Thomas Randriamahazaka

De Morgan-Płonka Sums

Abstract. This paper develops De Morgan-Płonka sums, which generalise Płonka sums to contexts in which negation is not topically transparent but still respects De Morgan duality. We give a general theory of De Morgan-Płonka sums, on the model of the general theory of Płonka sums. Additionally, we describe free De Morgan-Płonka sums and apply our construction to give an algebraic proof of completeness for Kit Fine's truthmaker semantics for Angell's logic of analytic containment.

Keywords: Płonka sums, De Morgan duality, Nonclassical logics, Algebraic logic, Truthmaker semantics.

Introduction

Most formal semantics rely on the idea that two formulas express the same proposition if they are true in the same cases (e.g., possible worlds). However, it has recently been argued that truth-conditional equivalence is necessary but not sufficient for propositional identity, since it does not guarantee sameness of subject-matter [2,19]. One can then distinguish between thin propositions—individuated only through truth-conditions—and thick propositions—individuated through both truth-conditions and subject-matter [3]. There is no consensus as to how to model the subject-matter of a thick proposition, but most agree that whatever is to play this role—let's call them topics—must have a mereological structure, i.e., they form a semilattice [12]. For instance, it is broadly accepted that the topic of a conjunction is the mereological fusion of the topics of each conjuncts, and the same goes with disjunction. As such, conjunction and disjunction are said to be topically transparent.

Interestingly, there is less agreement regarding negation [13,16]. The tradition of containment logics—following the seminal work of [14] and linked to the Weak Kleene logics— take negation to be topically transparent, i.e., a proposition and its negation have the same subject-matter. However, an alternative tradition—following the work of [1] on analytic containment—rejects the topical transparency of negation. Consequently, the semilattice

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of topics must be endowed with a further operation which lifts at the level of topics what the operation of negation does at the level of propositions. Under minimal assumptions, the space of topics is then an involutive semilattice.

The disagreement on the topical status of negation gives rise to a disagreement on the correct syntactic approximation of sameness of subject-matter. Parry's followers treat two uninterpreted formulas φ and ψ as topically equivalent if the same propositional letters occur in φ and ψ. Let's call that the Parry Condition. It tracks the idea that all logical connectives, and therefore negation, do not contribute to the topic of a proposition. By contrast, what we can call the Angell Condition demands more, namely that every propositional letter occurs in φ under the scope of an even (respectively odd) number of negations if and only if it occurs in ψ under the scope of an even (respectively odd) number of negation. This corresponds to the idea that negation does contribute to subject-matter, though double negation does not.

Theories of propositions which follow Parry in taking negation to be transparent are linked to the algebraic theory of Płonka sums [4–6,15]. The Płonka sum construction allows one to "glue" together several algebras following the pattern given by a semilattice. This allows us to understand the algebra of thick propositions (under a theory that follows Parry) as the Płonka sum of algebras of thin propositions over the semilattice of topics. This provides an algebraic understanding to the idea that thick propositions are obtained from thin propositions once subject-matter is taken into account—under the assumption that topics are to be modelled by a semilattice. In addition, the link between Płonka sums and theories of propositions following Parry is particularly useful from a model-theoretic perspective because we know how to build free algebras from Płonka sums [17] and these free algebras can be used to easily prove completeness results.

Moreover, Płonka sums are closely linked to the Parry Condition. Under minimal assumptions, a Płonka sum only satisfies regular equations, i.e., equations that satisfies the Parry Condition. Moreover, let $\mathfrak{V}$ be an algebraic variety and suppose that it is strongly irregular, i.e., that it satisfies an equation of the form $p(x, y) = x$. Then, the regularisation of $\mathfrak{V}$, namely the class of algebras satisfying the regular equations of satisfied in $\mathfrak{V}$, corresponds—up to isomorphism—to the Płonka sums of members of $\mathfrak{V}$.

The goal of this talk is to develop an analog to Płonka sums for theories of propositions following Angell. The motivation is to be able to understand the algebra of thick propositions as a sum of algebras of thin propositions over the involutive semilattice of topics. The kind of sum we are looking for can no longer treat negation as the other logical connectives but must integrate it in the way algebras are glued together.

Some steps towards such an algebraic tool have been taken in the literature. In particular, [9] have developed involutorial Płonka sums, where algebras are glued over an involutorial semilattice. Unfortunately, their construction is not applicable to logical contexts because the assumptions they put on negation are too strong. The requirement that negation commutes with all other logical connectives, more precisely, goes against the accepted idea that negation does not commute with conjunction ($\neg(A \wedge B)$ is not the same as $\neg A \wedge \neg B$) or even disjunction.

To overcome the shortcoming of Dolinka and Vinčić's approach, we develop a more general construction—De Morgan-Płonka sums—which uses a weaker requirement on negation, inspired by De Morgan duality between conjunction, disjunction and negation. We first develop a general theory of De Morgan duality and characterise the class of varieties which fall under it—that we call the symmetric varieties. This allows us to define a general procedure to add a De Morgan negation to a symmetric variety (whose type does not necessarily already contain a negation symbol), thus obtaining its De Morganification. This generalises the link between distributive lattices and De Morgan lattices, which provide a natural algebraic understanding of the negation of many non-classical logics. We then define our notion of De Morgan-Płonka sums of involutive semilattice systems of algebras. Just like Płonka sums satisfy regular equations, De Morgan-Płonka sums satisfy Angellic equations, i.e., equations which satisfy the Angell Condition. This motivates us to define the Angellicisation of a variety $\mathfrak{V}$, namely the class of algebras satisfying the Angellic equations satisfied in $\mathfrak{V}$. Our main theorem is that the Angellicisation of the De Morganification of a strongly irregular symmetric variety $\mathfrak{V}$ corresponds—up to isomorphism—to the De Morgan-Płonka sums of members of $\mathfrak{V}$. Interestingly, this class can also be described as the De Morganification of the regularisation of $\mathfrak{V}$.

To demonstrate the usefulness of our construction, we provide a general procedure to build free De Morgan-Płonka sums. This allows us to characterise the free algebras in Angellicisation of De Morganification of strongly irregular symmetric varieties. We apply our result to the Angellicisation of the variety of De Morgan lattices—which we call Angellic algebras –, allowing us to give a purely algebraic completeness proof for [10]'s truthmaker semantics for Angell's logic of analytic containment.

1. Preliminaries

1.1. Universal Algebra

We introduce the basic concepts of universal algebra, mostly to fix the notation. Overall, we direct the reader towards classic textbooks like [7].

A plural type is composed of a set F of function symbols such that a positive integer n has been assigned to each member f of F and where at least one member of F is assigned a number above 1. The integer n is called the arity of f and we call f an n-ary function symbol. Since we do not allow n to be 0, we do not consider in this paper types with constant symbols. For the rest of the paper, we fix a plural type F.

An algebra of type F is a pair $\mathcal{A} = \langle A, (.)^{\mathcal{A}} \rangle$ where A is a non-empty set and, for all n-ary $f \in F$, we have $f^{\mathcal{A}} : A^n \to A$. The set A is called the domain of $\mathcal{A}$ and the function $f^{\mathcal{A}}$ is called the interpretation of f in $\mathcal{A}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras of type F. A homomorphism $h : \mathcal{A} \to \mathcal{B}$ of type F is a function $h : A \to B$ such that, for all n-ary $f \in F$ and for all $a_1, ..., a_n \in A$, we have $h(f^{\mathcal{A}}(a_1, ..., a_n)) = f^{\mathcal{B}}(h(a_1), ..., h(a_n))$.

Let Var be a countably infinite set of variables. The set T_F of terms of type F (over Var) is defined recursively as follows:

- If $x \in Var$, then $x \in T_F$,
- If $f \in F$ is n-ary and $t_1, ..., t_n \in T_F$ then $f(t_1, ..., t_n) \in T_F$.

Let $\mathcal{T}_F = \langle T_F, (.)^{\mathcal{T}_F} \rangle$ where, for an n-ary $f \in F$ and $t_1, ..., t_n \in T_F$, we have $f^{\mathcal{T}_F}(t_1, ..., t_n) = f(t_1, ..., t_n)$. The algebra $\mathcal{T}_F$ is called the algebra of terms of type F (over Var). Unless specified, we drop the reference to the type F and talk of algebras, homomorphisms, etc.

Let $\mathcal{A}$ be an algebra. A valuation is a homomorphism $v : \mathcal{T}_F \to \mathcal{A}$. An equation is an expression of the form $t_1 \approx t_2$ where $t_1, t_2 \in T_F$. We say that the algebra $\mathcal{A}$ satisfies the equation $t_1 \approx t_2$ if $v(t_1) = v(t_2)$ for all valuations v.

Any set E of equations forms an equational theory. We define $Mod(E)$ as the class of algebras which satisfy all the members of E. A class K of algebra is called an equational class if there exists an equational theory E such that $Mod(E) = K$. If K is a class of algebras, we call $Th(K)$ the set of equations satisfied by all the members of K. An equational theory E entails an equation if it is a member of $Th(Mod(E))$, i.e., if there is no algebra satisfying all members of E without satisfying that equation. Since Birkhoff, we know that equational classes correspond to varieties, i.e., classes of algebras closed under subalgebra, homomorphic image and product.

Let S be the type composed of a unique binary function symbol $\sqcup$. Let $\mathfrak{Sem}$ be the variety of type S defined by the following equational theory:

- $x \sqcup x \approx x$ (Idempotence)
- $x \sqcup y \approx y \sqcup x$ (Commutativity)
- $x \sqcup (y \sqcup z) \approx (x \sqcup y) \sqcup z$ (Associativity)

Members of $\mathfrak{Sem}$ are called semilattices. Where $\mathcal{I}$ is a semilattice, we usually write $\langle I, \sqcup^{\mathcal{I}} \rangle$ instead of $\langle I, (.)^{\mathcal{I}} \rangle$ and, for $i, j \in I$, we usually write $i \sqcup^{\mathcal{I}} j$ instead of $\sqcup^{\mathcal{I}}(i, j)$. For $i, j \in I$, we write $i \sqsubseteq^{\mathcal{I}} j$ if $i \sqcup^{\mathcal{I}} j = j$. The binary relation $\sqsubseteq^{\mathcal{I}}$ is a partial order and $i \sqcup^{\mathcal{I}} j$ corresponds to the least upper bound of x and y w.r.t. that order. A semilattice is said to be complete if every subset of its domain has a least upper bound.

1.2. Płonka Sums and Regular Varieties

We base our exposition on [5]. Further references can be found there.

Let F be a plural type and let $\mathfrak{V}$ be variety of type F. A semilattice system of members of $\mathfrak{V}$ consists of:

- A semilattice $\mathcal{I} = \langle I, \sqcup^{\mathcal{I}} \rangle$,
- For all $i \in I$, an algebra $\mathcal{A}_i$ of $\mathfrak{V}$,
- For all $i, j \in I$ such that $i \sqsubseteq^{\mathcal{I}} j$, a homomorphism $p_i^j : \mathcal{A}_i \to \mathcal{A}_j$ such that p_i^i is the identity for all $i \in J$ and $p_j^k \circ p_i^j = p_i^k$ for all $i, j, k \in I$ such that $i \sqsubseteq^{\mathcal{I}} j \sqsubseteq^{\mathcal{I}} k$.

Let $\mathcal{X}$ be such a semilattice system. The Płonka sum of $\mathcal{X}$, written $\mathbb{P}(\mathcal{X})$ is the algebra whose domain is $\bigcup_{i \in I} A_i \times \{i\}$ and such that, for an n-ary $f \in F$, we have:

$$f^{\mathbb{P}(\mathcal{X})} : \langle \langle a_1, i_1 \rangle, ..., \langle a_n, i_n \rangle \rangle \mapsto \langle f^{\mathcal{A}_i}(p_{i_1}^i(a_1), ..., p_{i_n}^i(a_n)), i \rangle$$

where $i = i_1 \sqcup^{\mathcal{I}} ... \sqcup^{\mathcal{I}} i_n$. So, the interpretation of f in $\mathbb{P}(\mathcal{X})$ just uses the functions p_i^j to move all of its arguments in a single algebra and then uses the interpretation of f in that algebra. The Płonka sum operation $\mathbb{P}$ can be used to glue algebras over a semilattice.

Conversely, some algebras can be decomposed into a semilattice system. A capital notion here is that of partition function. Let $\mathcal{A}$ be an algebra. A partition function on $\mathcal{A}$ is a function $\cdot : A^2 \to A$ such that, for all n-ary $f \in F$ and $a, b, c, a_1, ..., a_n \in A$:

- $a \cdot a = a$,
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
- $a \cdot (b \cdot c) = a \cdot (c \cdot b)$,
- $f^{\mathcal{A}}(a_1, ..., a_n) \cdot b = f^{\mathcal{A}}(a_1 \cdot b, ..., a_n \cdot b)$,
- $b \cdot f^{\mathcal{A}}(a_1, ..., a_n) = b \cdot a_1 \cdot ... \cdot a_n$.

Let $\preceq$ be the binary relation on A defined by $a \preceq b$ if $b \cdot a = b$. Let $\sim$ be the binary relation on A defined by $a \sim b$ if $a \preceq b$ and $b \preceq a$. Because of the conditions put on $\cdot$, we get that $\sim$ is a congruence on $\mathcal{A}$. We define the algebra $\mathcal{I}_{\mathcal{A}} = \langle A/{\sim}, \sqcup^{\mathcal{I}_{\mathcal{A}}} \rangle$ of type S where, for $[a]_\sim, [b]_\sim \in A/{\sim}$, we have $[a]_\sim \sqcup^{\mathcal{I}_{\mathcal{A}}} [b]_\sim = [f^{\mathcal{A}}(a, b, ..., b)]/\sim$ where f is an n-ary member of F such that $n > 1$. For all $a, b \in A$, we have that $[a]_\sim \sqsubseteq^{\mathcal{I}_{\mathcal{A}}} [b]_\sim$ if and only if $a \preceq b$. Moreover, each equivalence class $[a]_\sim$ forms a subalgebra $\mathcal{A}_{[a]_\sim}$ of $\mathcal{A}$.

Let $\mathbb{D}.(\mathcal{A})$ be the semilattice system consisting of:

- The semilattice $\mathcal{I}_{\mathcal{A}}$,
- For all $[a]_\sim \in A/{\sim}$, the algebra $\mathcal{A}_{[a]_\sim}$,
- For all $a \preceq b$ in A, the homomorphism $p^{[b]_\sim}_{[a]_\sim} : x \mapsto x \cdot b$.

We call it the Płonka decomposition of $\mathcal{A}$ relative to $\cdot$.

Theorem 1. *([5]) Let $\mathcal{A}$ be an algebra and let $\cdot : A^2 \to A$ be a partition function on $\mathcal{A}$. Then, the Płonka sum of the Płonka decomposition of $\mathcal{A}$ relative to $\cdot$ is isomorphic to $\mathcal{A}$, i.e., $\mathbb{P}(\mathbb{D}.(\mathcal{A})) \cong \mathcal{A}$.*

The theory of Płonka sums is linked to that of regular equations. We define the variables $Var(t)$ of a term t recursively as follows:

- If $x \in Var$, then $Var(x) = \{x\}$,
- If $f \in F$ is n-ary and $t_1, ..., t_n \in T_F$ then $Var(f(t_1, ..., t_n)) = Var(t_1) \cup ... \cup Var(t_n)$.

An equation $t_1 \approx t_2$ is said to be regular if $Var(t_1) = Var(t_2)$.

Theorem 2. *([5]) The Płonka sum of a semilattice system satisfies all the regular equations satisfied in all algebras of the semilattice system.*

Theorem 3. *([5]) If an equation is satisfied by the Płonka sum of a semilattice system, then it is satisfied by all the algebras contained in the semilattice systems.*

Let $\mathbf{2}_s$ be the semilattice $\langle \{0, 1\}, \sqcup^{\mathbf{2}_s} \rangle$ where $0 \sqcup^{\mathbf{2}_s} 1 = 1$.

Theorem 4. *([5]) If $\mathbf{2}_s$ is a subalgebra of a semilattice $\mathcal{I}$, then the Płonka sum of any semilattice system based on $\mathcal{I}$ satisfies only regular equations.*

A variety $\mathfrak{V}$ is called regular if there is a set E of regular equations such that $\mathfrak{V} = Mod(E)$. A variety $\mathfrak{V}$ is called strongly irregular if $Th(\mathfrak{V})$ contains an equation of the form $t \approx x$ where $Var(t) = \{x, y\}$ for distinct $x, y \in Var$. To express this condition on t, we usually write $t(x, y)$ instead of t. Interestingly, strongly irregular varieties are always of the form $Mod(E \cup \{t(x, y) \approx x\})$ where E is a set of regular equations. If $\mathfrak{V}$ is a variety, we call $R(\mathfrak{V})$ its regularisation, namely the variety axiomatised by the regular equations of $Th(\mathfrak{V})$.

Theorem 5. *([5]) Let $\mathfrak{V}$ be a strongly irregular variety. The regularisation $R(\mathfrak{V})$ of $\mathfrak{V}$ is composed, up to isomorphism, of Płonka sums of semilattice systems of members of $\mathfrak{V}$.*

2. De Morgan-Płonka Sums

2.1. De Morgan Duality and Symmetric Variety

A dualised type is a pair $\langle F, d\rangle$ where F is a plural type and $d : F \to F$ is such that $d(d(f)) = f$ and $d(f)$ is n-ary for all n-ary $f \in F$.

If $\mathcal{A} = \langle A, (.)^{\mathcal{A}}\rangle$ is an algebra of type F, we define the d-symmetry of $\mathcal{A}$ by $\mathcal{A}^d = \langle A, (.)^{\mathcal{A}^d}\rangle$ where $f^{\mathcal{A}^d} = d(f)^{\mathcal{A}}$ for all $f \in F$.

We define the d-translation t^d of a term t of type F (w.r.t. d) as follows:

- If $x \in Var$, then $x^d = x$,
- If $f \in F$ is n-ary and $t_1, ..., t_n \in T_F$, then $f(t_1, ..., t_n)^d = d(f)(t_1^d, ..., t_n^d)$.

One easily checks the following proposition.

Proposition 1. *An algebra $\mathcal{A}$ of type F satisfies an equation $t_1 \approx t_2$ if and only it $\mathcal{A}^d$ satisfies $t_1^d \approx t_2^d$.*

We say that an equational theory E of type F is symmetric (w.r.t. d) in case it entails an equation if and only it entails its d-translation.

Proposition 2. *An equational theory E is symmetric (w.r.t. d) if and only if $Mod(E)$ is closed under d-symmetry, i.e., in case an algebra $\mathcal{A}$ is in $Mod(E)$ if and only if $\mathcal{A}^d$ is in $Mod(E)$.*

Proof. *Suppose E is symmetric and $\mathcal{A}$ is in $Mod(E)$. We prove that $\mathcal{A}^d$ is in $Mod(E)$. Let $t_1 \approx t_2 \in E$. So $\mathcal{A}$ satisfies $t_1^d \approx t_2^d$ and therefore $\mathcal{A}^d$ satisfies $(t_1^d)^d \approx (t_2^d)^d$ and thus $t_1 \approx t_2$. Consequently, $\mathcal{A}^d$ is in $Mod(E)$.*

Conversely, suppose E is not symmetric. So there is some equation $t_1 \approx t_2$ in E such that there is some algebra $\mathcal{A}$ in $Mod(E)$ which does not satisfy

$t_1^d \approx t_2^d$. *Consequently,* $\mathcal{A}^d$ *does not satisfy* $t_1 \approx t_2$ *and so is not in* $Mod(E)$. *Thus,* $Mod(E)$ *is not closed under d-symmetry.* ■

A variety $\mathfrak{V}$ of type F is said to be symmetric (w.r.t. d) if $Th(\mathfrak{V})$ is symmetric (w.r.t. d).

Starting from a plural type F, we define another plural type F^* which extends F by adding a unary function symbol $\neg$.

If $\delta = \langle F, d\rangle$ is a dualised type, we can define the following equational theory of type F^*:

$$DM_\delta = \{\neg\neg x \approx x\} \cup \{\neg f(x_1, ..., x_n) \approx d(f)(\neg x_1, ..., \neg x_n) \mid f \in F \text{ is } n\text{-ary}\}$$

Let $\mathfrak{V}$ be a symmetric variety of type F. The De Morganification of $\mathfrak{V}$ (w.r.t. d) is the variety $DM(\mathfrak{V})$ of type F^* defined as $Mod(Th(\mathfrak{V}) \cup DM_\delta)$.

Let $\mathcal{A}$ be an algebra of type F. We define the bilateralisation[1] $b\mathcal{A}$ of $\mathcal{A}$ as the algebra of type F^* whose domain is $A \times A$ and such that:

- For all n-ary $f \in F$ and $a_1, b_1, ..., a_n, b_n \in A$, we have $f^{b\mathcal{A}}(\langle a_1, b_1\rangle, ..., \langle a_n, b_n\rangle) = \langle f^{\mathcal{A}}(a_1, ..., a_n), d(f)^{\mathcal{A}}(b_1, ..., b_n)\rangle$,
- For $a, b \in A$, we have $\neg^{b\mathcal{A}}(a, b) = \langle b, a\rangle$.

Proposition 3. *Let* $\mathcal{A}$ *be an algebra of type* F *of* $\mathfrak{V}$. *Then,* $b\mathcal{A}$ *is in* $DM(\mathfrak{V})$.

Proof. *Let* $t_1 \approx t_2$ *be in* $Th(\mathfrak{V})$. *We prove that it is satisfied in* $b\mathcal{A}$. *Let* v *be a valuation on* $b\mathcal{A}$. *We define the projections* $v_1, v_2 : T_F \to A$ *such that, for all* $t \in T_F$, *we have* $v(t) = \langle v_1(t), v_2(t)\rangle$. *Note that* v_1 *is a valuation on* $\mathcal{A}$ *and that* v_2 *is a* F*-valuation on* $\mathcal{A}^d$. *We need to prove that* $v(t_1) = v(t_2)$. *Since* $t_1 \approx t_2$ *is in* $Th(\mathfrak{V})$, *we know that* $v_1(t_1) = v_1(t_2)$. *Moreover, since* $\mathfrak{V}$ *is symmetric, we know that* $\mathcal{A}^d$ *is in* $\mathfrak{V}$ *so* $v_2(t_1) = v_2(t_2)$. *Consequently* $v(t_1) = v(t_2)$ *and, thus,* $b\mathcal{A}$ *satisfies all members of* $Th(\mathfrak{V})$.

Now we need to check that $b\mathcal{A}$ *satisfies the members of* DM_δ. *Again, let* v *be a valuation on* $b\mathcal{A}$ *and we define* v_1 *and* v_2 *as previously. We have:*

$$\begin{aligned} v(\neg\neg x) &= \neg^{b\mathcal{A}}(\neg^{b\mathcal{A}}(v(x))) \\ &= \neg^{b\mathcal{A}}(\neg^{b\mathcal{A}}(v_1(x), v_2(x))) \\ &= \neg^{b\mathcal{A}}(v_2(x), v_1(x)) \\ &= \langle v_1(x), v_2(x)\rangle \end{aligned}$$

[1]The bilateralisation construction is a generalisation of the twist-product construction from the bilattice literature (e.g., [11]). Moreover, as suggested by an anonymous referee, it is closely connected to the very general construction of *duplication* introduced in [8]. However, it is not an instance of duplication since our notion of bilateralisation does not satisfy the condition (M) of the definition of duplication.

$$= v(x)$$

Consequently, $b\mathcal{A}$ satisfies $\neg\neg x \approx x$.

Moreover, for an n-ary $f \in F$, we have:

$$\begin{aligned}
v(\neg f(x_1, ..., x_n)) &= \neg^{b\mathcal{A}}(v(f(x_1, ..., x_n))) \\
&= \neg^{b\mathcal{A}}(f^{b\mathcal{A}}(v(x_1), ..., v(x_n))) \\
&= \neg^{b\mathcal{A}}(f^{\mathcal{A}}(v_1(x_1), ..., v_1(x_n)), d(f)^{\mathcal{A}}(v_2(x_1), ..., v_2(x_n))) \\
&= \langle d(f)^{\mathcal{A}}(v_2(x_1), ..., v_2(x_n)), f^{\mathcal{A}}(v_1(x_1), ..., v_1(x_n)) \rangle \\
&= \langle d(f)^{\mathcal{A}}(v_2(x_1), ..., v_2(x_n)), d(d(f))^{\mathcal{A}}(v_1(x_1), ..., v_1(x_n)) \rangle \\
&= d(f)^{b\mathcal{A}}(\langle v_2(x_1), v_1(x_1) \rangle, ..., \langle v_2(x_n), v_1(x_n) \rangle) \\
&= d(f)^{b\mathcal{A}}(\neg^{b\mathcal{A}}(v(x_1)), ..., \neg^{b\mathcal{A}}(v(x_n))) \\
&= d(f)^{b\mathcal{A}}((v(\neg x_1)), ..., (v(\neg x_n))) \\
&= v(d(f)(\neg x_1, ..., \neg x_n))
\end{aligned}$$

Consequently, $b\mathcal{A}$ satisfies $\neg f(x_1, ..., x_n) \approx d(f)(\neg x_1, ..., \neg x_n)$ for all n-ary $f \in F$.

As a result, $b\mathcal{A}$ is a member of $DM(\mathfrak{V})$. ■

2.2. De Morgan-Płonka Sums and De Morgan Partition Functions

Recall that S is the type of semilattices. Let $\iota = \langle F, l \rangle$ be the dualised type where $l(\sqcup) = \sqcup$. We define $\mathfrak{ISem} = DM(\mathfrak{Sem})$. In other words, the variety $\mathfrak{ISem}$ of involutive semilattices is the De Morganification of the variety of semilattices (w.r.t. l). An involutive semilattice $\mathcal{I}$ therefore contains a function $\neg^{\mathcal{I}} : I \to I$ such that $\neg^{\mathcal{I}}(\neg^{\mathcal{I}}(i)) = i$ and $\neg^{\mathcal{I}}(i \sqcup^{\mathcal{I}} j) = \neg^{\mathcal{I}}(i) \sqcup^{\mathcal{I}} \neg^{\mathcal{I}}(j)$ for all $i, j \in I$.

Let $\delta = \langle F, d \rangle$ be a dualised type. An involutive semilattice system of members of a variety $\mathfrak{V}$ of type F consists of:

- An involutive semilattice $\mathcal{I} = \langle I, \sqcup^{\mathcal{I}}, \neg^{\mathcal{I}} \rangle$,
- For all $i \in I$, an algebra $\mathcal{A}_i$ of $\mathfrak{V}$,
- For all $i, j \in I$ such that $i \sqsubseteq^{\mathcal{I}} j$, a homomorphism $p_i^j : \mathcal{A}_i \to \mathcal{A}_j$ such that p_i^i is the identity for all $i \in J$ and $p_j^k \circ p_i^j = p_i^k$ for all $i, j, k \in I$ such that $i \sqsubseteq^{\mathcal{I}} j \sqsubseteq^{\mathcal{I}} k$,
- For all $i \in I$, an isomorphism $n_i : \mathcal{A}_i \to \mathcal{A}^d_{\neg^{\mathcal{I}}(i)}$ such that $n_{\neg^{\mathcal{I}}(i)}(n_i(a)) = a$ and $p_{\neg^{\mathcal{I}}(i)}^{\neg^{\mathcal{I}}(j)}(n_i(a)) = n_j(p_i^j(a))$ for all $a \in A_i$ and $i, j \in I$ such that $i \sqsubseteq^{\mathcal{I}} j$.

Let $\mathcal{X}$ be such an involutive semilattice system. The De Morgan-Płonka sum of $\mathcal{X}$, written $\mathbb{DMP}(\mathcal{X})$ is the algebra of type F^* whose domain is $\bigcup_{i \in I} A_i \times \{i\}$ and such that $\neg^{\mathbb{DMP}(\mathcal{X})}(a, i) = \langle n_i(a), \neg^{\mathcal{I}}(i)\rangle$ and, for all n-ary $f \in F$, we have:

$$f^{\mathbb{DMP}(\mathcal{X})} : \langle\langle a_1, i_1\rangle, ..., \langle a_n, i_n\rangle\rangle \mapsto \langle f^{\mathcal{A}_i}(p^i_{i_1}(a_1), ..., p^i_{i_n}(a_n)), i\rangle$$

where $i = i_1 \sqcup^{\mathcal{I}} ... \sqcup^{\mathcal{I}} i_n$.

We now generalise the notion of partition function. Let $\mathcal{A}$ be an algebra of type F^* which satisfies DM_δ. A De Morgan partition function on $\mathcal{A}$ is a binary function $\cdot : A^2 \to A$ that, for all n-ary $f \in F$ such and $a, b, c, a_1, ..., a_n \in A$:

- $a \cdot a = a$,
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,
- $a \cdot (b \cdot c) = a \cdot (c \cdot b)$,
- $f^{\mathcal{A}}(a_1, ..., a_n) \cdot b = f^{\mathcal{A}}(a_1 \cdot b, ..., a_n \cdot b)$,
- $b \cdot f^{\mathcal{A}}(a_1, ..., a_n) = b \cdot a_1 \cdot ... \cdot a_n$,
- $\neg^{\mathcal{A}}(a \cdot b) = \neg^{\mathcal{A}}(a) \cdot \neg^{\mathcal{A}}(b)$.

Note that this is not in general a partition function on $\mathcal{A}$, which demands that $\neg^{\mathcal{A}}(a \cdot b) = \neg^{\mathcal{A}}(a) \cdot b$.

The relations $\preceq$ and $\sim$ are defined as previously. We define the algebra $\mathcal{I}_{\mathcal{A}} = \langle A/{\sim}, (.)^{\mathcal{I}_{\mathcal{A}}}\rangle$ of type S^* where, for $[a]_\sim, [b]_\sim \in A/{\sim}$, we have $\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim) = [\neg^{\mathcal{A}}(a)]_\sim$ and $[a]_\sim \sqcup^{\mathcal{I}_{\mathcal{A}}} [b]_\sim = [g^{\mathcal{A}}(a, b, ..., b)]_\sim$ where g is any n-ary member of F such that $n > 1$.

Proposition 4. *The algebra $\mathcal{I}_{\mathcal{A}}$ is an involutive semilattice.*

Proof. *We first check that $\mathcal{A}_{\mathcal{I}}$ is well-defined. The fact that $\sqcup^{\mathcal{I}^{\mathcal{A}}}$ is well-defined follows from the usual theory of Płonka sums. We just need to check that $\neg^{\mathcal{I}^{\mathcal{A}}}$ is well-defined.*

Let $a, b \in A$ such that $a \preceq b$. We prove that $\neg^{\mathcal{A}}(a) \preceq \neg^{\mathcal{A}}(b)$. Since $b \cdot a = b$, we have $\neg^{\mathcal{A}}(b \cdot a) = \neg^{\mathcal{A}}(b)$. Consequently, $\neg^{\mathcal{A}}(b) \cdot \neg^{\mathcal{A}}(a) = \neg^{\mathcal{A}}(b)$ and so $\neg^{\mathcal{A}}(a) \preceq \neg^{\mathcal{A}}(b)$. It follows that $a \sim b$ entails $\neg^{\mathcal{A}}(a) \sim \neg^{\mathcal{A}}(b)$. So $\neg^{\mathcal{I}_{\mathcal{A}}}$ is well-defined.

So $\mathcal{I}_{\mathcal{A}}$ is an algebra of type S^*. Let us check that it is an involutive semilattice. The fact that $\sqcup^{\mathcal{I}_{\mathcal{A}}}$ satisfies the equations defining the variety $\mathfrak{Sem}$ of semilattices follows from the usual theory of Płonka sums. So we just need to check that $\mathcal{I}_{\mathcal{A}}$ satsifies DM_ι.

Let $a \in A$. We have:

$$\begin{aligned} \neg^{\mathcal{I}_{\mathcal{A}}}(\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)) &= \neg^{\mathcal{I}_{\mathcal{A}}}([\neg^{\mathcal{A}}(a)]_\sim) \\ &= [\neg^{\mathcal{A}}(\neg^{\mathcal{A}}(a))]_\sim \\ &= [a]_\sim \end{aligned}$$

Consequently, $\mathcal{I}_{\mathcal{A}}$ satisfies $\neg\neg x \approx x$.

Let $a, b \in A$. We have:

$$\begin{aligned} \neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim \sqcup^{\mathcal{I}_{\mathcal{A}}} [b]_\sim) &= \neg^{\mathcal{I}_{\mathcal{A}}}([g^{\mathcal{A}}(a, b, ..., b)]_\sim) \\ &= [\neg^{\mathcal{A}}(g^{\mathcal{A}}(a, b, ..., b))]_\sim \\ &= [d(g)^{\mathcal{A}}(\neg^{\mathcal{A}}(a), \neg^{\mathcal{A}}(b), ..., \neg^{\mathcal{A}}(b))]_\sim \\ &= [\neg^{\mathcal{A}}(a)]_\sim \sqcup^{\mathcal{I}_{\mathcal{A}}} [\neg^{\mathcal{A}}(b)]_\sim \\ &= \neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim) \sqcup^{\mathcal{I}_{\mathcal{A}}} \neg^{\mathcal{I}_{\mathcal{A}}}([b]_\sim) \end{aligned}$$

Consequently, $\mathcal{I}_{\mathcal{A}}$ satisfies $\neg(x \sqcup y) \approx \neg x \sqcup \neg y$. ■

Proposition 5. *Each equivalence class $[a]_\sim$ forms a subalgebra $\mathcal{A}_{[a]_\sim}$ of the F-reduct of $\mathcal{A}$.*

Proof. *See the general theory of Płonka sums in [5].* ■

Let $\mathbb{DMD}.(\mathcal{A})$ be the involutive semilattice system consisting of:

- The semilattice $\mathcal{I}_{\mathcal{A}}$,
- For all $[a]_\sim \in A/\sim$, the algebra $\mathcal{A}_{[a]_\sim}$ of type F,
- For all $a \preceq b$ in A, the homomorphism $p_{[a]_\sim}^{[b]_\sim} : x \mapsto x \cdot b$,
- For $[a]_\sim \in A/\sim$, the isomorphism $n_{[a]_\sim} : a \mapsto \neg^{\mathcal{A}}(a)$.

Proposition 6. $\mathbb{DMD}.(\mathcal{A})$ *is well-defined.*

Proof. *It follows from the usual theory of Płonka sums that the homomorphisms $p_{[a]_\sim}^{[b]_\sim}$ are well-defined and satisfy the conditions of semilattice systems. So we just need to show that the isomorphisms $n_{[a]_\sim}$ are well-defined and satisfy the conditions of involutive semilattice systems.*

We should have $n_{[a]_\sim} : \mathcal{A}_{[a]_\sim} \to {\mathcal{A}_{\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)}}^d$. This is indeed the case because $\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim) = [\neg^{\mathcal{A}}(a)]_\sim$ and $n_{[a]_\sim}(x) = \neg^{\mathcal{A}}(x) \sim \neg^{\mathcal{A}}(a)$ for all $x \in [a]_\sim$. Moreover, for $x_1, ..., x_n \in [a]_\sim$, we have:

$$\begin{aligned} n_{[a]_\sim}(f^{\mathcal{A}_{[a]_\sim}}(x_1, ..., x_n)) &= \neg^{\mathcal{A}}(f^{\mathcal{A}}(x_1, ..., x_n)) \\ &= d(f)^{\mathcal{A}}(\neg^{\mathcal{A}}(x_1), ..., \neg^{\mathcal{A}}(x_n)) \\ &= d(f)^{\mathcal{A}_{\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)}}(\neg^{\mathcal{A}}(x_1), ..., \neg^{\mathcal{A}}(x_n)) \\ &= f^{{\mathcal{A}_{\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)}}^d}(\neg^{\mathcal{A}}(x_1), ..., \neg^{\mathcal{A}}(x_n)) \\ &= f^{{\mathcal{A}_{\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)}}^d}(n_{[a]_\sim}(x_1), ..., n_{[a]_\sim}(x_n)) \end{aligned}$$

Moreover, it is clear that $n_{\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)}(n_{[a]_\sim}(x)) = \neg^{\mathcal{A}}(\neg^{\mathcal{A}}(x)) = x$ *for all* $x \in [a]_\sim$*. Consequently,* $n_{[a]_\sim}$ *is an isomorphism.*

Now let $a \preceq b$ *in* A *and* $x \in [a]_\sim$*. We have that:*

$$\begin{aligned} p^{\neg^{\mathcal{I}_{\mathcal{A}}}([b]_\sim)}_{\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)}(n_{[a]_\sim}(x)) &= p^{[\neg^{\mathcal{A}}(b)]_\sim}_{[\neg^{\mathcal{A}}(a)]_\sim}(\neg^{\mathcal{A}}(x)) \\ &= \neg^{\mathcal{A}}(x) \cdot \neg^{\mathcal{A}}(b) \\ &= \neg^{\mathcal{A}}(x \cdot b) \\ &= \neg^{\mathcal{A}}(p^{[b]_\sim}_{[a]_\sim}(x)) \\ &= n_{[b]_\sim}(p^{[b]_\sim}_{[a]_\sim}(x)) \end{aligned}$$

This concludes the proof that $\mathbb{DMD}.(\mathcal{A})$ *is well-defined.* ∎

THEOREM 6. *Let* $\mathcal{A}$ *be an algebra of type* F^* *which satisfies* DM_δ *and let* $\cdot : A^2 \to A$ *be a De Morgan partition function on* $\mathcal{A}$*. Then, the De Morgan-Płonka sums of the De Morgan-Płonka decomposition of* $\mathcal{A}$ *relative to* $\cdot$ *is isomorphic to* $\mathcal{A}$*, i.e.,* $\mathbb{DMP}(\mathbb{DMD}.(\mathcal{A})) \cong \mathcal{A}$*.*

PROOF. *Let* $h : \mathcal{A} \to \mathbb{DMP}(\mathbb{DMD}.(\mathcal{A}))$ *be the function defined by* $a \mapsto \langle a, [a]_\sim \rangle$*. We prove that* h *is an isomorphism. Since it is very clearly a bijection so we only need to show that it is a homomorphism.*

Let $f \in F$ *be* n*-ary and let* $a_1, ..., a_n \in A$*. First, note that* $f^{\mathcal{A}}(a_1, ..., a_n) \in [a_1]_\sim \sqcup^{\mathcal{I}_{\mathcal{A}}} ... \sqcup^{\mathcal{I}_{\mathcal{A}}} [a_n]_\sim$*. Indeed, let* $z(a_1, ..., a_n) = g^{\mathcal{A}}(a_1, ..., (g^{\mathcal{A}}(a_2, ..., (..., g^{\mathcal{A}}(a_{n-1}, ..., a_n)...))))$*. We have:*

$$\begin{aligned} f^{\mathcal{A}}(a_1, ..., a_n) \cdot z(a_1, ..., a_n) &= f^{\mathcal{A}}(a_1, ..., a_n) \cdot a_1 \cdot ... \cdot a_n \\ &= f^{\mathcal{A}}(a_1, ..., a_n) \cdot f^{\mathcal{A}}(a_1, ..., a_n) \\ &= f^{\mathcal{A}}(a_1, ..., a_n) \end{aligned}$$

and

$$
\begin{aligned}
z(a_1,...,a_n)\cdot f^{\mathcal{A}}(a_1,...,a_n) &= z(a_1,...,a_n)\cdot a_1\cdot ...\cdot a_n\\
&= z(a_1,...,a_n)\cdot z(a_1,...,a_n)\\
&= z(a_1,...,a_n)
\end{aligned}
$$

Let $u = f^{\mathcal{A}}(a_1,...,a_n)$ *Consequently, we have:*

$$
\begin{aligned}
f^{\mathbb{DMP}(\mathbb{DMD}.(\mathcal{A}))}(h(a_1),...,h(a_n)) &= f^{\mathbb{DMP}(\mathbb{DMD}.(\mathcal{A}))}(\langle a_1,[a_1]_\sim\rangle,...,\langle a_n,[a_n]_\sim\rangle)\\
&= \langle f^{\mathcal{A}_{[u]_\sim}}(p^{[u]_\sim}_{[a_1]_\sim}(a_1),...,p^{[u]_\sim}_{[a_n]_\sim}(a_n)),[u]_\sim\rangle\\
&= \langle f^{\mathcal{A}_{[u]_\sim}}(a_1\cdot u,...,a_n\cdot u),[u]_\sim\rangle\\
&= \langle f^{\mathcal{A}_{[u]_\sim}}(a_1,...,a_n)\cdot u,[u]_\sim\rangle\\
&= \langle f^{\mathcal{A}}(a_1,...,a_n)\cdot u,[u]_\sim\rangle\\
&= \langle u\cdot u,[u]_\sim\rangle\\
&= \langle u,[u]_\sim\rangle\\
&= h(u)\\
&= h(f^{\mathcal{A}}(a_1,...,a_n))
\end{aligned}
$$

Moreover:

$$
\begin{aligned}
\neg^{\mathbb{DMP}(\mathbb{DMD}.(\mathcal{A}))}(h(a)) &= \neg^{\mathbb{DMP}(\mathbb{DMD}.(\mathcal{A}))}(\langle a,[a]_\sim\rangle)\\
&= \langle n_{[a]_\sim}(a),\neg^{\mathcal{I}_{\mathcal{A}}}([a]_\sim)\rangle\\
&= \langle \neg^{\mathcal{A}}(a),[\neg^{\mathcal{A}}(a)]_\sim\rangle\\
&= h(\neg^{\mathcal{A}}(a))
\end{aligned}
$$

Consequently, h is a homomorphism and therefore an isomorphism. ■

2.3. Angellic Equations

Just like the theory of Płonka sums is linked to that of regular equations, the theory of De Morgan Płonka sums is linked to that of Angellic equations.

We define the positive valence $Val^+(t)$ and the negative valence $Val^-(t)$ of a term t of type F^* as follows:

- If $x \in Var$, then $Val^+(x) = \{x\}$ and $Val^-(x) = \emptyset$,
- If $f \in F$ is n-ary and $t_1,...,t_n \in T_{F^*}$ then $Val^+(f(t_1,...,t_n)) = Val^+(t_1)\cup ...\cup Val^+(t_n)$,
- If $f \in F$ is n-ary and $t_1,...,t_n \in T_{F^*}$ then $Val^-(f(t_1,...,t_n)) = Val^-(t_1)\cup ...\cup Val^-(t_n)$,

- If $t \in T_{F^*}$, then $Val^+(\neg t) = Val^-(t)$ and $Val^-(\neg t) = Val^+(t)$.

We define $Val(t) = \langle Val^+(t), Val^-(t)\rangle$. Note that $Var(t) = Val^+(t) \cup Val^-(t)$.

An equation $t_1 \approx t_2$ is said to be Angellic if $Val(t_1) = Val(t_2)$.

THEOREM 7. *The De Morgan Płonka sum of an involutive semilattice system of members of a variety $\mathfrak{V}$ of type F satisfies all the Angellic equations of type F^* satisfied in the bilateralisations of all the algebras of that system.*

PROOF. *Let $\mathcal{X}$ be the involutive semilattice system consisting of:*

- *The involutive semilattice $\mathcal{I} = \langle I, \sqcup^{\mathcal{I}}, \neg^{\mathcal{I}}\rangle$,*
- *For all $i \in I$, an algebra $\mathcal{A}_i$ of $\mathfrak{V}$,*
- *For all $i, j \in I$ such that $i \sqsubseteq^{\mathcal{I}} j$, the homomorphism $p_i^j : \mathcal{A}_i \to \mathcal{A}_j$,*
- *For all $i \in I$, an isomorphism $n_i : \mathcal{A}_i \to \mathcal{A}^d_{\neg^{\mathcal{I}}(i)}$.*

We want to prove that an equation $t_1 \approx t_2$ of type F^* is satisfied in $\mathbb{DMP}(\mathcal{X})$ if it is Angellic and it is satisfied by $b\mathcal{A}_i$ for all $i \in I$.

Let $v : \mathcal{T}_{F^*} \to \mathbb{DMP}(\mathcal{X})$. We define $v_a : \mathcal{T}_{F^*} \to \bigcup_{i\in I} A_i$ and $v_s : \mathcal{T}_{F^*} \to I$ such that $v(t) = \langle v_a(t), v_s(t)\rangle$ for all $t \in T_{F^*}$.

One easily proves the following lemma by induction.

LEMMA 1. $v_s(t) = \bigsqcup^{\mathcal{I}}_{x\in Val^+(t)} v_s(x) \sqcup^{\mathcal{I}} \bigsqcup^{\mathcal{I}}_{x\in Val^-(x)} \neg^{\mathcal{I}}(v_x(s))$

It follows that $Val(t_1) = Val(t_2)$ entails $v_s(t_1) = v_s(t_2)$. Let $i = v_s(t_1) = v_s(t_2)$.

Let $v' : \mathcal{T}_{F^*} \to b\mathcal{A}_i$ such that $v'_1(x) = p^i_{v_s(x)}(v_a(x))$ for all $x \in Val^+(t_1)$ and $v'_2(x) = p^i_{\neg^{\mathcal{I}}(v_s(x))}(v_a(\neg x))$ for all $x \in Val^-(t_1)$ (for the notation v'_1 and v'_2, see the proof of Proposition 3).

LEMMA 2. *Let $u \in T_{F^*}$.*

(1) If $Val^+(u) \subseteq Val^+(t_1)$ and $Val^-(u) \subseteq Val^-(t_1)$, then $v'_1(u) = p^i_{v_s(u)}$ $(v_a(u))$,

(2) If $Val^-(u) \subseteq Val^+(t_1)$ and $Val^+(u) \subseteq Val^-(t_1)$, then $v'_2(u) = p^i_{\neg^{\mathcal{I}} v_s(u)}$ $(v_a(\neg u))$.

PROOF. *We proceed by induction on the construction of u.*

If $u \in Var$, then the results follow from the specification of v'.

Let $f \in F$ be n-ary and $u_1, ..., u_n \in \mathcal{T}_{F^*}$. Suppose $u = f(u_1, ..., u_n)$ and $Val^+(u) \subseteq Val^+(t_1)$ and $Val^-(u) \subseteq Val^-(t_1)$. It follows that for all $k \leq n$ we have $Val^+(u_k) \subseteq Val^+(t_1)$ and $Val^-(u_k) \subseteq Val^-(t_1)$. By induction *hypothesis, we have* $v_1'(u_k) = p^i_{v_s(u_k)}(v_a(u_k))$ *for all* $k \leq n$. *Consequently:*

$$\begin{aligned}
v_1'(u) &= f^{\mathcal{A}_i}(v_1'(u_1), ..., v_1'(u_n)) \\
&= f^{\mathcal{A}_i}(p^i_{v_s(u_1)}(v_a(u_1)), ..., p^i_{v_s(u_n)}(v_a(u_n))) \\
&= f^{\mathcal{A}_i}(p^i_{v_s(u)}(p^{v_s(u)}_{v_s(u_1)}(v_a(u_1))), ..., p^i_{v_s(u)}(p^{v_s(u)}_{v_s(u_n)}(v_a(u_n)))) \\
&= p^i_{v_s(u)}(f^{\mathcal{A}_{v_s(u)}}(p^{v_s(u)}_{v_s(u_1)}(v_a(u_1)), ..., p^{v_s(u)}_{v_s(u_n)}(v_a(u_n)))) \\
&= p^i_{v_s(u)}(v_a(u))
\end{aligned}$$

Similarly, suppose $Val^-(u) \subseteq Val^+(t_1)$ *and* $Val^+(u) \subseteq Val^-(t_1)$. *It follows that for all* $k \leq n$ *we have* $Val^-(u_k) \subseteq Val^+(t_1)$ *and* $Val^+(u_k) \subseteq Val^-(t_1)$. *By induction hypothesis, we have* $v_2'(u_k) = p^i_{v_s(\neg u_k)}(v_a(\neg u_k))$ *for all* $k \leq n$. *Consequently:*

$$\begin{aligned}
v_2'(u) &= d(f)^{\mathcal{A}_i}(v_2'(\neg u_1), ..., v_2'(\neg u_n)) \\
&= d(f)^{\mathcal{A}_i}(p^i_{v_s(\neg u_1)}(v_a(\neg u_1)), ..., p^i_{v_s(\neg u_n)}(v_a(\neg u_n))) \\
&= d(f)^{\mathcal{A}_i}(p^i_{v_s(\neg u)}(p^{v_s(\neg u)}_{v_s(\neg u_1)}(v_a(\neg u_1))), ..., p^i_{v_s(\neg u)}(p^{v_s(\neg u)}_{v_s(\neg u_n)}(v_a(\neg u_n)))) \\
&= p^i_{v_s(\neg u)}(d(f)^{\mathcal{A}_{v_s(\neg u)}}(p^{v_s(\neg u)}_{v_s(\neg u_1)}(v_a(\neg u_1)), ..., p^{v_s(\neg u)}_{v_s(\neg u_n)}(v_a(\neg u_n)))) \\
&= p^i_{v_s(\neg u)}(d(f)^{\mathcal{A}_{v_s(\neg u)}}(p^{v_s(\neg u)}_{v_s(\neg u_1)}(n_{v_s(u_1)}(v_a(u_1))), ..., p^{v_s(\neg u)}_{v_s(\neg u_n)}(n_{v_s(u_1)}(v_a(u_n))))) \\
&= p^i_{v_s(\neg u)}(d(f)^{\mathcal{A}_{v_s(\neg u)}}(n_{v_s(u)}(p^{v_s(u)}_{v_s(u_1)}v_a(u_1)))..., n_{v_s(u)}(p^{v_s(u)}_{v_s(u_n)}(v_a(u_n))))) \\
&= p^i_{v_s(\neg u)}(n_{v_s(u)}(f^{\mathcal{A}_{v_s(u)}}(p^{v_s(u)}_{v_s(u_1)}v_a(u_1)))..., p^{v_s(u)}_{v_s(u_n)}(v_a(u_n))) \\
&= p^i_{v_s(\neg u)}(n_{v_s(u)}(v_a(u)) \\
&= p^i_{v_s(\neg u)}(v_a(\neg u))
\end{aligned}$$

Now suppose that $u = \neg u'$ *for some* $u' \in \mathcal{T}_{F^*}$ *and* $Val^+(u) \subseteq Val^+(t_1)$ *and* $Val^-(u) \subseteq Val^-(t_1)$. *It follows that* $Val^-(u') \subseteq Val^+(t_1)$ *and* $Val^+(u') \subseteq Val^-(t_1)$. *By induction hypothesis, we have* $v_2'(u') = p^i_{\neg\mathcal{I}(v_s(u'))}(v_a(\neg u'))$. *Consequently, we have* $v_1'(u) = p^i_{v_s(u)}(v_a(u))$.

Now suppose that $Val^-(u) \subseteq Val^+(t_1)$ *and* $Val^+(u) \subseteq Val^-(t_1)$. *It follows that* $Val^+(u') \subseteq Val^+(t_1)$ *and* $Val^-(u') \subseteq Val^-(t_1)$. *By induction hypothesis, we have* $v_1'(u') = p^i_{v_s(u')}(v_a(u'))$. *Consequently, we have* $v_2'(u) = p^i_{\neg\mathcal{I}(v_s(u))}(v_a(\neg u))$. ■

From this lemma, it follows that $v_1'(t_1) = p^i_i(v_a(t_1)) = v_a(t_1)$ and $v_1'(t_2) = p^i_i(v_a(t_2)) = v_a(t_2)$. Since $t_1 \approx t_2$ is satisfied by $b\mathcal{A}_i$, we have $v'(t_1) = v'(t_2)$ and therefore $v_a(t_1) = v_a(t_2)$. Thus, we have $v(t_1) = v(t_2)$, as desired. ■

THEOREM 8. *If an equation of type F is satisfied by the De Morgan-Płonka sums of an involutive semilattice system, then it is satisfied in all the algebras of the involutive semilattice system.*

PROOF. *This follows directly from the fact that each algebra of the involutive semilattice system is a subalgebra of the F-reduct of its De Morgan-Płonka sum.* ■

Recall that $\mathbf{2}_s$ is the semilattice $\langle\{0,1\}, \sqcup^{\mathbf{2}_s}\rangle$ where $0 \sqcup^{\mathbf{2}_s} 1 = 1$.

THEOREM 9. *If $b\mathbf{2}_s$ is a subalgebra of the involutive semilattice $\mathcal{I}$, then the De Morgan-Płonka sum of any involutive semilattice system of members of a variety $\mathfrak{V}$ of type F based on $\mathcal{I}$ satisfies only Angellic equations of type F^*.*

PROOF. *Let $\mathcal{X}$ be the involutive semilattice system consisting of:*

- *An involutive semilattice $\mathcal{I} = \langle I, \sqcup^{\mathcal{I}}, \neg^{\mathcal{I}}\rangle$,*
- *For all $i \in I$, an algebra $\mathcal{A}_i$ of $\mathfrak{V}$,*
- *For all $i, j \in I$ such that $i \sqsubseteq^{\mathcal{I}} j$, a homomorphism $p_i^j : \mathcal{A}_i \to \mathcal{A}_j$,*
- *For all $i \in I$, an isomorphism $n_i : \mathcal{A}_i \to \mathcal{A}^d_{\neg^{\mathcal{I}}(i)}$.*

Moreover, let $d : b\mathbf{2}_s \to \mathcal{I}$ be an injective homomorphism.

Let $t_1, t_2 \in T_{F^*}$. Suppose $Val(t_1) \neq Val(t_2)$. Without loss of generality, suppose that there is some $x \in Val^+(t_1) \backslash Val^+(t_2)$. Let $v : \mathcal{T}_{F^*} \to \mathbb{DMP}(\mathcal{X})$ such that $v_s(x) = d(0,1)$ and $v_s(y) = d(0,0)$ for all $y \in Var \backslash \{x\}$. One easily proves the following lemma by induction.

LEMMA 3. *Let $s \in T_{F^*}$:*

(1) If $x \in Val^+(s)$ and $x \in Val^-(s)$, then $v_s(s) = d(1,1)$,

(2) If $x \in Val^+(s)$ and $x \notin Val^-(s)$, then $v_s(s) = d(0,1)$,

(3) If $x \notin Val^+(s)$ and $x \in Val^-(s)$, then $v_s(s) = d(1,0)$,

(4) If $x \notin Val^+(s)$ and $x \notin Val^-(s)$, then $v_s(s) = d(0,0)$.

It follows that $v_s(t_1)$ is either $d(1,1)$ or $d(0,1)$ whereas $v_s(t_2)$ is either $d(1,0)$ or $d(0,0)$. In any case, we have $v_s(t_1) \neq v_s(t_2)$ and therefore $v(t_1) \neq v(t_2)$. Consequently, $\mathbb{DMP}(\mathcal{X})$ does not satisfies $t_1 \approx t_2$. ■

2.4. Angellicisation

Let $\mathfrak{V}$ be a variety of type F^*. We call $A(\mathfrak{V})$ its Angellicisation, namely the variety of type F^* axiomatised by the Angellic equations of $Th(\mathfrak{V})$.

THEOREM 10. *Let $\mathfrak{V}$ be a strongly irregular symmetric variety of type F. The variety $A(DM(\mathfrak{V}))$, i.e., the Angellicisation of the De Morganification of $\mathfrak{V}$, is composed, up to isomorphism, of Płonka sums of involutive semilattice systems of members of $\mathfrak{V}$.*

PROOF. *Let $t_1 \approx t_2$ be an Angellic equations entailed by $Th(\mathfrak{V}) \cup DM_\delta$. Let $\mathcal{X}$ be an involutive semilattice systems of members of $\mathfrak{V}$. Suppose $t_1 \approx t_2$ is not satisfied by $\mathbb{DMP}(\mathcal{X})$. Since $t_1 \approx t_2$ is Angellic, it follows from Theorem 7 than there is an algebra $\mathcal{A}$ in $\mathfrak{V}$, contained in $\mathcal{X}$, such that $t_1 \approx t_2$ is not satisfied in $b\mathcal{A}$. But this contradicts Proposition 3, namely the fact that $b\mathcal{A}$ is in $DM(\mathfrak{V})$. Consequently, $\mathbb{DMP}(\mathcal{X})$ satisfies $t_1 \approx t_2$ and so $\mathbb{DMP}(\mathcal{X})$ is in $A(DM(\mathfrak{V}))$.*

Conversely, let $\mathcal{A}$ be an algebra of $A(DM(\mathfrak{V}))$. Since $\mathfrak{V}$ is strongly irregular, there is term $p(x,y)$ of type F such that $Var(p) = \{x, y\}$ where $x \neq y$ and $p(x,y) \approx x$ is in $Th(\mathfrak{V})$. Let $p^{\mathcal{A}} : A^2 \to A$ be the function defined by $\langle a, b\rangle \mapsto v(p(x,y))$ where $v : \mathcal{T}_{F^*} \to \mathcal{A}$ is any valuation such that $v(x) = a$ and $v(y) = b$.

LEMMA 4. *The function $p^{\mathcal{A}}$ is a De Morgan partition function.*

PROOF. *Note that the following equations of type F^* are Angellic and are in $Th(\mathfrak{V})$, and thus are satisfied by $\mathcal{A}$:*

(1) $p(x,x) \approx x$

(2) $p(x, p(y,z)) \approx p(p(x,y), z)$

(3) $p(x, p(y,z)) \approx p(x, p(z,y))$

(4) $p(f(x_1, ..., x_n), y) \approx f(p(x_1,y), ..., p(x_n,y))$ *for all n-ary $f \in F$*

(5) $p(x, f(y_1, ..., y_n) \approx p(x, p(y_1, p(..., y_n)))$

(6) $p(x,y) = p(x,y)^d$

Since DM_δ entails $\neg p(x,y) \approx p(\neg x, \neg y)^d$, we get that $\mathcal{A}$ satisfies $\neg p(x,y) \approx p(\neg x, \neg y)$. Put together, these facts imply that $p^{\mathcal{A}}$ is a De Morgan partition function. ■

Using Theorem 6, we know that $\mathcal{A}$ is isomorphic to $\mathbb{DMP}(\mathbb{DMD}_{p^{\mathcal{A}}}(\mathcal{A}))$.

Now we just need to show that, for every equivalence class $[a]_\sim \in A/\sim$, we have that $\mathcal{A}_{[a]_\sim}$ is in $\mathfrak{V}$. Recall that $\mathfrak{V}$ can be axiomatised by $E \cup \{p(x,y) \approx x\}$ where E is a set of regular equations of type F. Since regular equations of type F are Angellic equations of type F^*, we get that $\mathcal{A}$ satisfies all members of E and, since $\mathcal{A}_{[a]_\sim}$ is a subalgebra of the F-reduct of $\mathcal{A}$, we have that $\mathcal{A}_{[a]_\sim}$ satisfies all members of E. Moreover, by definition of $\sim$, it is clear that $p(x,y) \approx x$ is satisfied in $\mathcal{A}_{[a]_\sim}$. Consequently, $\mathcal{A}_{[a]_\sim}$ is in $\mathfrak{V}$. ■

THEOREM 11. *The Angellicisation of the De Morganification of a strongly irregular symmetric variety $\mathfrak{V}$ of type F coincides with the De Morganification of its regularisation, i.e., $A(DM(\mathfrak{V})) = DM(R(\mathfrak{V}))$.*

PROOF. *Let $\mathcal{A} \in A(DM(\mathfrak{V}))$. Since regular equations of type F are Angellic equations of type F^* and the members of DM_δ are Angellic, we have that $\mathcal{A} \in DM(R(\mathfrak{V}))$.*

Now let $\mathcal{A} \in DM(R(\mathfrak{V}))$. It is possible to reproduce the second part of the proof of the Theorem 10 to the effect that $\mathcal{A}$ is isomorphic to a De Morgan Płonka sum of an involutive semilattice system of members of $\mathfrak{V}$. So $\mathcal{A} \in A(DM(\mathfrak{A}))$. ■

3. Free Algebras

3.1. Preliminary

Let $\mathfrak{V}$ be a variety of type F and let X be a non-empty set. The free $\mathfrak{V}$-algebra over X, if it exists, is the algebra $\mathcal{L}_{\mathfrak{V}}(X) = \langle L_{\mathfrak{V}}(X), (.)^{\mathcal{L}_{\mathfrak{V}}(X)} \rangle$ such that there is an inclusion function $i_X^{\mathfrak{V}} : X \to \mathcal{L}_{\mathfrak{V}}(X)$ which satisfies the following property: for all algebras $\mathcal{B}$ in $\mathfrak{V}$ and functions $f : X \to B$, there is a unique homomorphism $\mathcal{L}_{\mathfrak{V}}f : \mathcal{L}_{\mathfrak{V}}(X) \to \mathcal{B}$ such that $f(x) = \mathcal{L}_{\mathfrak{V}}f(i_X^{\mathfrak{V}}(x))$ for all $x \in X$. If the free $\mathfrak{V}$-algebra over X exists, it is unique up to isomorphism.

Let us give the example of the free semilattice construction. Where X is a non-empty set, $\mathcal{L}_{\mathfrak{Sem}}(X)$ is the algebra of type S defined by:

- $L_{\mathfrak{Sem}}(X)$ is the set of non-empty finite subsets of X,
- For $K_1, K_2 \in L_{\mathfrak{Sem}}(X)$, we have $K_1 \sqcup^{\mathcal{L}_{\mathfrak{Sem}}(X)} K_2 = K_1 \cup K_2$.

The inclusion function is just $i_X^{\mathfrak{Sem}} : x \mapsto \{x\}$.

Indeed, suppose $\mathcal{B}$ is a semilattice and let $f : X \to B$. We define $\mathcal{L}_{\mathfrak{Sem}}f : K \mapsto \bigsqcup_{k\in K}^{\mathcal{B}} f(k)$. One easily checks that it is a homomorphism. Moreover, if $g : \mathcal{L}_{\mathfrak{Sem}}(X) \to \mathcal{B}$ such that $g(\{x\}) = f(x)$ for all x, then, for all $K \in L_{\mathfrak{Sem}}(X)$, we have $g(K) = g(\bigcup_{k\in K}\{k\}) = \bigsqcup_{k\in K}^{\mathcal{B}} g(\{k\}) = \bigsqcup_{k\in K}^{\mathcal{B}} f(k) = \mathcal{L}_{\mathfrak{Sem}}f(K)$.

Let us give another interesting example which will be useful in the rest of the paper. Let E by the type consisting in two binary function symbols $\wedge$ and $\vee$. We define the variety $\mathfrak{DLat}$ (of type E) of distributive lattices as axiomatised by the following equations:

- $x \wedge x \approx x$
- $x \wedge y \approx y \wedge x$
- $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
- $x \vee x \approx x$
- $x \vee y \approx y \vee x$
- $x \vee (y \vee z) \approx (x \vee y) \vee z$
- $x \vee (y \wedge x) \approx x$ (Absorption)
- $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$ (Meet-distributivity)
- $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$ (Join-distributivity)

We sometimes write $\mathcal{A} = \langle A, \wedge^{\mathcal{A}}, \vee^{\mathcal{A}} \rangle$.

We describe the free $\mathfrak{DLat}$-algebra over finite sets. We first need to define some notions. If $\mathcal{O} = \langle O, \leq, 0, 1 \rangle$ is a bounded partial order, we define $\mathcal{U}(\mathcal{O})$ as the algebra $\langle Up(\mathcal{O}), \cap, \cup \rangle$ of type E where $Up(\mathcal{O})$ is the set of non-empty proper upsets of $\mathcal{O}$, i.e., the sets $U \subseteq O$ such that $1 \in U$, $0 \notin U$ and, for all $x \in U$ and $y \in O$, if $x \leq y$ then $y \in U$. Moreover, if X is a set, then $\mathcal{P}_X$ is the bounded partial order $\langle \mathcal{P}(X), \subseteq, \emptyset, X \rangle$.

Now let X be a finite set. We define $\mathcal{L}_{\mathfrak{DLat}}(X)$ as $\mathcal{U}(\mathcal{P}_X)$ and the inclusion as $i_X^{\mathfrak{DLat}} : x \mapsto \{J \subseteq X \mid x \in J\}$. If $\mathcal{B}$ is a distributive lattice and $f : X \to B$, we define $\mathcal{L}_{\mathfrak{DLat}} f : U \in Up(\mathcal{P}_X) \mapsto \bigvee_{K \in U}^{\mathcal{B}} \bigwedge_{k \in K}^{\mathcal{B}} f(x)$. This follows from the fact that $U = \bigcup_{K \in U} \bigcap_{k \in K} \{J \subseteq X \mid k \in K\}$ for all $U \in Up(\mathcal{O})$.

3.2. Free De Morgan-Płonka Sums

Let $\mathfrak{V}$ be a strongly irregular variety of type F which has free algebras for all finite sets. For any two sets X_1, X_2 such that $X_1 \subseteq X_2$, we define the map $id_{X_1}^{X_2} : X_1 \to X_2$ such that $x \mapsto x$.

Romanowska [17] proved that for all sets X, we have that $\mathcal{L}_{R(\mathfrak{V})}(X)$ is the Płonka sum of the semilattice system which consists of:

- The semilattice $\mathcal{L}_{\mathfrak{Sem}}(X)$,
- For all $K \in L_{\mathfrak{Sem}}(X)$, the algebra $\mathcal{L}_{\mathfrak{V}}(K)$,
- For $K_1 \subseteq K_2$ in $\mathcal{L}_{\mathfrak{Sem}}(X)$, the homomorphism $\mathcal{L}_{\mathfrak{V}}(i_{K_2}^{\mathfrak{V}} \circ id_{K_1}^{K_2})$.

We aim to prove a similar result for De Morgan-Płonka sums.

Let $\mathfrak{V}$ be a strongly irregular symmetric variety of type F. We aim to describe $\mathcal{L}_{A(DM(\mathfrak{V}))}(X)$ for any set X.

First, we describe free involutive semilattices. Where X is a set, $\mathcal{L}_{\mathfrak{ISem}}(X)$ is the algebra of type S^* defined by:

- $L_{\mathfrak{IGem}}(X)$ is the set of non-empty finite subsets of $X \times \{0,1\}$,
- For $K_1, K_2 \in L_{\mathfrak{IGem}}(X)$, we have $K_1 \sqcup^{\mathcal{L}_{\mathfrak{IGem}}(X)} K_2 = K_1 \cup K_2$,
- For $K \in L_{\mathfrak{IGem}}(X)$, we have $\neg^{\mathcal{L}_{\mathfrak{IGem}}(X)} K = \{\langle x,0\rangle \mid \langle x,1\rangle \in K\} \cup \{\langle x,1\rangle \mid \langle x,0\rangle \in K\}$.

The inclusion function is just $i_X^{\mathfrak{IGem}} : x \mapsto \{\langle x,1\rangle\}$.

Indeed, suppose $\mathcal{B}$ is an involutive semilattice and let $f : X \to B$. We define $\mathcal{L}_{\mathfrak{IGem}} f : K \mapsto \bigsqcup^{\mathcal{B}}_{\langle k,1\rangle \in K} f(k) \sqcup^{\mathcal{B}} \bigsqcup^{\mathcal{B}}_{\langle k,0\rangle \in K} \neg^{\mathcal{B}} f(k)$. One easily checks that it is a homomorphism. Moreover, if $g : \mathcal{L}_{\mathfrak{IGem}}(X) \to \mathcal{B}$ is such that $g(\{\langle x,1\rangle\}) = f(x)$ for all x, then, for all $K \in L_{\mathfrak{IGem}}(X)$, we have:

$$\begin{aligned}
g(K) &= g(\bigcup_{\langle k,1\rangle \in K} \{\langle k,1\rangle\} \cup \bigcup_{\langle k,0\rangle \in K} \{\langle k,0\rangle\}) \\
&= g(\bigcup_{\langle k,1\rangle \in K} \{\langle k,1\rangle\} \cup \bigcup_{\langle k,0\rangle \in K} \neg^{\mathcal{L}}_{\mathfrak{IGem}}(\{\langle k,1\rangle\})) \\
&= \bigcup_{\langle k,1\rangle \in K} g(\{\langle k,1\rangle\}) \cup \bigcup_{\langle k,0\rangle \in K} \neg^{\mathcal{L}}_{\mathfrak{IGem}}(g(\{\langle k,1\rangle\})) \\
&= \bigcup_{\langle k,1\rangle \in K} f(k) \cup \bigcup_{\langle k,0\rangle \in K} \neg^{\mathcal{L}}_{\mathfrak{IGem}}(f(k)) \\
&= \mathcal{L}_{\mathfrak{IGem}} f(K)
\end{aligned}$$

Note that g commutes with union and negation because g is a homomorphism. Notice that $\mathcal{L}_{\mathfrak{IGem}}(X)$ is isomorphic to $b\mathcal{L}_{\mathfrak{Gem}}(X)$.

Let X be a non-empty set. We define the function $z : X \times \{0,1\} \to X \times \{0,1\}$ such that $z(x,1) = \langle x,0\rangle$ and $z(x,0) = \langle x,1\rangle$. Notice that, for $K \in L_{\mathfrak{IGem}}(X)$, we have $\neg^{\mathcal{L}_{\mathfrak{IGem}}(X)}(K) = z[K]$.

Let $\mathcal{X}_X$ be the involutive semilattice system consisting of:

- The involutive semilattice $L_{\mathfrak{IGem}}(X)$,
- For all $K \in L_{\mathfrak{IGem}}(X)$, the algebra $\mathcal{L}_{\mathfrak{V}}(K)$,
- For $K_1 \subseteq K_2$ in $\mathcal{L}_{\mathfrak{IGem}}(X)$, the homomorphism $h_{K_1}^{K_2} = \mathcal{L}_{\mathfrak{V}}(i_{K_2}^{\mathfrak{V}} \circ id_{K_1}^{K_2})$,
- For $K \in L_{\mathfrak{IGem}}(X)$, the isomorphism $n_K = \mathcal{L}_{\mathfrak{V}}(i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{IGem}}(X)}(K)} \circ z)$.

Note that it is immediate from the definition of $h_{K_1}^{K_2}$ that $h_{K_1}^{K_2}(i_{K_1}^{\mathfrak{V}}(x)) = i_{K_2}(x)$ for all $x \in K_1$. Similarly, $n_K(i_K^{\mathfrak{V}}(x)) = i^{\mathfrak{V}}_{\neg^{L_{\mathfrak{IGem}}(X)}(K)}(z(x))$ for all $x \in K$.

LEMMA 5. *The system $\mathcal{X}_X$ is well-defined.*

Proof. *The only difference between our system and Romanowska's concerns the isomorphisms n_K so we focus on them.*

First, note that $i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)} \circ z$ is a map from K to the domain of $\mathcal{L}_{\mathfrak{V}}(\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K))$ which is also the domain of $\mathcal{L}_{\mathfrak{V}}(\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K))^d$. So we use the free construction inside of $\mathcal{L}_{\mathfrak{V}}(\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K))^d$ and get $n_K : \mathcal{L}_{\mathfrak{V}} \to \mathcal{L}_{\mathfrak{V}}(\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K))^d$.

We prove that it is an isomorphism. Consider the map $i^{\mathfrak{V}}_K \circ z$ from $\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)$ to $\mathcal{L}^d_{\mathfrak{V}}$. We define $m_K : \mathcal{L}_{\mathfrak{V}}(\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)) \to \mathcal{L}_{\mathfrak{V}}(K)^d$ as $\mathcal{L}_{\mathfrak{V}}(i^{\mathfrak{V}}_K \circ z)$. Clearly, $m_K(i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}(x)) = i_K(z(x))$ for all $x \in \neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}$ (K). We prove that n_K and m_K are inverse to one another. It suffices to show that $m_K \circ n_k(i^{\mathfrak{V}}_K(x)) = i^{\mathfrak{V}}_K(x)$ for all $x \in K$ and that $n_K \circ m_k(i_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}}$ $(K)^{\mathfrak{V}}(x)) = i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}(x)$ for all $x \in z[K]$. For the former, we have:

$$
\begin{aligned}
m_K \circ n_k(i^{\mathfrak{V}}_K(x)) &= m_k(i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}(z(x))) \\
&= i^{\mathfrak{V}}_K(z(z(x))) \\
&= i^{\mathfrak{V}}_K(x)
\end{aligned}
$$

For the latter, we have:

$$
\begin{aligned}
n_K \circ m_k(i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}(x)) &= n_k(i^{\mathfrak{V}}_K(z(x))) \\
&= i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}(z(z(x))) \\
&= i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}(x)
\end{aligned}
$$

So n_K and m_K are isomorphisms. Moreover, notice that $m_K = n_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K)}$.

Finally, for $K_1 \subseteq K_2 \subseteq X$, we prove that $h^{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_2)}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_1)} \circ n_{K_1} = n_{K_2} \circ h^{K_2}_{K_1}$. Of course, we only need to prove that $h^{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_2)}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_1)} \circ n_{K_1}(i^{\mathfrak{V}}_{K_1}(x)) = n_{K_2} \circ h^{K_2}_{K_1}(i^{\mathfrak{V}}_{K_1}(x))$ for all $x \in K_1$. We have:

$$
\begin{aligned}
h^{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_2)}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_1)} \circ n_{K_1}(i^{\mathfrak{V}}_{K_1}(x)) &= h^{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_2)}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_1)}(i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_1)}(z(x))) \\
&= i^{\mathfrak{V}}_{\neg^{\mathcal{L}_{\mathfrak{JSem}}(X)}(K_2)}(z(x)) \\
&= n_{K_2}(i^{\mathfrak{V}}_{K_2}(x)) \\
&= n_{K_2} \circ h^{K_2}_{K_1}(i^{\mathfrak{V}}_{K_1}(x))
\end{aligned}
$$

As required. ■

THEOREM 12. *Let X be a non-empty set. The free algebra $\mathcal{L}_{A(DM(\mathfrak{V}))}(X)$ is $\mathbb{DMP}(\mathcal{X}_X)$ with the inclusion function $i_X^{A(DM(\mathfrak{V}))} : x \in X \mapsto \langle i^{\mathfrak{V}}_{\{\langle x,1\rangle\}}(x,1), \{\langle x,1\rangle\}\rangle$.*

PROOF. *Let $\mathcal{B}$ be an algebra of $A(DM(\mathfrak{V}))$ and let $f : X \to B$. Without loss of generality, we can assume that $\mathcal{B} = \mathbb{DMP}(\mathcal{Y})$ where $\mathcal{Y}$ is an involutive semilattice system composed of:*

- *An involutive semilattice $\mathcal{I}$,*
- *For all $i \in I$, an algebra $\mathcal{B}_i$ of $\mathfrak{V}$,*
- *For all $i \sqsubseteq^{\mathcal{I}} j$ in $\mathcal{I}$, a homomorphism $p_i^j : \mathcal{B}_i \to \mathcal{B}_j$,*
- *For all $i \in I$, an isomorphism $w_i : \mathcal{B}_i \to \mathcal{B}^d_{\neg^{\mathcal{I}}(i)}$.*

So, for $x \in X$, $f(x)$ is of the form $\langle f_a(x), f_s(x)\rangle$ where $f_s(x) \in I$ and $f_a(x) \in \mathcal{B}_{f_s(x)}$. In particular, f_s is a map from X to I.

Let $K \in L_{\mathfrak{ISem}}(X)$. We define $u_k : K \to \mathcal{B}_{\mathcal{L}_{ISem} f_s(K)}$ such that $\langle k,1\rangle \in K \mapsto p_{f_s(k)}^{\mathcal{L}_{ISem} f_s(K)}(f_a(k))$ and $\langle k,0\rangle \in K \mapsto p_{\neg^{\mathcal{I}}(f_s(k))}^{\mathcal{L}_{ISem} f_s(K)}(w_{f_s(k)}(f_a(k)))$. So $\mathcal{L}_{\mathfrak{V}} u_K : \mathcal{L}_{\mathfrak{V}}(K) \to \mathcal{B}_{\mathcal{L}_{ISem} f_s(K)}$.

We define $\mathcal{L}_{A(DM(\mathfrak{V}))} f$ as the map which associates $\langle y, K\rangle$, where $K \in L_{\mathfrak{ISem}}(X)$ and $y \in L_{\mathfrak{V}}(K)$, to $\langle \mathcal{L}_{\mathfrak{V}} u_K(y), \mathcal{L}_{\mathfrak{ISem}} f_s(K)\rangle$.

We prove that $\mathcal{L}_{A(DM(\mathfrak{V}))} f \circ i_X^{A(DM(\mathfrak{V}))} = f$. We have, for $x \in X$:

$$\begin{aligned}
\mathcal{L}_{A(DM(\mathfrak{V}))} f \circ i_X^{A(DM(\mathfrak{V}))}(x) &= \mathcal{L}_{A(DM(\mathfrak{V}))} f(i^{\mathfrak{V}}_{\{\langle x,1\rangle\}}(x,1), \{\langle x,1\rangle\}) \\
&= \langle \mathcal{L}_{\mathfrak{V}} u_{\{\langle x,1\rangle\}}(i^{\mathfrak{V}}_{\{\langle x,1\rangle\}}(x,1)), \mathcal{L}_{\mathfrak{ISem}} f_s(\{\langle x,1\rangle\})\rangle \\
&= \langle u_{\{\langle x,1\rangle\}}(x_1), \mathcal{L}_{\mathfrak{ISem}} f_s(i_X^{\mathfrak{ISem}}(x))\rangle \\
&= \langle p_{f_s(x)}^{\mathcal{L}_{ISem} f_s(\{\langle x,1\rangle\})}(f_a(x)), f_s(x)\rangle \\
&= \langle p_{f_s(x)}^{\mathcal{L}_{ISem} f_s(i_X^{\mathfrak{ISem}}(x))}(f_a(x)), f_s(x)\rangle \\
&= \langle p_{f_s(x)}^{f_s(x)}(f_a(x)), f_s(x)\rangle \\
&= \langle f_a(x), f_s(x)\rangle \\
&= f(x)
\end{aligned}$$

The uniqueness of $\mathcal{L}_{A(DM(\mathfrak{V}))} f$ follows from the uniqueness of $\mathcal{L}_{\mathfrak{K}} u_K$ and $\mathcal{L}_{\mathfrak{ISem}} f_s$ from which it is defined. ■

3.3. Angellic Algebras

The variety $\mathfrak{AA}$ of Angellic algebras is defined as the Angellicisation of DeMorganification of $\mathfrak{DLat}$. Consequently, it is defined over the type E^* with the following equations:

- $x \wedge x \approx x$
- $x \wedge y \approx y \wedge x$
- $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$
- $x \vee x \approx x$
- $x \vee y \approx y \vee x$
- $x \vee (y \vee z) \approx (x \vee y) \vee z$
- $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$
- $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$
- $\neg\neg x \approx x$
- $\neg(x \wedge y) \approx \neg x \vee \neg y$
- $\neg(x \vee y) \approx \neg x \wedge \neg y$

We use Theorem 12 to describe the free Angellic algebra over a non-empty set X.

We have that $\mathcal{L}_{\mathfrak{AA}}(X) = \langle L_{\mathfrak{AA}}(X), (.)^{\mathcal{L}_{\mathfrak{AA}}(X)} \rangle$ where:

- $L_{\mathfrak{AA}}(X) = \{\langle U, K\rangle \mid K \in L_{\mathfrak{ISem}}(X)$ and $U \in L_{\mathfrak{DLat}}(K)\}$,
- For $\langle U_1, K_1\rangle, \langle U_2, K_2\rangle \in L_{\mathfrak{AA}}(X)$, we have $\langle U_1, K_1\rangle \wedge^{\mathcal{L}_{\mathfrak{AA}}(X)} \langle U_2, K_2\rangle =$ $\langle \mathcal{L}_{\mathfrak{DLat}}(i^{\mathfrak{DLat}}_{K_1\cup K_2} \circ id^{K_1\cup K_2}_{K_1})(U_1) \wedge^{\mathcal{L}_{\mathfrak{DLat}}(K_1\cup K_2)} \mathcal{L}_{\mathfrak{DLat}}(i^{\mathfrak{DLat}}_{K_1\cup K_2} \circ id^{K_1\cup K_2}_{K_1})$ $(U_2), K_1 \cup K_2\rangle$,
- For $\langle U_1, K_1\rangle, \langle U_2, K_2\rangle \in L_{\mathfrak{AA}}(X)$, we have $\langle U_1, K_1\rangle \vee^{\mathcal{L}_{\mathfrak{AA}}(X)} \langle U_2, K_2\rangle =$ $\langle \mathcal{L}_{\mathfrak{DLat}}(i^{\mathfrak{DLat}}_{K_1\cup K_2} \circ id^{K_1\cup K_2}_{K_1})(U_1) \vee^{\mathcal{L}_{\mathfrak{DLat}}(K_1\cup K_2)} \mathcal{L}_{\mathfrak{DLat}}(i^{\mathfrak{DLat}}_{K_1\cup K_2} \circ id^{K_1\cup K_2}_{K_1})$ $(U_2), K_1 \cup K_2\rangle$,
- For $\langle U, K\rangle \in L_{\mathfrak{AA}}(X)$, we have $\neg^{\mathcal{L}_{\mathfrak{AA}}(X)}(U, K) = \langle \mathcal{L}_{\mathfrak{DLat}}(i^{\mathfrak{DLat}}_{\neg^{\mathcal{L}_{\mathfrak{ISem}}(X)}(K)} \circ z)(U), \neg^{\mathcal{L}_{\mathfrak{ISem}}(X)}(K)\rangle$.

Using the description of finite free distributive lattices, this simplifies to:

- $L_{\mathfrak{AA}}(X) = \{\langle U, K\rangle \mid K \subseteq X \times \{0, 1\}$ and $U \in Up(\mathcal{P}_K)\}$,
- For $\langle U_1, K_1\rangle, \langle U_2, K_2\rangle \in L_{\mathfrak{AA}}(X)$, we have $\langle U_1, K_1\rangle \wedge^{\mathcal{L}_{\mathfrak{AA}}(X)} \langle U_2, K_2\rangle =$ $\langle \{J \in \mathcal{P}_{K_1\cup K_2} \mid J \cap K_1 \in U_1$ and $K \cap K_2 \in U_2\}, K_1 \cup K_2\rangle$,

- For $\langle U_1, K_1\rangle, \langle U_2, K_2\rangle \in L_{\mathfrak{A}\mathfrak{A}}(X)$, we have $\langle U_1, K_1\rangle \vee^{\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)} \langle U_2, K_2\rangle = \langle\{J \in \mathcal{P}_{K_1\cup K_2} \mid J \cap K_1 \in U_1 \text{ or } K \cap K_2 \in U_2\}, K_1 \cup K_2\rangle$,
- for $\langle U, K\rangle \in L_{\mathfrak{A}\mathfrak{A}}(X)$, we have $\neg^{\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)}(U, K) = \langle\{J \in \mathcal{P}_{z[K]} \mid z[J] \cap O \neq \emptyset \text{ for all } O \in U\}, z[K]\rangle$.

We thus obtain a fairly simple description of free Angellic algebras.

Interestingly, it can be used to give an algebraic proof of completeness of [10]'s truthmaker semantics for [1]'s logic of Analytic Containment (AC). Indeed, an equation is derivable in AC if and only if it is satisfied by all Angellic algebras.

Let $\mathcal{S} = \langle S, \sqcup\rangle$ be a complete semilattice. For $K_1, K_2 \subseteq S$, we define $K_1 \bigsqcup K_2 = \{k_1 \sqcup k_2 \mid k_1 \in K_1 \text{ and } k_2 \in K_2\}$. For $K \subseteq S$, we define the convex closure of K as $K^c = \{k \in S \mid \exists k_1, k_2 \in K : k_1 \sqsubseteq k \sqsubseteq k_2\}$. The unilateral Fine algebra on $\mathcal{S}$ is the algebra $\mathcal{UF}(\mathcal{S}) = \langle UF(\mathcal{S}), \wedge^{\mathcal{UF}(\mathcal{S})}, \vee^{\mathcal{UF}(\mathcal{S})}\rangle$ of type E where[2]:

- $UF(\mathcal{S})$ is the set of non-empty complete convex subsets of S,
- For $K_1, K_2 \in UF(\mathcal{S})$, we define $K_1 \wedge^{\mathcal{UF}(\mathcal{S})} K_2$ as $(K_1 \bigsqcup K_2)^c$,
- For $K_1, K_2 \in UF(\mathcal{S})$, we define $K_1 \vee^{\mathcal{UF}(\mathcal{S})} K_2$ as $(K_1 \cup K_2 \cup (K_1 \bigsqcup K_2))^c$.

The bilateral Fine algebra on $\mathcal{S}$ is the bilateralisation of its unilateral Fine algebra, i.e., $\mathcal{BF}(\mathcal{S}) = b\mathcal{UF}(\mathcal{S})$. Fine proves that $\mathcal{UF}(\mathcal{S})$ is in $R(\mathfrak{DLat})$ (a variety known as the variety of distributive bisemilattices, or distributive Birkhoff systems) and therefore that $\mathcal{BF}(\mathcal{S})$ is in $DM(R(\mathfrak{DLat})) = \mathfrak{A}\mathfrak{A}$.

Note that this is not the way Fine presents his semantics but the difference is purely aesthetic. We use the algebraic language to present the structures and semantics Fine introduces in purely logical terms.

Let $\mathcal{CP}(X)$ be the complete semilattice $\langle \mathcal{P}(X \times \{0, 1\}), \cup\rangle$. Our goal is to embed $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ into $\mathcal{BF}(\mathcal{CP}(X))$. We start by defining the $h : \mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X) \to \mathcal{UF}(\mathcal{CP}(X))$ such that $\langle U, K\rangle \mapsto U$.

Proposition 7. *h is an embedding and, thus, $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ is a E-subalgebra of $\mathcal{UF}(\mathcal{CP}(X))$.*

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[2]As noted by an anonymous referee, this construction is very similar to that of [18]. In this paper, Romanowska and Smith study the bisemilattice of subsemilattices of a semilattice. They prove that it is the free meet-distributive bisemilattice over that semilattice. A simple modification of Fine's construction would be to consider to consider the bisemilattice of non-empty convex subsemilattices of a semilattice. This is in fact the free distributive bisemilattice over that semilattice, though a proof of that fact goes beyond the scope of this paper. Fine's semantics uses the complete version of that free construction.

Proof. *First, we prove that h is well-defined. Let $\langle U, K\rangle \in L_{\mathfrak{A}\mathfrak{A}}$. We need to show that U is a non-empty complete convex subset of $\mathcal{CP}(X)$. We know that U is an upset of $\mathcal{P}_K$. Since $K \subseteq X \times \{0,1\}$, we have that $\mathcal{P}_K$ is a subalgebra of $\mathcal{CP}(X)$ and so U is subset of $\mathcal{CP}(X)$. It is non-empty since $K \in U$. We show that it is convex and complete. Suppose $y_1 \subseteq y_2 \subseteq y_3$ in $\mathcal{CP}(X)$ such that $y_1, y_2 \in U$. Since $y_3 \in U$, we know that $y_3 \subseteq K$ and so $y_2 \subseteq K$. Since $y_1 \in U$ and $y_1 \subseteq y_2$ in $\mathcal{P}_K$, we have $y_2 \in U$. Similarly, suppose $Y \subseteq U$ is non-empty, so that there is some $y \in Y$. Then $y \subseteq \bigcup Y \subseteq K$. Since $y, K \in U$, we have $\bigcup Y \in U$ by convexity. Consequently, $U \in UF(\mathcal{CP}(X))$ for all $\langle U, K\rangle \in L_{\mathfrak{A}\mathfrak{A}}(X)$.*

Now we show that h is an homomorphism. Let $\langle U_1, K_1\rangle, \langle U_2, K_2\rangle \in L_{\mathfrak{A}\mathfrak{A}}(X)$. We prove $h(\langle U_1, K_1\rangle \wedge^{\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)} \langle U_2, K_2\rangle = h(U_1, K_1) \wedge^{\mathcal{UF}(\mathcal{CP}(X))} h(U_2, K_2)$. Concretely, we need to prove that $\{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ and $J \cap K_2 \in U_2\} = \{J_1 \cup J_2 \,|\, J_1 \in U_1$ and $J_2 \in U_1\}^c$. Let $J \in \{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ and $J \cap K_2 \in U_2\}^c$. Clearly, $J = (J \cap K_1) \cup (J \cap K_2)$ so $J \in \{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ and $J \cap K_2 \in U_2\}$. Conversely, let $J \in \{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ and $J \cap K_2 \in U_2\}^c$. So there is some $O_1, P_1 \in U_1$ and $O_2, P_2 \in U_2$ such that $O_1 \cup P_2 \subseteq J \subseteq O_2 \cup P_2$. Since $O_2 \cup P_2 \subseteq K_1 \cup K_2$, $J \in \mathcal{P}_{K_1 \cup K_2}$. Clearly, $O_1 \subseteq J \cap K_1$ and so $J \cap K_1 \in U_1$. Similarly, $J \cap K_2 \in U_2$, as desired.

Now, we prove $h(\langle U_1, K_1\rangle \vee^{\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)} \langle U_2, K_2\rangle = h(U_1, K_1) \vee^{\mathcal{UF}(\mathcal{CP}(X))} h(U_2, K_2)$. Concretely, we need to prove that $\{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ or $J \cap K_2 \in U_2\} = (U_1 \cup U_2 \cup \{J_1 \cup J_2 \,|\, J_1 \in U_1$ and $J_2 \in U_1\})^c$. Let $J \in \{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ or $J \cap K_2 \in U_2\}$. Without loss of generality, let us suppose that we have $J \cap K_1 \in U_1$. So $J \cap K_! \subseteq J \subseteq (J \cap K_1) \cup K_2$ and so $J \in (U_1 \cup U_2 \cup \{J_1 \cup J_2 \,|\, J_1 \in U_1$ and $J_2 \in U_1\})^c$. Conversely, suppose $J \in (U_1 \cup U_2 \cup \{J_1 \cup J_2 \,|\, J_1 \in U_1$ and $J_2 \in U_1\})^c$. Without loss of generality, let us suppose that there is some $O, P_1 \in U_1$ and $P_2 \in U_2$ such that $O \subseteq J \subseteq P_1 \cup P_2$. Clearly, $J \in \mathcal{P}_{K_1 \cup K_2}$. Moreover, $O \subseteq J \cap K_1 \subseteq K_1$ so $J \cap K_1 \in U_1$. So $J \in \{J \in \mathcal{P}_{K_1 \cup K_2} \,|\, J \cap K_1 \in U_1$ or $J \cap K_2 \in U_2\}$. ∎

Now we use a general fact about bilateralisation to transform h into an embedding from $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ into $\mathcal{BF}(\mathcal{CP}(X))$.

Proposition 8. *Let $\mathfrak{V}$ be a symmetric variety of type F (w.r.t. d), let $\mathcal{B}$ be an algebra of $\mathfrak{V}$ and let $\mathcal{A}$ be an algebra of type F^* of $DM(\mathfrak{V})$. If $k : \mathcal{A} \to \mathcal{B}$ is a homomorphism of type F, then $k^+ : \mathcal{A} \to b\mathcal{B}$ such that $a \mapsto \langle k(a), k(\neg^{\mathcal{A}}(a))\rangle$ is a homomorphism of type F^*. Moreover, if k is injective then k^+ is injective.*

Proof. *Let f be a connective of F. We prove, for $a_1, ..., a_n \in A$, that $k^+(f^{\mathcal{A}}(a_1, ..., a_n)) = f^{b\mathcal{B}}(k^+(a_1), ..., k^+(a_n))$. Indeed, we have:*

$$\begin{aligned} k^+(f^{\mathcal{A}}(a_1, ..., a_n)) &= \langle k(f^{\mathcal{A}}(a_1, ..., a_n), k(\neg^{\mathcal{A}}(f^{\mathcal{A}}(a_1, ..., a_n))\rangle \\ &= \langle f^{\mathcal{B}}(k(a_1), ..., k(a_n)), k(d(f)^{\mathcal{A}}(\neg^{\mathcal{A}}(a_1), ..., \neg^{\mathcal{A}}(a_n)))\rangle \\ &= \langle f^{\mathcal{B}}(k(a_1), ..., k(a_n)), d(f)^{\mathcal{B}}(k(\neg^{\mathcal{A}}(a_1)), ..., k(\neg^{\mathcal{A}}(a_n)))\rangle \\ &= f^{b\mathcal{B}}(k^+(a_1), ..., k^+(a_n)) \end{aligned}$$

Moreover, notice that:

$$\begin{aligned} k^+(\neg^{\mathcal{A}}(a)) &= \langle k(\neg^{\mathcal{A}}(a)), k(\neg^{\mathcal{A}}(\neg^{\mathcal{A}}(a)))\rangle \\ &= \langle k(\neg^{\mathcal{A}}(a)), k(a)\rangle \\ &= \neg^{b\mathcal{B}}(\langle k(a), k(\neg^{\mathcal{A}}(a))\rangle) \\ &= \neg^{b\mathcal{B}}(k^+(a)) \end{aligned}$$

So k^+ is a homomorphism.

Now suppose k is injective. So if $a \neq a'$ then $k(a) \neq k(a')$. Consequently, if $a \neq a'$ then $k^+(a) \neq k^+(a')$. ∎

Proposition 9. *$\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ is a subalgebra of $\mathcal{BF}(\mathcal{CP}(X))$.*

Proof. *h is an embedding from $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ into $\mathcal{UF}(\mathcal{CP}(X))$. Consequently, h^+ is an embedding of $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ into $\mathcal{BF}(\mathcal{CP}(X))$.* ∎

We can conclude that Fine's semantics is sound and complete for AC.

Theorem 13. *An equation is derivable in AC if and only if it is satisfied in $\mathcal{BF}(\mathcal{S})$ for all complete semilattices $\mathcal{S}$.*

Proof. *The left-to-right direction of the biconditional follows from the fact $\mathcal{BF}(\mathcal{S})$ is an Angellic algebra for all complete semilattices $\mathcal{S}$.*

For the right-to-left direction, suppose that an equation $t_1 \approx t_2$ is not derivable in AC. Thus, $t_1 \approx t_2$ is not satisfied by some Angellic algebra and so it is not satisfied by a free Angellic algebra of the form $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ for some non-empty set X (e.g., $Var(t_1) \cup Var(t_2)$). Since $\mathcal{L}_{\mathfrak{A}\mathfrak{A}}(X)$ is a subalgebra of $\mathcal{BF}(\mathcal{CP}(X))$, $t_1 \approx t_2$ is not satisfied in $\mathcal{BF}(\mathcal{CP}(X))$. ∎

Conclusion

The Płonka sum construction is a tool that allows one to glue multiple algebras over a semilattice. It gives a representation, via partition functions, of members of the regularisation of strongly irregular varieties. In this context, negation is treated like any other logical constant: it lives inside each algebras without interacting with the semilattice structure. This paper is an attempt to see how can we take negation out of the algebras and plug it into the semilattice structure. Accordingly, our De Morgan-Płonka sums allows one to glue multiple algebras over an involutive semilattice.

For the operation to go smoothly, one cannot apply it to any variety of algebras. It needs to be a welcoming environment for a negation. To this effect, we developed a general theory of De Morgan duality and symmetry and defined our De Morgan-Płonka sums on top of symmetric varieties. The achievements of the resulting theory are parallel to those of the original theory of Płonka sums. We get a representation, via De Morgan partition functions, of the Angellicisation of the De Morganification of strongly irregular symmetric varieties.

As mentioned in the introduction, the project to generalise the Płonka sum construction to allow for a non-transparent negation have been initiated in [9]. Our construction is a generalisation of theirs. In fact, what they call involutorial Płonka sums is the special case of De Morgan-Płonka sums in which the type is dualised by the identity function. In other terms, the approach of involutorial Płonka sums require that every logical constant is its own De Morgan dual. This forces the satisfaction of equations like $\neg(x \wedge y) \approx \neg x \wedge \neg y$ and $\neg(x \vee y) \approx \neg x \vee \neg y$. This restriction is not problematic for mathematicians interested in involutive algebras, but it prevents the application of involutorial Płonka sums to logical frameworks, where the commutativity of negation and conjunction (or disjunction) is almost never accepted. The main achievement of this paper is to free involutorial Płonka sum from this requirement, and thus to move to De Morgan-Płonka sums.

Moreover, our paper presents a general way to build free De Morgan-Płonka sums. This result is the natural counterpart of [17]'s description of free Płonka sums. Note that our construction immediately gives a way to build free involutorial Płonka sums and will be of interest for mathematicians like [9]. More importantly, it can be applied by logicians to produce completeness proofs. To illustrate how it can be used, we produced the first, to our knowledge, algebraic proof of completeness for [10]'s truthmaker semantics for AC. This creates a bridge between Fine's framework and the theory of Płonka sums, two spheres of philosophical logic which have interacted surprisingly little considering their shared thrive for hyperintensionality. We hope that this prefigures a convergence of these two traditions and, any case, offers a valuable opportunity for future works.

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T. Randriamahazaka
University of St Andrews
St Andrews
UK
tr52@st-andrews.ac.uk