Probabilistic generation of almost simple groups and their maximal subgroups

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Abstract

In this thesis we study $P_d(G)$ the probability of generating a non-abelian simple group G with d randomly chosen elements. We restrict our analysis to $G = \mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$ and $\mathrm{PSp}_n(q)$. Let $m(G)$ be the smallest value of $|G : M|$ over all maximal subgroups M of G. We show that $1 - \frac{1199}{243m(G)} \leq P_2(G)$ with equality occurring when $G = \mathrm{PSU}_4(3)$. We also provide sharper bounds for when we restrict our attention to $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_n(q)$ for $n \geq 3$ or $\mathrm{PSp}_n(q)$ on their own. We obtain the results $1 - \frac{38}{15m(\mathrm{PSL}_2(q))} \leq P_2(\mathrm{PSL}_2(q))$ where equality only occurs when $q = 11$, $1 - \frac{57}{20m(\mathrm{PSL}_n(q))} \leq P_2(\mathrm{PSL}_n(q))$ where equality only occurs when $(n, q) = (3, 4)$ and $1 - \frac{6067}{1440m(\mathrm{PSp}_n(q))} \leq P_2(\mathrm{PSp}_n(q))$ where equality only occurs when $(n, q) = (4, 4)$.

The values of $m(G)$ are known for the simple groups. However, in Chapter 5, we expand our scope to almost simple groups G with socle S, and calculate $m(G)$, the smallest value of $|G : M|$ over all maximal subgroups M of G such that $S \not\leq M$. We show which G satisfy $m(G) = m(S)$, and in addition provide the values of $m(G)$ for the cases where $m(G) \neq m(S)$.

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Chapter 1

Introduction

In this thesis we study two questions concerning finite groups: probabilistic generation of classical simple groups and the largest non-trivial maximal subgroup of almost-simple groups. At first blush these questions may seem unrelated, however both are actually questions about the orders of certain subgroups of simple groups.

In the first part of the thesis we study $P_d(G)$, the probability of generating a finite group G with d randomly chosen elements. We study the cases where G is a linear, symplectic or unitary simple group. A natural point for further investigations would be to include the orthogonals, however due to time constraints we have not covered this case, though we intend to do further work to correct this.

By the Classification of Finite Simple Groups all finite simple groups are 2-generated, consequently $P_2(G) > 0$ for all finite simple groups G.

In 1969 Dixon [15] showed that in fact $P_2(\mathrm{A}_n) \to 1$ as $n \to \infty$, settling a long standing conjecture by Netto [43]. Dixon would later provide a sharper statement in [16]. Upper and lower bounds for $P_2(\mathrm{A}_n)$ are given in work by Maróti and Tamburini [37] and more recently by Morgan and Roney-Dougal [41].

In [15], Dixon also conjectured that for all finite simple groups G we have $P_2(G) \to 1$ as $|G| \to \infty$. This was proved by Kantor and Lubotzky [22] for classical groups and some exceptional groups, with the result proved for the remaining exceptional groups by Liebeck and Shalev [31].

These results are asymptotic in nature, however concrete lower bounds for $P_2(G)$ were later obtained in [39]; it was shown that in fact $P_2(G) \geq \frac{53}{90}$ for all finite simple groups G with equality if and only if $G = \mathrm{A}_6$.

Let $m(G)$ be the index of the largest maximal subgroup of the simple group G. Liebeck and Shalev [32] showed that there exists constants $\alpha, \beta > 0$ such that for all finite simple groups G the following holds

$$1 - \frac{\alpha}{m(G)} \leq P_2(G) \leq 1 - \frac{\beta}{m(G)}. \tag{1.1}$$

For the case where $G = \mathrm{A}_n$, values for α and β are provided by Morgan and Roney-Dougal [41]. In Chapter 4 we continue such work and provide values for α for each of the cases where $G = \mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$ and $\mathrm{PSU}_n(q)$. In chapter 4 we prove the following

Theorem. *Let $n \geq 2$ and let q be a prime power. Furthermore let $G = \mathrm{PSL}_n(q), \mathrm{PSp}_n(q)$ or $\mathrm{PSU}_n(q)$. Then*

$$1 - \frac{1199}{243m(G)} \leq P_2(G).$$

Furthermore

- *Let $q \geq 4$. If $G = \mathrm{PSL}_2(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{38}{15} \approx 2.534$. Equality only occurs when $q = 11$.*
- *Let $n \geq 3$, where $(n, q) \neq (3, 2)$. If $G = \mathrm{PSL}_n(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{57}{20} = 2.85$. Equality only occurs when $(n, q) = (3, 4)$.*
- *Let $n \geq 4$ be even, and $(n, q) \neq (4, 2)$. If $G = \mathrm{PSp}_n(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{6067}{1440} \approx 4.214$. Equality only occurs when $(n, q) = (4, 4)$.*
- *Let $n \geq 3$, and $(n, q) \neq (3, 2)$. If $G = \mathrm{PSU}_n(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{1199}{243} \approx 4.935$. Equality only occurs when $(n, q) = (4, 3)$.*

All decimals above are rounded up values to three decimal places.

One related avenue of research lies in asking if it is possible to find a pair of generators of prescribed orders. We say that a finite group G is (a, b)-generated if there are two elements of G, one with order a, the other with order b, that together generate the whole group. It has been shown that sufficiently large non-abelian finite simple groups are $(2, 3)$-generated with the exception of $\mathrm{PSp}_4(2^f)$, $\mathrm{PSp}_4(3^f)$ and ${}^2\mathrm{B}_2(q)$. This is shown by Miller [40] for alternating groups, Liebeck and Shalev [33] for classical groups, Lübeck and Malle [34] for exceptional groups. In addition this problem has been covered for sporadic simple groups by Woldar [53]. An overview on this topic can be found in [7].

As is hinted in Equation (1.1), the probability $P_2(G)$ is actually related to the subgroup structure of G, in particular the maximal subgroups of G. Work from Cooperstein [10], with later corrections from Bray [5] and Kleidman and Liebeck [26, Theorem 5.2.2] provide us with information on the values of $m(G)$ for all classical simple groups G.

This leads us to the second part of the thesis, where we widen our field of view and consider all almost simple groups, groups G that satisfy $S \leq G \leq \mathrm{Aut}(S)$ for a non abelian simple group S. We calculate the index of the largest non-trivial maximal subgroup of G, that is, the largest maximal subgroup that does not contain S. We present our results in Theorems 5.2.1 and 5.2.2. This work builds on previous work on maximal subgroups of simple groups.

There is bountiful literature on this subject. In particular, we would like to point the reader to various results and sources on this topic. For example, in the cases of A_n and S_n the O'Nan-Scott Theorem gives us a classification of the subgroups; a subgroup either satisfies certain geometric conditions or is an almost simple group itself. Further work by Liebeck, Praeger and Saxl [28] has shown which of these groups are actually maximal.

An analogous result to the O'Nan Scott Theorem for classical groups is given by Aschbacher's Theorem [2]. More detailed information on the structure of certain non-trivial maximal subgroups of classical groups was provided by Kleidman and Liebeck [26] for dimension $n \geq 13$. In fact for dimension $n \leq 12$ Bray, Holt and Roney-Dougal [6] give us complete information on the maximal subgroups, and for dimension 13, 14 and 15 one may find information in work by Schröder [45].

For certain exceptional groups, the maximal subgroups are also known. For the Suzuki groups ${}^2\mathrm{B}_2(q)$ the maximal subgroups were determined by Suzuki himself [47]. For almost simple groups G with socle groups $\mathrm{G}_2(q)$, the maximal subgroups were determined by Kleidman [24] for q odd, by Cooperstein [11] for q even and G simple, and by Aschbacher [4] for the remaining cases. The small Ree groups ${}^2\mathrm{G}_2(q)$, and the groups ${}^3\mathrm{D}_4(q)$ were determined by Kleidman in [24] and [25]. Finally the maximal subgroups of almost simple groups with socle ${}^2\mathrm{F}_4(q)$ were determined by Malle [36].

For the remaining exceptional groups full classifications of maximal subgroups are currently incomplete. Nevertheless, there is still a large amount of information on these subgroups as outlined [30]. There is also further information on the maximal subgroups of the exceptional groups $F_4(q)$, $E_6(q)$ and ${}^2E_6(q)$ in recent work by Craven [12]. Finally, in the case of almost simple sporadic groups, a complete classification of the maximal subgroups, with the exception of the Monster, can be found in [52].

Chapter 2

Preliminaries

Let us first start with some fundamental definitions and notation. The following chapter discusses almost simple groups, classical groups and their subgroups in more detail. Throughout we will assume that all groups are finite unless otherwise stated.

2.1 General definitions and notation

Let us begin with a some general definitions.

Definition 2.1.1. Let A and B be groups. Then G is an *extension* of A by B if there exists an $N \trianglelefteq G$ such that N is isomorphic to A and G/N is isomorphic to B. In this case we use $A.B$ to denote G.

Furthermore if there also exists $M \leq G$ such that $MN = G$, $M \cap N = 1$ and M is isomorphic to B then this is a *split extension*. We denote this by $A : B$ while in the case where there is no such subgroup M then we have a *non-split extension* denoted by $A \,{}^{\cdot}\, B$.

The cyclic group of order n is denoted by C_n or when as a component of a group structure just by n. For example $S_3 = C_3 : C_2$, since there exists an element a of order 3 and another element b of order 2 such that the intersection of $\langle a \rangle$ and $\langle b \rangle$ meet trivially. Meanwhile $C_4 = C_2 \,{}^{\cdot}\, C_2$, since given a normal subgroup C_2 of C_4 there does not exist another C_2 subgroup that meets it trivially.

Definition 2.1.2. An *elementary abelian group* is an abelian group where all non-trivial elements have order p for some prime p.

An elementary abelian group of order p^n will be denoted by p^n. Furthermore groups of the form $p^a.p^b$ may be denoted by p^{a+b}. By $[n]$ we denote a group of order n of unspecified structure.

We denote by S_n and A_n the symmetric group and alternating group on n elements respectively. For even n we have that D_n denotes the dihedral group of order n and for n a power of 2 we write Q_n for the quaternion group of order n.

Definition 2.1.3. Let H and K such that there is an isomorphism ϕ from $Z_1 \leq Z(H)$ to $Z_2 \leq Z(K)$. Define $Z \leq H \times K$ by $Z = \{(x, \phi(x)) : x \in Z_1\}$. Then $G = (H \times K)/Z$ is a *central* *product* of H and K, often denoted by $H \circ K$. The groups H and K are called *central factors* of G. If Z_1 and Z_2 are not specified, then they are assumed to be the largest isomorphic subgroups of $Z(H)$ and $Z(K)$.

Definition 2.1.4. Given a group G, the *Frattini subgroup*, $\Phi(G)$ is the intersection of all maximal subgroups of G.

$\Phi(G)$ is equal to the set of all non-generating elements of G. A non-generating element of G is an element that can always be removed from a generating set. A group with a non-trivial Frattini subgroup is $G = C_{p^2}$ where p is prime and G is generated by a; here we have $\Phi(G) = \langle a^p \rangle$.

Definition 2.1.5. A p-group G is an *extraspecial* group if $G' = Z(G) = \Phi(G) \cong C_p$.

It can be shown that all extraspecial groups have order p^{1+2n} for $n \geq 1$. In addition for each such number there are exactly two extraspecial groups up to isomorphism. For r an odd prime, we write r_+^{1+2n} for an extraspecial group of order r^{1+2n} and exponent r, and r_-^{1+2n} for an extraspecial group of the same order, but exponent r^2. We write 2_+^{1+2m} for an extraspecial group of order 2^{1+2m} that is isomorphic to a central product of m copies of D_8, and we write 2_-^{1+2m} for an extraspecial group of the same order, but that is isomorphic to a central product of $m-1$ copies of D_8 and one of Q_8.

Definition 2.1.6. If G has a unique subgroup of index 2, then we denote this subgroup by $\frac{1}{2}G$.

Definition 2.1.7. By *shape* we mean a rough description of an isomorphism type. This is not a complete description of the group but allows the reader to read off the composition factors.

We may denote finite field of order $q = p^r$ by $\mathbb{F}_q$. We denote the multiplicative group of a field F by $F^\times$.

Definition 2.1.8. Let H be a subgroup of G then we define $\mathrm{Cl}_G(H)$ to be the set of subgroups of G that are conjugate to H.

2.2 Largest maximal subgroups

We first generalize the idea of a simple group.

Definition 2.2.1.
- For a group G, the *socle* is the subgroup of G generated by its minimal normal subgroups.
- An *almost simple group* G is a group satisfying $S \cong \mathrm{Inn}(S) \leq G \leq \mathrm{Aut}(S)$ for non-abelian simple group S. The socle of G is S.

Definition 2.2.2. A group G is *perfect* if $G = G'$ where G' is the derived subgroup of G.

Definition 2.2.3. A group G is *quasisimple* if it is perfect and if $G/Z(G)$ is simple.

Definition 2.2.4.
- If G is almost simple with socle S then we define $m(G)$ to be the minimum of $|G : M|$ over all maximal subgroups M of G such that $S \not\leq M$. In cases where we want to stress the subgroup S we use the notation $m_S(G)$.

- If G is quasisimple then we define $m(G)$ to be the minimum of $|G : M|$ over all maximal subgroups M of G.

Note that if G is non-abelian simple then it is also almost simple so the definition above also applies in such a case.

For some cases we would like to look beyond the largest order of maximal subgroups so for that end we have the following definitions.

Definition 2.2.5.

- Let G be almost simple with socle S. Let A be the set of integers arising as indices of maximal subgroups M of G that do not contain S. Then we define $m_n(G)$ to be the nth smallest element of A. In cases where we want to stress the subgroup S we use the notation $m_{n,S}(G)$.
- Let G be a quasisimple group. Let B be the set of integers arising as indices of maximal subgroups M of G. If G is quasisimple then we define $m_n(G)$ to be the nth smallest element of B.

Again, we note that if G is simple then it is also almost simple so the definition above also applies in such a case.

Lemma 2.2.6. *If G is a perfect group then its maximal subgroups are in one-to-one correspondence with the maximal subgroups of $G/Z(G)$.*

Proof. Suppose that M is a maximal subgroup of G and that $Z(G) \not\leq M$. Then $G = Z(G)M$. Therefore M must be normal in G and $G/M \cong Z(G)/(M \cap Z(G)) = 1$, which is a contradiction. We conclude that every maximal subgroup of G contains $Z(G)$ and the results follows from the correspondence theorem. $\square$

The following theorem is due to [10], with corrections from [26, Theorem 5.2.2] and further corrections by [5] as presented in [38].

Theorem 2.2.7 ([10], [5], [26])**.** *Let G be a simple classical group. Then the values of $m(G)$ as reproduced in Table 2.1.*

Corollary 2.2.8. *Let G be a quasisimple group such that $G/Z(G)$ is a simple classical group. Then $m(G) = m(G/Z(G))$ and as such can be obtained from Table 2.1 from the relevant row. In addition $m_n(G) = m_n(G/Z(G))$.*

2.3 Almost simple groups

Recall the definition of an almost simple group from Definition 2.2.1. For the rest of this section we shall assume that G is an almost simple group, and S is a simple group unless stated otherwise.

Note that as S is simple it is isomorphic to $\mathrm{Inn}(S)$. Since S is isomorphic to $\mathrm{Inn}(S)$ and G is a subgroup of $\mathrm{Aut}(S)$ we may consider G/S to be a set of outer automorphisms which act on the conjugacy classes of subgroups of S.

Table 2.1: $m(G)$ for simple classical groups

G	$m(G)$
$\mathrm{PSL}_n(q)$, $(n,q) \neq$ $(2,5),(2,7)(2,9),(2,11),(4,2)$	$\dfrac{q^n-1}{q-1}$
$\mathrm{PSL}_2(5), \mathrm{PSL}_2(7), \mathrm{PSL}_2(9),$ $\mathrm{PSL}_2(11), \mathrm{PSL}_4(2)$	5, 7, 6, 11, 8
$\mathrm{PSp}_{2m}(q), m \geq 2, q > 2$ $(m,q) \neq (2,3)$	$\dfrac{q^{2m}-1}{q-1}$
$\mathrm{Sp}_{2m}(2), m \geq 3$	$2^{m-1}(2^m-1)$
$\mathrm{Sp}_4(2)', \mathrm{PSp}_4(3)$	6, 27
$\mathrm{PSU}_3(q), q \neq 2, 5$	q^3+1
$\mathrm{PSU}_3(5)$	50
$\mathrm{PSU}_4(q)$	q^4+q^3+q+1
$\mathrm{PSU}_n(q), n \geq 5,$ $(n,q) \neq (2m,2)$	$\dfrac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{(q^2-1)}$
$\mathrm{PSU}_n(2), n \text{ even}, n \geq 6$	$\dfrac{2^{n-1}(2^n-1)}{3}$
$\Omega_{2m+1}(q), m \geq 3, q \geq 5 \text{ odd}$	$\dfrac{q^{2m}-1}{q-1}$
$\Omega_{2m+1}(3), m \geq 3$	$\dfrac{3^m(3^m-1)}{2}$
$\mathrm{P}\Omega^+_{2m}(q), m \geq 4, q \geq 4$	$\dfrac{(q^m-1)(q^{m-1}+1)}{q-1}$
$\mathrm{P}\Omega^+_{2m}(2), m \geq 4$	$2^{m-1}(2^m-1)$
$\mathrm{P}\Omega^+_{2m}(3), m \geq 4$	$\dfrac{3^{m-1}(3^m-1)}{2}$
$\mathrm{P}\Omega^-_{2m}(q), m \geq 4$	$\dfrac{(q^m+1)(q^{m-1}-1)}{q-1}$

Definition 2.3.1. A maximal subgroup M of G is called *trivial* if $\mathrm{soc}(G) \leq M$. Otherwise it is *non-trivial.*

In general we shall focus on the non-trivial case; the reason being that if $M < G$ is trivial, then by the Correspondence Theorem we can associate M with a subgroup of $\mathrm{Out}(\mathrm{soc}(G))$. In this case, the outer automorphism groups of simple groups are well known and therefore so are the trivial maximal subgroups M. In addition the trivial maximal subgroups of G have no relation with the maximal subgroups of $\mathrm{soc}(G)$.

Consequently the main focus will be on non-trivial maximal subgroups, which we can also split into two cases.

Definition 2.3.2. Let G satisfy $S \leq G \leq \mathrm{Aut}(S)$, where S is a non-abelian simple group, and let $M \leq G$. Then

- M is an *ordinary* maximal subgroup if $S \cap M$ is maximal in S.
- M is a *novel* maximal subgroup (or just a *novelty*) if $S \cap M$ is a proper non-maximal subgroup of S.

As can be seen from these definitions an important source of information about non-trivial maximal subgroups of G is their intersection with the $S = \mathrm{soc}(G)$. Consequently we introduce the following notation.

Definition 2.3.3. Let G be an almost simple group with socle S, and let H be a subgroup of G. Then we define H_S to be $S \cap H$.

Even though we can see how maximal subgroups of G lead us to subgroups of $S = \mathrm{soc}(G)$, in general we would prefer to be given a subgroup H of S and see if there exists a maximal subgroup M of G such that $M_S = H$.

Lemma 2.3.4. *Let G be an almost simple group with socle S. If K is a subgroup of G such that $K_S = H$ then $K \leq N_G(H)$.*

Proof. $S \trianglelefteq G$, so $H = S \cap K \trianglelefteq K$. Hence $K \leq N_G(H)$. $\square$

Lemma 2.3.5 ([6])**.** *Let G be a finite almost simple group with socle S. Suppose that M is a maximal subgroup of G. Then $S \cap M \neq 1$.*

Lemma 2.3.6. *Let G be an almost simple group with socle S. If M is a non-trivial maximal subgroup of G, then there exists a subgroup H in S such that $M = N_G(H)$ and $H = N_S(H)$. Furthermore $H = M_S$.*

Proof. Consider $M_S = M \cap S < S$. By Lemma 2.3.5 we know that $M_S \neq 1$. Also, by Lemma 2.3.4 we have that $M \leq N_G(M_S)$. However by the simplicity of S we have that $N_G(M_S) \ngeq S$, and by maximality of M we have that $N_G(M_S) = M$. So $M_S = N_G(M_S) \cap S = N_S(M_S)$. $\square$

This leads us to the fact that all non-trivial maximal subgroups of finite almost simple group G are in fact normalizers of subgroups of S in G. Furthermore we see that in order to find all non-trivial maximal subgroups of G we need only look at the normalizers of subgroups $H \leq S$ such that $N_S(H) = H$.

So now given a subgroup $H \leq S$ such that $H = N_S(H)$ and an almost simple group G with socle S, we would like to see if in fact $N_G(H)$ is actually a maximal subgroup of G. Firstly we cover some necessary conditions.

Lemma 2.3.7. *Let G be an almost simple group with socle S, and let $H \neq 1$ be a proper subgroup of S. If $N_G(H)$ is maximal in G then $N_G(H)S = G$.*

Proof. Suppose otherwise. We note that $H \ntrianglelefteq S$ and so $N_G(H) < G$ and also $S \nleq N_G(H)$. Then $N_G(H) < N_G(H)S < G$, a contradiction. $\square$

Recall that G/S can be considered to be a subgroup of the outer automorphisms of G, and thus acts on the conjugacy classes of subgroups of S.

Lemma 2.3.8. *Let G be an almost simple group with socle S, and let H be a subgroup of S so that $H \leq S = \mathrm{Inn}(S) < G < \mathrm{Aut}(S)$. Then $N_G(H)S = G$ if and only if G/S fixes the S-conjugacy class of H.*

Proof. If $N_G(H)S = G$ then $H^G = H^{N_G(H)S} = H^S$.

Let G/S fix the conjugacy class of H in S. Therefore for any $g \in G$, g maps H to a conjugate of H in S. Thus there exists an s such that $gs \in N_G(H)$. We therefore conclude that $g = gss^{-1} \in N_G(H)S$. $\square$

Consequently we may rephrase Lemma 2.3.7 as

Lemma 2.3.9. *Let G be an almost simple group with socle S, and let $H \neq 1$ be a proper subgroup of S. If $N_G(H)$ is maximal in G then G/S fixes the conjugacy class of H in S.*

The advantage of this formulation is that if we are given an almost simple group in the form of $G = S.T$ where $T \leq \mathrm{Out}(S)$ we can work out whether a subgroup H of S may lead to a maximal subgroup $N_G(H)$ without having to calculate $N_G(H)$ exactly.

It is also of value to understand what happens if G/S fixes the conjugacy class of a subgroup H in S. To this end we start with the following lemma

Lemma 2.3.10 ([3])**.** *Let G be transitive on X, $x \in X$ and $H \leq G$. Then H is transitive on X if and only if $G = G_xH$.*

Lemma 2.3.11. *Let G be an almost simple group with socle S, and let H be a subgroup of S. If G/S fixes the conjugacy class of H in S then there does not exist a subgroup M such that $N_G(H) \leq M < G$ and $S \leq M$.*

Proof. We have that $G = N_G(H)S$. By Lemma 2.3.10 S acts transitively on the set of G-conjugates of H.

Suppose otherwise, then there exists an M such that $S \leq M$ and $N_G(H) \leq M < G$ and so $SN_G(H) \leq M < G$. However, since G/S fixes the conjugacy class of H in S this means that $SN_G(H) = G$, which is a contradiction. $\square$

This in turn gives us the following:

Corollary 2.3.12. *Let G be an almost simple group with socle S, and let H be a subgroup of S. If G/S fixes the conjugacy class of H in S and there exists K such that $N_G(H) < K < G$, then $H < K_S < S$.*

Proof. By Lemma 2.3.11, $S \not\leq K$. Hence $K_S = K \cap S < S$. We note that $H \leq N_S(H) = N_G(H) \cap S \leq K \cap S = K_S$. However if $H = K_S$ then $K \leq N_G(H)$ by Lemma 2.3.4, a contradiction. □

We now look at our first necessary and sufficient condition.

Lemma 2.3.13. *Let G be an almost simple group with socle S, and M a maximal subgroup of S. Then $N_G(M)$ is maximal in G if and only if $N_G(M)S = G$.*

Proof. Suppose that $N_G(M)$ is not maximal in G. Then there exists a subgroup K satisfying $N_G(M) < K < G$. By Lemmas 2.3.8 and 2.3.12 that would imply that $M < K_S < S$ contradicting the maximality of M in S. The other direction is covered by Lemma 2.3.7. □

In the case where $H \leq S$ is not maximal in S then the problem is a bit more difficult. In fact it is due to the different nature of these two problems that we have Definition 2.3.2. We will focus primarily on the novelty case from now on in this section.

As noted in [6, p. 10], for $N_G(H)$ to be a novelty we require that $N_G(H)S = G$ which we shall assume to be the case. Here $N_G(H)$ fails to be maximal in G if and only if $N_G(H) < N_G(K)$ for some K such that $H < K < S$. Furthermore, by repeatedly replacing H with $N_S(H)$ and K with $N_S(K)$ we may assume that $N_S(H) = H$ and the analogue for K.

Definition 2.3.14. Let G be almost simple with socle S. If $H = N_S(H) < K = N_S(K) < S < G$ and $N_G(K)S \neq G$, then $N_G(H)$ is called a *type 1 novelty* with respect to K.

One example that was presented in [6, p. 10] is that of $G = \mathrm{PSL}_2(7)$, $S = \mathrm{PSL}_2(7)$, $H = D_6$, $M = D_12$. Here K can only be S_4, but that would give $N_G(K) = K$.

Another relevant example for later on in section 4.2, is that of $S = \mathrm{PSL}(3, q)$ and $G = \mathrm{Aut}(S)$. In this case we consider the two subgroups of S, P_1 and U, where U is the subgroup of upper diagonals and P_1 the subgroup of invertible matrices with the first column $\langle a, 0, 0\rangle^{\mathrm{T}}$ where a is non-zero. It turns out that $N_G(P_1)S \neq G$ and $N_G(U)S = G$, the first inequality resulting from the fact that the inverse transpose does not preserve the conjugacy class of P_1 in S. Here we have that $N_G(U)$ is a type 1 novelty with respect to P_1.

The other possibility that arises is:

Definition 2.3.15. Let G be almost simple with socle S. If we have $H < K < S < G$, with $N_S(H) = H$, $N_S(K) = K$, and $N_G(H)S = N_G(K)S = G$ and such that $M = N_G(H) \not\leq N_G(K)$ then M is *type 2 novelty* with respect to K.

We note that these subgroups H are defined to be novelties *with respect to* a K. This is because there is the possibility that there is a novelty H where there are multiple different K such that $H < K < S$ holds. We also note that a subgroup M being a novelty with respect to some group K does not imply that M is a maximal subgroup, only that it remains a candidate.

Again, we provide an example which comes from [6, p. 10]. Consider $G = \mathrm{PSp}_4(7) : 2$, $H = \mathrm{PSL}_2(7)$, and $M = \mathrm{PSL}_2(7) : 2$. The only possibility for K is A_7. The two conjugacy classes of groups isomorphic to H in K are fused in $N_G(K) = \mathrm{S}_7$.

We include the following proposition from [6, Proposition 1.3.10], the proof of which can be found there, which gives conditions as to when type 2 novelties occur.

Proposition 2.3.16. *Let G be almost simple with socle S. If we have $H < K < S < G$ with $N_S(H) = H$, $N_S(K) = K$ and $N_G(H)S = N_G(K)S = G$. Then $M = N_G(H) \nleq N_G(K)$ if and only if there exists a subgroup $H_0 < K$ with H and H_0 conjugate in $N_G(K)$ but not in K. In this situation H and H_0 are in fact also conjugate in S.*

We conclude with some results regarding the indices of subgroups of almost simple groups.

Lemma 2.3.17. *Let G be almost simple with socle S. Let $H < S$ be such that $H = N_S(H)$. Then if $N_G(H)S = G$ we have*

$$|G : N_G(H)| = |S : H|.$$

Proof. Since $H^G = H^{N_G(H)S} = H^S$ the conjugacy classes have the same size, i.e. $|G : N_G(H)| = |S : N_S(H)| = |S : H|$.

□

The above lemma leads us to the following lemma

Lemma 2.3.18. *Let G be an almost simple group with socle S. If $H \leq S$ is the largest maximal subgroup of S and $N_G(H)S = G$ then $N_G(H)$ is the largest non-trivial maximal subgroup of G. In particular $m(S) = m(G)$.*

Proof. By Lemma 2.3.13, $N_G(H)$ is a non-trivial maximal subgroup, and by Lemma 2.3.17 its index is equal to $|S : H|$. If there were a larger non-trivial maximal subgroup, let us call it M, then there would exist a proper subgroup K in S such that $M = N_G(K)$ and $K = N_S(K)$ by Lemma 2.3.6. By Lemma 2.3.17 the index of this group is $|S : K|$. So if there were a larger maximal non-trivial subgroup we would have $|G : M| < |G : N_G(H)|$ or alternatively $|S : K| < |S : H|$. This in turn implies that there would be a proper subgroup larger than H in S; a contradiction. By Lemma 2.3.17 we have

$$m(G) = |G : N_G(H)| = |S : H| = m(S).$$

□

2.4 The probability $P_d(G)$

Definition 2.4.1. Let G be a finite group.

- The *Eulerian function* $\phi_d(G)$ gives us the number of d-tuples $(g_1, \dots, g_d) \in G^d$ such that G is generated by the set $g_1, \dots, g_d$.

- Let $d(G)$ denote the minimal number of generators for a group G.

- Assume $d \geq d(G)$ then $P_d(G)$ is the probability that d independent and uniformly distributed random elements of G generate the whole group.
- For a simple group G such that $d(G) \leq 2$, we define c_G to be $(1 - P_2(G))m(G)$.

Rearranging the definition of c_G gives us

Lemma 2.4.2. $1 - c_G/m(G) = P_2(G)$.

Furthermore from Definition 2.4.1 we have that

$$P_d(G) = \frac{\phi_d(G)}{|G^d|}.$$

Theorem 2.4.3 ([32, Theorem 1.6])**.** *There exist constants $\alpha, \beta > 0$ such that*

$$1 - \frac{\alpha}{m(G)} \leq P_2(G) \leq 1 - \frac{\beta}{m(G)}$$

holds for all finite simple groups G.

Our aim is to provide actual values for α for specific families of simple groups. To this end let us derive some bounds for $P_d(G)$.

Lemma 2.4.4. *Assume that $d(G) \leq d$. Let $N \lhd G$. Then $P_d(G) \leq P_d(G/N)$.*

Proof. If $(g_1, \dots, g_d)$ is an d-tuple of elements of G that generate G then $(g_1N, \dots, g_dN)$ generates G/N. In addition if $(g_1N, \dots, g_dN)$ is an d-tuple of elements of G/N then there are $|N^d|$ pairs of elements of G that get mapped onto $(g_1N, \dots, g_dN)$ by the quotient map. Consequently we observe that $\phi_d(G/N) \geq \phi_d(G)/|N^d|$. Finally we have

$$P_d(G) = \frac{\phi_d(G)}{|G^d|} \leq \frac{\phi_d(G/N)|N^d|}{|G^d|} = \frac{\phi_d(G/N)}{|(G/N)^d|} = P_d(G/N).$$

□

Lemma 2.4.5. *Let $\mathcal{M}$ be a set of conjugacy class representatives for the maximal subgroups of the finite group G. If $d(G) \leq d$, then*

$$P_d(G) \geq 1 - \sum_{M \text{ max } G} |G : M|^{-d} \geq 1 - \sum_{M \in \mathcal{M}} |G : M|^{-d+1},$$

where the notation $\sum_{M \text{ max } G}$ means sum over all M where M is maximal subgroup of G.

Proof. Let Ω be the set of d-tuples that generate G. So $|\Omega| = \phi_d(G)$.

If a d-tuple $(g_1, \dots, g_d)$ does not generate G, then it must generate a subgroup of M where M is some maximal subgroup of G. Therefore we get

$$G^d \setminus \Omega = \bigcup_{M \text{ max } G} M^d.$$

So from there we see
$$|G^d \setminus \Omega| < \sum_{M \ \max \ G} |M^d|.$$
Hence
$$\phi_d(G) = |\Omega| = |G^d| - |G^d \setminus \Omega| \geq |G^d| - \sum_{M \ \max \ G} |M^d|$$
and so we get our first inequality
$$P_d(G) = \frac{\phi_d(G)}{|G^d|} \geq 1 - \sum_{M \ \max \ G} |G : M|^{-d}.$$
For the second inequality we notice that the number of conjugates of a maximal subgroup M of G is $|G : N_G(M)| \leq |G : M|$, and so
$$\sum_{M \max G} |G : M|^{-d} \leq \sum_{M \in \mathcal{M}} |G : M|^{-d+1}$$
and our inequality follows. □

Proposition 2.4.6. *Let $\mathcal{M}$ be a set whose elements are maximal subgroups of G and $d(G) \leq d$. Then*
$$P_d(G) \leq 1 - \sum_{M \in \mathcal{M}} |G : M|^{-d} + \sum_{\substack{M,N \in \mathcal{M} \\ M \neq N}} |G : M \cap N|^{-d}$$

Proof. Using the same notation as before we have
$$G^d \setminus \Omega = \bigcup_{M \ \max \ G} M^d \supseteq \bigcup_{M \in \mathcal{M}} M^d.$$
By Inclusion Exclusion Principle we observe that
$$|G^d \setminus \Omega| \geq \sum_{M \in \mathcal{M}} |M^d| - \sum_{\substack{M,N \in \mathcal{M} \\ M \neq N}} |(M \cap N)^d|.$$
Hence
$$\phi_d(G) \leq |G^d| - \sum_{M \in \mathcal{M}} |M^d| + \sum_{\substack{M,N \in \mathcal{M} \\ M \neq N}} |(M \cap N)^d|.$$
Dividing both sides by $|G^d|$ achieves the required inequality. □

Lemma 2.4.7. *Let G be a simple group, let $\mathcal{M}$ be a set of conjugacy class representatives for maximal subgroups of G and let $d(G) \leq 2$. Then we have*
$$c_G \leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|}.$$

Proof. By the previous Lemma 2.4.5 we have that

$$P_2(G) \geq 1 - \sum_{M \in \mathcal{M}} |G : M|^{-1},$$

and by definition we have that

$$c_G = (1 - P_2(G))m(G).$$

Therefore we have

$$c_G \leq (1 - 1 + \sum_{M \in \mathcal{M}} |G : M|^{-1})m(G) = \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|}.$$

□

Corollary 2.4.8. *Let G be a quasisimple group, let $\mathcal{M}_G$ be a set of conjugacy class representatives for maximal subgroups of G, let $d(G) \leq 2$. Then we have*

$$c_{G/Z(G)} \leq \sum_{M \in \mathcal{M}_G} \frac{|M|m(G)}{|G|}.$$

Proof. Let $\mathcal{M}_{G/Z(G)}$ be a set of conjugacy class representatives for maximal subgroups of $G/Z(G)$. By Lemma 2.2.6 we have

$$\sum_{M \in \mathcal{M}_{G/Z(G)}} |M| = \sum_{M \in \mathcal{M}_G} |M|/|Z(G)|.$$

We also have that $m(G/Z(G)) = m(G)$ by Corollary 2.2.8. Substituting this into the previous Lemma 2.4.7, we get

$$c_{G/Z(G)} \leq \sum_{M \in \mathcal{M}_{G/Z(G)}} \frac{|M|m(G/Z(G))}{|G/Z(G)|} = \sum_{M \in \mathcal{M}_G} \frac{|M||Z(G)|m(G)}{|Z(G)||G|} = \sum_{M \in \mathcal{M}_G} \frac{|M|m(G)}{|G|}$$

as required. □

By Lemma 2.4.5 we have a way of bounding $P_2(G)$ from below, which gives us a way of bounding α from above. In cases where we have conjugacy class representatives for the maximal subgroups of G we can use this bound directly, however in cases where we do not, for example large dimension classical groups, we may use Lemma 2.4.10 instead. However before we present the lemma we present the following definition.

Definition 2.4.9. Let G be a group, let $H \leq G$ be a maximal subgroup, and let $\mathcal{M}$ be a set of conjugacy class representatives of the maximal subgroups of G. We define $\mathcal{M}_H$ to be the set $\{M \in \mathcal{M} \mid |M| > |H|\}$.

Lemma 2.4.10. *Let G be a group, $d(G) \leq d$, let $H \leq G$ be a maximal subgroup, and let $\mathcal{M}$ be a set of conjugacy class representatives of the maximal subgroups of G. Furthermore let $\mathcal{M}_1$ be a set that satisfies $\mathcal{M}_H \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$, then*

$$P_d(G) \geq 1 - \sum_{M \in \mathcal{M}_1} |G : M|^{-d+1} - \sum_{M \in \mathcal{M}/\mathcal{M}_1} |G : H|^{-d+1}.$$

Proof.

$$\begin{aligned}
P_d(G) &\geq 1 - \sum_{M \text{ max } G} |G:M|^{-d} \geq 1 - \sum_{M \in \mathcal{M}} |G:M|^{-d+1} && \text{by Lemma 2.4.5} \\
&= 1 - \sum_{M \in \mathcal{M}_1} |G:M|^{-d+1} - \sum_{M \in \mathcal{M}/\mathcal{M}_1} |G:M|^{-d+1} \\
&\geq 1 - \sum_{M \in \mathcal{M}_1} |G:M|^{-d+1} - \sum_{M \in \mathcal{M}/\mathcal{M}_1} |G:H|^{-d+1} && \text{since } |M| \leq |H| \text{ for } M \in \mathcal{M}/\mathcal{M}_1.
\end{aligned}$$

□

Consequently, when calculating α we need only know the exact orders of the larger maximal subgroups. Instead of requiring the orders of representatives lying in $M \in \mathcal{M}/\mathcal{M}_1$ we now only require an upper bound.

2.5 The classical groups

Throughout let K be a finite field of characteristic p and let V be a vector space of finite dimension n over K.

Definition 2.5.1. We write $\mathrm{GL}(V,K)$ for the *general linear group* of V over K, the group of all non-singular K-linear transformations of V.

Definition 2.5.2. We write $\mathrm{SL}(V,K)$ for the *special linear group*, the group of elements of $\mathrm{GL}(V,K)$ with determinant 1.

Let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V. Then each element of $\mathrm{GL}(V,K)$ is determined by where it maps each vector v_i of β. If $g \in \mathrm{GL}(V,K)$, then g_β denotes the $n \times n$ matrix that satisfies $v_i g = \sum_{j=1}^{n} (g_\beta)_{ij} v_j$.

Definition 2.5.3. Let $\mathrm{GL}_n(K)$ denote the group of invertible $n \times n$ matrices with entries in K. In the case where $|K| = q$ we also denote this group by $\mathrm{GL}_n(q)$.

Under the map $g \mapsto g_\beta$ we have an isomorphism between $\mathrm{GL}(V,K)$ and $\mathrm{GL}_n(K)$.

Definition 2.5.4. For $\lambda \in K$ let $diag_\beta(\lambda) \in \mathrm{GL}(V,K)$ be the transformation satisfying $(v)diag_\beta(\lambda) = \lambda v$ for all $v \in V$. We call this a *scalar*.

The centre of $\mathrm{GL}(V,K)$ consists of all non-zero scalars, and is isomorphic to $K^\times$. In this section we shall denote the centre by Z.

Definition 2.5.5. A map $g : V \to V$ is a *K-semilinear transformation* if there is a field automorphism σ_g of K such that for all $v, w \in V$ and $\lambda \in K$,

$$(v+w)g = vg + wg \text{ and } (\lambda v)g = \lambda^{\sigma_g} vg.$$

A K-semilinear transformation g is *non-singular* if the only solution to $gv = 0$ is $v = 0$.

Definition 2.5.6. The *general semilinear group* of V over K, $\Gamma\mathrm{L}(V,K)$, is defined to be the group of all non-singular K-semilinear transformations. We may also denote this group by $\Gamma\mathrm{L}_n(K)$ and $\Gamma\mathrm{L}_n(q)$ when $|K| = q$.

It is relatively straightforward to verify that $\Gamma\mathrm{L}(V,K)$ is in fact closed under composition and is actually a group. Given a basis $\{v_1,\ldots,v_n\}$ of V we note that each element $g \in \Gamma\mathrm{L}(V,K)$ is determined by its action with respect to the basis along with σ_g.

We note that Z is a normal subgroup of $\Gamma\mathrm{L}(V,K)$.

Definition 2.5.7. The *projective semi-linear group* $\mathrm{P}\Gamma\mathrm{L}(V,K)$ is the group $\Gamma\mathrm{L}(V,K)/Z$.

Definition 2.5.8. Let σ be an automorphism of K. A map $\beta : V \times V \to K$ is a *σ-sesquilinear form* on V if, for all $u,v,w \in V$ and all $\lambda,\mu \in K$:

(a) $\beta(u, v+w) = \beta(u,v) + \beta(u,w)$.

(b) $\beta(u+v, w) = \beta(u,w) + \beta(v,w)$.

(c) $\beta(\lambda u, \mu v) = \lambda\mu^\sigma\beta(u,v)$.

Definition 2.5.9. (a) A σ-sesquilinear form is *bilinear* if $\sigma = 1$.

(b) A bilinear form is *alternating* if $\beta(v,v) = 0$ for all $v \in V$.

(c) A bilinear form is *symmetric* if $\beta(u,v) = \beta(v,u)$ for all $u,v \in V$.

(d) A σ-sesquilinear form is *Hermitian* if $\beta(u,v) = \beta(v,u)^\sigma$ for all $u,v \in V$ and σ has order 2.

Definition 2.5.10. The map $Q : V \to K$ is a *quadratic form* if

(a) $Q(\lambda v) = \lambda^2 Q(v)$ for all $\lambda \in K$, $v \in V$.

(b) The map β defined by $\beta(u,v) = Q(u+v) - Q(u) - Q(v)$ for all $u,v \in V$, is a symmetric bilinear form on V. Furthermore we call β the *polar form* of Q.

We note that $\beta(v,v) = 2Q(v)$ and so when the characteristic of K is not 2, Q is determined by β and vice-versa.

Definition 2.5.11. Let β be a σ-sesquilinear form on V, let Q be a quadratic form on V and let $g \in \mathrm{GL}(V)$. Then g is an *isometry* of β if $\beta(ug,vg) = \beta(u,v)$ for all $u,v \in V$. In turn, g is an *isometry* of Q if $Q(vg) = Q(v)$ for all $v \in V$.

Definition 2.5.12. Let β be a σ-sesquilinear form on V, let Q be a quadratic form on V and let $g \in \mathrm{GL}(V)$. Then g is a *similarity* of β if there exists a $\lambda \in K$ such that $\beta(ug,vg) = \lambda\beta(u,v)$ for all $u,v \in V$. In turn g is a *similarity* of Q if there exists a $\lambda \in K$ such that $Q(vg) = \lambda Q(v)$ for all $v \in V$.

Definition 2.5.13. Let β be a σ-sesquilinear form on V, let Q be a quadratic form on V and let $g \in \Gamma\mathrm{L}(V)$. Then g is a *semi-similarity* of β if there exist $0 \neq \lambda \in K$ and field automorphism ϕ of K such that $\beta(ug,vg) = \lambda\beta(u,v)^\phi$ for all $u,v \in V$. In turn, g is an *similarity* of Q if there exist $0 \neq \lambda \in K$ and a field automorphism ϕ of K such that $Q(vg) = \lambda Q(v)^\phi$ for all $v \in V$.

For a given σ-sesquilinear or quadratic form κ, the set of all isometries forms a group, the *isometry group*. The set of all similarities also forms a group, the *similarity group*. Note that if κ is identically zero then both groups are equal to $\mathrm{GL}(V, K)$. Furthermore the isometry group of κ is a normal subgroup of the similarity group of κ. We also note that the set of all semi-similarities forms a group, called the *semi-similarity* group. In the case where κ is identically zero this group is equal to $\Gamma\mathrm{L}(V, K)$.

Let κ be a σ-sesquilinear form or quadratic form on V. Then we have the following chain of subgroups

$$S \leq I \leq \Delta \leq \Gamma. \tag{2.1}$$

Here I denotes the isometry group of κ, Δ the similarity group of κ, and Γ the semi-similarity group of κ. Finally S denotes the *special* group, the intersection $I \cap \mathrm{SL}(V, K)$, or equivalently the set of isometries with determinant 1.

Definition 2.5.14. Two σ-sesquilinear forms α and β on V are *isometric* if there is a $g \in \mathrm{GL}(V, K)$ such that $\alpha(u, v) = \beta(ug, vg)$. The forms are similar if there is a $g \in \mathrm{GL}(V, K)$ such that $\alpha(u, v) = \lambda\beta(ug, vg)$. Two quadratic forms Q_1 and Q_2 on V are *isometric* if there exists a $g \in \mathrm{GL}(V, K)$ such that $Q_1(v) = Q_2(vg)$, the forms are *similar* if there exists a $g \in \mathrm{GL}(V, K)$ such that $Q_1(v) = \lambda Q_2(vg)$.

Definition 2.5.15. A σ-sesquilinear form β is *non-degenerate* if $\beta(u, v) = 0$ for a fixed $v \in V$ and all $u \in V$ implies that $v = 0$. A quadratic form Q with polar form β is *non-degenerate* if β is a non-degenerate bilinear form. The form Q is *non-singular* if $Q(v) \neq 0$ for all $v \in V$ such that $\beta(w, v) = 0$ for all $w \in V$.

Let us note that a non-degenerate quadratic form is by definition non-singular.

Definition 2.5.16. Let β be a σ-sesquilinear form on V and let W be a subspace of V. Then

(a) W is *non-degenerate* if the restriction of β to W is non-degenerate.

(b) W is *totally singular* of *totally isotropic* if β when restricted to W is identically 0.

Definition 2.5.17. Let Q be a quadratic form on V with polar form β, and let W be a subspace of V then

(a) W is *totally singular* if $Q(w) = 0$ for all $w \in W$.

(b) W is *totally isotropic* if $\beta(v, w) = 0$ for all $v, w \in W$.

(c) A vector $v \in V$ is *singular* if $Q(v) = 0$.

(d) A vector $v \in V$ is *isotropic* if $\beta(v, v) = 0$.

It turns out that all singular vectors are isotropic, however the converse is not necessarily true in characteristic 2. We now consider certain σ-sesquilinear forms.

Definition 2.5.18. Let β be a non-degenerate alternating form on V. Then the isometry group of β is called a *symplectic group*, and is denoted by $\mathrm{Sp}(V, K, \beta)$.

It turns out that up to isomorphism $\mathrm{Sp}(V, K, \beta)$ is independent of choice of alternating form β , and that V has even dimension, see [26, Proposition 2.4.1]. Consequently we may denote the group by $\mathrm{Sp}_{2m}(K)$ or even $\mathrm{Sp}_{2m}(q)$ when $K = \mathbb{F}_q$.

Definition 2.5.19. Let β be a non-degenerate hermitian form on V. Then the isometry group of β is called a *unitary group*, and is denoted by $\mathrm{GU}(V, K, \beta)$.

We note firstly, following the notation in Definition 2.5.9, that since the field automorphism σ has order 2 the underlying field K of V must be $\mathbb{F}_{q^2}$ for some prime power q. Much like before it turns out that the isomorphism type of $\mathrm{GU}(V, K, \beta)$ is in fact independent of choice of hermitian form β, see [26, Proposition 2.3.1]. Therefore, we denote the group by $\mathrm{GU}_n(K) \leq \mathrm{GL}_n(V)$ or even $\mathrm{GU}_n(q)$ when $K = \mathbb{F}_{q^2}$.

Definition 2.5.20. Let Q be a non-degenerate quadratic form on V. Then the isometry group of β is called an *orthogonal group*, and is denoted by $\mathrm{GO}(V, K, Q)$.

By [26, Proposition 2.5.1] if K is of even characteristic then $n = \dim(V)$ is even also. If the dimension of V is odd, then, up to isomorphism, there is only one group $\mathrm{GO}(V, K, Q)$, and so we denote this group as $\mathrm{GO}_n^\circ(K)$ or $\mathrm{GO}_n^\circ(q)$ when $K = \mathbb{F}_q$. In the case where there is no confusion as to n being odd, we may also denote the group as $\mathrm{GO}_n(K) = \mathrm{GO}_n(q)$.

In the case where the dimension of V is odd, then, up to isomorphism there are two orthogonal groups, which we refer to as *plus type* and *minus type*. We denote these groups by $\mathrm{GO}_n^+(K) = \mathrm{GO}_n^+(q)$ and $\mathrm{GO}_n^-(K) = \mathrm{GO}_n^-(q)$ respectively. Finally if we want to refer to orthogonal group without specifying what type it is we use the notation $\mathrm{GO}_n^\epsilon(K) = \mathrm{GO}_n^\epsilon(q)$. A much more thorough review of orthogonal groups can be found in [26, §§2.5-2.8].

Recall the chain of groups from Series 2.1, we extend the chain. If $I = \mathrm{GL}_n(q)$ with $n \geq 3$ then the group S possesses an inverse-transpose automorphism $\gamma = -\mathbf{T}$.

Definition 2.5.21. If $I = \mathrm{GL}_n(q)$ with $n \geq 3$ then A is defined to be the split extension $\Gamma\mathrm{L}_n(q) : \langle \gamma \rangle$. Otherwise A is defined to be equal to Γ.

Definition 2.5.22. In the case where $I = \mathrm{GO}_n^\epsilon(q)$ then S contains a certain subgroup of index 2, for more information on such a subgroup see [26, §2.5]. If $I = \mathrm{GO}_n^\epsilon(q)$ then we define Ω to be this subgroup, otherwise Ω is defined to be the same as S.

Consequently, we now have a chain of groups

$$\Omega \leq S \leq I \leq \Delta \leq \Gamma \leq A. \tag{2.2}$$

This chain is A invariant, that is, each group is normalized by A. In addition $Z \cap A \trianglelefteq A$.

Definition 2.5.23. If $H \leq A$ we denote by $\mathrm{P}H$ or $\overline{H}$ the group $H/(H \cap Z)$.

This gives us another chain of groups

$$\overline{\Omega} \leq \overline{S} \leq \overline{I} \leq \overline{\Delta} \leq \overline{\Gamma} \leq \overline{A}. \tag{2.3}$$

Definition 2.5.24. Let $I = \mathrm{GL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{GU}_n(q)$ or $\mathrm{GO}_n(q)$. Then the groups in the Series 2.2 and 2.3 are called *classical* groups.

It turns out that $\overline{\Omega}$ is non-abelian simple, save for a few cases.

Theorem 2.5.25. *Assume that $n = \dim_K(V)$ is at least $2, 3, 4$ and 7 in the cases where $I = \mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{Sp}_n(q)$ and $\mathrm{GO}^{\epsilon}_n(q)$. Then $\overline{\Omega}$ is non-abelian simple, except for when $\overline{\Omega} = \mathrm{PSL}_2(2)$, $\mathrm{PSL}_2(3)$, $\mathrm{PSU}_3(2)$, $\mathrm{Sp}_4(2)$.*

A proof of the above Theorem 2.5.25 can be found in [1] and [14].

Proposition 2.5.26 ([6, 1.10.3])**.** *Let Ω be $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, $\mathrm{Sp}_n(q)$ or $\Omega^{\epsilon}_n(q)$, with $n \geq 2$. Then if $\overline{\Omega}$ is simple, then the group Ω is quasisimple.*

It also turns out that $\overline{A} \cong \mathrm{Aut}(\overline{\Omega})$ except in a few cases. The following result is classical, a proof of which can be found in [8, Chapter 12] and [46, Chapters 10 & 11].

Theorem 2.5.27. *Assume that $\overline{\Omega}$ is simple and that n is as in Theorem 2.5.25. Then $\overline{A} = \mathrm{Aut}(\overline{\Omega})$ except when $\Omega = \mathrm{Sp}_4(q)$ with q even and when $\Omega = \Omega^{+}_8(q)$.*

Definition 2.5.28. Let $\overline{\Omega}$ be a simple classical group. Furthermore let, $\overline{G}$ be such that $\overline{\Omega} \leq \overline{G} \leq \mathrm{Aut}(\overline{\Omega})$, then $\ddot{G}$ is defined to be $\overline{G}/\overline{\Omega}$.

For $g \in G$, we have that $\overline{g}$ and $\ddot{g}$ are the corresponding elements in $\overline{G}$ and $\ddot{G}$. Note that since, $\overline{\Omega}$ is simple then $\overline{\Omega}$ is isomorphic to the group of its own inner automorphisms. Consequently a group $\overline{G}$ such that $\overline{\Omega} \leq \overline{G} \leq \mathrm{Aut}(\overline{\Omega})$ can be considered as a group of automorphisms of $\overline{\Omega}$ acting by conjugation. Furthermore $\ddot{G}$ can be considered as a group of outer automorphisms of Ω.

Let $\overline{\phi}$ be an automorphism of $\overline{S}$. Then we also have a corresponding outer automorphism $\ddot{\phi}$. Furthermore there are elements in $\Gamma\mathrm{L}(V, K) : \langle\gamma\rangle$ which are coset representatives of $\overline{\phi}$ modulo Z. However this coset representative is clearly not unique. We do, however, have preferred coset representatives, which may be found in $[6, 1.7.1]$.

Furthermore, in an abuse of notation, we will mostly use the notation ϕ for all three of these different objects; the automorphism, the outer automorphism, and our preferred coset-representative in $\Gamma\mathrm{L}(V, K) : \langle\gamma\rangle$. In general, we reserve the notation $\overline{\phi}$ and $\ddot{\phi}$ for situations where confusion may arise.

We have specific notation for the groups Ω, S, I, Δ, Γ depending on whether $I = \mathrm{GL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{GU}_n(q)$ or $\mathrm{GO}^{\epsilon}_n(q)$. We rewrite the chain $\Omega \leq S \leq I \leq \Delta \leq \Gamma$ using this notation for each case to obtain the following four series of groups.

$$\mathrm{SL}_n(q) \leq \mathrm{SL}_n(q) \leq \mathrm{GL}_n(q) \leq \mathrm{GL}_n(q) \leq \Gamma\mathrm{L}_n(q).$$

$$\mathrm{SU}_n(q) \leq \mathrm{SU}_n(q) \leq \mathrm{GU}_n(q) \leq \mathrm{CGU}_n(q) \leq \mathrm{C}\Gamma\mathrm{U}_n(q).$$

$$\mathrm{Sp}_n(q) \leq \mathrm{Sp}_n(q) \leq \mathrm{Sp}_n(q) \leq \mathrm{CSp}_n(q) \leq \mathrm{C}\Gamma\mathrm{Sp}_n(q).$$

$$\Omega^{\epsilon}_n(q) \leq \mathrm{SO}^{\epsilon}_n(q) \leq \mathrm{GO}^{\epsilon}_n(q) \leq \mathrm{CGO}^{\epsilon}_n(q) \leq \mathrm{C}\Gamma\mathrm{O}^{\epsilon}_n(q).$$

The orders of the classical groups are well known, for example see [8, 3.3.1, 3.5, 3.6 & 3.7.2] for derivations of the orders of the isometry groups I. [26, Tables 2.1.C and 2.1.D], reproduced here as Tables 2.3 and 2.4 provide a summary of the indices of the groups in Series 2.2 and 2.3. The proof of the following can be found in [26, §2.2-2.8].

Table 2.2: Order of I

I	$\|I\|$
$\mathrm{GL}_n(q)$	$q^{n(n-1)/2}\prod_{i=1}^{n}(q^i-1)$
$\mathrm{GU}_n(q)$	$q^{n(n-1)/2}\prod_{i=1}^{n}(q^i-(-1)^i)$
$\mathrm{Sp}_n(q)$	$q^{n^2/4}\prod_{i=1}^{n/2}(q^{2i}-1)$
$\mathrm{GO}_n^{\circ}(q)$	$2q^{(n-1)^2/4}\prod_{i=1}^{(n-1)/2}(q^{2i}-1)$
$\mathrm{GO}_n^{\pm}(q)$	$2q^{n(n-2)/4}(q^{n/2}\mp 1)\prod_{i=1}^{n/2-1}(q^{2i}-1)$

Theorem 2.5.29. *Let $q=p^f$, let $n\geq 2$ and let $I=\mathrm{GL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{GU}_n(q)$ or $\mathrm{GO}_n^{\epsilon}(q)$. Then the order of I is as in Table 2.2. Furthermore the indices between adjacent groups in the Series $\Omega\leq S\leq I\leq\Delta\leq\Gamma\leq A$ and $\overline{\Omega}\leq\overline{S}\leq\overline{I}\leq\overline{\Delta}\leq\overline{\Gamma}\leq\overline{A}$ can be found in the Tables 2.3 and 2.4. Finally $|I\cap Z|$ can be found in Table 2.4.*

Here $a_{\pm}$ is either 1 or 2, $a_+a_-=2^{(2,q)}$, and when q is odd, $a_+=2$ if and only if $\frac{1}{4}n(q-1)$ is even. If $I=\mathrm{GL}_n(q)$ and $n=2$ then $c=1$, otherwise $c=2$. If $I=\mathrm{GO}_n^{\circ}(q)$ and $n=1$ then $d=1$, otherwise $d=2$. If $I=\mathrm{GU}_n(q)$ and $n=2$ then $f'=f$ otherwise $f'=2f$.

We note that when $n\geq 7$, the different $\overline{\Omega}$ are always simple and give rise to always distinct groups. However for $n\leq 6$ the $\overline{\Omega}$ are not always pairwise non-isomorphic. However in certain cases there are exceptional isomorphism, these isomorphisms may be obtained from [26, Proposition 2.9.1.]. In light of [26, Proposition 2.9.1.], for groups in $\Omega\leq G\leq A$ and $\overline{\Omega}\leq\overline{G}\leq\overline{A}$, we will often assume that $n\geq 2$ when $\Omega=\mathrm{SL}_n(q)$, $n\geq 3$ when $\Omega=\mathrm{SU}_n(q)$, $n\geq 4$ when $\Omega=\mathrm{Sp}_n(q)$, and $n\geq 7$ when $\Omega=\Omega_n^{\epsilon}(q)$. Furthermore $\mathrm{P}\Omega_{2n+1}(2^m)$ is isomorphic to a symplectic group as noted in [26, Section 2.5]. Consequently if n is odd for $\mathrm{P}\Omega_n(q)$ we will assume that q is odd also.

Finally let us generalize Lemma 2.2.6 for the classical simple groups.

Lemma 2.5.30. *Let $\Omega=\mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$ or $\Omega_n^{\epsilon}(q)$ be such that $\overline{\Omega}$ is simple, and let $\Omega\leq G\leq A$. Then there is a one to one correspondence between the non trivial maximal subgroups of $\overline{G}$ and the maximal subgroups of G that do not contain Ω, where a maximal subgroup M of G corresponds to $M/(G\cap Z)$.*

Proof. Let us first show that if M is a maximal subgroup of G that does not contain Ω, then $Z\cap G\leq M$. Suppose otherwise. Then $M(Z\cap G)=G$, by maximality of M. Therefore $M\trianglelefteq G$ and G/M is abelian. Since Ω is perfect we have that $\Omega M/M=1$, but that would imply that $\Omega\leq M$, which is a contradiction.

Table 2.3: Indices of classical groups 1

I	$\lvert S:\Omega\rvert$	$\lvert I:S\rvert$	$\lvert C:I\rvert$	$\lvert\Gamma:C\rvert$	$\lvert A:\Gamma\rvert$
$\mathrm{GL}_n(q)$	1	$q-1$	1	f	c
$\mathrm{GU}_n(q)$	1	$q+1$	$q-1$	f'	1
$\mathrm{Sp}_n(q)$	1	1	$q-1$	f	1
$\mathrm{GO}_n^\circ(q)$	d	2	$\frac{1}{2}(q-1)$	f	1
$\mathrm{GO}_n^\pm(q)$	2	$(2,q-1)$	$q-1$	f	1

Table 2.4: Indices of classical groups 2

I	$\lvert I\cap Z\rvert$	$\lvert\overline{S}:\overline{\Omega}\rvert$	$\lvert\overline{I}:\overline{S}\rvert$	$\lvert\overline{C}:\overline{I}\rvert$	$\lvert\overline{\Gamma}:\overline{C}\rvert$	$\lvert\overline{A}:\overline{\Gamma}\rvert$
$\mathrm{GL}_n(q)$	$q-1$	1	$(q-1,n)$	1	f	c
$\mathrm{GU}_n(q)$	$q+1$	1	$(q+1,n)$	1	f'	1
$\mathrm{Sp}_n(q)$	$(2,q-1)$	1	1	$(2,q-1)$	f	1
$\mathrm{GO}_n^\circ(q)$	d	2	1	1	f	1
$\mathrm{GO}_n^\pm(q)$	$(2,q-1)$	$a_\pm$	$(2,q-1)$	$(2,q-1)$	f	1

We now note that

$$\frac{\Omega(Z\cap G)}{Z\cap G}\cong\frac{\Omega}{Z\cap G\cap\Omega}=\frac{\Omega}{Z\cap\Omega}=\overline{\Omega}\qquad\text{by the Second Isomorphism Theorem.}$$

By the Correspondence Theorem, every subgroup of $\overline{G}=G/(Z\cap G)$ corresponds to a unique maximal subgroup of G which contains $Z\cap G$. In particular $\Omega(Z\cap G)$ corresponds to $\overline{\Omega}$ and every maximal subgroup of $\overline{G}$ corresponds to a unique maximal subgroup of G which contains $Z\cap G$.

If M is a maximal subgroup of G that does not contain Ω then as $M\geq Z\cap G$, we know that $M\not\geq\Omega(Z\cap G)$. By the Correspondence Theorem $M/(Z\cap G)\not\geq\overline{\Omega}$.

If $\overline{M}\leq\overline{G}$ is a maximal subgroup that does not contain $\overline{\Omega}$, then by the Correspondence Theorem there is a subgroup M of G such that $(Z\cap G)\leq M$ and $M/(Z\cap G)=\overline{M}$. Furthermore, by the Correspondence Theorem, $M\not\geq\Omega(Z\cap G)$, and since $(Z\cap G)\leq M$ we conclude that $\Omega\not\leq M$.

Therefore every maximal subgroup of M of G that does not contain Ω corresponds with a subgroup of $\overline{M}$ of $\overline{G}$ that does not contain $\overline{\Omega}$. In addition every maximal subgroup $\overline{M}$ of $\overline{G}$ that does not contain $\overline{\Omega}$ corresponds with a subgroup M of G that does not contain Ω. □

2.6 Bounds for classical groups

In this section we bound the orders of the classical groups, as defined in the previous section. We start with a few bounds.

Lemma 2.6.1. *Let $n\geq 2$. Then $\prod_{i=2}^{n}(2^i-1)\geq 2^{\frac{n(n+1)}{2}-2}$.*

Proof. We first have

$$\prod_{i=1}^{n}(2^i-1)=2^{\sum_1^n i}\prod_{i=1}^{n}(1-2^{-i}).$$

By induction on n we can prove that

$$\prod_{i=1}^{n}(1-2^{-i})\geq\frac{1}{4}+\frac{1}{2^{n+1}}.$$

Let us expand further. This is true for $n=1$. Suppose that

$$\prod_{i=1}^{k}(1-2^{-i})\geq\frac{1}{4}+\frac{1}{2^{k+1}}.$$

Then

$$\prod_{i=1}^{k+1}(1-2^{-i})\geq(\frac{1}{4}+\frac{1}{2^{k+1}})(1-\frac{1}{2^{k+1}})$$

$$= \frac{1}{4} + \frac{1}{2^{k+1}} - \frac{1}{2^{k+3}} - \frac{1}{2^{2k+2}}$$
$$\geq \frac{1}{4} + \frac{1}{2^{k+1}} - \frac{1}{2^{k+2}} = \frac{1}{4} + \frac{1}{2^{k+2}}.$$

So

$$\prod_{i=2}^{n}(2^i - 1) = \prod_{i=1}^{n}(2^i - 1) = 2^{\sum_1^n i}\prod_{i=1}^{n}(1 - 2^{-i}) \geq 2^{\sum_1^n i}(\frac{1}{4} + \frac{1}{2^{n+1}}) \geq 2^{-2+\sum_1^n i} = 2^{-2+\frac{n(n+1)}{2}}.$$

$\square$

Lemma 2.6.2 ([13, Lemma 3.2]). *Let I be a finite set of positive integers, and let $q \geq 3$. Then $\prod_{i\in I}(q^i - 1) > q^{-1+\sum_{i\in I} i}$. In particular $\prod_1^n(q^{2i} - 1) > q^{-1+n(n+1)}$ for all $q \geq 2$; and $\prod_1^n(q^i - 1) > q^{-1+n(n+1)/2}$ for all $q \geq 3$.*

Corollary 2.6.3. *Let $n \geq 2$. Then $(2^n - 1)\prod_{i=1}^{n-1}(2^{2i} - 1) > 2^{n^2-1}$.*

The following lemma is an extension of [13, Proposition 3.3]. We give both upper and lower bounds for the orders of $\mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$ and $\mathrm{SO}_n^\epsilon(q)$ for $q \geq 2$. In addition we correct an error in [13, Proposition], the upper bound for $\mathrm{SO}_n^+(q)$ given there does not hold for q even.

Lemma 2.6.4. *Let q be a prime power.*

(a) If $G = \mathrm{SL}_n(q)$ then $q^{n^2-2} < |G| < q^{n^2-1}$.

(b) If $G = \mathrm{Sp}_n(q)$ then $q^{n^2/2+n/2-1} < |G| < q^{n^2/2+n/2}$.

(c) If $G = \mathrm{SU}_n(q)$ then $q^{n^2-2} < |G| < q^{n^2-1}$.

(d) If $G = \mathrm{SO}_n^\circ(q)$ then q is odd and $q^{n^2/2-n/2-1} < |G| < q^{n^2/2-n/2}$.

(e) If $n \geq 4$ and $G = \mathrm{SO}_n^\pm(q)$ then $q^{n^2/2-n/2-1} < |G| < (2, q)q^{n^2/2-n/2}$.

Proof. (a) By Theorem 2.5.29 we have that

$$|\mathrm{SL}_n(q)| = q^{n(n-1)/2}\prod_{i=2}^{n}(q^i - 1).$$

First note that

$$|\mathrm{SL}_n(2)| \geq 2^{\frac{n(n-1)}{2}+\frac{n(n+1)}{2}-2} = q^{n^2-2}$$

by Lemma 2.6.1. If $q > 2$ then the lower bound follows from Lemma 2.6.2, since

$$|\mathrm{SL}_n(q)| \geq q^{\frac{n(n-1)}{2}+\frac{n(n+1)}{2}-2} = q^{n^2-2}.$$

For $q \geq 2$ the upper bound follows from $q^i - 1 < q^i$, so

$$|\mathrm{SL}_n(q)| \leq q^{\frac{n(n-1)}{2}+\frac{n(n+1)}{2}-1} = q^{n^2-1}.$$

(b) By Theorem 2.5.29 we have that

$$|\mathrm{Sp}_n(q)| = q^{n^2/4}\prod_{i=1}^{n/2}(q^{2i} - 1).$$

Note that by Lemma 2.6.2

$$|\mathrm{Sp}_n(q)| \geq q^{n^2/4+n/2(n/2+1)-1} = q^{n^2/2+n/2-1}.$$

Here the upper bound follows since $q^{2i}-1 < q^{2i}$ and so

$$|\mathrm{Sp}_n(q)| \leq q^{n^2/4+n(n+2)/4} = q^{n^2/2+n/2}.$$

(c) By Theorem 2.5.29 we have that

$$|\mathrm{SU}_n(q)| = q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i).$$

Note here that $(q^{2k}-1)(q^{2k-1}+1) > q^{4k-1}$ for all $k \geq 1$. If n is even then

$$\begin{aligned}
|\mathrm{SU}_n(q)| &= q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)\\
&= q^{n(n-1)/2}(q^2-1)\prod_{i=3}^{n}(q^i-(-1)^i)\\
&= q^{n(n-1)/2}(q^2-1)\prod_{i=2}^{n/2}((q^{2i}-1)(q^{2i-1}+1))\\
&> q^{n(n-1)/2}(q^2-1)\prod_{i=2}^{n/2}q^{4i-1}\\
&= q^{n(n-1)/2}(q^2-1)q^{\sum_{i=2}^{n/2}(4i-1)}\\
&= q^{n(n-1)/2}(q^2-1)q^{(n/2-1)(2n+6)/2}\\
&> q^{n^2/2-n/2+1}q^{n^2/2+n/2-3} = q^{n^2-2}.
\end{aligned}$$

If n is odd then

$$\begin{aligned}
|\mathrm{SU}_n(q)| &= q^{n(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)\\
&= q^{n(n-1)/2}(q^2-1)(q^n+1)\prod_{i=3}^{n-1}(q^i-(-1)^i)\\
&= q^{n(n-1)/2}(q^2-1)(q^n+1)\prod_{i=2}^{(n-1)/2}((q^{2i}-1)(q^{2i-1}+1))\\
&> q^{n(n-1)/2}(q^2-1)(q^n+1)\prod_{i=2}^{(n-1)/2}q^{4i-1}\\
&\geq q^{n(n-1)/2+1+n}\prod_{i=2}^{(n-1)/2}q^{4i-1}
\end{aligned}$$

$$
\begin{aligned}
&= q^{n^2/2+n/2+1} q^{\sum_{i=2}^{(n-1)/2}(4i-1)} \\
&= q^{n^2/2+n/2+1} q^{(\frac{n-1}{2}-1)(2n+4)/2} \\
&= q^{n^2/2+n/2+1} q^{n^2/2-n/2-3} = q^{n^2-2}.
\end{aligned}
$$

Now note that $(q^{2k}-1)(q^{2k+1}+1) < q^{4k+1}$ for all $k \geq 1$. If n is odd then

$$
\begin{aligned}
|\mathrm{SU}_n(q)| &= q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - (-1)^i) \\
&= q^{n(n-1)/2} \prod_{i=1}^{(n-1)/2} ((q^{2i}-1)(q^{2i+1}+1)) \\
&< q^{n(n-1)/2} \prod_{i=1}^{(n-1)/2} q^{4k+1} \\
&= q^{n(n-1)/2} q^{\sum_{i=1}^{(n-1)/2}(4k+1)} \\
&= q^{n^2/2-n/2} q^{\frac{n-1}{2}(2n+4)/2} \\
&= q^{n^2/2-n/2} q^{n^2/2+n/2-1} = q^{n^2-1}.
\end{aligned}
$$

If n is even then

$$
\begin{aligned}
|\mathrm{SU}_n(q)| &= q^{n(n-1)/2} \prod_{i=2}^{n} (q^i - (-1)^i) \\
&= q^{n(n-1)/2} (q^n - 1) \prod_{i=2}^{n-1} (q^i - (-1)^i) \\
&= q^{n(n-1)/2} (q^n - 1) \prod_{i=1}^{(n-2)/2} ((q^{2i}-1)(q^{2i+1}+1)) \\
&< q^{n(n+1)/2} \prod_{i=1}^{(n-2)/2} q^{4k+1} \\
&= q^{n^2/2+n/2} q^{\sum_{i=1}^{(n-2)/2}(4k+1)} \\
&= q^{n^2/2+n/2} q^{\frac{n-2}{2}(2n+2)/2} \\
&= q^{n^2/2+n/2} q^{n^2/2-n/2-1} = q^{n^2-1}.
\end{aligned}
$$

(d) By Theorem 2.5.29 we have that

$$
|\mathrm{SO}_n^\circ(q)| = q^{(n-1)^2/4} \prod_{i=1}^{(n-1)/2} (q^{2i}-1).
$$

The lower bound follows from Lemma 2.6.2

$$
|\mathrm{SO}_n^\circ(q)| \geq q^{(n-1)^2/4} q^{-1+\frac{n-1}{2}(\frac{n-1}{2}+1)} = q^{n^2/4-n/2+1/4} q^{n^2/4-5/4} = q^{n^2/2-n/2-1}.
$$

Here the upper bound follows since $q^{2i}-1 < q^{2i}$, therefore

$$|\mathrm{SO}_n^\circ(q)| \leq q^{(n-1)^2/4} q^{\frac{n-1}{2}(\frac{n-1}{2}+1)} = q^{n^2/4-n/2+1/4} q^{n^2/4-1/4} = q^{n^2/2-n/2}.$$

(e) Let us assume that $n \geq 4$. By Theorem 2.5.29, $|\mathrm{SO}_n^-(q)| > |\mathrm{SO}_n^+(q)|$ and $|\mathrm{GO}_n^\pm(q) : \mathrm{SO}_n^\pm(q)| = (2, q-1)$. So

$$|\mathrm{SO}_n^\pm(q)| \geq q^{n(n-2)/4}(q^{n/2}-1) \prod_{i=1}^{n/2-1}(q^{2i}-1).$$

If $q = 2$ then by Corollary 2.6.3, taking $n/2$ in place of n, we have

$$|\mathrm{SO}_n^\pm(q)| \geq q^{n(n-2)/4} q^{n^2/4-1} = q^{n^2/2-n/2-1}.$$

If $q \geq 3$ then by the first part of Lemma 2.6.2

$$\begin{aligned}
|\mathrm{SO}_n^\pm(q)| &\geq q^{n(n-2)/4} q^{-1+n/2+\sum_{i=1}^{n/2-1} 2i} \\
&= q^{n(n-2)/4} q^{-1+n/2+(n/2-1)(n/2)} = q^{n^2/4-n/2} q^{n^2/4-1} = q^{n^2/2-n/2-1}.
\end{aligned}$$

By Theorem 2.5.29 we have that $|\mathrm{SO}_n^+(q)| < |\mathrm{SO}_n^-(q)| = \frac{1}{(2,q+1)}|\mathrm{GO}_n^-(q)|$. So

$$|\mathrm{SO}_n^\pm(q)| \leq (2,q) q^{n(n-2)/4}(q^{n/2}+1) \prod_{i=1}^{n/2-1}(q^{2i}-1).$$

If $n = 4$ then

$$|\mathrm{SO}_n^\pm(q)| \leq (2,q)q^2(q^2+1)(q^2-1) = (2,q)q^2(q^4-1) < (2,q)q^{2+4} = (2,q)q^{4^2/2-4/2}$$

If $n \geq 6$ then

$$\begin{aligned}
|\mathrm{SO}_n^\pm(q)| &\leq (2,q)q^{n(n-2)/4}(q^{n/2}+1)(q^2-1) \prod_{i=2}^{n/2-1}(q^{2i}-1) \\
&= (2,q)q^{n(n-2)/4}(q^{n/2+2}+q^2-q^{n/2}-1) \prod_{i=2}^{n/2-1}(q^{2i}-1) \\
&< (2,q)q^{n(n-2)/4}q^{n/2+2} \prod_{i=2}^{n/2-1}(q^{2i}) = (2,q)q^{n^2/4+2}q^{\sum_{i=2}^{n/2-1} 2i} \\
&= (2,q)q^{n^2/4+2}q^{(n/2-2)(n+2)/2} \\
&= (2,q)q^{n^2/4+2}q^{n^2/4-n/2-2} = (2,q)q^{n^2/2-n/2}.
\end{aligned}$$

□

2.7 Aschbacher's theorem

For this section we define $u = 1$ when talking about linear, symplectic or orthogonal groups, and $u = 2$ when talking about unitary groups. A rough description of the classes $\mathcal{C}_1$ to $\mathcal{C}_8$ from [26] is as such:

$\mathcal{C}_1$ Stabilisers of totally singular or non-singular subspaces.

$\mathcal{C}_2$ Stabilisers of decompositions of the form $V = \bigoplus_{i=1}^{t} V_i$ where the V_i all have the same dimension and are similar with respect to the underlying form.

$\mathcal{C}_3$ Stabilisers of extension fields of $\mathbb{F}_{q^u}$ of prime index.

$\mathcal{C}_4$ Stabilisers of tensor product decompositions $V = V_1 \otimes V_2$.

$\mathcal{C}_5$ Stabilisers of subfields of $\mathbb{F}_{q^u}$ of prime index.

$\mathcal{C}_6$ Normalisers of symplectic-type r-groups (where r is prime) in absolutely irreducible representations (a definition for absolutely irreducible may be found in [6, p.38.]).

$\mathcal{C}_7$ Stabilisers of tensor product decompositions $V = \bigotimes_{i=1}^{t} V_i$ where the V_i all have the same dimension.

$\mathcal{C}_8$ Classical subgroups.

Let $\Omega = \mathrm{SL}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$ or $\Omega_n^\epsilon(q)$. A more precise definition of these classes can be found in [6, Section 2.2] for the group Γ. We shall write these classes as $\mathcal{C}_i(\Gamma)$. If $A \neq \Gamma$ then we define $\mathcal{C}(A)$ to be the set $\{N_A(H) | H \in \mathcal{C}_i(\Gamma)\}$. For $\Omega \leq G \leq A$ we now define $\mathcal{C}(G)$ to be the set $\{H \cap G | H \in \mathcal{C}_i(\Gamma)\}$ if $G \leq \Gamma$ and the set $\{H \cap G | H \in \mathcal{C}_i(A)\}$ if $G \not\leq \Gamma$.

It is a fact that all members of $\mathcal{C}_i(\Gamma)$ contain the group Z of scalars. Therefore $\overline{H \cap G} = \overline{H} \cap \overline{G}$ whenever $H \in \mathcal{C}_i(\Gamma)$ and $X \leq \Gamma$. In fact the above holds even when replacing A with Γ. In light of the above we may define the classes for $\overline{\Omega} \leq \overline{G} \leq \overline{A}$. Here $\mathcal{C}_i(\overline{G})$ is the set $\{\overline{H} \cap \overline{G} | \overline{H} \in \mathcal{C}_i(\Gamma)\}$ if $G \leq \Gamma$ and the set $\{\overline{H} \cap \overline{G} | \overline{H} \in \mathcal{C}_i(A)\}$ if $G \not\leq \Gamma$.

In general we shall omit reference to the group and refer to the classes as $\mathcal{C}_i$, especially when the group we are referencing is clear.

Definition 2.7.1 ([6, Definition 2.1.2]). Let H be a subgroup of G where $\Omega \leq G \leq A$, or $\overline{\Omega} \leq G \leq \overline{A}$ with $\Omega = \mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, $\mathrm{Sp}_n(q)$ or $\Omega_n^\epsilon(q)$ with the restrictions that $n \geq 2$ if $\Omega = \mathrm{SL}_n(q)$, $n \geq 3$ if $\Omega = \mathrm{SU}_n(q)$, $n \geq 4$ if $\Omega = \mathrm{Sp}_n(q)$, and $n \geq 7$ if $\Omega = \Omega_n^\epsilon(q)$. If $H \leq G$ belongs in Class $\mathcal{C}_i$ for some $1 \leq i \leq 8$, then H is defined to be a *geometric* subgroup.

Definition 2.7.2 ([6, Definition 2.1.3]). Let H be a subgroup of G where $\Omega \leq G \leq A$ with Ω as in Definition 2.7.1. Then H lies in Class $\mathcal{S}$ of G if $H/(H \cap Z)$ is almost simple and the following all hold:

(a) H does not contain Ω.

(b) H^∞ acts absolutely irreducibly (a definition for an absolutely irreducible action may be found in [6, p.38.]).

(c) There does not exist a $g \in \mathrm{GL}_n(q^u)$ such that $(H^\infty)^g$ is defined over a proper subfield of $\mathbb{F}_{q^u}$.

(d) H^∞ preserves a non-zero unitary form if and only if $\Omega = \mathrm{SU}_n(q)$.

(e) H^∞ preserves a non-zero quadratic form if and only if $\Omega = \Omega_n^\epsilon(q)$.

(f) H^∞ preserves a non-zero symplectic form and no non-zero quadratic form if and only if $\Omega = \mathrm{Sp}_n(q)$.

(g) H^∞ preserves no non-zero classical form if and only if $\Omega = \mathrm{SL}_n(q)$.

For given group G in $\Omega \leq G \leq A$ if $H \leq G$ lies in $\mathcal{S}$, then we also say that the subgroup $\overline{H} \leq \overline{G}$ lies in $\mathcal{S}$. Note that $\overline{H}$ is almost simple.

Theorem 2.7.3 ([6, Theorem 2.2.19]). *Let Ω be a quasisimple classical group, and let G be any group such that $\Omega \leq G \leq A$.*

(a) Let H be a geometric subgroup of G that is maximal in G. Then:

(i) The group H is a member of $\mathcal{C}_i$ for some $1 \leq i \leq 8$ where the classes are defined in [6, Section 2.2].

(ii) The shape of $H \cap \Omega$ is as given in [6, Tables 2.3, 2.5-2.11].

(iii) The number of conjugacy classes in Ω of groups of the same type as H and their stabilisers in G, are as given in [26, Tables 3.5.A-F], except that in $\mathrm{SL}_n(q)$ with $n \geq 3$ the groups of type P_k and P_{n-k} are both conjugate to their groups of type P_k where $k < n/2$.

(b) If K is any other maximal subgroup of G, and K does not contain Ω, then K lies in Class $\mathcal{S}$.

2.8 Some basic estimates

Lemma 2.8.1. *The number of factors of an integer n is bounded above by $2\sqrt{n}$.*

Proof. Let d be a divisor of n. Then either $d = \sqrt{n}$ or precisely one of d and n/d is less than $\sqrt{n}$. Then all the divisors of n are of the form d or n/d for some $d \leq \sqrt{n}$. We conclude that the total number of divisors of n is bounded by $2\sqrt{n}$. □

Lemma 2.8.2. *Let $t \geq 1$ and $q \geq 2$. Then $t! \leq q^{t\log_2(t)}$.*

Proof. Here we notice that $t! \leq t^t = q^{t\log_q(t)} \leq q^{t\log_2(t)}$. □

Lemma 2.8.3. *The number of prime divisors of an integer n is bounded by $\log_2(n)$.*

Proof. Write $n = p_1^{\alpha_1}\dots p_k^{\alpha_k}$ where the p_i are distinct primes and $\alpha_i \geq 1$ for $1 \leq i \leq k$. We observe that $n \geq 2^{\alpha_1+\dots+\alpha_k} \geq 2^k$, leading us to conclude that the number of prime divisors $k \leq \log_2 n$. □

Lemma 2.8.4. *Let $n \geq 2$. Then $\log_2(n) < n^{0.55}$.*

Proof. Assume first that $n \geq 6$. Consider the function $f(n) = n^{0.55} - \log_2(n)$. Here

$$\frac{d}{dn}(f(n)) = -\frac{1}{n\ln(2)} + \frac{0.55}{n^{0.45}}.$$

The turning point of $f(n)$ occurs when $n = \frac{1}{(0.55\ln(2))^{\frac{1}{0.55}}} < 6$. Since $f(n)$ tends to infinity as n tends to infinity, we conclude that $f(n)$ is increasing for $n \geq 6$. Combining this with the fact that $f(6) > 0$ gives us our result for $n \geq 6$.

The cases where $2 \leq n \leq 5$ can be shown to hold by direct calculation. □

Lemma 2.8.5. *Fix $n \geq 4$, and let n_1, n_2 satisfy $n_1 n_2 = n$ and $2 \leq n_1 \leq \sqrt{n}$ then $n_1^2 + n_2^2 \leq n^2/4 + 4$.*

Proof. $n_1^2 + n_2^2 = n_1^2 + n^2/n_1^2$. Differentiating $f(n_1) = n_1^2 + n^2/n_1^2$ with respect to n_1 gives $f'(n_1) = 2n_1 - 2n^2/n_1^3$. We see that $f'(n_1) = 0$ for $\sqrt[4]{n_1} = n$ and no other positive n_1. Therefore the maximum value for $f(n_1)$ occurs either at $n_1 = 2$ or $n_1 = \sqrt{n}$. Finally $f(2) = n^2/4 + 4 \geq 2n = f(\sqrt{n})$. $\square$

Chapter 3

Preliminary data on maximal subgroups of classical groups

Since our main focus ends up on the $\mathcal{C}_1$ subgroups we shall introduce them with more detail. Subgroups in this class, roughly speaking, are ones which preserve certain subspaces of V. There are a couple of types of important subgroup here that we shall consider.

First recall the definition A and Ω from Definitions 2.5.21 and 2.5.22.

Definition 3.0.1. Let $\Omega \leq G \leq A$. If $\Omega = \mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$ or $\Omega^\epsilon(q)$ then a group of type P_k is the stabilizer of a totally singular subspace of dimension k. If $\Omega = \mathrm{SL}_n(q)$, it is the stabiliser of any k-space.

Definition 3.0.2. Let $\Omega \leq G \leq A$, and let $\Omega = \mathrm{SL}_n(q)$. A group of type $P_{k,n-k}$ is the stabiliser of two subspaces, one of dimension k and the other of dimension $n-k$ such that the $(n-k)$-space contains the k-space.

Definition 3.0.3. A group of type $A \oplus B$ is the stabiliser of a pair of subspaces with trivial intersection which span the space.

Definition 3.0.4. A group of type $A \perp B$ is the stabiliser of a pair of non-degenerate subspaces which are mutually orthogonal and span the space.

Definition 3.0.5. A subgroup H of $\Gamma\mathrm{L}_n(q)$ is *reducible* if the subgroup H stabilises a proper non-zero subspace of $\mathbb{F}_q^n$. Let G be a group satisfying $\Omega \trianglelefteq G \leq A$ as in Series (2.2), and let K be a subgroup of G. If $G \leq \Gamma\mathrm{L}_n(q)$, then K lies in Class $\mathcal{C}_1$ if $K = N_G(W)$ or $N_G(W,U)$ where W and U are subspaces described in [6, Table 2.2]. Otherwise, K lies in Class $\mathcal{C}_1$ if $K = N_A(H) \cap G$ where H is a $\mathcal{C}_1$-subgroup of Γ.

3.1 Types of maximal subgroups

Let $n \geq 2$, and q be a prime power, and let $G = \mathrm{SL}_n(q), \mathrm{Sp}_n(q)$, $\mathrm{SU}_n(q)$ or $\Omega_n^\epsilon(q)$ unless stated otherwise. Let us recall a few facts from Section 2.7. We have noted previously a special class of maximal subgroups of our classical simple groups, namely the geometric class of subgroups denoted by $C(G)$, which we shall now refer to as $\mathcal{C}_G$. We may split this class into 8 smaller classes

$\mathcal{C}_1$ to $\mathcal{C}_8$. More information about these classes $\mathcal{C}_i$ and the structure of the maximal subgroups of G belonging to them can be obtained from [6, Chapter 2]; information which we shall rely on to obtain bounds on the sizes of these groups.

We have also noted in Definition 2.7.2 the existence of other subgroups belonging to the class of subgroups $\mathcal{S}$, the ones that do not belong in the geometric class. To that end we provide the following.

Definition 3.1.1. Let $\overline{\Omega}$ be simple, and let $\overline{\Omega} \leq \overline{G} \leq \overline{A}$. By $\mathcal{E}_{\overline{G}}$ we denote the maximal subgroups $H \leq \overline{G}$ in $\mathcal{S}$ which are isomorphic to A_{n+1}, S_{n+1}, A_{n+2} or S_{n+2}, and which are embedded according to [27, Section 4]. We also denote by $\mathcal{E}_G$ the subgroups of H of G such that $\overline{H}$ lies in $\mathcal{E}_{\overline{G}}$.

We note that maximal subgroups of this form only occur when G is symplectic or orthogonal.

Lemma 3.1.2 ([38, Lemma 5.1.11])**.** *Let G_0 be a simple classical group, let $G_0 \leq G \leq \mathrm{Aut}(G_0)$, and let $m_{\mathcal{E}}(G)$ denote the number of conjugacy class representatives for maximal subgroups of G lying in $\mathcal{E}_G$. Then we may bound $m_{\mathcal{E}}(G)$ as follows.*

(a) If $G_0 = \mathrm{PSp}_n(q)$ for $n \geq 4$, then $m_{\mathcal{E}}(G) \leq 4$.

(b) If $G_0 = \Omega_n(q)$ for $n \geq 7$, then $m_{\mathcal{E}}(G) \leq 4$.

(c) If $G_0 = \mathrm{P}\Omega_n^{\pm}(q)$ for $n \geq 8$, then $m_{\mathcal{E}}(G) \leq 16$.

Theorem 3.1.3 ([27, Theorem 4.1])**.** *Let G_0 be a simple classical group with natural projective module V of dimension n over $\mathbb{F}_q$, and let G be a group such that $G_0 \leq G \leq \mathrm{Aut}(G_0)$. Let H be a maximal subgroup of G such that $G = HG_0$. Then one of the following holds:*

(a) $H \in \mathcal{C}_G$;

(b) $H \in \mathcal{E}_G$;

(c) $|H| \leq q^{3n}$.

Corollary 3.1.4. *Let Ω be a quasisimple classical group with natural projective module V of dimension n over $\mathbb{F}_q$ such that $\overline{A} = \mathrm{Aut}(\overline{\Omega})$. Let H be a maximal subgroup of Ω, then one of the following holds:*

(a) $H \in \mathcal{C}_\Omega$;

(b) $H \in \mathcal{E}_\Omega$;

(c) $|H| \leq q^{3n+1}$

Proof. By Lemma 2.2.6 , $\overline{H}$ is a maximal subgroup of $\overline{\Omega}$ and so we may apply Theorem 3.1.3 to it. If $H \notin \mathcal{C}_G$ and $H \notin \mathcal{E}_\Omega$ then $|\overline{H}| \leq q^{3n}$. Therefore $|H| \leq |Z|q^{3n}$ and so $|H| \leq |q^{3n+1}|$. □

Note that in the case where the classical group $G_0 = \mathrm{PSL}_n(q), \mathrm{PSp}_n(q)$ or $\mathrm{P}\Omega_n^{\pm}(q)$ the natural projective module V is defined over the field $\mathbb{F}_q$. If $G_0 = \mathrm{PSU}_n(q)$ the natural projective module V is actually defined over the field $\mathbb{F}_{q^2}$. Therefore, if $\Omega = \mathrm{SU}_n(q)$ then either $H \in \mathcal{C}_\Omega$, $H \in \mathcal{E}_\Omega$ or $|H| \leq q^{6n+2}$.

3.2 Linear groups

In this section our aim is to show, for sufficiently large n, that P_2 type subgroups of $\mathrm{PSL}_n(q)$, are larger than all other subgroups of $\mathrm{PSL}_n(q)$ other than P_1 type subgroups and their $\mathrm{P\Gamma L}_n(q)$-conjugates. In other terms, if H is a P_2 type subgroup of $\mathrm{PSL}_n(q)$ then $|\mathrm{PSL}_n(q) : H| = m_2(\mathrm{PSL}_n(q))$ where $m_2(G)$ is defined in Definition 2.2.5. Before we compare the orders of subgroups of $\mathrm{SL}_n(q)$ let us first categorise the geometric subgroups.

Theorem 3.2.1. *Let $n \geq 5$, let q be a prime power and let $G = \mathrm{SL}_n(q)$. Furthermore, let M be a maximal subgroup of G. Then either M is a geometric subgroup, and is one of the types given in Table 3.1 or M lies in $\mathcal{S}$. If M is in Table 3.1 then c, the number of $\mathrm{PSL}_n(q)$ conjugacy classes of that type of group, lies in Table 3.1 also. Furthermore if M is found in Table 3.1 then the structure of M can be found in Table 3.2.*

Proof. The theorem is a sub-case of [6, Theorem 2.2.19]. We obtain the data for Tables 3.1 and 3.2 from [6, Tables 2.2-2.11], which present conditions for the existence of and the shapes of the geometric subgroups of G. Information for c comes from [26, MAIN THEOREM], though note we are following the conventions for P_i from [6] which differs from those of [26] in that we consider P_i and P_{n-i} separately. □

Let us take this moment to note that although $P_{m,n-m}$ and $\mathrm{GL}_m(q) \oplus \mathrm{GL}_{n-m}(q)$ are not maximal in $\mathrm{SL}_n(q)$ we include them in Tables 3.2 and 3.1 as we will refer to them later on, when we look at almost simple groups with socle $\mathrm{PSL}_n(q)$.

Lemma 3.2.2. *Let G be $\mathrm{SL}_n(q)$, let M_1 and M_2 be subgroups of G, each respectively of P_k and P_{n-k} type. Then $|M_1| = |M_2|$.*

Proof. By [6, Theorem 2.2.19] both P_k and P_{n-k} type subgroups are $\mathrm{Aut}(\mathrm{PSL}_n(q))$-conjugate to subgroups of type P_k as presented in [26]. Therefore $|M_1| = |M_2|$. □

In light of Lemma 3.2.2, when doing calculations about the order of P_k type subgroups we will in general make the assumption that $1 \leq n \leq n/2$ to ease calculation.

We now compare the orders of P_i type subgroups.

Lemma 3.2.3. *Let G be $\mathrm{SL}_n(q)$. If $n/2 \geq i > j \geq 1$ then $|P_i| \leq |P_j|$.*

Proof. The size of an arbitrary P_k subgroup is

$$\begin{aligned}
|P_k| &= (q-1)q^{k(n-k)}|\mathrm{SL}_k(q)||\mathrm{SL}_{n-k}(q)| && \text{by Theorem 3.2.1}\\
&= (q-1)q^{k(n-k)}q^{k(k-1)/2}q^{(n-k)(n-k-1)/2}\prod_{i=2}^{k}(q^i-1)\prod_{i=2}^{n-k}(q^i-1) && \text{by Theorem 2.5.29}\\
&= (q-1)q^{(2kn-2k^2+k^2-k+n^2-nk-n-kn+k^2+k)/2}\prod_{i=2}^{k}(q^i-1)\prod_{i=2}^{n-k}(q^i-1)\\
&= (q-1)q^{n(n-1)/2}\prod_{i=2}^{k}(q^i-1)\prod_{i=2}^{n-k}(q^i-1).
\end{aligned}$$

Table 3.1: Geometric subgroups of $\mathrm{SL}_n(q)$

Class	Subgroup Type	Conditions	c
$\mathcal{C}_1$	P_m	$1 \leq m \leq n-1$	1
$\mathcal{C}_1$	$\mathrm{GL}_m(q) \oplus \mathrm{GL}_{n-m}(q)$	$1 \leq m < n/2$	1
$\mathcal{C}_1$	$P_{m,n-m}$	$1 \leq m < n/2$	1
$\mathcal{C}_2$	$\mathrm{GL}_m(q) \wr \mathrm{S}_t$	$n = mt, t \geq 2$	1
$\mathcal{C}_3$	$\mathrm{GL}_m(q^r)$	$n = mr$, r prime	1
$\mathcal{C}_4$	$\mathrm{GL}_{n_1}(q) \otimes \mathrm{GL}_{n_2}(q)$	$n = n_1 n_2, 2 \leq n_1 < \sqrt{n}$	$(q-1, n_1, n_2)$
$\mathcal{C}_5$	$\mathrm{GL}_n(q_0)$	$q = q_0^r$, r prime	$(\frac{q-1}{q_0-1}, n)$
$\mathcal{C}_6$	$r^{1+2m}.\mathrm{Sp}_{2m}(r)$	$n = r^m$, r prime	$(q-1, n)$
$\mathcal{C}_7$	$\mathrm{GL}_m(q) \wr \mathrm{S}_t$	$n = m^t, m \geq 3$	c_7
$\mathcal{C}_8$	$\mathrm{Sp}_n(q)$	n even	$(q-1, n/2)$
$\mathcal{C}_8$	$\mathrm{GO}_n(q)$	q, n odd	$(q-1, n)$
$\mathcal{C}_8$	$\mathrm{GO}_n^{\epsilon}(q)$	$\epsilon = \pm, q$ odd$, n$ even	$(q-1, n)/2$
$\mathcal{C}_8$	$\mathrm{GU}_n(q^{1/2})$	q square	$(q^{1/2}-1, n)$

Here $c_7 = \frac{1}{2}(q-1, m)$ if $t = 2$, $m \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$ and $c_7 = (q-1, \frac{n}{m})$ otherwise

Table 3.2: Geometric subgroups of $\mathrm{SL}_n(q)$

Class	Subgroup Type	Structure
$\mathcal{C}_1$	P_m	$[q^{m(n-m)}]:(\mathrm{SL}_m(q)\times\mathrm{SL}_{n-m}(q)):(q-1)$
$\mathcal{C}_1$	$\mathrm{GL}_m(q)\oplus\mathrm{GL}_{n-m}(q)$	$\mathrm{SL}_m(q)\times\mathrm{SL}_{n-m}(q)):(q-1)$
$\mathcal{C}_1$	$P_{m,n-m}$	$[q^{m(2n-3m)}]:(\mathrm{SL}_m(q)^2\times\mathrm{SL}_{n-2k}(q)):(q-1)^2$
$\mathcal{C}_2$	$\mathrm{GL}_m(q)\wr\mathrm{S}_t$	$\mathrm{SL}_m(q)^t.(q-1)^{t-1}.\mathrm{S}_t$
$\mathcal{C}_3$	$\mathrm{GL}_m(q^r)$	$\left(\frac{(q-1,m)(q^r-1)}{(q-1)}\circ\mathrm{SL}_m(q^r)\right).\frac{(q^r-1,m)}{(q-1,m)}.r$
$\mathcal{C}_4$	$\mathrm{GL}_{n_1}(q)\otimes\mathrm{GL}_{n_2}(q)$	$(\mathrm{SL}_{n_1}(q)\circ\mathrm{SL}_{n_2}(q)).[(q-1,n_1,n_2)^2]$
$\mathcal{C}_5$	$\mathrm{GL}_n(q_0)$	$\mathrm{SL}_n(q_0).[(\frac{q-1}{q_0-1},n)]$
$\mathcal{C}_6$	$r^{1+2m}.\mathrm{Sp}_{2m}(r)$	$((q-1,n)\circ r^{1+2m}).\mathrm{Sp}_{2m}(r)$
$\mathcal{C}_7$	$\mathrm{GL}_m(q)\wr\mathrm{S}_t$, $t=2$, $m\equiv 2 \mod 4$, $q\equiv 3 \mod 4$	$(q-1,m).\mathrm{PSL}_m(q)^2.[(q-1,m)^2]$
$\mathcal{C}_7$	$\mathrm{GL}_m(q)\wr\mathrm{S}_t$, otherwise	$(q-1,m).\mathrm{PSL}_m(q)^t.[(q-1,\frac{n}{m})(q-1,m)^{t-1}].\mathrm{S}_t$
$\mathcal{C}_8$	$\mathrm{Sp}_n(q)$, $(q-1,n/2)=(q-1,n)/2$	$(q-1,n)^{\cdot}\,\mathrm{PSp}_n(q)$
$\mathcal{C}_8$	$\mathrm{Sp}_n(q)$, $(q-1,n/2)=(q-1,n)$	$(q-1,n)^{\cdot}\,\mathrm{PCSp}_n(q)$
$\mathcal{C}_8$	$\mathrm{GO}_n^{\epsilon}(q)$	$\mathrm{SU}_n(q^{1/2}).(q^{1/2}-1,n)$
$\mathcal{C}_8$	$\mathrm{GU}_n(q^{1/2})$	$\mathrm{SO}_n^{\epsilon}(q).(q-1,n)$

For $1 \leq k < n/2$

$$\begin{aligned}|P_k| &= (q-1)q^{n(n-1)/2}\prod_2^k(q^i-1)\prod_2^{n-k}(q^i-1) \\ &\geq (q-1)q^{n(n-1)/2}\prod_2^{k+1}(q^i-1)\prod_2^{n-k-1}(q^i-1) = |P_{k+1}| \qquad \text{since } q^{n-k}-1 \geq q^{k+1}-1.\end{aligned}$$

Consequently we can see that

$$|P_1| \geq |P_2| \geq \ldots \geq |P_{\lfloor n/2 \rfloor}|$$

holds, thus giving us the desired result. □

Lemma 3.2.4. *Let $M \leq \mathrm{SL}_n(q)$ be a $\mathcal{C}_1$ subgroup of type $P_{1,n-1}$ or of type P_2. Furthermore let $H \leq \mathrm{SL}_n(q)$ be a maximal subgroup from the Aschbacher classes $\mathcal{C}_i$ with $2 \leq i \leq 8$. If $n \geq 7$ then $|M| \geq |H|$. Moreover:*

- *If $n \geq 5$ then $|M| \geq |H|$ for H in Aschbacher classes $\mathcal{C}_i$ for $i \in \{4,5\}$.*
- *If $n \geq 6$ then $|M| \geq |H|$ for H in Aschbacher classes $\mathcal{C}_i$ for $i \in \{3,4,5,7,8\}$.*
- *If $n \geq 7$ then $|M| \geq |H|$ for H in Aschbacher classes $\mathcal{C}_i$ for $i \in \{2,3,4,5,6,7,8\}$.*

Proof. Let M be a $P_{1,n-1}$ type subgroup. Then

$$\begin{aligned}|M| &= (q-1)^2q^{2n-3}|\mathrm{SL}_{n-2}(q)| && \text{by Theorem 3.2.1} \\ &\geq (q-1)^2q^{2n-3}q^{n^2-4n+2} && \text{by Lemma 2.6.4} \\ &= (q-1)^2q^{n^2-2n-1}.\end{aligned}$$

In the case where H belongs to $\mathcal{C}_2$ and $t \neq n$ we know that

$$\begin{aligned}|H| &\leq |\mathrm{SL}_m(q)|^t(q-1)^{t-1}|\mathrm{S}_t| && \text{from Theorem 3.2.1} \\ &\leq |\mathrm{SL}_m(q)|^t(q)^t t! \\ &\leq q^{(m^2-1)t}q^t t! = q^{m^2t}t! && \text{by Lemma 2.6.4} \\ &\leq q^{n^2/t}t! && \text{since } mt = n \\ &\leq q^{n^2/t}q^{t\log_2(t)} && \text{by Lemma 2.8.2} \\ &\leq q^{n^2/t+t^{1.55}} && \text{by Lemma 2.8.4} \\ &\leq q^{n^2/2+(\frac{n}{2})^{1.55}} && \text{since } 2 \leq t \leq n/2 \\ &\leq q^{n^2/2+\frac{n^2}{7^{0.45}\times 2^{1.55}}} && \text{since } (\frac{n}{7})^{0.45} \geq 1, \text{ assuming } n \geq 7 \\ &\leq q^{0.65n^2} \leq q^{n^2-2n-1} && \text{assuming } n \geq 7.\end{aligned}$$

In the case where $t = n$

$$\begin{aligned}|H| &\leq (q-1)^{n-1}n! \leq (q)^n n! && \text{from Theorem 3.2.1} \\ &\leq q^{n+n\log_2(n)} && \text{by Lemma 2.8.2}\end{aligned}$$

$$
\begin{aligned}
&\leq q^{n+n^{1.55}} && \text{by Lemma 2.8.4}\\
&\leq q^{n+\frac{n^2}{7^{0.45}}} \leq q^{n^2-2n-1} \leq |M| && \text{assuming } n \geq 7.
\end{aligned}
$$

In the case where H belongs to $\mathcal{C}_3$ then

$$
\begin{aligned}
|H| &\leq r\frac{(q^r-1,m)(q^r-1)}{q-1}|\mathrm{SL}_m(q^r)| && \text{from Theorem 3.2.1}\\
&\leq rmq^r q^{r(m^2-1)} \leq rmq^{rm^2} && \text{by Lemma 2.6.4}\\
&\leq q^{mn}n && \text{since } rm=n \text{, Theorem 3.2.1}\\
&\leq q^{mn+\log_2(n)} \leq q^{mn+n^{0.55}} && \text{by Lemma 2.8.4}\\
&\leq q^{n^2/2+n^{0.55}} && \text{since } m \leq n/2\\
&\leq q^{n^2/2+\frac{n^2}{6^{1.45}}} < q^{0.58n^2} \leq q^{n^2-2n-1} \leq |M| && \text{assuming } n \geq 6.
\end{aligned}
$$

In the case where H belongs to $\mathcal{C}_4$ then

$$
\begin{aligned}
|H| &\leq |\mathrm{SL}_{n_1}(q)||\mathrm{SL}_{n_2}(q)|(q-1,n_1,n_2)^2 && \text{from Theorem 3.2.1}\\
&\leq q^{n_1^2-1}q^{n_2^2-1}(q-1) \leq q^{n_1^2+n_2^2} && \text{by Lemma 2.6.4}\\
&\leq q^{n^2/4+4} && \text{by Lemma 2.8.5}\\
&\leq q^{n^2-2n-1} \leq |M| && \text{assuming } n \geq 5.
\end{aligned}
$$

In the case where H belongs to $\mathcal{C}_5$ then

$$
\begin{aligned}
|H| &\leq |\mathrm{SL}_n(q_0)|(\frac{q-1}{q_0-1},n) && \text{from Theorem 3.2.1}\\
&\leq q_0^{n^2-1}(q-1) \leq q_0^{n^2}(q-1) && \text{by Lemma 2.6.4}\\
&\leq q^{n^2/2}(q-1) && \text{since } q_0 \leq q^{1/2}\\
&\leq q^{n^2-2n-1}(q-1) \leq |M| && \text{assuming } n \geq 5.
\end{aligned}
$$

In the case where H belongs to $\mathcal{C}_6$ then

$$
\begin{aligned}
|H| &\leq (q-1)r^{1+2m}|\mathrm{Sp}_{2m}(r)| && \text{from Theorem 3.2.1}\\
&\leq (q-1)r^{1+2m}r^{2m^2+m} = (q-1)r^{2m^2+3m+1} && \text{by Lemma 2.6.4}\\
&\leq (q-1)n^{2m+3+1/m} && \text{as } n=r^m\\
&\leq (q-1)n^{2\log_2(n)+4} && \text{as } m \leq \log_2(n)\\
&\leq (q-1)q^{2\log_2(n)^2+4\log_2(n)} && \text{as } n \leq q^{\log_2(n)}\\
&\leq (q-1)q^{2n^{1.1}+4n^{0.55}} && \text{by Lemma 2.8.4}\\
&\leq (q-1)q^{\frac{2n^2}{7^{0.9}}+\frac{4n}{7^{0.45}}} \leq (q-1)q^{n^2-2n-1} \leq |M| && \text{assuming } n \geq 7.
\end{aligned}
$$

In the case where H belongs to $\mathcal{C}_7$ then there are two relevant cases. In the first case, i.e. where $t=2$, $m \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, we have

$$
|H| \leq (q-1)|\mathrm{PSL}_m(q)|^2(q-1)^2 \qquad \text{from Theorem 3.2.1}
$$

$$\begin{aligned}
&\le (q-1)^3|\mathrm{SL}_m(q)|^2 \le (q-1)^3 q^{2m^2-2} && \text{by Lemma 2.6.4}\\
&\le (q-1)^2 q^{2m^2-1} \le (q-1)^2 q^{2n-1} && \text{as } n = m^2\\
&\le (q-1)^2 q^{n^2-2n-1} \le |M| && \text{assuming } n \ge 5.
\end{aligned}$$

In the second case where H belongs to $\mathcal{C}_7$ we have

$$\begin{aligned}
|H| &\le (q-1)|\mathrm{PSL}_m(q)|^t (q-1,m)^t t! && \text{from Theorem 3.2.1}\\
&= (q-1)|\mathrm{SL}_m(q)|^t t! \le (q-1) q^{t(m^2-1)} t! && \text{by Lemma 2.6.4}\\
&\le (q-1) q^{t(m^2-1)} q^{t\log_2(t)} && \text{by Lemma 2.8.2}\\
&\le (q-1) q^{tm^2-t+t\log_2(t)} && \\
&\le (q-1) q^{tn+t\log_2(t)} && \text{since } n \ge m^2, \text{ Theorem 3.2.1}\\
&\le (q-1) q^{\log_2(n)n+\log_2(n)\log_2(\log_2(n))} && \text{since } t \le \log_2(n), \text{ Theorem 3.2.1}\\
&\le (q-1) q^{n^{1.55}+n^{0.55}\log_2(n^{0.55})} \le (q-1) q^{n^{1.55}+n^{0.55}n^{0.3025}} && \text{by Lemma 2.8.4}\\
&= (q-1) q^{n^{1.55}+n^{0.8525}} \le (q-1) q^{\frac{n^2}{6^{0.45}}+\frac{n}{6^{0.1475}}} && \text{assuming } n \ge 6\\
&< (q-1) q^{0.45n^2+0.77n} \le (q-1) q^{n^2-2n-1} \le |M| && \text{assuming } n \ge 6.
\end{aligned}$$

Finally let us consider the $\mathcal{C}_8$ case. There are three forms that H can take. We tackle each one in turn. If H is of the $\mathrm{Sp}(q)$ form then

$$\begin{aligned}
|H| &\le (q-1)|\mathrm{PCSp}_n(q)| && \text{from Theorem 3.2.1}\\
&\le (q-1)^2 q^{(n^2+n)/2} && \text{by Lemma 2.6.4}\\
&\le (q-1)^2 q^{n^2-2n-1} \le |M| && \text{assuming } n \ge 6.
\end{aligned}$$

If H is of the $\mathrm{GO}(q)$ form then

$$\begin{aligned}
|H| &\le (q-1)|\mathrm{SO}_n^-(q)| && \text{from Theorem 3.2.1}\\
&\le 2(q-1) q^{(n^2-n)/2} && \text{by Lemma 2.6.4}\\
&\le (q-1)^2 q^{n^2-2n-1} \le |M| && \text{assuming } n \ge 4.
\end{aligned}$$

Finally if H is of the shape $\mathrm{GU}(q)$ form then

$$\begin{aligned}
|H| &\le (q-1)|\mathrm{SU}_n(q^{1/2})| && \text{from Theorem 3.2.1}\\
&\le (q-1) q^{n^2/2} && \text{by Lemma 2.6.4}\\
&\le (q-1)^2 q^{n^2-2n-1} \le |M| && \text{assuming } n \ge 5.
\end{aligned}$$

By Theorem 3.2.1 we have considered all possible H. Therefore if M is a $P_{1,n-1}$ type subgroup then $|M| > |H|$.

Finally we show that if M_1 is a $P_{1,n-1}$ type subgroup and M_2 is a P_2 type then $|M_1| < |M_2|$.

Let $n \ge 3$. Then

$$|M_1| = (q-1)^2 q^{2n-3} |\mathrm{SL}_{n-2}(q)| \qquad \text{from Theorem 3.2.1}$$

$$
\begin{aligned}
&= q^{2n-4}q(q-1)^2|\mathrm{SL}_{n-2}(q)| \\
&< q^{2n-4}q(q^2-1)|\mathrm{SL}_{n-2}(q)| \\
&= q^{2n-4}|\mathrm{SL}_2(q)||\mathrm{SL}_{n-2}(q)| = |M_2| && \text{from Theorem 3.2.1.}
\end{aligned}
$$

Consequently if M is a P_2 type subgroup then $|M| > |H|$. □

Lemma 3.2.5. *Let G be $\mathrm{SL}_n(q)$ for $n \geq 6$, let $M \leq G$ be a P_2 type subgroup and let $H \leq G$ be subgroup of type $\mathcal{S}$. Then $|M| > |H|$.*

Proof. There are no subgroups which belong to $\mathcal{E}_G$, and so by Theorem 3.1.4 $|H| \leq q^{3n+1}$. This gives us

$$
\begin{aligned}
|M| &\geq q^{2(n-2)}|\mathrm{SL}_2(q)||\mathrm{SL}_{n-2}(q)| && \text{by Theorem 3.2.1} \\
&> q^{2(n-2)}q^2|\mathrm{SL}_{n-2}(q)| > q^{2(n-2)}q^{(n-2)^2-2+2} && \text{by Lemma 2.6.4} \\
&= q^{n^2-2n} \geq q^{3n+1} && \text{since } n \geq 6.
\end{aligned}
$$

□

Proposition 3.2.6. *Let G be $\mathrm{SL}_n(q)$ for $n \geq 7$, let $M_1 \leq G$ be a P_1 type subgroup, and let $M_2 \leq G$ be a P_2 type subgroup. Then $m(G) = |G|/|M_1|$. Furthermore $m_2(G) = |G|/|M_2|$ as defined in Definition 2.2.5.*

Proof. It is a straightforward calculation to calculate the index of M_1 in G, which is $\frac{q^n-1}{q-1}$. Theorem 2.2.7 shows us that $m(G) = |G|/|M_1|$. Lemma 3.2.2 also shows us that if $H \leq G$ was a P_{n-1} subgroup then $m(G) = |G|/|H|$ also.

Let $H \leq G$ be a maximal subgroup of G that is not a P_1 or P_{n-1} type subgroup. Then H either lies in $\mathcal{C}_G$ or in $\mathcal{S}$. If $H \in \mathcal{C}_1$ then H is of P_i type for $2 \leq i \leq n-2$, in this case Lemmas 3.2.2 and 3.2.3 show that $|M_2| \geq |H|$. If H lies in $\mathcal{C}_i$ with $2 \leq i \leq 8$ then Lemma 3.2.4 shows that $|M_2| \geq |H|$. Finally if $H \in \mathcal{S}$ then Lemma 3.2.5 shows that $|M| \geq |H|$. This shows that $m_2(G) = |G|/|M_2|$. □

3.3 Symplectic groups

In this section our aim is to show, for sufficiently large n, that P_2 type subgroups of $\mathrm{PSp}_n(q)$ are larger than all other subgroups of $\mathrm{PSp}_n(q)$ other than P_1 type subgroups or those lying in the $\mathcal{C}_8$ class of subgroups. Before we compare the orders of subgroups of $\mathrm{Sp}_n(q)$ let us first categorise the geometric subgroups.

Theorem 3.3.1. *Let $n \geq 4$, let q be a prime power and let $G = \mathrm{Sp}_n(q)$. Furthermore, let M be a maximal subgroup of G. Then either M is a geometric subgroup, and is one of the types given in Table 3.3 or M lies in $\mathcal{S}$. If M is in Table 3.1 then c, the number of $\mathrm{PSp}_n(q)$ conjugacy classes of that type of group, lies in Table 3.1 also. Furthermore if M is found in Table 3.3 then the shape of M can be found in Table 3.4.*

Proof. The theorem is a sub-case of [6, Theorem 2.2.19]. We obtain the data for Tables 3.3 and 3.4 from [6, Tables 2.2-2.11], which present conditions for the existence of and the shapes of the geometric subgroups of G. Information for c comes from [26, MAIN THEOREM]. □

Table 3.3: Geometric subgroups of $\mathrm{Sp}_n(q)$

Class	Subgroup Type	Conditions	c
$\mathcal{C}_1$	P_m	$1 \leq m \leq n/2$	1
$\mathcal{C}_1$	$\mathrm{Sp}_m(q) \perp \mathrm{Sp}_{n-m}(q)$	$2 \leq m < n/2$, m even	1
$\mathcal{C}_2$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$	$n = mt$, m even, $t \geq 2$	1
$\mathcal{C}_2$	$\mathrm{GL}_{n/2}(q).2$	q odd	1
$\mathcal{C}_3$	$\mathrm{Sp}_m(q^r)$	$n = mr$, r prime, m even	1
$\mathcal{C}_3$	$\mathrm{GU}_{n/2}(q)$	q odd	1
$\mathcal{C}_4$	$\mathrm{Sp}_{n_1}(q) \otimes \mathrm{GO}^{\epsilon}_{n_2}(q)$	$n = n_1 n_2$, q odd, $3 \leq n_2 \leq n/2$, n_1 even	1
$\mathcal{C}_5$	$\mathrm{Sp}_n(q_0)$	$q = q_0^r$, r prime	$(2, r, q-1)$
$\mathcal{C}_6$	$2_-^{1+2m}.\Omega^-_{2m}(2)$	$n = 2^m$, 2 prime	c_6
$\mathcal{C}_7$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$	$n = m^t$, qt odd, m even, $(m,q) \neq (2,3)$	1
$\mathcal{C}_8$	$\mathrm{GO}^{\pm}_n(q)$	q even	1

Here $c_5 = 2$ if $p \equiv \pm 1 \mod 8$ and $c_5 = 1$ otherwise

Table 3.4: Geometric subgroups of $\mathrm{Sp}_n(q)$

Class	Subgroup Type	Structure
$\mathcal{C}_1$	P_m	$[q^{m(n+\frac{1-3m}{2})}] : (\mathrm{GL}_m(q) \times \mathrm{Sp}_{n-2m}(q))$
$\mathcal{C}_1$	$\mathrm{Sp}_m(q) \perp \mathrm{Sp}_{n-m}(q)$	$\mathrm{Sp}_m(q) \times \mathrm{Sp}_{n-m}(q)$
$\mathcal{C}_2$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$	$\mathrm{Sp}_m(q)^t : \mathrm{S}_t$
$\mathcal{C}_2$	$\mathrm{GL}_{n/2}(q).2$	$\mathrm{GL}_{n/2}(q).2$
$\mathcal{C}_3$	$\mathrm{Sp}_m(q^r)$	$\mathrm{Sp}_m(q^r).r$
$\mathcal{C}_3$	$\mathrm{GU}_{n/2}(q)$	$\mathrm{GU}_{n/2}(q).2$
$\mathcal{C}_4$	$\mathrm{Sp}_{n_1}(q) \otimes \mathrm{GO}^{\epsilon}_{n_2}(q)$	$(\mathrm{Sp}_{n_1}(q) \circ \mathrm{GO}^{\epsilon}_{n_2}(q)).(n_2, 2)$
$\mathcal{C}_5$	$\mathrm{Sp}_n(q_0)$	$\mathrm{Sp}_n(q_0).(2, q-1, r)$
$\mathcal{C}_6$	$2^{1+2m}_{-}.\Omega^{-}_{2m}(2)$, $q \equiv \pm 1 \mod 8$	$2^{1+2m}_{-}.\mathrm{SO}^{-}_{2m}(2)$
$\mathcal{C}_6$	$2^{1+2m}_{-}.\Omega^{-}_{2m}(2)$, $q \equiv \pm 3 \mod 8$	$2^{1+2m}_{-}.\Omega^{-}_{2m}(2)$
$\mathcal{C}_7$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$	$2.\mathrm{PSp}_m(q)^t.2^{t-1}.\mathrm{S}_t$
$\mathcal{C}_8$	$\mathrm{GO}^{\pm}_n(q)$	$\mathrm{GO}^{\pm}_n(q)$

We start first with a bound.

Lemma 3.3.2. *Let $n \geq 10$, and let $3 \leq k \leq n/2$. Then*

$$\frac{3k^2 - 2kn - 3k + n^2 + n}{2} \leq \frac{n^2 - 3n + 4}{2}.$$

Proof. Let us fix n and differentiate $\frac{3k^2-2kn-3k+n^2+n}{2}$ with respect to k. This gives us $3k - n$. Therefore the local minimum turning point occurs at $n/3 = k$. The maximum in the range of $3 \leq k \leq n/2$ occurs either at $k = 3$ or $k = n/2$.

If $k = n/2$, then

$$\begin{aligned} \frac{3k^2 - 2kn - 3k + n^2 + n}{2} &= \frac{3n^2 - 2n}{8} \\ &\leq \frac{n^2 - 3n + 4}{2} && \text{since } n \geq 10. \end{aligned}$$

If $k = 3$, then

$$\begin{aligned} \frac{3k^2 - 2kn - 3k + n^2 + n}{2} &= \frac{n^2 - 5n + 18}{2} \\ &\leq \frac{n^2 - 3n + 4}{2} && \text{since } n \geq 8. \end{aligned}$$

□

Lemma 3.3.3. *Let $n \geq 10$ and q be a prime power, let $M \leq \mathrm{Sp}_n(q)$ be a P_2 type subgroup and let $H \leq \mathrm{Sp}_n(q)$ be a P_k type subgroup for $3 \leq k \leq n/2$. Then $|M| \geq |H|$.*

Proof.

$$\begin{aligned} |H| &= q^{k(n-\frac{1+3k}{2})}|\mathrm{GL}_k(q)||\mathrm{Sp}_{n-2k}(q)| && \text{by Theorem 3.3.1} \\ &\leq q^{k(n-\frac{1+3k}{2})}q^{k^2}q^{\frac{(n-2k)^2+(n-2k)}{2}} && \text{by Lemma 2.6.4} \\ &= q^{\frac{3k^2-2kn-3k+n^2+n}{2}} \leq q^{\frac{n^2-3n+4}{2}} && \text{by Lemma 3.3.2} \\ &= q^{2n-5}q^2q^{\frac{(n-4)^2+(n-4)}{2}-1} \\ &\leq q^{2n-5}(q^2-1)(q^2-q)q^{\frac{(n-4)^2+(n-4)}{2}-1} \\ &\leq q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| && \text{by Theorem 2.5.29 and Lemma 2.6.4} \\ &= |M| && \text{by Theorem 3.3.1.} \end{aligned}$$

□

Lemma 3.3.4. *Let $n \geq 10$, and let $4 \leq k \leq n/2$. Then*

$$\frac{2k^2 - 2kn + n^2 + n}{2} \leq \frac{n^2 - 3n + 4}{2}.$$

Proof. Let us fix n and differentiate $\frac{2k^2-2kn+n^2+n}{2}$ with respect to k. This gives us $2k-n$. Therefore the local minimum turning point occurs at $n/2=k$. The maximum in the range of $4 \le k \le n/2$ occurs at $k=4$.

If $k=4$, then

$$\begin{aligned}\frac{2k^2-2kn+n^2+n}{2} &= \frac{n^2-7n+32}{2} \\ &\le \frac{n^2-3n+4}{2} && \text{since } n \ge 8.\end{aligned}$$

□

Lemma 3.3.5. *Let $n \ge 2$ and let $q \ge 2$. Then*

$$q^{(n-2)^2/4+1}(q^{n-2}-1) \le (q^2-q)q^{n^2/4-1}.$$

Proof.

$$q^{(n-2)^2/4+1}(q^{n-2}-1) \le (q^2-q)q^{(n-2)^2/4}q^{n-2} = (q^2-q)q^{n^2/4-1}.$$

□

Lemma 3.3.6. *Let $n \ge 10$ and q be a prime power, let $M \le \mathrm{Sp}_n(q)$ be a P_2 type subgroup and let $H \le \mathrm{Sp}_n(q)$ have shape $\mathrm{Sp}_k(q) \times \mathrm{Sp}_{n-k}(q)$, where k is even and $2 \le k < n/2$. Then $|M| \ge |H|$.*

Proof. Let $k \ge 4$. Then

$$\begin{aligned}|H| &= |\mathrm{Sp}_k(q)||\mathrm{Sp}_{n-k}(q)| && \text{by Theorem 3.3.1} \\ &\le q^{\frac{k^2+k}{2}}q^{\frac{(n-k)^2+(n-k)}{2}} && \text{by Lemma 2.6.4} \\ &= q^{\frac{2k^2-2kn+n^2+n}{2}} \le q^{\frac{n^2-3n+4}{2}} && \text{by Lemma 3.3.4} \\ &= q^{2n-5}q^2q^{\frac{(n-4)^2+(n-4)}{2}-1} \\ &\le q^{2n-5}(q^2-1)(q^2-q)q^{\frac{(n-4)^2+(n-4)}{2}-1} \\ &\le q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| && \text{by Theorem 2.5.29 and Lemma 2.6.4} \\ &= |M| && \text{by Theorem 3.3.1.}\end{aligned}$$

If $k=2$, then

$$\begin{aligned}|H| &= |\mathrm{Sp}_k(q)||\mathrm{Sp}_{n-k}(q)| && \text{by Theorem 3.3.1} \\ &= (q^2-1)q^{(n-2)^2/4+1}\prod_{i=1}^{(n-2)/2}(q^{2i}-1) && \text{by Theorem 2.5.29} \\ &\le (q^2-1)(q^2-q)q^{n^2/4-1}\prod_{i=1}^{(n-4)/2}(q^{2i}-1) && \text{by Lemma 3.3.5}\end{aligned}$$

$$
\begin{aligned}
&= q^{2n-5}(q^2-1)(q^2-q)q^{(n-4)^2/4}\prod_{i=1}^{(n-4)/2}(q^{2i}-1) \\
&= q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| && \text{by Theorem 2.5.29} \\
&= |M| && \text{by Theorem 3.3.1.}
\end{aligned}
$$

□

Lemma 3.3.7. *Let $n \geq 8$, let c be a positive integer, and let H be a P_2 type subgroup of $\mathrm{Sp}_n(q)$. Then $|H| > q^{(n^2-4n+10)/2}$. Furthermore, if $n \geq c$ then $|H| > q^{\frac{1-4/c}{2}n^2}$.*

Proof.

$$
\begin{aligned}
|H| &= q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| && \text{by Theorem 3.3.1} \\
&\geq q^{2n-5}(q^2-q)(q^2-1)q^{((n-4)^2+n-4)/2-1} && \text{by Theorem 2.5.29 and Lemma 2.6.4} \\
&= q^{2n-5}(q^2-q)(q^2-1)q^{(n-4)^2/2+(n-6)/2} \\
&\geq q^{2n-5}(q^2-q)(q^2-1)q^{(n-4)^2/2} && \text{since } n \geq 8 \\
&\geq q^{(n^2-4n+10)/2} \\
&\geq q^{(n^2-4n)/2} \geq q^{\frac{1-4/c}{2}n^2} && \text{assuming } n \geq c.
\end{aligned}
$$

□

Lemma 3.3.8. *Let $n \geq 10$ and let $M \leq \mathrm{Sp}_n(q)$ be a $\mathcal{C}_1$ subgroup of type of type P_2. Furthermore let $H \leq \mathrm{SL}_n(q)$ be a maximal subgroup from the Aschbacher classes $\mathcal{C}_i$ with $2 \leq i \leq 7$. Then $|M| \geq |H|$.*

Proof. We will show that $|M|$ is larger than the order of maximal subgroups of $\mathcal{C}_i$ type for $2 \leq i \leq 7$ in order.

Let H be a geometric subgroup of type $\mathcal{C}_2$. Here there are two possibilities by Theorem 3.3.1, the first of which is where H has the shape $\mathrm{Sp}_m(q)^t : \mathrm{S}_t$ where $n = mt$. We note also that $n/2 \geq m \geq 2$ must be even. Assume that $n \geq 18$, then

$$
\begin{aligned}
|H| &= |\mathrm{Sp}_m(q)^t|t! \\
&\leq q^{t(m^2+m)/2}t! && \text{by Lemma 2.6.4} \\
&\leq q^{t(m^2+m)/2}q^{t\log_2(t)} && \text{by Lemma 2.8.2} \\
&= q^{(nm+n)/2+t\log_2(t)} && \text{as } n = mt \\
&\leq q^{(n^2/2+n)/2+n/2\log_2(n/2)} && \text{as } m,t \leq n/2 \\
&\leq q^{n^2/4+(n/2)^{1.55}+n/2} && \text{by Lemma 2.8.4} \\
&\leq q^{\frac{n^2}{4}+\frac{1}{2^{1.55}}\frac{n^2}{18^{0.45}}+\frac{n^2}{36}} = q^{(\frac{1}{4}+\frac{1}{2^{1.55}\times 18^{0.45}}+\frac{1}{36})n^2} && \text{as } n \geq 18 \\
&< q^{\frac{7}{18}n^2} \leq |M| && \text{by Lemma 3.3.7, taking } c = 18.
\end{aligned}
$$

By Lemma 3.3.7

$$|M| > q^{(n^2-4n+10)/2}$$

for $n \geq 8$. For the cases where $n = 10$, 12, 14 and 16, the possibilities for (m,t) are $(2,5)$, $(2,6)$, $(4,3)$, $(6,2)$, $(2,7)$, $(2,8)$, $(4,4)$ and $(8,2)$. Direct substitution of these values shows that

$$|H| \leq q^{t(m^2+m)/2} t! \leq q^{(nm+n)/2+t\log_2(t)} \leq q^{(n^2-4n+10)/2} \leq |M|.$$

Therefore, $|H| \leq |M|$ for $n \geq 10$.

In the second possibility for H, where H has shape $\mathrm{GL}_{n/2}(q).2$, we have

$$\begin{aligned}
|H| &\leq 2|\mathrm{GL}_{n/2}(q)| = 2(q-1)|\mathrm{SL}_{n/2}(q)| && \text{by Theorem 3.3.1}\\
&\leq 2(q-1)q^{n^2/4-1} \leq q^{n^2/4+1} && \text{by Lemma 2.6.4}\\
&\leq q^{\frac{3}{10}n^2} && \text{since } n \geq 5\\
&\leq |M| && \text{by Lemma 3.3.7, taking } c = 10.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_3$. Here there are two possibilities by Theorem 3.3.1, the first of which is where the H has shape $\mathrm{Sp}_m(q^r).r$ where $ms = n$ and r is prime. Note that $2 \leq m \leq n/2$ is even. Let $n \geq 12$. Then

$$\begin{aligned}
|H| &\leq |\mathrm{Sp}_m(q^r)|r \leq rq^{r(m^2+m)/2} && \text{by Lemma 2.6.4}\\
&\leq q^{\log_2(r)} q^{r(m^2+m)/2} && \text{since } r = q^{\log_q(r)} \text{ and } q \geq 2\\
&\leq q^{\log_2(r)+n^2/4+n/2} && \text{since } mr = n \text{ and } m \leq n/2\\
&\leq q^{\log_2(n/2)+n^2/4+n/2} && \text{since } r \leq n/2\\
&\leq q^{n^2/4+n/2+(n/2)^{0.55}} && \text{by Lemma 2.8.4}\\
&\leq q^{(\frac{1}{4}+\frac{1}{24}+\frac{1}{2^{0.55}\times 12^{1.45}})n^2} && \text{as } n \geq 12\\
&\leq q^{\frac{1}{3}n^2} \leq |M| && \text{by Lemma 3.3.7, taking } c = 12.
\end{aligned}$$

By Lemma 3.3.7

$$|M| > q^{(n^2-4n+10)/2}$$

for $n \geq 8$. For the cases where $n = 10$, the possibilities for (m,s) is $(2,5)$ and $(5,2)$. Direct substitution of these values shows that

$$|H| \leq q^{\log_2(s)+s(m^2+m)/2} \leq q^{(n^2-4n+10)/2} \leq |M|.$$

For the second possibility of H, where H has shape $\mathrm{GU}_{n/2}(q).2$

$$\begin{aligned}
|H| &\leq 2|\mathrm{GU}_{n/2}(q)| \leq q|\mathrm{GU}_{n/2}(q)| && \text{by Theorem 3.3.1}\\
&= q(q-1)|\mathrm{SU}_{n/2}(q)|\\
&\leq q^2|\mathrm{SU}_{n/2}(q)| \leq q^{n^2/4+1} && \text{by Lemma 2.6.4}\\
&\leq q^{\frac{3}{10}n^2} && \text{since } n \geq 5\\
&\leq |M| && \text{by Lemma 3.3.7, taking } c = 10.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_4$. Here, by Theorem 3.3.1, the shape of H is $(\mathrm{Sp}_{n_1}(q) \circ \mathrm{GO}^{\epsilon}_{n_2}(q)).(n_2, 2)$, where $n_1 n_2 = n$ and where $n/2 \geq n_2 \geq 3$. If n_2 is even then

$$|H| \leq 2|\mathrm{Sp}_{n_1}(q)||\mathrm{GO}^{-}_{n_2}(q)|$$

$$\begin{aligned}
&\le 4|\mathrm{Sp}_{n_1}(q)||\mathrm{SO}^-_{n_2}(q)| \\
&\le 4q^{(n_1^2+n_1)/2}q^{(n_2^2-n_2)/2+1} && \text{by Lemma 2.6.4.}
\end{aligned}$$

If n_2 is odd then

$$\begin{aligned}
|H| &\le 2|\mathrm{Sp}_{n_1}(q)||\mathrm{GO}_{n_2}(q)| \\
&\le 4|\mathrm{Sp}_{n_1}(q)||\mathrm{SO}_{n_2}(q)| \\
&\le 4q^{(n_1^2+n_1)/2}q^{(n_2^2-n_2)/2+1} && \text{by Lemma 2.6.4.}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
|H| &\le 4q^{(n_1^2+n_1)/2}q^{(n_2^2-n_2)/2+1} \\
&\le q^{(n_1^2+n_1+n_2^2-n_2+6)/2} && \text{since } q \ge 2 \\
&\le q^{(n^2/9+n/3+n^2/4-n/2+6)/2} && \text{since } 2 \le n_1 \le n/3 \text{ and } 3 \le n_2 \le n/2 \\
&= q^{13/72n^2-n/6+3} < q^{13/72n^2+3} \\
&\le q^{(\frac{13}{72}+\frac{3}{64})n^2} && \text{as } n \ge 8 \\
&< q^{\frac{1}{4}n^2} \le |M| && \text{by Lemma 3.3.7, taking } c = 8.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_5$. Here, by Theorem 3.3.1, H has shape $\mathrm{Sp}_n(q_0).(2, q-1, r)$ where $q_0^r = q$ and r is prime. Therefore we have

$$\begin{aligned}
|H| &\le q|\mathrm{Sp}_n(q_0)| \le q \times q_0^{(n^2+n)/2} && \text{by Lemma 2.6.4} \\
&\le q \times q^{(n^2+n)/4} = q^{(n^2+n+4)/4} && \text{as } q_0 \le q^{1/2} \\
&\le q^{(\frac{1}{4}+\frac{1}{40}+\frac{1}{100})n^2} && \text{as } n \ge 10 \\
&\le q^{\frac{3}{10}n^2} \le |M| && \text{by Lemma 3.3.7, taking } c = 10.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_6$. Here, by Theorem 3.3.1, H has shape $2_-^{1+2m}.\mathrm{SO}^-_{2m}(2)$ or $2_-^{1+2m}.\Omega^-_{2m}(2)$, where $n = 2^m$. Since, no such H occurs when $n = 10$ we may assume $n \ge 16$ here. Therefore we have

$$\begin{aligned}
|H| &\le 2_-^{1+2m}|\mathrm{SO}^-_{2m}(2)| \\
&\le 2_-^{1+2m}2^{(4m^2-2m+2)/2} \le 2^{2m^2+m+2} && \text{by Lemma 2.6.4} \\
&\le 2^{2\log_2(n)^2+\log_2(n)+2} && \text{as } n = 2m \\
&\le q^{2n^{1.1}+n^{0.55}+2} && \text{by Lemma 2.8.4} \\
&\le q^{(\frac{2}{16^{0.9}}+\frac{1}{16^{1.45}}+\frac{1}{128})n^2} && \text{as } n \ge 12 \\
&\le q^{\frac{1}{3}n^2} \le |M| && \text{by Lemma 3.3.7, taking } c = 12.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_7$. Here, by Theorem 3.3.1, the shape of H is $2.\mathrm{PSp}_m(q)^t.2^{t-1}.\mathrm{S}_t$ where $n = m^t$. Since, no such H occurs when $n = 10$ we may assume $n \ge 14$ here.

$$|H| \le 2^t|\mathrm{Sp}_m(q)|^t t! \le q^t|\mathrm{Sp}_m(q)|^t t!$$

$$
\begin{aligned}
&\leq q^t q^{t(m^2+m)/2} t! && \text{by Lemma 2.6.4}\\
&\leq q^t q^{t(m^2+m)/2} q^{t\log_2(t)} && \text{by Lemma 2.8.2}\\
&= q^{tm^2/2+tm/2+t\log_2(t)+t} &&\\
&\leq q^{\log_2(n)n/2+\log_2(n)n^{1/2}/2+\log_2(n)^2+\log_2(n)} && \text{as } m\leq n^{1/2} \text{ and } t\leq \log_2(n)\\
&\leq q^{n^{1.55}/2+n^{1.1}+n^{1.05}/2+n^{0.55}} && \text{by Lemma 2.8.4}\\
&\leq q^{(\frac{1}{2\times 14^{0.45}}+\frac{1}{14^{0.9}}+\frac{1}{2\times 14^{0.95}}+\frac{1}{14^{1.45}})n^2} && \text{as } n\geq 14\\
&\leq q^{\frac{5}{14}n^2} \leq |M| && \text{by Lemma 3.3.7, taking } c=14.
\end{aligned}
$$

By Theorem 3.2.1 we have considered all possible H. $\square$

Lemma 3.3.9. *Let $M \leq \mathrm{Sp}_n(q)$ be a $\mathcal{C}_1$ subgroup of type of type P_2. Furthermore let $H \leq \mathrm{SL}_n(q)$ be a non-geometric maximal subgroup lying in $\mathcal{S}$. If $n \geq 10$ then $|M| \geq |H|$.*

Proof. Let H be a maximal subgroup of $\mathrm{Sp}_n(q)$ of type $\mathcal{E}_G$ as defined in Definition 3.1.1. Assume first that $n \geq 16$.

$$
\begin{aligned}
|H| &\leq (2,q-1)(n+2)! < q(n+2)! \leq q^{(n+2)\log_2(n+2)+1} && \text{by Lemma 2.8.2}\\
&\leq q^{\frac{18}{16}n\log_2(\frac{18}{16}n)+1} && \text{as } n\geq 16,\ \frac{18}{16}n \geq n+2\\
&\leq q^{\frac{18^{1.55}}{16^{1.55}}n^{1.55}+1} && \text{by Lemma 2.8.4}\\
&\leq q^{\frac{18^{1.55}}{16^{1.55}}\frac{n^2}{16^{0.45}}+1} = q^{\frac{18^{1.55}}{16^2}n^2+1} && \text{as } n\geq 16\\
&\leq q^{\frac{3}{8}n^2} \leq |M| && \text{by Lemma 3.3.7, taking } c=16.
\end{aligned}
$$

In the case where $8 \leq n \leq 14$, we may verify by direct calculation that

$$
\begin{aligned}
|M| &\geq q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| && \text{by Theorem 3.3.1}\\
&\geq 2^{2n-5}|\mathrm{GL}_2(2)||\mathrm{Sp}_{n-4}(2)| &&\\
&\geq 2(n+2)! \geq |H|. &&
\end{aligned}
$$

Let K be a maximal subgroup of $\mathrm{Sp}_n(q)$ that is non-geometric but not of type $\mathcal{E}_G$. Furthermore, assume that $n \geq 6$. Then

$$
\begin{aligned}
|H| &\leq q^{3n+1} && \text{by Lemma 3.1.4}\\
&\leq q^{(n^2-4n+10)/2} && \text{since } n\geq 10\\
&\leq |M| && \text{by Lemma 3.3.7.}
\end{aligned}
$$

$\square$

Lemma 3.3.10. *Let G be $\mathrm{Sp}_n(q)$ for $n \geq 10$, let $M \leq G$ be a P_2 subgroup and let $H \leq G$ be a maximal subgroup that is not of P_1 or in the $\mathcal{C}_8$ class. Then $|M| \geq |H|$.*

Proof. If H lies in the class $\mathcal{C}_1$ then Theorem 3.3.1 shows that H is either of P_k type or has shape $\mathrm{Sp}_k(q) \times \mathrm{Sp}_{n-k}(q)$. If H is the former, then Lemma 3.3.3 shows us that $|H| \leq |M|$. If H is the latter, then Lemma 3.3.6 gives us the required result.

If H lies in the classes $\mathcal{C}_i$ for $2 \leq i \leq 7$ then the result follows from Lemma 3.3.8. Finally if H belongs to $\mathcal{S}$ then Lemma 3.3.9 gives us the result. $\square$

Table 3.5: Geometric subgroups of $\mathrm{SU}_n(q)$

Class	Subgroup Type	Conditions	c
$\mathcal{C}_1$	P_m	$1 \leq m \leq n/2$	1
$\mathcal{C}_1$	$\mathrm{GU}_m(q) \perp \mathrm{GU}_{n-m}(q)$	$1 \leq m < n/2$	1
$\mathcal{C}_2$	$\mathrm{GU}_m(q) \wr \mathrm{S}_t$	$n = mt$, $t \geq 2$	1
$\mathcal{C}_2$	$\mathrm{GL}_{n/2}(q^2).2$	q odd, n even	1
$\mathcal{C}_3$	$\mathrm{GU}_m(q^r)$	$n = mr$, r odd prime	1
$\mathcal{C}_4$	$\mathrm{GU}_{n_1}(q) \otimes \mathrm{GU}_{n_2}(q)$	$n = n_1 n_2$, $2 \leq n_1 < \sqrt{n}$	$(q+1, n_1, n_2)$
$\mathcal{C}_5$	$\mathrm{GU}_n(q_0)$	$q = q_0^r$, r odd prime	$(\frac{q+1}{q_0+1}, 5)$
$\mathcal{C}_5$	$\mathrm{Sp}_n(q)$	n even	$(q+1, n/2)$
$\mathcal{C}_5$	$\mathrm{GO}_n^{\epsilon}(q)$	q odd	$\frac{(q+1,n)}{(n,2)}$
$\mathcal{C}_6$	$r^{1+2m}.\mathrm{Sp}_{2m}(r)$	$n = r^m$, r prime	$(q+1, n)$
$\mathcal{C}_7$	$\mathrm{GU}_m(q) \wr \mathrm{S}_t$	$n = m^t$, $m \geq 3$, $(m,q) \neq (3,2)$	c_7

Here $c_7 = \frac{1}{2}(q+1, m)$ if $t = 2$, $m \equiv 2 \mod 4$ and $q \equiv 1 \mod 4$, and $c_7 = (q+1, n/m)$ otherwise

3.4 Unitary groups

In this section our aim is to show, for sufficiently large n, that P_2 type subgroups, of $\mathrm{PSU}_n(q)$ are larger than all other subgroups of $\mathrm{PSU}_n(q)$ other than P_1 type subgroups or those of type $\mathrm{GU}_1(q) \perp \mathrm{GU}_{n-1}(q)$. Before we compare the orders of subgroups of $\mathrm{SU}_n(q)$, let us first categorise the geometric subgroups.

Theorem 3.4.1. *Let $n \geq 5$, let q be a prime power and let $G = \mathrm{SU}_n(q)$. Furthermore, let M be a maximal subgroup of G. Then either M is a geometric subgroup, and is of one of the types given in Table 3.5 or M lies in $\mathcal{S}$. If M is in Table 3.1 then c, the number of $\mathrm{PSU}_n(q)$ conjugacy classes of that type of group, lies in Table 3.1 also. Furthermore if M is found in Table 3.5 then the shape of M can be found in Table 3.6.*

Proof. The theorem is a sub-case of [6, Theorem 2.2.19]. We obtain the data for Tables 3.5 and 3.6 from [6, Tables 2.2-2.11], which present conditions for the existence of and the shapes of the geometric subgroups of G. Information for c comes from [26, MAIN THEOREM]. □

Let us now derive an upper bound for the order of P_2 type subgroup $H \leq \mathrm{SU}_n(q)$.

Lemma 3.4.2. *Let H be a P_2 type subgroup of $\mathrm{SU}_n(q)$. Then $|H| > q^{n^2-4n+9}$. Furthermore, if $n \geq 11$ then $|H| > q^{\frac{9}{13}n^2}$.*

Table 3.6: Geometric subgroups of $\mathrm{SU}_n(q)$

Class	Subgroup Type	Structure
$\mathcal{C}_1$	P_m, $m < n/2$	$[q^{m(2n-3m)}] : (\mathrm{SL}_m(q^2) \times \mathrm{SU}_{n-2m}(q)).(q^2-1)$
$\mathcal{C}_1$	P_m, $m = n/2$	$[q^{n^2/4}] : \mathrm{SL}_m(q^2).(q-1)$
$\mathcal{C}_1$	$\mathrm{GU}_m(q) \perp \mathrm{GU}_{n-m}(q)$	$(\mathrm{SU}_m(q) \times \mathrm{SU}_{n-m}(q).(q+1)$
$\mathcal{C}_2$	$\mathrm{GU}_m(q) \wr \mathrm{S}_t$	$\mathrm{SU}_m(q)^t.(q-1)^{t-1}.\mathrm{S}_t$
$\mathcal{C}_2$	$\mathrm{GL}_{n/2}(q^2).2$	$\mathrm{SL}_{n/2}(q^2).(q+1).2$
$\mathcal{C}_3$	$\mathrm{GU}_m(q^r)$	$\left(\frac{(q+1,m)(q^r+1)}{q+1} \circ \mathrm{SU}_m(q^r)\right).\frac{(q^r+1,m)}{(q+1,m)}.r$
$\mathcal{C}_4$	$\mathrm{GU}_{n_1}(q) \otimes \mathrm{GU}_{n_2}(q)$	$(\mathrm{SU}_{n_1}(q) \circ \mathrm{SU}_{n_2}(q)).[(q+1,n_1,n_2)^2]$
$\mathcal{C}_5$	$\mathrm{GU}_n(q_0)$	$\mathrm{SU}_n(q_0).[(\frac{q+1}{q_0+1},n)]$
$\mathcal{C}_5$	$\mathrm{Sp}_n(q)$	$\mathrm{Sp}_n(q).[(q+1,n/2)]$
$\mathcal{C}_5$	$\mathrm{GO}_n^\epsilon(q)$	$\mathrm{SO}_n^\epsilon(q).[(q+1,n)]$
$\mathcal{C}_6$	$r^{1+2m}.\mathrm{Sp}_{2m}(r)$	$((q+1,n) \circ r^{1+2m}).\mathrm{Sp}_{2m}(q)$
$\mathcal{C}_7$	$\mathrm{GU}_m(q) \wr \mathrm{S}_t$, $t=2$, $m \equiv 2 \mod 4$, $q \equiv 1 \mod 4$	$(q+1,m).\mathrm{PSU}_m(q)^2.[(q+1,m)^2]$
$\mathcal{C}_7$	$\mathrm{GU}_m(q) \wr \mathrm{S}_t$, otherwise	$(q+1,m).\mathrm{PSU}_m(q)^t.[(q+1,\frac{n}{m})(q+1,m)^{t-1}].\mathrm{S}_t$

Proof.

$$\begin{aligned}
|H| &= q^{2(2n-6)}|\mathrm{SL}_2(q^2)||\mathrm{SU}_{n-4}(q)|(q^2-1) && \text{by Theorem 3.4.1}\\
&\geq q^{2(2n-6)}q^2(q^4-1)q^{(n-4)^2-1}(q^2-1) && \text{by Theorem 2.5.29 and Lemma 2.6.4}\\
&= q^{n^2-4n+5}(q^4-1)(q^2-1)\\
&> q^{n^2-4n+9} > q^{\frac{9}{13}n^2} && \text{since } n \geq 11.
\end{aligned}$$

□

Before we compare the order of P_2 type subgroups with that of other P_k type subgroups for $3 \leq k < n/2$ we obtain the following bound.

Lemma 3.4.3. *Let $n \geq 12$, let $3 \leq k < n/2$ and let $q \geq 2$. Then*

$$\prod_{i=n-2k+1}^{n-4}(q^i-(-1)^i) \geq \prod_{i=3}^{k}(q^{2i}-1).$$

Proof. For ease of reference we define $P(q)$ to be $\prod_{i=n-2k+1}^{n-4}(q^i-(-1)^i)$ and $Q(q)$ to be $\prod_{i=3}^{k}(q^{2i}-1)$. We split the proof into four cases, where $k \leq (n-4)/2$, where $k = (n-3)/2$, where $k = (n-2)/2$ and where $k = (n-1)/2$.

Let us first assume that $k \leq (n-4)/2$. Then the j'th largest factor of $Q(n)$, $q^{2(k+1-j)}-1$, is smaller than the j'th largest factor of $P(q)$, $q^{n-3-j}-(-1)^{n-3-j}$. This is because

$$\begin{aligned}
q^{n-3-j}-(-1)^{n-3-j} &\geq q^{n-3-j}-1 \geq q^{n-2-2j}-1\\
&= q^{2(n-4)/2+2-2j}-1 \geq q^{2k+2-2j}-1 \qquad \text{since } k \leq (n-4)/2.
\end{aligned}$$

In addition there are $2k-4$ factors in $P(n)$ while there are $k-2$ factors in $Q(q)$. Consequently we have that $P(q) \geq Q(q)$.

Let us assume that $k = (n-3)/2$. In this case, n is odd,

$$P(q) = \prod_{i=4}^{n-4}(q^i-(-1)^i)$$

and

$$Q(q) = \prod_{i=3}^{(n-3)/2}(q^{2i}-1).$$

The $(q^{n-3}-1)$ factor of $Q(q)$ is smaller than the product of the two factors $(q^{n-4}+1)$ and (q^5+1) of $P(q)$. Note that we are assured the existence of these two distinct factors $(q^{n-4}+1)$ and (q^5+1) as $n \geq 11$. In addition the remaining factors of $Q(q)$ are also factors of $P(q)$. Therefore $P(q) \geq Q(q)$.

Let us assume next that $k = (n-2)/2$. In this case n is even, we have

$$P(q) = \prod_{i=3}^{n-4}(q^i-(-1)^i)$$

and

$$Q(q) = \prod_{i=3}^{(n-2)/2} (q^{2i} - 1).$$

The $(q^{n-2} - 1)$ factor of $Q(n)$ is smaller than the product of the two factors $(q^{n-5} + 1)$ and $(q^3 + 1)$ of $P(q)$. Note that we are assured the existence of these two distinct factors, $(q^{n-5} + 1)$ and $(q^3 + 1)$, as $n \geq 10$. The remaining factors $(q^{2i} - 1)$ of $Q(q)$ are also factors of $P(q)$. Therefore $P(q) \geq Q(q)$.

Let us assume next that $k = (n-1)/2$. In this case n must be odd, we have

$$P(n) = \prod_{i=2}^{n-4} (q^i - (-1)^i)$$

and

$$Q(n) = \prod_{i=3}^{(n-1)/2} (q^{2i} - 1).$$

Here we have that

$$\begin{aligned}(q^{n-4} + 1)(q^{n-6} + 1)(q^2 - 1)(q^3 + 1)(q^4 - 1)(q^5 + 1) &> (q^{n-4})(q^{n-6})(q)(q^3)^2(q^5) \\ &> (q^{n-1})(q^{n-3}) > (q^{n-1} - 1)(q^{n-3} - 1).\end{aligned}$$

Consequently the product of factors $(q^{n-1} - 1)$ and $(q^{n-3} - 1)$ of $Q(n)$ is smaller than the product of the factors $(q^{n-4} + 1)$, $(q^{n-6} + 1)$, $(q^2 - 1)$, $(q^3 + 1)$, $(q^4 - 1)$ and $(q^5 + 1)$ of $P(n)$. Note that we are assured the existence of these distinct factors since $n \geq 13$. The remaining factors $(q^{2i} - 1)$ of $Q(n)$ are also factors of $P(n)$. Consequently we also have $P(q) \geq Q(q)$. □

Lemma 3.4.4. *Let $n \geq 12$ and let q be a prime power, let $M \leq \mathrm{SU}_n(q)$ be a P_2 type subgroup and let $H \leq \mathrm{SU}_n(q)$ be a P_k type subgroup for $3 \leq k < n/2$. Then $|M| \geq |H|$.*

Proof.

$$\begin{aligned}
|M| &= (q^2 - 1)q^{2(2n-6)}|\mathrm{SL}_2(q^2)||\mathrm{SU}_{n-4}(q)| && \text{by Theorem 3.4.1} \\
&= (q^2 - 1)q^{2(2n-6)+2+\frac{(n-4)(n-5)}{2}}(q^4 - 1)\prod_{i=2}^{n-4}(q^i - (-1)^i) && \text{by Theorem 2.5.29} \\
&= (q^2 - 1)q^{n(n-1)/2}(q^4 - 1)\prod_{i=2}^{n-4}(q^i - (-1)^i) \\
&\geq (q^2 - 1)q^{n(n-1)/2}\prod_{i=2}^{k}(q^{2i} - 1)\prod_{i=2}^{n-2k}(q^i - (-1)^i) && \text{by Lemma 3.4.3} \\
&= (q^2 - 1)q^{k(2n-3k)+k(k-1)+\frac{(n-2k)(n-2k-1)}{2}}\prod_{i=2}^{k}(q^{2i} - 1)\prod_{i=2}^{n-2k}(q^i - (-1)^i) \\
&= (q^2 - 1)q^{k(2n-3k)}|\mathrm{SL}_k(q^2)||\mathrm{SU}_{n-2k}(q)| && \text{by Theorem 2.5.29} \\
&= |H| && \text{by Theorem 3.4.1.}
\end{aligned}$$

□

Before we compare the order of P_2 type subgroups with that of $P_{n/2}$ type subgroups, we derive the following lemma.

Lemma 3.4.5. *Let $n \geq 14$ be even and let $q \geq 2$. Then*

$$(q+1)\prod_{i=2}^{n-4}(q^i - (-1)^i) \geq \prod_{i=3}^{n/2}(q^{2i}-1).$$

Proof. For ease of reference we define $P(q)$ to be $(q+1)\prod_{i=2}^{n-4}(q^i - (-1)^i)$ and $Q(q)$ to be $\prod_{i=3}^{n/2}(q^{2i}-1)$. First notice that

$$\begin{aligned}
&(q+1)(q^2-1)(q^4-1)(q^{n-5}+1)(q^{n-7}+1)(q^{n-9}+1)(q^{n-11}+1)\\
&\qquad \geq q^{1+1+3+(n-5)+(n-7)+(n-9)+(n-11)} = q^{4n-27}\\
&\qquad \geq q^{2n-2} \geq (q^n-1)(q^{n-2}-1) \qquad \text{as } n > 13.
\end{aligned}$$

Consequently the product of factors (q^n-1) and $(q^{n-2}-1)$ of $Q(n)$ is smaller than the product of the factors $(q+1)$, (q^2-1), (q^4-1), $(q^{n-5}+1)$, $(q^{n-7}+1)$, $(q^{n-9}+1)$ and $(q^{n-11}+1)$ of $P(n)$. Note that we are assured the existence of these distinct factors since $n \geq 14$. The remaining factors $(q^{2i}-q)$ for $3 \leq i \leq (n-4)/2$ of $Q(n)$ are also factors of $P(n)$. Consequently we also have $P(q) \geq Q(q)$. □

Lemma 3.4.6. *Let $n \geq 14$ be even and let q be a prime power, let $M \leq \mathrm{SU}_n(q)$ be a subgroup of P_2 type, and let $H \leq \mathrm{SU}_n(q)$ be a subgroup of $P_{n/2}$ type. Then $|M| \geq |H|$.*

Proof.

$$\begin{aligned}
|M| &= (q^2-1)q^{2(2n-6)}|\mathrm{SL}_2(q^2)||\mathrm{SU}_{n-4}(q)| && \text{by Theorem 3.4.1}\\
&= (q^2-1)q^{2(2n-6)+2+\frac{(n-4)(n-5)}{2}}\prod_{i=2}^{2}(q^{2i}-1)\prod_{i=2}^{n-4}(q^i-(-1)^i) && \text{by Theorem 2.5.29}\\
&= (q^2-1)q^{n(n-1)/2}\prod_{i=2}^{2}(q^{2i}-1)\prod_{i=2}^{n-4}(q^i-(-1)^i)\\
&\geq (q-1)q^{n(n-1)/2}\prod_{i=2}^{n/2}(q^{2i}-1) && \text{by Lemma 3.4.5}\\
&= (q-1)q^{n^2/4+(n/2)(n/2-1)}\prod_{i=2}^{n/2}(q^{2i}-1)\\
&= (q-1)q^{n^2/4}|\mathrm{SU}_{n/2}(q^2)| = |H| && \text{by Theorems 2.5.29 and 3.4.1.}
\end{aligned}$$

□

Lemma 3.4.7. *Let $n \geq 5$, let q be a prime power and let $2 \leq k < n/2$. Furthermore, let $H_1 \leq \mathrm{SU}_n(q)$ be a P_k type subgroup, and let $H_2 \leq \mathrm{SU}_n(q)$ be a $\mathrm{GU}_k(q) \perp \mathrm{GU}_{n-k}(q)$ type subgroup. Then $|H_1| \geq |H_2|$.*

Proof. By Theorem 3.4.1,

$$|H_1| = q^{k(2n-3k)}|\mathrm{SL}_k(q^2)||\mathrm{SU}_{n-2k}(q)|(q^2-1)$$

and

$$|H_2| = |\mathrm{SU}_k(q)||\mathrm{SU}_{n-k}(q)|(q+1).$$

Then

$$\begin{aligned}
|H_1| &= q^{k(2n-3k)}|\mathrm{SL}_k(q^2)||\mathrm{SU}_{n-2k}(q)|(q^2-1) \\
&\geq q^{k(2n-3k)}q^{2(k^2-2)}q^{(n-2k)^2-2}(q^2-1) && \text{by Lemma 2.6.4} \\
&= q^{3k^2-2kn+n^2-6}(q^2-1) \\
&\geq q^{2k^2-2kn+n^2-2}(q+1) && \text{since } k \geq 2 \\
&= q^{k^2-1}q^{(n-k)^2-1}(q+1) \\
&\geq |\mathrm{SU}_k(q)||\mathrm{SU}_{n-k}(q)|(q+1) = |H_2| && \text{by Lemma 2.6.4.}
\end{aligned}$$

□

Lemma 3.4.8. *Let $n \geq 13$, and let $M \leq \mathrm{SU}_n(q)$ be a $\mathcal{C}_1$ subgroup of type of type P_2. Furthermore let $H \leq \mathrm{SL}_n(q)$ be a maximal subgroup from the Aschbacher classes $\mathcal{C}_i$ with $2 \leq i \leq 7$. Then $|M| \geq |H|$.*

Proof. By Lemma 3.4.2

$$|M| > q^{\frac{9}{13}n^2}.$$

We will show that $|M|$ is larger than the order of maximal subgroups of $\mathcal{C}_i$ type for $2 \leq i \leq 8$ in order.

Let H be a geometric subgroup that lies in Class $\mathcal{C}_2$. Here there are two possibilities by Theorem 3.4.1, the first of which is where $H \cong \mathrm{SU}_m(q)^t.(q+1)^{t-1}.\mathrm{S}_t$ where $n = mt$.

In the case where $t \leq n/2$ we have

$$\begin{aligned}
|H| &\leq q^{tm^2-t}(q+1)^{t-1}t! && \text{by Theorem 3.4.1 and Lemma 2.6.4} \\
&\leq q^{tm^2-t}(3q/2)^{t-1}t! \\
&\leq q^{tm^2-t}(3q/2)^{t-1}q^{t\log_2(t)} && \text{by Lemma 2.8.2} \\
&\leq q^{tm^2-1}q^{t-1}q^{\log_2(3/2)(t-1)}q^{t\log_2(t)} \\
&= q^{tm^2+t\log_2(t)+t\log_2(3/2)+t-\log_2(3/2)-2} \\
&\leq q^{n^2/2+n/2\log_2(n/2)+n/2\log_2(3/2)+n/2-\log_2(3/2)-2} && \text{as } n = mt \text{ and } m,t \leq n/2 \\
&\leq q^{\frac{n^2}{2}+\frac{n^{1.55}}{2^{1.55}}+\frac{n\log_2(3/2)}{2}+\frac{n}{2}-\log_2(3/2)-2} && \text{by Lemma 2.8.4} \\
&\leq q^{\frac{n^2}{2}+\frac{n^{1.55}}{2^{1.55}}+\frac{n(\log_2(3/2)+1)}{2}} \\
&\leq q^{(\frac{1}{2}+\frac{1}{2^{1.55}\times 8^{0.45}}+\frac{\log_2(3/2)+1}{2\times 8})n^2} && \text{as } n \geq 8
\end{aligned}$$

$$< q^{0.71n^2} < q^{\frac{9}{13}n^2} \leq |M|.$$

In the case where $t = n$ we have

$$\begin{aligned}
|H| &\leq (q+1)^{n-1} n! && \text{by Theorem 3.4.1} \\
&\leq (3q/2)^{n-1} n! && \text{as } q \geq 2 \\
&\leq (3q/2)^{n-1} q^{n \log_2(n)} && \text{by Lemma 2.8.2} \\
&\leq q^{n \log_2(n) + n - 1} q^{(n-1)\log_2(3/2)} \\
&= q^{n \log_2(n) + n \log_2(3/2) + n - \log_2(3/2) - 1} \\
&\leq q^{n^{1.55} + (1+\log_2(3/2))n - \log_2(3/2) - 1} && \text{by Lemma 2.8.4} \\
&< q^{n^{1.55} + (1+\log_2(3/2))n} \\
&\leq q^{(\frac{1}{6^{0.45}} + \frac{1+\log_2(3/2)}{6})n^2} && \text{as } n \geq 6 \\
&< q^{0.711n^2} < q^{\frac{9}{13}n^2} \leq |M|.
\end{aligned}$$

The second possibility where H lies in Class $\mathcal{C}_2$ is where it has shape $\mathrm{SL}_{n/2}(q^2).(q-1).2$ by Theorem 3.4.1. Therefore

$$\begin{aligned}
|H| &= 2|\mathrm{SL}_{n/2}(q^2)|(q-1) \\
&\leq 2(q-1)q^{n^2/2-2} && \text{by Lemma 2.6.4} \\
&\leq q^{n^2/2} \leq q^{\frac{9}{13}n^2} \leq |M|.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_3$. Here, by Theorem 3.4.1, the order of H is bounded above by $r(q^r+1)^2|\mathrm{SU}_m(q^r)|$ where $mr = n$ and $r \geq 2$.

$$\begin{aligned}
|H| &\leq r(q^r+1)^2 |\mathrm{SU}_m(q^r)| \\
&\leq r(q^r+1)^2 q^{r(m^2-1)} && \text{by Lemma 2.6.4} \\
&\leq 4nq^{2n} q^{nm-2} \leq 4nq^{2n} q^{n^2/2-2} && \text{by } mr = n \text{ and } 2 \leq r \leq n \\
&\leq 2^{\log_2(n)+2} q^{n^2/2+2n-2} \leq q^{n^2/2+2n+\log_2(n)} \\
&\leq q^{n^2/2+2n+n^{0.55}} && \text{by Lemma 2.8.4} \\
&\leq q^{(\frac{1}{2}+\frac{2}{13}+\frac{1}{13^{1.45}})n^2} && \text{as } n \geq 13 \\
&< q^{0.68n^2} < q^{\frac{9}{13}n^2} \leq |M|.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_4$ subgroup. Here by Theorem 3.4.1, the order of K is bounded above by $(q+1)^2|\mathrm{SU}_{n_1}(q)||\mathrm{SU}_{n_2}(q)|$ with $n = n_1 n_2$. Note that $1 < n_1, n_2 \leq n/2$. Therefore

$$\begin{aligned}
|H| &\leq (q+1)^2 |\mathrm{SU}_{n_1}(q)||\mathrm{SU}_{n_2}(q)| \\
&\leq q^4 q^{n_1^2-1} q^{n_2^2-1} = q^{n_1^2+n_2^2+2} && \text{by Lemma 2.6.4} \\
&\leq q^{n^2/4+n^2/4+2} = q^{n^2/2+2} \leq q^{\frac{9}{13}n^2} \leq |M| && \text{since } n \geq 4.
\end{aligned}$$

Let K be a geometric subgroup of type $\mathcal{C}_5$. Here by Theorem 3.4.1 we have 3 possibilities for the structure of H. In the first case the order of K is bounded above by $(q+1)|\mathrm{SU}_n(q_0)|$ for where $q = q_0^r$ where r is an odd prime. Therefore

$$\begin{aligned}
|H| &\leq (q+1)|\mathrm{SU}_n(q_0)| \\
&\leq q^2 q_0^{n^2-1} \leq q^2 q^{(n^2-1)/2} && \text{by Lemma 2.6.4 and } q_0 \leq q^{1/2} \\
&\leq q^{(n^2+3)/2} \leq q^{\frac{9}{13}n^2} \leq |M| && \text{as } n \geq 11.
\end{aligned}$$

In the second case, the order of H is bounded above by $(q+1)|\mathrm{Sp}_n(q)|$. Therefore

$$\begin{aligned}
|H| &\leq (q+1)|\mathrm{Sp}_n(q)| \\
&\leq q^2 q^{(n^2+n)/2} = q^{(n^2+n+4)/2} && \text{by Lemma 2.6.4} \\
&\leq q^{\frac{9}{13}n^2} \leq |M| && \text{as } n \geq 11.
\end{aligned}$$

In the third case the order of H is bounded above by $(q+1)|\mathrm{SO}_n^-(q)|$. Therefore

$$\begin{aligned}
|H| &\leq (q+1)|\mathrm{SO}_n^-(q)| \\
&\leq 2q^2 q^{(n^2+n)/2} && \text{by Lemma 2.6.4} \\
&\leq q^3 q^{(n^2+n)/2} = q^{(n^2+n+6)/2} \\
&\leq q^{\frac{9}{13}n^2} \leq |M| && \text{as } n \geq 11.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_6$. Here by Theorem 3.4.1 we have that the order of H is bounded above by $(q+1)r^{1+2m}|\mathrm{Sp}_{2m}(r)|$ where $r^m = n$ with r prime. So

$$\begin{aligned}
|H| &\leq (q+1)r^{1+2m}|\mathrm{Sp}_{2m}(r)| \\
&\leq (q+1)r^{1+2m}r^{2m^2+m} = (q+1)r^{2m^2+3m+1} && \text{by Lemma 2.6.4} \\
&\leq (q+1)n^{2m+3+1/m} && \text{as } n = r^m \\
&< (q+1)n^{2m+4} \leq (q+1)n^{2\log_2(n)+4} && \text{as } m \leq \log_2(n) \\
&\leq (q+1)q^{2\log_2(n)^2+4\log_2(n)} && \text{as } n \leq q^{\log_2(n)} \\
&< q^{2\log_2(n)^2+4\log_2(n)+2} \leq q^{2n^{1.1}+4n^{0.55}+2} && \text{by Lemma 2.8.4} \\
&\leq q^{(\frac{2}{7^{0.9}}+\frac{4}{7^{1.45}}+\frac{2}{49})n^2} \leq (q-1)q^{n^2-2n-2} && \text{as } n \geq 7 \\
&< q^{0.63n^2} < q^{\frac{9}{13}n^2} \leq |M|.
\end{aligned}$$

Let H be a geometric subgroup of type $\mathcal{C}_7$. Here by Theorem 3.4.1, $|H|$ is bounded above by $(q+1,m)^t(q+1,\frac{n}{m})|\mathrm{PSU}_m(q)|^t t!$ where $n = m^t$ and $t \neq 1$. Therefore

$$\begin{aligned}
|H| &\leq (q+1,m)^t(q+1,\frac{n}{m})|\mathrm{PSU}_m(q)|^t t! \\
&= (q+1)|\mathrm{SU}_m(q)|^t t!
\end{aligned}$$

$$
\begin{aligned}
&\leq q^2 q^{tm^2-t} t! && \text{by Lemma 2.6.4}\\
&\leq q^2 q^{tm^2-t} q^{t\log_2(t)} < q^{m^2 t + t\log_2(t)+2} && \text{by Lemma 2.8.2}\\
&\leq q^{n\log_2(n)+\log_2(n)\log_2(\log_2(n))+2} && \text{since } t \leq \log_2(n) \text{ and } m \leq n^{1/2}\\
&\leq q^{n^{1.55}+n^{0.8525}+2} && \text{by Lemma 2.8.4}\\
&\leq q^{(\frac{1}{7^{0.45}}+\frac{1}{7^{1.1475}}+\frac{2}{49})n^2} && \text{as } n \geq 7.\\
&< q^{0.6n^2} < q^{\frac{9}{13}n^2} \leq |M|.
\end{aligned}
$$

By Theorem 3.4.1 we have considered all possible H. □

Lemma 3.4.9. *Let $n \geq 10$ and q be a prime power, let $M \leq \mathrm{SU}_n(q)$ be a subgroup of type P_2 and let $H \leq \mathrm{SU}_n(q)$ of type $\mathcal{S}$. Then $|M| \geq |H|$.*

Proof. Theorem 3.1.4 states that H cannot be a maximal subgroup of type $\mathcal{E}_G$, therefore the subgroups H in $\mathcal{S}$ must satisfy $|H| \leq q^{6n+2}$ by Lemma 3.1.4. Consequently we obtain

$$|M| \geq q^{n^2-4n+9} > q^{6n+2} \geq |H| \qquad \text{by Lemma 3.4.2 and since } n \geq 10.$$

□

Lemma 3.4.10. *Let G be $\mathrm{SU}_n(q)$ for $n \geq 13$, let $M \leq G$ be a P_2 subgroup and let $H \leq G$ be a maximal subgroup that is not of type P_1 or $\mathrm{GU}_1(q) \perp \mathrm{GU}_{n-1}(q)$. Then $|M| \geq |H|$.*

Proof. If H lies in $\mathcal{C}_1$ then Lemmas 3.4.4, 3.4.6 and 3.4.7 show the result. In further detail, if H is of P_i type, then Lemmas 3.4.4 and 3.4.6 cover that case, depending on whether i is $n/2$ or not. Finally if H is of $\mathrm{GU}_k(q) \perp \mathrm{GU}_{n-k}(q)$ type then Lemma 3.4.7 shows that the order of H is smaller than a that of a subgroup of type P_k, and therefore $|H| \leq |M|$ also.

If H lies in the classes $\mathcal{C}_i$ for $2 \leq i \leq 8$ then the result follows from Lemma 3.4.8. Note that by Theorem 3.4.1 the class $\mathcal{C}_8$ is empty. Finally, if H belongs to $\mathcal{S}$ then Lemma 3.4.9 gives us the result. □

Chapter 4

Calculating α for simple classical groups

The aim of this section is to calculate the values for α such that the inequality

$$1 - \alpha/m(G) \leq P_2(G)$$

holds for all simple groups $G = \mathrm{PSL}_n(q), \mathrm{PSp}_n(q)$ and $\mathrm{PSU}_n(q)$. More specifically we will prove the following theorem.

Theorem 4.0.1. *Let $n \geq 2$ and let q be a prime power. Furthermore let $G = \mathrm{PSL}_n(q), \mathrm{PSp}_n(q)$ or $\mathrm{PSU}_n(q)$. Then*

$$1 - (1199/243)/m(G) \leq P_2(G).$$

Furthermore

- *Let $q \geq 4$. If $G = \mathrm{PSL}_2(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{38}{15} \approx 2.534$. Equality only occurs when $q = 11$.*

- *Let $n \geq 3$, where $(n,q) \neq (3,2)$. If $G = \mathrm{PSL}_n(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{57}{20} = 2.85$. Equality only occurs when $(n,q) = (3,4)$.*

- *Let $n \geq 4$ be even, and $(n,q) \neq (4,2)$. If $G = \mathrm{PSp}_n(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{6067}{1440} \approx 4.214$. Equality only occurs when $(n,q) = (4,4)$.*

- *Let $n \geq 3$, and $(n,q) \neq (3,2)$. If $G = \mathrm{PSU}_n(q)$ then $1 - c/m(G) \leq P_2(G)$ where $c = \frac{1199}{243} \approx 4.935$. Equality only occurs when $(n,q) = (4,3)$.*

All decimals above are rounded up values to three decimal places.

The proof will be spread throughout the whole chapter. The cases for large dimension, i.e. $n \geq 12$ for $\mathrm{PSL}_n(q)$ and $\mathrm{PSp}_n(q)$ and $n \geq 14$ for $\mathrm{PSU}_n(q)$, are covered in Section 4.2. The remaining small cases are covered in Section 4.3.

4.1 Bounds on the number of maximal subgroups

We may bound the number of conjugacy classes of maximal subgroups of an almost simple classical group G via the following.

Theorem 4.1.1 ([20, Theorem 1.1])**.** *Let G be a finite almost simple group with socle G_0 a classical group of dimension n over the field $\mathbb{F}_q$. Let $max(G)$ denote the number of conjugacy classes of maximal subgroups of G not containing the socle. Then*

$$max(G) < 2n^{5.2} + n\log_2\log_2(q).$$

We also may estimate the number of conjugacy classes of maximal subgroups of the corresponding quasisimple group to G. There is in fact a one to one correspondence in this case. By Lemma 2.2.6 we obtain the following.

Corollary 4.1.2. *Let G be a quasisimple group, where $G/Z(G)$ is a classical simple group of dimension n over the field $\mathbb{F}_q$. Let $max(G)$ denote the number of conjugacy classes of maximal subgroups of G . Then*

$$max(G) < 2n^{5.2} + n\log_2\log_2(q).$$

We also obtain the following corollary.

Corollary 4.1.3. *Let G be a quasisimple group where $G/Z(G)$ is a classical group of dimension n over the field $\mathbb{F}_q$. Let $max_{\mathcal{S}}(G)$ denote the number of conjugacy classes of maximal subgroups $M \leq G$ which also lie in the class $\mathcal{S}$. Then*

$$max_{\mathcal{S}}(G) \leq max(G) < q^{5.2\log_2(n)+1.68}.$$

Proof.

$$\begin{aligned}
max_{\mathcal{S}}(G) &\leq max(G) \leq 2n^{5.2} + n\log_2\log_2(q) \\
&\leq 2n^{5.2} + n(q^{0.55})^{0.55} = 2n^{5.2} + nq^{0.3025} && \text{by Lemma 2.8.4} \\
&\leq q^{5.2\log_2(n)+1} + q^{\log_2(n)+0.3025} && \text{since } q \geq 2 \\
&\leq q^{5.2\log_2(n)+1} + 0.6q^{\log_2(n)+1} \\
&\leq 1.6q^{5.2\log_2(n)+1} \leq q^{5.2\log_2(n)+1+\log_2(1.6)} = q^{5.2\log_2(n)+1.68}.
\end{aligned}$$

□

At certain points we also require sharper bounds on the number of conjugacy classes of specific types of maximal subgroups.

Lemma 4.1.4 ([20, Table 16])**.** *Let G be a finite almost simple group with socle G_0 a classical group of dimension n over the field $\mathbb{F}_q$. Then the number of conjugacy classes of maximal $\mathcal{C}_5$ subgroups of G is bounded above by $n(\log_2\log_2(q)+1)$. The number of conjugacy classes of maximal subgroups of G belonging to other geometric families is bounded above by $\frac{3}{2}n^2 + 8n\log_2(n) + 8n$.*

Lemma 4.1.5. *Let $n \geq 10$ and let q be a prime power. Then*

$$5.1q^{0.3025}n^2 > n(\log_2\log_2(q)+1) + \frac{3}{2}n^2 + 8n\log_2(n) + 8n.$$

Proof. We have

$$n(\log_2\log_2(q)+1) + \frac{3}{2}n^2 + 8n\log_2(n) + 8n$$

$$
\begin{aligned}
&\leq n(q^{0.55^2}+9)+\frac{3}{2}n^2+8n^{1.55} && \text{by Lemma 2.8.4}\\
&\leq 10n(q^{0.55^2})+\frac{3}{2}n^2+8n^{1.55}\\
&\leq (q^{0.55^2}+\frac{3}{2}+\frac{8}{10^{0.45}})n^2\\
&\leq (q^{0.55^2}+5)n^2\\
&\leq (q^{0.55^2}+\frac{5q^{0.55^2}}{2^{0.55^2}})n^2 && \text{as } q\geq 2\\
&\leq (q^{0.55^2}+4.1q^{0.55^2})n^2 = 5.1q^{0.55^2}n^2 \leq 5.1q^{0.3025}n^2.
\end{aligned}
$$

□

Corollary 4.1.6. *Let G be a quasisimple group where $G/Z(G)$ is a classical group of dimension n over the field $\mathbb{F}_q$. Let $max_{\mathcal{C}}(G)$ denote the number of conjugacy classes of maximal geometric subgroups of G .Then $max_{\mathcal{C}}(G) < 5.1q^{0.3025}n^2$.*

Proof.

$$
\begin{aligned}
max_{\mathcal{C}}(G) &= max_{\mathcal{C}}(G/Z(G)) && \text{by Lemma 2.2.6}\\
&\leq n(\log_2\log_2(q)+1)+\frac{3}{2}n^2+8n\log_2(n)+8n && \text{by Lemma 4.1.4}\\
&< 5.1q^{0.3025}n^2 && \text{by Lemma 4.1.5.}
\end{aligned}
$$

□

4.2 Calculating α for large dimension

As noted earlier, Lemma 2.4.10 gives us a way to bound α without having complete knowledge of all the different maximal subgroups of G, instead we only need to know the exact orders of the larger maximal subgroups and an upper bound for the orders of the remaining subgroups. Using this lemma, we calculate α for $G = \mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$ and $\mathrm{PSU}_n(q)$ for sufficiently large n.

4.2.1 Linear groups

In this section we prove Theorem 4.0.1 for the case where $G = \mathrm{PSL}_n(q)$ and $n \geq 13$.

Lemma 4.2.1. *Let $n \geq 13$ and let $q \geq 2$ be a prime power. Then*

$$\frac{q^{5.2\log_2(n)+1.68}q^{3n+1}(q^n-1)}{|\mathrm{SL}_n(q)|(q-1)}+\frac{5.1q^{0.3025}n^2(q^2-1)}{q^{n-1}-1}\leq 17/20.$$

Proof. Let us first notice that

$$\frac{1}{q^{n-1}-1}<\frac{2^{n-2}}{q^{n-2}(2^{n-1}-1)} \qquad \text{since } q\geq 2$$

$$\leq \frac{2^{11}}{q^{n-2}(2^{12}-1)} < \frac{2048}{4095q^{n-2}} < \frac{0.501}{q^{n-2}} \qquad \text{since } n \geq 13.$$

So

$$\begin{aligned}
&\frac{q^{5.2\log_2(n)+1.68}q^{3n+1}(q^n-1)}{|\mathrm{SL}_n(q)|(q-1)} + \frac{5.1q^{0.3025}n^2(q^2-1)}{q^{n-1}-1} \\
&\leq \frac{q^{5.2\log_2(n)+1.68}q^{3n+1}(q^n-1)}{q^{n^2-2}(q-1)} + \frac{5.1q^{0.3025}n^2(q^2-1)}{q^{n-1}-1} && \text{by Lemma 2.6.4} \\
&\leq \frac{q^{5.2\log_2(n)+1.68}q^{4n+1}}{q^{n^2-2}} + \frac{5.1q^{2.3025}n^2}{q^{n-1}-1} \\
&= q^{-n^2+4n+5.2\log_2(n)+4.68} + \frac{5.1q^{2.3025}n^2}{q^{n-1}-1} \\
&< q^{-n^2+4n+5.2\log_2(n)+4.68} + \frac{2.56q^{2.3025}n^2}{q^{n-2}} \\
&\leq q^{-n^2+4n+5.2\log_2(n)+4.68} + 2.56q^{-n+2\log_2(n)+4.3025} && \text{since } n = q^{\log_q(n)} \leq q^{\log_2(n)}.
\end{aligned}$$

We recognize that $-n + 2\log_2(n) + 4.3025$ is negative when $n \geq 12$, and that its derivative $\frac{2}{n\ln(2)} - 1$ is negative for $n \geq 3$. We also notice that $-n^2 + 4n + 5.2\log_2(n) + 4.68$ is negative when $n \geq 7$, and that its derivative $\frac{5.2}{n\ln(2)} - 2n + 4$ is negative for $n \geq 4$.

Therefore if $n \geq 14$

$$\begin{aligned}
&q^{-n^2+4n+5.2\log_2(n)+4.68} + 2.56q^{-n+2\log_2(n)+4.3025} \\
&\leq 2^{-n^2+4n+5.2\log_2(n)+4.68} + 2.56 \times 2^{-n+2\log_2(n)+4.3025} && \text{since } n \geq 12 \text{ and } q \geq 2 \\
&\leq 2^{-14^2+4\times 14+5.2\log_2(14)+4.68} + 2.56 \times 2^{-14+2\log_2(14)+4.3025} \leq \frac{17}{20} && \text{since } n \geq 14.
\end{aligned}$$

If $n = 13$ and $q \geq 3$ then

$$\begin{aligned}
&q^{-n^2+4n+5.2\log_2(n)+4.68} + 2.56q^{-n+2\log_2(n)+4.3025} \\
&\leq 3^{-n^2+4n+5.2\log_2(n)+4.68} + 2.56 \times 3^{-n+2\log_2(n)+4.3025} && \text{since } n \geq 12 \text{ and } q \geq 2 \\
&\leq 3^{-13^2+4\times 13+5.2\log_2(13)+4.68} + 2.56 \times 3^{-13+2\log_2(13)+4.3025} \leq \frac{17}{20} && \text{since } n = 13.
\end{aligned}$$

If $n = 13$ and $q = 2$ then the required inequality is shown to hold via direct substitution. □

Proposition 4.2.2. *Let $S = \mathrm{PSL}_n(q)$ for $n \geq 13$. Then $1 - \frac{57}{20m(S)} < P_2(S)$.*

Proof. Let $G = \mathrm{SL}_n(q)$ and let $H \leq \mathrm{SL}_n(q)$ be a subgroup of type P_2. Let $\mathcal{M}_1$ be the set of $\mathrm{SL}_n(q)$-conjugacy class representatives of maximal subgroups of $\mathrm{SL}_n(q)$ of type P_1, P_{n-1} and the non-geometric subgroups $\mathcal{S}$. By Proposition 3.2.6 $\mathcal{M}_1 \supseteq \mathcal{M}_H$, where $\mathcal{M}_H$ is defined in Definition 2.4.9. We also note that if M is a maximal subgroup of type P_1 then by Proposition 3.2.6 $|G : M| = m(G)$ and Theorem 2.2.7 says that $m(G) = \frac{q^n-1}{q-1}$. It is a straightforward calculation, using Theorems 3.2.1 and 2.5.29, to show

$$|G : H|^{-1} = \frac{(q-1)(q^2-1)}{(q^{n-1}-1)(q^n-1)}.$$

Consequently

$$
\begin{aligned}
P_2(S) &\geq P_2(\mathrm{SL}_n(q)) && \text{by Lemma 2.4.4}\\
&\geq 1 - \sum_{M\in\mathcal{M}_1} |G:M|^{-1} - \sum_{M\in\mathcal{M}/\mathcal{M}_1} |G:H|^{-1} && \text{by Lemma 2.4.10}\\
&\geq 1 - \sum_{M\in\mathcal{M}_1} |G:M|^{-1} - 5.1q^{0.3025}n^2|G:H|^{-1} && \text{by Corollary 4.1.6}\\
&\geq 1 - 2\frac{q-1}{q^n-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}} |G:M|^{-1} - \frac{5.1q^{0.3025}n^2(q-1)(q^2-1)}{(q^{n-1}-1)(q^n-1)}\\
&\geq 1 - 2\frac{q-1}{q^n-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}} \frac{q^{3n+1}}{|\mathrm{SL}_n(q)|} - \frac{5.1q^{0.3025}n^2(q-1)(q^2-1)}{(q^{n-1}-1)(q^n-1)} && \text{by Lemma 3.1.4}\\
&\geq 1 - 2\frac{q-1}{q^n-1} - \frac{q^{5.2\log_2(n)+1.68}q^{3n+1}}{|\mathrm{SL}_n(q)|} - \frac{5.1q^{0.3025}n^2(q-1)(q^2-1)}{(q^{n-1}-1)(q^n-1)} && \text{by Lemma 4.1.3}\\
&\geq 1 - \Big(2 + \frac{q^{5.2\log_2(n)+1.68}q^{3n+1}(q^n-1)}{|\mathrm{SL}_n(q)|(q-1)} + \frac{5.1q^{0.3025}n^2(q^2-1)}{q^{n-1}-1}\Big)/m(G) && \text{by Theorem 2.2.7}\\
&\geq 1 - \frac{2+17/20}{m(G)} = 1 - \frac{57/20}{m(G)} && \text{by Lemma 4.2.1}\\
&= 1 - \frac{57/20}{m(S)} && \text{by Lemma 2.2.8.}
\end{aligned}
$$

□

4.2.2 Symplectic groups

In this section we prove Theorem 4.0.1 for the case of $G = \mathrm{PSp}_n(q)$ and $n \geq 14$.

Lemma 4.2.3. *Let $n \geq 14$ be even, and let $q \geq 3$. Then*

$$q^{-n^2/2+7n/2+5.2\log_2(n)+3.68} + (5.1q^{0.3025}n^2+4)q^{-n^2/2+5n/2-4}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| \leq 1747/1440.$$

Proof. Let us bound from above the first term. For the function $f(n) = -n^2/2 + 7n/2 + 5.2\log_2(n) + 3.68$ we have that $f'(n) = -n + 7/2 + \frac{5.2}{\ln(2)n}$. We note that $f'(n)$ is negative for $n \geq 6$. In addition since $f(13) < 0$, we have that $f(n) < 0$ for $n \geq 13$. Consequently

$$
\begin{aligned}
q^{-n^2/2+7n/2+5.2\log_2(n)+3.68} &\leq 3^{-n^2/2+7n/2+5.2\log_2(n)+3.68}\\
&\leq 3^{-14^2/2+7\times 14/2+5.2\log_2(14)+3.68} < 0.01.
\end{aligned}
$$

We now look at the second term,

$$
\begin{aligned}
&(5.1q^{0.3025}n^2+4)q^{-n^2/2+5n/2-4}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)|\\
&\leq (5.1q^{0.3025}n^2+4)q^{-n^2/2+5n/2}|\mathrm{Sp}_{n-4}(q)| && \text{by Lemma 2.6.4}\\
&\leq (5.1q^{0.3025}n^2+4)q^{-n^2/2+5n/2}q^{((n-4)^2+(n-4))/2} && \text{by Lemma 2.6.4}\\
&\leq (5.1q^{0.3025}n^2+4)q^{-n+6} = 5.1q^{-n+6.3025}n^2 + 4q^{-n+6}
\end{aligned}
$$

$$\leq 5.1q^{-n+2\log_3(n)+6.3025} + 4q^{-n+6} \qquad \text{since } q \geq 3.$$

We notice that function $f(n) = -n + 2\log_3(n) + 6.3025$ has derivative $f'(n) = \frac{2}{n\ln(3)} - 1$. Note that $f'(n) < 0$ for $n \geq 2$ and that $f(11) < 0$, therefore $f(n) < 0$ for $n \geq 11$. So

$$\begin{aligned} 5.1q^{-n+2\log_3(n)+6.3025} + 4q^{-n+6} &\leq 5.1 \times 3^{-n+2\log_3(n)+6.3025} + 4 \times 3^{-n+6} \\ &\leq 5.1 \times 3^{-14+2\log_3(14)+6.3025} + 4 \times 3^{-14+6} \qquad \text{since } n \geq 14 \\ &< 0.22. \end{aligned}$$

Therefore

$$q^{-n^2/2+7n/2+5.2\log_2(n)+3.68} + (5.1q^{0.3025}n^2 + 4)q^{-n^2/2+5n/2-4}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)| \leq 0.23 < \frac{1747}{1440}.$$

□

Proposition 4.2.4. *Let $n \geq 14$ be even, let $q \geq 3$, and let $S = \mathrm{PSp}_n(q)$. Then $1 - \frac{6067}{1440m(S)} < P_2(S)$.*

Proof. Let $G = \mathrm{Sp}_n(q)$, let $H \leq \mathrm{Sp}_n(q)$ be a subgroup of type P_2, let $\mathcal{M}_1$ be the set of conjugacy class representatives of maximal subgroups of type P_1, $\mathrm{GO}_n^{\pm}(q)$, and the non-geometric subgroups $\mathcal{S}\backslash\mathcal{E}_G$ where $\mathcal{E}_G$ is defined in Definition 3.1.1. By Lemma 3.3.10 $\mathcal{M}_1 \supseteq \mathcal{M}_H$, where $\mathcal{M}_H$ is defined in Definition 2.4.9. In addition, by Lemma 3.1.2 and Corollary 4.1.6 we have that $|\mathcal{M}\backslash\mathcal{M}_1| \leq 5.1 \times q^{0.3025}n^2 + 6$. We note that $|H| = q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)|$, as written in Theorem 3.3.1.

Let M_{max} be a P_1 subgroup, then M_{max} has shape $q^{n-1} : (\mathrm{GL}_1(q) \times \mathrm{Sp}_{n-2}(q))$ by Theorem 3.3.1. We also note here that since G is transitive on 1-spaces, then the $|G : M_{max}|$ is the number of 1-spaces, i.e. $\frac{q^{2n}-1}{q-1}$ We also note that $|G : M_{max}| = m(G)$ by Theorem 2.2.7. Furthermore by Lemma 2.6.4 we have $|M_{max}| \geq q^{n-1}q^{\frac{(n-2)^2+(n-2)}{2}-1} \geq q^{n^2/2-n/2-1}$.

$$\begin{aligned} P_2(S) &\geq P_2(\mathrm{Sp}_n(q)) && \text{by Lemma 2.4.4} \\ &\geq 1 - \sum_{M\in\mathcal{M}_1} |G : M|^{-1} - \sum_{M\in\mathcal{M}/\mathcal{M}_1} |G : H|^{-1} && \text{by Lemma 2.4.10} \\ &\geq 1 - \sum_{M\in\mathcal{M}_1} |G : M|^{-1} - (5.1q^{0.3025}n^2 + 4)|G : H|^{-1} \\ &= 1 - \sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} |G : M|^{-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}} |G : M|^{-1} - (5.1q^{0.3025}n^2 + 4)|G : H|^{-1} \\ &\geq 1 - \sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} |G : M|^{-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}} \frac{q^{3n+1}}{|G|} - (5.1q^{0.3025}n^2 + 4)|G : H|^{-1} && \text{by Lemma 3.1.4} \\ &\geq 1 - \sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} |G : M|^{-1} - \frac{q^{3n+5.2\log_2(n)+2.68}}{|G|} - (5.1q^{0.3025}n^2 + 4)|G : H|^{-1} && \text{by Lemma 4.1.3} \\ &= 1 - \Big(\sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} \frac{|M|}{|M_{max}|} + \frac{q^{3n+5.2\log_2(n)+2.68}}{|M_{max}|} + \frac{(5.1q^{0.3025}n^2 + 4)|H|}{|M_{max}|}\Big)/m(G) \end{aligned}$$

$$\geq 1 - \Big(3 + \frac{q^{3n+5.2\log_2(n)+2.68}}{q^{n^2/2-n/2-1}} + \frac{(5.1q^{0.3025}n^2+4)|H|}{q^{n^2/2-n/2-1}}\Big)/m(G).$$

We note that $|H| = q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)|$, as written in Theorem 3.3.1. So

$$\begin{aligned}
P_2(S) &\geq 1 - \Big(3 + \frac{q^{3n+5.2\log_2(n)+2.68}}{q^{n^2/2-n/2-1}} + \frac{(5.1q^{0.3025}n^2+4)q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)|}{q^{n^2/2-n/2-1}}\Big)/m(G) \\
&= 3 + q^{-n^2/2+7n/2+5.2\log_2(n)+3.68} + (5.1q^{0.3025}n^2+4)q^{-n^2/2+5n/2-4}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)|.
\end{aligned}$$

Finally by Lemmas 4.2.3 and 2.2.8 we have that

$$P(S) \leq 1 - \frac{6067}{1440m(G)} = 1 - \frac{6067}{1440m(S)}.$$

□

We now deal with the case $q = 2$ with $n \geq 14$ separately.

Lemma 4.2.5. *Let $n \geq 14$. Then*

$$\frac{2^{3n+5.2\log_2(n)+2.68}}{|\mathrm{GO}_n^-(2)|} + \frac{(6.3n^2+4)2^{2n-5}|\mathrm{GL}_2(2)||\mathrm{Sp}_{n-4}(2)|}{|\mathrm{GO}_n^-(2)|} \leq 1747/1440.$$

Proof. Firstly we bound the first fraction,

$$\begin{aligned}
\frac{2^{3n+5.2\log_2(n)+2.68}}{|\mathrm{GO}_n^-(2)|} &\leq \frac{2^{3n+5.2\log_2(n)+2.68}}{2^{(n^2-n)/2-1}} && \text{by Lemma 2.6.4} \\
&= 2^{-n^2/2+7n/2+5.2\log_2(n)+3.68}.
\end{aligned}$$

For the function $f(n) = -n^2/2+7n/2+5.2\log_2(n)+3.68$ we have that $f'(n) = -n+7/2+\frac{5.2}{\ln(2)n}$. We note that $f'(n)$ is negative for $n \geq 6$. Consequently

$$2^{-n^2/2+7n/2+5.2\log_2(n)+3.68} \leq 2^{-14^2/2+7\times 14/2+5.2\log_2(14)+3.68} < 0.001 \qquad \text{since } n \geq 14.$$

Finally let us bound the second fraction,

$$\begin{aligned}
&\frac{(6.3n^2+4)2^{2n-5}|\mathrm{GL}_2(2)||\mathrm{Sp}_{n-4}(2)|}{|\mathrm{GO}_n^-(2)|} \\
&< \frac{6(6.3n^2+4)2^{2n-5}|\mathrm{Sp}_{n-4}(2)|}{|\mathrm{GO}_n^-(2)|} \\
&= \frac{2^{\log_2(37.8n^2+24)+2n-5}|\mathrm{Sp}_{n-4}(2)|}{|\mathrm{GO}_n^-(2)|} \\
&\leq \frac{2^{\log_2(37.8n^2+24)+2n-5}2^{(n-4)^2/4}\prod_{i=1}^{(n-4)/2}(2^{2i}-1)}{2^{n(n-2)/4+1}(2^{n/2}+1)\prod_{i=1}^{(n-2)/2}(2^{2i}-1)} && \text{by Theorem 2.5.29} \\
&= \frac{2^{\log_2(37.8n^2+24)+2n-5}2^{(n-4)^2/4}}{2^{n(n-2)/4+1}(2^{n/2}+1)(2^{n-2}-1)} = \frac{2^{\log_2(37.8n^2+24)+n/2-2}}{(2^{n/2}+1)(2^{n-2}-1)}
\end{aligned}$$

$$\leq \frac{2^{\log_2(37.8n^2+24)-2}}{(2^{n-2}-1)} \leq \frac{2^{\log_2(37.8n^2+24)-2}}{2^{n-3}} = 2^{-n+\log_2(37.8n^2+24)+1}.$$

We note that the function $f(n) = -n+\log_2(37.8n^2+24)+1$ has derivative $f'(n) = \frac{75.6n}{\ln(2)(37.8n^2+24)} - 1$. Here $f'(n) < 0$ for $n \geq 3$. Furthermore, $f(14)$ is negative therefore $f(n) < 0$ for $n \geq 14$. So

$$-n + \log_2(37.8n^2 + 24) + 1 \leq 0.$$

Therefore

$$\frac{2^{3n+5.2\log_2(n)+2.68}}{|\mathrm{GO}_n^-(2)|} + \frac{(6.3n^2+4)2^{2n-5}|\mathrm{GL}_2(2)||\mathrm{Sp}_{n-4}(2)|}{|\mathrm{GO}_n^-(2)|} \leq 0.001 + 2^0 < 1747/1440.$$

□

Proposition 4.2.6. *Let $S = \mathrm{PSp}_n(2)$ for even $n \geq 14$. Then $1 - \frac{6067}{1440m(S)} < P_2(S)$.*

Proof. Let $G = \mathrm{Sp}_n(2)$, let $H \leq \mathrm{Sp}_n(2)$ be a subgroup of type P_2, let $\mathcal{M}_1$ be the set of conjugacy class representatives of maximal subgroups of type P_1, $\mathrm{GO}_n^{\pm}(2)$, and the non-geometric subgroups $\mathcal{S}\backslash\mathcal{E}_G$. By Lemma 3.3.10 $\mathcal{M}_1 \supseteq \mathcal{M}_H$. In addition, by Lemma 3.1.2 and Corollary 4.1.6, and taking $q = 2$, we have that $|\mathcal{M}\backslash\mathcal{M}_1| \leq 5.1 \times 2^{0.3025}n^2 + 6 < 6.3n^2 + 6$.

We note that

$$\begin{aligned}
|G : \mathrm{GO}_n^-(2)| &= \frac{|\mathrm{Sp}_n(2)|}{|\mathrm{GO}_n^-(2)|} \\
&= \frac{2^{n^2/4}\prod_{i=1}^{n/2}(2^{2i}-1)}{2 \times 2^{n(n-2)/4}(2^{n/2}+1)\prod_{i=1}^{n/2-1}(2^{2i}-1)} && \text{by Theorem 2.5.29} \\
&= \frac{2^{n^2/4}(2^n-1)}{2^{n(n-2)/4+1}(2^{n/2}+1)} = \frac{2^{n^2/4}(2^{n/2}-1)}{2^{n(n-2)/4+1}} \\
&= 2^{n/2-1}(2^{n/2}-1).
\end{aligned}$$

Furthermore note that $|G : \mathrm{GO}_n^-(2)| = m(G)$ by Theorem 2.2.7.

$$\begin{aligned}
P_2(S) &\geq P_2(\mathrm{Sp}_n(2)) && \text{by Lemma 2.4.4} \\
&\geq 1 - \sum_{M\in\mathcal{M}_1}|G:M|^{-1} - \sum_{M\in\mathcal{M}/\mathcal{M}_1}|G:H|^{-1} && \text{by Lemma 2.4.10} \\
&\geq 1 - \sum_{M\in\mathcal{M}_1}|G:M|^{-1} - (6.3n^2+4)|G:H|^{-1} \\
&= 1 - \sum_{M\in\mathcal{M}_1\backslash\mathcal{S}}|G:M|^{-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}}|G:M|^{-1} - (6.3n^2+4)|G:H|^{-1} \\
&\geq 1 - \sum_{M\in\mathcal{M}_1\backslash\mathcal{S}}|G:M|^{-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}}\frac{2^{3n+1}}{|G|} - (6.3n^2+4)|G:H|^{-1} && \text{by Lemma 3.1.4}
\end{aligned}$$

$$\geq 1 - \sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} |G:M|^{-1} - \frac{2^{5.2\log_2(n)+1.68}2^{3n+1}}{|G|} - (6.3n^2+4)|G:H|^{-1} \qquad \text{by Lemma 4.1.3}$$

$$= 1 - \Big(\sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} \frac{|M|}{|\mathrm{GO}_n^-(2)|} + \frac{2^{5.2\log_2(n)+1.68}2^{3n+1}}{|\mathrm{GO}_n^-(2)|} + \frac{(6.3n^2+4)|H|}{|\mathrm{GO}_n^-(2)|}\Big)/m(G).$$

Since $|G:\mathrm{GO}_n^-(2)| = m(G)$ we know that all other maximal subgroups M have smaller order than $\mathrm{GO}_n^-(2)$. Consequently,

$$\sum_{M\in\mathcal{M}_1\backslash\mathcal{S}} \frac{|M|}{|\mathrm{GO}_n^-(2)|} \leq |\mathcal{M}_1\backslash\mathcal{S}| \leq 3.$$

Finally since H is a P_2 type subgroup $|H| = q^{2n-5}|\mathrm{GL}_2(q)||\mathrm{Sp}_{n-4}(q)|$ by Theorem 3.3.1. So

$$P(S) \leq 1 - \Big(3 + \frac{2^{3n+5.2\log_2(n)+2.68}}{|\mathrm{GO}_n^-(2)|} + \frac{(6.3n^2+6)2^{2n-5}|\mathrm{GL}_2(2)||\mathrm{Sp}_{n-4}(2)|}{|\mathrm{GO}_n^-(2)|}\Big)/m(G).$$

Finally by Lemmas 4.2.5 and 2.2.8 we have that

$$P(G) \leq 1 - \frac{6067}{1440m(G)} = 1 - \frac{6067}{1440m(S)}.$$

□

4.2.3 Unitary groups

In this section we prove Theorem 4.0.1 for the case of $G = \mathrm{PSU}_n(q)$ and $n \geq 11$.

Proposition 4.2.7. *Let $S = \mathrm{PSU}_n(q)$ for $n \geq 11$. If n is even then assume that $q \neq 2$. Then* $1 - \frac{1199}{243m(S)} < P_2(S)$

Proof. Let $G = \mathrm{SU}_n(q)$, let $H \leq \mathrm{SU}_n(q)$ be a subgroup of type P_2, and let $\mathcal{M}_1$ be the set of conjugacy class representatives of maximal subgroups of type P_1, $\mathrm{GU}_1(q) \perp \mathrm{GU}_{n-1}(q)$, and the non-geometric subgroups $\mathcal{S}$. By Lemma 3.4.10 $\mathcal{M}_1 \supseteq \mathcal{M}_H$, where $\mathcal{M}_H$ is defined in Definition 2.4.9.

Let M_{max} be a P_1 subgroup. Then M_{max} has shape $q^{2n-3} : (\mathrm{SL}_1(q^2) \times \mathrm{SU}_{n-2}(q)).(q^2-1)$ by Theorem 3.4.1. We also note here that

$$\begin{aligned}
|G:M_{max}| &= \frac{|\mathrm{SU}_n(q)|}{(q^2-1)q^{2n-3}|\mathrm{SU}_{n-2}(q)|} \\
&= \frac{q^{(n)(n-1)/2}\prod_{i=2}^{n}(q^i-(-1)^i)}{(q^2-1)q^{2n-3}q^{(n-2)(n-3)/2}\prod_{i=2}^{n-2}(q^i-(-1)^i)} \qquad \text{by Theorem 2.5.29} \\
&= \frac{\prod_{i=2}^{n}(q^i-(-1)^i)}{(q^2-1)\prod_{i=2}^{n-2}(q^i-(-1)^i)} \\
&= \frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{(q^2-1)}.
\end{aligned}$$

Note that $|G : M_{max}| = m(G)$ by Theorem 2.2.7.

$$\begin{aligned}
P_2(S) &\geq P_2(\mathrm{SU}_n(q)) && \text{by Lemma 2.4.4}\\
&\geq 1 - \sum_{M\in\mathcal{M}_1} |G:M|^{-1} - \sum_{M\in\mathcal{M}/\mathcal{M}_1} |G:H|^{-1} && \text{by Lemma 2.4.10}\\
&\geq 1 - \sum_{M\in\mathcal{M}_1} |G:M|^{-1} - (5.1q^{0.3025}n^2)|G:H|^{-1} && \text{by Corollary 4.1.6}\\
&= 1 - \sum_{M\in\mathcal{M}_1\setminus\mathcal{S}} |G:M|^{-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}} |G:M|^{-1} - (5.1q^{0.3025}n^2)|G:H|^{-1}\\
&\geq 1 - \sum_{M\in\mathcal{M}_1\setminus\mathcal{S}} |G:M|^{-1} - \sum_{M\in\mathcal{M}_1\cap\mathcal{S}} \frac{q^{6n+2}}{|G|} - (5.1q^{0.3025}n^2)|G:H|^{-1} && \text{by Lemma 3.1.4}\\
&\geq 1 - \sum_{M\in\mathcal{M}_1\setminus\mathcal{S}} |G:M|^{-1} - \frac{q^{6n+5.2\log_2(n)+3.68}}{|G|} - (5.1q^{0.3025}n^2)|G:H|^{-1} && \text{by Lemma 4.1.3}\\
&= 1 - \Big(\sum_{M\in\mathcal{M}_1\setminus\mathcal{S}} \frac{|M|}{|M_{max}|} + \frac{q^{6n+5.2\log_2(n)+3.68}}{|M_{max}|} + \frac{(5.1q^{0.3025}n^2)|H|}{|M_{max}|}\Big)/m(G)\\
&= 1 - \Big(2 + \frac{q^{6n+5.2\log_2(n)+3.68}}{|M_{max}|} + \frac{(5.1q^{0.3025}n^2)|H|}{|M_{max}|}\Big)/m(G).
\end{aligned}$$

By Theorem 2.2.7, the group H has order $q^{4n-12}|\mathrm{SU}_2(q^2)||\mathrm{SU}_{n-4}(q)|(q^2-1)$. As noted before, M_{max} has order $q^{2n-3}|\mathrm{SU}_{n-2}(q)|(q^2-1)$. Therefore

$$\begin{aligned}
|M_{max}| &= q^{2n-3}|\mathrm{SU}_{n-2}(q)|(q^2-1)\\
&\geq q^{2n-3}q^{(n-2)^2-1}(q^2-1) && \text{by Lemma 2.6.4}\\
&= q^{n^2-2n}(q^2-1) > q^{n^2-2n}.
\end{aligned}$$

In addition

$$\begin{aligned}
\frac{|H|}{|M_{max}|} &= \frac{q^{4n-12}|\mathrm{SU}_2(q^2)||\mathrm{SU}_{n-4}(q)|(q^2-1)}{q^{2n-3}|\mathrm{SU}_{n-2}(q)|(q^2-1)}\\
&\leq \frac{q^{4n-12}|\mathrm{SU}_2(q^2)||\mathrm{SU}_{n-4}(q)|(q^2-1)}{q^{n^2-2n}}\\
&\leq \frac{q^{4n-12}q^6q^{(n-4)^2-1}q^2}{q^{n^2-2n}} && \text{by Lemma 2.6.4}\\
&= q^{-2n+11}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
P_2(S) &\geq 1 - (2 + q^{6n+5.2\log_2(n)+3.68}q^{-n^2+2n} + 5.1q^{0.3025}n^2q^{-2n+11})/m(G)\\
&\geq 1 - (2 + q^{-n^2+8n+5.2\log_2(n)+3.68} + q^{-2n+2\log_2(n)+11.3025+\log_2(5.1)})/m(G)\\
&> 1 - (2 + q^{-n^2+8n+5.2\log_2(n)+3.68} + q^{-2n+2\log_2(n)+13.66})/m(G).
\end{aligned}$$

Finally, we note that the derivative of $f(n) = -n^2 + 8n + 5.2\log_2(n) + 3.68$ is $f'(n) = -2n + \frac{5.2}{\ln(2)n} + 8$. Since $f'(n) < 0$ for $n \geq 5$ and that $f(11) < 0$, we conclude that $f(n)$ is negative for $n \geq 11$. Similarly, one can deduce that $-2n + 2\log_2(n) + 12.66 < 0$ for $n \geq 11$ also. Therefore, by Lemma 2.2.8

$$P_2(S) \geq 1 - (2 + q^0 + q^0)/m(G) \geq 1 - \frac{1199}{243m(G)} = 1 - \frac{1199}{243m(S)}.$$

□

Proposition 4.2.8. *Let $S = \mathrm{PSU}_n(2)$ for $n \geq 12$ even. Then $1 - 1199/(243m(S)) < P_2(S)$*

Proof. For $n = 12$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_{12}(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.9. Note that a complete list of the maximal subgroups of $\mathrm{PSU}_{12}(2)$ may be found in [6, Tables 4.78 & 4.79], both geometric and non-geometric. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $n \geq 14$, let $G = \mathrm{SU}_n(2)$, let $H \leq \mathrm{SU}_n(2)$ be a subgroup of type P_2, let $\mathcal{M}_1$ be the set of conjugacy class representatives of maximal subgroups of type P_1, $\mathrm{GU}_1(2) \perp \mathrm{GU}_{n-1}(2)$, and the non-geometric subgroups $\mathcal{S}$. By Lemma 3.4.10 $\mathcal{M}_1 \supseteq \mathcal{M}_H$.

Let M_{max} be a $\mathrm{GU}_1(2) \perp \mathrm{GU}_{n-1}(2)$ subgroup. Then M_{max} has shape $(\mathrm{SU}_1(2) \times \mathrm{SU}_{n-1}(2)).(2+1)$ by Theorem 3.4.1. We also note here that

$$\begin{aligned} |G : M_{max}| &= \frac{|\mathrm{SU}_n(2)|}{3|\mathrm{SU}_{n-1}(2)|} \\ &= \frac{2^{n(n-1)/2}\prod_{i=1}^{n}(2^i - (-1)^i)}{3 \times 2^{(n-1)(n-2)/2}\prod_{i=1}^{n-1}(2^i - (-1)^i)} && \text{by Theorem 2.5.29} \\ &= \frac{2^{n-1}\prod_{i=1}^{n}(2^i - (-1)^i)}{\prod_{i=1}^{n-1}(2^i - (-1)^i)} \\ &= \frac{2^{n-1}(2^n - 1)}{3}. \end{aligned}$$

Note that $|G : M_{max}| = m(G)$ by Theorem 2.2.7.

$$\begin{aligned} P_2(S) &\geq P_2(\mathrm{SU}_n(2)) && \text{by Lemma 2.4.4} \\ &\geq 1 - \sum_{M \in \mathcal{M}_1} |G : M|^{-1} - \sum_{M \in \mathcal{M}/\mathcal{M}_1} |G : H|^{-1} && \text{by Lemma 2.4.10} \\ &\geq 1 - \sum_{M \in \mathcal{M}_1} |G : M|^{-1} - (5.1 \times 2^{0.3025}n^2)|G : H|^{-1} && \text{by Corollary 4.1.6} \\ &\geq 1 - \sum_{M \in \mathcal{M}_1 \setminus \mathcal{S}} |G : M|^{-1} - \sum_{M \in \mathcal{M}_1 \cap \mathcal{S}} |G : M|^{-1} - 6.29n^2|G : H|^{-1} \end{aligned}$$

$$\geq 1 - \sum_{M \in \mathcal{M}_1 \setminus \mathcal{S}} |G:M|^{-1} - \sum_{M \in \mathcal{M}_1 \cap \mathcal{S}} \frac{2^{6n+2}}{|G|} - 6.29n^2|G:H|^{-1} \qquad \text{by Lemma 3.1.4}$$

$$\geq 1 - \sum_{M \in \mathcal{M}_1 \setminus \mathcal{S}} |G:M|^{-1} - \frac{2^{6n+5.2\log_2(n)+3.68}}{|G|} - 6.29n^2|G:H|^{-1} \qquad \text{by Lemma 4.1.3}$$

$$= 1 - \Big(\sum_{M \in \mathcal{M}_1 \setminus \mathcal{S}} \frac{|M|}{|M_{max}|} + \frac{2^{6n+5.2\log_2(n)+3.68}}{|M_{max}|} + \frac{6.29n^2|H|}{|M_{max}|} \Big)/m(G)$$

$$= 1 - \Big(2 + \frac{2^{6n+5.2\log_2(n)+3.68}}{|M_{max}|} + \frac{6.29n^2|H|}{|M_{max}|} \Big)/m(G).$$

By Theorem 2.2.7, the group H has order $3 \times 2^{4n-12}|\mathrm{SU}_2(4)||\mathrm{SU}_{n-4}(2)|$. As noted before M_{max} has order $3|\mathrm{SU}_{n-1}(2)|$. Therefore

$$|M_{max}| = 3|\mathrm{SU}_{n-1}(2)| \geq 3 \times 2^{(n-1)^2-3} = 3 \times 2^{n^2-2n-2}$$

Also

$$\begin{aligned} \frac{|H|}{|M_{max}|} &= \frac{3 \times 2^{4n-12}|\mathrm{SU}_2(4)||\mathrm{SU}_{n-4}(2)|}{3|\mathrm{SU}_{n-1}(2)|} \\ &= \frac{180 \times 2^{4n-12}|\mathrm{SU}_{n-4}(2)|}{3 \times 2^{n^2-2n-2}} \\ &\leq \frac{60 \times 2^{4n-12}2^{(n-4)^2-1}}{2^{n^2-2n-2}} = 60 \times 2^{-2n+5} \qquad \text{by Lemma 2.6.4.} \end{aligned}$$

Therefore, we have that

$$\begin{aligned} P_2(S) &\geq 1 - (2 + \frac{2^{7n+5.2\log_2(n)+3.68}2^{-n^2+2n+2}}{3} + 377.4n^2 2^{-2n+5})/m(G) \\ &= 1 - (2 + \frac{2^{-n^2+9n+5.2\log_2(n)+5.68}}{3} + 2^{-2n+5+2\log_2(n)+\log_2(377.4)})/m(G) \end{aligned}$$

Finally, we note that $f(n) = -n^2+9n+5.2\log_2(n)+5.68$ has derivative $f'(n) = -2n+9+\frac{5.2}{n\ln(2)}$. Here $f'(n) < 0$ for $n \geq 6$, and furthermore $f(11) < 0$. Therefore $f(n) < 0$ for $n \geq 11$. A similar approach shows us that $-2n+5+2\log_2(n)+\log_2(377.4)$ is negative for $n \geq 11$. So we conclude, by Lemma 2.2.8, that

$$P_2(S) \geq 1 - (2 + \frac{2^0}{3} + 2^0)/m(G) > 1 - \frac{1199}{243m(G)} = 1 - \frac{1199}{243m(S)}.$$

□

We conclude this section noting that we have shown Theorem 4.0.1 to be true for the cases of $\mathrm{PSL}_n(q)$ for $n \geq 13$, $\mathrm{PSp}_n(q)$ for even $n \geq 14$ and $\mathrm{PSU}_n(q)$ for $n \geq 11$.

4.3 Calculating α for small dimensions

In this section we prove Theorem 4.0.1 for the cases of $\mathrm{PSL}_n(q)$ for $n \leq 12$, $\mathrm{PSp}_n(q)$ for even $n \leq 12$ and $\mathrm{PSU}_n(q)$ for $n \leq 12$.

Since we want to find values for α such that

$$1 - \alpha/m(G) \leq P_2(G)$$

for all $G = \mathrm{PSL}_n(q), \mathrm{PSp}_n(q)$ and $\mathrm{PSU}_n(q)$, it would be valuable to calculate $(1 - P_2(G))m(G)$ for small G. Recall the definition of c_G from Definition 2.4.1. We notice that a requirement of α is that it must satisfy $c_G \leq \alpha$ for all G. Also recall from Lemma 2.4.2 that $1 - c_G/m(G) = P_2(G)$.

Theorem 4.3.1. *Let G satisfy one of the following cases*

1. $G = \mathrm{PSL}_2(q)$ *for* $5 \leq q \leq 25$.
2. $G = \mathrm{PSL}_3(q)$ *for* $3 \leq q \leq 8$.
3. $G = \mathrm{PSL}_4(q)$ *for* $q = 2, 3$.
4. $G = \mathrm{PSL}_5(2)$.
5. $G = \mathrm{PSp}_4(4)$.
6. $G = \mathrm{PSp}_6(2)$.
7. $G = \mathrm{PSU}_3(q)$ *for* $3 \leq q \leq 5$.
8. $G = \mathrm{PSU}_4(q)$ *for* $q = 2, 3$.
9. $G = \mathrm{PSU}_5(2)$.

Then $P_2(G)$, $m(G)$, c_G and c_G rounded up to 3 d.p may be found in Tables 4.1 and 4.2.

Proof. From the work of N. Menezes [38, Tables 3.1 & 3.2 & 5.1] we obtain $P_2(G)$ for various groups in Tables 4.1 and 4.2: the groups $\mathrm{PSL}_2(q)$ for $4 \leq q \leq 19$, $\mathrm{PSL}_3(q)$ for $q = 2$ and 3, $\mathrm{PSp}_6(2)$, $\mathrm{PSU}_3(3)$ and $\mathrm{PSU}_4(2)$. The remaining results are obtained from Magma, by computing the value of $P_2(G)$. We then calculate c_G using the values of $m(G)$ obtained from Theorem 2.2.7. $\square$

For the remaining $\mathrm{PSL}_n(q), \mathrm{PSp}_n(q)$ and $\mathrm{PSU}_n(q)$ we require information about the maximal subgroups. Luckily this information can be found in [6] and so we provide the following theorem, a subcase of [6, Main Theorem 2.1.1.].

Theorem 4.3.2 ([6, Main Theorem 2.1.1.]). *Let q be a prime power, let $n \leq 12$, and let $G = \mathrm{PSp}_n(q), \mathrm{PSp}_n(q)$ or $\mathrm{PSU}_n(q)$. Then representatives of the conjugacy classes of maximal subgroups of G are as specified in the appropriate table in [6, Chapter 8].*

We will liberally use the above theorem; in each of the following sections we will present a table with a selection of information gathered from tables in [6, Chapter 8] to aid with the calculation in that given section. However we note that this information is an incomplete version of those in [6], ours containing only the information needed for our calculations. We will state in each section the tables of [6] we are citing, for ease of reference.

Table 4.1: c_G for small $\mathrm{PSL}_n(q)$

G	$P_2(G)$	$m(G)$	c_G	c_G to 3 d.p
$\mathrm{PSL}_2(5)$	$19/30$	5	$11/6$	1.834
$\mathrm{PSL}_2(7)$	$19/28$	7	$9/4$	2.250
$\mathrm{PSL}_2(8)$	$71/84$	9	$39/28$	1.393
$\mathrm{PSL}_2(9)$	$53/90$	6	$37/15$	2.467
$\mathrm{PSL}_2(11)$	$127/165$	11	$38/15$	2.534
$\mathrm{PSL}_2(13)$	$165/182$	14	$17/13$	1.308
$\mathrm{PSL}_2(16)$	$313/340$	17	$27/20$	1.350
$\mathrm{PSL}_2(17)$	$283/306$	18	$23/17$	1.353
$\mathrm{PSL}_2(19)$	$157/171$	20	$280/171$	1.638
$\mathrm{PSL}_2(23)$	$2881/3036$	24	$310/253$	1.226
$\mathrm{PSL}_2(25)$	$911/975$	26	$128/75$	1.707
$\mathrm{PSL}_3(3)$	$101/117$	13	$16/9$	1.778
$\mathrm{PSL}_3(4)$	$121/140$	21	$57/20$	2.850
$\mathrm{PSL}_3(5)$	$175087/186000$	31	$10913/6000$	1.819
$\mathrm{PSL}_3(7)$	$302467/312816$	57	$10349/5488$	1.886
$\mathrm{PSL}_3(8)$	$892163/915712$	73	$23549/12544$	1.878
$\mathrm{PSL}_4(2)$	$133/180$	8	$94/45$	2.089
$\mathrm{PSL}_4(3)$	$706709/758160$	40	$51451/18954$	2.715
$\mathrm{PSL}_5(2)$	$310801/333312$	31	$22511/10752$	2.094

Table 4.2: c_G for small $\mathrm{PSU}_n(q)$ and $\mathrm{PSp}_n(q)$

G	$P_2(G)$	$m(G)$	c_G	c_G to 3 d.p
$\mathrm{PSp}_4(4)$	$116333/122400$	85	$6067/1440$	4.214
$\mathrm{PSp}_6(2)$	$219703/241620$	28	$153419/60405$	2.540
$\mathrm{PSU}_3(3)$	$58/63$	28	$20/9$	2.223
$\mathrm{PSU}_3(4)$	$5089/5200$	65	$111/80$	1.388
$\mathrm{PSU}_3(5)$	$19483/21000$	50	$1517/420$	3.612
$\mathrm{PSU}_4(2)$	$767/864$	27	$97/32$	3.032
$\mathrm{PSU}_4(3)$	$26017/27216$	112	$1199/243$	4.935
$\mathrm{PSU}_5(2)$	$3370951/3421440$	165	$50489/20736$	2.435

4.3.1 Linear groups

Let us first tackle the case of $\mathrm{PSL}_n(q)$ for $n \leq 12$. We will show that $c_S \leq 38/15 \approx 2.534$ rounded up to 3 decimal places, for $S = \mathrm{PSL}_2(q)$ with equality if and only if $q = 11$. Secondly we show $c_S \leq 57/20 = 2.85$ for $S = \mathrm{PSL}_n(q)$ for $3 \leq n \leq 12$ with equality occurring only when $n = 3$ and $q = 4$. Since we have ready information about the subgroup structure of $\mathrm{SL}_n(q)$ for $n \leq 12$ in [6] we may find an upper bound for

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of the conjugacy class representatives of maximal subgroups of $\mathrm{SL}_n(q)$. By then applying Lemma 2.4.8 we may find appropriate bounds for $c_{\mathrm{PSL}_n(q)}$.

Calculating α for $\mathrm{PSL}_2(q)$

In this section we show that $c_S \leq 38/15 \approx 2.534$ rounded up to 3 decimal places, for $S = \mathrm{PSL}_2(q)$ where $q \geq 4$. We also show that $c_S = 38/15$ occurs if and only if q is 11. This certainly holds for $q \leq 25$, by Theorem 4.3.1. We now show that $c_S < 38/15$ does in fact holds for larger q. We also note here that even though our main reference for the maximal subgroups is [6], the subgroups of $\mathrm{PSL}_2(q)$ were originally worked out by Dickson [17].

Theorem 4.3.3. *Let $q \geq 4$ and let $S = \mathrm{PSL}_2(q)$. Then $1 - \frac{c}{m(S)} \leq P_2(S)$, where $c = \frac{38}{15} \approx 2.534$ rounded up to 3 decimal places. Furthermore equality happens if and only if $q = 11$.*

Proof. The cases of $4 \leq q \leq 25$ are covered by Theorem 4.3.1. We make special note of when $q = 11$, here $c_S = 38/15$ and so $1 - \frac{38}{15m(S)} = P_2(S)$.

For $27 \leq q \leq 156$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_2(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 38/15. Finally by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 157$ and let $G = \mathrm{SL}_2(q)$. Here $m(G) = q + 1$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q(q-1) \qquad \text{by Theorem 2.5.29.} \tag{4.1}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.3 describes the types of maximal subgroups that may appear in $\mathrm{SL}_2(q)$ as listed in [6, Table 8.1 & 8.2]. We first aim to bound from above $\sum_{M \in \mathcal{M}} |M|$.

If $H \leq G$ lies in $\mathcal{C}_1$ then $|H| = q(q-1)$. Furthermore, there is at most 1 conjugacy class of $\mathcal{C}_1$ subgroups in G.

If $H \leq G$ lies in $\mathcal{C}_2$ then $|H| = 2(q-1)$. Here also is at most one conjugacy class of $\mathcal{C}_2$ subgroups in G.

If $H \leq G$ lies in $\mathcal{C}_3$ then $|H| = 2(q+1)$. There is at most one conjugacy class of $\mathcal{C}_3$ subgroups in G.

If $H \leq G$ lies in $\mathcal{C}_5$ then

$$\begin{aligned} |H| &\leq 2|\mathrm{SL}_2(q_0)| = 2q_0(q_0^2 - 1) && \text{by Theorem 2.5.29} \\ &\leq 2q^{1/2}(q-1) && \text{since } q_0 \leq q^{0.5}. \end{aligned}$$

Furthermore there are at most 2 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$ such that r is prime. In other words, if $q = p^e$ for prime p, it is the number of prime factors of e. We may bound e above by $\log_2(q)$ and bound the number of prime factors of e by $\log_2(e)$ also. Consequently, we may bound the number of r by $\log_2(\log_2(q))$. This in turn can be bounded above by $q^{0.3025}$ by Lemma 2.8.4.

If $H \leq G$ lies in $\mathcal{C}_6$ then $|H| \leq 48$. We note that there are at most 2 conjugacy classes of such subgroups in G.

Finally, if $H \in \mathcal{S}$ then $|H| = 120$. Here there are at most 2 conjugacy classes of $\mathcal{S}$ subgroups in G.

Consequently

$$\begin{aligned} \sum_{M \in \mathcal{M}} |M| &\leq q(q-1) + 2(q-1) + 2(q+1) + 4q^{0.3025}q^{1/2}(q-1) + 2 \times 48 + 2 \times 120 \\ &= q^2 + 4q^{1.8025} + 3q - 4q^{0.8025} + 336 < q^2 + 4q^{1.8025} + 3q + 336. \end{aligned}$$

Therefore

$$\begin{aligned} c_{\mathrm{PSL}_2(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq \frac{(q^2 + 4q^{1.8025} + 3q + 336)m(G)}{|G|} \\ &\leq \frac{q^2 + 4q^{1.8025} + 3q + 336}{q(q-1)} && \text{by Eq. (4.1)} \\ &= \frac{q}{q-1} + \frac{4q^{0.8025}}{q-1} + \frac{3}{q-1} + \frac{336}{q(q-1)} \\ &\leq \frac{157}{156} + \frac{4 \times 157^{0.8025}}{156} + \frac{3}{156} + \frac{336}{157 \times 156} < 2.53 && \text{since } q \geq 157 \\ &< \frac{38}{15} \approx 2.534 && \text{rounded up to 3 d.p.} \end{aligned}$$

The result now follows from Lemma 2.4.2. $\square$

Calculating α for $\mathrm{PSL}_3(q)$

Theorem 4.3.4. *Let $q \geq 3$ and let $S = \mathrm{PSL}_3(q)$. Then $1 - \frac{c}{m(S)} \leq P_2(S)$, where $c = \frac{57}{20} = 2.85$. Furthermore, equality occurs if and only if $q = 4$.*

Table 4.3: Maximal subgroups of $\mathrm{SL}_2(q)$ for $q > 3$

Class	Subgroup	Details	Number of conjugacy classes	Stab
$\mathcal{C}_1$	$[q] : (q-1)$	P_1 type	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{Q}_{2(q-1)}$	$q \neq 5, 7, 11; q$ odd	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$\mathrm{D}_{2(q-1)}$	q even	1	$\langle \phi \rangle$
$\mathcal{C}_3$	$\mathrm{Q}_{2(q+1)}$	$q \neq 7, 9; q$ odd	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$\mathrm{D}_{2(q+1)}$	q even	1	$\langle \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SL}_2(q_0).2$	$q = q_0^r$, q odd, $r = 2$	2	$\langle \phi \rangle$
$\mathcal{C}_5$	$\mathrm{SL}_2(q_0)$	$q = q_0^r$, q odd, r odd prime	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_5$	$\mathrm{PSL}_2(q_0)$	$q = q_0^r$, q even, $q_0 \neq 2$, r prime	1	$\langle \phi \rangle$
$\mathcal{C}_6$	$2_-^{1+2}.\mathrm{S}_3$	$q = p \equiv \pm 1 \mod 8$	2	1
$\mathcal{C}_6$	$2_-^{1+2} : 3$	$q = p \equiv \pm 2, 5, \pm 13 \mod 40$	1	$\langle \delta \rangle$
$\mathcal{S}$	$2^{\cdot}\mathrm{A}_5$	$q = p \equiv \pm 1 \mod 10$	2	1
		$q = p^2, p \equiv \pm 3 \mod 10$	2	$\langle \phi \rangle$

Novelties

Class	Subgroup	Details	Number of conjugacy classes	Stab
$\mathcal{C}_2$	$\mathrm{Q}_{2(q-1)}$	N1 if $q = 7, 11$	1	$\langle \delta \rangle$
$\mathcal{C}_2$	$\mathrm{Q}_{2(q-1)}$	N2 if $q = 9$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_3$	$\mathrm{Q}_{2(q+1)}$	N1 if $q = 7$	1	$\langle \delta \rangle$
$\mathcal{C}_3$	$\mathrm{Q}_{2(q+1)}$	N2 if $q = 9$	1	$\langle \delta, \phi \rangle$
$\mathcal{C}_6$	$2_-^{1+2} : 3$	N1 if $q = p \equiv \pm 11, \pm 19 \mod 40$	1	$\langle \delta \rangle$

N1 denotes maximal under $\langle \delta \rangle$

N2 denotes maximal under subgroups not contained in $\langle \phi \rangle$

Table 4.4: Maximal subgroups of $\mathrm{SL}_3(q)$

Class	Subgroup	Details	Number of conjugacy classes	Stabilizer
$\mathcal{C}_1$	$[q^2] : \mathrm{GL}_2(q)$	P_1, P_2 type	2	$\langle \delta, \phi \rangle$
$\mathcal{C}_2$	$(q-1)^2 : \mathrm{S}_3$	$q \geq 5$	1	$\langle \delta, \phi, \gamma \rangle$
$\mathcal{C}_3$	$(q^2+q+1) : 3$	$q \neq 4$	1	$\langle \delta, \phi, \gamma \rangle$
$\mathcal{C}_5$	$\mathrm{SL}_3(q_0).(\frac{q-1}{q_0-1}, 3)$	$q = q_0^r$, r prime	$(\frac{q-1}{q_0-1}, 3)$	$\langle \delta^c, \phi, \gamma \rangle$
$\mathcal{C}_6$	$3_+^{1+2} : \mathrm{Q}_8.\frac{(q-1,9)}{3}$	$q = p \equiv 1 \mod 3$	$(q-1,9)/3$	$\langle \delta^c, \gamma \rangle$
$\mathcal{C}_8$	$(q-1,3) \times \mathrm{SO}_3(q)$	q odd	$(q-1,3)$	$\langle \phi, \gamma \rangle$
$\mathcal{C}_8$	$(q_0-1,3) \times \mathrm{SU}_3(q_0)$	$q = q_0^2$	$(q_0-1,3)$	$\langle \delta^c, \phi, \gamma \rangle$
$\mathcal{S}$	$(q-1,3) \times \mathrm{PSL}_2(7)$	$q = p \equiv 1,2,4 \mod 7, p \neq 2$	$(q-1,3)$	$\langle \gamma \rangle$
$\mathcal{S}$	$3{\cdot}\mathrm{A}_6$	$q = p \equiv \pm 1, 4 \mod 15$	3	$\langle \gamma \rangle$
		$q = p^2, p \equiv \pm 2 \mod 5, p \neq 3$	3	$\langle \phi, \gamma \rangle$
		Novelties		
$\mathcal{C}_1$	$[q^{1+2}] : (q-1)^2$	N1	1	$\langle \delta, \phi, \gamma \rangle$
$\mathcal{C}_1$	$\mathrm{GL}_2(q)$	N1	1	$\langle \delta, \phi, \gamma \rangle$
$\mathcal{C}_3$	$(q^2+q+1) : 3$	N2 if $q = 4$	1	$\langle \delta, \phi, \gamma \rangle$

N1 denotes maximal under subgroups not contained in $\langle \delta, \phi \rangle$

N2 denotes maximal under subgroups not contained in $\langle \phi, \gamma \rangle$

Proof. The case of $3 \leq q \leq 8$ follows from Theorem 4.3.1. We make special note of when $q = 4$, here $c_S = 57/20$ and so $1 - \frac{57}{20m(S)} = P_2(S)$.

For $9 \leq q \leq 23$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_3(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 25$, let $G = \mathrm{SL}_3(q)$. Here $m(G) = \frac{q^3-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned} |G|/m(G) = q^2|\mathrm{GL}_2(q)| &= q^3(q^2-1)(q-1) && \text{by Theorem 2.5.29} \\ &\geq q^3 \frac{q^2}{2}\frac{q}{2} = q^6/4. && (4.2) \end{aligned}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.4 describes the types of maximal subgroups that may appear in $\mathrm{SL}_3(q)$ as listed in [6, Tables 8.3 & 8.4]. Also, let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of subgroups that do not lie in $\mathcal{C}_1$, do not lie in $\mathcal{C}_5$ and are also not isomorphic to $(q_0 - 1, 3) \times \mathrm{SU}_3(q_0)$. One can show that if $H \in \mathcal{M}_1$ then $|H| \leq 6q^3$ for $q \geq 13$. To see this the case of $H \cong (q-1,3) \times \mathrm{SO}_3(q)$ then

$$|H| \leq 3q(q^2-1) \leq 3q^2$$

by Theorem 2.5.29. Furthermore, by Table 4.4,

$$|\mathcal{M}_1| \leq 5 + 2(q-1,3) + (q-1,9)/3 = 5 + 3 \times 3 = 14.$$

If H lies in $\mathcal{C}_1$ then $|H| = q^2(q^2-1)(q-1)$. Note that there are 2 conjugacy classes of such subgroups in $\mathrm{SL}_3(q)$. If H lies in $\mathcal{C}_5$ then $H \cong \mathrm{SL}_3(q_0).(\frac{q-1}{q_0-1}, 3)$, so

$$\begin{aligned} |H| &\leq 3q_0^3(q_0^3-1)(q_0^2-1) && \text{by Theorem 2.5.29} \\ &\leq 3q^{1.5}(q^{1.5}-1)(q-1) < 3q^4 && \text{since } q_0 \leq q^{1/2}. \end{aligned}$$

Note that there are at most 3 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. If $H \in \mathcal{C}_8$ where $H \cong (q_0-1,3) \times \mathrm{SU}_3(q_0)$ we note that there are at most 3 conjugacy classes of subgroups H. Then

$$|H| \leq 3q_0^3(q_0^3+1)(q_0^2-1) < 6q^4 \qquad \text{since } q = q^{1/2}.$$

Consequently

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (14 \times 6q^3) + (3q^{0.55} \times 3q^4) + (3 \times 6q^4)$$

$$= 2\frac{|G|}{m(G)} + 9q^{4.55} + 18q^4 + 84q^3.$$

So

$$\begin{aligned}
c_{\mathrm{PSL}_3(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8}\\
&\leq 2 + \frac{(9q^{4.55} + 18q^4 + 85q^3)m(G)}{|G|}\\
&\leq 2 + \frac{4(9q^{4.55} + 18q^4 + 84q^3)}{q^6} && \text{by Eq. (4.2)}\\
&= 2 + 36q^{-1.45} + 72q^{-2} + 336q^{-3}\\
&\leq 2 + 3(6 \times 25^{-1.45}) + (72 \times 25^{-2}) + (336 \times 25^{-3}) < 2.5 \leq \frac{57}{20} = 2.85 && \text{since } q \geq 25.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_4(q)$

Theorem 4.3.5. *Let $q \geq 2$ and let $S = \mathrm{PSL}_4(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. The cases of $q = 2, 3$ follow from Theorem 4.3.1.

For $4 \leq q \leq 11$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_4(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 13$ and let $G = \mathrm{SL}_4(q)$. Here $m(G) = \frac{q^4-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned}
|G|/m(G) &= q^3|\mathrm{GL}_3(q)| = q^6(q^3-1)(q^2-1)(q-1) && \text{by Theorem 2.5.29}\\
&\geq q^6 \frac{q^3}{2}\frac{q^2}{2}\frac{q}{2} = q^{12}/8. && (4.3)
\end{aligned}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.5 describes the types of maximal subgroups that may appear in $\mathrm{SL}_4(q)$ as listed in [6, Table 8.8 & 8.9]. Here c denotes the number of conjugacy classes of groups of a given type. Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of subgroups that do not lie in $\mathcal{C}_1$ and are also not isomorphic to $\mathrm{Sp}_4(q).(q-1, 2)$.

One can show that if $H \in \mathcal{M}_1$ then $|H| \leq q^8$ for $q \geq 17$. To see this we note cases where this may not be obvious. If $H \in \mathcal{C}_i$ for $i = 2, 3, 5, 8$ we apply Lemma 2.6.4. Moreover in the case of $H \in \mathcal{C}_5$, we note that $q_0 \leq q^{1/2}$ and so in particular $q \geq 16$.

There are at most 4 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can

Table 4.5: Maximal subgroups of $\mathrm{SL}_4(q)$

Class	Subgroup	Details	c	Stab
$\mathcal{C}_1$	$[q^3]:\mathrm{GL}_3(q)$	P_1, P_3 type	2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$[q^4]:\mathrm{SL}_2(q)^2:(q-1)$	P_2 type	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_2$	$(q-1)^3:\mathrm{S}_4$	$q\geq 7$	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_2$	$\mathrm{SL}_2(q)^2:(q-1).2$	$q\geq 4$	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_3$	$\mathrm{SL}_2(q^2):(q+1).2$		1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_5$	$\mathrm{SL}_4(q_0).[(\frac{q-1}{q_0-1},4)]$	$q=q_0^r$, r prime	$(\frac{q-1}{q_0-1},4)$	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_6$	$(4\circ 2^{1+4})^{\cdot}\mathrm{S}_6$	$p=q\equiv 1 \mod 8$	4	$\langle\gamma\rangle$
$\mathcal{C}_6$	$(4\circ 2^{1+4}).\mathrm{A}_6$	$p=q\equiv 5 \mod 8$	2	$\langle\delta^2,\gamma\rangle$
$\mathcal{C}_8$	$\mathrm{SO}_4^+(q).[(q-1,4)]$	q odd	$(q-1,4)/2$	$\langle\delta^c,\phi,\gamma\rangle$
$\mathcal{C}_8$	$\mathrm{SO}_4^-(q).[(q-1,4)]$	q odd	$(q-1,4)/2$	S1
$\mathcal{C}_8$	$\mathrm{Sp}_4(q).(q-1,2)$		$(q-1,2)$	$\langle\delta^c,\phi,\gamma\rangle$
$\mathcal{C}_8$	$\mathrm{SU}_4(q_0).(q_0-1,4)$	$q=q_0^2$	$(q_0-1,4)$	$\langle\delta,\phi^c,\gamma\rangle$
$\mathcal{S}$	A_7	$q=2$	1	$\langle\gamma\rangle$
$\mathcal{S}$	$(q-1,4)\circ 2^{\cdot}\mathrm{A}_7$	$q=p\equiv 1,2,4$ mod $7, q\neq 2$	$(q-1,4)$	$\langle\gamma\rangle$
$\mathcal{S}$	$(q-1,4)\circ 2^{\cdot}\mathrm{PSU}_4(2)$	$q=p\equiv 1 \mod 6$	$(q-1,4)$	$\langle\gamma\rangle$
		Novelties		
$\mathcal{C}_1$	$[q^{1+4}]:(\mathrm{GL}_2(q)\times(q-1))$	N1	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_1$	$\mathrm{GL}_3(q)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_2$	$(q-1)^3:\mathrm{S}_4$	N2 if $q=5$	1	$\langle\delta,\gamma\rangle$
$\mathcal{C}_2$	$\mathrm{SL}_2(q)^2:(q-1).2$	N3 if $q=3$	1	$\langle\delta,\gamma\rangle$
$\mathcal{S}$	$(q-1,4)\circ 2.\mathrm{PSL}_2(7)$	N4, $q=p\equiv 1,2,4$ mod $7, q\neq 2$	$(q-1,4)$	S2

N1 denotes Maximal under subgroups not contained in $\langle\delta,\phi\rangle$

N2 denotes Maximal under subgroups not contained in $\langle\delta^2,\gamma\rangle$

N3 denotes Maximal under subgroups not contained in $\langle\gamma\rangle$

N4 denotes Maximal under S2

S1 denotes $\langle\delta^c,\phi\delta^{(p-1)/2},\gamma\delta\rangle$

S2 denotes $\langle\gamma\rangle$ if $p\equiv\pm 1 \mod 8$, $\langle\delta\gamma\rangle$ if $p\equiv\pm 3 \mod 8$

bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. So, by Table 4.5,

$$|\mathcal{M}_1| \leq 8 + 4q^{0.55} + (q_0 - 1, 4) + 3(q-1,4) + (q-1,2) = 8 + (4\times 4) + 2 + 4q^{0.55} = 26 + 4q^{0.55}.$$

If H is a P_1 type subgroup, then we notice that there are 2 conjugacy classes. It is also straightforward to calculate that $|H| = |G|/m(G)$ using Theorem 2.5.29. If H is a P_2 type subgroup then $|H| = q^6(q^2-1)^2(q-1)$ by Theorem 2.5.29. Note that there is only one conjugacy class for P_2 subgroups.

If $H \cong \mathrm{Sp}_4(q).(q-1,2)$ then $|H| \leq 2q^4(q^4-1)(q^2-1)$ by Theorem 2.5.29. We note that there are at most 2 different conjugacy classes of such $\mathcal{C}_8$ subgroups.

Consequently

$$\begin{aligned}
\sum_{M\in\mathcal{M}} |M| &\leq 2\frac{|G|}{m(G)} + q^6(q^2-1)^2(q-1) + 2(2q^4(q^4-1)(q^2-1)) + (26+4q^{0.55})q^8 \\
&= 2\frac{|G|}{m(G)} + q^{11} + 3q^{10} - 2q^9 + 4q^{8.55} + 24q^8 + q^7 - 5q^6 + 4q^4 \\
&\leq 2\frac{|G|}{m(G)} + q^{11} + 3q^{10} + 4q^{8.55} + 24q^8 + q^7 + 4q^4 \\
&\leq 2\frac{|G|}{m(G)} + q^{11} + 3q^{10} + 4q^{8.55} + 25q^8 \leq 2\frac{|G|}{m(G)} + q^{11} + 4q^{10}
\end{aligned}$$

since $q \geq 13$. Therefore

$$\begin{aligned}
c_{\mathrm{PSL}_4(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 2 + \frac{(q^{11}+4q^{10})m(G)}{|G|} \leq 2 + \frac{8(q^{11}+4q^{10})}{q^{12}} && \text{by Eq. (4.3)} \\
&\leq 2 + \frac{8(13^{11}+4\times 13^{10})}{13^{12}} < 2 + 0.81 < 2.85 && \text{since } q \geq 13.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_5(q)$

Theorem 4.3.6. *Let $q \geq 2$ and let $S = \mathrm{PSL}_5(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. First we note that the case of $q = 2$ follows from Theorem 4.3.1.

For the small cases of $3 \leq q \leq 4$ we calculate

$$\sum_{M\in\mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_5(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally by Lemma 2.4.2 we have that $1 - c_S/m(S) = P_2(S)$.

Table 4.6: Maximal subgroups of $\mathrm{SL}_5(q)$

Class	Subgroup	Details	No. of conj. classes - c	Stab
$\mathcal{C}_1$	$[q^4]:\mathrm{GL}_4(q)$	P_1, P_4 type	2	$\langle\delta,\phi\rangle$
$\mathcal{C}_1$	$[q^6]:(\mathrm{SL}_2(q)\times\mathrm{SL}_3(q)):(q-1)$	P_2, P_3 type	2	$\langle\delta,\phi\rangle$
$\mathcal{C}_2$	$(q-1)^4:\mathrm{S}_5$	$q\geq 5$	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_3$	$\frac{q^5-1}{q-1}:5$		1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_5$	$\mathrm{SL}_5(q_0).(\frac{q-1}{q_0-1},5)$	$q=q_0^r$, r prime	$(\frac{q-1}{q_0-1},5)$	$\langle\delta^c,\phi,\gamma\rangle$
$\mathcal{C}_6$	$5_+^{1+2}:\mathrm{Sp}_2(5)$	$p=q$	5	$\langle\gamma\rangle$
$\mathcal{C}_8$	$(q-1,5)\times\mathrm{SO}_5(q)$	q odd	$(q-1,5)$	$\langle\phi,\gamma\rangle$
$\mathcal{C}_8$	$(q_0-1,5)\times\mathrm{SU}_5(q_0)$	$q=q_0^2$	$(q_0-1,5)$	$\langle\delta^c,\phi,\gamma\rangle$
$\mathcal{S}$	$(q-1,5)\times\mathrm{PSL}_2(11)$	$q=p$	$(q-1,5)$	$\langle\gamma\rangle$
$\mathcal{S}$	M_{11}	$q=3$	2	$\langle\gamma\rangle$
$\mathcal{S}$	$(q-1,5)\times\mathrm{PSU}_4(2)$	$q=p$	$(q-1,5)$	$\langle\gamma\rangle$
		Novelties		
$\mathcal{C}_1$	$[q^{1+6}]:(\mathrm{GL}_3(q)\times(q-1))$	N1	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_1$	$[q^{4+4}]:\mathrm{GL}_2(q)^2$	N1	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_1$	$\mathrm{GL}_4(q)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{C}_1$	$(\mathrm{SL}_2(q)\times\mathrm{SL}_3(q)):(q-1)$	N1	1	$\langle\delta,\phi,\gamma\rangle$
$\mathcal{S}$	$d\times\mathrm{PSL}_2(11)$	N2, $q=3$	1	1

N1 denotes maximal under subgroups not contained in $\langle\delta,\phi\rangle$

N2 denotes maximal under $\langle\gamma\rangle$

Let $q \geq 5$ and let $G = \mathrm{SL}_5(q)$. Here $m(G) = \frac{q^5-1}{q-1}$, by Theorem 2.2.7. We have that

$$\begin{aligned} |G|/m(G) &= q^4|\mathrm{GL}_4(q)| = q^{10}(q^4-1)(q^3-1)(q^2-1)(q-1) && \text{by Theorem 2.5.29} \\ &\geq q^{10}\frac{624q^4}{625}\frac{124q^3}{125}\frac{24q^2}{25}\frac{4q}{5} > 0.76q^{20} && \text{since } q \geq 5. \end{aligned} \tag{4.4}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.6 describes the types of maximal subgroups that may appear in $\mathrm{SL}_5(q)$ as listed in [6, Table 8.18 & 8.19].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of subgroups that do not lie in $\mathcal{C}_1$. One can show that if $H \in \mathcal{M}_1$ then $|H| \leq q^{13}$ for $q \geq 5$. To see this note that if $H \in \mathcal{C}_i$ for $i = 5, 6, 8$ we apply Lemma 2.6.4. Moreover in the case of $H \in \mathcal{C}_5$, we note that $q_0 \leq q^{1/2}$. Finally we note that if $H \cong \mathrm{M}_{11}$ then $|H| = 7920$.

Note that there are at most 5 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. Therefore, by Table 4.6

$$\begin{aligned} |\mathcal{M}_1| &\leq 9 + 5q^{0.55} + (q_0-1,5) + 3(q-1,5) \\ &= 9 + 4 \times 5 + 5q^{0.55} = 29 + 5q^{0.55} < 29q && \text{since } q \geq 5. \end{aligned}$$

If H is a P_1 type subgroup, then we notice that there are 2 conjugacy classes. As noted before, $|H| = |G|/m(G)$. If H is a P_2 type subgroup then $H \cong [q^6] : (\mathrm{SL}_3(q) \times \mathrm{SL}_2(q)) : (q-1)$. Then

$$|H| = q^{10}(q^3-1)(q^2-1)^2(q-1) \qquad \text{by Theorem 2.5.29.}$$

Consequently

$$\begin{aligned} \sum_{M \in \mathcal{M}} |M| &\leq 2\frac{|G|}{m(G)} + 2q^{10}(q^3-1)(q^2-1)^2(q-1) + (29q \times q^{13}) \\ &\leq 2\frac{|G|}{m(G)} + 2q^{18} + 29q^{14}. \end{aligned}$$

Therefore

$$\begin{aligned} c_{\mathrm{PSL}_5(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 2 + \frac{(2q^{18} + 29q^{14})m(G)}{|G|} \leq 2 + \frac{2q^{18} + 29q^{14}}{0.76q^{20}} && \text{by Eq. (4.4)} \\ &\leq 2 + \frac{(2 \times 5^{18}) + (29 \times 5^{14})}{0.76 \times 5^{20}} < 2 + 0.11 < 2.85 && \text{since } q \geq 3. \end{aligned}$$

The result follows from Lemma 2.4.2. $\square$

Table 4.7: Maximal subgroups of $\mathrm{SL}_6(q)$

Class	Subgroup	Details	Number of conjugacy classes
$\mathcal{C}_1$	$[q^5] : \mathrm{GL}_5(q)$	P_1, P_5 type	2
$\mathcal{C}_1$	$[q^8] : (\mathrm{SL}_4(q) \times \mathrm{SL}_2(q)) : (q-1)$	P_2, P_4 type	2
$\mathcal{C}_1$	$[q^9] : (\mathrm{SL}_3(q) \times \mathrm{SL}_3(q)) : (q-1)$	P_3 type	1
$\mathcal{C}_2$	$(q-1)^5 : \mathrm{S}_6$	$q \geq 5$	1
$\mathcal{C}_2$	$\mathrm{SL}_2(q)^3 : (q-1)^2.\mathrm{S}_3$	$q \geq 3$	1
$\mathcal{C}_2$	$\mathrm{SL}_3(q)^2 : (q-1).\mathrm{S}_2$		1
$\mathcal{C}_3$	$\mathrm{SL}_3(q^2).(q+1).2$		1
$\mathcal{C}_3$	$\mathrm{SL}_2(q^3).(q^2+q+1).3$		1
$\mathcal{C}_4$	$\mathrm{SL}_2(q) \times \mathrm{SL}_3(q)$	$q \geq 3$	1
$\mathcal{C}_5$	$\mathrm{SL}_6(q_0).[(\frac{q-1}{q_0-1}, 6)]$	$q = q_0^r$, r prime	$(\frac{q-1}{q_0-1}, 6)$
$\mathcal{C}_8$	$(q-1,3) \times \mathrm{SO}_6^+(q).2$	q odd	$(q-1,6)/2$
$\mathcal{C}_8$	$(q-1,3) \times \mathrm{SO}_6^-(q).2$	q odd	$(q-1,6)/2$
$\mathcal{C}_8$	$(q-1,3) \times \mathrm{Sp}_6(q)$		$(q-1,3)$
$\mathcal{C}_8$	$\mathrm{SU}_6(q_0).(q_0-1,6)$	$q = q_0^2$	$(q_0-1,6)$
$\mathcal{S}$	$(q-1,6) \circ 2^{\cdot}\mathrm{PSL}_2(11)$		$(q-1,6)$
$\mathcal{S}$	$6^{\cdot}\mathrm{A}_7$		12
$\mathcal{S}$	$6^{\cdot}\mathrm{PSL}_3(4)^{\cdot}2_1^-$		6
$\mathcal{S}$	$2^{\cdot}\mathrm{M}_{12}$		2
$\mathcal{S}$	$6_1{}^{\cdot}\mathrm{PSU}_4(3)^{\cdot}2_2^-$		6
$\mathcal{S}$	$6_1{}^{\cdot}\mathrm{PSU}_4(3)$		3
$\mathcal{S}$	$(q-1,6) \circ \mathrm{SL}_3(q)$		2

Calculating α for $\mathrm{PSL}_6(q)$

Theorem 4.3.7. *Let $q \geq 2$ and let $S = \mathrm{PSL}_6(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. For the small cases of $2 \leq q \leq 4$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_6(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 5$ and let $G = \mathrm{SL}_6(q)$. Here $m(G) = \frac{q^6-1}{q-1}$, by Theorem 2.2.7. If H is a P_1 type subgroup, then we may show that $|H| = |G|/m(G)$ using Theorem 2.5.29. Furthermore

$$|H| = q^5|\mathrm{GL}_5(q)| \geq q^5 q^{5^2-2} = q^{28} \qquad \text{by Lemma 2.6.4.} \tag{4.5}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.7 describes the types of maximal subgroups that may appear in $\mathrm{SL}_6(q)$ as listed in [6, Table 8.24& 8.25].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of subgroups that do not lie in $\mathcal{C}_1$. Then we can show, using Lemma 2.6.4 and the fact that $q_0 \leq q^{1/2}$ that $|H| \leq q^{22}$ for $q \geq 5$.

There are at most 6 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. Therefore

$$\begin{aligned} |\mathcal{M}_1| &\leq 37 + 6q^{0.55} + (q_0 - 1, 6) + 2(q-1,6) + (q-1,3) \\ &\leq 37 + (3 \times 6) + 3 + 6q^{0.55} = 58 + 6q^{0.55} < 12q + 6q^{0.55} < 18q \qquad \text{since } q \geq 5. \end{aligned}$$

There is 1 conjugacy classes each for P_1 and P_5 type subgroups As noted before, $|H| = |G|/m(G)$ using Theorem 2.5.29.

If H is a P_2 type subgroup then then

$$|H| \leq q^{26}(q-1) \qquad \text{by Table 4.7.}$$

There are at most 2 conjugacy classes of P_2 type subgroups.

If H is a P_3 type subgroup, then $H \cong [q^9] : (\mathrm{SL}_3(q) \times \mathrm{SL}_3(q)) : (q-1)$. Then

$$|H| \leq q^{26} \qquad \text{by Table 4.7.}$$

There is at most 1 conjugacy class of P_3 type subgroups. Consequently

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + 2q^{26}(q-1) + q^{26} + (18q)q^{22}$$

$$
\begin{aligned}
&\leq 2\frac{|G|}{m(G)} + 2q^{27} - q^{26} + 18q^{23} \\
&\leq 2\frac{|G|}{m(G)} + 2q^{27} && \text{since } q \geq 5.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
c_{\mathrm{PSL}_6(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 2 + \frac{2q^{27}m(G)}{|G|} = 2 + \frac{2q^{27}}{q^{28}} && \text{by Eq. (4.5)} \\
&= 2 + \frac{2}{q} < 2 + \frac{2}{5} < 2.85 && \text{since } q \geq 5.
\end{aligned}
$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_7(q)$

Theorem 4.3.8. *Let $q \geq 2$ and let $S = \mathrm{PSL}_7(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. For $2 \leq q \leq 8$ we calculate

$$\sum_{M\in\mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_7(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 9$ and let $G = \mathrm{SL}_7(q)$. Here $m(G) = \frac{q^7-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^6|\mathrm{GL}_6(q)| \geq q^6 q^{6^2-2} = q^{40} \qquad \text{by Lemma 2.6.4.} \tag{4.6}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Tables 8.35 & 8.36] describe the types of maximal subgroups that may appear in $\mathrm{SL}_7(q)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 or P_6 type. We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 for the geometric subgroups and [6, Table 8.36] for the non-geometrics. There are at most 4 conjugacy classes of $\mathcal{C}_1$ subgroup not of P_1 or P_6 type. There is only 1 of $\mathcal{C}_2$ and only 1 of $\mathcal{C}_3$ subgroups since 7 is prime. There are at most 7 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 7 conjugacy classes of $\mathcal{C}_6$ subgroups and at most 14 of $\mathcal{C}_7$ subgroups. Finally from [6, Table 8.36] we see that there are at most 7 non-geometric subgroups of $\mathrm{SL}_7(q)$. Consequently

$$|\mathcal{M}_1| \leq 4 + 1 + 1 + 7q^{0.55} + (4 \times 7) = 34 + 7q^{0.55}.$$

By Proposition 3.2.6, if $H \in \mathcal{M}_1$ then H has order at most that of a subgroup of P_2 type. That means that

$$|H| \leq q^{10}(q-1)|\mathrm{SL}_5(q)||\mathrm{SL}_2(q)| \qquad \text{by Theorem 3.2.1}$$

$$\leq q^{10+1+24+3} = q^{38} \qquad \text{by Lemma 2.6.4.}$$

There is 1 conjugacy class each for P_1 and P_6 type subgroups. By Proposition 3.2.6 $|H| = \frac{|G|}{m(G)}$. Hence

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (33 + 7q^{0.55})q^{38}.$$

Therefore

$$\begin{aligned}
c_{\mathrm{PSL}_7(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 2 + \frac{((34 + 7q^{0.55})q^{38})m(G)}{|G|} \leq 2 + \frac{(34 + 7q^{0.55})q^{38}}{q^{40}} && \text{by Eq. (4.6)} \\
&\leq 2 + \frac{34 + (7 \times 9^{0.55})}{9^2} < 2 + 0.71 < 2.85 && \text{since } q \geq 9.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_8(q)$

Theorem 4.3.9. *Let $q \geq 2$ and let $S = \mathrm{PSL}_8(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2, 3, 4$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_8(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than $57/20$. Note that [6, Tables 8.44 & 8.45] list the maximal subgroups $\mathrm{PSL}_8(q)$ which aids in our calculation. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 5$ and let $G = \mathrm{SL}_8(q)$. Here $m(G) = \frac{q^8-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned}
|G|/m(G) &= q^7|\mathrm{GL}_7(q)| && \text{by Proposition 3.2.6} \\
&\geq q^7 q^{7^2-2} = q^{54} && \text{by Lemma 2.6.4.}
\end{aligned} \tag{4.7}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Tables 8.44 & 8.45] describe the types of maximal subgroups that may appear in $\mathrm{SL}_8(q)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 or P_7 type. We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.45] for the non-geometric subgroups.

We note that there are 5 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1 or P_7 type. Since $n = 8$ there are three divisors of n larger than 1, so there are 3 conjugacy classes of $\mathcal{C}_2$ subgroups. There is only one prime factor of 8 and so there is only 1 conjugacy class of $\mathcal{C}_3$ subgroups. There only case where there is a $\mathcal{C}_4$ subgroup is when $n_1 = 2$ and then there are at most 2 conjugacy classes of such subgroups. There are at most 8 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 8 conjugacy classes of $\mathcal{C}_6$ subgroups, and at most 20 of $\mathcal{C}_8$ subgroups. Finally we note that, by [6, Table 8.45], there are at most 16 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 5 + 3 + 1 + 2 + 8q^{0.55} + 8 + 20 + 16 = 55 + 8q^{0.55}.$$

By Proposition 3.2.6, if $H \in \mathcal{M}_1$ then H has order less than a subgroup of P_2 type. That means that

$$\begin{aligned} |H| &\leq q^{12}(q-1)|\mathrm{SL}_6(q)||\mathrm{SL}_2(q)| && \text{by Theorem 3.2.1} \\ &\leq q^{12+1+35+3} = q^{51} && \text{by Lemma 2.6.4.} \end{aligned}$$

There is 1 conjugacy classes each for P_1 and P_7 type subgroups. By Proposition 3.2.6 $|H| = \frac{|G|}{m(G)}$. Hence

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (55 + 8q^{0.55})q^{51}.$$

Therefore

$$\begin{aligned} c_{\mathrm{PSL}_8(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 2 + \frac{((55 + 8q^{0.55})q^{51})m(G)}{|G|} \leq 2 + \frac{(55 + 8q^{0.55})q^{51}}{q^{54}} && \text{by Eq. (4.7)} \\ &\leq 2 + \frac{55 + (8 \times 5^{0.55})}{5^3} < 2 + 0.6 < 2.85 && \text{since } q \geq 5. \end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_9(q)$

Theorem 4.3.10. *Let $q \geq 2$ and let $S = \mathrm{PSL}_9(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2, 3$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_9(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 4$ and let $G = \mathrm{SL}_9(q)$. Here $m(G) = \frac{q^9-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^8|\mathrm{GL}_8(q)| \qquad \text{by Proposition 3.2.6}$$

$$\geq q^8 q^{8^2-2} = q^{70} \qquad \text{by Lemma 2.6.4.} \tag{4.8}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Tables 8.54& 8.55] describe the types of maximal subgroups that may appear in $\mathrm{SL}_9(q)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 or P_8 type. We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.55] for the non-geometric subgroups.

We note that there are 6 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1 or P_8 type. Since $n = 9$ there are two divisors of n larger than 1, so there are 2 conjugacy classes of $\mathcal{C}_2$ subgroups. There is only one prime factor of 9 and so there is only 1 conjugacy class of $\mathcal{C}_3$ subgroups. There are no $\mathcal{C}_4$ subgroups. There are at most 9 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 9 conjugacy classes of $\mathcal{C}_6$ subgroups, at most 3 conjugacy classes of $\mathcal{C}_7$ subgroups, and at most 18 of $\mathcal{C}_8$ subgroups. Finally we note that, by [6, Table 8.55], there are at most 15 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 6 + 1 + 2 + 9q^{0.55} + 9 + 3 + 18 + 15 = 54 + 9q^{0.55}.$$

By Proposition 3.2.6, if $H \in \mathcal{M}_1$ then H has order a subgroup of P_2 type. That means that

$$\begin{aligned} |H| &\leq q^{14}(q-1)|\mathrm{SL}_7(q)||\mathrm{SL}_2(q)| && \text{by Theorem 3.2.1} \\ &\leq q^{14+1+48+3} = q^{66} && \text{by Lemma 2.6.4.} \end{aligned}$$

There is 1 conjugacy classes each for P_1 and P_8 type subgroups. As noted before $|H| = \frac{|G|}{m(G)}$. Consequently

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (54 + 9q^{0.55})q^{66}.$$

Therefore

$$\begin{aligned} c_{\mathrm{PSL}_9(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 2 + \frac{((54 + 9q^{0.55})q^{66})m(G)}{|G|} \leq 2 + \frac{(54 + 9q^{0.55})q^{66}}{q^{70}} && \text{by Eq. (4.8)} \\ &\leq 2 + \frac{54 + (9 \times 4^{0.55})}{4^4} < 2 + 0.29 < 2.85 && \text{since } q \geq 4. \end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_{10}(q)$

Theorem 4.3.11. *Let $q \geq 2$ and let $S = \mathrm{PSL}_{10}(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_{10}(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 3$ and let $G = \mathrm{SL}_{10}(q)$. Here $m(G) = \frac{q^{10}-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned} |G|/m(G) &= q^9|\mathrm{GL}_9(q)| && \text{by Proposition 3.2.6} \\ &\geq q^9 q^{9^2-2} = q^{88} && \text{by Lemma 2.6.4.} \end{aligned} \tag{4.9}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Tables 8.60 & 8.61] describe the types of maximal subgroups that may appear in $\mathrm{SL}_{10}(q)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 or P_9 type. We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.61] for the non-geometric subgroups.

We note that there are 7 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1 or P_9 type. Since $n = 10$ there are three divisors of n larger than 1, so there are 3 conjugacy classes of $\mathcal{C}_2$ subgroups. There are two prime factors of 10 and so there are 2 conjugacy classes of $\mathcal{C}_3$ subgroups. In the case of $\mathcal{C}_4$ there is only possibility for n_1, and in that case there are is only one conjugacy class of such a subgroup. There are most 10 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 25 conjugacy classes of $\mathcal{C}_8$ subgroups. There are no $\mathcal{C}_6$ nor $\mathcal{C}_7$ subgroups, since 10 is not a prime power. Finally we note that, by [6, Table 8.61], there are at most 67 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 7+3+2+1+10q^{0.55}+25+89 \leq 15+(10.5\times 10)+5+2+10q^{0.55} = 105+10q^{0.55}.$$

By Proposition 3.2.6, if $H \in \mathcal{M}_1$ then H has order at most that of a subgroup of P_2 type. That means that

$$\begin{aligned} |H| &\leq q^{16}(q-1)|\mathrm{SL}_8(q)||\mathrm{SL}_2(q)| && \text{by Theorem 3.2.1} \\ &\leq q^{16+1+63+3} = q^{83} && \text{by Lemma 2.6.4.} \end{aligned}$$

There is 1 conjugacy class each for P_1 and P_9 type subgroups. As noted before $|H| = \frac{|G|}{m(G)}$. So,

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (105+10q^{0.55})q^{83}.$$

Therefore

$$c_{\mathrm{PSL}_{10}(q)} \leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} \qquad \text{by Lemma 2.4.8}$$

$$\leq 2 + \frac{((105 + 10q^{0.55})q^{83})m(G)}{|G|} \leq 2 + \frac{(105 + 10q^{0.55})q^{83}}{q^{88}} \qquad \text{by Eq. (4.9)}$$

$$\leq 2 + \frac{105 + (10 \times 3^{0.55})}{3^5} < 2 + 0.51 < 2.85 \qquad \text{since } q \geq 4.$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_{11}(q)$

Theorem 4.3.12. *Let $q \geq 2$ and let $S = \mathrm{PSL}_{11}(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSL}_{11}(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 57/20. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 3$ and let $G = \mathrm{SL}_{11}(q)$. Here $m(G) = \frac{q^{11}-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned} |G|/m(G) &= q^{10}|\mathrm{GL}_{10}(q)| && \text{by Proposition 3.2.6} \\ &\geq q^{10}q^{10^2-2} = q^{108} && \text{by Lemma 2.6.4.} \end{aligned} \tag{4.10}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Tables 8.70 & 8.71] describe the types of maximal subgroups that may appear in $\mathrm{SL}_{11}(q)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 or P_{10} type. We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.71] for the non-geometric subgroups.

We note that there are 8 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1 or P_{10} type. Since $n = 11$ there is only one divisor of n larger than 1, so there is 1 conjugacy class of $\mathcal{C}_2$ subgroups. There is only one prime factor of n and so there is only 1 conjugacy class of $\mathcal{C}_3$ subgroups. There are no $\mathcal{C}_4$ subgroups since $n = 11$ is prime. There are at most 11 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 11 conjugacy classes of $\mathcal{C}_6$ subgroups. There are no $\mathcal{C}_7$ subgroups since n is prime. There are at most 22 conjugacy classes of $\mathcal{C}_8$ subgroups. Finally we note that, by [6, Table 8.71], there are at most 22 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 8 + 1 + 1 + 11q^{0.55} + 11 + 22 + 22 = 65 + 11q^{0.55}.$$

By Proposition 3.2.6, if $H \in \mathcal{M}_1$ then H has order at most that of a subgroup of P_2 type. That means that

$$|H| \leq q^{18}(q-1)|\mathrm{SL}_9(q)||\mathrm{SL}_2(q)| \qquad \text{by Theorem 3.2.1}$$

$$\leq q^{18+1+80+3} = q^{102} \qquad \text{by Lemma 2.6.4.}$$

There is 1 conjugacy class each for P_1 and P_{10} type subgroups. As noted before $|H| = \frac{|G|}{m(G)}$. So,

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (65 + 11q^{0.55})q^{102}.$$

Therefore

$$\begin{aligned}
c_{\mathrm{PSL}_{11}(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 2 + \frac{((65 + 11q^{0.55})q^{102})m(G)}{|G|} \leq 2 + \frac{(65 + 11q^{0.55})q^{102}}{q^{108}} && \text{by Eq. (4.10)} \\
&\leq 2 + \frac{65 + (11 \times 3^{0.55})}{3^6} < 2 + 0.12 < 2.85 && \text{since } q \geq 3.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSL}_{12}(q)$

Theorem 4.3.13. *Let $q \geq 2$ and let $S = \mathrm{PSL}_{12}(q)$. Then $1 - \frac{2.85}{m(S)} \leq P_2(S)$.*

Proof. Let $q \geq 2$ and let $G = \mathrm{SL}_{12}(q)$. Here $m(G) = \frac{q^{12}-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned}
|G|/m(G) &= q^{11} |\mathrm{GL}_{11}(q)| && \text{by Proposition 3.2.6} \\
&\geq q^{11} q^{11^2-2} = q^{130} && \text{by Lemma 2.6.4.} \qquad (4.11)
\end{aligned}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Tables 8.76 & 8.77] describe the types of maximal subgroups that may appear in $\mathrm{SL}_{12}(q)$.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 or P_{11} type. We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.77] for the non-geometric subgroups.

We note that there are 9 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1 or P_{11} type. Since $n = 12$ there are five divisors of n larger than 1, so there are 5 conjugacy classes of $\mathcal{C}_2$ subgroups. There are only two distinct prime factor of 12 and so there are only 2 conjugacy classes of $\mathcal{C}_3$ subgroups. In the case of $\mathcal{C}_4$ there are two possibilities for n_1, either $n_1 = 2$ or $n_1 = 3$. In the first case there are at most 2 conjugacy classes of such $\mathcal{C}_4$ type subgroups, in the second there is at most 1. Therefore there are at most 3 conjugacy classes of $\mathcal{C}_4$ subgroups. There are at most 12 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are no $\mathcal{C}_6$ nor $\mathcal{C}_7$ subgroups, since $n = 12$ is not a proper power. There are at most 30 conjugacy classes of $\mathcal{C}_8$ subgroups. Finally we note that, by [6, Table 8.77], there are at most 24 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 9 + 5 + 2 + 3 + 12q^{0.55} + 30 + 24 = 73 + 12q^{0.55}.$$

By Proposition 3.2.6, if $H \in \mathcal{M}_1$ then H has order at most that of a subgroup of P_2 type. That means that

$$\begin{aligned} |H| &\leq q^{20}(q-1)|\mathrm{SL}_{10}(q)||\mathrm{SL}_2(q)| && \text{by Theorem 3.2.1} \\ &\leq q^{20+1+99+3} = q^{123} && \text{by Lemma 2.6.4.} \end{aligned}$$

There is 1 conjugacy classes each for P_1 and P_{11}. As noted before $|H| = \frac{|G|}{m(G)}$. Therefore,

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (73 + 12q^{0.55})q^{123}.$$

and

$$\begin{aligned} c_{\mathrm{PSL}_{12}(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 2 + \frac{((73 + 12q^{0.55})q^{123})m(G)}{|G|} \leq 2 + \frac{((73 + 12q^{0.55})q^{123}}{q^{130}} && \text{by Eq. (4.6)} \\ &\leq 2 + \frac{73 + (12 \times 2^{0.55})}{2^7} < 2 + 0.71 < 2.85. \end{aligned}$$

The result follows from Lemma 2.4.2. □

4.3.2 Symplectic groups

Let us now tackle the case of $\mathrm{PSp}_n(q)$ for even $n \leq 12$, here we will show that $c_S \leq 6067/1440 \approx 4.214$ rounded up to 3 decimal places, for $S = \mathrm{PSp}_n(q)$ for $4 \leq n \leq 12$. The approach will be uniform throughout: since we have ready information about the subgroup structure $\mathrm{Sp}_n(q)$ for $n \leq 12$ in [6] we are able to find an upper bound for

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of the conjugacy class representatives of maximal subgroups of $\mathrm{Sp}_n(q)$. By then applying Lemma 2.4.8 we may find appropriate bounds for $c_{\mathrm{PSp}_n(q)}$.

Calculating α for $\mathrm{PSp}_4(q)$

Theorem 4.3.14. *Let $q \geq 3$ and let $S = \mathrm{PSp}_4(q)$. Then $1 - \frac{c}{m(S)} \leq P_2(S)$, where $c = \frac{6067}{1440} \approx 4.214$ rounded up to 3 decimal places. Equality occurs if and only if $q = 4$.*

Proof. Both cases of $q = 3$ and 4 follow from Theorem 4.3.1.

For $5 \leq q \leq 31$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

Table 4.8: Maximal subgroups of $\mathrm{Sp}_4(q)$ for $q > 2$

Class	Subgroup	Details	Number of conjugacy classes
$\mathcal{C}_1$	$[q^{1+2}] : ((q-1)\times \mathrm{Sp}_2(q))$		1
$\mathcal{C}_1$	$[q^3] : \mathrm{GL}_2(q)$		1
$\mathcal{C}_2$	$\mathrm{Sp}_2(q)^2 : 2$		1
$\mathcal{C}_2$	$\mathrm{GL}_2(q).2$	$q \geq 5$ odd	1
$\mathcal{C}_3$	$\mathrm{Sp}_2(q^2) : 2$		1
$\mathcal{C}_3$	$\mathrm{GU}_2(q).2$	$q \geq 5$ odd	1
$\mathcal{C}_5$	$\mathrm{Sp}_4(q_0).(2, r, q-1)$	$q = q_0^r$, r prime	$(2, r)$
$\mathcal{C}_6$	$2_-^{1+4}.\mathrm{S}_5$	$q = p \equiv \pm 1 \mod 8$	2
$\mathcal{C}_6$	$2_-^{1+4}.\mathrm{A}_5$	$q = p \equiv \pm 3 \mod 8$	2
$\mathcal{C}_8$	$\mathrm{SO}_4^+(q)$	q even	1
$\mathcal{C}_8$	$\mathrm{SO}_4^-(q)$	q even	1
$\mathcal{S}$	$2^{\cdot}\mathrm{A}_6$	$q = p \equiv 5, 7 \mod 12$	1
$\mathcal{S}$	$2^{\cdot}\mathrm{S}_6$	$q = p \equiv 1, 11 \mod 12$	2
$\mathcal{S}$	$2^{\cdot}\mathrm{A}_7$	$q = 7$	1
$\mathcal{S}$	$\mathrm{SL}_2(q)$	$p \geq 5$, $q \geq 7$	1
$\mathcal{S}$	$\mathrm{Sz}(q)$	$q \geq 8$ even	1

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSp}_4(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 6067/1440. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 32$ and let $G = \mathrm{Sp}_4(q)$. Then $m(G) = \frac{q^4-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^4(q-1)(q^2-1) \geq q^4 \frac{31q}{32} \frac{1023q^2}{1024} > 0.96q^7 \qquad \text{since } q \geq 32. \tag{4.12}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.8 describes the types of maximal subgroups that appear in $\mathrm{SU}_4(q)$ as listed in [6, Tables 8.12 & 8.13].

If H lies in $\mathcal{C}_1$ then we note that

$$|H| = \frac{|G|}{m(G)}.$$

There are at most two conjugacy classes of $\mathcal{C}_1$ type groups.

If $H \cong \mathrm{Sp}_2(q)^2 : 2$ then by Lemma 2.6.4 we have that $|H| < 2q^6$. There is at most one conjugacy class of H in $\mathrm{Sp}_4(q)$.

If $H \cong \mathrm{GL}_2(q).2$ then by Lemma 2.6.4 $|H| < 2q^4$. This subgroup only appears when q is odd and there is at most one conjugacy class of H in $\mathrm{Sp}_4(q)$.

If $H \cong \mathrm{Sp}_2(q^2) : 2$ then $|H| < 2q^6$ by Lemma 2.6.4. Furthermore there is at most one conjugacy class of such an H in $\mathrm{Sp}_4(q)$.

If $H \cong \mathrm{GU}_2(q).2$ type group then by Lemma 2.6.4 and Theorem 2.5.29 we have that $|H| < 2(q+1)q^3$. Since $q > 3$ we have that $|H| \leq 3q^4$. This subgroup only appears when q is odd and there is at most one conjugacy class of such an H in $\mathrm{Sp}_4(q)$.

If H lies in $\mathcal{C}_5$ then

$$\begin{aligned} |H| &\leq (2, q-1)q_0^4(q_0^4-1)(q_0^2-1) && \text{by Theorem 2.5.29} \\ &= (2, q-1)q^2(q^2-1)(q-1) < (2, q-1)q^5 && \text{since } q_0 \leq q^{1/2}. \end{aligned}$$

We note that there are at most 2 conjugacy classes of such subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4.

If H lies in $\mathcal{C}_6$, then $|H| \leq 3840$. Note that there are at most 2 conjugacy classes of $\mathcal{C}_6$ subgroups, and that these only occur when q is odd.

If H lies in $\mathcal{C}_8$ then we note that

$$|H| \leq \frac{|G|}{m(G)}.$$

Note that H only exists if q is even, and if q is even there are at most 2 classes of $\mathcal{C}_8$ subgroups.

Finally, if H lies in $\mathcal{S}$ and q is odd, then we have four possibilities for the shape of H. However for all four cases, we notice that $|H| \leq q^4$ since $q \geq 32$. If $H \cong \mathrm{SL}_2(q)$ then $|H| \leq q^4$ follows from Lemma 2.6.4. There are at most 3 different conjugacy classes of subgroups of $\mathrm{Sp}_4(q)$ belonging to $\mathcal{S}$.

If q is even, and H lies in $\mathcal{S}$, then $H \cong Sz(q)$. Then from [47] we have

$$|H| = q^2(q^2+1)(q-1) < q^5.$$

Here there is at most one conjugacy class of such a subgroup.

Now suppose that q is odd, then by the above we have

$$\begin{aligned}\sum_{M\in\mathcal{M}} |M| &\leq 2\frac{|G|}{m(G)} + 2q^6 + 2q^4 + 2q^6 + 3q^4 + (2q^{0.55} \times 2q^5) + (2 \times 3840) + 3q^4 \\ &= 2\frac{|G|}{m(G)} + 4q^6 + 4q^{5.55} + 8q^4 + 7680 < 2\frac{|G|}{m(G)} + 9q^6 && \text{since } q \geq 32.\end{aligned}$$

Consequently

$$\begin{aligned} c_{\mathrm{PSp}_4(q)} &\leq \sum_{M\in\mathcal{M}_G} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 2 + \frac{9q^6 m(G)}{|G|} \leq 2 + \frac{9q^6}{0.96q^7} && \text{by Eq. (4.12)} \\ &\leq 2 + \frac{9}{0.96 \times 32} < 2 + 0.3 < \frac{6067}{1440} && \text{since } q \geq 32.\end{aligned}$$

Suppose that q is even, then we have

$$\sum_{M\in\mathcal{M}} |M| \leq 4\frac{|G|}{m(G)} + 2q^6 + 2q^6 + (q^{0.55} \times q^5) + q^5 < 4\frac{|G|}{m(G)} + 6q^6 \qquad \text{since } q \geq 32.$$

Consequently

$$\begin{aligned} c_{\mathrm{PSp}_4(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 4 + \frac{6q^6 m(G)}{|G|} \leq 4 + \frac{6q^6}{0.96q^7} && \text{by Eq. (4.12)} \\ &\leq 4 + \frac{6}{0.96 \times 32} < 4 + 0.20 < \frac{6067}{1440} && \text{since } q \geq 32.\end{aligned}$$

The result now follows from Lemma 2.4.2. $\square$

Calculating α for $\mathrm{PSp}_6(q)$

Theorem 4.3.15. *Let $q \geq 2$ and let $S = \mathrm{PSp}_6(q)$. Then $1 - \frac{4.2}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$, the result follows from Theorem 4.3.1.

For $3 \leq q \leq 5$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSp}_6(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.2. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 7$ and let $G = \mathrm{Sp}_6(q)$. Here $m(G) = \frac{q^6-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned} |G|/m(G) &= q^5(q-1)|\mathrm{Sp}_4(q)| = q^9(q-1)(q^2-1)(q^4-1) \\ &\geq q^9 \frac{6q}{7}\frac{48q^2}{49}\frac{2400q^4}{2401} = \frac{691200q^{16}}{823543} \geq 0.83q^{16} \qquad \text{since } q \geq 7. \end{aligned} \tag{4.13}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. A table of the non-geometric maximal subgroups may be found in [6, Table 8.29].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that do not lie in $\mathcal{C}_1$, $\mathcal{C}_5$ or $\mathcal{C}_8$, nor are of the shape $\mathrm{G}_2(q)$.

If $H \in \mathcal{C}_i$ for $2 \leq i \leq 7$, then we know the shape of H and the number of conjugacy classes of subgroups H by Theorem 3.3.1. Applying Lemma 2.6.4 one can show that $|H| \leq q^{10}$ when $q \geq 7$. There are at most 2 conjugacy classes of subgroups in $\mathcal{C}_2$ since there is only one even proper divisor of 6. There are at most 2 conjugacy classes of subgroups in $\mathcal{C}_3$, one of type $\mathrm{Sp}_2(q^3)$ and one of type $\mathrm{GU}_3(q)$. There is at most 1 conjugacy class of subgroups in $\mathcal{C}_4$. Finally we note that there are no subgroups belonging to $\mathcal{C}_6$ and $\mathcal{C}_7$ since 6 is not a prime power. Therefore $|\mathcal{M}_1 \cap \bigcup_{i=2}^{7} \mathcal{C}_i| \leq 5$.

If $H \in \mathcal{M}_1 \cap \mathcal{S}$ then H is listed in [6, Table 8.29]. One can show that $|H| \leq 1209600 < q^8$ for $q \geq 7$, when H does not have shape $2^{\cdot}\mathrm{PSL}_2(q)$ or $\mathrm{G}_2(q)$.Of particular note we have that if H has shape $2^{\cdot}\mathrm{J}$ then $|H| = 1209600$ which is the largest of such subgroups. In the case where H has shape $2^{\cdot}\mathrm{PSL}_2(q)$ we may use Lemma 2.6.4 to show that the $|H| \leq q^8$. By [6, Table 8.29], $|\mathcal{M}_1 \cap \mathcal{S}| \leq 17$. Therefore

$$\sum_{M \in \mathcal{M}_1} |M| \leq (5q^{10} + 17q^8).$$

If H is a P_1 subgroup or lies in $\mathcal{C}_8$ then we bound their order above by $\frac{|G|}{m(G)}$. Here there is only 1 conjugacy class of P_1 type subgroup H in $\mathrm{Sp}_6(q)$, while there are 2 conjugacy classes of subgroup H lying in $\mathcal{C}_8$.

If H is a P_2 type group then

$$|H| = q^9(q^2-1)^2(q-1) < q^{14} \qquad \text{by Theorem 2.5.29.}$$

If H is a P_3 type group then

$$|H|q^9(q^3-1)(q^2-1)(q-1) < q^{15} \qquad \text{by Theorem 2.5.29.}$$

If $H \cong \mathrm{Sp}_2(q) \times \mathrm{Sp}_4(q)$ then

$$|H| \leq q^3 q^{10} < q^{13} \qquad \text{by Theorem 2.6.4.}$$

If H lies in $\mathcal{C}_5$ then

$$\begin{aligned} |H| &\leq 2q_0^{21} && \text{by Lemma 2.6.4} \\ &= 2q^{10.5} && \text{since } q_0 \leq q^{1/2}. \end{aligned}$$

We note that there are at most 2 conjugacy classes of such subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. Now, if $H \cong \mathrm{G}_2(q)$ then according to the ATLAS [9] we have

$$|H| = q^6(q^6-1)(q^2-1) < q^{14}.$$

Furthermore by [6, Table 8.29], there is only one conjugacy class of H. Consequently

$$\begin{aligned} \sum_{M \in \mathcal{M}} |M| &\leq 3\frac{|G|}{m(G)} + 5q^{10} + 17q^8 + q^{14} + q^{15} + q^{13} + (2q^{0.55} \times 2q^{10.5}) + q^{14} \\ &= 3\frac{|G|}{m(G)} + q^{15} + 2q^{14} + q^{13} + 4q^{11.05} + 5q^{10} + 17q^8 \\ &< 3\frac{|G|}{m(G)} + 2q^{15} && \text{since } q \geq 7. \end{aligned}$$

And so

$$\begin{aligned} c_{\mathrm{PSp}_6(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 3 + \frac{2q^{15}m(G)}{|G|} \leq 3 + \frac{2q^{15}}{0.83q^{16}} && \text{by Eq. (4.13)} \\ &\leq 3 + \frac{2}{0.83 \times 7} < 3 + 0.35 < 4.2 && \text{since } q \geq 7. \end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSp}_8(q)$

Theorem 4.3.16. *Let $q \geq 2$ and let $S = \mathrm{PSp}_8(q)$. Then $1 - \frac{4.2}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2, 3$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSp}_8(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.2. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 4$ and let $G = \mathrm{Sp}_8(q)$. Here $m(G) = \frac{q^8-1}{q-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^7(q-1)|\mathrm{Sp}_6(q)| \geq q^7 q^{\frac{36+6}{2}-1} = q^{27} \tag{4.14}$$

by Lemma 2.6.4.

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. A table of the non-geometric maximal subgroups may be found in [6, Table 8.49].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, and do not lie in $\mathcal{C}_8$. Note that by Theorem 3.3.1, $|\mathcal{M}\backslash\mathcal{M}_1| = 3$. For $H \in \mathcal{M}\backslash\mathcal{M}_1$, we bound the order of H by $\frac{|G|}{m(G)}$.

If H is a P_4 type group then by Lemma 2.6.4 one can calculate that $|H| \leq q^{26}$. We note that there is at most one $\mathrm{Sp}_8(q)$-conjugacy class of such subgroups. Otherwise for $H \in \mathcal{M}_1 \cap \mathcal{C}_1$ we have that there are at most 3 other conjugacy classes in $\mathrm{Sp}_8(q)$: they are of P_2, P_3 and $\mathrm{Sp}_6(q) \perp \mathrm{Sp}_2(q)$ type subgroups. By Theorem 3.3.1 and Lemma 2.6.4, we have for these cases that $|H| < q^{25}$. In further detail, if H is a P_2 type subgroup then, $|H| < q^{11+4+10} = q^{25}$. If H is a P_3 type subgroup then $|H| < q^{12+9+3} = q^{24}$. Finally if H is a $\mathrm{Sp}_6(q) \perp \mathrm{Sp}_2(q)$ type subgroup then $|H| < q^{21+3} = q^{24}$.

If $H \in \mathcal{C}_i$ for $2 \leq i \leq 7$, then by Theorem 3.3.1 one can show that $|H| \leq q^{21}$ where $q \geq 4$. Of particular note is the case where H lies in $\mathcal{C}_6$, here $|H| \leq 6635520 \leq 4^{21}$, for the remaining cases we obtain the result by applying Lemma 2.6.4.

By Theorem 3.3.1 there are at most 3 conjugacy classes of $\mathcal{C}_2$ subgroups, since there are two even proper divisors of 8. There are at most 2 conjugacy classes of $\mathcal{C}_3$, one of type $\mathrm{Sp}_4(q^2)$ and one of type $\mathrm{GU}_4(q)$. There is at most 1 conjugacy class of $\mathcal{C}_4$ subgroups since the only possibility for n_2 is $n_2 = 4$. We note that there are at most 2 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 2 conjugacy classes of $\mathcal{C}_6$ subgroups. Also, there is at most one conjugacy class of $\mathcal{C}_7$ subgroups since the only possibility for m is $m = 2$.

Finally if $H \in \mathcal{S}$ then again one can show, with reference to [6, Table 8.49] that $|H| \leq q^{11}$ for $q \geq 4$. Note that if H has the shape $2\,{}^{\cdot}\mathrm{PSL}_2(q)$ or $2\,{}^{\cdot}\mathrm{PSL}_2(q^3).3$ then the result follows from Lemma 2.6.4. For the remaining cases, we note that the order of H must divide $4|\mathrm{PSL}_2(7)|$, $2|\mathrm{S}_6|$, $|\mathrm{PSL}_2(17)|$ or $|\mathrm{S}_{10}|$, and that therefore $|H| \leq 4^{11}$. By [6, Table 8.49], $|\mathcal{M}_1 \cap \mathcal{S}| \leq 8$.

Therefore

$$\sum_{M \in \mathcal{M}} |M| \leq 3\frac{|G|}{m(G)} + q^{26} + 3q^{25} + (9 + 2q^{0.55})q^{21} + 8q^{11} < 3\frac{|G|}{m(G)} + 2q^{26} \qquad \text{since } q \geq 4.$$

Consequently

$$\begin{aligned}
c_{\mathrm{PSp}_8(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8}\\
&= 3 + \frac{(2q^{26})m(G)}{|G|} \leq 3 + \frac{2q^{26}}{q^{27}} && \text{by Eq. (4.14)}\\
&\leq 3 + \frac{2}{4} = 3 + 0.5 < 4.2 && \text{since } q \geq 4.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSp}_{10}(q)$

Theorem 4.3.17. *Let $q \geq 2$ and let $S = \mathrm{PSp}_{10}(q)$. Then $1 - \frac{4.2}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$ we calculate

$$\sum_{M\in\mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSp}_{10}(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.2. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 3$ and let $G = \mathrm{Sp}_{10}(q)$. Here $m(G) = \frac{q^{10}-1}{q-1}$, by Theorem 2.2.7. Furthermore one can calculate that

$$|G|/m(G) = q^9(q-1)|\mathrm{Sp}_8(q)| \geq q^9 q^{\frac{64+8}{2}-1} = q^{44} \qquad \text{by Lemma 2.6.4.} \tag{4.15}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. [6, Table 8.64] describes the types of geometric maximal subgroups that may appear in $\mathrm{Sp}_{10}(q)$. A table of the non-geometric maximal subgroups may also be found in [6, Table 8.65].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, and do not lie in $\mathcal{C}_8$. Note that by Table 3.3.1, $|\mathcal{M}\backslash\mathcal{M}_1| = 3$. We bound the order of $H \in \mathcal{M}\backslash\mathcal{M}_1$, by $\frac{|G|}{m(G)}$.

By Lemma 3.3.10, if H is a maximal subgroup of $\mathrm{Sp}_n(q)$ that is not of P_1 type or in the $\mathcal{C}_8$ class then H has order at most that of a subgroup of P_2 type. That means that

$$\begin{aligned}
|H| &\leq q^{15}|\mathrm{GL}_2(q)||\mathrm{Sp}_6(q)| && \text{by Theorem 3.3.1}\\
&\leq q^{15+4+21} = q^{40} && \text{by Lemma 2.6.4.}
\end{aligned}$$

We may bound $|\mathcal{M}_1|$ using Theorem 3.3.1 for the geometric subgroups and [6, Table 8.81] for the non-geometrics. There are at most 6 conjugacy classes of $\mathcal{C}_1$ subgroup not of P_1 type. There are at most 2 conjugacy classes of $\mathcal{C}_2$ subgroups, since there is one even proper divisor of 10. There are at most 2 conjugacy classes of $\mathcal{C}_3$, one of type $\mathrm{Sp}_2(q^5)$ and one of type $\mathrm{GU}_5(q)$. There is at most 1 conjugacy class of $\mathcal{C}_4$ subgroups since the only possibility for n_2 is $n_2 = 5$. We note that there are at most 2 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by

$q^{0.55}$ by Lemma 2.8.4. There are no $\mathcal{C}_6$ nor $\mathcal{C}_7$ subgroups since 8 is not a prime power. Finally from [6, Table 8.65] we see that there are at most 11 non-geometric subgroups of $\mathrm{Sp}_{10}(q)$.

Therefore
$$|\mathcal{M}_1| \leq 6+2+2+1+2q^{0.55}+11 = 2q^{0.55}+22.$$
Consequently
$$\sum_{M\in\mathcal{M}} |M| \leq 3\frac{|G|}{m(G)} + (22+2q^{0.55})q^{40} < 3\frac{|G|}{m(G)} + 10q^{41} \qquad \text{since } q \geq 3.$$
Therefore
$$\begin{aligned} c_{\mathrm{PSp}_{10}(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 3 + \frac{10q^{41}m(G)}{|G|} \leq 3 + \frac{10q^{41}}{q^{44}} && \text{by Eq. (4.15)} \\ &\leq 3 + \frac{10}{3^3} < 3 + 0.38 < 4.2 && \text{since } q \geq 3. \end{aligned}$$
The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSp}_{12}(q)$

Theorem 4.3.18. *Let $q \geq 2$ and let $S = \mathrm{PSp}_{12}(q)$. Then $1 - \frac{4.2}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$ we calculate
$$\sum_{M\in\mathcal{M}} |M|,$$
the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSp}_{12}(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.2. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 3$ and let $G = \mathrm{Sp}_{12}(q)$. Here $m(G) = \frac{q^{12}-1}{q-1}$, by Theorem 2.2.7. Furthermore
$$|G|/m(G) = q^{11}(q-1)|\mathrm{Sp}_{10}(q)| \geq q^{11}q^{\frac{100+10}{2}-1} = q^{65} \qquad \text{by Lemma 2.6.4.} \tag{4.16}$$
Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. A table of the non-geometric maximal subgroups may be found in [6, Table 8.81].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, and do not lie in $\mathcal{C}_8$. Note that by Table 3.3.1, $|\mathcal{M}\backslash\mathcal{M}_1| = 3$. We bound the order of $H \in \mathcal{M}\backslash\mathcal{M}_1$, by $\frac{|G|}{m(G)}$.

By Lemma 3.3.10, if H is a maximal subgroup of $\mathrm{Sp}_n(q)$ that is not of P_1 type or in the $\mathcal{C}_8$ class then H has order at most that of a subgroup of P_2 type. That means that
$$\begin{aligned} |H| &\leq q^{19}|\mathrm{GL}_2(q)||\mathrm{Sp}_8(q)| && \text{by Theorem 3.3.1} \\ &\leq q^{19+4+36} = q^{59} && \text{by Lemma 2.6.4.} \end{aligned}$$

We may bound $|\mathcal{M}_1|$ using Theorem 3.3.1 for the geometric subgroups and [6, Table 8.81] for the non-geometrics. There are at most 7 conjugacy classes of $\mathcal{C}_1$ subgroup not of P_1 type. There are at most 4 conjugacy classes of $\mathcal{C}_2$ subgroups, since there are three even proper divisors of 12. There are at most 3 conjugacy classes of $\mathcal{C}_3$, one of type $\mathrm{Sp}_6(q^2)$, one of type $\mathrm{Sp}_4(q^3)$ and one of type $\mathrm{GU}_6(q)$. Also, there are at most 3 conjugacy classes of $\mathcal{C}_4$ subgroups, since there are only three possibilities for n_1, which are $n_1 = 2$, 4 and 6 . We note that there are at most 2 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are no $\mathcal{C}_6$ nor $\mathcal{C}_7$ subgroups since 12 is not a prime power. Finally from [6, Table 8.81] we see that there are at most 15 non-geometric subgroups of $\mathrm{Sp}_{12}(q)$.

Therefore

$$|\mathcal{M}_1| \leq 7 + 4 + 3 + 3 + 2q^{0.55} + 15 = 2q^{0.55} + 32.$$

Consequently

$$\sum_{M \in \mathcal{M}} |M| \leq 3\frac{|G|}{m(G)} + (32 + 2q^{0.55})q^{59}.$$

Therefore

$$\begin{aligned}
c_{\mathrm{PSp}_{12}(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 3 + \frac{(32 + 2q^{0.55})q^{59} m(G)}{|G|} \\
&\leq 3 + \frac{(32 + 2q^{0.55})q^{59}}{q^{65}} && \text{by Eq. (4.16)} \\
&\leq 3 + \frac{32 + (2 \times 3^{0.55})}{3^6} < 3 + 0.049 < 4.2 && \text{since } q \geq 3.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

4.3.3 Unitary groups

Let us now tackle the case of $\mathrm{PSU}_n(q)$ for $3 \leq n \leq 12$, here we will show that $c_S \leq 1199/243 \approx 4.935$ rounded up to 3 decimal places, for $S = \mathrm{PSU}_n(q)$ for $4 \leq n \leq 12$. The approach will be uniform throughout: since we have ready information about the subgroup structure of $\mathrm{SU}_n(q)$ for $n \leq 12$ in [6] we may find an upper bound for

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of the conjugacy class representatives of maximal subgroups of $\mathrm{SU}_n(q)$. By then applying Lemma 2.4.8 we may find appropriate bounds for $c_{\mathrm{PSU}_n(q)}$.

Calculating α for $\mathrm{PSU}_3(q)$

Theorem 4.3.19. *Let $q \geq 3$ and let $S = \mathrm{PSU}_3(q)$. Then $1 - \frac{4.9}{m(S)} \leq P_2(S)$.*

Proof. If $3 \leq q \leq 5$ then the result follows from Theorem 4.3.1.

For $7 \leq q \leq 13$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_3(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.9. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 16$ and let $G = \mathrm{SU}_3(q)$. Then $m(G) = q^3 + 1$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^3(q^2 - 1) = q^3 \frac{48q^2}{49} > 0.97q^5 \qquad \text{since } q \geq 7. \tag{4.17}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.9 describes the types of maximal subgroups that may appear in $\mathrm{SU}_3(q)$ as listed in [6, Tables 8.5 & 8.6]. We note that earlier results listing the maximal subgroups of $\mathrm{SU}_3(q)$ can be found in [19] for q even, and [42] for q odd.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that do not lie in $\mathcal{C}_1$. Note that by Table 4.9, $|\mathcal{M} \backslash \mathcal{M}_1| = 2$. We bound the order of $H \in \mathcal{M} \backslash \mathcal{M}_1$ by $\frac{|G|}{m(G)}$. We first note that if $H \in \mathcal{C}_5$ then either $H \cong \mathrm{SU}_3(q_0).[(\frac{q+1}{q_0+1}, 3)]$ or $H \cong (q+1, 3) \times \mathrm{SO}_3(q)$. In the first case

$$\begin{aligned} |H| &\leq 3q_0^3(q_0^3 + 1)(q_0^2 - 1) && \text{by Theorem 2.5.29} \\ &\leq 3q^1(q^1 + 1)(q^{2/3} - 1) < 6q^3 && \text{since } q_0 \leq q^{1/3}. \end{aligned}$$

Furthermore there are at most 3 conjugacy classes of $\mathcal{C}_5$ subgroups for each q_0 such that $q_0^r = q$ such that r is prime. In other words, if $q = p^e$ for prime p, it is the number of prime factors of e. We may bound e above by $\log_2(q)$ and bound the number of prime factors of e by $\log_2(e)$ also. Consequently, we may bound the number of r by $\log_2(\log_2(q))$. This in turn can be bounded above by $q^{0.3025}$ by Lemma 2.8.4.

In the second case

$$|H| \leq 3q(q^2 - 1) < 3q^3 \qquad \text{by Theorem 2.5.29.}$$

Here there are at most 3 conjugacy classes of such subgroups in $\mathrm{SU}_3(q)$. If $H \in \mathcal{M}_1 \backslash \mathcal{C}_5$ then by Lemma 2.6.4 one can show that $|H| \leq q^3$ since $q \geq 23$. Also from Table 4.9 we may see that $|\mathcal{M}_1 \backslash \mathcal{C}_5| \leq 5 + \frac{(q+1,9)}{3} + (q+1, 3) \leq 5 + 3 + 3 = 11$.

Therefore

$$\sum_{M \in \mathcal{M}} |M| \leq 2\frac{|G|}{m(G)} + (3q^{0.3025} \times 6q^3) + (3 \times 3q^3) + 11q^3$$

Table 4.9: Maximal subgroups of $\mathrm{SU}_3(q)$ for $q > 5$

Class	Subgroup	Details	Number of conjugacy classes
$\mathcal{C}_1$	$[q^{1+2}] : (q^2 - 1)$		1
$\mathcal{C}_1$	$\mathrm{GU}_2(q)$		1
$\mathcal{C}_2$	$(q+1)^2 : \mathrm{S}_3$		1
$\mathcal{C}_3$	$(q^2 - q + 1) : 3$		1
$\mathcal{C}_5$	$\mathrm{SU}_3(q_0).[(\frac{q+1}{q_0+1}, 3)]$	$q = q_0^r$, r odd prime	$(\frac{q+1}{q_0+1}, 3)$
$\mathcal{C}_5$	$(q+1, 3) \times \mathrm{SO}_3(q)$		$(q+1, 3)$
$\mathcal{C}_6$	$3_+^{1+2} : \mathrm{Q}_8.\frac{(q+1,9)}{3}$		$\frac{(q+1,9)}{3}$
$\mathcal{S}$	$(q+1, 3) \times \mathrm{PSL}_2(7)$		$(q+1, 3)$
$\mathcal{S}$	$3^{\cdot}\mathrm{A}_6$		3

$$= 2\frac{|G|}{m(G)} + 18q^{3.3025} + 20q^3.$$

Consequently

$$\begin{aligned}
c_{\mathrm{PSU}_3(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 2 + \frac{(18q^{3.3025} + 20q^3)m(G)}{|G|} \leq 2 + \frac{18q^{3.3025} + 20q^3}{0.97q^5} && \text{by Eq. (4.17)} \\
&\leq 2 + \frac{(18 \times 7^{3.3025}) + (20 \times 7^3)}{0.97 \times 7^5} < 2 + 1.2 < 4.9 && \text{since } q \geq 7.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSU}_4(q)$

Theorem 4.3.20. *Let $q \geq 2$ and let $S = \mathrm{PSU}_4(q)$. Then $1 - \frac{c}{m(S)} \leq P_2(S)$, where $c = \frac{1199}{243} \approx 4.935$ rounded up to 3 decimal places. Equality occurs if and only if $q = 3$.*

Proof. The case of $q = 2, 3$ the result follows from Theorem 4.3.1.

For $4 \leq q \leq 9$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_4(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 1199/243. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 11$ and $G = \mathrm{SU}_4(q)$. Then $m(G) = q^4 + q^3 + q + 1$, by Theorem 2.2.7. Furthermore

$$\begin{aligned}|G|/m(G) &= q^4|\mathrm{SL}_2(q^2)|(q-1) = q^6(q^4-1)(q-1) && \text{by Theorem 2.5.29}\\ &= q^6\frac{14640q^4}{14641}\frac{10q}{11} \geq 0.9q^{11} && \text{since } q \geq 11. \end{aligned} \tag{4.18}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Table 4.10 describes the types of maximal subgroups that may appear in $\mathrm{SU}_4(q)$, as listed in [6, Tables 8.10 & 8.11].

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of subgroups that do not lie in $\mathcal{C}_1$, nor are isomorphic to $\mathrm{Sp}_4(q).(q+1,2)$. If $H \in \mathcal{M}_1$, then $|H| \leq q^7$ for $q \geq 11$. In particular we note some useful bounds that are a consequence of Lemma 2.6.4; note that $|\mathrm{SU}_2(q)| \leq q^3$, $|\mathrm{SL}_2(q^2)| \leq q^6$, $|\mathrm{SU}_4(q_0)| \leq q_0^{15} \leq q^5$, and $|\mathrm{SO}^-(q)| \leq 2q^6$.

We note that for each q_0 such that $q_0^r = q$ there exists at most 1 conjugacy classes of subgroups $H \cong \mathrm{SU}_4(q_0)$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. Therefore

$$|\mathcal{M}_1| \leq 3 + q^{0.55} + (q+1,4) + 4 + 2(q+1,4) = 7 + (3\times 4) + q^{0.55} = 19 + q^{0.55}.$$

If H is a P_1 type subgroup then

$$|H| = q^5|\mathrm{SU}_2(q)|(q^2-1) = q^6(q^2-1)^2 < q^{10} \qquad \text{by Theorem 2.5.29.}$$

In this case there is a maximum of one conjugacy class of such a subgroup of $\mathrm{SU}_4(q)$.

If H lies in $\mathcal{C}_1$ but not a P_1 type subgroup, or if $H \cong \mathrm{Sp}_4(q).(q+1,2)$ then we bound their order by $\frac{|G|}{m(G)}$. We notice that there are at most four $\mathrm{SU}_4(q)$ conjugacy classes of such subgroups.

Therefore

$$\begin{aligned}\sum_{M\in\mathcal{M}}|M| &\leq 4\frac{|G|}{m(G)} + (19+q^{0.55})q^7 + q^{10}\\ &= 4\frac{|G|}{m(G)} + q^{10} + q^{7.55} + 19q^7 < 4\frac{|G|}{m(G)} + 2q^{10} && \text{since } q \geq 11.\end{aligned}$$

Consequently

$$\begin{aligned}c_{\mathrm{PSU}_4(q)} &\leq \sum_{M\in\mathcal{M}}\frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8}\\ &\leq 4 + \frac{2q^{10}m(G)}{|G|} \leq 4 + \frac{2q^{10}}{0.9q^{11}} && \text{by Eq. (4.18)}\end{aligned}$$

Table 4.10: Maximal subgroups of $\mathrm{SU}_4(q)$ for $q > 3$

Class	Subgroup	Details	Number of conjugacy classes
$\mathcal{C}_1$	$q^{1+4} : \mathrm{SU}_2(q) : (q^2-1)$	P_1 type	1
$\mathcal{C}_1$	$q^4 : \mathrm{SL}_2(q^2) : (q-1)$	P_2 type	1
$\mathcal{C}_1$	$\mathrm{GU}_3(q)$		1
$\mathcal{C}_2$	$(q+1)^3.\mathrm{S}_4$		1
$\mathcal{C}_2$	$\mathrm{SU}_2(q)^2 : (q+1).2$		1
$\mathcal{C}_2$	$\mathrm{SL}_2(q^2).(q-1).2$		1
$\mathcal{C}_5$	$\mathrm{SU}_4(q_0)$	$q = q_0^r$, r odd prime	1
$\mathcal{C}_5$	$\mathrm{Sp}_4(q).(q+1,2)$		$(q+1,2)$
$\mathcal{C}_5$	$\mathrm{SO}_4^+(q).[(q+1,4)]$		$(q+1,4)/2$
$\mathcal{C}_5$	$\mathrm{SO}_4^-(q).[(q+1,4)]$		$(q+1,4)/2$
$\mathcal{C}_6$	$(4 \circ 2^{1+4}).\mathrm{S}_6$	$p = q \equiv 7 \bmod 8$	4
$\mathcal{C}_6$	$(4 \circ 2^{1+4}).\mathrm{A}_6$	$p = q \equiv 3 \bmod 8$	2
$\mathcal{S}$	$(q+1,4) \circ 2^{\cdot}\mathrm{A}_7$		$(q+1,4)$
$\mathcal{S}$	$(q+1,4) \circ 2^{\cdot}\mathrm{PSU}_4(2)$		$(q+1,4)$

$$\le 4 + \frac{2}{0.9 \times 11} \qquad \text{since } q \ge 11$$
$$< 4 + 0.11 < \frac{1199}{243}.$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSU}_5(q)$

Theorem 4.3.21. *Let $q \ge 2$ and let $S = \mathrm{PSU}_5(q)$. Then $1 - \frac{4.9}{m(S)} \le P_2(S)$.*

Proof. The case of $q = 2$ follows from Theorem 4.3.1.

The case of $q = 3, 4$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_5(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.9. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 5$ and let $G = \mathrm{SU}_5(q)$. Then $m(G) = \frac{(q^5+1)(q^4-1)}{q^2-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^7|\mathrm{SU}_3(q)|(q^2-1) \geq q^8 q^{3^2-2} = q^{15} \qquad \text{by Lemma 2.6.4.} \tag{4.19}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. The shape and number of conjugacy classes of the subgroups of $\mathrm{SU}_5(q)$ can be found in Theorem 3.3.1 for the geometric type subgroups and in [6, Tables 8.21] for the non-geometric subgroups.

If H lies in $\mathcal{C}_1$ then we bound its order by $\frac{|G|}{m(G)}$. Tthere are at most 4 conjugacy classes of maximal $\mathcal{C}_1$ groups in G.

Note that $\mathcal{C}_4, \mathcal{C}_7, \mathcal{C}_8$ are empty. If H lies in $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_6$, then one can show using Lemma 2.6.4 that $|H| \leq q^9$ for $q \geq 5$. In further detail if H lies in $\mathcal{C}_2$ then $|H| \leq 5!(q+1)^4$, if H lies in $\mathcal{C}_3$ then $|H| < 5(q^5+1)$ and if H lies in $\mathcal{C}_6$ then $|H| = 15000$. Furthermore there are at most 7 conjugacy classes of such groups.

If $H \in \mathcal{C}_5$ then one can calculate using Lemma 2.6.4 and the fact that $q_0 \leq q^{1/3}$ that $|H| \leq q^{11}$ for $q \geq 5$. We note also that for each q_0 such that $q_0^r = q$ there exists at most 5 conjugacy classes of subgroups H. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. Therefore

$$|\mathcal{M} \cap \mathcal{C}_5| \leq 5q^{0.55} + 5.$$

Finally if $H \in \mathcal{S}$ then [6, Table 8.21] shows that $|H| \leq 5|\mathrm{PSU}_4(2)| = 129600 \leq q^8$ for $q \geq 5$. Furthermore, note that there are at most 10 classes of such subgroups H. Therefore

$$\begin{aligned}
\sum_{M \in \mathcal{M}} |M| &\leq 4\frac{|G|}{m(G)} + 7q^9 + (5q^{0.55}+5)q^{11} + 10q^8 \\
&= 4\frac{|G|}{m(G)} + 5q^{11.55} + 5q^{11} + 7q^9 + 10q^8 < 4\frac{|G|}{m(G)} + 5q^{11.55} + 6q^{11} \qquad \text{since } q \geq 5.
\end{aligned}$$

Consequently

$$\begin{aligned}
c_{\mathrm{PSU}_5(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 4 + \frac{(5q^{11.55}+6q^{11})m(G)}{|G|} \leq 4 + \frac{5q^{11.55}+6q^{11}}{q^{15}} && \text{by Eq. (4.19)} \\
&\leq 4 + \frac{(5 \times 5^{11.55}) + (6 \times 5^{11})}{5^{15}} < 4 + 0.03 < 4.9 && \text{since } q \geq 5.
\end{aligned}$$

The result follows from Lemma 2.4.2.

$\square$

Calculating α for $\mathrm{PSU}_6(q)$

Theorem 4.3.22. *Let $q \geq 2$ and let $S = \mathrm{PSU}_6(q)$. Then $1 - \frac{4.9}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2, 3$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_6(q)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.9. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 4$ and let $G = \mathrm{SU}_6(q)$. Then $m(G) = \frac{(q^6-1)(q^5+1)}{q^2-1}$, by Theorem 2.2.7. Furthermore

$$\begin{aligned} |G|/m(G) &= q^9|\mathrm{SU}_4(q)|(q^2-1) = q^{15}(q^4-1)(q^3+1)(q^2-1)^2 \qquad \text{by Theorem 2.5.29} \\ &\geq q^{15}q^7q^3 = q^{25}. \end{aligned} \tag{4.20}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. From Theorem 3.4.1 we know the shapes of the geometric subgroups of $\mathrm{SU}_6(q)$. We also know the shapes of the non-geometric subgroups from [6, Tables 8.27] .

If H is a P_2 type group then one may calculate using Theorem 3.4.1 and Lemma 2.6.4 that $|H| \leq q^{23}$. Here, $|H| \leq q^{2(12-6)}q^6q^3q^{=}q^{23}$. By Theorem 3.4.1, there is at most 1 conjugacy class of such a group in $\mathrm{SU}_6(q)$.

If $H \cong \mathrm{SU}_4(q) \times \mathrm{SU}_2(q) : (q+1)$ then $|H| \leq q^{20}$ by Lemma 2.6.4. Here also there is at most 1 such conjugacy class.

If $H \in \mathcal{C}_1$, but not a P_2 type subgroup nor $H \cong \mathrm{SU}_4(q) \times \mathrm{SU}_2(q) : (q+1)$ then we bound the order of H by $\frac{|G|}{m(G)}$. Here we notice that there are at most 3 such conjugacy classes of maximal subgroups in $\mathrm{SU}_6(q)$: the classes of P_1, P_3 and for $\mathrm{GU}_4(q) \times \mathrm{GU}_2(q)$ type subgroups.

If H lies in $\mathcal{C}_2$, $\mathcal{C}_3$ or $\mathcal{C}_4$, then we may show using Lemma 2.6.4 that $|H| \leq q^{20}$. Expanding further, we note that $|\mathrm{SU}_3(q)| \leq q^8$ and that $|\mathrm{SU}_2(q)| \leq q^3$, and in particular $|H| \leq 2q^{2\times 9}(q+1) < q^{20}$. Furthermore, there are at most 4 conjugacy classes of $\mathcal{C}_2$ subgroups, and only one conjugacy class each of $\mathcal{C}_3$ and $\mathcal{C}_4$ subgroups, therefore $|\mathcal{M} \cap \bigcup_{i=2}^{4} \mathcal{C}_i| \leq 6$.

If $H \in \mathcal{C}_5$ then, by Lemma 2.6.4 and the fact that $q_0 \leq q^{1/3}$, we show that $|H| \leq 3q^{21}$. Note here that, $|\mathrm{SU}_6(q_0)| \leq q_0^{35} < q^{12}$, $|\mathrm{Sp}_6(q)| \leq q^{(36+6)/2} = q^{21}$ and $|\mathrm{SO}_6^-(q)| \leq q^{15}$.

We note also, that for each q_0 such that $q_0^r = q$ there exists at most 6 conjugacy classes of subgroups $H \cong \mathrm{SU}_6(q_0).[(\frac{q+1}{q_0+1}, 6)]$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. So

$$|\mathcal{M} \cap \mathcal{C}_5| \leq (\frac{q+1}{q_0+1}, 6)q^{0.55} + 3(q+1, 3) \leq 6q^{0.55} + 9.$$

Finally if $H \in \mathcal{S}$ then we can show that $|H| \leq q^{13}$ for $q \geq 4$. The order of M_{22} is $443520 < 4^13$ by the ATLAS [9]. Also note that if $H \cong 6_1{}^{\cdot}\mathrm{PSU}_4(3) : 2_2^-$, then $|H| = 39191040 < 4^{13} \leq q^{13}$.

Furthermore $|\mathcal{M} \cap \mathcal{S}| \leq 26 + (q+1, 6) \leq 32$.

Therefore

$$\sum_{M \in \mathcal{M}} |M| \leq 3\frac{|G|}{m(G)} + q^{23} + q^{20} + 6q^{20} + ((6q^{0.55} + 9) \times 3q^{21}) + 32q^{13}$$
$$= \leq 3\frac{|G|}{m(G)} + q^{23} + 18q^{21.55} + 27q^{21} + 7q^{20} + 32q^{13} < 5q^{23} \qquad \text{since } q \geq 4.$$

Consequently

$$c_{\mathrm{PSU}_6(q)} \leq \sum_{M \in \mathcal{M}} \frac{|M|m(G)}{|G|} \qquad \text{by Lemma 2.4.8}$$
$$\leq 3 + \frac{5q^{23}m(G)}{|G|} \leq 3 + \frac{5q^{23}}{q^{25}} \qquad \text{by Eq. (4.20)}$$
$$\leq 3 + \frac{5}{16} < 4.9 \qquad \text{since } q \geq 4.$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSU}_7(q)$

Theorem 4.3.23. *Let $q \geq 2$ and let $S = \mathrm{PSU}_7(q)$. Then $1 - \frac{4.9}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$ we calculate
$$\sum_{M \in \mathcal{M}} |M|,$$
the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_8(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.9. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 3$ and let $G = \mathrm{SU}_7(q)$. Here $m(G) = \frac{(q^7+1)(q^6-1)}{q^2-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^{11}|\mathrm{SU}_5(q)|(q^2-1) \geq q^{11+(5^2-2)+1} = q^{35} \qquad \text{by Lemma 2.6.4.} \tag{4.21}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G. Theorem 3.4.1 describes the types of geometric maximal subgroups that may appear in $\mathrm{SU}_7(q)$, and [6, Tables 8.38] describes the non-geometric maximal subgroups.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, not of P_3 type, nor of shape $\mathrm{GU}_6(q)$, nor of $\mathrm{GU}_7(q_0)$ type.

We note that there are 3 conjugacy classes of $\mathcal{C}_1$ subgroups that are not P_1 type, not of P_3 type, nor of shape $\mathrm{GU}_6(q)$: they are subgroups of type P_2, type $\mathrm{GU}_2(q) \times \mathrm{GU}_5(q)$ and type $\mathrm{GU}_3(q) \times \mathrm{GU}_4(q)$. If H is of one of these types then, applying Theorem 3.4.1 and Lemma 2.6.4, we have that $|H| \leq q^{2(14-6)}|\mathrm{SL}_2(q^2)||\mathrm{SU}_3(q)|(q^2-1) \leq q^{16}q^{2\times 3}q^8q^2 = q^{32}$.

Since $n = 7$ there is only one divisor of n larger than 1, so there is 1 conjugacy class of $\mathcal{C}_2$ subgroups. There is only one odd prime factor of 7 and so there is only 1 conjugacy class of $\mathcal{C}_3$ subgroups. If H lies in $\mathcal{C}_2$ or $\mathcal{C}_3$ we can show that $|H| \leq q^{32}$.

There are no $\mathcal{C}_4$ subgroups. There are at most 7 conjugacy classes of $\mathrm{GO}_7(q)$ type subgroups. Note that $|\mathrm{SO}_7(q)| \leq q^{\frac{49-7}{2}} = q^{21}$, therefore if H is of $\mathrm{GO}_7(q)$ type then $|H| \leq q^{32}$.

There are at most 7 conjugacy classes of $\mathcal{C}_6$ subgroups. Here also, if H lies in $\mathcal{C}_6$ then $|H| \leq q^32$.

There are no $\mathcal{C}_7$ subgroups. Finally we note that there are at most 7 conjugacy classes of non-geometric subgroups, and they have order at most $7|\mathrm{PSU}_3(3)| = 42336 < q^{32}$.

In conclusion, we have that

$$|\mathcal{M}_1| \leq 3 + 1 + 1 + 7 + 7 + 7 = 26.$$

We also have that if H lies in $\mathcal{M}_1$ then $|H| \leq q^{32}$.

We note that for each q_0 such that $q_0^r = q$ there exists at most 7 conjugacy classes of subgroups $H \cong \mathrm{SU}_7(q_0).(\frac{q+1}{q_0+1}, 7)$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. We also note that if $H \cong \mathrm{SU}_7(q_0).[(\frac{q+1}{q_0+1}, 7)]$ then $|H| \leq 7q_0^{48} \leq 7q^{16}$. Finally if H does not lie in $\mathcal{M}_1$ then we bound its order from above by $\frac{|G|}{m(G)}$. There are 3 conjugacy classes of maximal subgroups not lying in $\mathcal{M}_1$.

Therefore

$$\sum_{M \in \mathcal{M}} |M| \leq 3\frac{|G|}{m(G)} + 26q^{32} + (7q^{0.55} \times 7q^{16}) < 27q^{32} \qquad \text{since } q \geq 3.$$

Consequently

$$\begin{aligned} c_{\mathrm{PSU}_7(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8} \\ &\leq 3 + \frac{27q^{32} m(G)}{|G|} \leq 3 + \frac{27q^{32}}{q^{35}} && \text{by Eq. (4.21)} \\ &\leq 3 + \frac{27}{3^3} = 4 < 4.9 && \text{since } q \geq 3. \end{aligned}$$

The result follows from Lemma 2.4.2.

□

Calculating α for $\mathrm{PSU}_8(q)$

Theorem 4.3.24. *Let $q \geq 2$ and let $S = \mathrm{PSU}_8(q)$. Then $1 - \frac{4.9}{m(S)} \leq P_2(S)$.*

Proof. For $q = 2$ we calculate

$$\sum_{M \in \mathcal{M}} |M|,$$

the sum of the orders of representatives of the conjugacy classes of maximal subgroups in $\mathrm{PSU}_8(2)$, in MAGMA, and then apply Lemma 2.4.7 to get an upper bound for c_S smaller than 4.9. Finally, by Lemma 2.4.2 we have $1 - c_S/m(S) = P_2(S)$.

Let $q \geq 3$ and let $G = \mathrm{SU}_8(q)$. Then $m(G) = \frac{(q^8-1)(q^7+1)}{q^2-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^{13}|\mathrm{SU}_6(q)|(q^2-1) \geq q^{13+(6^2-2)+1} = q^{48} \qquad \text{by Lemma 2.6.4.} \tag{4.22}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, not of P_4 type, nor of shape $\mathrm{GU}_7(q)$, nor of $\mathrm{GU}_8(q_0)$ type. From Theorem 3.4.1 we know the shapes of the geometric subgroups of $\mathrm{SU}_8(q)$. From [6, Table 8.47] we also know the shapes of the non-geometric subgroups. From Lemma 2.6.4 we may see that for geometric subgroups $H \in \mathcal{M}_1$ we have

$$|H| \leq q^{43}.$$

Let us look at some specific large cases; if H is of P_2 type, then $|H| \leq q^{20}q^{2\times 3}q^{15}q^2 = q^{43}$. If H is of P_3 type then $|H| \leq q^{21}q^{2\times 8}q^3q^2 = q^{42}$. If H is of $\mathrm{GU}_6(q) \times \mathrm{GU}_2(q)$ type, then $|H| \leq q^{35}q^3q^2 = q^{40}$. Finally if H is of $\mathrm{GU}_5(q) \times \mathrm{GU}_3(q)$ type, then $|H| \leq q^{24}q^8q^2 = q^{34}$. Also in the case where H lies in $\mathcal{C}_5$ then we note that $|\mathrm{SO}_8^{\pm}(q)| \leq |\mathrm{Sp}_8(q)| \leq q^{(64+8)/2} = q^{36}$. If $H \in \mathcal{M}_1$ is a non-geometric subgroup then $|H| \leq 16|\mathrm{PSL}_3(4)| = 322560 < q^{43}$.

We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.47] for the non-geometric subgroups.

We note that there are 4 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1, P_4, nor of shape $\mathrm{GU}_7(q)$. Since $n = 8$ there are three divisors of n larger than 1, so there are 4 conjugacy classes of $\mathcal{C}_2$ subgroups. Since n is even there are no $\mathcal{C}_3$ subgroups. In the case of $\mathcal{C}_4$ the only possibility is when $n_1 = 2$. Therefore there are at most 2 conjugacy classes of $\mathcal{C}_4$ subgroups. There are at most 4 conjugacy classes each of $\mathrm{Sp}_8(q)$, $\mathrm{GO}_8^+(q)$ and $\mathrm{GO}_8^-(q)$ type subgroups. There are at most 8 conjugacy classes of $\mathcal{C}_6$ subgroups. Also, there are no $\mathcal{C}_7$ subgroups. Finally there are at most 16 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 4 + 4 + 2 + 12 + 8 + 16 = 46.$$

If $H \cong \mathrm{SU}_8(q_0)$ we note that for each q_0 such that $q_0^r = q$ there exists at most 8 conjugacy classes of subgroups $H \cong \mathrm{SU}_8(q_0)$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. Furthermore, by Lemma 2.6.4 and the fact that $q_0 \leq q^{1/3}$ we may deduce that $|H| \leq q_0^{63} \leq q^{21}$.

If H is of P_1 type, or of P_4 type, or has shape $\mathrm{GU}_7(q)$ then we bound the order of H by $\frac{|G|}{m(G)}$. Furthermore there are 3 conjugacy classes of maximal subgroups not in $\mathcal{M}_1$.

Therefore

$$\begin{aligned}
c_{\mathrm{PSU}_8(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8}\\
&\leq 3+\frac{(46q^{43}+(8q^{0.55}\times q^{31.5}))m(G)}{|G|}\\
&\leq 3+\frac{46q^{43}+8q^{32.05}}{q^{48}} && \text{by Eq. (4.22)}\\
&< 3+\frac{47q^{43}}{q^{48}} < 4 < 4.9 && \text{since } q\geq 3.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSU}_9(q)$

Theorem 4.3.25. *Let $q \geq 2$ and let $S = \mathrm{PSU}_9(q)$. Then $1 - \frac{4.9}{m(S)} \leq P_2(S)$.*

Proof. Let $q \geq 2$ and let $G = \mathrm{SU}_9(q)$. Here $m(G) = \frac{(q^9+1)(q^8-1)}{q^2-1}$, by Theorem 2.2.7. Furthermore

$$|G|/m(G) = q^{15}|\mathrm{SU}_7(q)|(q^2-1) \geq q^{15+(7^2-2)+1} = q^{63} \qquad \text{by Lemma 2.6.4.} \tag{4.23}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, nor of shape $\mathrm{GU}_8(q)$. From Theorem 3.4.1 we know the shapes of the geometric subgroups of $\mathrm{SU}_9(q)$. From Lemma 2.6.4 and the fact that $q_0 \leq q^{1/2}$ we may see that for geometric subgroups $H \in \mathcal{M}_1$ we have

$$|H| \leq q^{56}.$$

From [6, Tables 8.57] the bound holds for the orders of non-geometric subgroups too. Let us look at some of the larger groups in more detail. If H is a P_2 type subgroup then $|H| \leq q^{24}q^{2\times 3}q^{24}q^2 = q^{56}$. If H is a P_3 type subgroup then $|H| \leq q^{27}q^{2\times 8}q^8q^2 = q^{53}$. If H is a P_4 type subgroup then $|H| \leq q^{24}q^{2\times 15} = q^{54}$. If H is a $\mathrm{GU}_7(q)\times\mathrm{GU}_2(q)$ type subgroup then $|H| \leq q^{48}q^3q2 = q^{53}$. If H is a $\mathrm{GU}_6(q)\times\mathrm{GU}_3(q)$ type subgroup then $|H| \leq q^{35}q^8q2 = q^{45}$. In the case where H is of $\mathrm{GU}_9(q_0)$ type then $|H| \leq 9q_0^{80} < 9q^{27} < q^{31}$ and if H is of $\mathrm{GO}_9(q)$ type then $|H| \leq 9q^{45} < q^{49}$.

From [6, Table 8.57] we may also show, for a non-geometric subgroup H, that $|H| \leq q^{56}$. Note that here, the order of H divides one of the following, $9|\mathrm{PSL}_2(19)|$, $|3^{\cdot}\mathrm{J}_3|$ or $6|\mathrm{SL}_3(q^2)|$. From the Atlas [9], if the order of H divides one of the two first cases then $|H| \leq 150698880 < q^{56}$. If the order of H divides the latter case, then applying Lemma 2.6.4 we may show that $|H| \leq q^{56}$.

We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.57] for the non-geometric subgroups.

We note that there are 6 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1 or $\mathrm{GU}_8(q)$ type. Since $n = 9$ there are two divisors of n larger than 1, so there are 2 conjugacy classes of $\mathcal{C}_2$ subgroups. There is only one odd prime factor of 9 and so there is only 1 conjugacy class of $\mathcal{C}_3$

subgroups. There are no $\mathcal{C}_4$ subgroups. There are at most 9 conjugacy classes of $\mathcal{C}_5$ subgroups of type $\mathrm{GU}_9(q_0)$ for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 6 conjugacy classes of $\mathrm{GO}_9(q)$ type subgroups. There are at most 9 conjugacy classes of $\mathcal{C}_6$ subgroups, and at most 3 conjugacy classes of $\mathcal{C}_7$ subgroups. Finally we note that, by [6, Table 8.57], there are at most 15 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 6 + 2 + 1 + 9q^{0.55} + 6 + 9 + 3 + 15 = 42 + 9q^{0.55}.$$

If H is of P_1 type, or of shape $\mathrm{GU}_8(q)$ we bound their order from above by $\frac{|G|}{m(G)}$. There are at most 2 conjugacy classes of such H.

Consequently

$$\begin{aligned}
c_{\mathrm{PSU}_9(q)} &\leq \sum_{M\in\mathcal{M}} \frac{|M|m(G)}{|G|} && \text{by Lemma 2.4.8}\\
&\leq 2 + \frac{(42+9q^{0.55})q^{56}m(G)}{|G|}\\
&\leq 2 + \frac{(42+9q^{0.55})q^{56}}{q^{63}} && \text{by Eq. (4.23)}\\
&= 2 + \frac{42+9q^{0.55}}{q^7} < 3 < 4.9 && \text{since } q \geq 3.
\end{aligned}$$

The result follows from Lemma 2.4.2. □

Calculating α for $\mathrm{PSU}_{10}(q)$

Theorem 4.3.26. *Let $q \geq 2$ and let $S = \mathrm{PSU}_{10}(q)$. Then $1 - \frac{4.9}{m(S)} \leq P_2(S)$.*

Proof. Let $q \geq 2$ and let $G = \mathrm{SU}_{10}(q)$. If $q = 2$ then $m(G) = \frac{2^9(2^{10}-1)}{3}$, by Theorem 2.2.7, and

$$|G|/m(G) = \mathrm{GU}_9(2) \geq 2^{9^2-2} = 2^{79} \qquad \text{by Lemma 2.6.4.} \tag{4.24}$$

If $q \geq 3$ then $m(G) = \frac{(q^{10}-1)(q^9+1)}{q^2-1}$, by Theorem 2.2.7, and

$$|G|/m(G) = q^{17}|\mathrm{SU}_8(q)|(q^2-1) \geq q^{17+(8^2-2)+1} = q^{80} > q^{79} \qquad \text{by Lemma 2.6.4.} \tag{4.25}$$

Let $\mathcal{M}$ be a set of representatives for the conjugacy classes of maximal subgroups of G.

Let $\mathcal{M}_1$ be the subset of $\mathcal{M}$ consisting of groups that are not of P_1 type, not of P_5 type, nor of shape $\mathrm{GU}_9(q)$. From Theorem 3.4.1 we know the shapes of the geometric subgroups of $\mathrm{SU}_{10}(q)$. From Lemma 2.6.4 and the fact that $q_0 \leq q^{1/3}$ we may see that for geometric subgroups $H \in \mathcal{M}_1$ we have

$$|H| \leq q^{71}.$$

Let us look at some larger cases in more detail; if H is a P_2 type subgroup then $|H| \leq q^{28}q^{2\times 3}q^{35}q^2 = q^{71}$. If H is a P_3 type subgroup then $|H| \leq q^{33}q^{2\times 8}q^{15}q^2 = q^{66}$. If H is a P_4 type subgroup then $|H| \leq q^{32}q^{2\times 15}q^3q^2 = q^{67}$. If H is a $\mathrm{GU}_8(q) \times \mathrm{GU}_2(q)$ type subgroup then $|H| \leq q^{63}q3q2 = q^{68}$. If H is a $\mathrm{GU}_{10}(q_0)$ type subgroup then $|H| \leq 10q_0^{99} = 10q^{33}$. Finally for the remaining cases where H lies in $\mathcal{C}_5$ note that $|\mathrm{SO}_{10}^{\pm}(q)| < |\mathrm{Sp}_{10}(q) \leq q^{55}$.

From [6, Tables 8.63] the bound holds for the orders of non-geometric subgroups too. Note here the order of H must divide one of the following $10|\mathrm{PSL}_2(19)|$, $20|\mathrm{M}_{12}|$, $20|\mathrm{M}_2|$, $30|\mathrm{PSU}_3(q)|$, $20|\mathrm{PSU}_4(q)|$ or $10|\mathrm{SU}_5(q)|$. From the ATLAS [9] we have that order of M_{22} is 443520 and the order of M_{12} is 95040. By Lemma 2.6.4 and the fact that $q \geq 2$ one can show that $|H| < q^{71}$.

We may bound $|\mathcal{M}_1|$ using Theorem 3.2.1 to count the number of conjugacy classes of geometric subgroups and [6, Table 8.63] for the non-geometric subgroups.

We note that there are 6 conjugacy classes of $\mathcal{C}_1$ subgroups that are not of P_1, P_5 or $\mathrm{GU}_9(q)$ type. Since $n = 10$ there are three divisors of n larger than 1, so there are 4 conjugacy classes of $\mathcal{C}_2$ subgroups. There is only one odd prime factor of 12 and so there is only 1 conjugacy class of $\mathcal{C}_3$ subgroups. In the case of $\mathcal{C}_4$ we have that $n_1 = 2$ and there is at most 1 conjugacy class of $\mathcal{C}_4$ subgroups. There are at most 10 conjugacy classes of $\mathcal{C}_5$ subgroups of type $\mathrm{GU}_{10}(q_0)$ for each q_0 such that $q_0^r = q$. We can bound the number of distinct q_0 satisfying $q_0^r = q$ by $\log_2(q)$. This in turn can be bounded above by $q^{0.55}$ by Lemma 2.8.4. There are at most 5 conjugacy classes each of $\mathrm{Sp}_{10}(q)$, $\mathrm{GO}_{10}^{+}(q)$ and $\mathrm{GO}_{10}^{-}(q)$ type subgroups. There are no $\mathcal{C}_6$ nor $\mathcal{C}_7$ subgroups, since $n = 10$ is not a prime power. Finally we note that, by [6, Table 8.63], there are at most 67 conjugacy classes of non-geometric subgroups.

Therefore

$$|\mathcal{M}_1| \leq 6 + 4 + 1 + 1 + 10q^{0.55} + 15 + 67 = 94 + 10q^{0.55}.$$

If H is of P_1 type, of P_5 type, or of shape $\mathrm{GU}_9(q)$ we bound the order of H above by $\frac{|G|}{m(G)}$. Also there are at most 3 conjugacy classes of such H.

Consequently

$$\begin{aligned}
c_{\mathrm{PSU}_{10}(q)} &\leq \sum_{M \in \mathcal{M}} \frac{|M| m(G)}{|G|} && \text{by Lemma 2.4.8} \\
&\leq 3 + \frac{(94 + 10q^{0.55})q^{71} m(G)}{|G|} \\
&\leq 3 + \frac{(94 + 10q^{0.55})q^{71}}{q^{79}} && \text{by Eqs. (4.24) and (4.25)} \\
&= 3 + \frac{94 + 10q^{0.55}}{q^8} < 4 < 4.9.
\end{aligned}$$

The result follows from Lemma 2.4.2 □

Chapter 5

Largest non-trivial maximal subgroups of almost simple groups

5.1 Preliminaries about groups of Lie type

5.1.1 Untwisted groups of Lie type

The following constitutes a brief summary of the basic information needed to understand simple groups of Lie type and their parabolic subgroups. However for a more thorough approach and further information we recommend as a reference [8]. The information presented here comes from [8] and [48].

Let $\mathcal{L}$ be a simple Lie algebra over field $\mathbb{C}$, with product operation denoted by $[xy]$, and with $\mathcal{L} = \mathcal{K} \oplus \mathcal{L}_{r_1} \oplus \mathcal{L}_{r_2} \oplus \ldots \oplus \mathcal{L}_{r_k}$ its Cartan decomposition. In this case $\mathcal{K}$ is a Cartan subalgebra as defined in [8, 3.2] and $\mathcal{L}_{r_1}, \ldots, \mathcal{L}_{r_k}$ are one-dimensional subspaces. We pick a non-zero element e_r of each 1-dimensional subspace $\mathcal{L}_r$. For each $k \in \mathcal{K}$, $[ke_r]$ is a scalar multiple of e_r, and so we may write

$$[ke_r] = r(k)e_r$$

This map $r : \mathcal{K} \to \mathbb{C}$ is an element of the dual space of $\mathcal{K}$. The maps $r_1, r_2, \ldots, r_k$ obtained this way are called the *roots* of $\mathcal{L}$ and the related subspaces $\mathcal{L}_{r_i}$ are the *root subspaces* of L.

It can be shown that the Killing form $(,) : \mathcal{K} \to \mathbb{C}$, the definition of which can be found in [8, 3.1], of a semi-simple Lie algebra is non-singular. It follows that the Killing form of $\mathcal{L}$ remains non-singular when restricted to the Cartan subalgebra $\mathcal{K}$. Therefore each element of the dual space of $\mathcal{K}$ may be expressed as $h \mapsto (x, h)$ for a unique element x in $\mathcal{K}$. The element x associated with the map $h \mapsto r(h)$ therefore may be identified with the root r. Let Φ be the subset of $\mathcal{K}$ obtained by considering the roots as elements x of $\mathcal{K}$. We may then consider $\mathcal{K}_{\mathbb{R}}$, the set of all elements of $\mathcal{K}$ which are linear combinations of elements of Φ with real coefficients. It turns out that $\mathcal{K}_{\mathbb{R}}$ may be considered as a Euclidean space, a space isomorphic to the inner product space $\mathbb{R}^n$, and $(,)$ may be taken as the inner product, and there exist corresponding notions of length and angle. The subset Φ of $\mathcal{K}_{\mathbb{R}}$ forms a root system in the sense of [8, 2.1]. More details on the information in the above two paragraphs can be found in [8, Chapter 3].

Since Φ is a root system, there exists a subset $\Pi = \{p_1, \ldots, p_l\} \subseteq \Phi$ such that the p_i are a *system of simple roots* (*fundamental system of roots*); that is, a set of linearly independent elements such that all elements of Φ are linear combinations of the elements of Π where the coefficients are either all non-negative integers or all non-positive integers. These p_i form a basis of the vector space $\mathcal{K}_{\mathbb{R}}$, and the elements of Π are called *simple roots*. The number l is called the *rank* of $\mathcal{L}$. An element x of Φ which is a non-negative linear integral combination of the Π is called a positive root, respectively x is a negative root if x is a non-positive linear integral combination. By Φ^+ we denote the set of positive roots corresponding to the system Π, by Φ^- we denote the set of negative roots. For more information, we recommend [8, 2.1].

We may also define the length of a root a as $(a, a)^{1/2}$. In the cases which we consider, where Φ is irreducible, then at most two root lengths occur, as shown in [21, p.53 Lemma C]. Consequently, we may refer to roots as *long* and *short*.

We can encode information about a simple Lie algebra in a Dynkin diagram, by letting the vertices be the simple roots Π and the edges represent the angle between the roots. For example, if between two vertices there is no edge, then it means that the corresponding simple roots are orthogonal. For convenience we label the point associated with the simple root p_k by k. More information can be found in [8, 3.4].

More details about the simple Lie Algebras can be found in Table 5.1, including the number $N = |\Phi^+|$ of positive roots, and the related Dynkin diagram. The d_i and d values are invariants associated with the simple Lie algebra, which are useful for the following Lemma 5.1.13.

Definition 5.1.1. Given $x \in \mathcal{K}_{\mathbb{R}}$ we define w_x to be the map $\mathcal{K}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$ satisfying

$$w_x(y) = y - \frac{2(x, y)}{(x, x)} x,$$

for all $y \in \mathcal{K}_{\mathbb{R}}$. In other words the reflection of $\mathcal{K}_{\mathbb{R}}$ in a hyperplane orthogonal to x.

The Weyl group W of the root system Φ, as defined in [8, 2.1], is the subgroup of $GL(\mathcal{K}_{\mathbb{R}})$ generated by the set $\{w_r | r \in \Phi\}$. By [8, p.26] the set $\{w_r | r \in \Pi\}$ also generates W.

Let J be some subset of Π. We denote by $\mathcal{K}_{\mathbb{R},J}$ the $\mathbb{R}$-linear span of vectors in J. In turn Φ_J is defined to be the set $\Phi \cap \mathcal{K}_{\mathbb{R},J}$, and W_J the subgroup of W given by $\langle w_r | r \in J\rangle$. In the case where $J = \Pi \backslash \{p_i\}$, where p_i is a simple root, we denote J in this case by J_i, Φ_J by Φ_i, and W_J by W_i. Further information may be found in [8, 2.5].

According to [8, p.61], for $t \in \mathbb{C}$ and $r \in \Phi$ we may define an automorphism $x_r(t)$ of the Lie algebra $\mathcal{L}$ by

$$x_r(t) = \exp(t \cdot \operatorname{ad} e_r).$$

Given a simple Lie algebra $\mathcal{L}$ over $\mathbb{C}$ with Chevalley basis, defined in [8, 4.2], we denote by $\mathcal{L}_{\mathbb{Z}}$ the subset of $\mathcal{L}$ of all linear combinations of the basis elements with coefficients in the ring $\mathbb{Z}$. As stated in [8, 4.4], this $\mathcal{L}_{\mathbb{Z}}$ is a Lie algebra over $\mathbb{Z}$.

Given any field K, we can form the tensor product of the additive group of K with the additive group of $\mathcal{L}_{\mathbb{Z}}$. We denote this tensor product by $\mathcal{L}_K$. Further information regarding this

Lie algebra may be found in [8, 4.4].

One may define automorphisms of $\mathcal{L}_K$ analogous to the $x_r(t)$ of $\mathcal{L}$, as defined in [8, 4.4]. A *Chevalley group of type* $\mathcal{L}$ over a field K is the automorphism group of the Lie algebra $\mathcal{L}_K$ generated by these $x_r(t)$. The group is denoted by $\mathcal{L}(K)$, however if the field K is of order q we may also denote the group by $\mathcal{L}(q)$.

Definition 5.1.2. Given $G = \mathcal{L}(K)$ and $r \in \Phi$ we denote by X_r the set $\{x_r(t) \mid t \in K\}$. We call X_r a *root subgroup* of G.

Indeed X_r is a subgroup of G as shown in [8, 5.1]. Furthermore it is abelian and isomorphic to the additive group of K under the isomorphism $i : t \mapsto x_r(t)$. As a consequence $x_r(t)^{-1} = x_r(-t)$.

Definition 5.1.3 ([8, 5.1])**.** The subgroups of $G = \mathcal{L}(K)$ defined by $\langle X_r | r \in \Phi^+\rangle$ and $\langle X_r | r \in \Phi^-\rangle$ are denoted by U and V respectively.

Definition 5.1.4 ([8, Lem. 6.4.4])**.**

- $n_r(t) \in G$ is defined to be $x_r(t) \cdot x_{-r}(-t^{-1}) \cdot x_r(t)$ for $t \in K^\times$.
- $h_r(t) \in G$ to be $n_r(t) \cdot n_r(-1)$ for $t \in K^\times$.
- n_r is defined to be $n_r(1)$.

We note that $n_r(-t) = n_r(t)^{-1}$.

Definition 5.1.5 ([8, 7.1])**.** The subgroup $\langle h_r(t) | r \in \Phi, t \in K^\times\rangle$ of G is denoted by H.

The subgroup H can be shown to be also abelian. The order of H may be found in [8, p.121]. Proofs of the fact that H normalizes the subgroups U and V, and the fact that $U \cap H = V \cap H = 1$ may also be found there [8, p.100 & p.101]. For more information on H we point to [8, 7.1].

Definition 5.1.6 ([8, 8.1])**.** The subgroup UH of G is denoted by B.

Definition 5.1.7 ([8, 7.2])**.** The subgroup of G generated by the subgroup H and the elements n_r for all $r \in \Phi$ is denoted by N.

Lemma 5.1.8 ([8, Thm. 7.2.2])**.** *There exists a homomorphism ϕ from N onto the Weyl group W such that $\phi(n_r) = w_r$, and $\ker(\phi) = H$.*

Corollary 5.1.9. *Let J be a subset of the system of simple roots Π, let $W_J \leq W$ be the subgroup $\langle w_r \mid r \in J\rangle$, and let N_J be the full pre-image of W_J under ϕ. Then $N_J = \langle H, n_r \mid r \in J\rangle$.*

Lemma 5.1.10 ([8, Prop. 8.2.2 & 8.2.3])**.** *For all $w \in W$, let n_w be a preimage of an element $w \in W$ under the natural homomorphism ϕ in Lemma 5.1.8. Then*

(a) $G = \bigcup_{w \in W} B n_w B$.

(b) If $B n_w B = B n_{w'} B$, then $w = w'$.

(c) Let J be a subset of the system of simple roots Π, let W_J be the subgroup of W generated by the reflections w_r for $r \in J \leq \Pi$, and let N_J be the full preimage of W_J under ϕ. Then $P_J = B N_J B$ is a subgroup of G.

Definition 5.1.11.

- A subgroup of G which is conjugate to a P_J as defined in Lemma 5.1.10 is called *parabolic*.
- Let $J \subset \Pi$, then we may denote P_J by $P_{\{k|p_k \in J\}}$ instead.
- Let $p_k \in \Pi$ and $J = \Pi \backslash \{p_k\}$ then we refer to P_J as P_k.

We remark that a subgroup of G is parabolic if and only if it contains a conjugate of B, by [8, 8.3].

Lemma 5.1.12. *For an Chevalley group $G = \mathcal{L}(K)$, each subgroup P_J of G is equal to its normalizer.*

Proof. [8, 8.2.1] shows that G has a (B, N)-pair which is defined in [8, 8.2]. [8, 8.3.3] then states that if a group has a (B, N) pair then P_J is equal to its normalizer. □

Let us now look at the order of a parabolic subgroup of P_J of a Chevalley group G. For a proper subset J of Π, we call a subset of simple roots, $I \subseteq J$, a *connected component* if the part of the Dynkin diagram corresponding to the roots in I is a connected graph and if $(r, s) = 0$ for any two roots $r \in I$ and $s \in J \backslash I$. From here we see that J can be uniquely represented as the union $I_1 \cup I_2 \cup \ldots \cup I_t$ of pairwise disjoint connected components.

Lemma 5.1.13 ([48, Prop. 1])**.** *Let P_J be a parabolic subgroup of G, a Chevalley group of type $\mathcal{L}$ over a field K. Let $J \subset \Pi$ be represented as the union $I_1 \cup I_2 \cup \ldots \cup I_t$ of pairwise disjoint connected components. Furthermore let $\mathcal{L}_m$ be a simple Lie algebra over the same field K, however with Dynkin diagram the connected component I_m.*

Let $l = |\Pi|$, and let $l_m = |I_m|$ and $l_0 = \sum_{m=1}^{t} l_m = |J|$. Let $d_{m,i}$ be the invariant d_i of $\mathcal{L}_m$ as found in Table 5.1. Let $N = |\Phi^+|$, which can also be found in Table 5.1. Finally let d be the invariant of $\mathcal{L}$ found in Table 5.1.

Then the order of P_J in $G = \mathcal{L}(K)$ is equal to

$$\frac{1}{d} q^N (q-1)^{l-l_0} \prod_{m=1}^{t} (q^{d_{m,1}} - 1)(q^{d_{m,2}} - 1) \ldots (q^{d_{m,l_m}} - 1).$$

We end this section with a look at certain automorphisms of $\mathcal{L}(K)$.

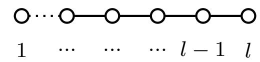
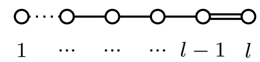
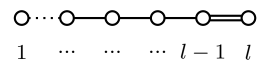
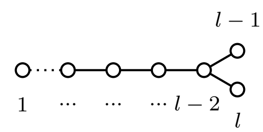
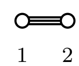
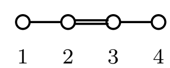
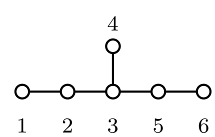
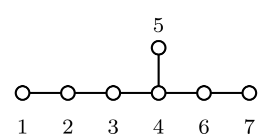
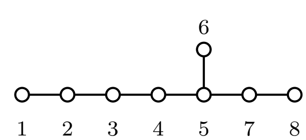
Definition 5.1.14. Let $R = \mathbb{Z}\Phi$ be the *root lattice*, the abelian group of $\mathbb{Z}$-linear combinations of elements of Φ, and $\chi : R \to K^\times$ be a K-character, in other words a group homomorphism from the additive group R to $K^\times$ (for more information see [8, Section 7.1, p. 97]). Then χ gives rise to an automorphism, $h(\chi)$, of the Lie algebra $\mathcal{L}(K)$ whose action is given in [8, Section 7.1, p. 98]. We denote the group $\{h(\chi) | \chi : \mathbb{Z}\Phi \to K^\times\}$ by $\widehat{H}$.

We note $h_r(t)$ introduced previously are examples of $h(\chi)$, more precisely

Lemma 5.1.15 ([8, Prop. 8.4.6.])**.** $\widehat{H} \cap G = H$.

Definition 5.1.16. We call elements in $\widehat{H} \backslash H$ *diagonal automorphisms.*

Table 5.1: Information on Simple Lie Algebras

Type	$N = \|\Phi^+\|$	$d_1, d_2, \ldots, d_l$	d	Undirected Dynkin Diagram
A_l	$l(l+1)/2$	$2, 3, \ldots, l+1$	$(l+1, q-1)$	
B_l	l^2	$2, 4, 6, \ldots, 2l$	$(2, q-1)$	
C_l	l^2	$2, 4, 6, \ldots, 2l$	$(2, q-1)$	
D_l	$l(l-1)$	$2, 4, 6, \ldots, 2l-2, l$	$(4, q^l - 1)$	
G_2	6	$2, 6$	1	
F_4	24	$2, 6, 8, 12$	1	
E_6	36	$2, 5, 6, 8, 9, 12$	$(3, q-1)$	
E_7	63	$2, 6, 8, 10, 12, 14, 18$	$(2, q-1)$	
E_8	120	$2, 8, 12, 14, 18, 20, 24, 30$	1	

5.1.2 Twisted groups of Lie type

We follow up the previous section with a brief summary of the basic information needed for the twisted groups of Lie type and their parabolic subgroups. For a more thorough approach and further information we recommend as a reference [8, Ch. 13-14]. The information presented here comes from [8] and [50].

For this section let $\mathcal{L}$ be a simple Lie algebra, let $\mathcal{K}$ be the Cartan subalgebra of $\mathcal{L}$. Let W be the Weyl group of $\mathcal{L}$. Let Π be a system of simple roots of $\mathcal{L}$ and let Φ be the root system of $\mathcal{L}$ spanned by Π, and Φ^+ the corresponding positive roots.

Let ρ be a nontrivial symmetry of the Dynkin diagram of $\mathcal{L}$, and let $\overline{r} = \rho(r)$ the image of $r \in \Pi$ under the action of ρ. Then there exists a unique isometry τ of the Euclidean space $\mathcal{V} = \mathcal{K}_{\mathbb{R}}$ such that $\tau(r) = c\overline{r}$. Here the value of c depends on r. Further information can be found in [8, Section 13.1 p. 217]

Denote by $\mathcal{V}^1$ the subspace of fixed vectors of $\mathcal{V}$ with respect to τ. We also denote by v^1 the orthogonal projection of $v \in \mathcal{V}$ onto $\mathcal{V}^1$. If $|\tau| = t$ then $v^1 = \frac{1}{t}\sum_{k=0}^{t-1}\tau^k(v)$ by [8, Section 13.1 p. 217]. For every $r \in \Pi$, we have $\tau w_r \tau^{-1} = w_{\overline{r}}$. So τ normalizes W in the group of isometries of $\mathcal{V}$.

We denote by W^1 the centralizer of the automorphism τ in W. Recall the definition of w_x from Definition 5.1.1. From [8, Cor. 13.1.4] we have that $W^1 = \langle w_{r^1} | r \in \Pi\rangle$.

We denote by Φ^1, Φ^{+1} and Π^1 the projections of Φ, Φ^+ and Π onto $\mathcal{V}^1$ respectively. In this case, $\mathcal{V}^1$ is a $\mathbb{R}$-linear span of Φ^1 and every element of Φ^{+1} is a linear combination of vectors in Π^1 with nonnegative coefficients. Also, for every r^1 and s^1 in Φ^1, $w_{r^1}(s^1)$ is an element of Φ^1. Therefore Φ^1 akin to a root system for W^1, and Π^1 acts similarly to a system of simple roots for Φ^1. To see the exact extent of the similarity we point the reader to [8, Prop. 13.2.2.]. We note that there may exist vectors in Φ^1 and Π such that one is a positive multiple of the other. For that we have the following result.

Lemma 5.1.17 ([8, Lemma 13.2.1])**.** *Let w run through the elements of W^1 and I run through the ρ-orbits of Π. Then the sets $w(\Phi_I^+) = \{w(r) | r \in \Phi_I^+\}$ form a partition of Φ. In addition, the roots r and s lie in the same set $w(\Phi_I^+)$ if and only if $r^1 = cs^1$ for some $c \in \mathbb{R}^+$. For such r and s, the reflections w_{r^1} and w_{s^1} coincide.*

Definition 5.1.18. Denote by $\overline{\Phi}^1$ the set of equivalence classes from Lemma 5.1.17. We also denote by $\overline{\Pi}^1$ the subset of $\overline{\Phi}^1$ consisting of classes containing elements from Π.

Definition 5.1.19. For $S \in \overline{\Phi}^1$ we denote by w_S a reflection w_{r^1} where $r \in S$.

If $\{I_1, I_2, \ldots, I_k\}$ is the set of all ρ-orbits of Π then $\overline{\Pi}^1 = \{\Phi_{I_i}^+ | i = 1, \ldots, k\}$. Let $r_1, r_2, \ldots, r_k$ be a set of roots, one from $\Phi_{I_i}^+$. Then this set forms a basis for $\mathcal{V}^1$.

Let $G = \mathcal{L}(K)$ be a finite Chevalley group of type $\mathcal{L}$ over finite field K of order $q = p^s$, where p is the characteristic of K, such that its Dynkin diagram has a nontrivial symmetry ρ of order

t. In the cases where $\mathcal{L} = \mathrm{B}_2$ or F_4 then let $p = 2$, if $\mathcal{L} = \mathrm{G}_2$ then let $p = 3$, in the other cases we have no restriction on p. Then, as noted in [8, Section 13.4] with further details in [8, 12.2.3, 12.3.3 & 12.4.1], there exists an automorphism γ of G such that $\gamma(X_r) = X_{\overline{r}}$ for every $r \in \Pi$. We call this automorphism a *graph automorphism*. We look at two specific cases, F_4 and E_6.

Lemma 5.1.20. *Let Φ be an indecomposable self-dual root system of type F_4, let $r \in \Phi$, let $\Pi = \{p_1, \ldots, p_4\}$ and let ρ be the non-trivial symmetry of the Dynkin diagram. Suppose the co-root, defined to be $h_r = \frac{2r}{(r,r)}$, satisfies*

$$h_r = \sum_{i=1}^{4} n_i h_{p_i}.$$

Then the element

$$\overline{r} = \sum_{i=1}^{4} n_i p_{\rho(i)}$$

is in Φ.

Proof. See [8, Lemmas 12.3.1 and 12.3.2]. □

We also note, according to the proof of [8, Lemma 12.3.1], that $p_{\rho(i)}$ is a long root if and only if p_i is a short one.

Lemma 5.1.21. *Let G be the Chevalley group $\mathrm{F}_4(K)$, where K is a perfect field of characteristic 2. For each root $r \in \Phi$ define $\lambda(r)$ to be 1 if r is a short root and 2 if r is a long root. Then the map*

$$x_r(t) \mapsto x_{\overline{r}}(t^{\lambda(\overline{r})})$$

where $r \in \Phi$ and $t \in K$ can be extended to an automorphism of G. Here $\overline{r}$ is as defined in the previous lemma.

Proof. See [8, Prop. 12.3.3]. □

Lemma 5.1.22. *Let G be a Chevalley group of type E_6, let $\Pi = \{p_1, \ldots, p_6\}$, and let $r \mapsto \overline{r}$ be a map of Φ into itself arising from a symmetry of the Dynkin diagram that swaps p_1 and p_6, p_2 and p_5 and fixes p_3 and p_4. Then there exists numbers $\gamma_r = \pm 1$ such that the map*

$$x_r(t) \mapsto x_{\overline{r}}(\gamma_r t)$$

can be extended to an automorphism of G. The γ_r can be chosen so that $\gamma_r = 1$ if $r \in \Pi$ or $-r \in \Pi$.

Proof. All the roots of E_6 have the same length as can be seen in [8, Section 3.6 (ix) p.49]. Therefore we may appeal to [8, Prop. 12.2.3.]. □

Furthermore we also note, from [8, p. 202], that for $r \in \Phi$ we have $\gamma_r \gamma_{-r} = 1$. Equivalently, we have $\gamma_r = \gamma_{-r}$.

Further details about such automorphisms may be found in [8, Sections 12.2-12.4].

For every field automorphism ϕ of K there corresponds an automorphism ϕ_G of the group $G = \mathcal{L}(K)$ such that $f(x_r(t)) = x_r(\phi(t))$ for all $r \in \Phi$. This is called a *field automorphism* of G. When it is clear that we are talking about the group automorphism and there is no risk of confusion we often omit the reference to G and denote the group automorphism by just ϕ.

Definition 5.1.23. Suppose that G has an associated Dynkin diagram with a non-trivial symmetry ρ of order t. If there exists a field automorphism ϕ of G such that the automorphism $\sigma = \gamma\phi$ has order t, we call such an automorphism a *Steinberg automorphism*.

If there exists a Steinberg automorphism σ, then all the subgroups U, V, H and N of G are σ-invariant, and σ acts on $N/H \cong W$ by the rule $\sigma(w_r) = w_{\overline{r}}$ for every $r \in \Pi$ [8, Prop. 13.4.1].

There exists an automorphism ϕ where $\sigma = \gamma\phi$ has order t if the following conditions on the field K hold:

(a) if $\mathcal{L} = \mathrm{A}_l$ for $l \geq 2$, D_l for $l \geq 4$, or E_6, then $|K|$ is a square of a prime power.

(b) if $\mathcal{L} = \mathrm{D}_4$, then $|K|$ is a cube of a prime power.

(c) if $\mathcal{L} = \mathrm{B}_2$ or F_4, then $|K| = 2^{2m+1}$.

(d) if $\mathcal{L} = \mathrm{G}_2$ then $|K| = 3^{2m+1}$.

Further details of this may be found in [8, p. 225 & 250-251].

Definition 5.1.24 ([8, 13.4.2.]).

- $U^1 = \langle x \in U | \sigma(x) = x\rangle$, $V^1 = \langle x \in V | \sigma(x) = x\rangle$, where U and V are defined in Definition 5.1.3.
- G^1 is defined to be $\langle U^1, V^1\rangle$.
- H^1 is defined to be $G^1 \cap H$ where H is defined in Definition 5.1.5.
- N^1 is defined to be $G^1 \cap N$ where N is defined in Definition 5.1.7.
- B^1 is defined to be $G^1 \cap B$ where B is defined in Definition 5.1.6.

Also note that $G^1 = \{x \in G | \sigma(x) = x\}$ as noted in [44, p.242], and consequently H^1, N^1, B^1 are also the fixed point sets of their corresponding group H, N, B under σ. We may denote this group G^1 by ${}^t\mathcal{L}(K)$, and it is called the *twisted group of type* ${}^t\mathcal{L}$ *over* K.

Definition 5.1.25. If ${}^t\mathcal{L} = {}^2\mathrm{A}_l$ for $l \geq 2$, ${}^2\mathrm{D}_l$ for $l \geq 4$, or ${}^2\mathrm{E}_6$, then let $q^2 = |K|$. If ${}^t\mathcal{L} = {}^3\mathrm{D}_4$, then let $q^3 = |K|$. If ${}^t\mathcal{L} = {}^2\mathrm{B}_2$ or ${}^2\mathrm{F}_4$, then let $q = |K| = 2^{2m+1}$. If $\mathcal{L} = {}^2\mathrm{G}_2$ then let $q = |K| = 3^{2m+1}$. Then we may denote ${}^t\mathcal{L}(K)$ by ${}^t\mathcal{L}(q)$.

Definition 5.1.26. Let S be an equivalence class on Φ, as defined in Lemma 5.1.17. Define X_S to be $\langle X_r \mid r \in S\rangle = \prod_{r\in S} X_r$.

Definition 5.1.27 ([44, p.243])**.** Define X_S^1 to be the set of fixed points of σ acting on X_S.

By [8, 13.5.2.] we have that for every element $w \in W^1$ there exists an element $n_w \in N^1$ corresponding to w under the natural homomorphism from N onto W. Furthermore, the factor group N^1/H^1 is isomorphic to W^1.

Recall the definition for w_S in Definition 5.1.19. Let J be a subset of $\overline{\Pi}^1$, let $W_J^1 = \langle w_S | S \in J\rangle$ and let N_J^1 be the preimage of W_J^1 in N^1. Then $P_J^1 = B^1 N_J^1 B^1$ is subgroup of G^1.

Definition 5.1.28. A subgroup conjugate to P_J^1 in G^1 is defined to be *parabolic.*

Further justification of this may be found in [8, Section 13.5]. Also, as previously noted $W^1 = \langle w_{r^1} | r \in \Pi \rangle$ and so W_J^1 is indeed a subgroup of W^1.

Lemma 5.1.29. *For G^1 a twisted group of Lie type, each subgroup P_J^1 of G^1 is equal to its normalizer.*

Proof. [8, 13.5.4] shows that G^1 has a (B, N)-pair. [8, 8.3.3] then states that if a group has a (B, N) pair then P_J is equal to its normalizer. □

We end this section with the following formulation of the Levi decomposition for twisted groups from [50].

Lemma 5.1.30 ([50, Lemma 2])**.** *Let $\overline{\Phi}_J^0 = \overline{\Phi^+}^1 \cap \{\overline{\Phi}^1 \backslash \overline{\Phi}_J^1\}$, $U_J^1 = \langle X_S^1 \mid S \in \overline{\Phi}_J^0 \rangle$, and let $L_J^1 = \langle H^1, X_S^1 \mid S \in \overline{\Phi}_J^1 \rangle$. Then $P_J^1 = U_J^1 L_J^1$.*

5.1.3 Further information on groups of Lie type

It is worth noting that the simple classical groups discussed previously are examples of groups of Lie type.

Lemma 5.1.31. *Let q be a prime power.*

- *Then* $\mathrm{A}_n(q) \cong \mathrm{PSL}_{n+1}(q)$.
- *If $n \geq 3$ then* $\mathrm{B}_n(q) \cong \mathrm{P\Omega}_{2n+1}(q)$.
- *If $n \geq 2$ then* $\mathrm{C}_n(q) \cong \mathrm{PSp}_{2n}(q)$.
- *If $n \geq 4$ then* $\mathrm{D}_n(q) \cong \mathrm{P\Omega}_{2n}^+(q)$.
- *If $n \geq 2$ then* ${}^2\mathrm{A}_n(q) \cong \mathrm{PSU}_{n+1}(q)$.
- *If $n \geq 4$ then* ${}^2\mathrm{D}_n(q) \cong \mathrm{P\Omega}_{2n}^-(q)$.

We note that definitions of parabolic subgroups coincide under these two approaches, and results regarding groups of Lie type in general will also apply to the classical simple groups. In addition, information on the outer automorphisms of the classical groups can be found in [6, Sections 1.7.1 & 1.7.2]. For convenience we reproduce the information of [6, Section 1.7.2].

Lemma 5.1.32. *Let G be a classical simple group. Let $q = p^e$ with p prime. Let $\delta, \phi, \gamma, \tau, \delta'$ all be the elements of* $\mathrm{Aut}(G)$ *as defined in [6, 1.7.1]. Furthermore, let $\ddot{\delta}, \ddot{\phi}, \ddot{\gamma}, \ddot{\tau}, \ddot{\delta}'$ denote their images in* $\mathrm{Out}(G)$*. Then the following are presentations of* $\mathrm{Out}(G)$.

- $\mathrm{PSL}_2(q)$*:*
$$\langle \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}^{(q-1,2)} = \ddot{\phi}^e = [\ddot{\delta}, \ddot{\phi}] = 1 \rangle.$$
- $\mathrm{PSL}_n(q)$*, $n \geq 3$:*
$$\langle \ddot{\delta}, \ddot{\phi}, \ddot{\gamma} \mid \ddot{\delta}^{(q-1,n)} = \ddot{\gamma}^2 = \ddot{\phi}^e = [\ddot{\gamma}, \ddot{\phi}] = 1, \ddot{\delta}^{\ddot{\gamma}} = \ddot{\delta}^{-1}, \ddot{\delta}^{\ddot{\phi}} = \ddot{\delta}^p \rangle.$$

- $\mathrm{PSU}_n(q)$, $n \geq 3$:

$$\langle \ddot{\delta}, \ddot{\phi}, \ddot{\gamma} \mid \ddot{\delta}^{(q+1,n)} = \ddot{\gamma}^2 = 1, \ddot{\phi}^e = \ddot{\gamma}, \ddot{\delta}^{\ddot{\gamma}} = \ddot{\delta}^{-1}, \ddot{\delta}^{\ddot{\phi}} = \ddot{\delta}^p \rangle.$$

- $\mathrm{PSp}_n(p^e)$, $n \geq 2$, $(n,p) \neq (4,2)$:

$$\langle \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}^{(q-1,2)} = \ddot{\phi}^e = [\ddot{\delta}, \ddot{\phi}] = 1 \rangle.$$

- $\mathrm{PSp}_4(p^e)$, $p = 2$:

$$\langle \ddot{\gamma}, \ddot{\phi} \mid \ddot{\gamma}^2 = \ddot{\phi}, \ddot{\phi}^e = 1 \rangle.$$

- $\mathrm{P}\Omega_n^{\circ}(q)$, $n \geq 3$ *odd:*

$$\langle \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}^2 = \ddot{\phi}^e = [\ddot{\delta}, \ddot{\phi}] = 1 \rangle.$$

- $\mathrm{P}\Omega_n^+(q)$, $n \geq 6$ *even,* $n \neq 8$, q *even:*

$$\langle \ddot{\gamma}, \ddot{\phi} \mid \ddot{\gamma}^2 = \ddot{\phi}^e = [\ddot{\gamma}, \ddot{\phi}] = 1 \rangle.$$

- $\mathrm{P}\Omega_8^+(q)$, q *even:*

$$\langle \ddot{\tau}, \ddot{\gamma}, \ddot{\phi} \mid \ddot{\tau}^3 = \ddot{\gamma}^2 = (\ddot{\gamma}\ddot{\tau})^2 = \ddot{\phi}^e = [\ddot{\tau}, \ddot{\phi}] = [\ddot{\gamma}, \ddot{\phi}] = 1 \rangle.$$

- $\mathrm{P}\Omega_n^-(q)$, $n \geq 4$ *even,* q *even:*

$$\langle \ddot{\gamma}, \ddot{\phi} \mid \ddot{\gamma}^2 = 1, \ddot{\phi}^e = \ddot{\gamma} \rangle.$$

- $\mathrm{P}\Omega_8^+(q)$, q *odd:*

$$\begin{aligned}\langle \ddot{\delta}', \ddot{\tau}, \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}'^2 = \ddot{\tau}^3 = \ddot{\gamma}^2 = (\ddot{\gamma}\ddot{\tau})^2 = \ddot{\delta}^2 = 1, \ddot{\delta}^{\ddot{\tau}} = \ddot{\delta}', \ddot{\delta}'^{\ddot{\tau}} = \ddot{\delta}\ddot{\delta}',\\ (\ddot{\delta}\ddot{\gamma})^2 = \ddot{\delta}', \ddot{\phi}^e = [\ddot{\delta}, \ddot{\phi}] = [\ddot{\tau}, \ddot{\phi}] = [\ddot{\gamma}, \ddot{\phi}] = 1 \rangle.\end{aligned}$$

- $\mathrm{P}\Omega_n^+(q)$, $n \geq 12$, $4|n$, q *odd:*

$$\langle \ddot{\delta}', \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}'^2 = \ddot{\gamma}^2 = \ddot{\delta}^2 = 1, (\ddot{\delta}\ddot{\gamma})^2 = \ddot{\delta}', \ddot{\phi}^e = [\ddot{\delta}, \ddot{\phi}] = [\ddot{\gamma}, \ddot{\phi}] = 1 \rangle.$$

- $\mathrm{P}\Omega_n^+(q)$, $n \geq 6$, $n \equiv 2 \mod 4$, $q \equiv 1 \mod 4$:

$$\langle \ddot{\delta}', \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}'^2, \ddot{\gamma}^2 = 1, \ddot{\delta}^2 = \ddot{\delta}' = \ddot{\delta}^{\ddot{\gamma}} = \ddot{\delta}^{-1}, \ddot{\phi}^e = [\ddot{\gamma}, \ddot{\phi}] = 1, \ddot{\delta}^{\ddot{\phi}} = \ddot{\delta}^p \rangle.$$

- $\mathrm{P}\Omega_n^+(q)$, $n \geq 6$, $n \equiv 2 \mod 4$, $q \equiv 3 \mod 4$:

$$\langle \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \mid \ddot{\gamma}^2 = \ddot{\delta}^2 = [\ddot{\delta}, \ddot{\gamma}] = \ddot{\phi}^e = [\ddot{\gamma}, \ddot{\phi}] = [\ddot{\delta}, \ddot{\phi}] = 1 \rangle.$$

- $\mathrm{P}\Omega_n^-(q)$, $n \geq 4$, $4|n$ *or* $q \equiv 1 \mod 4$, q *odd:*

$$\langle \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \mid \ddot{\gamma}^2 = \ddot{\delta}^2 = [\ddot{\delta}, \ddot{\gamma}] = [\ddot{\delta}, \ddot{\phi}] = 1, \ddot{\phi}^e = \ddot{\gamma} \rangle.$$

- $\mathrm{P}\Omega_n^-(q)$, $n \geq 4$, $n \equiv 2 \mod 4$, $q \equiv 3 \mod 4$:

$$\langle \ddot{\delta}', \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \mid \ddot{\delta}'^2 = \ddot{\gamma}^2 = 1, \ddot{\delta}^2 = \ddot{\delta}', \ddot{\delta}^{\ddot{\gamma}} = \ddot{\delta}^{-1}, \ddot{\phi}^e = [\ddot{\gamma}, \ddot{\phi}] = [\ddot{\delta}, \ddot{\phi}] = 1 \rangle.$$

Since we will be dealing with parabolic subgroups we would also like to know how some outer automorphisms act on the conjugacy classes of parabolic subgroups. We continue with the notation introduced in the two previous sections. Let us start with the following lemma.

Lemma 5.1.33 ([8, Prop 8.5.1.]). *Let G be a Chevalley group and let $J \subset \Pi$. Furthermore let Φ_J be the set of roots which are integral combinations of roots in J. Then P_J is the subgroup of G generated by H and the root subgroups X_r for $r \in \Phi^+ \cup \Phi_J$.*

Theorem 5.1.34. *Let G be a simple group of Lie type over a finite field K, let $J \subset \Pi$ and let a be an automorphism of G. If a is a diagonal or a field automorphism, both defined in Section 5.1.2, then*

(a) $a(X_r) = X_r$, where X_r is the root subgroup $\{x_r(t)|t \in K\}$.

(b) $a(H) = H$, where H is defined in 5.1.5.

(c) $a(P_J) = P_J$.

(d) a preserves the G-conjugacy classes of parabolic subgroups.

Proof. First, let a be a diagonal automorphism. Then a is, by [8, p.200], defined to be one of the form $a(g) = h(\chi)gh(\chi)^{-1}$ for some $h(\chi)$ as defined in Definition 5.1.14. Remember from Definition 5.1.14 that $\widehat{H}$ is the group of all $h(\chi)$. From [8, p. 100] we have that $a(X_r) = X_r$.

Remember the definition for H from Definition 5.1.5. By Lemma 5.1.15 we have $a(H) = a(\widehat{H} \cap G) = \widehat{H} \cap G = H$.

Let a be a field automorphism. Then by [8, Section 12.2] we have that $a(x_r(t)) = x_r(f(t))$, where ϕ is an automorphism of the field K, for all $r \in \Phi$ and $t \in K$. We may see that we $a(X_r) = X_r$.

Since $h_r(t) = n_r(t) \cdot n_r(-1)$ and $n_r(t) = x_r(t) \cdot x_{-r}(-t^{-1}) \cdot x_r(t)$, we have that $a(n_r(t)) = n_r(f(t))$ and $a(h_r(t)) = h_r(f(t))$ for the given field automorphism ϕ of the underlying field of G.

Therefore, in both cases we have that
$$a(H) = H.$$
By Lemma 5.1.33 we have that $P_J = \langle H, X_r \mid r \in \Phi^+ \cup \Phi_J \rangle$ therefore we have

$$a(P_J) = a(\langle H, X_r \mid r \in \Phi^+ \cup \Phi_J \rangle) = \langle a(H), a(X_r) \mid r \in \Phi^+ \cup \Phi_J \rangle = \langle H, X_r \mid r \in \Phi^+ \cup \Phi_J \rangle = P_J.$$

The fact that a preserves the G-conjugacy classes of parabolic subgroups follows from the fact that $a(P_J) = P_J$ for all $J \subset \Pi$ and that all parabolics are conjugate to a P_J for some $J \in \Pi$.

□

As a supplement to the above Theorem 5.1.34 we have

Lemma 5.1.35 ([26, Prop. 4.1.17-20]). *Let G be a classical simple group of dimension $n \geq 9$ and $H < G$ be a parabolic subgroup of type P_m. Then all outer automorphisms of G fix the conjugacy class of H in G, except in the following cases:*

(a) $G = \mathrm{PSL}_n(q)$ *with* $m \neq n/2$. *In this case the outer automorphisms of* G *which fix the conjugacy class of* H *in* G *lie in* $\langle \ddot{\delta}, \ddot{\phi} \rangle$.

(b) $G = \mathrm{O}_n^+(q)$ *with* $m = n/2$.

Proof. In the four propositions cited from [26], a value c appears in the statement. The value c , defined on [26, p. 62], is the length of the orbit of the conjugacy class of H in G under the action of $\mathrm{Out}(G)$.

The π that appears in the four propositions cited from [26] is also defined on [26, p. 62]. This is the corresponding permutation representation on this set of c conjugacy classes.

If $c = 1$ then the premutation representation π is trivial and the conjugacy class of H in G is fixed by all elements of $\mathrm{Out}(G)$. In particular from the statements in [26] we conclude that, with the exception of cases (a) and (b) listed, $c = 1$ and so all outer automorphisms of G fix the conjugacy class of H in G.

For the two exceptions where $c \neq 1$ information on the action π can be found in [26, Action Table 3.5.G] under $\mathrm{L}; \pi_3$ and $\mathrm{O}; \pi_2$. □

Lemma 5.1.36. *Suppose that* $q = 2^n$*, for some positive integer* n*, and let* γ *be a graph automorphism of* $G = \mathrm{F}_4(q)$*. If* $J \subset \Pi$ *then the parabolic subgroup* P_J *is mapped by* γ *to* $P_{\overline{J}}$ *where* $\overline{J}$ *is the image of* J *under the related map* $r \mapsto \overline{r}$ *of* Φ.

The conjugacy class of a parabolic subgroup P_J *for* $J \subset \Pi$ *is fixed by* γ *if and only if* $J = \overline{J}$.

Proof. We know how γ acts on the $x_r(t)$ due to Lemma 5.1.21. By Lemma 5.1.20 can see that the related map $r \mapsto \overline{r}$ of Φ permutes the $p_i \in \Pi$ and so maps all roots in Φ^+ to Φ^+. To expand a bit more, if $r \in \Phi^+$ the coefficients when written as a sum of elements p_i are all positive. Since the co-root h_r is a positive scalar multiple of r we have that h_r, when written as a sum of h_{p_i}, has positive coefficients also. Finally that implies that $\overline{r}$ also has positive coefficients when written as a sum of p_i. By Lemma 5.1.3, U is generated by the $x_r(t)$ for $r \in \Phi^+$, and consequently U is mapped to U under γ.

Since $n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$, the image under γ is $x_{\overline{r}}(t^{\lambda(\overline{r})})x_{\overline{-r}}((-t)^{-\lambda(\overline{-r})})x_{\overline{r}}(t^{\lambda(\overline{r})})$. We note first that since $t \in \mathbb{F}_{2^n}$ that $t = -t$. Let us also recall that $p_{\rho(i)}$ is a long root if and only if p_i is a short one. Therefore if r is a short root, then the image is $x_{\overline{r}}(t^2)x_{\overline{-r}}(t^2)x_{\overline{r}}(t^2) = n_{\overline{r}}(t^2)$. If r is a long root then the image is $x_{\overline{r}}(t)x_{\overline{-r}}(-t)x_{\overline{r}}(t) = n_{\overline{r}}(t)$. In either case $\gamma(n_r) = \gamma(n_r(1)) = n_{\overline{r}}(1)$.

Since H is generated by $h_r(t)$ for $r \in \Phi$ and $t \in K^\times$ according to Definition 5.1.5, we look at the image of $h_r(t)$ under γ. In this case since $h_r(t) = n_r(t)n_r(-1)$ the image is $n_{\overline{r}}(t^{\lambda(\overline{r})})n_{\overline{r}}(1) = h_{\overline{r}}(t^{\lambda(\overline{r})})$. Therefore $\gamma(H) = H$. Coupled with the fact that $\gamma(U) = U$, we get that $\gamma(B) = B$.

From Corollary 5.1.9

$$\gamma(N_J) = \gamma(\langle H, n_r \mid r \in J \rangle) = \langle H, \gamma(n_r) \mid r \in J \rangle = \langle H, n_{\overline{r}} \mid r \in J \rangle = N_{\overline{J}}.$$

From Lemma 5.1.10(c) we see that therefore P_J gets mapped to $P_{\overline{J}}$. By [8, Theorem 8.3.3.] no two P_J are conjugate to each other, consequently the conjugacy class of P_J is fixed by g if and only if $J = \overline{J}$. □

Lemma 5.1.37. *Let γ be a graph automorphism of $G = \mathrm{E}_6(q)$. If $J \subset \Pi$ then the parabolic subgroup P_J is mapped to $P_{\overline{J}}$ where $\overline{J}$ is the image of J under the related map $r \mapsto \overline{r}$ of Φ as introduced in Lemma 5.1.22.*

Furthermore, a conjugacy class of a parabolic subgroup P_J for $J \subset \Pi$ is fixed by γ if and only if $J = \overline{J}$. In particular γ fixes the conjugacy class of P_3 and P_4, but does not fix the conjugacy class of any other P_i where $i = 1, 2, 5, 6$.

Proof. We know how γ acts on the $x_r(t)$ due to Lemma 5.1.22. The related map $r \mapsto \overline{r}$ of Φ arising from the symmetry of the Dynkin diagram maps all roots in Π to Π. Consequently, it also maps all roots in Φ^+ to Φ^+. By Lemma 5.1.3, U is generated by the $x_r(t)$ for $r \in \Phi^+$. Consequently U is mapped to U under γ.

Since $n_r(t) = x_r(t)x_{-r}(-t^{-1})x_r(t)$, if $\gamma_r = 1$ we have that

$$\gamma(n_r(t)) = x_{\overline{r}}(t)x_{\overline{-r}}(-t^{-1})x_{\overline{r}}(t) = n_{\overline{r}}(t).$$

If $\gamma_r = -1$ then

$$\gamma(n_r(t)) = x_{\overline{r}}(-t)x_{\overline{-r}}(t^{-1})x_{\overline{r}}(-t) = n_{\overline{r}}(-t).$$

If $r \in \Pi$ then $\gamma_r = 1$ and so $\gamma(n_r(t)) = n_{\overline{r}}(t)$. In particular, $\gamma(n_r) = n_{\overline{r}}$ for $r \in \Pi$.

Remember that $h_r(t) = n_r(t)n_r(-1)$. Therefore if $\gamma_r = 1$ then

$$\gamma(h_r(t)) = n_{\overline{r}}(t)n_{\overline{r}}(-1) = h_{\overline{r}}(t).$$

If $\gamma_r = -1$ then

$$\gamma(h_r(-t)h_r(-1)^{-1}) = \gamma(n_r(-t)n_r(-1)^{-1}) = n_{\overline{r}}(t)n_{\overline{r}}(1)^{-1} = n_{\overline{r}}(t)n_{\overline{r}}(-1) = h_{\overline{r}}(t).$$

Since H is generated by $h_r(t)$ for $r \in \Phi$ and $t \in K^\times$ according to Definition 5.1.5, we conclude that $H \leq \gamma(H)$. Since γ is an automorphism we have $\gamma(H) = H$. Coupled with the fact that $\gamma(U) = U$, we get that $\gamma(B) = B$.

From Corollary 5.1.9 we have that

$$\gamma(N_J) = \gamma(\langle H, n_r \mid r \in J\rangle) = \langle H, \gamma(n_r) \mid r \in J\rangle = \langle H, n_{\overline{r}} \mid r \in J\rangle = N_{\overline{J}}.$$

From Lemma 5.1.10(c) we see that therefore P_J is mapped to $P_{\overline{J}}$. By [8, Theorems 8.3.3.] no two P_J are conjugate to each other, consequently the conjugacy class of P_J is fixed by γ if and only if $J = \overline{J}$.

In addition we see that γ maps P_1 and P_6 to each other, it also maps P_2 and P_5 to each other, while $\gamma(P_3) = P_3$ and $\gamma(P_4) = P_4$. □

Let G be a Chevalley group with Steinberg automorphism σ and graph automorphism γ. A field automorphism ϕ of G commutes with γ according to [8, p. 225], and so also commutes with σ.

We notice that an automorphism of G that commutes with σ is an automorphism of G^1 when restricted to G^1. Since G^1 is the fixed-point set in G of σ, if automorphism a of G commutes with σ we have

$$a(x) = a(\sigma(x)) = \sigma(a(x))$$

for all x in G^1. That therefore implies that $a(x)$ is fixed by σ and thus lies in G^1 also. This leads us to the following definitions

Definition 5.1.38. Let ϕ be a field automorphism of G, then we call such an automorphism when restricted to G^1 a *field automorphism* of G^1.

Definition 5.1.39 ([44, p.244])**.** Let d be a diagonal automorphism of untwisted Chevalley group G, then if d commutes with Steinberg automorphism σ then it a *diagonal automorphism* of G^1.

Theorem 5.1.40. *Let G^1 be a simple twisted group of Lie type, let $J \subset \Pi$, and let a be an automorphism of G^1. If a is a diagonal or a field automorphism then $a(P_J) = P_J$. Furthermore they preserve the G^1-conjugacy classes of parabolic subgroups.*

Proof. Let G be the corresponding untwisted group of G^1. Let X_r be the root subgroup $\{x_r(t) \mid t \in K\}$ for $r \in \Phi$, and for $S \subset \Phi$ let X_S be the group generated by X_r for $r \in S$.

$$\begin{aligned} a(H^1) &= a(H \cap G^1) = a(H) \cap a(G^1) && \text{by Definition 5.1.24} \\ &= H \cap a(G^1) && \text{from Theorem 5.1.34} \\ &= H \cap G^1 = H^1. \end{aligned}$$

Also,

$$\begin{aligned} a(X_S^1) &= a(X_S \cap G^1) = a(X_S) \cap a(G^1) && \text{by Definition 5.1.27} \\ &= a(\langle X_i \mid i \in S\rangle) \cap a(G^1) && \text{by Definition 5.1.26} \\ &= \langle X_i \mid i \in S\rangle \cap a(G^1) && \text{from Theorem 5.1.34} \\ &= \langle X_i \mid i \in S\rangle \cap G^1 = X_S \cap G^1 = X_S^1. \end{aligned}$$

Consequently, using the same notation as Lemma 5.1.30,

$$\begin{aligned} a(P_J^1) &= a(U_J^1 L_J^1) = a(\langle H^1, X_S^1 \mid S \in \overline{\Phi}_J^0 \cup \overline{\Phi}_J^1\rangle) && \text{according to Lemma 5.1.30} \\ &= \langle a(H^1), a(X_S^1) \mid S \in \overline{\Phi}_J^0 \cup \overline{\Phi}_J^1\rangle \\ &= \langle H^1, X_S^1 \mid S \in \overline{\Phi}_J^0 \cup \overline{\Phi}_J^1\rangle = U_J^1 L_J^1 = P_J^1. \end{aligned}$$

The fact that a preserves the G^1-conjugacy classes of parabolic subgroups follows from the fact that $a(P_J^1) = P_J^1$ for all $J \subset \Pi$ and that all parabolics are conjugate to a P_J^1 for some $J \in \Pi$. □

Now that we have introduced, diagonal, field and graph automorphisms, let us consider $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G^1)$.

Theorem 5.1.41 ([46, Theorem 30]). *Let G be a group of Lie type, then every automorphism of G can be expressed as the product of an inner, a diagonal, a graph and a field automorphism.*

Theorem 5.1.42 ([46, Theorem 36]). *Let G^1 be a twisted group of Lie type, then every automorphism of G^1 is a product of an inner, a diagonal and a field automorphism*

Due to us making arguments depending on the choice of simple group it would be expedient to collate some information about the different exceptional groups. For that end we reproduce the relevant information from the ATLAS [9] and [51], presented in the same format as Tables 2.8 and 2.9 from [38]. Table 5.2 gives the orders of the simple exceptional groups of Lie type; and Table 5.3 presents some information as to the shape of their outer automorphism groups respectively.

Theorem 5.1.43. *Let G be a simple exceptional groups of Lie type then the order of G may be found in Table 5.2.*

Theorem 5.1.44. *Let G be a simple exceptional group group of Lie type, let d be the order of the subgroup consisting of elements in $\mathrm{Out}(G)$ corresponding to diagonal automorphisms, let f be the order of the subgroup consisting of elements in $\mathrm{Out}(G)$ corresponding to field automorphisms. Also, let g be the order of the subgroup consisting of elements in $\mathrm{Out}(G)$ corresponding to graph automorphisms (modulo outer field automorphisms) as discussed Section 5.1.2. Then the orders d, f and g can be found in Table 5.3.*

We note that the outer automorphism group has order dfg. Also we note that the subgroups of $\mathrm{Out}(G)$ consisting of elements corresponding to diagonal automorphisms referenced in Theorem 5.1.44 is cyclic. Let ϕ be the Frobenius automorphism $\phi : x \mapsto x^p$ of the field $\mathbb{F}_{p^n}$. This Frobenius automorphism generates all field automorphisms, and so the corresponding field automorphism ϕ_G of G generates all field automorphisms of G.

5.2 The main theorem

Our aim to is to calculate the largest non-trivial maximal subgroup of the almost simple groups A with socle being a classical simple group. When $A = \mathrm{soc}(A)$ this information is already known as noted in the following theorem, as previously stated in Section 2.2, Theorem 2.2.7. Recall that if G is a classical simple group, then the values of $m(G)$ are reproduced on Table 2.1.

The following chapters will be dedicated to proving the next result.

Theorem 5.2.1. *Let A be almost simple, with socle S a classical simple group. Then the index of the largest non-trivial maximal subgroup of A equals $m(S)$, except for the cases noted in Tables 5.4, 5.5 and 5.6.*

In addition to the above theorem we will prove in Section 5.6 the following theorem .

Theorem 5.2.2. *Let A be almost simple, with socle S a simple exceptional group of Lie type. Then the index of the largest non-trivial maximal subgroup of A equals $m(S)$ are as found in Table 5.7, except for the cases noted in Table 5.10.*

We note that the information from Table 5.7 is obtained from [48], [49] and [50], where we note in the table which resource each one comes from.

Table 5.2: Orders of exceptional groups

Group	Order
$\mathrm{G}_2(q)$	$q^6(q^2-1)(q^6-1)$
$\mathrm{F}_4(q)$	$q^{24}\prod_{i\in\{2,6,8,12\}}(q^i-1)$
$\mathrm{E}_6(q)$	$\frac{1}{(3,q-1)}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-1)$
$\mathrm{E}_7(q)$	$\frac{1}{(2,q-1)}q^{63}\prod_{i\in\{2,6,8,10,12,14,18\}}(q^i-1)$
$\mathrm{E}_8(q)$	$q^{120}\prod_{i\in\{2,8,12,14,18,20,24,30\}}(q^i-1)$
${}^2\mathrm{B}_2(q)$	$q^2(q^2+1)(q-1)$
${}^2\mathrm{G}_2(q)$	$q^3(q^3+1)(q-1)$
${}^2\mathrm{F}_4(q)$	$q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$
${}^3\mathrm{D}_4(q)$	$q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$
${}^2\mathrm{E}_6(q)$	$\frac{1}{(3,q+1)}q^{36}\prod_{i\in\{2,5,6,8,9,12\}}(q^i-(-1)^i)$

Table 5.3: Outer automorphisms of exceptional groups

Group	d	f	g
$\mathrm{G}_2(q)$	1	$q = p^f$	2 if $p = 3$, 1 otherwise
$\mathrm{F}_4(q)$	1	$q = p^f$	2 if $p = 2$, 1 otherwise
$\mathrm{E}_6(q)$	$(3, q-1)$	$q = p^f$	2
$\mathrm{E}_7(q)$	$(2, q-1)$	$q = p^f$	1
$\mathrm{E}_8(q)$	1	$q = p^f$	1
${}^2\mathrm{E}_6(q)$	$(3, q+1)$	$q^2 = p^f$	1
${}^3\mathrm{D}_4(q)$	1	$q^3 = p^f$	1
${}^2\mathrm{B}_2(q)$	1	$q = 2^f$	1
${}^2\mathrm{F}_4(q)$	1	$q = 2^f$	1
${}^2\mathrm{G}_2(q)$	1	$q = 3^f$	1

Table 5.4: Largest non-trivial maximals for almost simple groups with socle $\mathrm{PSL}_n(q)$

n	q	Conditions on G	Subgroup Type	$m(G)$
2	7	$G = \mathrm{PSL}_2(7).\langle\ddot{\delta}\rangle$	P_1	8
2	9	$G \not\leq \mathrm{PSL}_2(9).\langle\ddot{\phi}\rangle$	P_1	10
2	11	$G = \mathrm{PSL}_2(11).\langle\ddot{\delta}\rangle$	P_1	12
3	3	$G = \mathrm{PSL}_3(3).\langle\ddot{\gamma}\rangle$	$P_{1,2}$	52
3	4	$G \not\leq \mathrm{PSL}_3(4).\langle\ddot{\delta},\ddot{\phi}\rangle$	$3^{\cdot}\mathrm{A}_6$	56
3	≥ 5	$G \not\leq \mathrm{PSL}_3(q).\langle\ddot{\delta},\ddot{\phi}\rangle$	$P_{1,2}$	$\frac{(q^3-1)(q+1)}{(q-1)}$
4	3	$G = \mathrm{PSL}_4(3).\langle\ddot{\gamma}\rangle$	$\mathrm{Sp}_4(q)$	117
4	3	$G \not\leq \mathrm{PSL}_4(3).\langle\ddot{\gamma}\rangle$ and $G \not\leq \mathrm{PSL}_4(3).\langle\ddot{\delta}\rangle$	P_2	130
4	≥ 4	$G \not\leq \mathrm{PSL}_4(q).\langle\ddot{\delta},\ddot{\phi}\rangle$	P_2	$\frac{(q^3-1)(q^2+1)}{(q-1)}$
5	all	$G \not\leq \mathrm{PSL}_5(q).\langle\ddot{\delta},\ddot{\phi}\rangle$	$P_{1,4}$	$(q^5-1)(q^2+1)$
6	all	$G \not\leq \mathrm{PSL}_6(q).\langle\ddot{\delta},\ddot{\phi}\rangle$	P_3	$\frac{(q^5-1)(q^3+1)(q^2+1)}{(q-1)}$
≥ 7	all	$G \not\leq \mathrm{PSL}_n(q).\langle\ddot{\delta},\ddot{\phi}\rangle$	$P_{1,n-1}$	$\frac{(q^{n-1}-1)(q^n-1)}{(q-1)^2}$

For completion we also include the following result which is a collection of information from the ATLAS [9].

Theorem 5.2.3. *Let A be almost simple, with socle S a sporadic simple group. Then the index of the largest non-trivial maximal subgroup of A equals $m(S)$ are as found in Table 5.8 and 5.9, except for the cases noted in Table 5.10.*

5.2.1 A recap of relevant information

Before we tackle the proofs of Theorems 5.2.1, 5.2.2 and 5.2.3, we would like to remind the reader of certain key concepts which play an important role in the following sections.

For this subsection let $\overline{\Omega}$ be a simple group. In addition let G be a group satisfying $\overline{\Omega} \leq G \leq$

Table 5.5: Largest non-trivial maximals for almost simple groups with socle $\mathrm{PSp}_n(q)$ and $\mathrm{PSU}_n(q)$

n	q	Conditions on G	Subgroup Type	$m(G)$
$\mathrm{PSp}_n(q)$				
4	$q \geq 4$	q even with $G \not\leq \mathrm{PSp}_4(q).\langle\ddot{\phi}\rangle$	$\mathcal{A}_1$	$\frac{(q^4-1)(q+1)}{(q-1)}$
$\mathrm{PSU}_n(q)$				
3	5	$G = \mathrm{PSU}_3(5).T$, T is not contained in any $\mathrm{Out}(\mathrm{PSU}_3(5))$-conjugates of $\langle\ddot{\phi}\rangle$	P_1	126

$\mathrm{Aut}(\overline{\Omega})$. Our aim in the following sections is to compare the indices of the non-trivial maximal subgroups M of G (subgroups such that $M \not\geq \overline{\Omega}$).

For a given non-trivial maximal subgroup M of G by Lemma 2.3.6 we have $M_{\overline{\Omega}} = N_{\overline{\Omega}}(M_{\overline{\Omega}})$ and $M = N_G(M_{\overline{\Omega}})$. So, by Lemma 2.3.17 we have that $G : M = |\overline{\Omega} : \overline{M_{\overline{\Omega}}}|$. In other words the problem of comparing the indices of non-trivial maximal subgroups of $\overline{X}$ becomes one of comparing the indices of subgroups of $\overline{\Omega}$ (note that these subgroups are not necessarily maximal).

To this end we will employ work from [26] and [6] in the cases where $\overline{\Omega}$ is a classical simple group, i.e. where $\overline{\Omega}$ equals $\mathrm{PSL}_n(q)$, $\mathrm{PSU}_n(q)$, $\mathrm{PSp}_n(q)$ or $\mathrm{O}_n^\epsilon(q)$. For the other cases we will be using a variety of sources, information of which will be given at the beginning of the appropriate subsection.

However as a note, in the cases where $\overline{\Omega}$ is a classical simple group we may sometimes instead compare the indices of subgroups $H \leq \Omega$ where $Z(\Omega) \leq H$. Here, Ω is the quasisimple group $\mathrm{SL}_n(q)$, $\mathrm{SU}_n(q)$, $\mathrm{Sp}_n(q)$ or $\Omega_n^\epsilon(q)$. We note that by the isomorphism theorem $|\Omega : H| = |\frac{\Omega}{Z(\Omega)} : \frac{H}{Z(\Omega)}|$, and so this change of viewpoint is justified. An important fact to note here also, is that the maximal subgroups of Ω do contain $Z(\Omega)$ and so are in one-to-one correspondence with the maximal subgroups of $\overline{\Omega}$ by Lemma 2.2.6.

Table 5.6: Largest non-trivial maximals for almost simple groups with socle $P\Omega_n(q)$

n	q	Conditions on G	Subgroup type	$m(G)$
$P\Omega_n^+(q)$				
8	2	$G = P\Omega_8^+(2).T$, T is not contained in any $\mathrm{Out}(P\Omega_8(2))$-conjugate of $\langle\ddot{\gamma}\rangle$	P_2	1575
8	3	$G = P\Omega_8^+(3).T$, T is contained in a conjugate of $\langle\ddot{\delta}, \ddot{\gamma}, \ddot{\delta}'\rangle$ and T is not contained in any $\mathrm{Out}(P\Omega_8(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$	P_1	1120
8	3	$G = P\Omega_8^+(3).T$, T is contained in a conjugate of $\langle\ddot{\gamma}, \ddot{\tau}\rangle$ and T is not contained in any $\mathrm{Out}(P\Omega_8(3))$-conjugate of $\langle\ddot{\delta}, \ddot{\gamma}, \ddot{\delta}'\rangle$	$2^{\cdot}\Omega_8^+(2)$	28431
8	3	$G = P\Omega_8^+(3).T$, T is not contained in any conjugate of $\langle\ddot{\gamma}, \ddot{\tau}'\rangle$ nor any $\mathrm{Out}(P\Omega_8(3))$-conjugate of $\langle\ddot{\delta}, \ddot{\gamma}, \ddot{\delta}'\rangle$	P_2	36400
8	≥ 4	$G = P\Omega_8^+(q).T$, T is not contained in any $\mathrm{Out}(P\Omega_8(q))$-conjugate of $\langle\ddot{\delta}, \ddot{\gamma}, \ddot{\delta}', \ddot{\phi}\rangle$, where $\ddot{\delta}'$ and $\ddot{\delta}$ are non-trivial only if q is odd	P_2	$\frac{(q^2+1)^2(q^6-1)}{q-1}$
≥ 10	3	$G = P\Omega_n^+(q).T$, T is not contained in any $\mathrm{Out}(P\Omega_n(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$, where $\ddot{\delta}'$ is non-trivial only if $4\|n$	P_1	$\frac{(3^m+1)(3^{m-1}-1)}{2}$

Table 5.7: $m(G)$ for exceptional simple groups

G	$m(G)$	Reference
$\mathrm{G}_2(3)$	351	[48]
$\mathrm{G}_2(4)$	416	[48]
$\mathrm{G}_2(q)$, $q \neq 3, 4$	$\frac{q^6-1}{q-1}$	[48]
F_4	$\frac{(q^{12}-1)(q^4-1)}{(q-1)}$	[48]
E_6	$\frac{(q^9-1)(q^8+q^4+1)}{(q-1)}$	[49]
E_7	$\frac{(q^{14}-1)(q^9+1)(q^5+1)}{(q-1)}$	[49]
E_8	$\frac{(q^{30}-1)(q^{12}+1)(q^{10}+1)(q^6+1)}{(q-1)}$	[49]
${}^2\mathrm{E}_6(q^2)$	$\frac{(q^{12}-1)(q^6-q^3+1)(q^4+1)}{(q-1)}$	[50]
${}^3\mathrm{D}_4(q^3)$	$\frac{(q^8+q^4+1)(q+1)}{(q-1,2)}$	[50]
${}^2\mathrm{B}_2(2^{2n+1})$	(q^2+1)	[50]
${}^2\mathrm{F}_4(2^{2n+1})$	$(q^6+1)(q^3+1)(q+1)$	[50]
${}^2\mathrm{F}_4(2)'$	1600	[50]
${}^2\mathrm{G}_2(3^{2n+1})$	(q^3+1)	[50]

Table 5.8: $m(G)$ for sporadic simple groups

G	$m(G)$
M_{11}	11
M_{12}	12
M_{22}	22
M_{23}	23
M_{24}	24
J_1	266
J_2	100
J_3	6156
J_4	$173,067,389$
Co_1	$98,280$
Co_2	$2,300$
Co_3	276
Fi_{22}	$3,510$
Fi_{23}	$31,671$
Fi'_{24}	$306,936$

Table 5.9: $m(G)$ for sporadic simple groups (cont.)

G	$m(G)$
HS	100
McL	275
He	$2,058$
Ru	$4,060$
Suz	$1,782$
O'N	$122,760$
HN	$1,140,000$
Ly	$8,835,156$
Th	$143,127,000$
B	$13,571,955,000$
M	$97,239,461,142,009,186,000$

Table 5.10: Largest non-trivial maximals for almost simple exceptional groups and sporadics

$Soc(G)$	Conditions on G	$m(G)$
$\mathrm{E}_6(q)$	$G \not\leq \mathrm{E}_6(q).\langle\ddot{\phi},\ddot{\delta}\rangle$	$\frac{(q^9-1)(q^6+1)(q^4+1)(q^3+1)}{(q-1)}$
$\mathrm{F}_4(q)$	$G \not\leq \mathrm{F}_4(q).\langle\ddot{\phi}\rangle$	$\frac{(q^{12}-1)(q^6-1)(q^4+1)}{(q-1)^2}$
$\mathrm{G}_2(q)$	$G \not\leq \mathrm{G}_2(q).\langle\ddot{\phi}\rangle$, $q=3^e$, $e \neq 1$	$\frac{(q+1)(q^6-1)}{(q-1)}$
${}^2\mathrm{F}_4(2)'$	$G = {}^2\mathrm{F}_4(2)$	1755
M_{12}	$G = \mathrm{M}_{12}:2$	144
O'N	$G = \mathrm{O'N}:2$	$2,624,832$

5.3 Orders of non-trivial maximal subgroups of almost simple groups with socle $\mathrm{PSL}_n(q)$

Before we begin this section it will be valuable to revisit the concepts of parabolic subgroups in the context of linear groups. For further information on the structure of parabolic subgroups see Theorem 3.2.1.

The group $\mathrm{SL}_n(q)$ acts naturally on a vector space V of dimension n. Subgroups which fix subspaces of V are called *parabolic* subgroups. Of particular note, with their own notation, we have P_k subgroups, which we call *maximal parabolic*, which are the stabilisers of k-subspaces of V. Also of note we have $P_{k,n-k}$ subgroups, which are the stabilisers of two subspaces, one of dimension k and the other of dimension $n-k$, such that the $n-k$-dimensional subspace contains the k-space.

The projective image of a parabolic P_k subgroup $H \leq \mathrm{SL}_n(q)$ is called a parabolic P_k subgroup of $\mathrm{PSL}_n(q)$. Similarly the projective image of a parabolic $P_{k,n-k}$ subgroup of $H \leq \mathrm{SL}_n(q)$ is called a parabolic $P_{k,n-k}$ subgroup of $\mathrm{PSL}_n(q)$.

5.3.1 Sizes of non-trivial maximal subgroups of almost simple groups with socle $\mathrm{PSL}_n(q)$ for $n \geq 7$

We start by noting the following

Lemma 5.3.1. *Let $n \geq 3$, let G be almost simple with socle $\mathrm{PSL}_n(q)$.*

- *If $H \leq \mathrm{PSL}_n(q)$ is a P_i subgroup for $i \neq n/2$, then $N_G(H)\mathrm{PSL}_n(q) = G$ if and only if $G \leq \mathrm{P\Gamma L}_n(q)$.*

- *If $H \leq \mathrm{PSL}_n(q)$ is a $P_{i,n-i}$ subgroup for $i < n/2$, then for all G we have $N_G(H)\mathrm{PSL}_n(q) = G$.*

Proof. In the first case where H is a P_i type subgroup the result follows from [26, 4.1.17]. Recall the definition of c and π introduced in the proof of Lemma 5.1.35. Information on the permutation representation π of the action of $\mathrm{Out}(\mathrm{PSL}_n(q))$ on the conjugacy classes of H can be found in [26, Table 3.5.G]. We notice that the point stabilizer of this action, and kernel, is $\ddot{\Gamma} = \mathrm{P\Gamma L}_n(q)/\mathrm{PSL}_n(q) \leq \mathrm{Out}(\mathrm{PSL}_n(q))$. Consequently, by Lemma 2.3.8, $N_G(H)\mathrm{PSL}_n(q) = G$ if and only if $G \leq \mathrm{P\Gamma L}_n(q)$.

For the second case, where H is a $P_{i,n-i}$ type subgroup we refer to [26, 4.1.22]. Here c is equal to 1, therefore all elements of $\mathrm{Out}(\mathrm{PSL}_n(q))$ fix the $\mathrm{PSL}_n(q)$-conjugacy classes of H. Therefore $N_G(H)\mathrm{PSL}_n(q) = G$. □

We now aim to compare the order of a $P_{1,n-1}$ of $\mathrm{SL}_n(q)$ with other maximals of $\mathrm{SL}_n(q)$.

Lemma 5.3.2. *Let $n \geq 7$, let $m \geq 3$ and let q be a prime power. If $H_1 \leq \mathrm{SL}_n(q)$ is a $P_{m,n-m}$ type subgroup and $H_2 \leq \mathrm{SL}_n(q)$ is a $\mathrm{GL}_m(q) \oplus \mathrm{GL}_{n-m}(q)$ type subgroup with the same m and n, then $|H_1| \geq |H_2|$.*

Proof.

$$\begin{aligned}
|H_1| &= (q-1)^2 q^{m(2n-3m)}|\mathrm{SL}_m(q)|^2|\mathrm{SL}_{n-2m}(q)| && \text{from Theorem 3.2.1}\\
&\geq (q-1)|\mathrm{SL}_m(q)|q^{m(2n-3m)}q^{m^2-2}q^{n^2-4nm+4m^2-2} && \text{by Lemma 2.6.4}\\
&= (q-1)|\mathrm{SL}_m(q)|q^{n^2-2mn+2m^2-4} \geq (q-1)|\mathrm{SL}_m(q)|q^{n^2-2mn+m^2} && \text{as } m \geq 3\\
&\geq (q-1)|\mathrm{SL}_m(q)||\mathrm{SL}_{n-m}(q)| && \text{by Lemma 2.6.4}\\
&= |H_2| && \text{from Theorem 3.2.1.}
\end{aligned}$$

□

Lemma 5.3.3. *Let $n \geq 5$ and q be a prime power. If $H_1 \leq \mathrm{SL}_n(q)$ is a $P_{2,n-2}$ type subgroup and $H_2 \leq \mathrm{SL}_n(q)$ is a $\mathrm{GL}_2(q) \oplus \mathrm{GL}_{n-2}(q)$ type subgroup, then $|H_1| \geq |H_2|$.*

Proof.

$$\begin{aligned}
|H_1| &= (q-1)^2|\mathrm{SL}_2(q)|q^{2(2n-6)}|\mathrm{SL}_2(q)||\mathrm{SL}_{n-4}(q)| && \text{from Theorem 3.2.1}\\
&= (q-1)^2|\mathrm{SL}_2(q)|q^{2(2n-6)}q(q^2-1)q^{(n-4)(n-5)/2}\prod_{i=2}^{n-4}(q^i-1)\\
&\geq (q-1)|\mathrm{SL}_2(q)|q^{2(2n-6)+1+(n-4)(n-5)/2}\prod_{i=2}^{n-4}(q^i-1)\\
&= (q-1)|\mathrm{SL}_2(q)|q^{(n^2-n-2)/2}\prod_{i=2}^{n-4}(q^i-1)
\end{aligned}$$

$$
\begin{aligned}
&= (q-1)|\mathrm{SL}_2(q)|q^{(n-2)(n-3)/2}q^{2n-4}\prod_{i=2}^{n-4}(q^i-1) \\
&\geq (q-1)|\mathrm{SL}_2(q)|(q^{n-3}-1)(q^{n-2}-1)q^{(n-2)(n-3)/2}\prod_{i=2}^{n-4}(q^i-1) \\
&= (q-1)|\mathrm{SL}_2(q)|q^{(n-2)(n-3)/2}\prod_{i=2}^{n-2}(q^i-1) \\
&= (q-1)|\mathrm{SL}_2(q)||\mathrm{SL}_{n-2}(q)| = |H_2| && \text{from Theorem 3.2.1.}
\end{aligned}
$$

□

Lemma 5.3.4. *Let $n \geq 3$ and q be a prime power. If $H_1 \leq \mathrm{SL}_n(q)$ is a $P_{1,n-1}$ type subgroup and $H_2 \leq \mathrm{SL}_n(q)$ is a $\mathrm{GL}_1(q) \oplus \mathrm{GL}_{n-1}(q)$ type subgroup, then $|H_1| \geq |H_2|$.*

Proof.

$$
\begin{aligned}
|H_1| &= (q-1)^2q^{2n-3}|\mathrm{SL}_{n-2}(q)| && \text{from Theorem 3.2.1} \\
&= (q-1)^2q^{2n-3}q^{(n-2)(n-3)/2}\prod_{i=2}^{n-2}(q^i-1) && \text{by Theorem 2.5.29} \\
&= (q-1)^2q^{n(n-1)/2}\prod_{i=2}^{n-2}(q^i-1) \\
&= (q-1)^2q^{n-1}q^{(n-2)(n-1)/2}\prod_{i=2}^{n-2}(q^i-1) \\
&\geq (q-1)(q^{n-1}-1)q^{(n-2)(n-1)/2}\prod_{i=2}^{n-2}(q^i-1) \\
&= (q-1)q^{(n-2)(n-1)/2}\prod_{i=2}^{n-1}(q^i-1) \\
&= (q-1)|\mathrm{SL}_{n-1}(q)| = |H_2| && \text{from Theorem 3.2.1.}
\end{aligned}
$$

□

Lemma 5.3.5. *Let $n \geq 3$ and q be a prime power. If $H \leq \mathrm{SL}_n(q)$ is a $P_{1,n-1}$ type subgroup then $|H| \geq (q-1)^2q^{n^2-2n-1}$.*

Proof.

$$
\begin{aligned}
|H| &= (q-1)^2q^{2n-3}|\mathrm{SL}_{n-2}(q)| && \text{from Theorem 3.2.1} \\
&\geq (q-1)^2q^{2n-3}q^{(n-2)^2-2} = (q-1)^2q^{n^2-2n-1} && \text{by Lemma 2.6.4.}
\end{aligned}
$$

□

Lemma 5.3.6. *Let $n \geq 7$ and $n/2 > m \geq 2$. If $H_1 \leq \mathrm{SL}_n(q)$ is a $P_{1,n-1}$ type subgroup and $H_2 \leq \mathrm{SL}_n(q)$ is a $P_{m,n-m}$ type subgroup, then $|H_1| \geq |H_2|$.*

Proof. In the case where $n/2 > m \geq 4$ we have

$$\begin{aligned}
|H_1| &\geq (q-1)^2 q^{n^2-2n-1} && \text{by Lemma 5.3.5} \\
&= (q-1)^2 q^{(n^2-2)+(2mn-2n)-2mn} \\
&\geq (q-1)^2 q^{(n^2-2)+(4m^2-4m)-2mn} && \text{since } n/2 \geq m \\
&= (q-1)^2 q^{(n^2-2)+(3m^2-1)-2mn} && \text{since } m \geq 4 \\
&\geq (q-1)^2 q^{3m^2-2mn+n^2-3} \\
&= (q-1)^2 q^{m(2n-3m)} q^{2(m^2-1)} q^{(n-2m)^2-1} \\
&\geq (q-1)^2 q^{m(2n-3m)} |\mathrm{SL}_m(q)|^2 |\mathrm{SL}_{n-2m}(q)| = |H_2| && \text{by Lemma 3.2.1.}
\end{aligned}$$

In the case where $m = 3$

$$\begin{aligned}
|H_1| &\geq (q-1)^2 q^{n^2-2n-1} && \text{by Lemma 5.3.5} \\
&\geq (q-1)^2 q^{24-6n+n^2} && \text{since } n \geq 7 \\
&= (q-1)^2 q^{3(2n-9)} q^{2(9-1)} q^{(n-6)^2-1} \\
&\geq (q-1)^2 q^{3(2n-9)} |\mathrm{SL}_3(q)|^2 |\mathrm{SL}_{n-6}(q)| = |H_2| && \text{by Lemma 2.6.4}
\end{aligned}$$

Finally in the case where $m = 2$

$$\begin{aligned}
|H_1| &\geq (q-1)^2 q^{n^2-2n-1} && \text{by Lemma 5.3.5} \\
&\geq (q-1)^2 q^{9-4n+n^2} && \text{since } n \geq 6 \\
&= (q-1)^2 q^{2(2n-6)} q^{2(4-1)} q^{(n-4)^2-1} \\
&\geq (q-1)^2 q^{2(2n-6)} |\mathrm{SL}_2(q)|^2 |\mathrm{SL}_{n-4}(q)| = |H_2| && \text{by Lemma 3.2.1.}
\end{aligned}$$

□

Finally we show that $P_{1,n}$ type subgroups are larger than $P_{n/2}$ type.

Lemma 5.3.7. *Let $n \geq 8$ be even and let q be a prime power. Let $H_1 \leq \mathrm{SL}_n(q)$ be a $P_{1,n-1}$ type subgroup, and let $H_2 \leq \mathrm{SL}_n(q)$ be a $P_{n/2}$ type subgroup. Then $|H_1| \geq |H_2|$.*

Proof. We first consider the case where $n \geq 10$. Then

$$\begin{aligned}
|H_1| &\geq (q-1)^2 q^{n^2-2n-1} && \text{by Lemma 5.3.5} \\
&\geq (q-1)^2 q^{\frac{3n^2}{4}} && \text{as } n \geq 9 \\
&\geq (q-1)^2 q^{n^2/4} q^{2n^2/4} \geq q^{n^2/4} |\mathrm{SL}_{n/2}(q)|^2 && \text{by Lemma 2.6.4} \\
&= |H_2| && \text{from Theorem 3.2.1.}
\end{aligned}$$

In the case where $n = 8$

$$\begin{aligned}
|H_1| &= (q-1)^2 q^{13} |\mathrm{SL}_6(q)| && \text{from Theorem 3.2.1} \\
&= (q-1) q^{13} q^{15} \prod_{i=2}^{6} (q^i - 1) = (q-1) q^{28} \prod_{i=2}^{6} (q^i - 1)
\end{aligned}$$

$$
\begin{aligned}
&= (q-1)(q^6-1)(q^5-1)q^{28}\prod_{i=2}^{4}(q^i-1) \\
&\geq (q-1)q^5q^4q^{28}\prod_{i=2}^{4}(q^i-1) \\
&\geq q^4q^3q^2q^{28}\prod_{i=2}^{4}(q^i-1) \\
&\geq (q^4-1)(q^3-1)(q^2-1)q^{28}\prod_{i=2}^{4}(q^i-1) = q^{16}q^{12}\prod_{i=2}^{4}(q^i-1)^2 \\
&= q^{16}|\mathrm{SL}_4(q)|^2 = |H_2| \qquad \text{from Theorem 3.2.1.}
\end{aligned}
$$

□

We finally look at the maximal subgroups which are not of geometric type (i.e. not of a type in Table 2).

Lemma 5.3.8. *Let $n \geq 6$ and let $H_1 \leq \mathrm{SL}_n(q)$ is a $|P_{1,n-1}|$ type subgroup. Then $|H_1| \geq q^{3n+1}$.*

Proof.

$$
\begin{aligned}
|H_1| &\geq (q-1)^2 q^{n^2-2n-1} && \text{by Lemma 5.3.5} \\
&\geq (q-1)^2 q^{3n+1} > q^{3n+1} && \text{since } n \geq 6\text{, by assumption.}
\end{aligned}
$$

□

Combining the above results we get the following result.

Proposition 5.3.9. *Let $n \geq 7$ and let q be a prime power. Furthermore, let $\overline{M} \leq \mathrm{PSL}_n(q)$ be a subgroup of type $P_{1,n-1}$. Let $\overline{H} \leq \mathrm{PSL}_n(q)$ be a subgroup in $\mathcal{C}_{\mathrm{PSL}_n(q)}$. Then, if $|\overline{H}| > |\overline{M}|$ then $\overline{H}$ is a P_i type subgroup where $1 \leq i < n/2$.*

Proof. For $\overline{H}$ and $\overline{M}$ there exist subgroups H and M of $\mathrm{SL}_n(q)$ of the same type. Note that $\overline{H} = H/(Z \cap \mathrm{SL}_n(q))$ and $\overline{M} = M/(Z \cap \mathrm{SL}_n(q))$. We compare the sizes of M and H.

By Theorem 3.2.1, the types of subgroups that H can be are listed in Table 3.1. In the case where the maximal subgroup H lies in $\mathcal{C}_i$ for $2 \leq i \leq 8$ then $|H| \leq |M|$ by Lemma 3.2.4.

If H lies in $\mathcal{C}_1$ and is of parabolic $P_{n/2}$ type then $|H| \leq |M|$ by Lemma 5.3.7. Finally, if H is a $P_{m,n-m}$ or $\mathrm{GL}_m(q) \oplus \mathrm{GL}_{n-m}(q)$ type subgroup then we have $|M| \geq |H|$. To elaborate, by Lemma 5.3.6, the order of a $P_{1,n-1}$ type subgroup is larger than that of a $P_{m,n-m}$ type subgroup for $1 < m < n/2$. By Lemmas 5.3.2 - 5.3.4, we have that a $P_{m,n-m}$ type subgroup, for $1 \leq m < n/2$ has larger order than a subgroup of type $\mathrm{GL}_1(q) \oplus \mathrm{GL}_{n-1}(q)$. Therefore unless H is a P_i type subgroup where $1 \leq i < n/2$ we have that $|H| \leq |M|$, and so $|\overline{H}| \leq |\overline{M}|$. □

Proposition 5.3.10. *Let $n \geq 7$, q be a prime power, and let G have socle $\mathrm{PSL}_n(q)$.*

- *If $G \leq \mathrm{P\Gamma L}_n(q)$ then $m(G) = m(\mathrm{PSL}_n(q)) = |\mathrm{PSL}_n(q) : H|$, where H is a P_1 subgroup of $\mathrm{PSL}_n(q)$.*

- *If $G \not\leq \mathrm{P\Gamma L}_n(q)$ then $m(G) = |\mathrm{PSL}_n(q) : H|$, where H is a $P_{1,n-1}$ subgroup of $\mathrm{PSL}_n(q)$.*

Proof. By Lemma 5.1.32, we note that $\mathrm{Out}(\mathrm{PSL}_n(q)) = \langle \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \rangle$.

Let H_1 be a P_1 subgroup of $\mathrm{PSL}_n(q)$. In this case $|\mathrm{PSL}_n(q) : H_1| = m(\mathrm{PSL}_n(q))$, as can be seen in Theorem 2.2.7. Therefore if $G \leq \mathrm{P\Gamma L}_n(q)$ then by Lemma 5.3.1 we have that $N_G(H_1)\mathrm{PSL}_n(q) = G$. By 2.3.18 we have that $m(G) = m(\mathrm{PSL}_n(q))$.

Now suppose that the image of G in $\mathrm{Out}(\mathrm{PSL}_n(q))$ is not contained in the stabilizer of the conjugacy class of H_1, or equivalently $G \not\leq \mathrm{P\Gamma L}_n(q)$. Let H_2 be a $P_{1,n-1}$ subgroup of $\mathrm{PSL}_n(q)$. By Lemma 5.3.1 we know that $N_G(H_2)\mathrm{PSL}_n(q) = G$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{PSL}_n(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{PSL}_n(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{PSL}_n(q) = G$.

By Theorem 3.1.4, either M belongs in $\mathcal{C}_G$ or $|K| \leq 3^{3n}$. If $|K| \leq 3^{3n}$ then $|K \cap \mathrm{PSL}_n(3)| \leq 3^{3n}$. By Lemma 5.3.8 a $P_{1,n-1}$ subgroup of $\Omega_n^+(3)$ has order larger than 3^{3n+1} and so therefore a $|H_2| \geq |K|$.

If $K \in \mathcal{C}_G$ then, by Lemma 5.3.9, $|H_2| \geq |K|$ unless K is a P_i type subgroup for $i \neq n/2$. However if K is a P_i type subgroup for $i \neq n/2$, by Lemma 5.3.1 we have that $N_G(K)\mathrm{PSL}_n(q) \neq G$. Therefore $N_G(K)$ is not a maximal subgroup of G.

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{PSL}_n(q)$ that satisfies $|K| \leq |H_2|$. Hence $|G : N_G(H_2)| = |\mathrm{PSL}_n(q) : H_2| \leq |\mathrm{PSL}_n(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_2)| = |\mathrm{PSL}_n(q) : H_2|$. □

5.3.2 Sizes of non-trivial maximal subgroups of almost simple groups with socle $\mathrm{PSL}_n(q)$ for $n \leq 6$

Here we aim to calculate the largest maximal subgroups for almost simple groups G with socle $\mathrm{PSL}_n(q)$ where $n \leq 6$. We aim to do this by considering each dimension separately, and heavy use of the tables from [6, Chapter 8].

Proposition 5.3.11. *Let $n = 2$, q be a prime power, and let G be an almost simple group satisfying $\mathrm{PSL}_n(q) \leq G \leq \mathrm{P\Gamma L}_n(q)$. The following all hold.*

- *If $q = 5$ then $m(G) = m(\mathrm{PSL}_2(5)) = |\mathrm{PSL}_2(5) : H|$, where H is of $\mathcal{C}_6$ type.*
- *If $q = 7$ and $G = \mathrm{PSL}_2(7)$ then $m(G) = |\mathrm{PSL}_2(7) : H|$, where H is of $\mathcal{C}_6$ type, where H is isomorphic to S_4.*
- *If $q = 7$ and $G = \mathrm{PSL}_2(7).\langle \ddot{\delta} \rangle$ then $m(G) = |\mathrm{PSL}_2(7) : H|$, where H is of P_1 type.*
- *If $q = 9$ and $G \leq \mathrm{PSL}_2(9).\langle \ddot{\phi} \rangle$ then $m(G) = m(\mathrm{PSL}_2(9)) = |\mathrm{PSL}_2(9) : H|$, where H is not of geometric type and H is the image of a subgroup of $\mathrm{SL}_2(9)$ with shape $2^{\cdot}\mathrm{A}_5$.*
- *If $q = 9$ and $G \not\leq \mathrm{PSL}_2(9).\langle \ddot{\phi} \rangle$ then $m(G) = |\mathrm{PSL}_2(9) : H|$, where H is of P_1 type.*

- *If* $q = 11$ *and* $G = \mathrm{PSL}_2(11)$ *then* $m(G) = |\mathrm{PSL}_2(11) : H|$, *where* H *is not of geometric type and* H *is the image of a subgroup of* $\mathrm{SL}_2(11)$ *with shape* $2^{\cdot}\mathrm{A}_5$.
- *If* $q = 11$ *and* $G = \mathrm{PSL}_2(11).\langle\ddot{\delta}\rangle$ *then* $m(G) = |\mathrm{PSL}_2(11) : H|$, *where* H *is of* P_1 *type.*
- *otherwise* $m(G) = m(\mathrm{PSL}_2(q)) = |\mathrm{PSL}_2(q) : H|$, *where* H *is of* P_1 *type.*

Proof. Let us first note that by Lemma 5.1.32 that $\mathrm{Out}(\mathrm{PSL}_2(q)) = \langle\ddot{\delta}, \ddot{\phi}\rangle$. For the cases where $q = 5, 7, 9, 11$ this is a direct calculation, either from reading the subgroups off Table 4.3, or calculating in MAGMA the subgroups with the largest order.

Theorem 2.2.7 and Table [6, Table 8.1] show us that for the cases where $q \neq 5, 7, 9, 11$ and $G = \mathrm{PSL}_2(q)$, a largest maximal subgroup is in fact of P_1 type; let us call this subgroup H. The stabilizer of the $\mathrm{PSL}_2(q)$-conjugacy class of H, when acted upon by $\mathrm{Out}(\mathrm{PSL}_2(q))$ is in fact $\langle\ddot{\phi}, \ddot{\delta}\rangle = \mathrm{Out}(\mathrm{PSL}_2(q))$. Therefore we have $N_G(H)\mathrm{PSL}_2(q) = G$. By Lemma 2.3.18 we have that $m(G) = m(\mathrm{PSL}_2(q))$. □

Before we tackle $3 \leq n \leq 6$ separately we derive the following general lemma.

Lemma 5.3.12. *Let* $n \geq 3$, q *be a prime power and* $(n, q) \neq (4, 2)$. *Let furthermore* G *satisfy* $\mathrm{PSL}_n(q) \leq G \leq \mathrm{PSL}_n(q).\langle\ddot{\delta}, \ddot{\phi}\rangle = \mathrm{P\Gamma L}_n(q)$. *Then* $m(G) = m(\mathrm{PSL}_n(q)) = |\mathrm{PSL}_n(q) : H|$, *where* H *is a* P_1 *type subgroup.*

Proof. The index of H in $\mathrm{PSL}_n(q)$ is $\frac{q^n-1}{q-1}$ and so by Theorem 2.2.7 we have $|\mathrm{PSL}_n(q) : H| = m(\mathrm{PSL}_n(q))$. By Lemma 5.3.1 we have that $N_G(H)\mathrm{PSL}_n(q) = G$. By Lemma 2.3.18 we get the required result. □

From here on we tackle the remaining cases for G with $3 \leq n \leq 6$.

Proposition 5.3.13. *Let* $n = 3$, q *be a prime power and let* G *be almost simple with socle* $\mathrm{PSL}_n(q)$.

- *If* $q = 3$ *and* $G = \mathrm{PSL}_3(3).\langle\ddot{\gamma}\rangle$ *then* $m(G) = |\mathrm{PSL}_3(3) : H|$, *where* H *is a* $P_{1,2}$ *type subgroup.*
- *If* $q = 4$ *and* $G = \mathrm{PSL}_n(q).T$, *where* T *lies in a* $\mathrm{Out}(\mathrm{PSL}_3(4))$*-conjugate of* $\langle\ddot{\gamma}, \ddot{\phi}\rangle$ *and* T *does not lie in* $\langle\ddot{\delta}, \ddot{\phi}\rangle$, *then* $m(G) = |\mathrm{PSL}_3(4) : H|$, *where* H *is non geometric, and* H *is the image of a subgroup of* $\mathrm{SL}_3(4)$ *with shape* $3^{\cdot}\mathrm{A}_6$.
- *If* $q = 4$ *and* $G = \mathrm{PSL}_3(4).T$, *where* T *does not lie in any* $\mathrm{Out}(\mathrm{PSL}_3(4))$*-conjugate of* $\langle\ddot{\gamma}, \ddot{\phi}\rangle$ *and* T *does not lie in* $\langle\ddot{\delta}, \ddot{\phi}\rangle$, *then* $m(G) = |\mathrm{PSL}_3(4) : H|$, *where* H *is a* $P_{1,2}$ *type subgroup.*
- *if* $q \geq 5$ *and* $G \not\leq \mathrm{PSL}_3(q).\langle\ddot{\delta}, \ddot{\phi}\rangle$ *then* $m(G) = |\mathrm{PSL}_3(q) : H|$, *where* H *is a* $P_{1,2}$ *type subgroup.*

Proof. Let us first note, by Lemma 5.1.32 that that $\mathrm{Out}(\mathrm{PSL}_3(q)) = \langle\ddot{\delta}, \ddot{\phi}, \ddot{\gamma}\rangle$. For the smaller cases $q = 3$ and $q = 4$ we appeal to Table 4.4, which provides us with a comprehensive list of the maximal subgroups of G.

Let H be a $P_{1,2}$ subgroup of $\mathrm{PSL}_3(q)$ and let $G \not\leq \mathrm{PSL}_3(q).\langle\ddot{\delta}, \ddot{\phi}\rangle$. Let $d = (q-1, 3) = |Z(\mathrm{SL}_3(q))|$ so that $d|H| = q^3(q-1)^2$. Note that the stabilizer of the conjugacy class of H in $\mathrm{PSL}_3(q)$ is the whole of $\mathrm{Out}(\mathrm{PSL}_3(q))$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{PSL}_3(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{PSL}_3(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{PSL}_3(q) = G$. The possibilities for K are found in the rows of Table 4.4 for which the stabilizer listed is not contained in $\langle \ddot{\delta}, \ddot{\phi} \rangle$. One may show that $|H| > |K|$.

There are two cases where this inequality is not necessarily obvious, so we elaborate upon this inequalities for when K is the image of a subgroup of $\mathrm{SL}_n(q)$ with shape $\mathrm{SL}_3(q_0).\big(\frac{q-1}{q_0-1}, 3\big)$ where $q = q_0^r$ and r is prime and K is the image of a subgroup of $\mathrm{SL}_n(q)$ with shape $(q_0 - 1, 3) \times \mathrm{SU}_3(q_0)$ where $q = q_0^n$.

Neither of these cases occur when $q = 5$ or 7, so we may assume that $q \geq 8$. We notice that

$$\begin{aligned} |\mathrm{SL}_3(q_0).\big(\frac{q-1}{q_0-1}, 3\big)| &\leq 3q_0^3(q_0^3-1)(q_0^2-1) & \\ &\leq 3q^{3/2}q^{3/2}q = 3q^4 & \text{since } q_0 \leq q^{1/2} \\ &< q^3(q-1)^2 = d|H| & \text{since } q \geq 8. \end{aligned}$$

We also notice that

$$\begin{aligned} |(q_0 - 1, 3) \times \mathrm{SU}_3(q_0)| &\leq 3q_0^3(q_0^2-1)(q_0^3+1) & \\ &\leq 6q^{3/2}qq^{3/2} = 6q^4 & \text{since } q_0 \leq q^{1/2} \\ &< q^3(q-1)^2 = d|H| & \text{since } q \geq 8. \end{aligned}$$

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{PSL}_3(q)$ that satisfies $|K| \leq |H|$. Hence $|G : N_G(H)| = |\mathrm{PSL}_3(q) : H| \leq |\mathrm{PSL}_3(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H)| = |\mathrm{PSL}_3(q) : H|$.

□

Proposition 5.3.14. *Let $n = 4$, q be a prime power, and let G be almost simple with socle $\mathrm{PSL}_n(q)$.*

- *If $q = 2$ and $G = \mathrm{PSL}_4(2).\langle \ddot{\gamma} \rangle$ then $m(G) = |\mathrm{PSL}_4(2) : H|$, where H is of non-geometric type and H is the image of a subgroup of $\mathrm{SL}_4(2)$ isomorphic to A_7.*
- *If $q = 3$ and $G = \mathrm{PSL}_4(3).\langle \ddot{\gamma} \rangle$ then $m(G) = |\mathrm{PSL}_4(3) : H|$, where H is a $\mathrm{Sp}_4(3)$ type subgroup.*
- *If $q = 3$ and $G \not\leq \mathrm{PSL}_4(3).\langle \ddot{\delta} \rangle$ and $G \not\leq \mathrm{PSL}_4(3).\langle \ddot{\gamma} \rangle$, then $m(G) = |\mathrm{PSL}_4(3) : H|$, where H is P_2 type subgroup.*
- *If $q \geq 4$ and $G \not\leq \mathrm{PSL}_4(q).\langle \ddot{\delta}, \ddot{\phi} \rangle$ then $m(G) = |\mathrm{PSL}_4(q) : H|$, where H is a P_2 type subgroup.*

Proof. The smaller cases $q = 2$ and 3 are covered by appealing to Table 4.5 which provides us with a comprehensive list of the maximal subgroups of G. Recall that for this table c is the number of conjugacy classes of groups of a given type.

Now let $q \geq 4$, let H be a P_2 subgroup of $\mathrm{PSL}_4(q)$ and let $G \not\leq \mathrm{PSL}_4(q).\langle \ddot{\delta}, \ddot{\phi} \rangle$. Let $d =$

$(q-1,4) = |Z(\mathrm{SL}_4(q))|$ so that $d|H| = q^6(q^2-1)^2(q-1)$. Note that the stabilizer of the conjugacy class of H in $\mathrm{PSL}_4(q)$ is the whole of $\mathrm{Out}(\mathrm{PSL}_4(q))$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{PSL}_4(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{PSL}_4(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{PSL}_4(q) = G$. The possibilities for K are found in the rows of Table 4.5 for which the stabilizer listed is not contained in $\langle \ddot{\delta}, \ddot{\phi} \rangle$. One may show that $|H| > |K|$ for such K.

Let us see this in more detail. If K is a $P_{1,3}$ subgroup then

$$d|K| = q^6(q^2-1)(q-1)(q-1) < d|H|.$$

Furthermore Lemma 5.3.4 now covers the case for where K is a $\mathrm{GL}_1 \oplus \mathrm{GL}_3$ subgroup.

If K is a $\mathcal{C}_2$ subgroup then we have two possibilities. If K stabilises a decomposition into four 1-spaces then

$$d|K| = 4!(q-1)^3 < q^6(q-1)^3 < q^6(q^2-1)^2(q-1) = d|H|.$$

If K stabilises a decomposition into two 2-spaces then

$$d|K| = 2|\mathrm{SL}_2(q)|^2(q-1) = 2(q^2-1)^2q^2(q-1) < d|H|.$$

If K is a $\mathcal{C}_3$ subgroup then

$$d|K| = 2|\mathrm{SL}_2(q^2)|(q+1) = 2q^2(q^4-1)(q+1) < 2q^6(q+1) < d|H|.$$

If K is a $\mathcal{C}_5$ subgroup then

$$\begin{aligned} d|K| &\leq 4\mathrm{SL}_4(q_0) = 4q_0^6(q_0^2-1)(q_0^3-1)(q_0^4-1) \\ &\leq 4\mathrm{SL}_4(q_0) = 4q^3(q-1)(q^{1.5}-1)(q^2-1) && \text{as } q^{1/2} \geq q_0 \\ &< q^6(q^2-1)^2(q-1) = d|H|. \end{aligned}$$

If K is a $\mathcal{C}_6$ subgroup then $q \geq 5$ and so

$$d|K| = 2 \times 2^5 \times 720 < q^6(q^2-1)^2(q-1) = d|H|.$$

If K is a $\mathrm{SO}_n^{\pm}(q)$ type subgroup then $q \geq 3$, and we have

$$\begin{aligned} d|K| &< |\mathrm{SO}_4^-(q).[(q-1,4)]| \leq 8q^2(q^2+1)(q^2-1) \\ &= 8q^2(q+1)(q^2+1)(q-1) \leq q^6(q^2+1)(q-1) < d|H| && \text{since } q \geq 3. \end{aligned}$$

If K is a $\mathrm{Sp}_n(q)$ type subgroup then

$$\begin{aligned} d|K| &= |\mathrm{Sp}_4(q)|.(q-1,2) \leq 2(q^4-1)q^4(q^2-1) \\ &< q^2(q^2-1)(q-1)q^4(q^2-1) = d|H| && \text{since } q \geq 4. \end{aligned}$$

Finally if K is a $\mathrm{SU}_n(q_0)$ type subgroup then, where $q = q_0^2 \geq 4$, we have

$$d|K| = |\mathrm{SU}_4(q_0)|.(q_0-1,4) \leq 4q_0^6(q_0^4-1)(q_0^3+1)(q_0^2-1)$$

$$= 4q^3(q^2-1)(q^{3/2}+1)(q-1)$$
$$\leq 8q^{15/2} < q^6(q^2-1)^2(q-1) = d|H|.$$

Finally, if K is a non-geometric maximal subgroup then by Table 4.5 we have

$$d|H| = q^6(q^2-1)^2(q-1) \geq 4^6(4^2-1)^2(4-1) = 2764800 > d|K|.$$

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{PSL}_4(q)$ that satisfies $|K| \leq |H|$. Hence $|G : N_G(H)| = |\mathrm{PSL}_4(q) : H| \leq |\mathrm{PSL}_4(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H)| = |\mathrm{PSL}_4(q) : H|$.

□

Proposition 5.3.15. *Let $n = 5$, q be a prime power, and let G be almost simple with socle $\mathrm{PSL}_5(q)$. If $q \geq 2$ and $G \not\leq \mathrm{P\Gamma L}_5(q)$ then $m(G) = |\mathrm{PSL}_5(q) : H|$, where H is a $P_{1,4}$ type subgroup.*

Proof. Let H be a $P_{1,4}$ subgroup of $\mathrm{PSL}_5(q)$ and let $G \not\leq \mathrm{PSL}_5(q).\langle\ddot{\delta}, \ddot{\phi}\rangle$. Let $d = (q-1, 5) = |Z(\mathrm{SL}_5(q))|$ so that $d|H| = q^7(q-1)|\mathrm{GL}_3(q)| = q^{10}(q^3-1)(q^2-1)(q-1)^2$. Note that the stabilizer of the conjugacy class of H in $\mathrm{PSL}_5(q)$ is the whole of $\mathrm{Out}(\mathrm{PSL}_5(q))$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{PSL}_5(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{PSL}_5(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{PSL}_5(q) = G$. The possibilities for K are found in the rows of Table 4.6 for which the stabilizer listed is not contained in $\langle\ddot{\delta}, \ddot{\phi}\rangle$. One may show that $|H| > |K|$ for such K.

Let us see this in more detail: Lemma 5.3.6 shows that when K is $P_{m,n-m}$ type for $m \neq 1$ then $|K| < |H|$. Furthermore Lemmas 5.3.3 and 5.3.4 cover the cases where K is $\mathrm{GL}_m(q) \oplus \mathrm{GL}_{n-m}(q)$ type subgroup for $m = 1, 2$. Finally we note that if K is a P_m type subgroup then $N_G(K)\mathrm{PSL}_5(q) = G$ by Lemma 5.3.1.

If K is a $\mathcal{C}_2$ subgroup then

$$d|K| = 5!(q-1)^4 < q^{10}(q-1)^4 < q^{10}(q^3-1)(q^2-1)(q-1)^2 = d|H|.$$

If K is a $\mathcal{C}_3$ subgroup then

$$d|K| = 5\frac{q^5-1}{q-1} < q^8 < d|H|.$$

Lemma 3.2.4 covers the cases where K is a $\mathcal{C}_5$ subgroup.

If K is a $\mathcal{C}_6$ subgroup we note that

$$d|K| = 15000 < 21504 = (2^3-1)(2^2-1)(2-1)^2 2^{10} \leq d|H|.$$

If K is a $\mathcal{C}_8$ subgroup we have two possibilities, if K is the image of a subgroup of $\mathrm{SL}_5(q)$ with shape $d \times \mathrm{SO}_5(q)$ then

$$d|K| \leq 5|\mathrm{SO}_5(q)| = 5q^4(q^4-1)(q^2-1) < 5q^{10} < d|H|.$$

If if K is the image of a subgroup of $\mathrm{SL}_5(q)$ with shape $(q_0-1,5)\times \mathrm{SU}_5(q_0)$ where $q_0^2=q$, then

$$\begin{aligned}
d|K| &\le 5|\mathrm{SU}_5(q_0)| = 5q_0^{10}\prod_{2}^{5}(q_0^i-(-1)^i)\\
&= 5(q^{3/2}+1)q^5(q-1)(q^2-1)(q^{5/2}+1)\\
&< q^{3/2}2q^{3/2}q^5(q-1)(q^2-1)(q^3-1)\\
&\le q^9(q-1)(q^2-1)(q^3-1) < d|H|.
\end{aligned}$$

In the case where K is non-geometric we refer to Table 4.6. Please note that the subgroup of form $d\times \mathrm{PSU}_4(2)$ does not occur until $q\ge 7$.

We have shown that if M is any non-trivial maximal subgroup of G then $M=N_G(K)$ for a subgroup $K\le \mathrm{PSL}_5(q)$ that satisfies $|K|\le|H|$. Hence $|G:N_G(H)|=|\mathrm{PSL}_5(q):H|\le|\mathrm{PSL}_5(q):K|=|G:M|$, by Lemma 2.3.17. Therefore $m(G)=|G:N_G(H)|=|\mathrm{PSL}_5(q):H|$. □

Proposition 5.3.16. *Let $n=6$, q be a prime power, and let G be almost simple with socle $\mathrm{PSL}_6(q)$. If $q\ge 2$ and $G\not\le \mathrm{P\Gamma L}_6(q)$ then $m(G)=|H:\mathrm{PSL}_6(q)|$ where H is P_3 type subgroup.*

Proof. Let H be a P_3 subgroup of $\mathrm{PSL}_6(q)$ and let $G\not\le \mathrm{PSL}_5(q).\langle\ddot{\delta},\ddot{\phi}\rangle$. Let $d=(q-1,6)=|Z(\mathrm{SL}_6(q))|$ so that $d|H|=(q-1)q^9|\mathrm{SL}_3(q)|^2=(q-1)q^{15}(q^3-1)^2(q^2-1)^2$, as can be seen in Table [6, Table 8.24 & 8.25]. Note that the stabilizer of the conjugacy class of H in $\mathrm{PSL}_6(q)$ is the whole of $\mathrm{Out}(\mathrm{PSL}_6(q))$.

Let M be a non-trivial maximal subgroup of G and let $K=M\cap\mathrm{PSL}_6(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K\ne\mathrm{PSL}_6(q)$, and $N_G(K)=M$, and $N_G(K)\mathrm{PSL}_6(q)=G$. The possibilities for K are found in the rows of [6, Tables 8.24 & 8.25] for which the stabilizer listed is not contained in $\langle\ddot{\delta},\ddot{\phi}\rangle$. One may show that $|H|>|K|$ for such K.

Let us expand on this statement further: if K is a $P_{1,5}$ subgroup then

$$(q^3-1)(q^2-1)-(q-1)(q^4-1)=q^4-q^3-q^2+q=(q^2-1)q(q-1)>0 \qquad \text{for } q\ge 2.$$

Therefore $(q^3-1)(q^2-1)>(q-1)(q^4-1)$. From Theorem 3.2.1 we have

$$\begin{aligned}
d|K| &= (q-1)^2q^9|\mathrm{SL}_4(q)| = (q-1)^2q^{15}(q^4-1)(q^3-1)(q^2-1)\\
&< (q-1)q^{15}(q^3-1)^2(q^2-1)^2 = d|H|.
\end{aligned}$$

Consequently, if K is a $P_{2,4}$ type subgroup then $|H|>|K|$ by Lemma 5.3.6. Furthermore Lemmas 5.3.3 and 5.3.4 cover the cases where K is $\mathrm{GL}_m(q)\oplus\mathrm{GL}_{n-m}(q)$ type subgroup for $m=1,2$. Finally we note that if K is a P_m type subgroup for $m\ne 3$ then $N_G(K)\mathrm{PSL}_6(q)=G$ by Lemma 5.3.1.

By Lemma 3.2.4 if K is a $\mathcal{C}_i$ subgroup where $i=3,4,5,7,8$ then we have $|H|>|K|$. Lemma 5.3.8 in turn covers the case where K is a non-geometric subgroup.

If K is a $\mathcal{C}_2$ subgroup that stabilizes a decomposition of six 1-spaces, then

$$d|K|=6!(q-1)^5<q^{15}(q-1)^5<d|H|.$$

If K is a $\mathcal{C}_2$ subgroup that stabilizes a decomposition of three 2-spaces, then

$$d|K| = 6|\mathrm{SL}_2(q)|^3(q-1)^2 = 6q^3(q^2-1)^3(q-1)^2 < q^{15}(q^2-1)^3(q-1)^2 < d|H|.$$

If K is a $\mathcal{C}_2$ subgroup that stabilizes a decomposition of two 3-spaces, then

$$d|K| = 2|\mathrm{SL}_3(q)|^2(q-1) = 2q^3(q^3-1)(q^2-1)(q-1) < q^{15}(q^3-1)(q^2-1)(q-1) < d|H|.$$

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{PSL}_6(q)$ that satisfies $|K| \leq |H|$. Hence $|G : N_G(H)| = |\mathrm{PSL}_6(q) : H| \leq |\mathrm{PSL}_6(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H)| = |\mathrm{PSL}_6(q) : H|$.

□

Combining the above Propositions 5.3.11, 5.3.12, 5.3.13, 5.3.14, 5.3.15 and 5.3.16 we prove the case where $n \leq 6$ and $S = \mathrm{PSL}_n(q)$ for Theorem 5.2.1. Lemma 5.3.10 covers the case where $n \geq 7$ and $S = \mathrm{PSL}_n(q)$ for Theorem 5.2.1. This completes the proof for Theorem 5.2.1 where $S = \mathrm{PSL}_n(q)$.

5.4 Orders of non-trivial maximal subgroups of almost simple groups with socle $\mathrm{PSp}_n(q)$ or $\mathrm{PSU}_n(q)$

This section's aim is to provide the proof for Theorem 5.2.1 for the cases where our simple group has socle $\mathrm{PSp}_n(q)$ or $\mathrm{PSU}_n(q)$.

Lemma 5.4.1. *Let $n \geq 6$ be even, $q \geq 3$ be a prime power, and let G have socle $\mathrm{PSp}_n(q)$. Then $m(G) = m(\mathrm{PSp}_n(q)) = |\mathrm{PSp}_n(q) : H|$ where H is a P_1 subgroup of $\mathrm{PSp}_n(q)$.*

Proof. Let H be a P_1 subgroup of $\mathrm{PSp}_n(q)$. The index of this subgroup is

$$\begin{aligned}
|\mathrm{PSp}_n(q) : H| &= \frac{|\mathrm{Sp}_n(q)|}{q^{n-1}(q-1)|\mathrm{Sp}_{n-2}(q)|} && \text{by Theorem 3.3.1} \\
&= \frac{q^{(n/2)^2}\prod_{i=1}^{n/2}(q^{2i}-1)}{q^{n-1}(q-1)q^{(n/2-1)^2}\prod_{i=1}^{n/2-1}(q^{2i}-1)} && \text{by Theorem 2.5.29} \\
&= \frac{q^{(n/2)^2}(q^n-1)}{q^{n-1}(q-1)q^{(n/2-1)^2}} = \frac{q^n-1}{q-1} = m(\mathrm{PSp}_n(q)) && \text{by Theorem 2.2.7.}
\end{aligned}$$

Recall the definition of c introduced in the proof of Lemma 5.1.35. [26, Prop. 4.1.19] states that the number c of $\mathrm{PSp}_n(q)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{PSp}_n(q))$-conjugacy class of H is 1. So $\mathrm{Out}(\mathrm{PSp}_n(q))$ fixes the $\mathrm{PSp}_n(q)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{PSp}_n(q) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{PSp}_n(q))$.

□

Lemma 5.4.2. *Let $n \geq 6$, $q = 2$ and let G have socle $\mathrm{PSp}_n(2)$. Then $m(G) = m(\mathrm{PSp}_n(2)) = |\mathrm{PSp}_n(2) : H|$, where H is a $\mathrm{GO}_n^-(2)$ type subgroup of $\mathrm{PSp}_n(2)$.*

Proof. Let H be a $\mathrm{GO}_n^-(2)$ type subgroup of $\mathrm{PSp}_n(2)$. The index of this subgroup is

$$\begin{aligned}
|\mathrm{PSp}_n(2):H| &= \frac{|\mathrm{Sp}_n(2)|}{|\mathrm{GO}_n^-(2)|} && \text{by Theorem 3.3.1}\\
&= \frac{2^{(n/2)^2}\prod_{i=1}^{n/2}(2^{2i}-1)}{2\times 2^{n(n-2)/4}(2^{n/2}+1)\prod_{i=1}^{n/2-1}(2^{2i}-1)} && \text{by Theorem 2.5.29}\\
&= \frac{2^{(n/2)^2}(2^n-1)}{2^{n^2/4-n/2+1}(2^{n/2}+1)}\\
&= \frac{2^{n/2-1}(2^n-1)}{2^{n/2}+1} = 2^{n/2-1}(2^{n/2}-1)\\
&= m(\mathrm{PSp}_n(2)) && \text{by Theorem 2.2.7.}
\end{aligned}$$

[26, Prop. 4.8.6] states that the number c of $\mathrm{PSp}_n(2)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{PSp}_n(2))$-conjugacy class of H is 1. Therefore $\mathrm{Out}(\mathrm{PSp}_n(2))$ fixes the $\mathrm{PSp}_n(2)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{PSp}_n(2) = G$. So, by Lemma 2.3.18 we have that $m(G) = m(\mathrm{PSp}_n(2))$.

□

Lemma 5.4.3. *Let $n = 4$, $q \geq 4$ be a prime power, and let G have socle $\mathrm{PSp}_4(q)$.*

- *If q is odd, or q is even with $G \leq \mathrm{PSp}_4(q).\langle\ddot{\phi}\rangle$, then $m(G) = m(\mathrm{PSp}_4(q)) = |\mathrm{PSp}_4(q) : H|$ where H is P_1 type subgroup.*
- *Otherwise $m(G) = |\mathrm{PSp}_4(q) : H| = \frac{(q^4-1)(q+1)}{(q-1)}$, where H is a $\mathcal{A}_1$ type subgroup as found in [6, Table 8.14].*

Proof. First, let us note, by Lemma 5.1.32, that if q is odd then $\mathrm{Out}(\mathrm{PSp}_4(q)) = \langle\ddot{\phi},\ddot{\delta}\rangle$. If q is even then $\mathrm{Out}(\mathrm{PSp}_4(q)) = \langle\ddot{\phi},\ddot{\gamma}\rangle$.

We make reference to [6, Tables 8.12-8.14]. Let H_1 be a P_1 subgroup of $\mathrm{PSp}_4(q)$. Here following the calculations in Lemma 5.4.1

$$\frac{|\mathrm{PSp}_4(q)|}{|H|} = \frac{q^n-1}{q-1} = m(\mathrm{PSp}_4(q)).$$

Let $G = \mathrm{PSp}_4(q).T$ where T lies in the $\mathrm{Out}(\mathrm{PSp}_4(q))$-stabilizer of $\mathrm{Cl}_{\mathrm{PSp}_4(q)}(H_1)$. In the case where q is odd then the stabilizer is the whole of $\mathrm{Out}(\mathrm{PSp}_4(q))$, in the case where q is even the stabilizer is $\langle\ddot{\phi}\rangle$. By Lemma 2.3.8 we have $N_G(H_1)\mathrm{PSp}_4(q) = G$. Note that in both cases the stabilizer is a normal subgroup of $\mathrm{Out}(\mathrm{PSp}_4(q))$. Therefore by Lemma 2.3.18 $m(G) = (\mathrm{PSp}_4(q))$.

Now suppose that $G = \mathrm{PSp}_4(q).T$ where T in the stabilizer of the $\mathrm{PSp}_4(q)$-conjugacy class of H_1. Note that this only occurs if q is even. Let H_2 be a $\mathcal{A}_1$ subgroup of $\mathrm{PSU}_3(5)$ as in [6, Table 8.14]. Here $\langle\ddot{\gamma}\rangle = \mathrm{Out}(\mathrm{PSp}_4(q))$ stabilizes the $\mathrm{PSp}_4(q)$-conjugacy class of H_2, and so by Lemma 2.3.8 we know that $N_G(H_2)\mathrm{PSp}_4(q) = G$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{PSp}_4(q)$. By Lemmas 2.3.6

and 2.3.7 we have that $K \neq \mathrm{PSp}_4(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{PSp}_4(q) = G$. The possibilities for K are found in the rows of [6, Table 8.14] for which the stabilizer listed does not lie in the stabilizer of the $\mathrm{PSp}_4(q)$ conjugacy class of H_1, in other words not contained in $\langle \ddot{\phi} \rangle$. The only subgroups from [6, Table 8.14] that satisfy this conditions are those where K is $\mathcal{A}_i$ type for $1 \leq i \leq 5$. We compare the orders of K, for K not conjugate to H_2, with the order of H_2.

If K is a subgroup of $\mathcal{A}_2$ type, then

$$|K| = 8(q+1)^2 < (q-1)^2 q^4 = |H|, \qquad \text{since } q \geq 4.$$

If K is a subgroup of $\mathcal{A}_3$ type, then

$$|K| = 4(q^2+1) < (q-1)^2 q^4 = |H|, \qquad \text{since } q \geq 4.$$

If K is a subgroup of $\mathcal{A}_4$ type, then

$$|K| = |\mathrm{Sp}_4(q_0)| = q_0^4(q_0^4-1)(q_0^2-1) \leq q^2(q^2-1)(q-1) < |H| \qquad \text{since } q \geq q_0^2.$$

If K is a subgroup of $\mathcal{A}_5$ type, then

$$|K| = |\mathrm{Sz}(q)| = q^2(q^2+1)(q-1) = q^4(q-1)^2 = |H| \qquad \text{since } q \geq 4.$$

Note that the order of $\mathrm{Sz}(q)$ may be found in the ATLAS [9].

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{PSp}_4(q)$ that satisfies $|K| \leq |H_2|$. Hence $|G : N_G(H_2)| = |\mathrm{PSp}_4(q) : H_2| \leq |\mathrm{PSp}_4(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_2)| = |\mathrm{PSp}_4(q) : H_2|$. □

Lemma 5.4.4. *Let $n = 4$, $q = 3$, and let G have socle* $\mathrm{PSp}_4(3)$*. Then $m(G) = m(\mathrm{PSp}_4(3)) = |\mathrm{PSp}_4(3) : H|$, where H is a $\mathcal{C}_6$ subgroup of* $\mathrm{PSp}_4(3)$.

Proof. First let us note, by Lemma 5.1.32, that in general $\mathrm{Out}(\mathrm{PSp}_n(q)) = \langle \ddot{\phi}, \ddot{\delta} \rangle$. However since $q = 3$, we have that $\mathrm{Out}(\mathrm{PSp}_4(3)) = \langle \ddot{\delta} \rangle$.

Let H be a $\mathcal{C}_6$ subgroup of $\mathrm{PSp}_4(3)$. With reference to [6, Tables 8.12], one may show that $|\mathrm{PSp}_4(3) : H| = 27$, which by Theorem 2.2.7 equals $m(G)$. Furthermore the stabilizer of the $\mathrm{PSp}_4(3)$-conjugacy class of H is $\mathrm{Out}(\mathrm{PSp}_4(3))$. Therefore by Lemma 2.3.8 we get that $N_G(H)\mathrm{PSp}_4(3) = G$. The result now follows from Lemma 2.3.18. □

Lemma 5.4.5. *Let $n \geq 6$ be even, $q = 2$, and let G have socle* $\mathrm{PSU}_n(2)$*. Then $m(G) = m(\mathrm{PSU}_n(2)) = |\mathrm{PSU}_n(2) : H|$, where H is a $\mathrm{GU}_1(2) \perp \mathrm{GU}_{n-1}(2)$ type subgroup of* $\mathrm{PSU}_n(2)$.

Proof. Let H be a $\mathrm{GU}_1(2) \perp \mathrm{GU}_{n-1}(2)$ type subgroup of $\mathrm{PSU}_n(2)$. The index of this subgroup is

$$\begin{aligned} |\mathrm{PSU}_n(2) : H| &= \frac{|\mathrm{SU}_n(2)|}{|\mathrm{GU}_n(2)|} && \text{by Theorem 3.4.1} \\ &= \frac{2^{(n)(n-1)/2} \prod_{i=2}^{n}(2^i - (-1)^i)}{2^{(n-1)(n-2)/2} \prod_{i=1}^{n-1}(2^i - (-1)^i)} && \text{by Theorem 2.5.29} \end{aligned}$$

$$= \frac{2^{(n)(n-1)/2}(2^n - 1^n)}{2^{(n-1)(n-2)/2}(3)}$$

$$= \frac{2^{n-1}(2^n - 1)}{3} = m(\mathrm{PSU}_n(2)) \qquad \text{by Theorem 2.2.7.}$$

[26, Prop. 4.1.4] states that the number c of $\mathrm{PSU}_n(2)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{PSU}_n(2))$-conjugacy class of H is 1. In other words, $\mathrm{Out}(\mathrm{PSU}_n(2))$ fixes the $\mathrm{PSU}_n(2)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{PSU}_n(2) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{PSU}_n(2))$.

□

Lemma 5.4.6. *Let $n \geq 5$, q be a prime power, $(n, q) \neq (2m, 2)$, and let Ghave socle $\mathrm{PSU}_n(q)$. Then $m(G) = m(\mathrm{PSU}_n(q)) = |\mathrm{PSU}_n(q) : H|$, where H is a P_1 type subgroup of $\mathrm{PSU}_n(q)$.*

Proof. Let H be a P_1 subgroup of $\mathrm{PSU}_n(q)$. The index of this subgroup is

$$|\mathrm{PSU}_n(q) : H| = \frac{|\mathrm{SU}_n(q)|}{q^{2n-3}|\mathrm{SU}_{n-2}(q)|(q^2-1)} \qquad \text{by Theorem 3.4.1}$$

$$= \frac{q^{(n)(n-1)/2}\prod_{i=2}^{n}(q^i - (-1)^i)}{q^{2n-3}(q^2-1)q^{(n-2)n-3)/2}\prod_{i=2}^{n-2}(q^i - (-1)^i)} \qquad \text{by Theorem 2.5.29}$$

$$= \frac{q^{n(n-1)/2}(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)}{q^{2n-3}(q^2-1)q^{(n-2)n-3)/2}}$$

$$= \frac{(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)}{(q^2-1)} = m(\mathrm{PSU}_n(q)) \qquad \text{by Theorem 2.2.7.}$$

[26, Prop. 4.1.19] states that the number c of $\mathrm{PSU}_n(q)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{PSU}_n(q))$-conjugacy class of H is 1. Thus $\mathrm{Out}(\mathrm{PSU}_n(q))$ fixes the $\mathrm{PSU}_n(q)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{PSU}_n(q) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{PSU}_n(q))$. □

Lemma 5.4.7. *Let $n = 4$, q be a prime power, $(n, q) \neq (2m, 2)$, and let G have socle $\mathrm{PSU}_4(q)$. Then $m(G) = m(\mathrm{PSU}_4(q)) = |\mathrm{PSU}_4(q) : H|$, where H is a P_2 type subgroup of $\mathrm{PSU}_4(q)$.*

Proof. First let us note, by Lemma 5.1.32, that $\mathrm{Out}(\mathrm{PSU}_4(q)) = \langle \ddot{\phi}, \ddot{\delta} \rangle$.

Let H be a P_2 subgroup of $\mathrm{PSU}_4(q)$. The index of this subgroup is

$$|\mathrm{PSU}_4(q) : H| = \frac{|\mathrm{SU}_4(q)|}{q^4|\mathrm{SL}_2(q^2)|(q-1)} \qquad \text{by [6, Table 8.10]}$$

$$= \frac{q^6(q^4-1)(q^3+1)(q^2-1)}{q^4q^2(q^4-1)(q-1)} \qquad \text{by Theorem 2.5.29}$$

$$= \frac{(q^3+1)(q^2-1)}{(q-1)}$$

$$= (q^3+1)(q+1) = q^4 + q^3 + q + 1 = m(\mathrm{PSU}_4(q)) \qquad \text{by Theorem 2.2.7.}$$

By [6, Table 8.10] we know that the stabilizer of the $\mathrm{PSU}_4(q)$-conjugacy class of H is $\langle \ddot{\delta}, \ddot{\phi} \rangle = \mathrm{Out}(\mathrm{PSU}_n(q))$. Consequently all outer automorphisms fix the $\mathrm{PSU}_4(q)$-conjugacy class of and by Lemma 2.3.8 we have that $N_G(H)\mathrm{PSU}_4(q) = G$. Therefore by Lemma 2.3.18 we have our result. □

Lemma 5.4.8. *Let $n = 3$, $q \neq 2, 5$ be a prime power, and let G have socle* $\mathrm{PSU}_3(q)$. *Then* $m(G) = m(\mathrm{PSU}_3(q)) = |\mathrm{PSU}_3(q) : H|$, *where H is a P_1 type subgroup of* $\mathrm{PSU}_n(q)$.

Proof. First let us note, by Lemma 5.1.32, that $\mathrm{Out}(\mathrm{PSU}_n(q)) = \langle \delta, \ddot{\phi} \rangle$.

Let H be a P_1 subgroup of $\mathrm{PSU}_3(q)$. The index of this subgroup is the number of singular 1-spaces, i.e

$$|\mathrm{PSU}_3(q) : H| = (q^3 + 1) = m(\mathrm{PSU}_3(q)) \qquad \text{by Theorem 2.2.7.}$$

By [6, Table 8.5] we know that the stabilizer of the $\mathrm{PSU}_n(q)$-conjugacy class of H is $\langle \ddot{\delta}, \ddot{\phi} \rangle$. Consequently all outer automorphisms fix the $\mathrm{PSU}_3(q)$-conjugacy class of H and so by Lemma 2.3.8 we have that $N_G(H)\mathrm{PSU}_3(q) = G$. Therefore by Lemma 2.3.18 we have our result. □

Lemma 5.4.9. *Let $n = 3$, $q = 5$, and let G have socle* $\mathrm{PSU}_3(5)$.

- *If $G = \mathrm{PSU}_3(5).T$ where T lies in some* $\mathrm{Out}(\mathrm{PSU}_3(5))$*-conjugate of $\langle \ddot{\phi} \rangle$. Then $m(G) = m(\mathrm{PSU}_3(5)) = |\mathrm{PSU}_3(5) : H|$, where H is the image of a subgroup of* $\mathrm{SU}_3(5)$ *with shape* $3^{\cdot}\mathrm{A}_7$.
- *Otherwise $m(G) = |\mathrm{PSU}_3(5) : H| = 126$ where H is a P_1 type subgroup of* $\mathrm{PSU}_3(5)$.

Proof. First let us note, by Lemma 5.1.32, that $\mathrm{Out}(\mathrm{PSU}_3(5)) = \langle \delta, \ddot{\phi} \rangle$. We also note that $\ddot{\gamma} = \ddot{\phi}$ here.

We appeal to [6, Table 8.5 & 8.6]. Let $H_1 \leq \mathrm{PSU}_3(5)$ be the image of a subgroup of $\mathrm{SU}_3(5)$ isomorphic to $3^{\cdot}\mathrm{A}_7$. One may show that $|\mathrm{PSU}_3(5) : H_1| = m(\mathrm{PSU}_3(5))$, where $m(\mathrm{PSU}_3(5))$ can be found in Table 2.2.7.

Let $G = \mathrm{PSU}_3(5).T$ where T lies in a conjugate of the $\mathrm{Out}(\mathrm{PSU}_3(5))$-stabilizer of $\mathrm{Cl}_{\mathrm{PSU}_3(5)}(H_1)$. We note that here the stabilizer is $\langle \ddot{\gamma} \rangle = \langle \ddot{\phi} \rangle$. Therefore, T also fixes the $\mathrm{PSU}_3(5)$-conjugacy class of a $\mathrm{Aut}(\mathrm{PSU}_3(5))$-conjugate of H_1, let us call this conjugate H_T. By Lemma 2.3.8 we have $N_G(H_T)\mathrm{PSU}_3(5) = G$. Therefore by Lemma 2.3.18 $m(G) = (\mathrm{PSU}_3(5))$.

Now suppose that $G = \mathrm{PSU}_3(5).T$ where T is not contained in any $\mathrm{Out}(\mathrm{PSU}_3(5))$-conjugate of the stabilizer of the $\mathrm{PSU}_3(5)$-conjugacy class of H_1. Let H_2 be a P_1 subgroup of $\mathrm{PSU}_3(5)$. [6, Table 8.50] shows that the whole of $\mathrm{Out}(\mathrm{PSU}_3(5))$ stabilizes the $\mathrm{PSU}_3(5)$-conjugacy class of H_2, and so by Lemma 2.3.8 we know that $N_G(H_2)\mathrm{PSU}_3(5) = G$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{PSU}_3(5)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{PSU}_3(5)$, and $N_G(K) = M$, and $N_G(K)\mathrm{PSU}_3(5) = G$. The possibilities for K are found in the rows of [6, Table 8.5 & 8.6] for which the stabilizer listed does not lie in any $\mathrm{Out}(\mathrm{PSU}_3(5))$-conjugate of the stabilizer of the $\mathrm{PSU}_3(5)$ conjugacy class of H_1. One may show that $|K| \leq |H_2|$ for these cases.

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{PSU}_3(5)$ that satisfies $|K| \leq |H_2|$. Hence $|G : N_G(H_2)| = |\mathrm{PSU}_3(5) : H_2| \leq |\mathrm{PSU}_3(5) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_2)| = |\mathrm{PSU}_3(5) : H_2|$. □

We conclude our proof of Theorem 5.2.1 for $S = \mathrm{PSp}_n(q)$ or $\mathrm{PSU}_n(q)$. Lemmas 5.4.1-5.4.4 show that if $\mathrm{PSp}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSp}_n(q))$ then $m(G) = m(\mathrm{PSp}_n(q))$ except when (as shown in Lemma 5.4.3) $n = 4$, $q \geq 4$ is an even prime power and $G \not\leq \mathrm{PSp}_n(q).\langle\ddot{\phi}\rangle$ where $m(G) = \frac{(q^4-1)(q+1)}{(q-1)}$.

Lemmas 5.4.1-5.4.4 show that if $\mathrm{PSp}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSp}_n(q))$ then $m(G) = m(\mathrm{PSp}_n(q))$ except when $n = 3$, $q = 5$ is an even prime power and $G \not\leq \mathrm{PSp}_n(q).\langle\ddot{\phi}\rangle$. In this case $m(G) = 126$, as shown in Lemma 5.4.9.

Meanwhile Lemmas 5.4.5-5.4.9 show that if $\mathrm{PSU}_n(q) \leq G \leq \mathrm{Aut}(\mathrm{PSU}_n(q))$ then $m(G) = m(\mathrm{PSU}_n(q))$ except in the case where $n = 3$, and $q = 5$ and $G = \mathrm{PSU}_n(q).T$ where T is not contained in any $\mathrm{Out}(\mathrm{PSU}_n(q))$-conjugates of $\langle\ddot{\phi}\rangle$.

5.5 Orders of non-trivial maximal subgroups of almost simple groups with socle $\mathrm{P}\Omega_n^{\epsilon}(q)$

This section aims to provide the proof for Theorem 5.2.1 for the cases where our simple group has socle $\mathrm{P}\Omega_n^{\epsilon}(q)$. We attack these cases by considering different n and g in addition to the type of orthogonal group. Here once more the small cases in particular make heavy reference to [6, Chapter 8]. In particular, due to $\mathrm{P}\Omega_n^{+}(q)$ being a more involved case, we present information from [6] about its geometric subgroups.

Theorem 5.5.1. *Let $n \geq 6$ be even, let q be a prime power and let $G = \Omega_n^{+}(q)$. Furthermore, let M be a non-trivial maximal subgroup of G. Then either M is a geometric subgroup, and is one of the types given in Table 5.11 or M lies in $\mathcal{S}$. Furthermore if M is found in Table 5.11 then the structure of M can be found in Tables 5.12 and 5.13.*

Proof. The theorem is a sub-case of [6, Theorem 2.2.19]. We obtain the data for Tables 5.11, 5.12 and 5.13 from [6, Tables 2.2-2.11], which present some conditions for the existence of and the shapes of the geometric subgroups of G. □

Please note, that for clarity, that not all of the subgroups in Table 5.11 are necessarily maximal. For a complete list of necessary and sufficient conditions, we point the reader to [6, Section 2.2] and [26, MAIN THEOREM].

We also present some information as to the structure of $\mathcal{C}_1$ subgroups of $\Omega_n^{\epsilon}(q)$, since we will often make reference to them.

Lemma 5.5.2 ([26, Propositions 4.1.6, 4.1.7 & 4.1.20], [6, Theorem 2.2.19])**.** *If $G = \Omega_n^{\epsilon}(q)$ then:*

- *a P_k type subgroup has the shape $[q^d] : (\mathrm{GL}_k(q) \times \Omega_{n-2k}^{\epsilon}(q))$ if q even.*
- *a P_k type subgroup has the shape $[q^d] : \frac{1}{2}\mathrm{GL}_k(q)$ if $k = \lfloor n/2 \rfloor$, q odd.*
- *a P_k type subgroup has the shape $[q^d] : (\frac{1}{2}\mathrm{GL}_k(q) \times \Omega_{n-2k}^{\epsilon}(q)).2$ otherwise.*
- *a $\mathrm{GO}_k^{\epsilon_1}(q) \perp \mathrm{GO}_k^{\epsilon_2}(q)$ type subgroup has the shape. $(\Omega_k^{\epsilon_1}(q) \times \Omega_k^{\epsilon_2}(q)).2$ if $k = 1$ or q even.*
- *a $\mathrm{GO}_k^{\epsilon_1}(q) \perp \mathrm{GO}_k^{\epsilon_2}(q)$ type subgroup has the shape $(\Omega_k^{\epsilon_1}(q) \times \Omega_k^{\epsilon_2}(q)).2^2$ otherwise.*

If $G = \Omega_n^{\pm}(q)$ then:

- *a* $\mathrm{Sp}_{n-2}(q)$ *type subgroup has the shape* $\mathrm{Sp}_{n-2}(q)$ *if* q *even.*

Where $d = k(n - \frac{1+3k}{2})$. ϵ_1 *and* ϵ_2 *are defined in [6, Table 2.2].*

Lemma 5.5.3. *Let* $n \geq 7$ *odd,* $q \geq 5$ *be an odd prime power, and let* G *have socle* $\mathrm{P}\Omega_n(q)$. *Then* $m(G) = m(\mathrm{P}\Omega_n(q)) = |\mathrm{P}\Omega_n(q) : H|$, *where* H *is a* P_1 *type subgroup of* $\mathrm{P}\Omega_n(q)$.

Proof. Let H be a P_1 subgroup of $\mathrm{P}\Omega_n(q)$. The index of this subgroup is the number of singular 1-spaces, in other words

$$|\mathrm{P}\Omega_n(q) : H| = \frac{q^{n-1}-1}{q-1} = m(\mathrm{P}\Omega_n(q)) \qquad \text{by Theorem 2.2.7.}$$

Recall the definition of c introduced in the proof of Lemma 5.1.35. [26, Prop. 4.1.20] states that the number c of $\mathrm{P}\Omega_n(q)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{P}\Omega_n(q))$-conjugacy class of H is 1. $\mathrm{Out}(\mathrm{P}\Omega_n(q))$ fixes the $\mathrm{P}\Omega_n(q)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{P}\Omega_n(q) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n(q))$.

□

Lemma 5.5.4. *Let* $n \geq 7$ *odd,* $q = 3$, *and let* G *have socle* $\mathrm{P}\Omega_n(3)$. *Then* $m(G) = m(\mathrm{P}\Omega_n(3)) = |\mathrm{P}\Omega_n(q) : H|$, *where* H *is a* $\mathrm{GO}_{n-1}^-(3) \perp \mathrm{GO}_1(3)$ *type subgroup of* $\mathrm{P}\Omega_n(3)$.

Proof. Let H be a $\mathrm{GO}_{n-1}^-(q) \perp \mathrm{GO}_1(q)$ type subgroup of $\mathrm{P}\Omega_n(3)$. The index of this subgroup is

$$\begin{aligned}
|\mathrm{P}\Omega_n(3) : H| &= \frac{|\Omega_n(3)|}{2|\Omega_{n-1}^-(3)||\Omega_1(3)|} = \frac{|\Omega_n(3)|}{2|\Omega_{n-1}^-(3)|} && \text{by [6, Table 8.10]} \\
&= \frac{3^{(n-1)^2/4}\prod_{i=1}^{(n-1)/2}(3^{2i}-1)}{2 \times 3^{(n-1)(n-3)/4}(3^{(n-1)/2}+1)\prod_{i=1}^{(n-3)/2}(3^{2i}-1)} && \text{by Theorem 2.5.29} \\
&= \frac{3^{(n-1)^2/4}(3^{n-1}-1)}{2 \times 3^{(n-1)(n-3)/4}(3^{(n-1)/2}+1)} \\
&= \frac{3^{(n-1)/2}(3^{(n-1)/2}-1)}{2} = m(\mathrm{P}\Omega_n(3)) && \text{by Theorem 2.2.7.}
\end{aligned}$$

[26, Prop. 4.1.6] states that the number c of $\mathrm{P}\Omega_n(3)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{P}\Omega_n(3))$-conjugacy class of H is 1. Therefore $\mathrm{Out}(\mathrm{P}\Omega_n(3))$ fixes the $\mathrm{P}\Omega_n(3)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{P}\Omega_n(3) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n(3))$. □

Lemma 5.5.5. *Let* $n \geq 10$ *even,* $q \geq 4$ *be a prime power, and let* G *have socle* $\mathrm{P}\Omega_n^+(q)$. *Then* $m(G) = m(\mathrm{P}\Omega_n^+(q)) = |\mathrm{P}\Omega_n^+(q) : H|$, *where* H *is a* P_1 *type subgroup of* $\mathrm{P}\Omega_n^+(q)$.

Proof. Let H be a P_1 subgroup of $\mathrm{P}\Omega_n^+(q)$. The index of this subgroup is

$$|\mathrm{P}\Omega_n^+(q) : H| = \frac{|\Omega_n^+(q)|}{q^{n-2}(q-1)|\Omega_{n-2}^+(q)|} \qquad \text{by Theorem 5.5.1}$$

Table 5.11: Conditions for geometric subgroups of $\Omega_n^+(q)$

Class	Subgroup Type	Conditions
$\mathcal{C}_1$	P_m	$m \leq n/2$
$\mathcal{C}_1$	$\mathrm{GO}_m^\epsilon(q) \perp \mathrm{GO}_{n-m}^\epsilon(q)$	$1 \leq m < n/2$; q even $\implies m$ even
$\mathcal{C}_1$	$\mathrm{Sp}_{n-2}(q)$	q even
$\mathcal{C}_2$	$\mathrm{GO}_1(p) \wr \mathrm{S}_n$	$q = p > 2$; $n \not\equiv 2 \mod 4$ or $q \not\equiv 3 \mod 4$
$\mathcal{C}_2$	$\mathrm{GO}_m^{\epsilon_1}(q) \wr \mathrm{S}_t$	$mt = n, m > 1$; m even $\implies \epsilon_1^t = 1$; m odd with t even $\implies (-1)^{(q-1)n/4} = 1$
$\mathcal{C}_2$	$\mathrm{GL}_{n/2}(q).2$	
$\mathcal{C}_2$	$\mathrm{GO}_{n/2}^\circ(q)^2$	$(-1)^{(q+1)/2} = 1$, q odd
$\mathcal{C}_3$	$\mathrm{GO}_m^+(q^s)$	s prime, $m \geq 3, m = n/s$
$\mathcal{C}_3$	$\mathrm{GU}_{n/2}(q)$	$n \equiv 0 \mod 4$, $s = 2$
$\mathcal{C}_3$	$\mathrm{GO}_{n/2}^\circ(q^2)$	$qn/2$ odd, $s = 2$
$\mathcal{C}_4$	$\mathrm{Sp}_{n_1}(q) \otimes \mathrm{Sp}_{n_2}(q)$	$n_1 < \sqrt{n}, n = n_1 n_2$
$\mathcal{C}_4$	$\mathrm{GO}_{n_1}^\circ(q) \otimes \mathrm{GO}_{n_2}^+(q)$	q odd, $n_1 \geq 3$, $n_2 \geq 4, n = n_1 n_2$
$\mathcal{C}_4$	$\mathrm{GO}_{n_1}^\epsilon(q) \otimes \mathrm{GO}_{n_2}^\epsilon(q)$	$n_1, n_2 \geq 4$, n_i even, $n = n_1 n_2$, q odd, $n_1 < \sqrt{n}$
$\mathcal{C}_5$	$\mathrm{GO}_n^+(q_0)$	$q = q_0^r$, r prime
$\mathcal{C}_5$	$\mathrm{GO}_n^-(q_0)$	$q = q_0^2$
$\mathcal{C}_6$	$2_+^{1+2m}.\Omega_{2m}^+(2)$	$n = r^m, r$ prime, $q \equiv \pm 1 \mod 8$ or $q \equiv \pm 3 \mod 8$
$\mathcal{C}_7$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$	$n = m^t$, qt even, $(m, q) \neq (2, 2), (2, 3)$
$\mathcal{C}_7$	$\mathrm{GO}_m^\epsilon(q) \wr \mathrm{S}_t$	$n = m^t$, q odd, $\epsilon = + \implies m \geq 6$, $\epsilon = - \implies m \geq 4$

Table 5.12: Geometric subgroups of $\Omega_n^+(q)$

Class	Subgroup Type	Structure
$\mathcal{C}_1$	P_m, q even	$[q^{m(n-\frac{1+3m}{2})}]:(\mathrm{GL}_m(q)\times\Omega^+_{n-2m}(q))$
$\mathcal{C}_1$	$P_{n/2}$, q odd	$[q^{n/2(n-\frac{1+3(n/2)}{2})}]:\frac{1}{2}\mathrm{GL}_{n/2}(q)$
$\mathcal{C}_1$	P_m, otherwise	$[q^{m(n-\frac{1+3m}{2})}](\frac{1}{2}\mathrm{GL}_m(q)\times\Omega^+_{n-2m}(q)).2$
$\mathcal{C}_1$	$\mathrm{GO}^\epsilon_m(q)\perp\mathrm{GO}^\epsilon_{n-m}(q)$, $m=1$ or q is even	$(\Omega^\epsilon_m(q)\times\Omega^\epsilon_{n-m}(q)).2$
$\mathcal{C}_1$	$\mathrm{GO}^\epsilon_m(q)\perp\mathrm{GO}^\epsilon_{n-m}(q)$, otherwise	$(\Omega^\epsilon_m(q)\times\Omega^\epsilon_{n-m}(q)).2^2$
$\mathcal{C}_1$	$\mathrm{Sp}_{n-2}(q)$	$\mathrm{Sp}_{n-2}(q)$
$\mathcal{C}_2$	$\mathrm{GO}_1(p)\wr\mathrm{S}_n$, $q\equiv\pm3 \mod 8$	$2^{n-1}.\mathrm{A}_n$
$\mathcal{C}_2$	$\mathrm{GO}_1(p)\wr\mathrm{S}_n$, $q\equiv\pm1 \mod 8$	$2^{n-1}.\mathrm{S}_n$
$\mathcal{C}_2$	$\mathrm{GO}^{\epsilon_1}_m(q)\wr\mathrm{S}_t$	$\Omega^{\epsilon_1}_m(q)^t.2^{(2,q-1).(t-1)}.\mathrm{S}_t$
$\mathcal{C}_2$	$\mathrm{GL}_{n/2}(q).2$	$\mathrm{SL}_{n/2}(q).\frac{q-1}{(q-1,2)}.(n/2,2)$
$\mathcal{C}_2$	$\mathrm{GO}^\circ_{n/2}(q)^2$	$\mathrm{SO}^\circ_{n/2}(q)^2$
$\mathcal{C}_3$	$\mathrm{GO}^+_m(q^s)$	$\Omega^+_m(q^s)[(s,2)s]$
$\mathcal{C}_3$	$\mathrm{GO}^\circ_{n/2}(q^2)$	$(q-1,4)/2\times\Omega^\circ_{n/2}(q^2).2$
$\mathcal{C}_3$	$\mathrm{GU}_{n/2}(q)$	$((q+1)\circ\mathrm{SU}_{n/2}(q)).[(q,2)(q+1,\frac{n}{2})]$

Table 5.13: Geometric subgroups of $\Omega_n^+(q)$

Class	Subgroup Type	Structure
$\mathcal{C}_4$	$\mathrm{Sp}_{n_1}(q) \otimes \mathrm{Sp}_{n_2}(q)$	$(\mathrm{Sp}_{n_1}(q) \circ \mathrm{Sp}_{n_2}(q)).(2, q-1, n/4)$
$\mathcal{C}_4$	$\mathrm{GO}^\circ_{n_1}(q) \otimes \mathrm{GO}^+_{n_2}(q)$	$\mathrm{SO}^\circ_{n_1}(q) \times \Omega^+_{n_2}(q)$
$\mathcal{C}_4$	$\mathrm{GO}^\epsilon_{n_1}(q) \otimes \mathrm{GO}^\epsilon_{n_2}(q)$	$(\mathrm{SO}^\epsilon_{n_1}(q) \circ \mathrm{SO}^\epsilon_{n_2}(q)).[c]$
$\mathcal{C}_5$	$\mathrm{GO}^+_n(q_0)$, r odd or q even	$\Omega^+_n(q_0)$
$\mathcal{C}_5$	$\mathrm{GO}^+_n(q_0)$, $r = 2$ and q odd	$\mathrm{SO}^+_n(q_0).b$
$\mathcal{C}_5$	$\mathrm{GO}^-_n(q_0)$, q even	$\Omega^-_n(q_0)$
$\mathcal{C}_5$	$\mathrm{GO}^-_n(q_0)$, q odd	$\mathrm{SO}^-_n(q_0).b$
$\mathcal{C}_6$	$2_+^{1+2m}.\Omega^+_{2m}(2)$, $q \equiv \pm 1 \mod 8$	$2_+^{1+2m}.\mathrm{SO}^+_{2m}(2)$
$\mathcal{C}_6$	$2_+^{1+2m}.\Omega^+_{2m}(2)$, $q \equiv \pm 3 \mod 8$	$2_+^{1+2m}.\Omega^+_{2m}(2)$
$\mathcal{C}_7$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$, $t = 2$, $m \equiv 2 \mod 4$	$(q-1, 2).\mathrm{PSp}_m(q)^2$
$\mathcal{C}_7$	$\mathrm{Sp}_m(q) \wr \mathrm{S}_t$, otherwise	$(q-1, 2).\mathrm{PSp}_m(q)^t.(q-1, 2)^{t-1}.\mathrm{S}_t$
$\mathcal{C}_7$	$\mathrm{GO}^\epsilon_m(q) \wr \mathrm{S}_t$, $t = 2$, $m \equiv 2 \mod 4$	$2.\mathrm{PSO}^\epsilon_m(q)^2.[4]$
$\mathcal{C}_7$	$\mathrm{GO}^\epsilon_m(q) \wr \mathrm{S}_t$, $t = 2$, $m \equiv 0 \mod 4$	$2.\mathrm{PSO}^-_m(q)^2.[8]$
$\mathcal{C}_7$	$\mathrm{GO}^\epsilon_m(q) \wr \mathrm{S}_t$, $t = 3$, $m \equiv 2 \mod 4$, V_1 non-square discriminant	$2.\mathrm{PSO}^\epsilon_m(q)^3.[2^5].3$
$\mathcal{C}_7$	$\mathrm{GO}^\epsilon_m(q) \wr \mathrm{S}_t$, otherwise	$2.\mathrm{PSO}^\epsilon_m(q)^t.[2^{2t-1}].\mathrm{S}_t$

Note: $c = 4, 8$, the precise conditions for the value of c may be found in [6, 2.2.4]

$b = 1, 2$, the precise conditions for the value of b may be found in [6, 2.2.5]

Information on V_1 may be found in [6, 2.2.7]

$$\begin{aligned}
&= \frac{q^{n(n-2)/4}(q^{n/2}-1)\prod_{i=1}^{n/2-1}(q^{2i}-1)}{q^{n-2}(q-1)q^{(n-2)(n-4)/4}(q^{(n-2)/2}-1)\prod_{i=1}^{n/2-2}(q^{2i}-1)} && \text{by Theorem 2.5.29}\\
&= \frac{q^{n(n-2)/4}(q^{n/2}-1)(q^{n-2}-1)}{q^{n-2}(q-1)q^{(n-2)(n-4)/4}(q^{(n-2)/2}-1)}\\
&= \frac{(q^{n/2}-1)(q^{n-2}-1)}{(q-1)(q^{(n-2)/2}-1)}\\
&= \frac{(q^{n/2}-1)(q^{(n-2)/2}+1)}{q-1} = m(\mathrm{P}\Omega_n^+(q)) && \text{by Theorem 2.2.7.}
\end{aligned}$$

[26, Prop. 4.1.20] states that the number c of $\mathrm{P}\Omega_n^+(q)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{P}\Omega_n^+(q))$-conjugacy class of H is 1. Therefore, $\mathrm{Out}(\mathrm{P}\Omega_n^+(q))$ must fix the $\mathrm{P}\Omega_n^+(q)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{P}\Omega_n^+(q) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n^+(q))$. □

Lemma 5.5.6. *Let $q = 3$, let n be even, and let $M \leq \Omega_n^+(3)$ be a $\mathcal{C}_1$ subgroup of type P_1. Furthermore let $H \leq \Omega_n^+(3)$ be a non-trivial maximal subgroup from the Aschbacher classes $\mathcal{C}_i$ with $2 \leq i \leq 8$. If $n \geq 10$ then $|M| \geq |H|$. Furthermore*

- *If $n \geq 6$ then M has larger order than every maximal subgroup from the Aschbacher class $\mathcal{C}_6$ and $\mathcal{C}_3$.*
- *If $n \geq 8$ then M has larger order than every maximal subgroup from the Aschbacher class $\mathcal{C}_4$.*
- *If $n \geq 10$ then M has larger order than every maximal subgroup from the Aschbacher classes $\mathcal{C}_2$ and $\mathcal{C}_7$.*

Proof. Let M be a P_1 type subgroup. We begin by deriving a lower bound for $|M|$.

$$\begin{aligned}
|M| &= 3^{n-2}(3-1)|\Omega_{n-2}^+(3)| && \text{by Theorem 5.5.1}\\
&= 3^{n-2}|\mathrm{SO}_{n-2}^+(3)| && \text{by Theorem 2.5.29}\\
&> 3^{n-2}3^{(n-2)^2/2-(n-2)/2-1} = 3^{(n^2-3n)/2} && \text{by Lemma 2.6.4.}
\end{aligned}$$

Let us now compare M with the different possibilities for H.

If H belongs to $\mathcal{C}_2$ according to Theorem 5.5.1 we have four cases to consider. If H is a $\mathrm{GO}_1(p)\wr \mathrm{S}_n$ type subgroup

$$\begin{aligned}
|H| &\leq 2^{n-1}n! && \text{by Theorem 5.5.1}\\
&\leq 2^{n-1}2^{n\log_2(n)} && \text{by Lemma 2.8.2}\\
&\leq 2^{n-1}2^{n^{1.55}} = 3^{\log_3(2)(n^{1.55}+n-1)} && \text{by Lemma 2.8.4}\\
&< 3^{0.65(n^{1.55}+n-1)} < 3^{0.65(\frac{n^2}{9^{0.45}}+n-1)} && \text{assuming } n \geq 9\\
&< 3^{(n^2-3n)/2} < |M| && \text{assuming } n \geq 9.
\end{aligned}$$

If H is a $\mathrm{GO}^{\epsilon_1}_m(q) \wr \mathrm{S}_t$ type subgroup. Then we have the conditions $m = n/t$ and $1 < t \leq n/2$. If $n \geq 12$ then

$$\begin{aligned}
|H| &\leq |\Omega^{\epsilon}_m(3)|^t 2^{2(t-1)} t! && \text{by Theorem 5.5.1}\\
&\leq 3^{t(m^2-m)/2} 2^{2(t-1)} t! && \text{by Lemma 2.6.4 and Theorem 2.5.29}\\
&\leq 3^{t(m^2-m)/2} 2^{2t-2} 2^{t\log_2(t)} && \text{by Lemma 2.8.2}\\
&\leq 3^{t(m^2-m)/2} 2^{2t-2} 2^{t^{1.55}} && \text{by Lemma 2.8.4}\\
&= 3^{(nm-n)/2} 2^{t^{1.55}+2t-2} && \text{as } tm = n\\
&\leq 3^{(n^2/2-n)/2} 2^{(n/2)^{1.55}+n-2} && \text{as } m, t \leq n/2\\
&= 3^{(n^2/2-n)/2+\log_3(2)((n/2)^{1.55}+n-2)}\\
&< 3^{(n^2/2-n)/2+0.64((n/2)^{1.55}+n-2)}\\
&< 3^{n^2/4+0.22n^{1.55}+0.14n-1.28} \leq 3^{n^2/4+\frac{0.22n^2}{12^{0.45}}+0.14n-1.28}\\
&< 3^{0.33n^2+0.14n-1.28} < 3^{(n^2-3n)/2} < |M| && \text{as } n \geq 10.
\end{aligned}$$

If H is a $\mathrm{GL}_{n/2}(q).2$ type subgroup then

$$\begin{aligned}
|H| &\leq 2|\mathrm{SL}_{n/2}(3)| && \text{by Theorem 5.5.1}\\
&\leq 2 \times 3^{n^2/4-1} && \text{by Lemma 2.6.4}\\
&< 3^{n^2/4} < 3^{(n^2-3n)/2} < |M| && \text{since } n \geq 6.
\end{aligned}$$

If H is a $\mathrm{GL}^{\circ}_{n/2}(q)^2$ type subgroup then

$$\begin{aligned}
|H| &= |\mathrm{SO}^{\circ}_{n/2}(3)|^2 && \text{by Theorem 5.5.1}\\
&\leq 3^{n^2/4-n/2} && \text{by Lemma 2.6.4.}\\
&< 3^{n^2/4} < |M| && \text{since } n \geq 6.
\end{aligned}$$

If H belongs to $\mathcal{C}_3$, according to by Theorem 5.5.1 we have three cases to consider. Firstly if H is a $\mathrm{GO}^{+}_m(q^s)$ type subgroup. We have that, that $m = n/s$, s prime and $m \geq 3$. Furthermore,

$$\begin{aligned}
|H| &\leq 2s|\Omega^{+}_m(3^s)| && \text{by Theorem 5.5.1}\\
&\leq 2s3^{s(m^2-m)/2} && \text{by Lemma 2.6.4}\\
&\leq 2s3^{(nm-n)/2} && \text{as } m = n/s\\
&\leq 2n3^{(n^2/2-n)/2-1} < 3^{\log_2(n)+n^2/4-n/2} && \text{as } n/s \geq 3 \text{ and } m \leq n/2\\
&\leq 3^{(n)^{0.55}+n^2/4-n/2} < 3^{n^2/4-n/2+n^{0.55}} && \text{by Lemma 2.8.4}\\
&\leq 3^{n^2/4-n/2+\frac{n}{6^{0.45}}} \leq 3^{n^2/4} && \text{since } n \geq 0\\
&\leq 3^{(n^2-3n)/2} < |M| && \text{since } n \geq 6.
\end{aligned}$$

If H is a $\mathrm{GO}^{\circ}_{n/2}(q^2)$ type subgroup then

$$|H| < |\Omega^{\circ}_{n/2}(3^2)| \qquad \text{by Theorem 5.5.1}$$

$$
\begin{aligned}
&\leq 3^{(n^2/4-n/2)/2} && \text{by Lemma 2.6.4}\\
&< 3^{n^2/8} \leq 3^{(n^2-3n)/2} < |M| && \text{since } n \geq 6.
\end{aligned}
$$

If H is a $\mathrm{GU}^{\circ}_{n/2}(q)$, type subgroup and $n \geq 8$ then

$$
\begin{aligned}
|H| &\leq 16|\mathrm{SU}_{n/2}(3)| && \text{by Theorem 5.5.1}\\
&\leq 16 \times 3^{(n^2/4-1)} && \text{by Lemma 2.6.4}\\
&< 3^{n^2/4+2} \leq 3^{(n^2-3n)/2} < |M| && \text{assuming } n \geq 8.
\end{aligned}
$$

If H is a $\mathrm{GU}^{\circ}_{n/2}(q)$ type subgroup and $n = 6$, we can show that $|H| \leq |M|$ by substitution.

If K belongs to $\mathcal{C}_4$, where $n_1 n_2 = n$ and $n_1, n_2 \neq n$ as in Table 5.11, then we note $|\mathrm{Sp}_n(q)| \geq \mathrm{SO}^{\epsilon}_n(q)$. Therefore in all cases we have that

$$
\begin{aligned}
|H| &\leq 4|\mathrm{Sp}_{n_1}(3)||\mathrm{Sp}_{n_2}(3)| && \text{by Theorem 5.5.1}\\
&\leq 4 \times 3^{(n_1^2+n_1)/2} 3^{(n_2^2+n_2)/2} && \text{by Lemma 2.6.4}\\
&\leq 4 \times 3^{(n^2/4+n/2)/2} 3^{(n^2/4+n/2)/2} && \text{since } n_1, n_2 \leq n/2\\
&\leq 4 \times 3^{n^2/4+n/2} < 3^{n^2/4+n/2+2} \leq 3^{(n^2-3n)/2} < |M| && \text{assuming } n \geq 8.
\end{aligned}
$$

We note that H cannot belong to $\mathcal{C}_5$ in this case as $q = 3$

If H is belongs to $\mathcal{C}_6$, then $n = r^m$ where r is prime.

$$
\begin{aligned}
|H| &\leq 2^{1+2m}|\Omega^{+}_{2m}(2)| && \text{by Theorem 5.5.1}\\
&\leq 2^{1+2m} 2^{2m^2-m} && \text{by Lemma 2.6.4}\\
&\leq 2^{1+2\log_2(n)} 2^{2\log_2(n)^2-\log_2(n)} && \text{as } m \leq \log_2(n)\\
&\leq 2^{1+2n^{0.55}} 2^{2n^{1.1}-n^{0.55}} && \text{by Lemma 2.8.4}\\
&= 2^{2n^{1.1}+n^{0.55}+1} = 3^{\log_3(2)(2n^{1.1}+n^{0.55}+1)}\\
&\leq 3^{\log_3(2)(\frac{2n^2}{6^{0.9}}+\frac{n}{6^{0.45}}+1)} \leq 3^{(n^2-3n)/2} \leq |M| && \text{since } n \geq 6.
\end{aligned}
$$

If H belongs to $\mathcal{C}_7$ and $n = m^t$ then we have two possibilities. If H is a $\mathrm{Sp}_m(q) \wr S_t$ type subgroup then

$$
\begin{aligned}
|H| &\leq 2^t |\mathrm{PSp}_m(3)|^t t! && \text{by Theorem 5.5.1}\\
&\leq 2^t 3^{t(m^2+m)/2} t! < 2^{2t} 3^{t(m^2+m)/2} t! && \text{by Lemma 2.6.4.}
\end{aligned}
$$

If H is a $\mathrm{GO}^{\epsilon}_m(q) \wr S_t$ type subgroup then

$$
\begin{aligned}
|H| &\leq 2^t |\mathrm{PSO}^{-}_m(3)|^t t! && \text{by Theorem 5.5.1}\\
&\leq 2^{2t} 3^{t(m^2-m)/2} t! < 2^{2t} 3^{t(m^2+m)/2} t! && \text{by Lemma 2.6.4.}
\end{aligned}
$$

In either case

$$
|H| \leq 2^{2t} 3^{t(m^2+m)/2} t!
$$

$$
\begin{aligned}
&\leq 2^{2t}3^{t(m^2+m)/2}2^{t\log_2(t)} && \text{by Lemma 2.8.2}\\
&\leq 2^{2\log_2(n)+\log_2(n)\log_2(\log_2(n))}3^{\log_2(n)(n+\sqrt{n})/2} && \text{as } m \leq \sqrt{n} \text{ and } t \leq \log_2(n)\\
&\leq 2^{2n^{0.55}+n^{0.8525}}3^{n^{0.55}(n+\sqrt{n})/2} && \text{by Lemma 2.8.4}\\
&= 3^{\log_3(2)(2n^{0.55}+n^{0.8525})+n^{1.55}/2+n^{1.05}/2} &&\\
&\leq 3^{\log_3(2)(\frac{2n}{10^{0.45}}+\frac{n}{10^{0.1475}})+\frac{n^2}{2\times 10^{0.45}}+\frac{n^2}{2\times 10^{0.95}}} && \text{since } n \geq 10\\
&\leq 3^{0.24n^2+0.9n} \leq 3^{(n^2-3n)/2} \leq |M| && \text{since } n \geq 10.
\end{aligned}
$$

□

Lemma 5.5.7. *Let $n \geq 8$, and $q = 3$. If $M \leq \Omega_n^+(3)$ is a maximal subgroup of P_1 and $H \leq \Omega_n^+(3)$ is a P_k type subgroup where $k \neq 1$. Then $|M| \geq |H|$.*

Proof. Let M be a P_1 type subgroup. As in the previous lemma

$$|M| > 3^{(n^2-3n)/2}.$$

We split the different possibilities of H into two cases: P_k type for $k = n/2$ and then for $1 < k < n/2$. We compare the orders of these groups with H_1 separately.

If H is a $P_{n/2}$ type subgroup and $n \geq 10$ then

$$
\begin{aligned}
|H| &= \frac{1}{2}3^{\frac{n}{2}(n-\frac{1+3n/2}{2})}|\mathrm{GL}_{n/2}(3)| && \text{by Theorem 5.5.1}\\
&\leq 3^{\frac{n}{2}(n-\frac{1+3n/2}{2})}3^{\frac{n^2}{4}-1} && \text{by Lemma 2.6.4}\\
&= 3^{\frac{3n^2}{8}-\frac{n}{4}-1} \leq 3^{(n^2-3n)/2} = |M| && \text{since } n \geq 10.
\end{aligned}
$$

If H is a $P_{n/2}$ type subgroup and $n = 8$ then $|M| \geq |H|$ can be show via direct substitution.

If H is P_k type for $1 < k < n/2$ then

$$
\begin{aligned}
|H| &= 3^{k(n-\frac{1+3k}{2})}|\mathrm{GL}_k(3)||\Omega_{n-2k}^+(3)| && \text{by Theorem 5.5.1}\\
&\leq 3^{k(n-\frac{1+3k}{2})}3^{k^2}3^{\frac{(n-2k)^2-(n-2k)}{2}} && \text{by Lemma 2.6.4}\\
&= 3^{\frac{1}{2}(3k^2-2kn+k+n^2-n)}.
\end{aligned}
$$

To get an upper bound of $|H|$ independent of k we analyse the exponent of 3. We define the function $f(x) = (3x^2-2xn+x+n^2-n)$ and find the $x \in [2, n/2-1]$ that maximises $f(x)$ for this range.

We differentiate $f(x)$ with respect to x giving us $6x - 2n + 1$. Thus the turning point happens at $x = \frac{2n-1}{6}$. We see that this turning point occurs in the range of $[2, n/2-1]$ and also that it is a local minimum. Consequently the largest value for $f(x)$ for $x \in [2, n/2-1]$ occurs either at $x = 2$ or $x = n/2-1$.

$$f(n/2-1) = \frac{3}{8}(n^2-2n) \leq n^2-5n+12 = f(2) \qquad \text{since } n \geq 8.$$

Therefore we have an upper bound on $f(x)$ for the range $[2, n/2-1]$. Furthermore this upper bound is achieved when $x = 2$. Therefore

$$|H| \leq 3^{\frac{1}{2}(3k^2-2kn+k+n^2-n-2)} \leq 3^{\frac{1}{2}(n^2-5n+12)} \leq 3^{(n^2-3n)/2} = |M| \qquad \text{since } n \geq 6.$$

$\square$

Lemma 5.5.8. *Let $n \geq 6$, and $q = 3$. If $M \leq \Omega_n^+(3)$ is a maximal subgroup of type P_1 and $H \leq \Omega_n^+(3)$ is a $\mathrm{GO}_k^\epsilon(3) \perp \mathrm{GO}_{n-k}^\epsilon(3)$ type subgroup for $k \neq 1$. Then $|M| \geq |H|$.*

Proof. Let M be a P_1 type subgroup. As before

$$|M| > 3^{(n^2-3n)/2}.$$

We now derive an upper bound of H.

$$\begin{aligned}
|H| &= 4|\Omega_k^\epsilon(3)||\Omega_{n-k}^\epsilon(3)| && \text{by Theorem 5.5.1}\\
&\leq 4|\Omega_k^-(3)||\Omega_{n-k}^-(3)| \\
&\leq 4 \times 3^{\frac{k^2-k}{2}} 3^{\frac{(n-k)^2-(n-k)}{2}} && \text{by Lemma 2.6.4}\\
&= 4 \times 3^{\frac{1}{2}(2k^2-2kn+n^2-n)}.
\end{aligned}$$

To get an upper bound of $|H|$ independent of k we analyse the exponent of q. We define the function $f(x) = (2x^2 - 2xn + n^2 - n)$ and find the $x \in [2, n/2]$ that maximises $f(x)$ for this range.

The turning point of $f(x)$ occurs at $x = n/2$, so $f(x)$ is decreasing on $[2, n/2]$. Hence $f(2)$ is the maximum value. Since

$$f(2) = n^2 - 5n + 8,$$

we have that

$$|H| \leq 4^{\frac{1}{2}(2k^2-2kn+n^2-n)} \leq 4 \times 3^{\frac{1}{2}(n^2-5n+8)} \leq 3^{(n^2-3n)/2} < |M| \qquad \text{since } n \geq 6.$$

$\square$

Recall that $\mathcal{C}_G$ is the class of geometric subgroups of G. Combining the above we obtain

Proposition 5.5.9. *Let $n \geq 10$, $q = 3$, let $\overline{M}, \overline{H} \leq \mathrm{P}\Omega_n^+(3)$ both lie in $\mathcal{C}_{\mathrm{P}\Omega_n^+(3)}$. Then, if $|\overline{H}| > |\overline{M}|$ then $\overline{H}$ is a $\mathrm{GO}_1(3) \perp \mathrm{GO}_{n-1}(3)$ type group.*

Proof. For $\overline{H}$ and $\overline{M}$ there correspond subgroups H and M of $\Omega_n^+(q)$ of the same type. Note that $\overline{H} = H/(Z \cap \Omega_n^+(3))$ and $\overline{M} = M/(Z \cap \Omega_n^+(3))$. We compare the sizes of M and H.

By Theorem 5.5.1, the types of subgroups that H can be are listed in Table 5.11. In the case where the maximal subgroup H lies in $\mathcal{C}_i$ for $2 \leq i \leq 8$ then $|H| \leq |M|$ by Lemma 5.5.6.

If H lies in $\mathcal{C}_1$ and is of parabolic P_k type for $k \neq 1$ then $|H| \leq |M|$ by Lemma 5.5.7. Finally, if H is a $\mathrm{GO}_k^\epsilon(3) \perp \mathrm{GO}_{n-k}^\epsilon(3)$ type subgroup for $k \neq 1$ then $|H| \leq |M|$ by Lemma 5.5.8.

In all above cases we have that $|H| \leq |M|$ and so we have that $|\overline{H}| \leq |\overline{M}|$. $\square$

Before we tackle the next lemma, recall the definition of $\mathcal{E}_G$ found in Definition 3.1.1.

Lemma 5.5.10. *Let $n \geq 10$, and $q = 3$. If $\overline{M} \leq \mathrm{P}\Omega_n^+(3)$ is a subgroup of P_1 type and $\overline{H} \leq \mathrm{P}\Omega_n^+(3)$ is a subgroup of $\mathcal{E}_{\mathrm{P}\Omega_n^+(3)}$ type, then $|\overline{M}| \geq |\overline{H}|$.*

Proof. For $\overline{H}$ and $\overline{M}$ there correspond subgroups H and M of $\Omega_n^+(q)$ of the same type. Note that $\overline{H} = H/(Z \cap \Omega_n^+(q))$ and $\overline{M} = M/(Z \cap \Omega_n^+(q))$. We compare the sizes of M and H.

As in previous lemmas $|M| > 3^{(n^2-3n)/2}$.

If $H \in \mathcal{E}_{\Omega_n^+(q)}$ then

$$\begin{aligned}
|H| &\leq 2(n+2)! && \text{by Definition 3.1.1}\\
&\leq 2^{(n+2)\log_2(n+2)+1} && \text{by Lemma 2.8.2}\\
&\leq 2^{(n+2)^{1.55}+1} && \text{by Lemma 2.8.4}\\
&\leq 3^{\log_3(2)((n+2)^{1.55}+1)} \leq 3^{\frac{\log_3(2)}{11^{0.45}}(n+2)^2+\log_3(2)} && \text{because } n \geq 10\\
&\leq 3^{0.22(n+2)^2+\log_3(2)} = 3^{0.22(n^2+4n+4)+\log_3(2)} \leq 3^{(n^2-3n)/2} \leq |M| && \text{for } n \geq 10.
\end{aligned}$$

Consequently, $|\overline{H}| \leq |\overline{M}|$. ☐

Lemma 5.5.11. *Let $n \geq 10$, and $q = 3$. If $M \leq \Omega_n^+(3)$ is a subgroup of P_1 type then $|M| \geq 3^{3n+1}$. Furthermore if $\overline{M} \leq \mathrm{P}\Omega_n^+(3)$ is a subgroup of P_1 type then $|\overline{M}| \geq 3^{3n+1}$.*

Proof. As in the previous lemma $|M| > 3^{(n^2-3n)/2}$. So $|M| > 3^{(n^2-3n)/2} \geq 3^{3n+1}$ since $n \geq 10$. The second half now follows from Theorem 2.5.29. ☐

Lemma 5.5.12. *Let $n \geq 10$ even, $q = 3$, and let G have socle $\mathrm{P}\Omega_n^+(3)$.*

- *If $4|n$ and $G \leq \mathrm{P}\Omega_n^+(3).\langle\ddot{\gamma}, \ddot{\delta}'\rangle$ then $m(G) = m(\mathrm{P}\Omega_n^+(3)) = |\mathrm{P}\Omega_n^+(3) : H|$, where H is a $\mathrm{GO}_1(3) \perp \mathrm{GO}_{n-1}(3)$ type subgroup of $\mathrm{P}\Omega_n^+(3)$.*
- *If $n \equiv 2 \mod 4$ and $G \leq \mathrm{P}\Omega_n^+(3).\langle\ddot{\gamma}\rangle$ then $m(G) = m(\mathrm{P}\Omega_n^+(3)) = |\mathrm{P}\Omega_n^+(3) : H|$, where H is a $\mathrm{GO}_1(3) \perp \mathrm{GO}_{n-1}(3)$ type subgroup of $\mathrm{P}\Omega_n^+(3)$.*
- *Otherwise $m(G) = |\mathrm{PSL}_n(3) : H| = \frac{(3^m+1)(3^{m-1}-1)}{2}$, where H is a P_1 type subgroup of $\mathrm{P}\Omega_n^+(q)$.*

Proof. By Lemma 5.1.32, we note that if $4|n$ then $\mathrm{Out}(\mathrm{P}\Omega_n^+(3)) = \langle\ddot{\gamma}, \ddot{\delta}', \ddot{\delta}\rangle$. If $n \equiv 2 \mod 4$ then $\mathrm{Out}(\mathrm{P}\Omega_n^+(3)) = \langle\ddot{\gamma}, \ddot{\delta}\rangle$. Let $H_1 \leq \mathrm{P}\Omega_n^+(3)$ be the image of a subgroup of $\Omega_n^+(3)$ of $\mathrm{GO}_1(q) \perp \mathrm{GO}_{n-1}(q)$ type. The index of this subgroup is

$$\begin{aligned}
|\mathrm{P}\Omega_n^+(3) : H_1| &= \frac{|\Omega_n^+(3)|}{|\Omega_{n-1}(3).2|} && \text{by Theorem 5.5.1}\\
&= \frac{3^{(n^2-2n)/4}(3^{n/2}-1)\prod_{i=1}^{n/2-1}(3^{2i}-1)}{2 \times 3^{(n-2)^2/4}\prod_{i=1}^{n/2-1}(3^{2i}-1))} && \text{by Theorem 2.5.29}\\
&= \frac{3^{(n^2-2n)/4}(3^{n/2}-1)}{2 \times 3^{(n-2)^2/4}}
\end{aligned}$$

$$= \frac{(3^{n/2}-1)3^{n/2-1}}{2} = m(\mathrm{P}\Omega_n^+(3)) \qquad \text{by Theorem 2.2.7.}$$

[26] gives us the stabilizer of the $\mathrm{P}\Omega_n^+(3)$-conjugacy class of H_1, however this is given in their notation. We will translate the outer automorphisms $\ddot{r_\square}$, $\ddot{r_\boxtimes}$, $\ddot{\delta}$ and $\ddot{\phi}$ of $\mathrm{P}\Omega_n^+(3)$ presented in [26] into the notation used in Lemma 5.1.32 and by extension into that used in [6, Sections 1.7.1 & 1.7.2]. We do this to match up the assertions of two different sources about what the stabiliser of various subgroups are.

As shown in [26, Prop. 4.1.6.] and [26, Table 3.5.G], $\ker_{\ddot{\Gamma}}(\ddot{\tau})$ is the stabilizer of the $\mathrm{P}\Omega_n^+(3)$-conjugacy class of H_1. Let Γ be $\mathrm{C\Gamma O}_n^+(q)$. Here τ in [26, p.12] is a map from Γ to $F_q^\times$, and $\ddot{\tau}$, a map from $\ddot{\Gamma}$ to $F_q^\times/(F_q^\times)^2$, is defined further at the end of [26, §2.1]. Following their notation, [26, p.36] tells us that $\tau(\phi) = 1$ and $\tau(\delta) = \mu$ where $\langle\mu\rangle = F_q^\times$. Furthermore $r_\square$ and $r_\boxtimes$ lie in $\mathrm{GO}_n(3)$, where $r_\square, r_\boxtimes$ are defined in [26, p.30], therefore $\tau(r_\square) = \tau(r_\boxtimes) = 1$. Consequently $\ddot{\tau}(\ddot{r_\square}) = \ddot{\tau}(\ddot{r_\boxtimes}) = \ddot{\tau}(\ddot{\phi}) = 1$ and $\ddot{\tau}(\ddot{\delta}) \neq 1$ since $\mu \notin (F_q^\times)^2$. Therefore $\ker_{\ddot{\Gamma}}(\ddot{\tau}) = \langle\ddot{\phi}, \ddot{r_\square}, \ddot{r_\boxtimes}\rangle$, in the notation of [26].

Let us translate the above results into one using the outer automorphisms $\ddot{\delta}, \ddot{\delta}', \ddot{\phi}, \ddot{\gamma}$ as found in Lemma 5.1.32 and [6]. Following [6, 1.7.1. Case $\mathrm{O}^\pm$], $\ddot{\gamma} = \ddot{r_\square}$, and if $\ddot{\delta}'$ is non-trivial then $\ddot{\delta}' = \ddot{r_\square}\ddot{r_\boxtimes}$, while the definition of $\ddot{\delta}$ and $\ddot{\phi}$ coincides in both sources. Since $q = 3$ remember that $\ddot{\phi}$ is trivial. Consequently, by Lemma 5.1.32, if $4|n$ then $\langle\ddot{\delta}', \ddot{\gamma}\rangle = \ker_{\ddot{\Gamma}}(\ddot{\tau})$ is the stabilizer of the $\mathrm{P}\Omega_n^+(3)$-conjugacy class of H_1. In addition, if $n \equiv 2 \mod 4$ then $\langle\ddot{\gamma}\rangle = \ker_{\ddot{\Gamma}}(\ddot{\tau})$. Note here that in both cases the stabilizer is a normal subgroup of $\mathrm{Out}(\mathrm{P}\Omega_n^+(3))$.

If $4|n$ and $G \leq \mathrm{P}\Omega_n^+(3).\langle\ddot{\gamma}, \ddot{\delta}'\rangle$, then by Lemma 2.3.8 we have that $N_G(H_1)\mathrm{P}\Omega_n^+(3) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n^+(3))$.

If $n \equiv 2 \mod 4$ and $G \leq \mathrm{P}\Omega_n^+(q).\langle\ddot{\gamma}\rangle$, then by Lemma 2.3.8 we once again have that $N_G(H_1)\mathrm{P}\Omega_n^+(3) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n^+(3))$.

Now suppose that the image of G in $\mathrm{Out}(\mathrm{P}\Omega_n^+(3))$ is not contained in $\ker_{\ddot{\Gamma}}(\ddot{\tau})$, the stabilizer of the $\mathrm{P}\Omega_n^+(3)$-conjugacy class of H_1. Let H_2 be a P_1 subgroup of $\mathrm{P}\Omega_n^+(3)$. [26, Prop. 4.1.20] states that the number c of $\mathrm{P}\Omega_n^+(3)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{P}\Omega_n^+(3))$-conjugacy class of H_2 is 1. Thus, $\mathrm{Out}(\mathrm{P}\Omega_n^+(3))$ fixes $\mathrm{P}\Omega_n^+(3)$-conjugacy class of H_2, and so by Lemma 2.3.8 we know that $N_G(H_2)\mathrm{P}\Omega_n^+(3) = G$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{P}\Omega_n^+(3)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{P}\Omega_n^+(3)$, $N_G(K) = M$, and $N_G(K)\mathrm{P}\Omega_8^+(3) = G$.

By Theorem 3.1.3, either M belongs in $\mathcal{C}_G$, lies in $\mathcal{E}_G$ or $|M| \leq 3^{3n}$. If $|M| \leq 3^{3n}$ then $|K| \leq 3^{3n}$. By Lemma 5.5.11 a P_1 subgroup of $\mathrm{P}\Omega_n^+(3)$ has order larger than 3^{3n} and so therefore $|H_2| \geq |K|$.

If M belongs in $\mathcal{C}_G$ or in $\mathcal{E}_G$ then K lies in $\mathcal{C}_{\mathrm{P}\Omega_n^+(3)}$ or $\mathcal{E}_{\mathrm{P}\Omega_n^+(3)}$. By Lemmas 5.5.9 and 5.5.10, the only possibility for K where $|H_2| < |K|$ is where K is $\mathrm{Out}(\mathrm{P}\Omega_n^+(3))$-conjugate to H_1, a $\mathrm{GO}_1(q) \perp \mathrm{GO}_{n-1}(q)$ type subgroup. However in this case, the image of G in $\mathrm{Out}(\mathrm{P}\Omega_n^+(3))$ is not contained in the stabilizer of the $\mathrm{P}\Omega_n^+(3)$-conjugacy class of H_1 by our assumption of G, and so by Lemma 2.3.8 we have that $N_G(H_1).\mathrm{P}\Omega_n^+(3) \neq G$; a contradiction. Therefore $|H_2| \geq |K|$.

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{P}\Omega_n^+(3)$ that satisfies $|K| \leq |H_2|$. Hence $|G : N_G(H_2)| = |\mathrm{P}\Omega_n^+(3) : H_2| \leq |\mathrm{P}\Omega_n^+(3) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_2)| = |\mathrm{P}\Omega_n^+(3) : H_2|$.

$\square$

Lemma 5.5.13. *Let $n \geq 10$ even, $q = 2$, and let G have socle $\mathrm{P}\Omega_n^+(2)$. Then $m(G) = m(\mathrm{P}\Omega_n^+(2)) = |\mathrm{PSp}_n(2) : H|$, where H is a $\mathrm{Sp}_{n-2}(2)$ type subgroup of $\mathrm{P}\Omega_n^+(2)$.*

Proof. Let H be a $\mathrm{Sp}_{n-2}(q)$ type subgroup of $\mathrm{P}\Omega_n^+(q)$. The index of this subgroup is

$$\begin{aligned}
|\mathrm{P}\Omega_n^+(2) : H| &= \frac{|\Omega_n^+(2)|}{|\mathrm{Sp}_{n-2}(2)|} && \text{by Theorem 5.5.1}\\
&= \frac{2^{n(n-2)/4}(2^{n/2}-1)\prod_{i=1}^{n/2-1}(2^{2i}-1)}{2^{(n-2)^2/4}\prod_{i=1}^{n/2-1}(2^{2i}-1)} && \text{by Theorem 2.5.29}\\
&= \frac{2^{n(n-2)/4}(2^{n/2}-1)}{2^{(n-2)^2/4}}\\
&= 2^{n/2-1}(2^{n/2}-1) = m(\mathrm{P}\Omega_n^+(2)) && \text{by Theorem 2.2.7.}
\end{aligned}$$

[26, Prop. 4.1.7] states that the number c of $\mathrm{P}\Omega_n^+(2)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{P}\Omega_n^+(2))$-conjugacy class of H is 1. Thus, $\mathrm{Out}(\mathrm{P}\Omega_n^+(2))$ must fix the $\mathrm{P}\Omega_n^+(2)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{P}\Omega_n^+(2) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n^+(2))$.

$\square$

Lemma 5.5.14. *Let $n = 8$, $q \geq 4$ be a prime power, and let G have socle $\mathrm{P}\Omega_8^+(q)$.*

- *If q is odd, and $G \leq \mathrm{P}\Omega_8^+(q).T$, for $T = \langle \ddot{\delta}, \ddot{\gamma}, \ddot{\delta}', \ddot{\phi} \rangle$ or any of its $\mathrm{Out}(\mathrm{P}\Omega_8^+(q))$-conjugates, then $m(G) = m(\mathrm{P}\Omega_8^+(q)) = |\mathrm{P}\Omega_8^+(q) : H|$, where H is a P_1 type subgroup of $\mathrm{P}\Omega_8^+(q)$.*
- *If q is even, and $G \leq \mathrm{P}\Omega_8^+(q).T$, for $T = \langle \ddot{\gamma}, \ddot{\phi} \rangle$ or any of its $\mathrm{Out}(\mathrm{P}\Omega_8^+(q))$-conjugates, then $m(G) = m(\mathrm{P}\Omega_8^+(q)) = |\mathrm{P}\Omega_8^+(q) : H|$, where H is a P_1 type subgroup of $\mathrm{P}\Omega_8^+(q)$.*
- *Otherwise $m(G) = |\mathrm{P}\Omega_8^+(q) : H| = \frac{(q^2+1)^2(q^6-1)}{q-1}$, where H is a P_2 subgroup of $\mathrm{P}\Omega_8^+(q)$.*

Proof. We will make reference to [6, Table 8.50] throughout, however that table is taken from [23].

First let us note, by Lemma 5.1.32, that $\mathrm{Out}(\Omega_8^+(q)) = \langle \ddot{\tau}, \ddot{\gamma}, \ddot{\phi} \rangle$ if q is even and $\mathrm{Out}(\Omega_8^+(q)) = \langle \ddot{\delta}', \ddot{\tau}, \ddot{\gamma}, \ddot{\delta}, \ddot{\phi} \rangle$ if q is odd.

Let H_1 be a P_1 type subgroup of $\mathrm{P}\Omega_8^+(q)$. In this case, one may calculate directly from Theorems 5.5.1 and 2.5.29 that $|\mathrm{P}\Omega_8^+(q) : H| = m(\mathrm{P}\Omega_8^+(q))$, where $m(\mathrm{P}\Omega_8^+(q))$ can be found in Table 2.2.7.

Recall the definition for $\mathrm{CL}_G(H)$ from Definition 2.1.8. Now let $G = \mathrm{P}\Omega_8^+(q).T$ for T contained in a conjugate of the $\mathrm{Out}(\mathrm{P}\Omega_8^+(q))$-stabilizer of $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(3)}(H_1)$. For q odd this stabilizer is $\langle \ddot{\delta}, \ddot{\gamma}, \ddot{\delta}', \ddot{\phi} \rangle$, for q even the stabilizer is $\langle \ddot{\gamma}, \ddot{\phi} \rangle$. Therefore, T also fixes the $\mathrm{P}\Omega_8^+(q)$-conjugacy class of a $\mathrm{Aut}(\mathrm{P}\Omega_8^+(q))$-conjugate of H_1; let us call this conjugate H_T. By Lemma 2.3.8 we have $N_G(H_T)\mathrm{P}\Omega_8^+(q) = G$. Therefore by Lemma 2.3.18 $m(G) = m(\mathrm{P}\Omega_8^+(q))$.

Now suppose that $G = \mathrm{P}\Omega_8^+(q).T$ where T is not contained in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(q))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(q)$-conjugacy class of H_1. Let H_2 be a P_2 subgroup of $\mathrm{P}\Omega_8^+(q)$. The 4th row of [6, Table 8.50] shows that the whole of $\mathrm{Out}(\mathrm{P}\Omega_8^+(q))$ stabilizes the $\mathrm{P}\Omega_8^+(q)$-conjugacy class of H_2, and so by Lemma 2.3.8 we know that $N_G(H_2)\mathrm{P}\Omega_8^+(q) = G$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{P}\Omega_8^+(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{P}\Omega_8^+(q)$, $N_G(K) = M$, and $N_G(K)\mathrm{P}\Omega_8^+(q) = G$. The possibilities for K are found in the rows of [6, Table 8.50] for which the stabilizer listed does not lie in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(q))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(q)$ conjugacy class of H_1. In this case, these are the groups where the stabilizer listed contains τ. One may show that $|K| \leq |H_2|$.

Let us expand on this further. We will look at the orders of the corresponding groups $K_{\Omega_8^+(q)}$ in $\Omega_8^+(q)$ and show that the order of $K_{\Omega_8^+(q)}$ is smaller than that of a P_2 type subgroup of $\Omega_8^+(q)$. Note that the order of a P_2 type subgroup of $\Omega_8^+(q)$ is

$$\begin{aligned} q^9|\mathrm{GL}_2(q)||\Omega_4^+(q)| &\geq q^9q^3q^5 = q^{17} && \text{by Lemma 2.6.4} \quad (5.1)\\ &\geq 4^{17} = 17179869184 && \text{since } q \geq 4. \quad (5.2)\end{aligned}$$

We will compare this with the orders of $K_{\Omega_8^+(q)}$. For most of these we will apply Lemma 2.6.4. Let $d = (q-1, 2)$. We now consider the $K_{\Omega_8^+(q)}$ lying in $\mathcal{C}_1$. If $K_{\Omega_8^+(q)}$ has shape $[q^{11}] : [\frac{q-1}{d}]^2.\frac{1}{d}\mathrm{GL}_2(q).d^2$ then $|K_{\Omega_8^+(q)}| < q^{11}q^2q^4 = q^{17}$. If $K_{\Omega_8^+(q)}$ has shape $d \times \mathrm{G}_2(q)$ then by Theorem 5.1.43 we have that $|K_{\Omega_8^+(q)}| < 2q^6(q^6-1)(q^2-1) < q^{15}$. If $K_{\Omega_8^+(q)}$ has shape $(\Omega_2^+(q) \times \frac{1}{d}\mathrm{GL}_3(q)).[2d]$ then $|K_{\Omega_8^+(q)}| \leq 2q^{1+9} < q^{11}$. If $K_{\Omega_8^+(q)}$ has shape $(\Omega_2^-(q) \times \frac{1}{d}\mathrm{GU}_3(q)).[2d]$ then $|K_{\Omega_8^+(q)}| \leq 4q^{1+9} < q^{11}$.

Let us look now at the case where $K_{\Omega_8^+(q)}$ lies in $\mathcal{C}_2$. Here the order of $K_{\Omega_8^+(q)}$ may equal $2^{10}|\mathrm{PSL}_3(2)|$, $192d^3|\Omega_2^+(q)^4|$, $192d^3|\Omega_2^-(q)^4|$, $4d|\Omega_4^+(q)^2|$ or $4d(\frac{2(q^2+1)}{d})^2$. In all of these cases, since $d \leq 2$ and $q \geq 4$, we have that $|K_{\Omega_8^+(q)}| \leq q^{17}$.

Now consider the case where $K_{\Omega_8^+(q)}$ lies in $\mathcal{C}_5$. $K_{\Omega_8^+(q)}$ has shape $\Omega_8^+(q_0)$ or $\mathrm{SO}_8^+(q_0).2$, and therefore $|K_{\Omega_8^+(q)}| \leq 4q_0^{28} \leq q^{16}$ since $q_0 \leq q^{1/2}$.

Finally in the cases where $K_{\Omega_8^+(q)}$ is non-geometric then the order of $K_{\Omega_8^+(q)}$ is one of the following $3d|\mathrm{PSL}_3(q)|$, $3d|\mathrm{PSU}_3(q)|$, $d|{}^3\mathrm{D}_4(q_0)|$, $2|\Omega_8^+(2)|$ or $2|\mathrm{Sz}(8)|$ where $q_0 = q^{1/3}$. Recall that, by Theorem 5.1.43, the order of ${}^3\mathrm{D}_4(q_0)$ is $q_0^{12}(q_0^8+q_0^4+1)(q_0^6-1)(q_0^2-1) < q_0^{12+9+6+2} = q^{29/3} < q^{10}$. In addition by the ATLAS [9], the order of $\mathrm{Sz}(8)$ is 29120. In all of the above case one can show that $|K_{\Omega_8^+(q)}| \leq q^{17}$.

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{P}\Omega_8^+(q)$ that satisfies $|K| \leq |H_2|$. Hence $|G : N_G(H_2)| = |\mathrm{P}\Omega_8^+(q) : H_2| \leq |\mathrm{P}\Omega_8^+(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_2)| = |\mathrm{P}\Omega_8^+(q) : H_2|$.

$\square$

Lemma 5.5.15. *Let $n = 8$, $q = 3$, and let G have socle $\mathrm{P}\Omega_8^+(3)$.*

(a) If $G = \mathrm{P}\Omega_8^+(3).T$ where T lies in a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$ then $m(G) = m(\mathrm{P}\Omega_8^+(3)) = |\mathrm{P}\Omega_8^+(3) : H|$, where $H \leq \mathrm{P}\Omega_8^+(3)$ is a $\mathrm{GO}_1(3) \perp \mathrm{GO}_7(3)$ type subgroup.

(b) If $G = \mathrm{P}\Omega_8^+(3).T$ where T does not lie in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$ but lies in a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}', \ddot{\delta}\rangle$ then $m(G) = |\mathrm{P}\Omega_8^+(3) : H| = 1120$, where $H \leq \mathrm{P}\Omega_8^+(3)$ is a P_1 type subgroup of $\mathrm{P}\Omega_8^+(3)$.

(c) If $G = \mathrm{P}\Omega_8^+(3).T$ where T does not lie in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}', \ddot{\delta}\rangle$ but lies in a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\tau}\rangle$ then $m(G) = |\mathrm{P}\Omega_8^+(3) : H| = 28431$, where $H \leq \mathrm{P}\Omega_8^+(3)$ is a $\mathcal{S}$ subgroup and is the image of a subgroup of $\Omega_8^+(3)$ with shape $2^{\cdot}\Omega_8^+(2)$.

(d) Otherwise $m(G) = |\mathrm{P}\Omega_8^+(3) : H| = 36400$, where H is a P_2 type subgroup of $\mathrm{P}\Omega_8^+(3)$.

Proof. First let us note, by Lemma 5.1.32, that $\mathrm{Out}(\Omega_8^+(3)) = \langle\ddot{\delta}', \ddot{\tau}, \ddot{\gamma}, \ddot{\delta}\rangle$ as $\ddot{\phi}$ is trivial since $q = 3$ is prime.

Let $G = \mathrm{P}\Omega_8^+(3).T$ for T lying in some $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$. Let H_1 be a $\mathrm{GO}_1(q) \perp \mathrm{GO}_7(q)$ type subgroup of $\mathrm{P}\Omega_8^+(3)$. One may calculate from [6, Table 8.50] that $|\mathrm{P}\Omega_8^+(3) : H_1| = m(\mathrm{P}\Omega_8^+(3))$, where $m(\mathrm{P}\Omega_8^+(3))$ can be found in Theorem 2.2.7. From [6, Table 8.50] we see that for $X \in \mathrm{Cl}_{\mathrm{Aut}(\mathrm{P}\Omega_8^+(3))}(H_1)$ the stabilizer of $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(3)}(X)$ is a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$. So T fixes $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(3)}(X)$ for some $X \in \mathrm{Cl}_{\mathrm{Aut}(\mathrm{P}\Omega_8^+(3))}(H_1)$. Let us denote this conjugate of H_1 by H_T. By Lemma 2.3.8 we have $N_G(H_T)\mathrm{P}\Omega_8^+(3) = G$. Therefore by Lemma 2.3.18 $m(G) = |G : N_G(H_T)| = |\mathrm{P}\Omega_8^+(3) : H_1| = m(\mathrm{P}\Omega_8^+(3))$. This proves *(a)*.

Let us now show *(b)*. Let H_2 be a P_1 type subgroup of $\mathrm{P}\Omega_8^+(3)$. The stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_2 is $\langle\ddot{\gamma}, \ddot{\delta}', \ddot{\delta}\rangle$. Let $G = \mathrm{P}\Omega_8^+(3).T$ where T is contained in a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_2 but not contained in any of the $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugates of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_1.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{P}\Omega_8^+(3)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{P}\Omega_8^+(3)$, $N_G(K) = M$, and $N_G(K)\mathrm{P}\Omega_8^+(3) = G$. Since T is not contained in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_1, any maximal subgroup of G has the form $N_G(K)$ where K appears in [6, Table 8.50] and the stabilizer (defined up to conjugate in $\mathrm{Out}(S)$) listed cannot lie in any $\mathrm{Out}(S)$-conjugate of $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$. Translating the notation of [6], this means that the stabilizer is not an $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate to $\langle\ddot{\gamma}, \ddot{\delta}'\rangle$ or $\langle\ddot{\gamma}\rangle$. In particular we have that $N_G(H_1)$ is not a maximal subgroup of G. By direct substitution of the value of $q = 3$ one may calculate that $|K| \leq |H_2|$.

Since T is contained in a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_2, we have that T also fixes $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(3)}(X)$ for some $\mathrm{Aut}(\mathrm{P}\Omega_8^+(3)$ conjugate of H_2. Denote this conjugate of H_2 by H_T. By Lemma 2.3.8 we have $N_G(H_T)\mathrm{P}\Omega_8^+(3) = G$.

Consequently, for any maximal subgroup $M \leq G$ we have that $K = M \cap \mathrm{P}\Omega_8^+(3)$ satisfies

$M = N_G(K)$ and that $|K| \le |H_2|$. Hence $|G : N_G(H_T)| = |\mathrm{P}\Omega_8^+(3) : H_T| = |\mathrm{P}\Omega_8^+(3) : H_2| \le |\mathrm{P}\Omega_8^+(3) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_T)| = |\mathrm{P}\Omega_8^+(3) : H_2| = 1120$.

Let us now tackle *(c)*. Let H_3 be the image of a non-geometric subgroup of $\Omega_8^+(3)$ of shape $2^{\cdot}\Omega_8^+(2)$. The stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_3 is $\langle \ddot{\gamma}, \ddot{\tau} \rangle$. Suppose T is contained in a $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_3 but not contained in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_1 nor of H_2. Therefore, T fixes $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(3)}(X)$ for some $\mathrm{Aut}(\mathrm{P}\Omega_8^+(3))$-conjugate of H_3. Let us denote this conjugate of H_3 by H_T. By Lemma 2.3.8 we have $N_G(H_T)\mathrm{P}\Omega_8^+(3) = G$.

Following the same arguments as in the *(b)* case we may show that for any maximal subgroup $M \le G$, we have that $K = M \cap \mathrm{P}\Omega_8^+(3)$ satisfies $M = N_G(K)$ and that $|K| \le |H_3|$. Note that the possibilities for K are found in the rows of [6, Table 8.42] for which the stabilizer listed does not lie in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$ conjugacy class of H_1 nor of H_2. Hence $|G : N_G(H_T)| = |\mathrm{P}\Omega_8^+(3) : H_T| = |\mathrm{P}\Omega_8^+(3) : H_3| \le |\mathrm{P}\Omega_8^+(3) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_T)| = |\mathrm{P}\Omega_8^+(3) : H_3| = 28431$.

Let us now tackle *(d)*. Let H_4 be a P_2 subgroup of $\mathrm{P}\Omega_8^+(3)$. The stabilizer of the $\mathrm{P}\Omega_8^+(3)$ conjugacy class of H_4 is $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$.

Suppose that T is not contained in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$-conjugacy class of H_1 nor of H_2 nor of H_3. Then following the same arguments as above we may show that for any maximal subgroup $M \le G$, we have that $K = M \cap \mathrm{P}\Omega_8^+(3)$ satisfies $M = N_G(K)$ and that $|K| \le |H_4|$. Note that the possibilities for K are found in the rows of [6, Table 8.42] for which the stabilizer listed does not lie in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(3))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(3)$ conjugacy class of H_1 nor of H_2 nor of H_3. Hence $|G : N_G(H_4)| = \mathrm{P}\Omega_8^+(3) : H_4| \le |\mathrm{P}\Omega_8^+(3) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_4)| = |\mathrm{P}\Omega_8^+(3) : H_4| = 36400$. □

Lemma 5.5.16. *Let $n = 8$, $q = 2$, and let G have socle $\mathrm{P}\Omega_8^+(2)$.*

- *If $G = \mathrm{P}\Omega_8^+(2).T$ where T lies in some $\mathrm{Out}(\mathrm{P}\Omega_8^+(2))$-conjugate of $\langle \ddot{\gamma} \rangle$, then $m(G) = m(\mathrm{P}\Omega_8^+(2)) = |\mathrm{P}\Omega_8^+(2) : H|$, where $H \le \mathrm{P}\Omega_8^+(2)$ is a $\mathrm{Sp}_6(2)$ type subgroup.*
- *Otherwise $m(G) = |\mathrm{P}\Omega_8^+(2) : H| = 1575$, where H is a P_2 type subgroup of $\mathrm{P}\Omega_8^+(2)$.*

Proof. First let us note, by Lemma 5.1.32, that $\mathrm{Out}(\mathrm{P}\Omega_8^+(2)) = \langle \ddot{\gamma}, \ddot{\tau} \rangle$.

Let $G = \mathrm{P}\Omega_8^+(2).T$ for T lying in some $\mathrm{Out}(\mathrm{P}\Omega_8^+(2))$-conjugate of $\langle \ddot{\gamma} \rangle$. Let H_1 be a $\mathrm{Sp}_6(q)$ type subgroup of $\mathrm{P}\Omega_8^+(2)$. One may calculate from Theorem 5.5.1 that $|\mathrm{P}\Omega_8^+(2) : H_1| = m(\mathrm{P}\Omega_8^+(2))$, where $m(\mathrm{P}\Omega_8^+(2))$ can be found in Table 2.2.7. From [6, Table 8.50] we see that there exists a $X \in \mathrm{Cl}_{\mathrm{Aut}(\mathrm{P}\Omega_8^+(2))}(H_1)$ such that the stabilizer of $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(2)}(2)$ is $\langle \ddot{\gamma} \rangle$. So T also fixes $\mathrm{Cl}_{\mathrm{P}\Omega_8^+(2)}(X)$ for some $X \in \mathrm{Cl}_{\mathrm{Aut}(\mathrm{P}\Omega_8^+(2))}(H_1)$, let us denote this conjugate of H_1 by H_T. By Lemma 2.3.8 we have $N_G(H_T)\mathrm{P}\Omega_8^+(2) = G$. Therefore by Lemma 2.3.18 $m(G) = m(\mathrm{P}\Omega_8^+(2))$.

Now suppose that $G = \mathrm{P}\Omega_8^+(2).T$ where T is not contained in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(2))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(2)$-conjugacy class of H_1. Let H be a P_2 subgroup of $\mathrm{P}\Omega_8^+(2)$. The stabilizer of the $\mathrm{P}\Omega_8^+(2)$-conjugacy class of H_2 is the whole of $\mathrm{Out}(\mathrm{P}\Omega_8^+(2))$. Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{P}\Omega_8^+(2)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{P}\Omega_8^+(2)$, $N_G(K) = M$, and $N_G(K)\mathrm{P}\Omega_8^+(2) = G$. The possibilities for K are found in the rows of [6, Table 8.42] for which the stabilizer listed does not lie in any $\mathrm{Out}(\mathrm{P}\Omega_8^+(2))$-conjugate of the stabilizer of the $\mathrm{P}\Omega_8^+(2)$ conjugacy class of H_1. We can calculate the orders of these groups directly by substituting in $q = 2$ and so one may show that $|K| \leq |H_2|$.

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{P}\Omega_8^+(2)$ that satisfies $|K| \leq |H_2|$. Hence $|G : N_G(H_2)| = |\mathrm{P}\Omega_8^+(2) : H_2| \leq |\mathrm{P}\Omega_8^+(2) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H_2)| = |\mathrm{P}\Omega_8^+(2) : H_2| = 1575$. □

Lemma 5.5.17. *Let $n \geq 8$, q be a prime power, and let G have socle $\mathrm{P}\Omega_n^-(q)$. Then $m(G) = m(\mathrm{P}\Omega_n^-(q)) = |\mathrm{P}\Omega_n^-(q) : H|$, where H is a P_1 type subgroup of $\mathrm{P}\Omega_n^-(q)$.*

Proof. Let H be a P_1 subgroup of $\mathrm{P}\Omega_n^-(q)$. The index of this subgroup is

$$\begin{aligned}
\frac{|\mathrm{P}\Omega_n^-(q)|}{|H|} &= \frac{|\Omega_n^-(q)|}{q^{n-2}|\mathrm{GL}_1(q)||\Omega_{n-2}^-(q)|} && \text{by [6, Table 2.3]}\\
&= \frac{q^{n(n-2)/4}(q^{n/2}+1)\prod_{i=1}^{n/2-1}(q^{2i}-1)}{q^{n-2}(q-1)q^{(n-2)(n-4)/4}(q^{n/2-1}+1)\prod_{i=1}^{n/2-2}(q^{2i}-1)} && \text{by Theorem 2.5.29}\\
&= \frac{q^{n(n-2)/4}(q^{n/2}+1)(q^{n-2}-1)}{q^{n-2}(q-1)q^{(n-2)(n-4)/4}(q^{n/2-1}+1)}\\
&= \frac{(q^{n/2}+1)(q^{n-2}-1)}{(q-1)(q^{n/2-1}+1)}\\
&= \frac{(q^{n/2}+1)(q^{n/2-1}-1)}{(q-1)} = m(\mathrm{P}\Omega_n^-(q)) && \text{by Theorem 2.2.7.}
\end{aligned}$$

[26, Prop. 4.1.20] states that the number c of $\mathrm{P}\Omega_n^-(q)$-conjugacy classes that lie in the $\mathrm{Aut}(\mathrm{P}\Omega_n^-(q))$-conjugacy class of H is 1. So, $\mathrm{Out}(\mathrm{P}\Omega_n^-(q))$ fixes the $\mathrm{P}\Omega_n^-(q)$-conjugacy class of H, and so by Lemma 2.3.8 we know that $N_G(H)\mathrm{P}\Omega_n^-(q) = G$. Therefore by Lemma 2.3.18 we have that $m(G) = m(\mathrm{P}\Omega_n^-(q))$. □

We conclude our proof of Theorem 5.2.1 for $S = \mathrm{P}\Omega_n(q)$. Let us note here that we can assume $n \geq 7$ and that if n is odd then q is odd otherwise S is isomorphic to other classical groups. Lemmas 5.5.3-5.5.4 show for n odd, and Lemma 5.5.5 shows for n even, that if $\mathrm{P}\Omega_n(q) \leq G \leq \mathrm{Aut}(\mathrm{P}\Omega_n(q))$ then $m(G) = m(\mathrm{P}\Omega_n(q))$. Lemmas 5.5.12-5.5.16 show us that if $\mathrm{P}\Omega_n^+(q) \leq G \leq \mathrm{Aut}(\mathrm{P}\Omega_n^+(q))$ then $m(G) = m(\mathrm{P}\Omega_n^+(q))$ except in the cases outlined in Table 5.6, with the values of $m(G)$ also as given in Table 5.6. Finally Lemma 5.5.17 shows us that if $\mathrm{P}\Omega_n^-(q) \leq G \leq \mathrm{Aut}(\mathrm{P}\Omega_n^-(q))$ then $m(G) = m(\mathrm{P}\Omega_n^-(q))$.

5.6 Largest maximal subgroups of almost simple groups with exceptional simple group socle

5.6.1 $\mathrm{G}_2(q)$

The information about the subgroups of $\mathrm{G}_2(q)$ comes from [6], more specifically Tables 8.30, 8.41 and 8.42. However we note that this information comes from originally [4], [11] and [24]. Considering this information we pinpoint the largest maximal subgroups. In addition these tables contain information as to the outer automorphisms of the groups, and the stabilizers of the $\mathrm{G}_2(q)$-conjugacy classes of the subgroups of $\mathrm{G}_2(q)$.

For this section, if $q = p^e$ where $e \geq 2$ let, let ϕ denote the field automorphism of $\mathrm{G}_2(q)$ corresponding to the Frobenius automorphism of $\mathbb{F}_q$, and also denote by $\ddot{\phi}$ the image of ϕ in $\mathrm{Out}(\mathrm{G}_2(q))$. If $q = p$ then define $\ddot{\phi}$ to be the identity element of $\mathrm{Out}(\mathrm{G}_2(q))$. If $\mathrm{G}_2(q)$ has a graph automorphism, which only occurs when $q = 3^e$, we denote it by γ and we denote the image of γ in $\mathrm{Out}(\mathrm{G}_2(q))$ by $\ddot{\gamma}$. Finally we note that in the tables of [6, Chapter 8] the authors there will omit the $\ddot{}$ when talking about the outer automorphisms of $\mathrm{G}_2(q)$ which may be a point of potential confusion.

Lemma 5.6.1. *Let $q \geq 4$ be a prime power not divisible by 3, and let G be almost simple with socle $\mathrm{G}_2(q)$.*

- *If $q = 4$ then $m(G) = m(\mathrm{G}_2(4)) = |\mathrm{G}_2(4) : H| = 416$, where $H \cong \mathrm{J}_2$.*
- *Otherwise $m(G) = m(\mathrm{G}_2(q)) = |\mathrm{G}_2(q) : H| = \frac{(q^6-1)}{(q-1)}$, where H is of shape $[q^5] : \mathrm{GL}_2(q)$.*

Proof. Let $q = 4$ and $H \cong \mathrm{J}_2$. Thus $|\mathrm{G}_2(q)| = 251596800$. By [48, Theorem 1] $m(\mathrm{G}_2(4)) = |\mathrm{G}_2(4) : H| = 416$. Also by [6, Table 8.30] we know that the conjugacy class for H in $\mathrm{G}_2(q)$ is stabilized by $\mathrm{Out}(\mathrm{G}_2(q))$ and consequently $N_G(H)\mathrm{G}_2(q) = G$. Therefore by Lemma 2.3.18 we have our result.

Now let $q > 4$, from Theorem 5.1.43 we get $|\mathrm{G}_2(q)| = q^6(q^6-1)(q^2-1)$. Furthermore from Theorem 5.1.44 we have that $\mathrm{Out}(\mathrm{G}_2(q)) = \langle\ddot{\phi}\rangle$. Let H be a of $\mathrm{G}_2(q)$ with shape $[q^5] : \mathrm{GL}_2(q)$ as provided by [6, Table 8.30]. So $|H| = q^6(q^2-1)(q-1)$ and $|\mathrm{G}_2(q) : H|\frac{(q^6-1)}{(q-1)}$. By [48, Theorem 1] we know that $m(\mathrm{G}_2(q)) = \frac{(q^6-1)}{(q-1)}$. Finally, by [6, Table 8.30] we know that the conjugacy classes for each subgroup of $\mathrm{G}_2(q)$ are stabilized by $\mathrm{Out}(\mathrm{G}_2(q))$, in particular the conjugacy class of H, so $N_G(H)\mathrm{G}_2(q) = G$. Therefore by Lemma 2.3.18 we have our result. □

Lemma 5.6.2. *Let $q = 3^e$, and let G be almost simple with socle $\mathrm{G}_2(q)$.*

- *If $q = 3$, and $G = \mathrm{G}_2(3)$ then $m(G) = m(\mathrm{G}_2(3)) = |\mathrm{G}_2(3) : H| = 351$, where $H \cong \mathrm{SU}_3(3) : 2$.*
- *If $q > 3$, and $G \leq \mathrm{G}_2(q).\langle\ddot{\phi}\rangle$ then $m(G) = m(\mathrm{G}_2(q)) = |\mathrm{G}_2(q) : H| = \frac{(q^6-1)}{(q-1)}$, where $H \cong [q^5] : \mathrm{GL}_2(q)$.*
- *Otherwise $m(G) = |\mathrm{G}_2(q) : H| = \frac{(q^6-1)(q+1)}{(q-1)}$, where H is of shape $[q^6] : (q-1)^2$.*

Proof. As before $|\mathrm{G}_2(q)| = q^6(q^6-1)(q^2-1)$, however by the ATLAS [9] we have that $\mathrm{Out}(\mathrm{G}_2(q)) = \langle\ddot{\gamma}\rangle$ for all q. Note that when $q = 3$ where $\ddot{\phi} = 1$;

Let $q = 3$, here we have that $|\mathrm{G}_2(3)| = 4245696$. Let $H \leq \mathrm{G}_2(3)$ be of shape $\mathrm{SU}_3(3) : 2$ as provided by [6, Table 8.42]. Then $|H| = 12096$ and thus $|\mathrm{G}_2(3) : H| = 351$. By [48, Theorem 1] $m(\mathrm{G}_2(3)) = 351$.

Let $q > 3$, and let $G \leq \mathrm{G}_2(q).\langle\ddot{\phi}\rangle$. Furthermore let $H \leq \mathrm{G}_2(q)$ be of shape $[q^5] : \mathrm{GL}_2(q)$ as provided by [6, Table 8.42]. Following the same argument as in Lemma 5.6.1 we have that $|G : H| = \frac{(q^6-1)}{(q-1)} = m(\mathrm{G}_2(q))$. By [6, Table 8.42] we know that the conjugacy class of H is stabilized by $\langle\ddot{\phi}\rangle$, and so $N_G(H)\mathrm{G}_2(q) = G$. Therefore by Lemma 2.3.18 we have our result.

Now let $G \not\leq \mathrm{G}_2(q).\langle\ddot{\phi}\rangle$. Consider a maximal subgroup $N_G(H) \leq G$ where H is a subgroup of $\mathrm{G}_2(q)$ of shape $[q^6] : (q-1)^2$ as provided by [6, Table 8.42]. Note here that the stabilizer of the $\mathrm{G}_2(q)$-conjugacy class of H is $\langle\ddot{\gamma}\rangle = \mathrm{Out}(\mathrm{G}_2(q))$.

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{G}_2(q)$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{G}_2(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{G}_2(q) = G$. The possibilities for K are found in the rows of [6, Table 8.42] for which the stabilizer is listed as $\langle\gamma\rangle$. Let us compare the order of these K with that of H.

If $K \cong (\mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)).2$ then

$$|H| = q^6(q-1)^2 > 2q^2(q-1)^2(q+1)^2 = |K| \qquad \text{since } q \geq 3.$$

If $K \cong (q-1)^2 : \mathrm{D}_{12}$, $K \cong (q+1)^2 : \mathrm{D}_{12}$, $K \cong (q^2+q+1).6$ or $K \cong (q^2-q+1).6$ then one can show that

$$|H| \geq 4q^6 \geq |K|.$$

In the case where $K \cong 2^3 \cdot \mathrm{PSL}_3(2)$, we note that

$$|H| \geq 2196 > 1344 = |K| \qquad \text{since } q \geq 3.$$

If $K \cong \mathrm{G}_2(q_0)$ then

$$\begin{aligned} |H| = q^6(q-1)^2 &> q^3(q^3-1)(q-1) && \text{since } q \geq 3 \\ &\geq q_0^6(q_0^6-1)(q_0^2-1) = |K| && \text{since } q \geq q_0^2. \end{aligned}$$

Finally if $K \cong {}^2\mathrm{G}_2(q)$ by Theorem 5.1.43 we have

$$|H| = q^6(q-1)^2 > q^3(q^4-q^3)(q-1) > q^3(q^3+1)(q-1) = |K| \qquad \text{since } q \geq 3.$$

We have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{G}_2(q)$ that satisfies $|K| \leq |H|$. Hence $|G : N_G(H)| = |\mathrm{G}_2(q) : H| \leq |\mathrm{G}_2(q) : K| = |G : M|$, by Lemma 2.3.17. Therefore $m(G) = |G : N_G(H)| = |\mathrm{G}_2(q) : H|$. □

5.6.2 $\mathbf{F}_4(q)$

Before we tackle the case where $G = \mathrm{F}_4(q)$ recall the definition of a parabolic subgroup P_J from Definition 5.1.11, where here $\Pi = \{p_1, p_2, p_3, p_4\}$ is the set of simple roots. Recall also that we may encode information of these roots in a Dynkin Diagram as seen in Table 5.1. There is a symmetry of this graph, which induces a graph automorphism γ of G in certain cases. Information on this automorphism can be found in Lemmas 5.1.20 and 5.1.21.

For this section, if $q = p^e$ where $e \geq 2$ let ϕ denote the field automorphism of $\mathrm{F}_4(q)$ corresponding to the Frobenius automorphism of $\mathbb{F}_q$, and also denote by $\ddot{\phi}$ the image of ϕ in $\mathrm{Out}(\mathrm{F}_4(q))$. If $q = p$ then define $\ddot{\phi}$ to be the identity element of $\mathrm{Out}(\mathrm{F}_4(q))$. If $\mathrm{F}_4(q)$ has a graph automorphism, which only occurs when $q = 2^e$, we denote it by γ and we denote its image in $\mathrm{Out}(\mathrm{F}_4(q))$ by $\ddot{\gamma}$.

Lemma 5.6.3. *Let $q \geq 3$ be a prime power, and let G be almost simple with socle* $\mathrm{F}_4(q)$.

- *If $G \leq \mathrm{F}_4(q).\langle\ddot{\phi}\rangle$ then $m(G) = |\mathrm{F}_4(q) : H| = m(\mathrm{F}_4(q)) = (q^{12}-1)(q^4-1)/(q-1)$, where H is a parabolic subgroup P_1 or P_4 of $\mathrm{F}_4(q)$ corresponding to the roots p_1 or p_4 respectively.*
- *Otherwise $m(G) = |\mathrm{F}_4(q) : H| = (q^{12}-1)(q^6-1)(q^4+1)/(q-1)^2$, where H is a parabolic subgroup $P_{\{2,3\}}$ of $\mathrm{F}_4(q)$. Note that here the underlying field has characteristic* 2.

Proof. By Theorem 5.1.44 the outer automorphism group $\mathrm{Out}(\mathrm{F}_4(q)) = \langle\ddot{\phi}\rangle$ if q is odd, and $\mathrm{Out}(\mathrm{F}_4(q)) = \langle\ddot{\gamma}, \ddot{\phi}\rangle$ if q is even. However, in the case where q is even, we have from the ATLAS [9] that $\ddot{\gamma}^2 = \ddot{\phi}$ so $\mathrm{Out}(\mathrm{F}_4(q)) = \langle\ddot{\gamma}\rangle$.

Let $G \leq \mathrm{F}_4(q).\langle\ddot{\phi}\rangle$ and let H be a parabolic subgroup P_1 or P_4 of G. By [48, Theorem 2] we have that $m(\mathrm{F}_4(q)) = |\mathrm{F}_4(q) : H|$. By Theorem 5.1.34 we know that the outer automorphism $\ddot{\phi}$ stabilizes the conjugacy class of H. Consequently we have that $N_G(H)\mathrm{F}_4(q) = G$. Therefore by Lemma 2.3.18 we have our result.

Let $G \not\leq \mathrm{F}_4(q).\langle\ddot{\phi}\rangle$. This is only possible if socle $\mathrm{F}_4(q)$ possesses a graph automorphism, and so $q = 2^e$. Let H be a parabolic subgroup $P_{\{2,3\}}$ of $\mathrm{F}_4(q)$. By Lemma 5.1.36 we see that γ fixes the conjugacy class of H in $\mathrm{F}_4(q)$ and so the whole of $\mathrm{Out}(\mathrm{F}_4(q))$ fixes the conjugacy class of H also. Consequently we have that $N_G(H)\mathrm{F}_4(q) = G$. We note that $N_G(H)$ is in fact a maximal subgroup of G, a fact that will follow from later calculations.

We compute the order of the parabolic subgroup $P_{\{2,3\}}$. In Lemma 5.1.13 $N = 24$, there is a single connected component I_1 for J and $\mathcal{L}_1 = \mathrm{B}_2$. Hence, using the information from Table 5.1, we conclude that

$$|H| = q^{24}(q-1)^2(q^2-1)(q^4-1).$$

Using the order of $\mathrm{F}_4(q)$ obtained from Theorem 5.1.43, we deduce that

$$|\mathrm{F}_4(q) : H| = \frac{(q^{12}-1)(q^6-1)(q^4+1)}{(q-1)^2}.$$

From the main theorem of [29] it follows that if M is a non-trivial maximal subgroup of G then either $|M \cap \mathrm{F}_4(q)| \leq q^{24}$ or $\mathrm{F}_4(q) \cap M$ is a parabolic subgroup of $\mathrm{F}_4(q)$ or $\mathrm{F}_4(q) \cap M$ is isomorphic to one of the following: $\mathrm{B}_4(q)$, $\mathrm{D}_4(q).\mathrm{S}_3$, ${}^3\mathrm{D}_4(q).3$, $\mathrm{F}_4(q^{1/2})$ or ${}^2\mathrm{F}_4(q)$.

Let M be a non-trivial maximal subgroup of G, and let $K = M \cap \mathrm{F}_4(q)$ such that K is not $\mathrm{Aut}(\mathrm{F}_4(q))$-conjugate to H. We will show that $|K| \leq |H|$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{F}_4(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{F}_4(q) = G$.

Suppose K is conjugate to the parabolic subgroup P_J for $J \subset \Pi$. We know by Lemma 5.1.34 that $\ddot{\phi}$ fixes the conjugacy class of K in $\mathrm{F}_4(q)$. Notice here that the image of G in $\mathrm{Out}(\mathrm{F}_4(q))$ contains an element of the form $\ddot{\phi}^n\ddot{\gamma}$ since $\ddot{\gamma}^2 = \ddot{\phi}$. Therefore we see that $\ddot{\gamma}$ also fixes the conjugacy class of K in $\mathrm{F}_4(q)$. By Lemma 5.1.36, for the graph automorphism to fix the conjugacy class of K we require that $J = \overline{J}$ under the map described in Lemma 5.1.20. Under this map p_1 is mapped to p_4 and vice versa, while p_2 is mapped to p_3 and vice versa. Therefore K is a parabolic subgroup conjugate to $P_{\{1,4\}}$ or $P_\emptyset$. However $P_\emptyset$ is a subgroup of H so $|P_\emptyset| \leq |H|$.

We compute the order of the parabolic subgroup $P_{\{1,4\}}$. In Lemma 5.1.13 $N = 24$, there are two connected component $I_1 = \{p_1\}$ and $I_2 = \{p_4\}$ for J and $\mathcal{L}_1 = \mathcal{L}_2 = \mathrm{A}_1$. Hence, using the information from Table 5.1, we conclude that

$$|K| = q^{24}(q-1)^2(q^2-1)^2 < q^{24}(q-1)^2(q^2-1)(q^4-1) = |H|.$$

Therefore if K is a parabolic subgroup of $\mathrm{F}_4(q)$ such that $N_G(K)\mathrm{F}_4(q) = G$, then $|K| \leq |H|$.

Suppose that K is not a parabolic subgroup, then as noted before $|K| \leq q^{24}$ or is isomorphic to one of the following: $\mathrm{B}_4(q)$, $\mathrm{D}_4(q).\mathrm{S}_3$, ${}^3\mathrm{D}_4(q).3$, $\mathrm{F}_4(q^{1/2})$ or ${}^2\mathrm{F}_4(q).3$.

If $|K| \leq q^{24}$ then we have $|K| < |H|$.

If $K \cong \mathrm{B}_4(q)$, $\mathrm{D}_4(q).\mathrm{S}_3$ or ${}^3\mathrm{D}_4(q).3$ then by [29, Prop 7.1-7.3] $N_G(K)\mathrm{F}_4(q) \neq G$. If $K \cong \mathrm{F}_4(q^{\frac{1}{2}})$ or $K \cong {}^2\mathrm{F}_4(q)$ then by Theorem 5.1.43 one can show that $|K| < |H|$.

Therefore, we have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{F}_4(q)$ that satisfies $|K| \leq |H|$. By Lemma 5.1.12 we have that $N_{\mathrm{F}_4(q)}(H) = H$, and so $|G : N_G(H)| = |\mathrm{F}_4(q) : H| \leq |\mathrm{F}_4(q) : K| = |G : M|$, by Lemma 2.3.17. We note that $N_G(H)$ is therefore not contained in any other non-trivial maximal subgroup of G, and combining this with Lemma 2.3.11, we have that $N_G(H)$ is actually a maximal subgroup of G. Therefore $m(G) = |G : N_G(H)| = |\mathrm{F}_4(q) : H|$. □

5.6.3 $\mathrm{E}_6(q)$

Before we tackle the case where $G = \mathrm{E}_6(q)$ recall the definition of a parabolic subgroup P_J from Definition 5.1.11, where here $\Pi = \{p_1, \ldots, p_6\}$ is the set of simple roots. Recall also that we may encode information of these roots in a Dynkin Diagram as seen in Table 5.1. There is a symmetry of this graph, which induces a graph automorphism γ of G, as introduced in Lemma 5.1.22.

From Theorem 5.1.44 the order of the subgroup of $\mathrm{Out}(\mathrm{E}_6(q))$ corresponding to the diagonal automorphisms of $\mathrm{E}_6(q)$ is $(3, q-1)$. In the case where this subgroup is not trivial define $\ddot{\delta}$ to be a generator of this subgroup, in the case where this subgroup is trivial define $\ddot{\delta}$ to be the identity element of $\mathrm{Out}(\mathrm{E}_6(q))$. If $q = p^e$ where $e \geq 2$ let ϕ denote the field automorphism of

$\mathrm{E}_6(q)$ corresponding to the Frobenius automorphism of $\mathbb{F}_q$, and also denote by $\ddot{\phi}$ the image of ϕ in $\mathrm{Out}(\mathrm{E}_6(q))$. If $q = p$ define $\ddot{\phi}$ to be the identity element of $\mathrm{Out}(\mathrm{E}_6(q))$. Finally let $\ddot{\gamma}$ be the image of the graph automorphism γ in $\mathrm{Out}(\mathrm{E}_6(q))$.

Lemma 5.6.4. *Let q be a prime power, and let G be almost simple with socle $\mathrm{E}_6(q)$.*

- *If $G \leq \mathrm{E}_6(q).\langle\ddot{\phi}, \ddot{\delta}\rangle$ then $m(G) = m(\mathrm{E}_6(q)) = |\mathrm{E}_6(q) : H| = \frac{(q^9-1)(q^8+q^4+1)}{q-1}$, where H is a parabolic subgroup P_1 or P_6 of $\mathrm{E}_6(q)$ corresponding to the roots p_1 or p_6 respectively.*
- *Otherwise $m(G) = |E_6(q) : H| = \frac{(q^9-1)(q^6+1)(q^4+1)(q^3+1)}{q-1}$, where H is a parabolic subgroup P_4 of $E_6(q)$.*

Proof. By Theorem 5.1.44 the outer automorphism group $\mathrm{Out}(\mathrm{E}_6(q)) = \langle\ddot{\delta}, \ddot{\phi}, \ddot{\gamma}\rangle$.

Let $G \leq \mathrm{E}_6(q).\langle\ddot{\phi}, \ddot{\delta}\rangle$, and let H be a parabolic subgroup P_1 or P_6 of G. By [49, Theorem 1] we have that $m(\mathrm{E}_6(q)) = |\mathrm{E}_6(q) : H| = (q^9 - 1)(q^8 + q^4 + 1)/(q - 1)$. By Theorem 5.1.34 we know that since H is parabolic that the outer automorphisms $\ddot{\phi}$ and $\ddot{\delta}$ fix the conjugacy class of H. Consequently $N_G(H)\mathrm{E}_6(q) = G$. Therefore by Lemma 2.3.18 we have our result.

Now let $G \not\leq \mathrm{E}_6(q).\langle\ddot{\phi}, \ddot{\delta}\rangle$. Let H be a parabolic subgroup $P_4 = P_{\{1,2,3,5,6\}}$ of G. By Theorem 5.1.34 once again we know that the outer automorphisms $\ddot{\phi}$ and $\ddot{\delta}$ fix the conjugacy class of H. The associated map of γ as defined in Lemma 5.1.22 maps $\{1, 2, 3, 5, 6\}$ to itself. Lemma 5.1.37 shows that $\ddot{\gamma}$ fixes the conjugacy class of H also. So the whole of $\mathrm{Out}(\mathrm{E}_6(q))$ fixes the conjugacy of H. Therefore we have that $N_G(H)\mathrm{E}_6(q) = G$. We note that $N_G(H)$ is in fact a maximal subgroup of G, a fact that will follow from later calculations.

We compute the order of the parabolic subgroup P_4. In Lemma 5.1.13 $N = 36$, there is a single connected component I_1 for J and $\mathcal{L}_1 = \mathrm{A}_5$. Hence, using the information from Table 5.1, we conclude that

$$|H| = \frac{1}{(3, q-1)} q^{36}(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1).$$

Using the order of $\mathrm{E}_6(q)$ obtained from Theorem 5.1.43, we deduce that

$$|\mathrm{E}_6(q) : H| = \frac{(q^9-1)(q^6+1)(q^4+1)(q^3+1)}{q-1}.$$

From the main theorem of [29] it follows that if M is a non-trivial maximal subgroup of G then either $|K| \leq q^{37}$, or $\mathrm{E}_6(q) \cap M$ is a parabolic subgroup of $\mathrm{E}_6(q)$ or $\mathrm{E}_6(q) \cap M$ is isomorphic to one of the following $\mathrm{F}_4(q)$, $(\mathrm{SL}_2(q) \circ \mathrm{A}_5(q)).d$, $\mathrm{E}_6(q^{1/2})$, ${}^2\mathrm{E}_6(q^{1/2})$ or $(\mathrm{D}_5(q) \circ (q-1)/e_{+1}).f_{+1}$, where d, e_{+1} and f_{+1} are constants defined in [29, Table 1].

Let M be a non-trivial maximal subgroup of G and let $K = M \cap \mathrm{E}_6(q)$ such that K is not $\mathrm{Aut}(\mathrm{E}_6(q))$-conjugate to H. We will show that $|K| \leq |H|$. By Lemmas 2.3.6 and 2.3.7 we have that $K \neq \mathrm{E}_6(q)$, and $N_G(K) = M$, and $N_G(K)\mathrm{E}_6(q) = G$.

Suppose K is conjugate to the parabolic subgroup P_J for $J \subset \Pi$. Note that by assumption we have that K is not conjugate to $P_{\{1,2,3,5,6\}}$. We know by Lemma 5.1.34 that $\ddot{\phi}$ and $\ddot{\delta}$ fix the conjugacy class of K in $\mathrm{E}_6(q)$. We notice that the image of G in $\mathrm{Out}(\mathrm{E}_6(q))$ contains an element of the form $\ddot{\delta}^{e_1}\ddot{\phi}^{e_2}\ddot{\gamma}$. Therefore we see that $\ddot{\gamma}$ also fixes the conjugacy class of K in $\mathrm{E}_6(q)$. By Lemma 5.1.37, for the graph automorphism to fix the conjugacy class of K we require that $J = \overline{J}$ under the map described in Lemma 5.1.22. Under this map p_1 is mapped to p_6 and vice versa, while p_2 is mapped to p_5 and vice versa, while p_3 and p_4 are fixed. We note that if $J_1 \subset J_2 \subset \Pi$, then P_{J_1} is contained in P_{J_2} and so $|P_{J_1}| \leq |P_{J_2}|$. $|K| \leq |P_{\{1,2,4,5,6\}}|$, or $|K| \leq |P_{\{1,3,4,6\}}|$ or $|K| \leq |P_{\{2,3,4,5\}}|$.

We compute the order of the parabolic subgroups $P_{\{1,2,4,5,6\}}$, $P_{\{1,3,4,6\}}$ and $P_{\{2,3,4,5\}}$ in turn using Lemma 5.1.13.

Suppose K is a $P_{\{1,2,4,5,6\}}$ subgroup. Then in Lemma 5.1.13 $N = 36$, there are three connected components $I_1 = \{1,2\}$, $I_2 = \{4\}$ and $I_3 = \{5,6\}$ for J and $\mathcal{L}_1 = \mathcal{L}_3 = \mathrm{A}_2$, and $\mathcal{L}_2 = \mathrm{A}_1$. Hence, using the information from Table 5.1, we conclude that

$$\begin{aligned}|K| &= \frac{1}{(3,q-1)}q^{36}(q-1)(q^2-1)^3(q^3-1)^2 \\ &< \frac{1}{(3,q-1)}q^{36}(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1) = |H|.\end{aligned}$$

Suppose K is a $P_{\{1,3,4,6\}}$ subgroup. Then in Lemma 5.1.13 $N = 36$, there are three connected components $I_1 = \{1\}$, $I_2 = \{3,4\}$ and $I_3 = \{6\}$ for J and $\mathcal{L}_1 = \mathcal{L}_3 = \mathrm{A}_1$, $\mathcal{L}_2 = \mathrm{A}_2$. Hence, using the information from Table 5.1, we conclude that

$$|K| = \frac{1}{(3,q-1)}q^{36}(q-1)^2(q^2-1)^3(q^3-1) < |H|.$$

Suppose K is a $P_{\{2,3,4,5\}}$ subgroup. Then in Lemma 5.1.13 $N = 36$, there is one connected component I_1 for J and $\mathcal{L}_1 = \mathrm{D}_4$. Hence, using the information from Table 5.1, we conclude that

$$|K| = \frac{1}{(3,q-1)}q^{36}(q-1)^2(q^2-1)(q^4-1)^2(q^6-1) < |H|.$$

Therefore if K is a parabolic subgroup not $\mathrm{Aut}(\mathrm{E}_6(q))$-conjugate to H of $\mathrm{E}_6(q)$ such that $N_G(K)\mathrm{E}_6(q) = G$, then $|K| < |H|$.

Suppose that K is not a parabolic subgroup, then as noted before $|K| \leq q^{37}$ or is isomorphic to one of the following $\mathrm{F}_4(q)$, $(\mathrm{SL}_2(q) \circ \mathrm{A}_5(q)).d$, $\mathrm{E}_6(q^{1/2})$, ${}^2\mathrm{E}_6(q^{1/2})$ or $(\mathrm{D}_5(q) \circ (q-1)/e_{+1}).f_{+1}$, where d, e_{+1} and f_{+1} are constants defined in [29, Table]. We go through these cases in turn.

If $|K| \leq q^{37}$ then $|K| < |H|$.

If $K \cong \mathrm{F}_4(q)$ then

$$\begin{aligned}|K| &= q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1) < q^{52} && \text{from Theorem 5.1.43} \\ &< \frac{q^{36}(q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1)(q^6-1)}{(3,q-1)} = |H| && \text{By Lemma 2.6.1 or Lemma 2.6.2.}\end{aligned}$$

If $K \cong (\mathrm{SL}_2(q) \circ \mathrm{A}_5(q)).d$, one can show by the fact that $\mathrm{A}_5(q) \cong \mathrm{PSL}_6(q)$ and applying Lemma 2.6.4 that $|K| < |H|$.

If $K \cong \mathrm{E}_6(q^{\frac{1}{2}})$, or $K \cong {}^2\mathrm{E}_6(q^{\frac{1}{2}})$ then by Theorem 5.1.43 we have that $|K| < |H|$.

Finally if $K \cong (\mathrm{D}_5(q) \circ (q-1)/e_{+1}).f_{+1}$, where e_{+1} and f_{+1} are defined in [29, Table 1], then by the fact that $\mathrm{D}_5(q) \cong \mathrm{P}\Omega_{10}^{+}(q)$ and applying Lemma 2.6.4 we may also show that $|K| < |H|$.

Therefore, we have shown that if M is any non-trivial maximal subgroup of G then $M = N_G(K)$ for a subgroup $K \leq \mathrm{E}_6(q)$ that satisfies $|K| \leq |H|$. By Lemma 5.1.12 we have that $N_S(H) = H$, and so $|G : N_G(H)| = |\mathrm{E}_6(q) : H| \leq |\mathrm{E}_6(q) : K| = |G : M|$, by Lemma 2.3.17. We note that $N_G(H)$ is therefore not contained in any other non-trivial maximal subgroup of G, and combining this with Lemma 2.3.11, we have that $N_G(H)$ is actually a maximal subgroup of G. Therefore $m(G) = |G : N_G(H)| = |\mathrm{E}_6(q) : H|$. □

5.6.4 $\mathbf{E}_7(q)$ and $\mathbf{E}_8(q)$

Lemma 5.6.5. *Let q be a prime power and let G be almost simple with socle $\mathrm{E}_7(q)$. Then $m(G) = m(\mathrm{E}_7(q)) = |\mathrm{E}_7(q) : H| = \frac{(q^{14}-1)(q^9+1)(q^5+1)}{q-1}$, where H is a parabolic subgroup P_1.*

Proof. Let $H \leq \mathrm{E}_7(q)$ be a parabolic subgroup P_1. [49, Section 2B] shows that H is a subgroup of least index in $\mathrm{E}_7(q)$. By [49, Theorem 2]

$$|\mathrm{E}_7(q) : H| = \frac{(q^{14}-1)(q^9+1)(q^5+1)}{q-1} = m(\mathrm{E}_7(q)).$$

By Theorem 5.1.44 the only outer automorphisms of $\mathrm{E}_7(q)$ are diagonal and field type. Therefore, by Theorem 5.1.34 we know that $\mathrm{Out}(\mathrm{E}_7(q))$ preserves the conjugacy class of H, and therefore $N_G(H)\mathrm{E}_7(q) = G$. Therefore by Lemma 2.3.18 we have our result. □

Lemma 5.6.6. *Let q be a prime power and let G be almost simple with socle $\mathrm{E}_8(q)$. Then $m(G) = m(\mathrm{E}_8(q)) = |\mathrm{E}_8(q) : H| = \frac{(q^{30}-1)(q^{12}+1)(q^{10}+1)(q^6+1)}{q-1}$, where H is a parabolic subgroup P_1.*

Proof. Let $H \leq \mathrm{E}_8(q)$ be a parabolic subgroup P_1. [49, Section 3B] shows that H is a subgroup of least index in $\mathrm{E}_8(q)$. By [49, Theorem 3]

$$|\mathrm{E}_8(q) : H| = \frac{(q^{30}-1)(q^{12}+1)(q^{10}+1)(q^6+1)}{q-1} = m(\mathrm{E}_8(q)).$$

By Theorem 5.1.44 the only outer automorphisms of $\mathrm{E}_8(q)$ are field type. Therefore, by Theorem 5.1.34 we know that $\mathrm{Out}(\mathrm{E}_7(q))$ preserves the conjugacy class of H, and so $N_G(H)\mathrm{E}_8(q) = G$. Therefore by Lemma 2.3.18 we have our result. □

5.6.5 Twisted groups of Lie type

In this section we will prove Theorem 5.2.2 for the cases where S a simple twisted group of Lie type.

Lemma 5.6.7. *Let $q = p^{2n}$ be a prime power, and let G be almost simple with socle ${}^2\mathrm{E}_6(q)$. Then $m(G) = m({}^2\mathrm{E}_6(q)) = |{}^2\mathrm{E}_6(q) : H| = \frac{(q^{12}-1)(q^6-q^3+1)(q^4+1)}{q-1}$ where*

$$H \cong [q^{21}] : (d_+ \cdot {}^2\mathrm{A}_5(q) \times (q-1)/d'_+) \cdot d'_+,$$

and $d_+ = (2, q+1)$ and $d'_+ = (3, q+1)$.

Proof. Let $H \leq {}^2\mathrm{E}_6(q)$ be the parabolic subgroup of shape $[q^{21}] : (d_+ \cdot {}^2\mathrm{A}_5(q) \times (q-1)/d'_+) \cdot d'_+$, denoted by P^1 in [50, Section 4B]. By [50, Theorem 4] we have that

$$|{}^2\mathrm{E}_6(q) : H| = \frac{(q^{12}-1)(q^6-q^3+1)(q^4+1)}{q-1} = m({}^2\mathrm{E}_6(q)).$$

By Theorem 5.1.44 the only outer automorphisms of ${}^2\mathrm{E}_6(q)$ are diagonal and field type. Therefore, by Theorem 5.1.40 we know that $\mathrm{Out}({}^2\mathrm{E}_6(q))$ preserves the conjugacy class of H, and so $N_G(H){}^2\mathrm{E}_6(q) = G$. Therefore by Lemma 2.3.18 $m(G) = m({}^2\mathrm{E}_6(q))$ and we have our result. □

Lemma 5.6.8. *Let $q = p^{3n}$ be a prime power, and let G be almost simple with socle ${}^3\mathrm{D}_4(q)$. Then $m(G) = m({}^3\mathrm{D}_4(q)) = |{}^3\mathrm{D}_4(q) : H| = (q^8 + q^4 + 1)(q + 1)$, where $H \cong [q^9] : (d \cdot (\mathrm{A}_1(q^3) \times (q-1)/d)) \cdot d$, and $d = (2, q-1)$.*

Proof. Let $H \leq {}^3\mathrm{D}_4(q)$ be the parabolic subgroup of shape $[q^9] : (d \cdot (\mathrm{A}_1(q^3) \times (q-1)/d)) \cdot d$, denoted by P_2^1 in [50, Section 3B]. By [50, Theorem 3] we have

$$|{}^3\mathrm{D}_4(q) : H| = (q^8 + q^4 + 1)(q + 1) = m({}^3\mathrm{D}_4(q)).$$

By Theorem 5.1.44 the only outer automorphisms of ${}^3\mathrm{D}_4(q)$ are field type. Therefore, by Theorem 5.1.40 we know that $\mathrm{Out}({}^3\mathrm{D}_4(q))$ preserves the conjugacy class of H in ${}^3\mathrm{D}_4(q)$, and so $N_G(H){}^3\mathrm{D}_4(q) = G$. Therefore by Lemma 2.3.18 $m(G) = m({}^3\mathrm{D}_4(q))$ and we have our result. □

Lemma 5.6.9. *Let $q = 2^{2n+1}$ for $n \geq 1$, and let G be almost simple with socle ${}^2\mathrm{B}_2(q)$. Then $m(G) = m({}^2\mathrm{B}_2(q)) = |{}^2\mathrm{B}_2(q) : H| = q^2 + 1$, where $H \cong [q^2] : (q-1)$.*

Proof. Let $H \leq {}^2\mathrm{B}_2(q)$ be the parabolic subgroup of shape $[q^2] : (q-1)$ as provided by [50, Section 2 & Theorem 1]. [50, Theorem 1] shows that

$$|{}^2\mathrm{B}_2(q) : H| = (q^2 + 1) = m({}^2\mathrm{B}_2(q)).$$

By Theorem 5.1.44 the only outer automorphisms of ${}^2\mathrm{B}_2(q)$ are field type. Therefore, by Theorem 5.1.40 we know that $\mathrm{Out}({}^2\mathrm{B}_2(q))$ preserves the conjugacy class of H in ${}^2\mathrm{B}_2(q)$ and therefore $N_G(H){}^2\mathrm{B}_2(q) = G$. Therefore by Lemma 2.3.18 $m(G) = m({}^2\mathrm{B}_2(q))$ and we have our result. □

Lemma 5.6.10. *Let $q = 2^{2n+1}$ for $n \geq 1$, and let G be almost simple with socle ${}^2\mathrm{F}_4(q)$. Then $m(G) = m({}^2\mathrm{F}_4(q)) = |{}^2\mathrm{F}_4(q) : H| = (q^6+1)(q^3+1)(q+1)$, where $H \cong [q^{10}] : ({}^2\mathrm{B}_2(q) \times (q-1))$.*

Proof. Let $H \leq {}^2\mathrm{F}_4(q)$ be the parabolic subgroup of shape $[q^{10}] : ({}^2\mathrm{B}_2(q) \times (q-1))$, denoted by P_1 in [50, Section 5B]. [50, Theorem 5] shows that

$$|{}^2\mathrm{F}_4(q) : H| = (q^6 + 1)(q^3 + 1)(q + 1) = m({}^2\mathrm{F}_4(q)).$$

By Theorem 5.1.44 the only outer automorphisms of ${}^2\mathrm{F}_4(q)$ are field type. Therefore, by Theorem 5.1.40 we know that $\mathrm{Out}({}^2\mathrm{F}_4(q))$ preserves the conjugacy class of H in ${}^2\mathrm{F}_4(q)$ and therefore $N_G(H){}^2\mathrm{F}_4(q) = G$. Therefore by Lemma 2.3.18 $m(G) = m({}^2\mathrm{F}_4(q))$ and we have our result. □

In the case of $^2\mathrm{F}_4(2)'$ we obtain the following result from the Malle [36].

Lemma 5.6.11. *Let G be almost simple with socle $^2\mathrm{F}_4(2)'$.*

- *if $G = {}^2\mathrm{F}_4(2)'$ then $m(G) = m({}^2\mathrm{F}_4(2)') = |{}^2\mathrm{F}_4(2)' : H| = 1600$ where $H \cong \mathrm{PSL}_3(3) : 2$.*
- *if $G = {}^2\mathrm{F}_4(2)'.2$ then $m(G) = |G : H| = 1755$ where $H \cong 2.[2^9].5.4$.*

Lemma 5.6.12. *Let $q = 3^{2n+1}$ for $n \geq 1$ and let G be almost simple with socle $^2\mathrm{G}_2(q)$. Then $m(G) = m({}^2\mathrm{G}_2(q)) = |{}^2\mathrm{G}_2(q) : H| = q^3 + 1$, where $H \cong [q^3] : (q-1)$.*

Proof. Let $H \leq {}^2\mathrm{G}_2(q)$ be the parabolic subgroup of shape $[q^3] : (q-1)$ as provided by by [50, Section 2 & Theorem 2]. By [50, Theorem 2]

$$|{}^2\mathrm{G}_2(q) : H| = q^3 + 1 = m({}^2\mathrm{G}_2(q)).$$

By Theorem 5.1.44 the only outer automorphisms of $^2\mathrm{G}_2(q)$ are field type. Therefore, by Theorem 5.1.40 we know that $\mathrm{Out}({}^2\mathrm{G}_2(q))$ preserves the conjugacy class of H in $^2\mathrm{G}_2(q)$. Therefore $N_G(H){}^2\mathrm{G}_2(q) = G$ and by Lemma 2.3.18 $m(G) = m({}^2\mathrm{G}_2(q))$ and we have our result. $\square$

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