*

Proof. Let $U = \langle (s,t) \rangle \leq S \times T$ and let $(x,y) \in (S \times T) \setminus U$. We split into cases.

Suppose $\langle s \rangle \leq S$ is finite. As T is periodic, we also have that $\langle t \rangle$ is finite. Therefore U is also finite. As S is WSS it is also residually finite by Proposition 1.2.9 and Lemma 1.2.11, and therefore $S \times T$ is residually finite. Hence we can separate (x,y) from U.

Suppose that $x \notin \pi_S(U)$. Then $x \notin \langle s \rangle \leq S$. So we can separate (x,y) from U by factoring through S and invoking the weak subsemigroup separability of S.

Now suppose that $\langle s \rangle \cong \mathbb{N}$ and that $x \in \pi_S(U)$. Then $x = s^i$ for some $i \in \mathbb{N}$. As S is WSS, we can separate s^i from $\langle s^{i+1}, s^{i+2}, \ldots, s^{2i-1} \rangle = \{s^k \mid k \geq i+1\}$. Hence we can separate (x,y) from $\{(s^k, t^k) \mid k \geq i+1\}$ by factoring through S and using the weak subsemigroup separability of S. As S is WSS, it is residually finite by Proposition 1.2.9 and Lemma 1.2.11. Hence $S \times T$ is residually finite by [22, Theorem 2]. Then we can separate (x,y) from the finite set $\{(s^k, t^k) \mid k \leq i\}$. As we can separate (x,y) from both $\{(s^k, t^k) \mid k \leq i\}$ and $\{(s^k, t^k) \mid k \geq i+1\}$, we can separate (x,y) from their union by Proposition 1.2.20. Hence we can separate (x,y) from U as required. □

Remark 4.1.45. The statement of Proposition 4.1.44 is reminiscent of Theorem 4.1.27. If every residually finite periodic semigroup is WSS, then Proposition 4.1.44 becomes a specific instance of Theorem 4.1.27. However, it is not known whether every residually finite periodic semigroup is WSS. This is left as an open question at the end of this section.

We conclude this subsection with some open problems.

Open Problem 4.1.46. Is there a characterisation of when the direct product of two monogenic subsemigroup separable semigroups is itself monogenic subsemigroup separable?

Open Problem 4.1.47. If the direct product of two semigroups is monogenic subsemigroup separable, is at least one of the factors monogenic subsemigroup separable?

Open Problem 4.1.48. Is every residually finite periodic semigroup weakly subsemigroup separable?

4.2 Free Products

In this section we investigate how another semigroup construction, the free product, interacts with our separability properties. After defining the free product, we will discuss what is already known regarding free products and residual finiteness. We will also review the results for group separability properties in relation to the group free product. As with the direct product, we establish the equivalent results for semigroups. We find that the property of complete separability is preserved by arbitrary free products (Theorem 4.2.5). However, neither the property of strong subsemigroup separability nor weak subsemigroup separability are preserved by this construction (Example 4.2.6 and Example 4.2.8 respectively). On the other hand, the arbitrary free product of MSS semigroups is itself MSS (Theorem 4.2.10).

Definition 4.2.1. Let $\{S_i \mid i \in I\}$ be a family of pairwise disjoint semigroups. For $a \in \bigcup_{i\in I} S_i$, there is a unique $k \in I$ such that $a \in S_k$. We call k the *index* of a. Let $\sigma : \bigcup_{i\in I} S_i \to I$ be the map which takes an element to its index. The underlying set of the *free product* $F = \prod_{i \in I} *S_i$ is the set of all finite strings

$$a_1 a_2 \dots a_m,$$

where $m \geq 1$, $a_k \in \bigcup_{i \in I} S_i$ for $k \in \{1, 2, \dots, m\}$ and $\sigma(a_k) \neq \sigma(a_{k+1})$ for $k \in \{1, 2, \dots, m-1\}$. We call m the *length* of the string and denote it by $||a_1 a_2 \dots a_m|| = m$. We define a multiplication on F by

$$a_1 a_2 \dots a_m \cdot b_1 b_2 \cdots b_n = \begin{cases} a_1 a_2 \dots a_m b_1 b_2 \cdots b_n & \text{if } \sigma(a_m) \neq \sigma(b_1), \\ a_1 a_2 \dots a_{m-1} c b_2 \cdots b_n & \text{if } \sigma(a_m) = \sigma(b_1), \end{cases}$$

where $c = a_m b_1 \in S_{\sigma(a_m)}$. Under this multiplication F is a semigroup. When the index set I is finite, say $I = \{1, 2, \dots, k\}$, we write $F = S_1 * S_2 * \dots * S_k$. For more on free products, see [28, Section 8.2].

Example 4.2.2. Let $S = \langle\, a \mid \,\rangle$ and $T = \langle\, b \mid \,\rangle$ both be isomorphic to $\mathbb{N}$. Then $S * T = \langle a, b \mid \,\rangle$ is the free semigroup on the set $\{a, b\}$. In general, if we have a semigroup S' with presentation $\langle\, X \mid R\,\rangle$ and a semigroup T' with presentation $\langle\, Y \mid Q\,\rangle$, where X and Y are disjoint, then $\langle\, X \cup Y \mid R \cup Q\,\rangle$ is a presentation for $S' * T'$.

It is known that the free product of residually finite semigroups is residually finite, see [19]. Our aim is then to investigate which of our four separability properties are preserved under the free product. The group-theoretic versions of these separability properties have been studied in relation to the group free product. We do not formally define the group free product, but note that it is essentially the same as the semigroup free product with one notable difference, the identity elements are identified with each other. This means that whilst the semigroup free product of two copies of the trivial (semi)group is infinite, the group free product of two copies of the trivial group is again the trivial group. For more on the group free product, see [29, Section 9.6].

The group free product of two groups preserves the following properties: residual finiteness (shown in [26, Theorem 4.1]), monogenic subgroup separability (shown in [52, Theorem 5]), and weak subgroup separability (shown

Group Property $\mathcal{P}$	**CS**	**SSS**	**WSS**	**MSS**
G, H are $\mathcal{P}$	✗	✗	✓	✓
$\implies G * H$ is $\mathcal{P}$	Thm 1.2.19	Ex 4.2.3	[7, Cor 1.2]	[52, Thm 5]

Table 4.3: Separability properties of free products of groups

in [7, Corollary 1.2]). It is not true that the free product of two strongly subgroup separable groups need itself be strongly subgroup separable. This is folklore but we provide an example.

Example 4.2.3. Let $\overline{\mathbb{Z}}$ be an isomorphic copy of $\mathbb{Z}$. Then $\overline{\mathbb{Z}}$ is strongly subgroup separable by Theorem 3.1.4. However, $\mathbb{Z} * \overline{\mathbb{Z}}$ is isomorphic to the free group on a set of size two, and hence is not strongly subgroup separable by Lemma 2.1.11.

For trivial reasons, it is also the case that the free product of two completely separable groups need not be completely separable. This is because a group is completely separable if and only if it is finite (Theorem 1.2.19). But the free product of two non-trivial finite groups is infinite, and so the group free product does not preserve complete separability. It follows that the free product of two groups is CS if and only if they are both finite and at least one of them is trivial. The group results are recorded in Table 4.3. For the rest of this section we will provide the analogous results for the semigroup free product, as described in the introduction to this section and recorded in Table 4.4. Unlike the situation for the direct product, given a family of semigroups $\{S_i\}_{i\in I}$, for each $i \in I$ the semigroup S_i is isomorphic to a subsemigroup of $\Pi_{i\in I} * S_i$. Hence, if $\Pi_{i\in I} * S_i$ has any one of our separability properties, then so does each S_i by Proposition 1.2.13. We now show that the free product preserves complete separability, starting with a special case.

Lemma 4.2.4. *The free product of finitely many finite semigroups is completely separable.*

Proof. Let $S = S_1 * S_2 * \cdots * S_n$, where $\{S_i \mid 1 \leq i \leq n\}$ is a finite collection of pairwise disjoint finite semigroups. Let $s = s_1 s_2 \ldots s_k$, where $s_i \in S_{\alpha_i}$ for some $1 \leq \alpha_i \leq n$ and for $1 \leq i \leq k-1$ we have $S_{\alpha_i} \neq S_{\alpha_{i+1}}$.

Semigroup Property $\mathcal{P}$	CS	SSS	WSS	MSS
S, T are $\mathcal{P}$	✓	✗	✗	✓
$\implies S * T$ is $\mathcal{P}$	Thm 4.2.5	Ex 4.2.6	Ex 4.2.8	Thm 4.2.10

Table 4.4: Separability properties of free products of semigroups

Consider the set $I = \{i \in S : ||i|| > k\}$. We claim that I is an ideal. To see this let $i \in I$ and let $t \in S$. Denote $||i|| = p$ and $||t|| = r$ and note that $p > k$ and $r \geq 1$. Then $||it||, ||ti|| \geq ||i|| + ||t|| - 1 = p + r - 1 > k$. Hence $it, ti \in I$, completing the proof of the claim.

Now consider $S \setminus I$, which is the set of all elements of S whose length is at most k. As each S_i is finite, we conclude that $S \setminus I$ is finite. Note that $s \in S \setminus I$. Then the Rees quotient S/I is a finite semigroup and we have that $[s]_I = \{s\}$. Hence S is completely separable, as desired. □

We use Lemma 4.2.4 in the proof of the general result.

Theorem 4.2.5. *A free product of completely separable semigroups is completely separable.*

Proof. Let $S = \prod_{i \in I} * S_i$, where $\{S_i \mid i \in I\}$ is a family of pairwise disjoint completely separable semigroups. Let $s = s_1 s_2 \dots s_n \in S$, where $s_i \in S_{\alpha_i}$ for some $\alpha_i \in I$ and for $1 \leq i \leq n-1$ we have $S_{\alpha_i} \neq S_{\alpha_{i+1}}$. Let $K = I \setminus \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. If $K = \emptyset$, then $S \cong (S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})$. As $(S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})$ embeds into $(S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0$, it is sufficient to show that $(S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0$ is completely separable. In the case that K is non-empty, consider the ideal $J_s \leq S$ generated (as an ideal) by the set $\bigcup_{k \in K} S_k$. Observe that $s \in S \setminus J_s$. Then $S/J_s \cong (S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0$. As $[s]_{J_s} = \{s\}$, we will identify s as an element of $(S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0$. Again to show that S is completely separable, it is sufficient to show that $(S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0$ is completely separable.

Let $\overline{S} = (S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0$. For $1 \leq i \leq n$, each S_{α_i} is completely separable so there exists a finite semigroup P_i and homomorphism $\phi_i : S_{\alpha_i} \to P_i$ such that $\phi_i(s_i) \notin \phi_i(S_{\alpha_i} \setminus \{s_i\})$. We will assume without loss of generality that the family $\{P_i\}_{1\leq i\leq n}$ are pairwise disjoint. Then we can define a homomorphism $\phi : (S_{\alpha_1} * S_{\alpha_2} * \cdots * S_{\alpha_n})^0 \to (P_1 * P_2 * \cdots * P_n)^0$ by

$$\begin{aligned} t_1t_2\dots t_m &\mapsto p_1p_2\dots p_m, \text{ where } t_i \in S_{\alpha_{j_i}} \text{ and } \phi_{j_i}(t_i) = p_i; \\ 0 &\mapsto 0. \end{aligned}$$

Then $\phi(s) \notin (\phi(\overline{S}) \setminus \{s\})$. Note that $P_1 * P_2 * \cdots * P_n$ is completely separable by Lemma 4.2.4. We will see in Theorem 4.3.12 that $(P_1 * P_2 * \cdots * P_n)^0$ is also completely separable. Hence we can separate $\phi(s)$ from $(P_1 * P_2 * \cdots * P_n)^0 \setminus \{\phi(s)\}$. So by first factoring through $\overline{S}$, and then by factoring through $(P_1 * P_2 * \cdots * P_n)^0$, we can separate s from $S \setminus \{s\}$. Hence S is completely separable. □

We now provide two examples; the first shows that the property of strong subsemigroup separability is not preserved under the free product, and the second shows that the property of weak subsemigroup separability is not preserved under the free product.

Example 4.2.6. Let G be an infinite group with identity element 1. Let T be any semigroup and let $t \in T$. We show that $G * T$ is not SSS.

Let $U = \langle\{tgt \mid g \in G \setminus \{1\}\}\rangle$. As every element of the generating set has length three (in the free product sense), observe that a product of n generators will have length $3 + 2(n - 1)$. From this observation we conclude that $t1t \notin U$. Let $\sim$ be a finite index congruence on $G * T$. Then there exist $g, h \in G$, with $g \neq h$, such that $tg \sim th$. Then

$$t1t = tg \cdot g^{-1}t \sim th \cdot g^{-1}t = thg^{-1}t \in U.$$

Hence $G * T$ is not strongly subsemigroup separable.

Remark 4.2.7. In Example 4.2.6, if we choose T to be SSS and G to be an infinite SSS group (see Example 3.4.7), then we have shown that the free product of two SSS semigroups need not itself be SSS. However, the argument that $G * T$ is not SSS does not depend on T being SSS, so in fact we conclude that $G * S$ is not SSS for any choice of semigroup S. For this reason, we do not define a property of SSS-preserving in free products, as this class of semigroups would be empty.

Example 4.2.8. Let $S = \mathcal{S}(\mathrm{F}_2, \mathbb{Z}, \phi)$ be as in Example 3.4.13, let T be any semigroup and let $t \in T$. We show that $S * T$ is not WSS.

Let $U = \langle \{tx_o,\ a\} \rangle \leq S \times T$. Then $tx_i = tx_0 \cdot a^i \in U$ for all $i \geq 0$. These are the only elements of length two in U, and so we conclude $tx_{-1} \notin U$. Let $\sim$ be a finite index congruence on $S * T$. Then there exist $i, j \in \mathbb{N}$, with $i < j$, such that $tx_i \sim tx_j$. Then

$$tx_{-1} = tx_i b^{i+1} \sim tx_j b^{i+1} = tx_{j-i-1} \in U.$$

Hence $S * T$ is not weakly subsemigroup separable.

Remark 4.2.9. In the above example, if we choose T to be WSS then we have shown that the free product of two WSS semigroups is not necessarily WSS itself. Just as before, we do not need that T is WSS in order to show that $S * T$ is not WSS. So we can conclude that the class of WSS-preserving (in free products) semigroups is empty.

We conclude this section by showing that the free product preserves the property of monogenic subsemigroup separability.

Theorem 4.2.10. *A free product of monogenic subsemigroup separable semigroups is itself monogenic subsemigroup separable.*

Proof. Let $\{S_i\}_{i \in I}$ be a family of MSS semigroups and let $S = \Pi_{i \in I} * S_i$. Let $U \leq S$ be generated by $\{u\}$ and let $s \in S \backslash U$. As both u and s are finite length strings, there exists a finite subset $K \subseteq I$ such that $U \cup \{s\} \subseteq \Pi_{k \in K} * S_k$. If $I \setminus K = \emptyset$, then $S = (\Pi_{k \in K} * S_k)$. As S embeds in S^0, it is sufficient to show that S^0 is MSS. If $I \setminus K \neq \emptyset$, let J be the ideal of S generated (as an ideal) by $\bigcup_{i \in I \setminus K} S_i$. Then $U \cup \{s\} \subseteq S \setminus J$. As $S/J \cong (\Pi_{k \in K} * S_k)^0$, for each $n \in \mathbb{N}$ we will identify u^n as an element as $(\Pi_{k \in K} * S_k)^0$. We also identify s as an element of $(\Pi_{k \in K} * S_k)^0$. Therefore, in both cases, it is sufficient to show that in $\overline{S} = (\Pi_{k \in K} * S_k)^0$, we can separate s from U. We split into cases based on the length of both u and s.

Case 1. First we deal with the case that $||u|| = 1$ and $||s|| = 1$. Then $u \in S_\alpha$ and $s \in S_\beta$ for some $\alpha, \beta \in K$. We split into two subcases: $\alpha = \beta$ or $\alpha \neq \beta$.

Case 1a. First we assume that $u, s \in S_\alpha$. Then U is a subsemigroup of S_α. As S_α is MSS then there exists a finite semigroup P and a homomorphism $\phi : S_\alpha \to P$ such that $\phi(s) \notin \phi(U)$. Then we define a map $\overline{\phi} : \overline{S} \to P^0$ by

$$w \mapsto \begin{cases} \phi(w) & \text{if } w \in S_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism that separates s from U as desired.

Case 1b. Now we assume that $\alpha \neq \beta$. Let $\phi : S_\alpha \to \{e\}$ be the homomorphism from S_α into the trivial semigroup. Then we define a map $\overline{\phi} : \overline{S} \to \{e\}^0$ by

$$w \mapsto \begin{cases} \phi(w) & \text{if } w \in S_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism. As $\overline{\phi}(U) = \{e\}$ but $\overline{\phi}(s) = 0$, we have separated s from U.

Case 2. Now consider the case that $||u|| = 1$ and $||s|| > 1$. Then $u \in S_\alpha$ for some $\alpha \in K$. Furthermore $U \subseteq S_\alpha$. As $||s|| > 1$, it cannot be the case that $s \in S_\alpha$. Let $\phi : S_\alpha \to \{e\}$ be the homomorphism from S_α into the trivial semigroup. Then we define a map $\overline{\phi} : \overline{S} \to \{e\}^0$ by

$$w \mapsto \begin{cases} \phi(w) & \text{if } w \in S_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\overline{\phi}$ is a homomorphism. As $\overline{\phi}(U) = \{e\}$ but $\overline{\phi}(s) = 0$, we have separated s from U.

Case 3. The final case to consider is that $||u|| > 1$. Then it must be the case that $||u^{n+1}|| > ||u^n||$ for all $n \in \mathbb{N}$. Let $||s|| = m$. Then the set $Z_s = \{t \in \overline{S} \mid ||t|| > m\} \cup \{0\}$ is an ideal of $\overline{S}$. To see this, let $z \in Z_S$ and

$t \in \overline{S}$. If either $z = 0$ or $t = 0$ then $zt = tz = 0$ and $zt, tz \in Z_s$. Otherwise $||z|| > m$ and so $||zt||, ||tz|| \geq ||z|| + ||t|| - 1 > m$ and again $zt, tz \in Z_s$. Hence Z_s is an ideal.

As $||u^{n+1}|| > ||u^n||$ for all $n \in \mathbb{N}$, there exists a minimal $N \in \mathbb{N}$ such that for $k \geq N$ we have that $u^k \in Z_s$. Let $U_N = \{u, u^2, \ldots, u^{N-1}\}$. From [19] we have that $\Pi_{k \in K} * S_k$ is residually finite. Hence, $\overline{S} = (\Pi_{k \in K} * S_k)^0$ is residually finite by [50, Corolllary 4.6] (which is reproduced as Theorem 4.3.3). As $\overline{S}$ is residually finite, there exists a finite semigroup P and homomorphism $\phi : \overline{S} \to P$ such that $\phi(s) \notin \phi(U_N)$ (if $U_N = \emptyset$, then let $P = \{e\}$ be the trivial semigroup and $\phi : \overline{S} \to P$ be the trivial homomorphism).

We now separate s from $U \setminus U_N$. For $k \in K$, let $\{e_k\}$ be a copy of the trivial semigroup. Consider $T = \Pi_{k \in K} * \{e_k\}$. Define $J_m = \{t \in T \mid ||t|| > m\}$. It is clear that J_m is an ideal of T. Furthermore, T/J_m is a finite semigroup of size k^m+1, and has a zero element which will denote as 0. Define $\psi : \overline{S} \to T/J_m$ to be the extension of the map given $w \mapsto e_k$, when $w \in S_k$. Then $\psi(s)$ is some non-zero element of T/J_m as $||s|| = m$. However $\psi(U \setminus U_N) = \{0\}$. That is, ψ separates s from $U \setminus U_N$. As ϕ separates s from U_N, by Proposition 1.2.20 we conclude that s can be separated from U, completing the proof. □

The monoid free product could form the basis for a future topic of research. Like the group free product, the identity elements in an monoid free product are identified with each other. A line of enquiry could be made investigating the preservation of both semigroup separability properties and monoid separability properties under the monoid free product, and contrasting these results to group and semigroup cases.

4.3 Large Subsemigroups

Proposition 1.2.13 showed that subsemigroups inherit separability properties from the oversemigroup. In this section we investigate conditions under which knowing that a subsemigroup has a separability property may guarantee that the oversemigroup also has that separability property. In general it is not true that if a semigroup has a separability property, that this property will pass to the oversemigroup. For example, the subsemigroup $\{0\} \leq \mathbb{Z}$ is CS, yet we have seen in Example 1.2.5 that $\mathbb{Z}$ is not even MSS. Therefore, we restrict our attention to a certain type of subsemigroup, known as large subsemigroups. We first define large subsemigroups, providing examples, and then explain the motivation for studying them.

Definition 4.3.1. A subsemigroup $T \leq S$ is known as *large* if $|S \setminus T|$ is finite.

Example 4.3.2. Any semigroup is a large subsemigroup of itself. Any semigroup S is a large subsemigroup of both S^1 and S^0. Any subsemigroup of a finite semigroup is large. Consider $\mathbb{N} \times \mathbb{Z}$. Choose any finite subset $X \subseteq \mathbb{Z}$. Then we claim $T = (\mathbb{N} \times \mathbb{Z}) \setminus (\{1\} \times X)$ is a large subsemigroup. It is clear that T has finite complement. It is also a semigroup as each the set $\{1\} \times X$ is a set of indecomposable elements, and so it must be that $T^2 \subseteq T$.

The motivation for studying how the separability properties of large subsemigroups affects those of the oversemigroup is motivated by the following theorem of Ruškuc and Thomas.

Theorem 4.3.3. [50, Corollary 4.6] *Let T be a large subsemigroup of S. Then S is residually finite if and only if T is residually finite.*

Our aim is then to say for which of our separability properties we can find analogous results. In all cases except for that of monogenic subsemigroup separability, it is true that if a large subsemigroup has a separability property then so does the oversemigroup (Theorem 4.3.12). Proposition 4.3.10 demonstrates that a semigroup which is not MSS can have a large subsemigroup which is MSS.

Example 4.3.4. Let A be the semigroup given by the presentation $\langle\, a, b, c \mid ab^2c = b \,\rangle$, and let $S = A \cup \{d\}$ with multiplication inherited from A, and for

$w \in A$ we define the following multiplication:

$$\begin{aligned} d^i &= b^i, \text{ for } i \geq 2, \\ wd &= wb, \\ dw &= bw. \end{aligned}$$

An exhaustive check confirms that S is a semigroup. Then A is a large subsemigroup of S. We know that A is monogenic subsemigroup separable by Proposition 4.1.40.

However, S is not monogenic subsemigroup separable. We show that the element b cannot be separated from the subsemigroup $\langle d \rangle$. From the definition of the multiplication, we can see that $\langle d \rangle = \{d\} \cup \{b^i \mid i \geq 2\}$. But in the proof of Proposition 4.1.40, when showing that A is not WSS, we showed that we cannot separate b from $\langle b^2, b^3 \rangle = \{b^i \mid i \geq 2\}$. Hence, in S we cannot separate b from $\langle d \rangle$ and S is not MSS, as required.

When it comes to MSS semigroups, there is some good news. We are able to show that when T is a large subsemigroup of a semigroup S such that $S \setminus T$ is also a subsemigroup of S, monogenic subsemigroup separability will pass from T to S. This and other future results rely on theory developed in [50]. Before we give a review of this theory, we first define the notion of a left congruence and of a right congruence on a semigroup.

Definition 4.3.5. Let S be a semigroup and let π be a partition on S. We say that π is a *right congruence* on S if $(x, y) \in \pi$ implies that $(xs, ys) \in \pi$ for all $s \in S$. We say that π is a *left congruence* on S if $(x, y) \in \pi$ implies that $(sx, sy) \in \pi$ for all $s \in S$. We say that a left congruence or a right congruence has *finite index* if the underlying partition π has finitely many blocks.

Example 4.3.6. Let G be a group and let H be a subgroup of G. The set of right cosets $\{Hg \mid g \in G\}$ form a partition of G. Furthermore, if $x, y \in Hg$ for some $g \in G$, then $xk, yk \in H(gk)$ for any $k \in G$. Hence the right cosets of H form a right congruence of G. Analogously, the set of left cosets $\{gH \mid g \in G\}$ form a left congruence of G.

We note that a semigroup congruence is a partition which is both a left congruence and a right congruence. With the concept of right congruences in mind, we can adopt notation from [50].

Notation 4.3.7. For a partition π on S define

- $\Sigma_R(\pi) = \{(x, y) \in S \times S \mid (xs, ys) \in \pi \text{ for all } s \in S\}$;
- $\Sigma(\pi) = \{(x, y) \in S \times S \mid (txs, tys) \in \pi \text{ for all } s, t \in S\}$.

Note that $\Sigma_R(\pi)$ is the maximal right congruence contained within π and $\Sigma(\pi)$ is the maximal congruence contained within π.

Now we introduce two lemmas that are crucial to understanding how the separability properties of large subsemigroups influence those of the over-semigroup. We use Δ_X to denote the diagonal partition on a set X and ∇_X to denote the universal partition on a set X.

Lemma 4.3.8. [50, Theorem 4.3] *Let S be a semigroup, let T be a large subsemigroup of S, and let λ be a left congruence on T having finite index. Then the right congruence $\Sigma_R(\lambda \cup \Delta_{S \setminus T})$ has finite index in S.*

Lemma 4.3.9. [50, Theorem 2.4] *If ρ is a right congruence of finite index in S, then the congruence $\Sigma(\rho)$ has finite index in S.*

We now turn our attention to the aforementioned result concerning monogenic subsemigroup separability.

Proposition 4.3.10. *Let T be a large subsemigroup of a semigroup S such that $U = S \setminus T$ is a subsemigroup of S. Then S is monogenic subsemigroup separable if and only if T is monogenic subsemigroup separable.*

Proof. ($\Rightarrow$) This follows as subsemigroups inherit monogenic subsemigroup separability by Proposition 1.2.13.

($\Leftarrow$) Assume that T is MSS. Let $V \leq S$ be generated by the set $\{v\}$ and let $y \in S \setminus V$. We split into two cases: $v \in U$ and $v \in T$.

Case 1. Assume that $v \in U$. Note that by assumption U is a finite subsemigroup of S. Then it must be the case that V is a finite subsemigroup of U. As T is MSS, it is residually finite by Lemma 1.2.11. Hence S is residually finite by Theorem 4.3.3. Therefore we can separate y from the finite set V, as desired.

Case 2. Now assume that $v \in T$. We split into two subcases: $y \in U$ or $y \in T$.

Case 2a. Assume that $y \in U$. Consider the congruence $\xi = \Sigma(\nabla_T \cup \Delta_U)$. The congruence ξ will have finite index by Lemma 4.3.8 and Lemma 4.3.9. Furthermore, $[y]_\xi = \{y\}$. This follows as ξ is contained within $\nabla_T \cup \Delta_U$ and $y \in U$. Hence y is separated form V.

Case 2b. Now assume that $y \in T$. As we assuming that $V = \langle v \rangle$ is a monogenic subsemigroup of T and that T is MSS, there exists a finite index congruence η on T such that $[y]_\eta \neq [v^n]_\eta$ for all $n \in \mathbb{N}$. By Lemma 4.3.8 we have that $\Sigma_R(\eta \cup \Delta_U)$ has finite index in S, and hence so does the congruence $\xi = \Sigma(\eta \cup \Delta_U)$. Furthermore, $[y]_\xi \neq [v^n]_\xi$ for all $n \in \mathbb{N}$. This follows as ξ is contained within $\eta \cup \Delta_U$. Hence ξ separates y from V, as desired. □

As an immediate consequence of Proposition 4.3.10, we have the following corollary which shows that monogenic subsemigroup separability is preserved under two important semigroup constructions: the adjoining of an identity element and the adjoining of a zero element.

Corollary 4.3.11. *Let S be a semigroup. Then the following are equivalent:*

(1) *S is monogenic subsemigroup separable;*
(2) *S^1 is monogenic subsemigroup separable;*
(3) *S^0 is monogenic subsemigroup separable.*

We conclude this section by showing that our other separability properties pass from large subsemigroups to their overgroups.

Theorem 4.3.12. *Let $\mathscr{P}$ be one of the following properties: complete separability, strong subsemigroup separability or weak subsemigroup separability.*

Let T be a large subsemigroup of a semigroup S. Then S has property $\mathscr{P}$ if and only if T has property $\mathscr{P}$.

Proof. ($\Rightarrow$) Each of the properties is inherited by subsemigroups by Proposition 1.2.13.

($\Leftarrow$) Assume that T has property $\mathscr{P}$. Let $U \subseteq S$ be a subset of the type associated with property $\mathscr{P}$ and let $v \in S \setminus U$.

First consider the case where $v \in T$. If $U \cap T = \emptyset$, then the congruence $\xi = \Sigma(\nabla_T \cup \Delta_{S\setminus T})$ separates v from U. The congruence ξ will have finite index by Lemma 4.3.8 and Lemma 4.3.9. Furthermore $[u]_\xi = \{u\}$ for all $u \in U$. This follows as ξ is contained within $\nabla_T \cup \Delta_{S\setminus T}$ and $U \subseteq S \setminus T$. Hence v is separated from U.

Now we deal with the situation when $U \cap T \neq \emptyset$. Then $U \cap T$ is a subset of T of the type associated with property $\mathscr{P}$. In the case that $\mathscr{P}$ is complete separability or strong subsemigroup separability, this is clear. When U is a finitely generated subsemigroup, we have $U \cap T$ is a finitely generated subsemigroup of T by [9, Corollary 3.2]. As T has property $\mathscr{P}$, there exists a congruence η of finite index in T such that $[v]_\eta \notin [U \cup T]_\eta$. By Lemma 4.3.8 $\Sigma_R(\eta \cup \Delta_{S\setminus T})$ has finite index in S, and hence so does the congruence $\xi = \Sigma(\eta \cap \Delta_{S\setminus T})$ by Lemma 4.3.9. As ξ is contained in $\eta \cup \Delta_{S\setminus T}$, each η-class is a union of ξ-classes. Hence ξ separates v from U.

The final case to consider is when $v \in S \setminus T$. In this case the congruence $\xi = \Sigma(\nabla_T \cup \Delta_{S\setminus T})$ separates v from U. The congruence ξ will have finite index by Lemma 4.3.8 and Lemma 4.3.9. Furthermore $[v]_\xi = \{v\}$. This follows as ξ is contained within $\nabla_T \cup \Delta_{S\setminus T}$ and $v \in S \setminus T$. Hence v is separated from U, completing the proof. □

Remark 4.3.13. The reason why the argument from the proof of Theorem 4.3.12 does not the extend to the property of monogenic subsemigroup separability is because it is not necessarily true that the intersection of a large subsemigroup with a monogenic subsemigroup is itself monogenic. We can see this in Proposition 4.3.10, where $\langle d \rangle \cap A = \{b^i \mid i \geq 2\}$ is not monogenic.

Concluding Remarks and Future Work

The work in this thesis has demonstrated that the study of separability properties of semigroups and algebras is a rich area of mathematical interest. By considering these properties as part of a systematic framework, we have been able to undertake a comparison of different properties in various contexts. Our investigations have revealed the deep links between separability properties and structural theory. This is demonstrated by results such as Theorem 3.3.22, which states that a finitely generated commutative semigroup is MSS if and only every subgroup is finite, and Theorem 3.3.23, which establishes the equivalence of complete separability, strong subsemigroup separability and weak subsemigroup separability for finitely generated commutative semigroups by giving a characterisation of these properties in terms of Green's relation $\mathcal{H}$. The well-established link between separability properties and classical decision problems, such as the word problem and the generalised word problem, further motivates the drive to understand these properties. An intriguing and complex picture of how separability properties interact with algebraic constructions has emerged from the work in Chapter 4, but this picture is far from complete. Many open problems have been recorded throughout this thesis, and these have the potential of forming the basis for future research.

We conclude this work by bringing together some of the open questions already stated in this thesis. Although these questions have arisen from very different lines of investigation, it is possible to find a link between them. In

Chapter 2, we investigated the separability properties of free objects in various semigroup varieties. Our investigations into the free completely regular semigroup left us with the following open problem.

Open Problem 2.6.13. For $|X| \geq 2$, is the free completely regular semigroup FCR_X weakly subalgebra separable or monogenic subalgebra separable?

Recall that the free completely regular semigroup on a set of size two consists of two copies of $\mathbb{Z}$ which act upon an Rees matrix semigroup over the free group on a countable basis. Therefore, an understanding of the of the separability properties of Rees matrix semigroups over groups may aid our investigations into the free completely regular semigroup. We have already made some steps in this area with our results concerning the free completely simple semigroup; the class of Rees matrix semigroups over groups is precisely the class of completely simple semigroups. In the literature, some separability properties of both completely simple semigroups and completely 0-simple semigroup have been classified. Golubov was able to classify when both a completely simple semigroup and a 0-simple semigroup are strongly subsemigroup separable ([18, Theorems 1 and 2]), as well as considering when a completely zero-simple semigroup is residually finite ([18, Theorem 3]). These classifications rely upon the separability properties of the group used in the constructions of the Rees matrix semigroup, as well as some conditions being placed on the matrix. Of course Open Problem 2.6.13 is referring not to subsemigroups, but to subalgebras (which are subsemigroups which are also closed under the inversion map). Furthermore, we are restricting our attention to finitely generated and monogenic subalgebras. Nevertheless, Golubov's work may prove a useful starting point; it seems reasonable to expect these separability properties to be reliant upon the group separability properties of the underlying group as well as properties of the matrix.

In Section 3.5, we investigated separability properties of semigroups with finitely many $\mathcal{H}$-classes. In particular, the following questions remain open.

Open Problem 3.5.6. Is it true that a semigroup with only finitely many $\mathcal{H}$-

classes is weakly subsemigroup separable if and only if all its Schützenberger groups are weakly subsemigroup separable?

Open Problem 3.5.20. Let G be the Grigorchuk group, let I and Λ finite sets, and let $P = (p_{\lambda i})$ a $\Lambda \times I$ matrix with entries from G^0 such that no row or column consists entirely of zeros. Is the semigroup $M^0[G; I, \Lambda; P]$ weakly subsemigroup separable?

Again, Open Problem 3.5.20 makes reference to a completely 0-simple semigroup. The reason for this is because Rees' construction allows one to easily build a semigroup with a specified number of $\mathcal{H}$-classes, each having the same Schützenberger group. This time we do want to know about the semigroup separability properties of the semigroup $M^0[G; I, \Lambda; P]$. Therefore, by investigating weak semigroup separability properties of both completely simple and completely 0-simple semigroups, we would increase the chance of being able to answer Open Problem 3.5.20 and also Open Problem 3.5.6.

Finally, we can consider Rees' constructions in the light of the work of carried out in Chapter 4. The constructions allows us to build new semigroups from any base semigroup, not just groups. It is then natural to ask, given a semigroup S, what can we say about the separability properties of a Rees matrix semigroup over S. In particular, if S has a separability property $\mathcal{P}$, then under what conditions will the Rees matrix semigroup have $\mathcal{P}$? The work done by Golubov strongly suggests that the answer to this question will rely upon the matrix used in the construction.

The line of investigation outlined above is just one of the many possible paths future research into separability properties could take. Other lines of research could explore the topological interpretation of separability properties, or use separability properties to tackle decision problems. The study of separability promises to continue to be an exciting and productive area of mathematical research.

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