# CONTINUED FRACTIONS WHICH CORRESPOND TO TWO SERIES EXPANSIONS AND THE STRONG HAMBURGER MOMENT PROBLEM 

## A. Sri Ranga

A Thesis Submitted for the Degree of PhD at the University of St. Andrews


1984

Full metadata for this item is available in Research@StAndrews:FulIText
at:
http://research-repository.st-andrews.ac.uk/

Please use this identifier to cite or link to this item: http://hdl.handle.net/10023/2977

This item is protected by original copyright

# Continued fractions which correspond to two series expansions, and the strong Hamburger moment problem. 

A. SRI RANGA


A Thesis Submitted for the Degree of Doctor of Philosophy

To my parents

## ACKNOWLEDGEMENTS

First and most of all I would like to acknowledge, with sincere thanks, my supervisor Dr. J.H.hcCabe for making my stay in St. Andrews extremely pleasant, both academically and socially. I would also like to express my gratitude to the authorities of the University of St. Andrews, in particular Dr. J.J.Sanderson (applied maths) and Lt. Col. J.B.Smith, for the generous financial support I have been given for the past tinree years. Finally my thanks go also to Miss. S.Wilson for her efforts in typing this thesis.

I declare that the following thesis is a record of research work carried out by me, that the thesis is my own composition, and that it has not been previously presented in application for a higher degree.
A. Sri Ranga

## POSTGRADUATE CAREER

I was admitted into the University of St Andrews as a research student under Ordinance General No. 12 in October 1980 to work on Two-Point Padé Approximants and Continued Fractions under the supervision of Dr J.H. McCabe. I was admitted under the above resolution as a candidate for the degree of Ph.D. in October 1981.
A. Sri Ranga

## CERTIFICATE

I certify that A. Sri Ranga has satisfied the conditions of the Ordinance and Regulations and is qualified to submit the accompanying application for the Degree of Doctor of Philosophy.


J.H. McCabe

## SUMMARY

Just as the denominator polynomials of a J-fraction are orthogonal polynomials with respect to some moment functional, the denominator polynomials of an M-fraction are shown to satisfy a skew orthogonality relation with respect to a stronger moment functional. Many of the properties of the numerators and denominators of an Mfraction are also studied using this pseudo orthogonality relation of the denominator polynomials. Properties of the zeros of the denominator polynomials when the associated moment functional is positive definite are also considered.

A type of continued fraction, referred to as a $\hat{J}$-fraction, is shown to correspond to a power series about the origin and to another power series about infinity such that the successive convergents of this fraction include two more additional terms of any one of the power series. Given the power series expansions, a method of obtaining such a $\hat{J}$-fraction, whenever it exists, is also looked at. The first complete proof of the so called strong Hamburger moment problem using a continued fraction is given. In this case the continued fraction is a $\hat{J}$-fraction.

Finally a special class of $\hat{J}$-fraction, referred to as positive definite $\hat{J}$-fractions, is studied in detail.

The four chapters of this thesis are divided into sections. Each section is given a section number which is made up of the chapter number followed by the number of the section within the chapter. The equations in the thesis have an equation number consisting of the section number followed by the number of the equation within that section.

In Chapter One, in addition to looking at some of the historical and recent developments of corresponding continued fractions and their applications, we also present some preliminaries.

Chapter Two deals with a different approach of understanding the properties of the numerators and denominators of corresponding (two point) rational functions and continued fractions. This approach, which is based on a pseudo orthogonality relation of the denominator polynomials of the corresponding rational functions, provides an insight into understanding the moment problems. In particular, results are established which suggest a possible type of continued fraction for solving the strong Hamburger moment problem.

In the third chapter we study in detail the existence conditions and corresponding properties of this new type of continued fraction, which we call $\hat{J}$-fractions. A method of derivation of one of these $\hat{J}$-fractions is also considered. In the same chapter we also look at the all important application of solving the strong Hamburger moment problem, using these $\hat{J}$-fractions.

The fourth and final chapter is devoted entirely to the study of the convergence behaviour of a certain class of $\hat{J}$-fractions,
namely positive definite $\widehat{J}$-fractions. This study also provides some interesting convergence criteria for a real and regular $\hat{J}$-fraction.

Finally a word concerning the literature on continued fractions and moment problems. The more recent and up-to-date exposition on the analytic theory of continued fractions and their applications is the text of Jones and Thron [1980]. The two volumes of Baker and Graves-Morris [1981] provide a very good treatment on one of the computational aspects of the continued fractions, namely Padé approximants. There are also the earlier texts of Wall [1948] and Khovanskii [1963], in which the former gives an extensive insight into the analytic theory of continued fractions while the latter, being simpler, remains the ideal book for the beginner. In his treatise on Applied and Computational Complex Analysis, Henrici [1977] has also included an excellent chapter on continued fractions. Wall [1948] also includes a few chapters on moment problems and related areas. A much wider treatment of the classical moment problems is provided in the excellent texts of Shonat and Tamarkin [1943] and Akhieser [1965].

```
a A a is an element of the set A; a belongs to A.
A\subseteqB A is a subset of B.
A\subsetB A is a proper subset of B.
\mathbb{R}}\mathrm{ Set of all real numbers.
Re(z) Real part of z.
Im(z) Imaginary part of z.
z}\quad\mathrm{ Complex conjugate of z.
|z| Modulus or absolute value of z.
Domain Open connected subset of the set of all complex numbers.
```

Summary
Preface
Chapter 1 : Introduction
1.1 History ..... 1
1.2 Preliminaries ..... 5
Chapter 2 : Pseudo Orthogonal Polynomịals
2.1 Introduction ..... 12
2.2 General Pseudo Orthogonal Polynomials ..... 22
2.3 Some Properties of Pseudo Orthogonality
Polynomials and Associated Functions ..... 29
2.4 Zeros and the Quadrature Formula ..... 38
Chapter 3 : $\hat{J}$-Fractions and the Strong Hamburger Moment Problem
3.1 The Moment Problem ..... 54
3.2 $\hat{J}$-Fractions and Their Correspondence ..... 58
3.3 Methods of Derivation ..... 77
3.4 $\hat{J}$-Fractions and the Strong Hamburger Moment Problem ..... 83
Chapter 4 : Convergence Behaviour of a Class of $\hat{J}$-Fractions
4.1 Positive Definite $\hat{J}$-Fractions ..... 94
4.2 Convergence Circle ..... 105
4.3 Limit Point Case ..... 111
4.4 Limit Circle Case ..... 120
4.5 Uniform Convergence ..... 131

CHAPTER ONE

INTRODUCTION

### 1.1 HISTORY

Investigations into the problem of transforming arbitrary power series into continued fractions began in the early nineteenth century. Among those who contributed to the development of these so called corresponding continued fractions were Stern [1832], Heilermann [1846], Frobenius [1881], Stielrjes [1889] and Pade [1892]. With the exception of Stieltjes, they were mainly interested in two particuiar types of corresponding fractions, namely regular $C$-fractions and associated fractions.

The significance of finding such corresponding continued fractions was only realised when Frobenius and Padé arrived at general and elaborate techniques of obtaining rational approximants, given as convergents of continued fraction expansions of analytic functions in the complex plane. Padé in particular developed the rational approximants of such analytic functions and then organised them in tables. Such tables and their contents are now referred to as Padé tables and Padé approximants. With the recent advent of high speed computers these techniques of Padé and Frobenius have become powerful computational tools in mathematics and physical sciences, under the general name of Padé approximants.

During the last thirty years or so many algorithms have been developed for arriving at corresponding continued fractions of arbitrary power series. The most important of these techniques is perhaps the quotient-difference (q-d) algorithm of Rutishauser [1954], which can be used to find the regular C-fraction expansion.

M-fraction can be made to correspond simultaneously to a power series about the origin and another power series about infinity. They pointed out that for such power series expansions, rational approximants of various degrees of correspondence can be obtained as convergents of M-fractions. McCabe [1975] in particular, showed that in a similar way to that of ordinary Padé approximants, these rational approximants can also be organised in a table which he called an M-table. M-tables are also now sometimes referred to as two-point Padé tables.

For deriving M-fractions from given two power series, McCabe [1975] developed an algorithm similar in nature to the q-d algorithm of Rutishauser. The corresponding sequence algorithm of Murphy and $0^{\prime}$ Donohoe [1977] proved to be a valuable altemative algorithm as it works on many occasions when the q-d algorithm breaks down.

Independently of the above authors Jones and Thron [1977] and Thron [1977] also discovered the corresponding properties of an Mfraction. It is true to say that the continued fraction studied by Jones and Thron was not an M-fraction but a continued fraction equivalent to an M-fraction. They called their continued fraction the general T-fraction in contrast to the ordinary T-fraction of Thron [1948]. They also developed many convergence criteria (see for example Jones and Thron [1980]) for the general T-fractions.

Drew and Murphy [1977] described some ways of constructing Mfractions from two given power series (not necessarily expansions about the origin and infinity), and their derivatives and their integrals. They also described some ways of constructing other continued fractions when the M-fraction does not exist, but in these
cases the continued fractions are not as regular as the M-fraction, in the sense the partial quotients are not all of the same form. Practical applications of M-fractions such as the numerical. inversion of Laplace transforms, were studied by Grundy starting in [1977]. Jones and Magnus [1980] showed how M-fractions can be used to find the poles of certain functions.

One of the most interesting properties of M-fractions is their connection with the so called strong Stieltjes moment problem. This was brought to Iight when Jones, Thron and Waadeland [1980] proposed and solved this problem by making it dependent upon a positive M-fraction (or T-fraction).

Other two point continued fractions and algorithms were also looked at by McCabe [1981,1983]. A detailed study of the properties of one of these continued fractions, referred to as the $\hat{j}$-fraction, forms the basis of Chapters 3 and 4 of this thesis.

### 1.2 PRELIMINARIES

An infinite or finite mathematical expression of the form

$$
\begin{equation*}
b_{0}+\frac{a_{1}}{b_{1}+\frac{a_{2}}{b_{2}}+\frac{a_{3}}{b_{3}+\frac{a_{4}}{b_{4}+\ldots \ldots}}} \tag{1.2.1}
\end{equation*}
$$

where $a_{n}$ and $b_{n}$ are complex constants, complex variables or even functions of complex variables, is known as a continued fraction. For convenience other representations such as

$$
b_{0}+K\left(a_{n} / b_{n}\right),
$$

or

$$
b_{0}+\frac{a_{1}}{b_{1}}+\frac{a_{2}}{b_{2}}+\frac{a_{3}}{b_{3}}+\frac{a_{4}}{b_{4}}+\ldots
$$

are also adopted. The factors $a_{n}$ and $b_{n}$ are sometimes referred to as the coefficients or elements of the fraction but more frequently they are called the partial numerators and partial denominators respectively.

The truncated continued fraction

$$
\begin{equation*}
\mathrm{R}_{\mathrm{n}}=\mathrm{b}_{0}+\frac{a_{1}}{b_{1}}+\frac{a_{2}}{b_{2}}+\ldots+\frac{a_{n}}{b_{n}}, \tag{1.2.2}
\end{equation*}
$$

is called the $n$-th convergent or $n$-th approximant of the continued fraction (1.2.1). The limit of the sequence $\left\{R_{n}\right\}$, when it exists, is the value of the continued fraction.

The continued fraction (1.2.1) can also be defined in terms of linear fractional transformations (l.f.t.) by.

$$
\begin{array}{lll}
s_{0}(\omega)=b_{0}+\omega, & s_{n}(\omega)=\frac{a_{n}}{b_{n}+\omega}, & n=1,2, \ldots, \\
s_{0}(\omega)=s_{0}(\omega), & s_{n}(\omega)=s_{n-1}\left(s_{n}(\omega)\right), & n=1,2, \ldots,
\end{array}
$$

so that

$$
S_{n}(\omega)=b_{0}+\frac{a_{1}}{b_{1}}+\frac{a_{2}}{b_{2}}+\ldots+\frac{a_{n}}{b_{n}+\omega}
$$

Hence, from above the $n-t h$ convergent $R_{n}$ of (1.2.1) can be given by $S_{n}(0)$.

If we write

$$
R_{n}=\frac{A_{n}}{B_{n}}, \quad n=0,1,2, \ldots,
$$

where

$$
\begin{aligned}
& A_{0}=b_{0}, \quad B_{0}=1 \\
& A_{1}=b_{0} b_{1}+a_{1}, \quad B_{1}=b_{1}, \\
& A_{2}=b_{0} b_{1} b_{2}+b_{0} a_{2}+a_{1} b_{2}, \quad B_{2}=b_{1} b_{2}+a_{2},
\end{aligned}
$$

and in general $A_{n}$ and $B_{n}$ are polynomials in $a_{i}, b_{j}$, then these polynomials satisfy the three term recurrence formulas

$$
\begin{align*}
& A_{n}=b_{n} A_{n-1}+a_{n} A_{n-2}, \quad n \geqslant 2,  \tag{1.2.3}\\
& B_{n}=b_{n} B_{n-1}+a_{n} B_{n-2}, \quad n \geqslant 2,
\end{align*}
$$

The proof of these formulas, first established by Wallis in 1655, can be found for example in Khovanskii [1963]. $A_{n}$ and $B_{n}$ are called the numerator and denominator of the $n$-th approximant, respectively.

Any two continued fractions having their n-th approximants the same for all $n$ are said to be equivalent. Any equivalent continued fractions of (1.2.1) can be obtained by the equivalence transformation

$$
b_{0}+\frac{\sigma_{1} a_{1}}{\sigma_{1} b_{1}}+\frac{\sigma_{1} \sigma_{2} a_{2}}{\sigma_{2} b_{2}}+\frac{\sigma_{2} \sigma_{3} a_{3}}{\sigma_{3} b_{3}}+\frac{\sigma_{3} \sigma_{4} a_{4}}{\sigma_{4} b_{4}}+\ldots,
$$

where the sequence of numbers $\left\{\sigma_{n}\right\}$ are chosen appropriately.
Any continued fraction of the form

$$
\frac{F_{1}}{z+G_{1}}+\frac{F_{2} z}{z+G_{2}}+\frac{F_{3} z}{z+G_{3}}+\frac{F_{4} z}{z+G_{4}}+\ldots
$$

is said to be an M-fraction. The general $T$-fraction of Jones and Thron [1977] is equivalent to the M-fraction and takes the form

$$
\frac{1}{e_{1} z+f_{1}}+\frac{z}{e_{2} z+f_{2}}+\frac{z}{e_{3} z+f_{3}}+\frac{z}{e_{4} z+f_{4}}+\ldots
$$

where these coefficients $e_{n}$ and $f_{n}$ satisfy

$$
F_{1}=1 / e_{1}, \quad G_{n}=f_{n} / e_{n}, \quad F_{n+1}=1 /\left(e_{n} e_{n+1}\right), \quad n \geqslant 1
$$

Now, for the Hankel determinant $H_{r}^{(m)}$ defined by

$$
\mathrm{H}_{-1}^{(\mathrm{m})}=0, \quad \mathrm{H}_{0}^{(\mathrm{m})}=1
$$

and

$$
H_{r}^{(m)}=\left|\begin{array}{llll}
c_{m} & c_{m+1} & \cdots \cdots \cdots & c_{m+r-1}  \tag{1.2.4}\\
c_{m+1} & c_{m+2} & \cdots \cdots \cdots & c_{m+r} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
c_{m+r-1} & c_{m+r} & \cdots \cdots \cdots & c_{m+2 r-2}
\end{array}\right|
$$

for all m and all $\mathrm{r} \geqslant 1$, let us choose for a given $k$, the two sets of elements $\left\{n_{r+1}^{(k)}\right\}_{r=1}^{\infty}$ and $\left\{d_{r}^{(k)}\right\}_{r=1}^{\infty}$ as

$$
\begin{align*}
& \mathrm{n}_{\mathrm{r}+1}^{(\mathrm{k})}=\frac{-\mathrm{H}_{\mathrm{r}+1}^{(k-r)} \mathrm{H}_{\mathrm{r}-1}^{(\mathrm{k}-(\mathrm{r}-1))}}{\mathrm{H}_{\mathrm{r}}^{(\mathrm{k}-(\mathrm{r}-1))} \mathrm{H}_{\mathrm{r}}^{(\mathrm{k}-\mathrm{r})}}, \quad \mathrm{r} \geqslant 1,  \tag{1.2.5}\\
& \mathrm{~d}_{\mathrm{r}+1}^{(\mathrm{k})}=\frac{-\mathrm{H}_{\mathrm{r}+1}^{(k-r)} \mathrm{H}_{\mathrm{r}}^{(\mathrm{k}-\mathrm{r})}}{\mathrm{H}_{\mathrm{r}}^{(\mathrm{k}-(\mathrm{r}-1))} \mathrm{H}_{\mathrm{r}+1}^{(\mathrm{k}-(\mathrm{r}+1))}}, \mathrm{r} \geqslant 0 .
\end{align*}
$$

Then for

$$
M_{0}^{(k)}(z)=\left\{\begin{array}{ll}
\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\ldots+\frac{c_{k-1}}{z^{k}}, & k \geqslant 0 \\
-c_{-1}-c_{-2} z-\ldots-c_{k} z^{-(k+1)}, & k<0
\end{array},\right.
$$

the $r$-th convergent $M_{r}^{(k)}(z)$ of the $M$-fraction

$$
\begin{equation*}
M_{0}^{(k)}(z)+\frac{c_{k} / z^{k}}{z+d_{1}^{(k)}}+-\frac{n_{2}^{(k)} z}{z+d_{2}^{(k)}}+\frac{n_{3}^{(k)} z}{z+d_{3}^{(k)}}+\frac{n_{4}^{(k)} z}{z+d_{4}^{(k)}}+\ldots \tag{1.2.6}
\end{equation*}
$$

corresponds to $\Phi(r+k)$ terms of the power series

$$
f_{0}(z)=\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\frac{c_{2}}{z^{3}}+\ldots,
$$

and $\Phi(r-k)$ terms of the power series

$$
g_{0}(z)=-c_{-1}-c_{-2} z-c_{-3} z^{2}-\cdots
$$

Here, the integer function $\Phi(N)$ is given by

$$
\Phi(N)= \begin{cases}N, & N \geqslant 0  \tag{1.2.7}\\ 0, & N<0\end{cases}
$$

The coefficients or elements $n_{n}^{(k)}$ and $d_{n}^{(k)}$, given by (1.2.5), also satisfy the quotient difference relations (see McCabe [1975])

$$
\begin{align*}
& n_{n+1}^{(k)}-d_{n}^{(k+1)}=n_{n}^{(k+1)}-d_{n}^{(k)}, \quad n \geqslant 1,  \tag{1.2.8}\\
& d_{n+1}^{(k+1)} / n_{n+1}^{(k+1)}=d_{n}^{(k)} / n_{n+1}^{(k)}, \quad n \geqslant 1,
\end{align*}
$$

with

$$
n_{l}^{(k)}=0, d_{1}^{(k)}=-c_{k} / c_{k-1} \text { for all } k
$$

Denoting the numerator and denominator of the $n$-th convergent of the $M^{(k)}$-fraction (1.2.6) by $A_{n}^{(k)}(z)$ and $B_{n}^{(k)}(z)$ respectively, then we have

$$
\begin{align*}
& A_{n+1}^{(k)}(z)=\left(z+d_{n+1}^{(k)}\right) A_{n}^{(k)}(z)+n_{n+1}^{(k)} z A_{n-1}^{(k)}(z), \\
& B_{n+1}^{(k)}(z)=\left(z+d_{n+1}^{(k)}\right) B^{(k)}(z)+n_{n+1}^{(k)} z B_{n-1}^{(k)}(z), \tag{1.2.9}
\end{align*}
$$

and

$$
\begin{aligned}
& A_{n}^{(k+1)}(z)=A_{n}^{(k)}(z)+n_{n+1}^{(k)} A_{n-1}^{(k)}(z), \\
& B_{n}^{(k+1)}(z)=B_{n}^{(k)}(z)+n_{n+1}^{(k)} B_{n-1}^{(k)}(z),
\end{aligned}
$$

$$
n \geqslant 1, \quad(1.2 .10)
$$

with initial conditions

$$
\begin{aligned}
& A_{0}^{(k)}(z)=M_{0}^{(k)}(z), \quad B_{0}^{(k)}(z)=1 \\
& A_{1}^{(k)}(z)=\left(z+d_{1}^{(k)}\right) A_{0}^{(k)}(z)+c_{k} / z^{k}, \quad B_{1}^{(k)}(z)=\left(z+d_{1}^{(k)}\right) .
\end{aligned}
$$

The three term relation (1.2.9) follows from (1.2.3). The relation (1.2.10) can be proved inductively, using (1.2.9) and the q-d relation (1.2.8).

The M-table (or two point Pade table) of McCabe [1975] is the following table where the entry $M_{r}^{(k)}(z)$ is the r-th convergent of the $M^{(k)}$-fraction (1.2.6).


The M-table can be divided into three sections as indicated. The entries lying in the upper section are also the entries of the ordinary Padé table of the power serjes $g_{0}(z)$, and the entries lying in the lower section are also entries belonging to the E-array of Wynn [1960], for the series $f_{0}(z)$.

The elements of the M-table also satisfy many of the identities satisfied by the elements of the Pade table, for example the Wynn's identity

$$
\begin{aligned}
\left\{M_{n+1}^{(k)}(z)-M_{n}^{(k)}(z)\right\}^{-1} & +\left\{M_{n-1}^{(k)}(z)-M_{n}^{(k)}(z)\right\}^{-1} \\
& =\left\{M_{n}^{(k+1)}(z)-M_{n}^{(k)}(z)\right\}^{-1}+\left\{M_{n}^{(k-1)}(z)-M_{n}^{(k)}(z)\right\}^{-1}
\end{aligned}
$$

for all $k$ and all $n \geqslant 1$.

Finally for later reference, an identity relating the Hankel determinants (1.2.4), is

$$
\begin{equation*}
\left\{H_{r}^{(m)}\right\}^{2}-H_{r}^{(m-1)} H_{r}^{(m+1)}+H_{r+1}^{(m-1)} H_{r-1}^{(m+1)}=0 \tag{1,2.11}
\end{equation*}
$$

This identity is known as the Jacobi identity, and the proof of it can be found in Henrici [1974].

CHAPTER THO

```
PSEUDOORTHOGONALOOLYNOMIALS
```

$$
\begin{equation*}
\int_{a}^{b} \frac{1}{z-t} \phi(t) d t \tag{2.1.2}
\end{equation*}
$$

where $\phi(t)$ is non-negative in $(a, b)$. Heine gives brief discussions of the J-fraction associated with (2.1.2) and also an application of the related orthogonal polynomials, namely approximate quadrature.

Later, in his most celebrated paper of [1894-95], Stieltjes brought forward a new concept of integrals by replacing the $\phi(t) d t$ in (2.1.2) by $d \psi(t)$, where $\psi(t)$ is a bounded non-decreasing function in the interval ( $\mathrm{a}, \mathrm{b}$ ). This new idea of integrals, now known as the "Stieltjes integral", covers both integrals and sums such that, if $\psi(t)$ is continuous and differentiable in ( $a, b$ ) then

$$
\int_{a}^{b} \frac{1}{z-t} d \psi(t)=\int_{a}^{b} \frac{1}{z-t} \psi^{\prime}(t) d t
$$

where $\psi^{\prime}(t)$ is the derivative of $\psi(t)$. If $\psi(t)$ is step-wise with increments only at the distinct points $t=z_{i}$ then

$$
\int_{a}^{b} \frac{1}{z-t} d \psi(t)=\sum_{i} \frac{1}{z-z_{i}}\left\{\psi\left(z_{i}+\right)-\psi\left(z_{i}-\right)\right\}
$$

NOTE : A good explanation of the Stieltjes integrals is given in Rudin [1976].

Introducing this new analytical tool, the Stieltjes-integral, Stieltjes was able to not only widen the scope of the theory of orthogonal polynomials, but also to use continued fractions for the
treatment of his new problem, the Stieltjes moment problem in its most general form. More details of such moment problems are considered in the next chapter.

The achievements to the present day in this field of orthogonal polynomials and continued fractions can be summarised as follows:

Given a sequence of finite valued complex numbers $\left\{c_{n}\right\}_{n=0}^{\infty}$, we define a linear complex valued functional $I[\cdot]$ on the vector space of all polynomials on the real variable $t$, such that

$$
\begin{equation*}
I\left[t^{n}\right]=c_{n}, \quad n=0,1,2, \ldots \tag{2.3.4}
\end{equation*}
$$

Then a sequence of orthogonal polynomials $\left\{Q_{n}(z)\right\}$ can be defined by

$$
I\left[t^{s} Q_{n}(t)\right]=\left\{\begin{array}{cc}
0, & 0 \leqslant s \leqslant n-1  \tag{2.1.5}\\
v_{n} \neq 0, & s=n
\end{array}\right.
$$

for all $n \geqslant 0$. The existence of this orthogonal sequence such that $Q_{n}(z)$ is of degree $n$ precisely depends on the condition

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}^{(0)} \neq 0 \quad \text { for } \quad \mathrm{n}=0,1,2, \ldots \tag{2.1.6}
\end{equation*}
$$

where $H_{n}^{(k)}$ are the Hankel determinants given by (1.2.4). For the proof of this see Chihara [1978].

The functional $I[\cdot]$ which is determined by the moment sequence $\left\{c_{n}\right\}$ is referred to as a moment functional. If the moments also satisfy condition (2.1.6) then the functional is called a quasidefinite moment functional. Hence, for every quasi-definite moment functional $I_{q}[\cdot]$ there exists an orthogonal polynomial sequence $\left\{Q_{n}(z)\right\}$.

In the special case where all the moments $c_{n}$ are real and where all the determinants $H_{n}^{(0)}$ are positive then the corresponding moment functional is called a positive-definite moment functional. A positive definite moment functional, denoted by $I_{p}[\cdot]$ is also a quasi-definite moment functional, but it further satisfies the property

$$
I_{p}[\pi(t)]>0
$$

for all polynomials $\pi(t)$ which are non-negative but not identically zero for all $t \in E \subset(-\infty, \infty)$. The set $E$ is called the supporting set of the positive definite moment functional $I_{p}[\cdot]$.

From above it also follows that for any $I_{p}[\cdot]$, the corresponding sequence of orthogonal polynomials $\left\{Q_{n}(t)\right\}$ are all real and, further

$$
I_{p}\left[\left(Q_{n}(t)\right)^{2}\right]>0, \quad n=0,1,2, \ldots
$$

Stieltjes [1894-95], Hamburger [1921] and others have shown that any positive definite moment functional $I_{p}[\cdot]$ can be given in terms of a bounded non-decreasing function $\psi(t)$ with infinitely many points of increase. That is

$$
\begin{equation*}
I_{p}\left[t^{n}\right]=\int_{-\infty}^{\infty} t^{n} d \psi(t), \quad n=0,1,2, \ldots \tag{2.1.7}
\end{equation*}
$$

Later R.P. Boas [1939] pointed out that any real quasi-definite moment functional $I_{q}[\cdot]$ can be given in terms of a function $\phi(t)$ of bounded variation, by

$$
\begin{equation*}
I_{q}\left[t^{n}\right]=\int_{-\infty}^{\infty} t^{n} d \varphi(t), \quad n=0,1,2, \ldots \tag{2.1.8}
\end{equation*}
$$

The proof of (2.1.8), which follows from (2.1.7), can also be found in Chihara [1978].

Having established the existence of the sequence of orthogonal polynomials $\left\{Q_{\mathrm{n}}(z)\right\}$ for $\mathrm{I}_{\mathrm{q}}[\cdot]$, we can now define a second sequence of polynomials $\left\{P_{n}(z)\right\}$, where $P_{n}(z)$ is of degree ( $n-1$ ). Specifically

$$
P_{n}(z)=I_{q}\left[\frac{Q_{n}(z)-Q_{n}(t)}{z-t}\right], \quad n=0,1,2, \ldots .
$$

These polynomials $\mathrm{F}_{\mathrm{n}}(z)$ are the associated polynomials appearing in the theory of orthogonal polynomials. For convenience, setting the leading coefficient of $Q_{n}(z)$ to be unity, we find the following three term relations are satisfied. (See Erdelyi [1953], Szegö [1959] and Chihara [1978].)

$$
\begin{aligned}
& P_{n+1}(z)=\left(z+b_{n+1}\right) P_{n}(z)-a_{n+1} P_{n-1}(z), \\
& Q_{n+1}(z)=\left(z+b_{n+1}\right) Q_{n}(z)-a_{n+1} Q_{n-1}(z),
\end{aligned}
$$

with $Q_{0}(z)=1, \quad P_{0}(z)=0, Q_{1}(z)=\left(z+q_{1}, 0\right), P_{1}(z)=c_{0}$.

The coefficients $a_{n+1}$ are non-zero for all $n \geqslant 1$. If the moment functional $\mathrm{I}_{\mathrm{q}}[\cdot]$ is positive definite, then all the $\mathrm{a}_{\mathrm{n}+1}$ are positive and all the $b_{n+1}$ are real.

These three term relations immediately suggest that the quotient $P_{n}(z) / Q_{n}(z)$ must be the $n$-th convergent of the J-fraction

$$
\frac{c_{0}}{z+q_{1,0}}-\frac{a_{2}}{z+b_{2}}-\frac{a_{3}}{z+b_{3}}-\frac{a_{4}}{z+b_{4}}-\ldots
$$

It also follows from the definition of $P_{n}(z)$ that

$$
\begin{aligned}
\frac{P_{n}(z)}{Q_{n}(z)} & =\frac{1}{Q_{n}(z)} I_{q}\left[\frac{Q_{n}(z)-Q_{n}(t)}{z-t}\right] \\
& =I_{q}\left[\frac{1}{z-t}\right]-\frac{1}{Q_{n}(z)} I_{q}\left[\frac{Q_{n}(t)}{z-t}\right] .
\end{aligned}
$$

Expanding the second term on the right hand side above in terms of negative powers of $z$, we find

$$
\frac{P_{n}(z)}{Q_{n}(z)}=I_{q}\left[\frac{1}{z-t}\right]-\left\{z^{-n}+\alpha_{1}^{(n)} z^{-n-1}+\ldots\right\} I_{q}\left[\sum_{s=0}^{\infty} Q_{n}(t) t^{s} z^{-s-1}\right]
$$

Hence, applying the orthogonality property (2.1.5), it follows that

$$
\frac{P_{n}(z)}{Q_{n}(z)}=I_{q}\left[\frac{1}{z-t}\right]+\gamma_{1}^{(n)} z^{-2 n-1}+\text { lower order terms. }
$$

That is, the quotient $P_{n}(z) / Q_{n}(z)$ corresponds to the function given by

$$
S(z)=I_{q}\left[\frac{1}{z-t}\right]
$$

The function $S(z)$ is often referred to as the Stieltjes function in the case where the moment functional is positive definite. Expanding the function $S(z)$ about infinity and using (2.1.4) shows that it has the formal expansion

$$
S(z)=\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\frac{c_{2}}{z^{3}}+\cdots
$$

This establishes the relation between corresponding J-fractions and orthogonal polynomials.

As do those of the J-fractions, the denominator polynomials of M-fractions also exhibit some interesting properties. The remainder of this chapter is devoted to looking at some of these properties.

Given a double sequence $\left\{c_{n}\right\}_{n=-\infty}^{\infty}$ of finite valued complex numbers, we can define a linear moment functional I[•] on the vector space of functions spanned by $\ldots, t^{-2}, t^{-1}, 1, t, t^{2}, \ldots$, over a complex field as

$$
\begin{equation*}
I\left[t^{n}\right]=c_{n}, \quad n=\ldots,-2,-1,0,1,2, \ldots \tag{2.1.9}
\end{equation*}
$$

Hence, the Stieltjes function given by

$$
\begin{equation*}
S(z)=I\left[\frac{1}{z-t}\right], \tag{2.1.10}
\end{equation*}
$$

has the following two formal power series expansions

$$
\begin{equation*}
f_{0}(z)=\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\frac{c_{2}}{z^{3}}+\cdots, \tag{2.1.11}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{0}(z)=-c_{-1}-c_{-2} z-c_{-3} z^{2}-\ldots . \tag{2.1.12}
\end{equation*}
$$

Brezinski [1980] has shown that for any polynomial $B_{n}(z)$ of degree $n$ then, with the associated polynomial $A_{n}(z)$ defined by

$$
A_{n}(z)=I\left[\frac{B_{n}(z)-B_{n}(t)}{z-t}\right]
$$

the quotient $A_{n}(z) / B_{n}(z)$, which is referred to as a Pade type approximant of $S(z)$, satisfies

$$
f_{0}(z)-\frac{A_{n}(z)}{B_{n}(z)}=\mu_{1}^{(n)} z^{-n-1}+\text { lower order terms. }
$$

Hence, if we take $B_{n}(z)$ to be the special $n$-th degree polynomial satisfying the relation

$$
I\left[t^{-n+s} B_{n}(t)\right]=\left\{\begin{array}{lc}
0, & 0 \leqslant s \leqslant n-1  \tag{2.1.13}\\
\omega_{n}, & s=n
\end{array}\right.
$$

then the quotient $A_{n}(z) / B_{n}(z)$ also corresponds to $g_{0}(z)$ such that

$$
g_{0}(z)-\frac{A_{n}(z)}{B_{n}(z)}=\mu_{2}^{(n)} z^{n}+\text { higher order terms, }
$$

provided $B_{n}(0) \neq 0$. This can be seen as follows.
We have from the definition of $A_{n}(z)$ that

$$
\begin{aligned}
\frac{A_{n}(z)}{B_{n}(z)} & =\frac{1}{B_{n}(z)} I\left[\frac{B_{n}(z)-B_{n}(t)}{z-t}\right] \\
& =I\left[\frac{1}{z-t}\right]-\frac{1}{B_{n}(z)} I\left[\frac{B_{n}(t)}{z-t}\right] .
\end{aligned}
$$

Expanding the right hand side about the origin, we find

$$
\frac{A_{n}(z)}{B_{n}(z)}=g_{0}(z)-\left\{\frac{1}{B_{n}(0)}+\gamma_{n, 1} z+\ldots\right\} I\left[\sum_{s=0}^{\infty} B_{n}(t) t^{-s-1} z^{s}\right]
$$

Hence, using the relation (2.1.13) immediately gives the required result. Conditions for the existence of the polynomials $B_{n}(z)$, $(\mathrm{n} \geqslant 0)$ will be considered in Section 2.2 .

To show that the quotient $A_{n}(z) / B_{n}(z)$ is the $n$-th convergent of the M-fraction that corresponds to the formal expansions $f_{0}(z)$ and $g_{0}(z)$, it is only necessary to show that $A_{n}(z)$ and $B_{n}(z)$ satisfy the three term relations

$$
\begin{aligned}
& A_{n+1}(z)=\left(z+d_{n+1}\right) A_{n}(z)+n_{n+1} z A_{n-1}(z), \\
& \therefore \quad B_{n+1}(z)=\left(z+d_{n+1}\right) B_{n}(z)+n_{n+1} z B_{n-1}(z), \\
& \text { with } B_{0}(z)=1, A_{0}(z)=0, B_{1}(z)=\left(z+b_{1}, 0\right) \text { and } A_{1}(z)=c_{0} .
\end{aligned}
$$

Here, the polynomials $B_{n}(z)$ are also assumed to be monic.
The polynomials $A_{n}(z)$ and $B_{n}(z)$ do satisfy these three term relations and the proof is given in Section 2.3.

The polynomials $B_{n}(z), n \geqslant 0$, are not orthogonal, but the functions defined for $n \geqslant 0$ by

$$
\mathbb{R}_{2 n}(z)=z^{-n} B_{2 n}(z)
$$

and

$$
\mathbb{R}_{2 n+1}(z)=z^{-n-1} B_{2 n+1}(z)
$$

do form an orthogonal sequence, with respect to the moment functional $I[\cdot]$. Some of the properties of these orthogonal functions $\mathbb{R}_{n}(z)$, and their relations to the M-fraction that correspond to the Stieltjes function S(z), were first considered by Jones and Thron [1981] in a study that was carried out in parallel to the work described in this chapter. In their study Jones and Thron only considered positive definite functionals.

Even though the polynomials $B_{n}(z)$ are not orthogonal it will be shown in later sections, that they do behave in many ways as orthogonal polynomials. In view of this we will refer to them as "pseudo orthogonal polynomials".

### 2.2 GENERAL PSEUDO ORTHOGONAL POLYNOMIALS

The polynomials $B_{n}^{(k)}(z), n=0,1,2, \ldots$, which we shall call the $k$-th order pseudo orthogonal polynomials, are given by

$$
\begin{equation*}
B_{n}^{(k)}(z)=\sum_{r=0}^{n} b_{n, n-r^{(k)}}^{z^{n-r}}, \quad b_{n, n}^{(k)} \neq 0 \tag{2.2.1a}
\end{equation*}
$$

and

Here, for convenience, the polynomials $B_{n}^{(k)}(z)$ are also considered to be monic. That is $b_{n, n}^{(k)}=1$, for all $n \geqslant 0$.

If $k=0$ then the resulting polynomials $B_{n}^{(0)}(z), n \geqslant 0$, are the polynomials $B_{n}(z), n \geqslant 0$, described in Section 2.1 .

Using (2.1.9) we can write (2.2.1) as a system of simultaneous equations in the coefficients of $B_{n}^{(k)}(z)$. That is,

$$
\begin{gather*}
c_{k-n} b_{n, 0}^{(k)}+c_{k-n+1} b_{n, 1}^{(k)}+\ldots+c_{k} b_{n, n}^{(k)}=0 \\
c_{k-n+1} b_{n, 0}^{(k)}+c_{k-n+2} b_{n, 1}^{(k)}+\ldots+c_{k+1} b_{n, n}^{(k)}=0, \\
c_{k-1} b_{n, 0}^{(k)}+c_{k} b_{n, 1}^{(k)}+\ldots+c_{k+n-1} b_{n, n}^{(k)}=0,  \tag{2.2.2}\\
c_{k} b_{n, 0}^{(k)}+c_{k+1} b_{n, 1}^{(k)}+\ldots+c_{k+n} b_{n, n}^{(k)}=\omega_{n}^{(k)} .
\end{gather*}
$$

Now using Cramer's rule, the coefficient $b_{n, n}^{(k)}$ can be expressed as

$$
b_{n, n}^{(k)}=\omega_{n}^{(k)} H_{n}^{(k-n)} / H_{n+1}^{(k-n)}, \quad n \geqslant 0
$$

where $H_{r}^{(m)}$ are the Hankel determinants given by (1.2.4).
Since, $\mathrm{b}_{\mathrm{n}, \mathrm{n}}^{(\mathrm{k})}$ is considered to be unity we must have

$$
\begin{equation*}
\omega_{n}^{(k)}=H_{n+1}^{(k-n)} / H_{n}^{(k-n)}, \quad n \geqslant 0 \tag{2,2.3}
\end{equation*}
$$

If we replace the last equation of (2.2.2) by

$$
1 \cdot b_{n, 0}^{(k)}+z b_{n, 1}^{(k)}+\ldots+z^{n} b_{n, n}^{(k)}=B_{n}^{(k)}(z)
$$

and again apply the Cramer's rule we obtain

$$
B_{n}^{(k)}(z)=\frac{1}{H_{n}^{(k-n)}}\left|\begin{array}{llll}
c_{k-n} & c_{k-n+1} & \cdots \cdot & c_{k}  \tag{2.2.4}\\
c_{k-n+1} & c_{k-n+2} & \cdots & c_{k+1} \\
\cdots & \cdots & \cdots & \cdots \\
c_{k-1} & c_{k} & \cdots & \cdots \\
1 & z & \cdots & c_{k+n-1} \\
1
\end{array}\right|, \quad n \geqslant 0
$$

Thus, we see that a condition necessary and sufficient for the existence of the pseudo orthogonal polynomial $B_{n}^{(k)}(z)$ is

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}^{(\mathrm{k}-\mathrm{n})} \neq 0 \tag{2.2.5}
\end{equation*}
$$

In (2.2.4) if we let $z=0$, then we have

$$
\begin{equation*}
B_{n}^{(k)}(0)=b_{n, 0}^{(k)}=(-1)^{n_{n}} H_{n}^{(k-(n-1))} / H_{n}^{(k-n)}, \quad n \geqslant 0 \tag{2.2.6}
\end{equation*}
$$

Hence, a condition necessary and sufficient for the existence of the polynomial $B_{n}^{(k)}(z)$ with $B_{n}^{(k)}(0) \neq 0$, is

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}^{(\mathrm{k}-\mathrm{n})} \neq 0 \text { and } \mathrm{H}_{\mathrm{n}}^{(\mathrm{k}-(\mathrm{n}-1))} \neq 0 \tag{2.2.7}
\end{equation*}
$$

Using the relation

$$
c_{k-n-1} b_{n, 0}^{(k)}+c_{k-n} b_{n, 1}^{(k)}+\ldots+c_{k-1} b_{n, n}^{(k)}=I\left[t^{k-(n+1)} B_{n}^{(k)}(t)\right]
$$

in (2.2.2) we also find

$$
\begin{equation*}
v_{n}^{(k)}=I\left[t^{k-(n+1)} B_{n}^{(k)}(t)\right]=(-1)^{n_{H}} H_{n+1}^{(k-(n+1))} / H_{n}^{(k-n)}, \quad n \geqslant 0 \tag{2.2.8}
\end{equation*}
$$

Let us denote the moment functional which satisfies condition (2.2.7) for all $n \geqslant 0$ as $I_{q, M}^{(k)}$ and refer to it as a " $k$-th order quasi M-definite moment functional"; following Chihara's [1978] use of the term quasi definite in the case of ordinary orthogonal polynomials. Thus it follows that for a $k$-th order quasi M-definite moment functional there always exists a sequence of polynomials $\left\{B_{n}^{(k)}(z)\right\}$, defined by $(2.2 .1)$, such that $B_{n}^{(k)}(0) \neq 0$ for all $n \geqslant 0$. It will also be shown that there always exists an $M^{(k)}$-fraction associated with the $I_{q, M}^{(k)}[\cdot]$.

We now define the two special functionals, namely the positive definite and the positive M-definite moment functionals as follows.

LEMMA 2.2.1 : The moment functional I[•], defined by (2.1.9), is said to be positive definite, if all the moments $c_{n}, n=\ldots,-2,-1,0$, $1,2, \ldots$, are real and, for any function $\pi(t)$ belonging to the set of functions spanned by $\ldots 1 / \mathrm{t}^{2}, 1 / \mathrm{t}, 1, \mathrm{t}, \mathrm{t}^{2}, \ldots$, over a real field,

$$
\mathrm{I}[\pi(\mathrm{t})]>0,
$$

provided that $\pi(t)$ is also non-negative but not identically equal to zero for $t \in E \subset(-\infty, \infty)$. Further, if the set $E$ lies entirely on the positive half of the real axis then the moment functional is also called a positive M-definite moment functional.

The set E in Lemma 2.2.1 is referred to as a supporting set of the positive definite or the positive M-definite moment functional. We shall denote the positive definite and the positive M-definite moment functionals as $I_{p}[\cdot]$ and $I_{p, M}[\cdot]$, respectively. Hence, for an $I_{p}[\cdot]$, we must have

$$
I_{p}\left[t^{2 m}\left\{\sum_{s=-n}^{n} \xi_{s} t^{s}\right\}^{2}\right]>0
$$

for

$$
\sum_{s=-n}^{n} \xi_{s}^{2} \neq 0,
$$

where n is any positive integer and m is any integer, positive or negative.

We note that this relation is equivalent to the positive definiteness of the matrix given by

$$
\left[\begin{array}{llll}
c_{2 m-2 n} & c_{2 m-2 n+1} & \cdots & c_{2 m} \\
c_{2 m-2 n+1} & c_{2 m-2 n+2} & \cdots & c_{2 m+1} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
c_{2 m} & c_{2 m+1} & & c_{2 m+2 n}
\end{array}\right]
$$

From this we obtain, a necessary condition for the existence of the positive definite moment functional $I_{p}[\cdot]$, is (see for example Wall [1948])

$$
\begin{equation*}
H_{2 n}^{(2 m-2 n)}>0 \text { and } H_{2 n+1}^{(2 m-2 n)}>0 \tag{2.2.9}
\end{equation*}
$$

for all m and $\mathrm{n} \geqslant 0$.
Similarly, for the positive M-definite moment functional
$I_{p, M}[\cdot]$, we must have

$$
I_{P, M}\left[t^{m}\left\{\sum_{s=l_{1}}^{\ell_{2}} \xi_{s} t^{s}\right\}^{2}\right]>0
$$

for

$$
\sum_{s=\ell_{1}}^{\ell_{2}} \xi_{s}^{2} \neq 0
$$

where $m, \ell_{1}, \ell_{2}$ are any positive or negative integers with $\ell_{1} \leqslant \ell_{2}$.
Thus, from this we obtain, a necessary condition for the existence of the positive $M$-definite moment functional $I_{p, M}[\cdot]$, is

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}^{(\mathrm{m}-\mathrm{n})}>0 \text { and } \mathrm{H}_{\mathrm{n}+1}^{(\mathrm{m}-\mathrm{n})}>0 \tag{2.2.10}
\end{equation*}
$$

for all $m$ and $n \geqslant 0$.
It is seen from condition (2.2.10) that a positive M-definite moment functional is also a $k$-th order quasi M-definite moment functional.

With respect to the moment functional $I[\cdot]$, we now define the two associated functions $0_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z)$ by

$$
\begin{align*}
& O_{n}^{(k)}(z)=I\left[t^{k}\left\{B_{n}^{(k)}(z)-B_{n}^{(k)}(t)\right\} /(z-t)\right], \quad n \geqslant 0,  \tag{2.2.11}\\
& A_{n}^{(k)}(z)=\frac{1}{z^{k}} I\left[\left\{z_{n}^{k_{n}} n_{n}^{(k)}(z)-t_{B}^{\left.\left.k_{n}^{(k)}(t)\right\} /(z-t)\right], \quad n \geqslant 0 .}\right.\right. \tag{2.2.12}
\end{align*}
$$

Then the quotients $O_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ satisfy for all $n \geqslant 0$

$$
\begin{aligned}
\frac{0_{n}^{(k)}(z)}{B_{n}^{(k)}(z)} & =\frac{c_{k}}{z}+\frac{c_{k+1}}{z^{2}}+\ldots+\frac{c_{k+n-1}}{z^{n}}+\text { lower order terms, } \\
& =-c_{k-1}-c_{k-2} z-\ldots-c_{k-n^{2}} z^{n-1}+\text { higher order terms, } \\
\frac{A_{n}^{(k)}(z)}{B_{n}^{(k)}(z)} & =f_{0}(z)+\frac{\lambda_{n}^{(k)}}{z^{n+k+1}}+\text { lower order terms }, \\
& =g_{0}(z)+\mu_{n}^{(k)} z^{n-k}+\text { higher order terms. }
\end{aligned}
$$

To prove these results, we only need to consider (2.2.11) and (2.2.12). From these it follows that

$$
\frac{O_{n}^{(k)}(z)}{B_{n}^{(k)}(z)}=I\left[\frac{t^{k}}{z-t}\right]-\frac{1}{B_{n}^{(k)}(z)} I\left[\frac{t^{k} B_{n}^{(k)}(t)}{z-t}\right]
$$

and

$$
\frac{A_{n}^{(k)}(z)}{B_{n}^{(k)}(z)}=I\left[\frac{1}{z-t}\right]-\frac{1}{z^{k} B_{n}^{(k)}(z)} I\left[\frac{t^{k_{B}(k)}(t)}{z-t}\right]
$$

Hence, expanding the right hand sides of these equations about the origin and also about infinity, and then using the relation (2.2.1), immediately gives the required results. .

From (2.2.11) it is easily seen that the associated function ${ }_{0}^{(k)}(z)$ is a polynomial of degree $(n-1)$. On the other hand the associated functions $A_{n}^{(k)}(z)$ are not always polynomials but, even though it may not be apparent from (2.2.12), they are polynomials of degree $(n-1)$, when $n \geqslant k$.
2.3 SOME PROPERTIES OF THE PSEUDO ORTHOGONAL POLYNOMIALS AND ASSOCIATED FUNCTIONS.

The most interesting and useful property of the pseudo orthogonal polynomials $B_{n}^{(k)}(z)$ and their associated functions $O_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z)$ is that when the corresponding moment functional is the k-th order quasi M-definite moment functional, they all satisfy the following three term relations.

$$
\begin{align*}
& B_{n+1}^{(k)}(z)=\left(z+d_{n+1}^{(k)}\right) B_{n}^{(k)}(z)+n_{n+1}^{(k)} z B_{n-1}^{(k)}(z),  \tag{2.3.1a}\\
& 0_{n+1}^{(k)}(z)=\left(z+d_{n+1}^{(k)}\right) O_{n}^{(k)}(z)+n_{n+1}^{(k)} z O_{n-1}^{(k)}(z),  \tag{2.3.1b}\\
& A_{n+1}^{(k)}(z)=\left(z+d_{n+1}^{(k)}\right) A_{n}^{(k)}(z)+n_{n+1}^{(k)} z A_{n-1}^{(k)}(z), \tag{2.3.1c}
\end{align*}
$$

for all $n \geqslant 1$. The coefficients $n_{n+1}^{(k)}$ and $d_{n+1}^{(k)}$ are given by

$$
\begin{equation*}
n_{n+1}^{(k)}=-\omega_{n}^{(k)} / \omega_{n-1}^{(k)}, \quad n \geqslant 1 \tag{2.3.2}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{n+1}^{(k)}=-n_{n+1}^{(k)} v_{n-1}^{(k)} / v_{n}^{(k)}, \quad n \geqslant 1 \tag{2.3.3}
\end{equation*}
$$

where $\omega_{n}^{(k)}$ and $v_{n}^{(k)}$ are defined by (2.2.3) and (2.2.8), respectively. To prove these results, one has to consider first the polynomials $B_{n}^{(k)}(z) \quad(n \geqslant 0)$. Since $B_{n}^{(k)}(z)$ is a monic polynomial of degree $n$, the polynomial given by $\left\{B_{n+1}^{(k)}(z)-z B_{n}^{(k)}(z)\right\}$ is al so at most of degree $n$. Hence, this polynomial can be expressed in the form

$$
\begin{aligned}
\left\{B_{n+1}^{(k)}(z)-z B_{n}^{(k)}(z)\right\} & =d_{n+1}^{(k)} B_{n}^{(k)}(z)+\left\{n_{n+1}^{(k)} z+p_{n-1}^{(n, k)}\right\} B_{n-1}^{(k)}(z) \\
& +p_{n-2}^{(n, k)} B_{n-2}^{(k)}(z)+\ldots+p_{0}^{(n, k)} B_{0}^{(k)}(z) .
\end{aligned}
$$

Multiplying by $z^{k+r}$ and using the functional $I[\cdot]$, gives

$$
\begin{align*}
I\left[t ^ { k + r } \left\{B_{n+1}^{(k)}(t)-\right.\right. & \left.\left.t B_{n}^{(k)}(t)\right\}\right] \\
= & I\left[t ^ { k + r } \left\{d_{n+1}^{(k)} B_{n}^{(k)}(t)+\left\{n_{n+1}^{(k)} t+p_{n-1}^{(n, k)}\right\} B_{n-1}^{(k)}(t)\right.\right. \\
& \left.\left.+p_{n-2}^{(n, k)} B_{n-2}^{(k)}(t)+\ldots+p_{0}^{(n, k)} B_{0}^{(k)}(t)\right\}\right] \tag{2.3.4}
\end{align*}
$$

Therefore, by letting $r=-1,-2, \ldots,-n$, we obtain

$$
\begin{aligned}
-\omega_{n}^{(k)} & =n_{n+1}^{(k)} \omega_{n-1}^{(k)}+0+\ldots \ldots \ldots \ldots \ldots \ldots+0+p_{0}^{(n, k)} v_{0}^{(k)} \\
0 & =0+\ldots \ldots \ldots+0+p_{1}^{(n, k)} v_{1}^{(k)}+p_{0}^{(n, k)} I\left[t^{k-2} B_{0}^{(k)}(t)\right]
\end{aligned}
$$

$$
0=0+p_{n-1}^{(n, k)} v_{n-1}^{(k)}+p_{n-2}^{(n, k)} I\left[t^{k-n_{B}(k)}(t)\right]+\ldots+p_{0}^{(n, k)} I\left[t^{k-n_{B}(k)}(t)\right]
$$

If the functional $I[\cdot]$ is the $k$-th order quasi M-definite moment functional then for all $n \geqslant 0$ we must have the condition (2.2.7) to hold. Thus, from (2.2.3) and (2.2.8), we see that the coefficients $\omega_{n}^{(k)}$ and $\nu_{n}^{(k)}$ are non-zero for all $n \geqslant 0$, and so the coefficients $p_{r}^{(n, k)}(r=0,1, \ldots, n-1)$ can be made equal to zero by taking

$$
-\omega_{n}^{(k)}=n_{n+1}^{(k)} \omega_{n-1}^{(k)}
$$

The result (2.3.2) follows at once from the fact $\omega_{r}^{(k)} \neq 0$ for all r . This also establishes the three term relation (2.3.1a).

To find $d_{n+1}^{(k)}$, we set $r=n+1$ in (2.3.4) to give

$$
0=d_{n+1}^{(k)} \nu_{n}^{(k)}+n_{n+1}^{(k)} \nu_{n-1}^{(k)}
$$

and from this (2.3.3) follows immediately.
The three term relation (2.3.1a) can now be used to prove (2.3.1b) and (2.3.1c). We have

$$
\begin{aligned}
\left\{B_{n+1}^{(k)}(z)-B_{n+1}^{(k)}(t)\right\}= & \left\{z+d_{n+1}^{(k)}\right\}\left\{B_{n}^{(k)}(z)-B_{n}^{(k)}(t)\right\}+n_{n+1}^{(k)} z\left\{B_{n-1}^{(k)}(z)-B_{n-1}^{(k)}(t)\right\} \\
& +(z-t)\left\{B_{n}^{(k)}(t)+n_{n+1}^{(k)} B_{n-1}^{(k)}(t)\right\}
\end{aligned}
$$

Multiplying by $t^{k} /(z-t)$ and using the functional $I_{q, M}[\cdot]$, we arrive at

$$
0_{n+1}^{(k)}(z)=\left(z+d_{n+1}^{(k)}\right) 0_{n}^{(k)}(z)+n_{n+1}^{(k)} z 0_{n-1}^{(k)}(z)+\omega_{n}^{(k)}+n_{n+1}^{(k)} \omega_{n-1}^{(k)}
$$

Thus, from (2.3.2), we obtain the three term relation (2.3.1b).
Now to show the relation (2.3.1c), it is only required to look at (2.2.11) and (2.2.12). From this we obtain

$$
A_{n}^{(k)}(z)=z^{-k} I\left[\frac{z^{k}-t^{k}}{z-t}\right] B_{n}^{(k)}(z)+z^{-k_{0}(k)}(z)
$$

Since, on the right hand side above the coefficients of $B_{n}^{(k)}(z)$ and $O_{n}^{(k)}(z)$ are independent of $n$, we immediately obtain from (2.3.1a) and (2.3.1b) the relation (2.3.1c).

We have for (2.3.1) the initial conditions

$$
\begin{array}{ll}
B_{0}^{(k)}(z)=1, & B_{1}^{(k)}(z)=z+d_{1}^{(k)},\left(d_{1}^{(k)}=b_{1,0}^{(k)}\right) \\
0_{0}^{(k)}(z)=0, & 0_{1}^{(k)}(z)=c_{k}, \\
A_{0}^{(k)}(z)=z^{-k} I\left[\frac{z^{k}-t^{k}}{z-t}\right], & A_{1}^{(k)}(z)=A_{0}^{(k)}(z) B_{1}^{(k)}(z)+z^{-k} 0_{1}^{(k)}(z) .
\end{array}
$$

The three term relations (2.3.1), together with the above initial conditions, show that the quotient $O_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ is the n-th convergent of the M -fraction

$$
\frac{c_{k}}{z+d_{1}^{(k)}}+\frac{n_{2}^{(k)} z}{z+d_{2}^{(k)}}+\frac{n_{3}^{(k)} z}{z+d_{3}^{(k)}}+\frac{n_{4}^{(k)} z}{z+d_{4}^{(k)}}+\ldots
$$

and that the quotient $A_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ is the $n$-th convergent of the $M^{(k)}$-fraction

$$
z^{-k} I\left[\frac{z^{k}-t^{k}}{z-t}\right]+\frac{z^{-k} c_{k}}{z+d_{1}^{(k)}}+\frac{n_{2}^{(k)} z}{z+d_{2}^{(k)}}+\frac{n_{3}^{(k)} z}{z+d_{3}^{(k)}}+\frac{n_{4}^{(k)} z}{z+d_{4}^{(k)}}+\ldots
$$

$$
\text { By substituting from }(2.2 .3) \text { and }(2.2 .8) \text { respectively in }
$$ (2.3.2) and (2.3.3), we can write

$$
\begin{gather*}
n_{n+1}^{(k)}=\frac{-H_{n+1}^{(k-n)} \cdot H_{n-1}^{(k-(n-1))}}{H_{n}^{(k-n)} \cdot H_{n}^{(k-(n-1))}}, \quad n \geqslant 1  \tag{2.3.5}\\
d_{n+1}^{(k)}=\frac{-H_{n+1}^{(k-n)} \cdot H_{n}^{(k-n)}}{H_{n}^{(k-(n-1))} \cdot H_{n+1}^{(k-(n+1))}}, \quad n \geqslant 1 \tag{2.3.6}
\end{gather*}
$$

The latter equation also holds for $n$ equal to zero in view of $d_{1}^{(k)}=b_{1,0}^{(k)}$.

As a consequence of the three term relation (2.3.1), many results concerning $B_{n}^{(k)}(z), O_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z)$ can also be found. For instance the so called determinant relations are
$\left\{0_{n}^{(k)}(z) B_{n-1}^{(k)}(z)-0_{n-1}^{(k)} B_{n}^{(k)}(z)\right\}=(-1)^{n} n_{n}^{(k)} n_{n-1}^{(k)} \ldots n_{2}^{(k)} c_{k^{2}}^{n-1}, \quad n \geqslant 2$,
$\left\{A_{n}^{(k)}(z) B_{n-1}^{(k)}(z)-A_{n-1}^{(k)}(z) B_{n}^{(k)}(z)\right\}=(-1)^{n} n_{n}^{(k)} n_{n-1}^{(k)} \ldots n_{2}^{(k)} c_{k} z^{n-k-1}, n \geqslant 2$,
and the recurrence relation

$$
\begin{equation*}
G_{n+1}^{(k)}(z)=\left\{B_{n}^{(k)}(z)\right\}^{2}+n_{n+1}^{(k)} d_{n}^{(k)}\left\{B_{n-1}^{(k)}(z)\right\}^{2}+n_{n+1}^{(k)} n_{n}^{(k)} z_{2}^{2} G_{n-1}^{(k)}(z) \tag{2.3.8}
\end{equation*}
$$

for $n>1$, where

$$
G_{n}^{(k)}(z)=\left\{B_{n}^{\prime(k)}(z) B_{n-1}^{(k)}(z)-B_{n-1}^{\prime(k)}(z) B_{n}^{(k)}(z)\right\}, \quad n \geqslant 1
$$

Here, the function $B_{n}^{\prime(k)}(z)$ is the derivative of $B_{n}^{(k)}(z)$.

From (2.3.8) in particular we obtain

$$
\begin{align*}
G_{2 n+1}^{(k)}(z) & =\left\{B_{2 n}^{(k)}(z)\right\}^{2}+n_{2 n+1}^{(k)} d_{2 n}^{(k)}\left\{B_{2 n-1}^{(k)}(z)\right\}^{2}+n_{2 n+1}^{(k)} n_{2 n}^{(k)} z^{2}\left\{B_{2 n-2}^{(k)}(z)\right\}^{2} \\
& +n_{2 n+1}^{(k)} n_{2 n}^{(k)} n_{2 n-1}^{(k)} d_{2 n-2}^{(k)} z^{2}\left\{B_{2 n-3}^{(k)}(z)\right\}^{2}+\ldots \\
& \ldots+n_{2 n+1}^{(k)} n_{2 n}^{(k)} \ldots \ldots n_{4}^{(k)} n_{3}^{(k)} d_{2}^{(k)} z^{2 n-2}\left\{B_{1}^{(k)}(z)\right\}^{2} \\
& +n_{2 n+1}^{(k)} n_{2 n}^{(k)} \ldots \ldots n_{4}^{(k)} n_{3}^{(k)} n_{2}^{(k)} z_{2 n}^{2 n}\left\{B_{0}^{(k)}(z)\right\}^{2}, \tag{2.3.9a}
\end{align*}
$$

and

$$
\begin{align*}
& G_{2 n}^{(k)}(z)=\left\{B_{2 n-1}^{(k)}(z)\right\}^{2}+n_{2 n}^{(k)} d_{2 n-1}^{(k)}\left\{B_{2 n-2}^{(k)}(z)\right\}^{2}+n_{2 n}^{(k)} n_{2 n-1}^{(k)} z^{2}\left\{B_{2 n-3}^{(k)}(z)\right\}^{2} \\
& +n_{2 n}^{(k)} n_{2 n-1}^{(k)} n_{2 n-2}^{(k)} d_{2 n-3}^{(k)} z^{2}\left\{B_{2 n-4}^{(k)}(z)\right\}^{2}+\ldots . \\
& \ldots+n_{2 n}^{(k)} n_{2 n-1}^{(k)} \cdots \cdots n_{4}^{(k)} n_{3}^{(k)} z^{2 n-2}\left\{B_{1}^{(k)}(z)\right\}^{2} \\
& +n_{2 n}^{(k)} n_{2 n-1}^{(k)} \cdots \cdots \cdot n_{4}^{(k)} n_{3}^{(k)} n_{2}^{(k)} d_{1}^{(k)} z^{2 n-2}\left\{B_{0}^{(k)}(z)\right\}^{2} . \tag{2.3.9b}
\end{align*}
$$

For the $k$-th order quasi M-definite moment functional, the polynomials $B_{n}^{(k)}(z)$ and the associated functions $A_{n}^{(k)}(z)$ also satisfy the following property.

$$
\begin{align*}
& B_{n}^{(k+1)}(z)=B_{n}^{(k)}(z)+n_{n+1}^{(k)} B_{n-1}^{(k)}(z), \quad n \geqslant 1,  \tag{2.3.10a}\\
& A_{n}^{(k+1)}(z)=A_{n}^{(k)}(z)+n_{n+1}^{(k)} A_{n-1}^{(k)}(z), \quad n \geqslant 1 . \tag{2.3.10~b}
\end{align*}
$$

These results can be easily proved by first considering the polynomial $\left\{B_{n}^{(k+1)}(z)-B_{n}^{(k)}(z)\right\}$ and then using similar analysis to that of the proof of (2.3.1).

From (2.2.5) and (2.2.7) we see that for the $k-t h$ order quasi M-definite moment functional, all the polynomials $\mathrm{B}_{\mathrm{n}}^{(\mathrm{k})}(\mathrm{z}),(\mathrm{n} \geqslant 1)$, exist and they also satisfy the property $\mathrm{B}_{\mathrm{n}}^{(\mathrm{k})}(0) \neq 0,(n \geqslant 1)$. On the other hand, for the same moment functional, all the polynomials $B_{n}^{(k+1)}(z),(n \geqslant 1)$ do exist, but they may not satisfy $B_{n}^{(k+1)}(0) \neq 0$, $(n \geqslant 1)$.

The relations (2.3.10) can be used together with the three term relation (2.3.1) to obtain other three term relations such as

$$
\begin{aligned}
& B_{2 n}^{(k)}(z)=\left(z+D_{2 n}^{(k)}\right) B_{2 n-1}^{(k+1)}(z)-N_{2 n}^{(k)} B_{2 n-2}^{(k)}(z), \\
& A_{2 n}^{(k)}(z)=\left(z+D_{2 n}^{(k)}\right) A_{2 n-1}^{(k+1)}(z)-N_{2 n}^{(k)} A_{2 n-2}^{(k)}(z),
\end{aligned}
$$

(2,3.11)

$$
\begin{array}{r}
B_{2 n+1}^{(k+1)}(z)=\left\{\left(1+N_{2 n+1}^{(k)}\right) z+D_{2 n+1}^{(k)}\right\} B_{2 n}^{(k)}(z)-N_{2 n+1}^{(k)} z^{2} B_{2 n-1}^{(k+1)}(z) \\
n \geqslant 1,
\end{array}
$$

$$
A_{2 n+1}^{(k+1)}(z)=\left\{\left(1+N_{2 n+1}^{(k)}\right\} z+D_{2 n+1}^{(k)}\right\} A_{2 n}^{(k)}(z)-N_{2 n+1}^{(k)} z^{2} A_{2 n-1}^{(k+1)}(z)
$$

where

$$
\begin{align*}
& N_{2 n}^{(k)}=n_{2 n}^{(k)} d_{2 n}^{(k)}, \quad N_{2 n+1}^{(k)}=n_{2 n+1}^{(k)} / d_{2 n}^{(k)}, \\
& D_{2 n}^{(k)}=d_{2 n}^{(k)}, \quad D_{2 n+1}^{(k)}=d_{2 n+1}^{(k)}+n_{2 n+2}^{(k)} . \tag{2.3.12}
\end{align*}
$$

Using the relations (2.3.5) and (2.3.6) and the Jacobi identity (1.2.11), we can also write the coefficients $N_{r}^{(k)}$ and $D_{r}^{(k)}$ as

$$
\begin{equation*}
N_{2 n}^{(k)}=\left\{\frac{H^{(k-(2 n-1))}}{H_{2 n-1}^{(k-(2 n-2))}}\right\}^{2} \frac{H_{2 n-2}^{(k-(2 n-2))}}{H_{2 n}^{(k-2 n)}}, \quad N_{2 n+1}^{(k)}=\frac{H_{2 n+1}^{(k-2 n)} H_{2 n-1}^{(k-(2 n-2))}}{\left\{H_{2 n}^{(k-(2 n-1))}\right\}^{2}}, \tag{2,3.13}
\end{equation*}
$$

$D_{2 n}^{(k)}=\frac{-H_{2 n}^{(k-(2 n-1))} H_{2 n-1}^{(k-(2 n-1))}}{H_{2 n-1}^{(k-(2 n-2))} H_{2 n}^{(k-2 n)}}, \quad D_{2 n+1}^{(k)}=\frac{-H_{2 n+1}^{(k-(2 n-1))} H_{2 n}^{(k-2 n)}}{H_{2 n+1}^{(k-2 n)} H_{2 n}^{(k-(2 n-1))}}$,

As a consequence of the three term relations (2.3.11) we find

$$
\begin{array}{r}
\left\{A_{2 n}^{(k)}(z) B_{2 n-1}^{(k+1)}(z)-A_{2 n-1}^{(k+1)}(z) B_{2 n}^{(k)}(z)\right\}=N_{2 n}^{(k)} N_{2 n-1}^{(k)} \ldots N_{2}^{(k)} c_{k} z^{2 n-2-k}, \\
n \geqslant 1, \tag{2.3.14}
\end{array}
$$

$\left\{A_{2 n+1}^{(k+1)}(z) B_{2 n}^{(k)}(z)-A_{2 n}^{(k)}(z) B_{2 n+1}^{(k+1)}(z)\right\}=N_{2 n+1}^{(k)} N_{2 n}^{(k)} \ldots N_{2}^{(k)} c_{k}^{\left(z^{2 n-k},\right.}$
$n \geqslant 1$,
and

$$
\begin{align*}
K_{2 n+1}^{(k)}(z)=\left\{B_{2 n}^{(k)}(z)\right\}^{2} & +N_{2 n+1}^{(k)}\left\{B_{2 n}^{(k)}(z)-z B_{2 n-1}^{(k+1)}(z)\right\}^{2}+N_{2 n+1}^{(k)} N_{2 n}^{(k)} z^{2}\left\{B_{2 n-2}^{(k)}(z)\right\}^{2} \\
& +N_{2 n+1}^{(k)} N_{2 n}^{(k)} N_{2 n-1}^{(k)} z^{2}\left\{B_{2 n-2}^{(k)}(z)-z B_{2 n-3}^{(k+1)}(z)\right\}^{2}+\ldots \\
& \ldots+N_{2 n+1}^{(k)} N_{2 n}^{(k)} \ldots \ldots N_{2}^{(k)} z^{2 n}\left\{B_{0}^{(k)}(z)\right\}^{2}, \\
K_{2 n+2}^{(k)}(z)= & \left\{B_{2 n+1}^{(k+1)}(z)\right\}_{2}^{2}+N_{2 n+2}^{(k)}(z) K_{2 n+1}^{(k)}(z), \tag{2,3,15b}
\end{align*}
$$

where

$$
K_{2 n+1}^{(k)}(z)=\left\{B_{2 n+1}^{\prime(k+1)}(z) B_{2 n}^{(k)}(z)-B_{2 n}^{\prime}(k)(z) B_{2 n+1}^{(k+1)}(z)\right\}, \quad n \geqslant 0
$$

and

$$
K_{2 n+2}^{(k)}(z)=\left\{B_{2 n+2}^{\prime(k)}(z) B_{2 n+1}^{(k+1)}(z)-B_{2 n+1}^{r(k+1)}(z) B_{2 n+2}^{(k)}(z)\right\}, \quad n \geqslant 0
$$

### 2.4 ZEROS AND THE QUADRATURE FORMULA

From (2.2.6) and (2.2.7) we see that if the moment functional is the k -th order quasi M -definite moment functional then $\mathrm{z}=0$ is not a zero of any of the polynomials $B_{n}^{(k)}(z),(n \geqslant 1)$. Furthermore, for this moment functional if we consider the determinant formula (2.3.7), then for $z=z_{i}^{(n, k)}$, a zero of $B_{n}^{(k)}(z)$, we have

$$
o_{n}^{(k)}\left(z_{i}^{(n, k)}\right) B_{n-1}^{(k)}\left(z_{i}^{(n, k)}\right)=(-1)^{n-1} n_{n}^{(k)} n_{n-1}^{(k)} \ldots n^{(k)} c_{k}\left\{z_{i}^{(n, k)}\right\}^{n-1},
$$

$$
A_{n}^{(k)}\left(z_{i}^{(n, k)}\right) B_{n-1}^{(k)}\left(z_{i}^{(n, k)}\right)=(-1)^{n-1} n_{n}^{(k)} n_{n-i}^{(k)} \ldots n^{(k)} c_{k}\left\{z_{i}^{(n, k)}\right\}^{n-k-1}
$$

Hence, we have the following result.

Theorem 2.4.] : If the moment functional.I[•], defined by (2.1.9), is the $k$-th order quasi M-definite moment functionat, then for all $n \geqslant 1$ the zeros of the psevdo orthogonal polynomial $B_{n}^{(k)}(z)$ are different from the zeros of $B_{n-1}^{(k)}(z), 0_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z)$.
Similarly, we can also show by using the formula of (2.3.14)
that the following hold.

Theorem 2.4.2 : For the $k$-th order quasi M-definite moment functional all the zeros of $\mathrm{B}_{2 \mathrm{n}}^{(\mathrm{k})}(\mathrm{z})$ are different from the zeros of $\mathrm{B}_{2 \mathrm{n}-1}^{(\mathrm{k}+1)}(\mathrm{z})$, $0_{2 n}^{(k)}(z)$ and $A_{2 n}^{(k)}(z)$. Furthermore, any zero of $B_{2 n+1}^{(k+1)}(z)$ which is not equal to zero is also different from the zeros of $\mathrm{B}_{2 \mathrm{n}}^{(\mathrm{k})}(\mathrm{z}), 0_{2 \mathrm{n}+1}^{(\mathrm{k}+1)}(\mathrm{z})$ and $A_{2 n+1}^{(k+1)}(z)$.

If the moment functional given by (2.1.9) is positive definite and its supporting set $E$ is an interval, then the zeros of $B_{n}^{(k)}(z)$ exhibit a certain regularity in their behaviour with respect to $E$. To discuss this behaviour let us consider from (2.2.1) the relations

$$
\begin{equation*}
I_{p}\left[t^{k-n} B_{n}^{(k)}(t)\right]=0, \quad n \geqslant 1 \tag{2.4.1a}
\end{equation*}
$$

and

$$
\begin{equation*}
I_{p}\left[t^{k-n+1} B_{n}^{(k)}(t)\right]=0, \quad n>1 \tag{2,4.1b}
\end{equation*}
$$

Thus from lemma 2.2.1 that the functions $\left\{t^{k=n_{B}}{ }_{n}^{(k)}(t)\right\}$ and $\left\{t^{k-n+1} B_{n}^{(k)}(t)\right\}$ are not non-negative in the interval $E$. Now, suppose E lies entirely on the positive half of the real axis, i.e., the functional is positive M-definite. Then $t^{k-n}$ does not change sign in $E$ and so, $B_{n}^{(k)}(t)$ must change sign at least once in $E$. Let there be $r$ such points (i.e., zeros) $z_{1}^{(n, k)}, z_{2}^{(n, k)}, \ldots, z_{r}^{(n, k)}(r \leqslant n)$ in $E$. Then the polynomial given by

$$
t^{k-n}\left(t-z_{1}^{(n, k)}\right)\left(t-z_{2}^{(n, k)}\right) \ldots\left(t-z_{r}^{(n, k)}\right) B_{n}^{(k)}(t)
$$

does not change sign in $E$, and we must have

$$
I_{P, M}\left[t^{k-n}\left(t-z_{1}^{(n, k)}\right)\left(t-z_{2}^{(n, k)}\right) \ldots\left(t-z_{r}^{(n, k)}\right) B_{n}^{(k)}(t)\right] \neq 0
$$

But if $\mathrm{r}<\mathrm{n}$, then the above result is a contradiction to (2.2.1). Hence, there must be at least $n$ points in the interval $E$
at which $B_{n}^{(k)}(t)$ is zero. Therefore, since $B_{n}^{(k)}(z)$ is a polynomial of degree $n$, we have the result:

Theorem 2.4.3: If the moment functional given by (2.1.9) is positive M-definite and its supporting set $E$, which lies on the positive half of the real axis, is also an interval, then all the zeros of the polynomial $B_{n}^{(k)}(z)$ are distinct and lie within $E$.

On the other hand if $E$ is not necessarily on the positive half of the real axis, then from (2.4.1a) for $(k-n)$ even we have a similar result. If $(k-n)$ is odd then $(k-n+1)$ is even. Hence, considering (2.4.1b), we find that at least ( $n-1$ ) of the zeros of $B_{n}^{(k)}(z)$ lie in E. Summarising these results we obtain

Theorem 2.4.4: If the moment functional given by (2.1.9) is positive definite over a supporting intervat $E$ and if $(\mathrm{k}-\mathrm{n})$ is even then att the zeros of $\mathrm{B}_{\mathrm{n}}^{(\mathrm{k})}(\mathrm{z})$ are distinct and lie within E . If $(\mathrm{k}-\mathrm{n})$ is odd then all the zeros of $B_{n}^{(k)}(z)$ are real and distinet, but only ( $n-1$ ) of these zeros are certain to lie within E.

We have for the $k$-th order quasi M-definite moment functional the relations (2.3.8) hold, Suppose this moment functional is positive M-definite. Then from (2.2.10), (2.3.5) and (2.3.6) we find that the coefficients $n_{n+1}^{(k)},(n \geqslant 1)$ and $d_{n+1}^{(k)},(n \geqslant 0)$ all take negative values, Thus, from (2.3.9) we have, if the moment functional is positive M-definite then

$$
\begin{equation*}
G_{n}(z)=\left\{B_{n}^{\prime(k)}(z) B_{n-1}^{(k)}(z)-B_{n-1}^{\prime(k)}(z) B_{n}^{(k)}(z)\right\}>0 \tag{2.4.2}
\end{equation*}
$$

for $n \geqslant 1$ and for all real values of $z$. An immediate consequence of this result is (see Szegö [1959]).

Theorem 2.4.5: If the moment functional given by (2.1.9) is positive M-definite then between any two zeros of $\mathrm{B}_{\mathrm{n}-1}^{(\mathrm{k})}(\mathrm{z})$ there is a zero of $B_{n}^{(k)}(z)$.

We also have for the $k$-th order quasi M-definite moment functional the relation $(2.3 .15)$ hold. Hence, if this moment functional is positive definite, then from (2.2.9) and (2.3.13) we note that the $N_{r}^{(k)}$ are all positive, provided $k$ is even. Thus, from (2.3.15) we find, if the $k$-th order quasi M-definite moment functional is positive definite then

$$
\begin{gather*}
K_{2 n+1}^{(k)}(z)=\left\{B_{2 n+1}^{\prime(k+1)}(z) B_{2 n}^{(k)}(z)-B_{2 n}^{\prime(k)}(z) B_{2 n+1}^{(k+1)}\right\}>0, \\
K_{2 n+2}^{(k)}(z)=\left\{B_{2 n+2}^{\prime(k)}(z) B_{2 n+1}^{(k+1)}(z)-B_{2 n+1}^{\prime(k+1)}(z) B_{2 n+2}^{(k)}(z)\right\}>0, \tag{2.4.3}
\end{gather*}
$$

for $n \geqslant 0$ and for all real values of $z$, provided $k$ is even. Again an immediate consequence of this result is (see Szegö [1959]):

Theorem 2.4.6: If the moment functional given by (2.1.9) is positive definite and $k$-th order quasi M-definite, where $k$ is even, then the zeros of each of the polynomiais $B_{2 n+1}^{(k+1)}(z), B_{2 n+2}^{(k)}(z),(n \geqslant 0)$ are all real and distinct. Furthermore, for all $n \geqslant 0$, between any two zeros of $\mathrm{B}_{2 \mathrm{n}+2}^{(\mathrm{k})}(\mathrm{z})$ there is a zero of $\mathrm{B}_{2 \mathrm{n}+1}^{(\mathrm{k}+1)}(\mathrm{z})$ and between any two zeros of $\mathrm{B}_{2 \mathrm{n}+3}^{(\mathrm{k}+1)}(z)$ there is a zero of $\mathrm{B}_{2 \mathrm{n}+2}^{(\mathrm{k})}(z)$.

From now on we only consider positive definite moment functionals so that the zeros $z_{i}^{(n, k)}(i=1,2, \ldots, n)$ of $B_{n}^{(k)}(z)$ are real and distinct.

- Let $h(t)$ be an arbitrary function. Then we can express $h(t)$ by interpolation at the zeros of $B_{n}^{(k)}(z)$. That is,

$$
h(t)=\mathbb{P}_{n}(t)+\mathbb{E}_{n}(t)
$$

where $\mathbb{P}_{n}(t)$ is the interpolating polynomial and $\mathbb{E}_{n}(t)$ is the resulting error.

Expressing $\mathbb{I P}_{n}(t)$ in Lagrangian form and the error $\mathbb{E}_{n}(t)$ by the divided difference formula, it follows that

$$
\begin{aligned}
h(t) & =\sum_{r=1}^{n}\left\{\frac{B_{n}^{(k)}(t)}{B_{n}^{\prime(k)}\left(z_{r}^{(n, k)}\right)\left(t-z_{r}^{(n, k)}\right)}\right\} h\left(z_{r}^{(n, k)}\right) \\
& +B_{n}^{(k)}(t) h\left[t, z_{1}^{(n, k)}, z_{2}^{(n, k)}, \ldots, z_{n}^{(n, k)}\right] .
\end{aligned}
$$

If $h(t)$ is a polynomial of degree less than $2 n$ then $h\left[t, z_{1}^{(n, k)}, \ldots, z_{n}^{(n, k)}\right]$ is a polynomial of degree less than $n$ (see for example Phillips and Taylor [1973]). Hence, multiplying by $t^{k-n}$ and using the functional $I_{p}[\cdot]$, we find the quadrature formula,

$$
\begin{equation*}
I_{p}\left[t^{k-n} h(t)\right]=\sum_{r=1}^{n} \lambda_{r}^{(n, k)} h\left(z_{r}^{(n, k)}\right), \tag{2.4.4}
\end{equation*}
$$

where

$$
\lambda_{r}^{(n, k)}=\frac{1}{B_{n}^{\prime(k)}\left(z_{r}^{(n, k)}\right)} I_{p}\left[\frac{t^{k-n_{B_{n}}(k)}(t)}{\left(t-z_{r}^{(n, k)}\right)}\right], \quad r=1,2, \ldots, n . \text { (2.4.5) }
$$

If we take

$$
h(t)=\left\{\frac{B_{n}^{(k)}(t)}{B_{n}^{\prime(k)}\left(z_{i}^{(n, k)}\right)\left(t-z_{i}^{(n, k)}\right)}\right\}^{2},
$$

a polynomial of degree $(2 n-2)$, then from (2.4.4)

$$
I_{p}\left[t^{k-n}\left\{\frac{B_{n}^{(k)}(t)}{B_{n}^{\prime(k)}\left(z_{i}^{(n, k)}\right)\left(t-z_{i}^{(n, k)}\right)}\right\}^{2}\right]=\lambda_{i}^{(n, k)},
$$

and so

$$
\begin{equation*}
\lambda_{r}^{(n, k)}=\frac{1}{\int_{\left.B_{n}^{\prime(k)}\left(z_{r}^{(n, k)}\right)\right\}^{2}}} I_{p}\left[t^{k-n}\left\{\frac{B_{n}^{(k)}(t)}{t-z_{r}^{(n, k)}}\right\}^{2}\right] \tag{2.4.6}
\end{equation*}
$$

for $r=1,2, \ldots, n$.
Similarly, taking

$$
h(t)=t\left\{\frac{B_{n}^{(k)}(t)}{B_{n}^{\prime(k)}\left(z_{r}^{(n, k)}\right)\left(t-z_{r}^{(n, k)}\right)}\right\}^{2},
$$

a polynomial of degree $(2 n-1)$, also gives

$$
z_{r}^{(n, k)} \lambda_{r}^{(n, k)}=\frac{1}{\left\{B_{n}^{\prime(k)}\left(z_{r}^{(n, k)}\right)\right\}^{2}} I_{p}\left[t^{k-n+1}\left\{\frac{B_{n}^{(k)}(t)}{t-z_{r}^{(n, k)}}\right\}^{2}\right],(2.4 .7)
$$

for $r=1,2, \ldots, n$.
Consequently we have the following.
Theorem 2.4.7: If $z_{1}^{(n, k)}, z_{2}^{(n, k)}, \ldots, z_{n}^{(n, k)}$ denote the zeros of $B_{n}^{(k)}(z)$ then there exist $n$ unique real numbers $\lambda_{1}^{(n, k)}, \lambda_{2}^{(n, k)}, \ldots$, $\lambda_{\mathrm{n}}^{(\mathrm{n}, \mathrm{k})}$ such that

$$
I_{p}\left[r^{k-n} h(t)\right]=\sum_{r=1}^{n} \lambda_{r}^{(n, k)} h\left(z_{r}^{(n, k)}\right),
$$

whenever $\mathrm{h}(\mathrm{t})$ is a polynomial of degree less than or equal to ( $2 \mathrm{n}-1$ ). In addition if $(k-n)$ is even then the numbers $\lambda_{r}^{(n, k)}$ are all positive and if $(k-n)$ is odd then they take the sign of the corresponding zero $z_{r}^{(n, k)}$.

Following the theory of ordinary orthogonal polynomials, we call these numbers $\lambda_{r}^{(n, k)}$ the Christoffel numbers of $B_{n}^{(k)}(z)$. Now the following result concerning the zeros of different pseudo orthogonal polynomial can be shown to hold

Theorem 2.4.8: If $(k-n)$ is even then between any two zeros of $\mathrm{B}_{\mathrm{n}}^{(\mathrm{k})}(\mathrm{z})$ there exists at least one zero of $\mathrm{B}_{\mathrm{n}+2 \mathrm{r}}^{(\mathrm{k})}(\mathrm{z})$ and $\mathrm{B}_{\mathrm{n}+2 \mathrm{r}-1}^{(\mathrm{k}+1)}(\mathrm{z})$, where $r \geqslant 1$.

Proof : Using the quadrature formula on $B_{n+2 r}^{(k)}(z)$, we have

$$
\begin{aligned}
& I_{p}\left[t^{k-(n+2 r)}\left\{t^{2 r} \rho_{n-1}(t) B_{n}^{(k)}(t)\right\}\right] \\
& \quad=\sum_{i=1}^{n+2 r} \lambda_{i}^{(n+2 r, k)}\left\{\left(z_{i}^{(n+2 r, k)}\right)^{2 r} \rho_{n-1}\left(z_{i}^{(n+2 r, k)}\right) B_{n}^{(k)}\left(z_{i}^{(n+2 r, k)}\right)\right\}
\end{aligned}
$$

where $\rho_{n-1}(t)$ is an arbitrary polynomial of degree $\leqslant(n-1)$.
By writing the left hand side above as $I_{p}\left[t^{k-n} \rho_{n-1}(t) B_{n}^{(k)}(t)\right]$ we see that it is equal to zero. Hence, we have

$$
\sum_{i=1}^{n+2 r} \lambda_{i}^{(n+2 r, k)}\left\{\left(z_{i}^{(n+2 r, k)}\right)^{2 r} \rho_{n-1}\left(z_{i}^{(n+2 r, k)}\right) B_{n}^{(k)}\left(z_{i}^{(n+2 r, k)}\right)\right\}=0
$$

Here, if $(k-n)$ is even then all the $\lambda_{i}^{(n+2 r, k)}$ are positive. Thus, by considering the fact that $\rho_{n-1}(t)$ is an arbitrary polynomial of degree less than $n$ we can easily show that the sequence given by

$$
\left\{B_{n}^{(k)}\left(z_{i}^{(n+2 r, k)}\right)\right\}_{i=1}^{n+2 r}
$$

changes sign at least $n$ times. But, $B_{n}^{(k)}(z)$ is a polynomial of degree $n$ and therefore, the proof of the first of the required results follows.

The remaining result can similarly be obtained by considering

$$
I_{p}\left[t^{k+1-(n+2 r-1)}\left\{t^{2 r-2} \rho_{n-1}(t) B_{n}^{(k)}(t)\right\}\right]=0
$$

Let us now consider the quotients $0_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z) / B_{n}^{(k)}(z)$. Since, under the $k$-th order quasi M-definite moment functional the zeros of $O_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z)$ are different from those of $B_{n}^{(k)}(z)$, these quotients have partial decompositions of the form

$$
\begin{align*}
& \frac{0_{n}^{(k)}(z)}{B_{n}^{(k)}(z)}=\sum_{i=1}^{n} \frac{e_{i}^{(n, k)}}{z-z_{i}^{(n, k)}}, \quad n \geqslant 1,  \tag{2.4.8}\\
& \frac{A_{n}^{(k)}(z)}{B_{n}^{(k)}(z)}=\sum_{i=1}^{n} \frac{m_{i}^{(n, k)}}{z-z_{i}^{(n, k)}}, \quad n \geqslant|k| \tag{2,4,9}
\end{align*}
$$

where

$$
\begin{align*}
& \ell_{i}^{(n, k)}=o_{n}^{(k)}\left(z_{i}^{(n, k)}\right) / B_{n}^{\prime(k)}\left(z_{i}^{(n, k)}\right), \quad i=1,2, \ldots, n,  \tag{2,4.10}\\
& m_{i}^{(n, k)}=A_{n}^{(k)}\left(z_{i}^{(n, k)}\right) / B^{\prime(k)}\left(z_{i}^{(n, k)}\right), \quad i=1,2, \ldots, n \tag{2.4.11}
\end{align*}
$$

In (2.4.9), $n$ is taken to be greater than or equal to $|k|$ because only then is $A_{n}^{(k)}(z)$ a polynomial of degree $(n-1)$. Referring to (2.2.11), it follows that

$$
\begin{aligned}
0_{n}^{(k)}(z) & =I\left[t^{k}\left\{B_{n}^{(k)}(z)-B_{n}^{(k)}(t)\right\} /(z-t)\right] \\
& =I\left[\frac{t^{k-n}\left\{t^{n} B_{n}^{(k)}(z)-z^{n} B_{n}^{(k)}(t)\right\}}{z-t}\right]+I\left[t^{k-n_{B}(k)}(t) \frac{z^{n}-t^{n}}{z-t}\right]
\end{aligned}
$$

But, from (2.2.1) the second term on the right hand side is equal to zero, and so

$$
0_{n}^{(k)}(z)=I\left[\frac{t^{k-n}\left\{\tau^{n} B_{n}^{(k)}(z)-z^{n} B_{n}^{(k)}(t)\right\}}{(z-t)}\right]
$$

Hence, substituting $z=z_{i}^{(n, k)}$, a zero of $B_{n}^{(k)}(z)$, we obtain

$$
o_{n}^{(k)}\left(z_{i}^{(n, k)}\right)=\left\{z_{i}^{(n, k)}\right\}^{n} I\left[\frac{t^{k-n_{B}(k)}(t)}{t-z_{i}^{(n, k)}}\right]
$$

Similarly, by considering (2.2.12), we find

$$
A_{n}^{(k)}\left(z_{i}^{(n, k)}\right)=\left\{z_{i}^{(n, k)}\right\}^{n-k} I\left[\frac{t^{k-n_{B}^{(k)}(t)}}{z-z_{i}^{(n, k)}}\right]
$$

Consequently, applying these results in (2.4.10) and (2.4.11) we immediately find

Theorem 2.4.9: If the moment functional given by (2.1.9) is positive definite and $k$-th order quasi $M$-dejinite, then the quotients $O_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ and $A_{n}^{(k)}(z) / B_{n}^{(k)}(z)$ have partial decompositions of the form

$$
\begin{aligned}
& \frac{0_{n}^{(k)}(z)}{B_{n}^{(k)}(z)}=\sum_{i=1}^{n} \frac{\ell_{i}^{(n, k)}}{z-z_{i}^{(n, k)}}, \quad n \geqslant 1, \\
& \frac{A_{n}^{(k)}(z)}{B_{n}^{(k)}(z)}=\sum_{i=1}^{n} \frac{m_{i}^{(n, k)}}{z-z_{i}^{(n, k)},} n \geqslant|k|,
\end{aligned}
$$

where

$$
\begin{align*}
& \ell_{i}^{(n, k)}=\left\{z_{i}^{(n, k)}\right\}^{n} \lambda_{i}^{(n, k)}, \quad \dot{i}=i, 2, \ldots, n,  \tag{2.4.12}\\
& m_{i}^{(n, k)}=\left\{z_{i}^{(n, k)}\right\}^{n-k} \lambda_{i}^{(n, k)}, \quad i=1,2, \ldots, n . \tag{2.4.13}
\end{align*}
$$

From (2.4.13), we also note that the $m_{i}^{(n, k)}$ are positive for $i=1,2, \ldots, n$.

It has been shown by Jones, Thron and Waadeland [1980] and Jones, Thron and Njasted [1983], that any positive definite moment functional $I_{p}[\cdot]$ can be given in terms of a bounded non-decreasing function $\psi(t)$ with infinitely many points of increase, by

$$
I_{p}\left[t^{n}\right]=\int_{-\infty}^{\infty} t^{n} d \psi(t), \quad n=\ldots,-2,-1,0,1,2, \ldots
$$

The proof of this result using continued fractions, is given in the next chapter. It would now be appropriate to end this chapter with the analogous theorems to those of the separation theorems of Tschebycheff, Markov and Stieltjes (see Szegö [1959]).

Theorem 2.4.10: If $(k-n)$ is even then the zeros $z_{r}^{(n, k)}$, $r=1,2, \ldots, n$ of $B_{n}^{(k)}(z)$ arranged in increasing order, alternate with a unique set of real numbers $u_{r}^{(n, k)}, r=0,1,2, \ldots, n$, i.e.,

$$
\begin{gathered}
u_{0}^{(n, k)}=-\infty, \quad u_{n}^{(n, k)}=\infty \\
z_{r}^{(n, k)}<u_{r}^{(n, k)}<z_{r+1}^{(n, k)}, \quad r=1,2, \ldots, n-1,
\end{gathered}
$$

where these numbers $u_{r}^{(n, k)}$ satisfy

$$
\lambda_{r}^{(n, k)}=\int_{u_{r-1}}^{u_{r}^{(n, k)}} t^{k-n} d \psi(t), \quad r=1,2, \ldots, n
$$

Proof : Let $\pi_{1}(z)$ and $\pi_{2}(z)$ be polynomials of degree ( $2 n-2$ ), such that for $r<n$

$$
\begin{align*}
& \pi_{1}\left(z_{k}^{(n, k)}\right)= \begin{cases}1, & \text { if } k=1,2, \ldots, r \\
0, & \text { if } k=r+1, r+2, \ldots, n\end{cases} \\
& \pi_{1}^{\prime}\left(z_{k}^{(n, k)}\right)=0, \text { if } k=1,2, \ldots, r-1, r+1, \ldots, n, \tag{2.4.14}
\end{align*}
$$

and

$$
\begin{align*}
& \pi_{2}\left(z_{k}^{(n, k)}\right)= \begin{cases}0, & \text { if } k=1,2, \ldots, r \\
1, & \text { if } k=r+1, r+2, \ldots, n\end{cases} \\
& \pi_{2}^{\prime}\left(z_{k}^{(n, k)}\right)=0, \text { if } k=1,2, \ldots, r, r+2, \ldots, n . \tag{2.4.15}
\end{align*}
$$

The graphs of $\pi_{1}(t)$ and $\pi_{2}(t)$ have the forms (Szegö [1959]).


Hence, in the quadrature formula (2.4.4), choosing $h(t)$ to be successively $\pi_{1}(t)$ and $\pi_{2}(t)$, we find that.

$$
\lambda_{1}^{(n, k)}+\lambda_{2}^{(n, k)}+\ldots+\lambda_{r}^{(n, k)}=\int_{-\infty}^{\infty} t^{k-n} \pi_{2}(t) d \psi(t),
$$

and

$$
\lambda_{r+1}^{(n, k)}+\lambda_{r+2}^{(n, k)}+\ldots+\lambda_{n}^{(n, k)}=\int_{-\infty}^{\infty} t^{k-n} \pi_{2}(t) d \psi(t)
$$

Since ( $k-n$ ) is even it follows immediately that

$$
\begin{equation*}
\lambda_{1}^{(n, k)}+\lambda_{2}^{(n, k)}+\ldots+\lambda_{r}^{(n, k)}>\int_{-\infty}^{z_{r}^{(n, k)}+} t^{k-n} \pi_{1}(t) d \psi(t)>\int_{-\infty}^{z_{r}^{(n, k)}+} t^{k-n} d \psi(t) \tag{2.4.16}
\end{equation*}
$$

and that
$\lambda_{r+1}^{(n, k)}+\lambda_{r+2}^{(n, k)}+\ldots+\lambda_{n}^{(n, k)}>\int_{z_{r+1}}^{\infty} t^{k-n} \pi_{2}(t) d \psi(t)>\int_{z_{r+1}}^{\infty}(n, k)-$

In (2.4.4) letting $h(t)=1$, we also find

$$
\lambda_{1}^{(n, k)}+\lambda_{2}^{(n, k)}+\ldots+\lambda_{n}^{(n, k)}=\int_{-\infty}^{\infty} t^{k-n} d \psi(t)
$$

Hence, subtracting the inequality (2.4.17) from the above relation yields

$$
\begin{equation*}
\lambda_{1}^{(n, k)}+\lambda_{2}^{(n, k)}+\ldots+\lambda_{r}^{(n, k)}<\int_{-\infty}^{2(n, k)}{ }_{r+1}^{(n-n} t^{k-n} d \psi(t) \tag{2.4.18}
\end{equation*}
$$

Thus we note from (2.4.16) and (2.4.18) that there exists a unique number $u_{r}^{(n, k)}$, lying between $z_{r}^{(n, k)}$ and $z_{r+1}^{(n, k)}$, such that

$$
\lambda_{1}^{(n, k)}+\lambda_{2}^{(n, k)}+\ldots+\lambda_{r}^{(n, k)}=\int_{0}^{u(n, k)} t_{-\infty}^{k-n} d \psi(t)
$$

and this concludes the proof of the theorem.
This result can be extended to the case where $(k-n)$ is odd, as follows

Theorem 2.4.11: If $(k-n)$ is odd then the zeros $z_{r}^{(n, k)}$, $\mathrm{r}=1,2, \ldots, \mathrm{n}$ of $\mathrm{B}_{\mathrm{n}}^{(\mathrm{k})}(\mathrm{z})$ arranged in increasing order, altemate with a unique set of real numbers $u_{r}^{(n, k)}, r=0,1,2, \ldots, n$, where

$$
u_{0}^{(n, k)}=-\infty, \quad u_{n}^{(n, k)}=\infty
$$

and

$$
z_{r}^{(n, k)} \lambda_{r}^{(n, k)}=\int_{u_{r-1}^{(n, k)}}^{u_{r}^{(n, k)}} t^{k-n+1} d \psi(t), \quad r=1,2, \ldots, n
$$

Proof : In (2.4.4), choosing $h(t)$ to be the polynomials $t \cdot \pi_{1}(t)$ and $t \cdot \pi_{2}(t)$, where $\pi_{1}(t)$ and $\pi_{2}(t)$ are the polynomials defined by (2.4.14) and (2.4.15), and then using an argument similar to that of the proof of the theorem 2.4 .10 gives at once the required result.

## CHAPTER THREE

$\hat{J}-F R A C T I O N S$ AND THE STRONG HAMBURGERMOMENT PROBLEM

### 3.1 THE MOMENT PROBLEM

In his paper Recherches sur les fractions continues of 189495, Stieltjes proposed and solved the following problem.

Given the prescribed set of real numbers $c_{n}, n=0,1,2, \ldots$, find conditions for the existence of a non-decreasing function $\psi(t)$ in the interval $[0, \infty)$ such that

$$
\int_{0}^{\infty} t^{n} d \psi(t)=c_{n}, \quad n=0,1,2, \ldots
$$

Stieltjes called this the problem of moments by referring to $\int_{0}^{\infty} t d \psi(t)$ and $\int_{0}^{\infty} t^{2} d \psi(t)$, respectively the first moment (statical moment) and the second moment (moment of inertia), appearing in Mechanics. To solve this problem, now known as the Stieltjes moment problem, Stieltjes makes extensive use of continued fractions of the type

$$
\frac{1}{a_{1} z}+\frac{1}{a_{2}}+\frac{1}{a_{3} z}+\frac{1}{a_{4}}+\ldots
$$

and

$$
\frac{\ell_{1}}{z+m_{1}}-\frac{\ell_{2}}{z+m_{2}}-\frac{\ell_{3}}{z+m_{3}}-\frac{\ell_{4}}{z+m_{4}}-\ldots,
$$

corresponding to the series expansion

$$
\int_{0}^{\infty} \frac{1}{z-t} d \psi(t)=\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\frac{c_{2}}{z^{3}}+\cdots
$$

In the Stieltjes moment problem the range of integration is the semi-axis $[0, \infty)$. If this range is extended to the whole axis $(-\infty, \infty)$, then the resulting moment problem is called the Hamburger moment problem, after Hamburger [1920,1921] who was the first to propose and solve this problem. Important results connecting the Hamburger moment problem and many branches of mathematics were achieved almost immediately by Nevanlinna [1922], Riesz [1921,1922, 1923], Hellinger [1922], Carleman $[1922,1923,1926]$ and Hausdorff [1923]. Hausdorff gives criteria for the solution to the moment problem in a finite interval.

The development of M-fractions by McCabe and Murphy [Eg, 1976] and the equivalent T-fractions by Jones and Thron [1977], enabled Jones and Thron and Waadeland [1980] to consider a further moment problem, which they called the strong Stieltjes moment problem. The problem can be stated as follows.

For a given double sequence $\left\{c_{n}\right\}_{n=-\infty}^{\infty}$ of finite valued real numbers, find conditions to ensure the existence of a bounded nondecreasing function $\psi(t)$ with infinitely many points of increase in the interval $[0, \infty)$ such that

$$
\begin{equation*}
\int_{0}^{\infty} t^{n} d \psi(t)=c_{n}, \quad n=\ldots-2,-1,0,1,2, \ldots . \tag{3.1.1}
\end{equation*}
$$

Jones, Thron and Waadeland [1980] showed that a non-decreasing function $\psi(t)$ satisfying (3.1.1) exists if and only if

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}^{(-\mathrm{n})}>0 \quad \text { and } \quad H_{\mathrm{n}+1}^{(-n)}>0, \tag{3.1.2}
\end{equation*}
$$

for $n=0,1,2, \ldots$, where the $H_{r}^{(m)}$ are the Hankel determinants. This condition can be obtained by making it equivalent to the existence of an M-fraction

$$
\frac{c_{0}}{z+d_{1}}+\frac{n_{2} z}{z+d_{2}}+\frac{n_{3} z}{z+d_{3}}+\frac{n_{4} z}{z+d_{4}}+\ldots,
$$

which corresponds to the power series expansions

$$
\begin{align*}
& f_{0}(z)=\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\frac{c_{2}}{z^{3}}+\ldots  \tag{3.1.3a}\\
& g_{0}(z)=-c_{-1}-c_{-2} z-c_{73} z^{2}-\ldots, \tag{3.1.3b}
\end{align*}
$$

and for which $c_{0}>0$ and all the $n_{r}$ and $d_{r}$ are negative.
By considering the convergence criteria of the $\dot{M}$-fraction, Jones, Thron and Waadeland [1980] also showed that under the condition (3.1.2) the strong Stieltjes moment problem has either a unique solution or two distinct solutions.

The next step was to look at the strong Hamburger moment problem, i.e., the strong problem in the extended interval ( $-\infty, \infty$ ). This problem although it seems a trivial extension to the strong Stieltjes moment problem, it is by no means straightforward to solve. The inability to find any suitable continued fraction made the problem even more difficult. However, following Riesz [1921,1922, 1923] (who solved the Hamburger moment problem on the basis of
quasi-orthogonal polynomials), Jones, Thron and Njastad [1983a] solved the strong Hamburger moment problem by considering a certain type of function, which they called the quasi-orthogonal Laurent polynomials. They showed that the necessary and sufficient condition for the existence of a solution is

$$
\begin{equation*}
\mathrm{H}_{2 \mathrm{n}}^{(-2 n)}>0 \quad \text { and } \quad \mathrm{H}_{2 n+1}^{(-2 n)}>0, \tag{3.1.4}
\end{equation*}
$$

for $n=0,1,2, \ldots$.
Jones, Thron and Njastad [1983b] made attempts to find this condition by means of using continued fractions, but they could only arrive at partial conditions (i.e., conditions which are sufficient but not necessary).

Subsequent work in this chapter is concerned with obtaining the condition (3.1.4) of the strong Hamburger moment problem via continued fractions.

## $3.2 \hat{\jmath}$-FRACTIONS AND THEIR CORRESPONDENCE

A $\widehat{J}$-fraction is a continued fraction of the form

$$
\begin{equation*}
\frac{\left\{a_{1}(z)\right\}^{2}}{z+b_{1}}-\frac{\left\{a_{2}(z)\right\}^{2}}{z+b_{2}}-\frac{\left\{a_{3}(z)\right\}^{2}}{z+b_{3}}-\frac{\left\{a_{4}(z)\right\}^{2}}{z+b_{4}}-\ldots, \tag{3.2.1}
\end{equation*}
$$

where all $b_{n}$ are complex constants and each $a_{n}(z)$ is a complex function taking either the form $\left(l_{n}+i m_{n}\right)$ or $\left(l_{n}+i m_{n}\right) z$. Here, $\ell_{n}, m_{n} \in \mathbb{R}$.

The $\hat{J}$-fraction (3.2.1) is a J-fraction if all the $a_{n}(z)$ are complex constants ( $l_{n}+i m_{n}$ ). In Chapter 4, a detailed study of a special type of $\hat{J}$-fractions (under the name positive definite $\hat{J}$ fractions) is considered. For the purpose of the strong Hamburger moment problem only real $\hat{J}$-fractions, in which all $m_{n}$ are zero and all $b_{n}$ are real, are looked at in this Chapter.

It is well known that given a power series $f_{0}(z)$ as in (3.1.3a) it is possible, under certain conditions, to obtain a J-fraction which corresponds to this power series. Likewise, it is also possible to find $\hat{J}$-fractions which correspond to a power series $f_{0}(z)$ and to another power series $g_{0}(z)$ given by (3.1.3b).

To find such a $\hat{J}$-fraction, first of all let us consider the $M^{(k)}$-fraction (1.2.6), with r-th convergent $M_{r}^{(k)}(z)$ corresponding to $\Phi(r+k)$ terms of the series $f_{0}(z)$ and $\Phi(r-k)$ terms of the series $g_{0}(z)$. The integer function $\Phi(N)$ is defined by (1.2.7).

Now let us define a new continued fraction, denoted by $L^{(k)}(z)$, such that its $r$-th convergent $L_{r}^{(k)}(z)$ is given by

$$
\begin{align*}
& L_{2 s}^{(k)}(z)=M_{2 s}^{(k)}(z) \\
& L_{2 s+1}^{(k)}(z)=M_{2 s+1}^{(k+1)}(z) \tag{3.2.2}
\end{align*}
$$

for $s=0,1,2, \ldots$. Then it follows that $L_{2 s}^{(k)}(z)$ corresponds to $\Phi(2 s+k)$ terms of the series $f_{0}(z)$ and $\Phi(2 s-k)$ terms of the series $g_{0}(z)$, while $L_{2 s+1}^{(k)}(z)$ corresponds to $\Phi(2 s+2+k)$ terms of the series $f_{0}(z)$ and $\Phi(2 s-k)$ terms of the series $g_{0}(z)$.

Denoting the numerator and denominator of $\mathrm{L}_{\mathrm{r}}^{(\mathrm{k})}(\mathrm{z})$ by $\mathrm{P}_{\mathrm{r}}^{(\mathrm{k})}(\mathrm{z})$ and $Q_{r}^{(k)}(z)$ respectively, and further assuming that $Q_{r}^{(k)}(z)$ is a monic polynomial, we find from (3.2.2) that

$$
\begin{align*}
& \mathrm{P}_{2 \mathrm{~s}}^{(\mathrm{k})}(z)=\mathrm{A}_{2 \mathrm{~s}}^{(\mathrm{k})}(z), \quad \mathrm{n} \geqslant 0, \\
& Q_{2 S}^{(k)}(z)=B_{2 S}^{(k)}(z), \quad n \geqslant 0, \\
& P_{2 s+1}^{(k)}(z)=A_{2 s+1}^{(k+1)}(z), \quad n \geqslant 0,  \tag{3.2.3}\\
& Q_{2 s+1}^{(k)}(z)=B_{2 s+1}^{(k+1)}(z), \quad n \geqslant 0, \quad
\end{align*}
$$

where $A_{r}^{(k)}(z)$ and $B_{r}^{(k)}(z)$ are the numerator and denominator of $M_{r}^{(k)}(z)$ as described in Chapter 1. Hence, using the three term recurrence relations (1.2.9) and (1.2.10) for $A_{n}^{(k)}(z)$ and $B_{n}^{(k)}(z)$, together with the relation (3.2.3), we arrive at the following three term relations.

$$
\begin{gather*}
P_{2 s}^{(k)}(z)=\left(z+D_{2 s}^{(k)}\right) P_{2 s-1}^{(k)}(z)-N_{2 s}^{(k)} P_{2 s-2}^{(k)}(z), \\
Q_{2 s}^{(k)}(z)=\left(z+D_{2 s}^{(k)}\right) Q_{2 s-1}^{(k)}(z)-N_{2 s}^{(k)} Q_{2 s-2}^{(k)}(z),  \tag{3.2.4a}\\
P_{2 s+1}^{(k)}(z)=\left\{\left(1+N_{2 s+1}^{(k)}\right) z+D_{2 s+1}^{(k)}\right\} P_{2 s}^{(k)}(z)-N_{2 s+1}^{(k)} z^{2} p_{2 s-1}^{(k)}(z), \\
Q_{2 s+1}^{(k)}(z)=\left\{\left(1+N_{2 s+1}^{(k)}\right\} z+D_{2 s+1}^{(k)}\right\} Q_{2 s}^{(k)}(z)-N_{2 s+1}^{(k)} z^{2} Q_{2 s-1}^{(k)}(z),
\end{gather*}
$$

where

$$
\begin{gather*}
N_{2 s}^{(k)}=n_{2 s}^{(k)} d_{2 s}^{(k)}, \quad D_{2 s}^{(k)}=d_{2 s}^{(k)}, \\
N_{2 s+1}^{(k)}=n_{2 s+1}^{(k)} / d_{2 s}^{(k)}, \quad D_{2 s+1}^{(k)}=d_{2 s+1}^{(k)}+n_{2 s+2}^{(k)} \tag{3.2.5}
\end{gather*}
$$

As initial conditions for (3.2:4), we also find that

$$
Q_{0}^{(k)}(z)=1, \quad P_{0}^{(k)}(z)=M_{0}^{(k)}(z)
$$

$Q_{1}^{(k)}(z)=\left(z+d_{1}^{(k+1)}\right)=\left(z+D_{1}^{(k)}\right), \quad p_{1}^{(k)}(z)=\left(z+D_{1}^{(k)}\right) P_{0}^{(k)}(z)+c_{k} / z^{k}$.

Thus, from the recurrence relations (3.2.4) and the initial conditions (3.2.6), it immediately follows that
$L^{(k)}(z)=M_{0}^{(k)}(z)+\frac{c_{k} / z^{k}}{z+D_{1}^{(k)}}-\frac{N_{2}^{(k)}}{z+D_{2}^{(k)}}-\frac{N_{3}^{(k)} z^{2}}{\left(1+N_{3}^{(k)}\right) z+D_{3}^{(k)}}-\frac{N_{4}^{(k)}}{z+D_{4}^{(k)}}-\ldots$.

In particular, when $k=0$, it follows that

$$
L^{(0)}(z)=\frac{c_{0}}{z+D_{1}}-\frac{N_{2}}{z+D_{2}}-\frac{N_{3} z^{2}}{\left(1+N_{3}\right) z+D_{3}}-\frac{N_{4}}{z+D_{4}}-\frac{N_{5} z^{2}}{\left(1+N_{5}\right) z+D_{5}}-\ldots,
$$

where

$$
N_{r+1}=N_{r+1}^{(0)} \text { and } D_{r}=D_{r}^{(0)} \text {, for all } r \geqslant 1
$$

After a suitable equivalence transformation the continued fraction $L^{(0)}(z)$ can be written as
$L^{(0)}(z)=\frac{c_{0}}{z+D_{1}}-\frac{N_{2}}{z+D_{2}}-\frac{\frac{N_{3}}{1+N_{3}} z^{2}}{z+\frac{D_{3}}{1+N_{3}}}-\frac{\frac{N_{4}}{1+N_{3}}}{z+D_{4}}-\frac{\frac{N_{5}}{1+N_{5}}}{z+\frac{D_{5}}{1+N_{5}}}-\ldots$

Comparing (3.2.9) with $(3.2 .1)$ we see that $L^{(0)}(z)$ is also a $\hat{J}$-fraction. If all the $N_{r}$ are positive and all the $D_{r}$ are real then $L^{(0)}(z)$ is also a real $\hat{J}$-fraction. In addition to being a $\hat{J}$-fraction, $L^{(0)}(z)$ also has another important property in that its partial numerators are alternately constants and variables. As a consequence, as is shown in the next Chapter, this continued fraction possesses
some interesting convergence properties.
In a J-fraction the denominator of the $n$-th convergent is always a monic polynomial of degree $n$. But in the case of the $\hat{J}$ fractions of the form (3.2.1) this is not so. However, in the continued fraction (3.2.8), which is in fact equivalent to a $\hat{J}$-fraction, all the denominator polynomials are monic. For this reason, continued fractions of this form which will also be referred to as $\hat{J}$-fractions, are found to be more practical.

For the derivation of the $\hat{J}$-fraction $L^{(0)}(z)$ it was assumed that the M-fractions $M^{(0)}(z)$ and $M^{(1)}(z)$ both exist. In other words, from (1.2.5), the following condition is satisfied

$$
\mathrm{H}_{\mathrm{n}}^{(-\mathrm{n})} \neq 0, \quad \mathrm{H}_{\mathrm{n}+1}^{(-n)} \neq 0 \quad \text { and } H_{n+1}^{(-(n-1))} \neq 0
$$

for $n \geqslant 0$.
But, we have shown in Chapter 2 that for the 0 -th order quasi M-definite moment functional, i.e., when the condition

$$
\begin{equation*}
\mathrm{H}_{\mathrm{n}}^{(-\mathrm{n})} \neq 0 \text { and } \mathrm{H}_{\mathrm{n}+1}^{(-\mathrm{n})} \neq 0, \quad \mathrm{n} \geqslant 0 \tag{3,2,10}
\end{equation*}
$$

is satisfied, then there exist polynomials $P_{n}(z), n \geqslant 0$ and $Q_{n}(z)$, $(n \geqslant 0)$ such that they satisfy three term relations of the form (3.2.4). Furthermore the quotient $P_{2 n}(z) / Q_{2 n}(z)$ is the $2 n-t h$ convergent of the $M^{(0)}$-fraction and the quotient $P_{2 n+1}(z) / Q_{2 n+1}(z)$ corresponds to $2 n+2$ terms of the series $f_{0}(z)$. Since, the condition (3.2.10) is not sufficient to ensure that $Q_{2 n+1}(0) \neq 0$, the correspondence of $P_{2 n+1}(z) / Q_{2 n+1}(z)$ to the power series $g_{0}(z)$ is not determined.

Thus we note that under the condition (3.2.10), the $\hat{J}$-fraction (3.2.8) still exists, but with a weaker correspondence behaviour than as given by (3.2.2).

The coefficients $N_{r}$ and $D_{r}$ of the $\hat{J}$-fraction (3.2.8) can be given in terms of the Hankel determinants as follows.

$$
\begin{align*}
& N_{2 n}=\left\{\frac{H_{2 n}^{(-(2 n-1))}}{H_{2 n-1}^{(-(2 n-2))}}\right\}^{2} \frac{H_{2 n-2}^{(-(2 n-2))}}{H_{2 n}^{(-2 n)}}, \quad n \geqslant 1, \\
& D_{2 n}=\frac{-H_{2 n}^{(-(2 n-1))} \cdot H_{2 n-1}^{(-(2 n-1))}}{H_{2 n-1}^{(-(2 n-2))} \cdot H_{2 n}^{(-2 n)}}, \quad n \geqslant 1, \\
& N_{2 n+1}=\frac{H_{2 n+1}^{(-2 n)} H_{2 n-1}^{(-(2 n-2))}}{\left\{H_{2 n}^{(-(2 n-1))}\right\}^{2}}, \quad n \geqslant 1,  \tag{3.2.11}\\
& D_{2 n+1}=\frac{-H_{2 n+1}^{(-(2 n-1))} \cdot H_{2 n}^{(-2 n)}}{H_{2 n+1}^{(-2 n)} \cdot H_{2 n}^{(-(2 n-1))}}, \quad n \geqslant 0 .
\end{align*}
$$

From (3.2.11) it follows that if

$$
\begin{equation*}
H_{2 n}^{(-2 n)} \neq 0, \quad H_{2 n+1}^{(-2 n)} \neq 0 \quad \text { and } \quad H_{2 n}^{(-(2 n-1))} \neq 0 \tag{3.2.12}
\end{equation*}
$$

for all $\mathrm{n} \geqslant 0$, then all the partial coefficients are finite and, further, the numerators $N_{r}$ are all non-zero.

The condition (3.2.12) is in fact sufficient for the existence of the $\hat{J}$-fraction (3.2.8). To understand this, we consider a
different approach to the evaluation of the coefficients of this fraction.

Given the $\hat{J}$-fraction (3.2.8), the numerator polynomials $P_{n}(z)$ and denominator polynomials $Q_{n}(z)$ of the $n-t h$ convergent $L_{n}(z)$ satisfy the recurrence relations
$P_{2 s}(z)=\left(z+D_{2 s}\right) P_{2 s-1}(z)-N_{2 s} P_{2 s-2}(z)$,

$$
s \geqslant 1
$$

$Q_{2 s}(z)=\left(z+D_{2 s}\right) Q_{2 s-1}(z)-N_{2 s} Q_{2 s-2}(z)$,
$P_{2 s+1}(z)=\left\{\left(1+N_{2 s+1}\right) z+D_{2 S+1}\right\} P_{2 S}(z)-N_{2 S+1} z^{2} P_{2 S-1}(z)$,
$s \geqslant 1$,
$Q_{2 s+1}(z)=\left\{\left(1+N_{2 s+1}\right) z+D_{2 s+1}\right\} Q_{2 s}(z)-N_{2 s+1} z^{2} Q_{2 s-1}(z)$,
with initial conditions

$$
\left.\begin{array}{ll}
Q_{0}(z)=1, & { }_{0} P_{0}(z)=0  \tag{3.2.13b}\\
Q_{1}(z)=z+D_{1}, & P_{1}(z)=c_{0},
\end{array}\right\}
$$

From (3.2.13) we note that the numerators $P_{n}(z)$ are polynomials of degree $(n-1)$ and the denominators $Q_{n}(z)$ are polynomials of the form

$$
\left.\begin{array}{l}
Q_{0}(z)=1, \quad Q_{1}(z)=z+D_{1}, \\
Q_{2 n}(z)=z^{2 n}+\ldots+\sum_{r=1}^{n}\left(D_{2 r-1} D_{2 r}-N_{2 r}\right), \quad n \geqslant 1,  \tag{3.2.14}\\
Q_{2 n+1}(z)=z^{2 n+1}+\ldots+\left(1+N_{2 n+1}\right) Q_{2 n}(0) z+D_{2 n+1} Q_{2 n}(0), \quad n \geqslant 1
\end{array}\right\}
$$

Furthermore, using the same recurrence relations, it can also be shown that

$$
L_{2}(z)-L_{0}(z)=\frac{\left(z+D_{2}\right) c_{0}}{Q_{2}(z) Q_{0}(z)}
$$

$$
L_{2 n+2}(z)-L_{2 n}(z)=\frac{\left(z+D_{2 n+2}\right) N_{2 n+1} N_{2 n} \cdots N_{2} c_{0} z^{2 n}}{Q_{2 n+2}(z) Q_{2 n}(z)}, \quad n \geqslant i
$$

$$
\begin{equation*}
L_{2 n+3}(z)-L_{2 n+1}(z)=\frac{\left\{\left(1+N_{2 n+3}\right) z+D_{2 n+3}\right\} N_{2 n+2} N_{2 n+1} \cdots N_{2} c_{0} z^{2 n}}{Q_{2 n+3}(z) Q_{2 n+1}(z)}, n \geqslant 0 \tag{3.2.15}
\end{equation*}
$$

$L_{2 n+2}(z)-L_{2 n+1}(z)=\frac{N_{2 n+2} N_{2 n+1} \cdots N_{2} c_{0} z^{2 n}}{Q_{2 n+2}(z) Q_{2 n+1}(z)}, \quad n \geqslant 0$.

Since $Q_{n}(z),(n \geqslant 0)$ are monic polynomials of degree $n$, expanding the right hand sides of the equations in (3.2.15) about infinity yields
$L_{2}(z)-L_{0}(z)=c_{0} z^{-1}+$ lower order terms, $L_{2 n+2}(z)-L_{2 n}(z)=\left\{N_{2 n+1} N_{2 n} \ldots N_{2} c_{0}\right\} z^{-2 n-1}+$ lower order terms, $n \geqslant 1$,

$$
\begin{array}{r}
L_{2 n \div 3}(z)-L_{2 n+1}(z)=\left\{\left(1+N_{2 n+3}\right) N_{2 n+2} \cdots N_{2} c_{0}\right\}^{-2 n-3}+\text { lower order terms } \\
n \geqslant 0 \\
L_{2 n+2}(z)-L_{2 n+1}(z)=\left\{N_{2 n+2} N_{2 n+1} \cdots N_{2} c_{0}\right\}^{z^{-2 n-3}+1 \text { ower order terms }} \\
\end{array}
$$

Hence from above we see that by choosing the coefficients $\mathrm{N}_{\mathrm{n}}(\mathrm{n} \geqslant 2)$ and $\mathrm{D}_{\mathrm{n}}(\mathrm{n} \geqslant 1)$ appropriately, the convergent $\mathrm{L}_{2 \mathrm{n}}(\mathrm{z})$ of the $\hat{J}$-fraction (3.2.8) can be made to correspond to $2 n$ terms of the series $f_{0}(z)$ and the convergent $L_{2 n+1}(z)$ can be made to correspond to $(2 n+2)$ terms of the same series $f_{0}(z)$. That is,

$$
\begin{align*}
f_{0}(z)-L_{2 n}(z) & =\gamma_{1}^{(2 n)} z^{-2 n-1}+\text { lower order terms, } n \geqslant 0 \\
f_{0}(z)-L_{2 n+1}(z) & =\gamma_{1}^{(2 n+1)} z^{-2 n-3}+\text { lower order terms, } n \geqslant 0, \tag{3.2.16}
\end{align*}
$$

where

$$
\begin{align*}
& \gamma_{1}^{(0)}=c_{0}, \quad \gamma_{1}^{(2 n)}=N_{2 n+1} N_{2 n} \cdots N_{2} c_{0}, \quad n \geqslant 1, \\
& \gamma_{1}^{(2 n+1)}=\left(1+N_{2 n+3}\right) N_{2 n+2} N_{2 n+1} \cdots N_{2} c_{0}, \quad n \geqslant 0 . \tag{3,2,17}
\end{align*}
$$

Multiplying the equations of $(3.2 .16)$ by $Q_{2 n}(z)$ and $Q_{2 n+1}(z)$ respectively, gives

$$
\begin{gather*}
Q_{2 n}(z) \cdot f_{0}(z)-P_{2 n}(z)=\gamma_{1}^{(2 n)} z^{-1}+\text { lower order terms, } n \geqslant 0,  \tag{3.2.18}\\
Q_{2 n+1}(z) \cdot f_{0}(z)-P_{2 n+1}(z)=\gamma_{1}^{(2 n+1)} z^{-2}+\text { lower order terms, } n \geqslant 0 .
\end{gather*}
$$

Now, going back to the equations of (3.2.15), consider the
expansions of the right hand sides about the origin, first making the assumption that

$$
\begin{equation*}
Q_{2 n}(0)=\sum_{r=1}^{n}\left(D_{2 r-1} D_{2 r}-N_{2 r}\right) \neq 0, \text { for all } n \geqslant 1 \tag{3.2.19}
\end{equation*}
$$

Then, we have

$$
L_{2}(z)-L_{0}(z)=\frac{D_{2} c_{0}}{Q_{2}(0)}+\text { higher order terms }
$$

$L_{2 n+2}(z)-L_{2 n}(z)=\frac{D_{2 n+2} N_{2 n+1} N_{2 n} \cdots N_{2} c_{0}}{Q_{2 n+2}(0) Q_{2 n}(0)} z^{2 n}+$ higher order terms,

$$
\mathrm{n} \geqslant 1
$$

From this it follows that the convergent $L_{2 n}(z)$ of the $\hat{J}$-fraction (3.2.8) may also be made to correspond to $2 n$ terms of the power series $g_{0}(z)$. Furthermore, we also have from (3.2.15)

$$
\begin{gathered}
Q_{0}(z) \cdot L_{2}(z)-P_{0}(z)=\frac{D_{2} c_{0}}{Q_{2}(0)}+\text { higher order terms, } \\
Q_{2 n}(z) \cdot L_{2 n+2}(z)-P_{2 n}(z)=\frac{D_{2 n+2} N_{2 n+1} N_{2 n} \cdots N_{2} c_{0}}{Q_{2 n+2}(0)} z^{2 n}+\text { higher order terms, }
\end{gathered}
$$

$Q_{2 n+1}(z) \cdot L_{2 n+2}(z)-P_{2 n+1}(z)=\frac{N_{2 n+2} N_{2 n+1} \cdots N_{2} c_{0}}{Q_{2 n+2}(0)} z^{2 n}+$ higher order terms,

$$
\mathrm{n} \geqslant 0
$$

Thus, if $L_{2 n+2}(z)$ corresponds to $(2 n+2)$ terms of the power series $g_{0}(z)$, we must also have from above that
$Q_{2 n}(z) \cdot g_{0}(z)-P_{2 n}(z)=\rho_{1}^{(2 n)} z^{2 n}+$ higher order terms, $n \geqslant 0$,
$Q_{2 n+1}(z) \cdot g_{0}(z)-P_{2 n+1}(z)=\rho_{1}^{(2 n+1)} z^{2 n}+$ higher order terms, $n \geqslant 0$,
where

$$
\begin{align*}
& \rho_{1}^{(0)}=\frac{D_{2} c_{0}}{D_{1} D_{2}-N_{2}}, \\
& \rho_{1}^{(2 n)}=\frac{D_{2 n+2} N_{2 n+1} N_{2 n} \cdots N_{2} c_{0}}{\sum_{r=1}^{n+1}\left(D_{2 r-1} D_{2 r}-N_{2 r}\right)}, n \geqslant 1,  \tag{3.2.21}\\
& \rho_{1}^{(2 n+1)}=\frac{N_{2 n+2} N_{2 n+1}^{N} N_{2 n} \cdots N_{2} c_{0}}{\sum_{r=1}^{\left(D_{2 r-1} D_{2 r}-N_{2 r}\right)},},
\end{align*}
$$

Therefore, writing $Q_{n}(z)$ and $P_{n}(z)$ in (3.2.18) and (3.2.20) as

$$
Q_{n}(z)=q_{n}^{(n)} z^{n}+q_{n-1}^{(n)} z^{n-1}+\ldots+q_{0}^{(n)}, \quad n \geqslant 0
$$

and

$$
P_{0}(z)=0, \quad P_{n}(z)=p_{n-1}^{(n)} z^{n-1}+p_{n-2}^{(n)} z^{n-2}+\ldots+p_{0}^{(n)}, \quad n \geqslant 1
$$

the following four systems of linear equations are obtained.

$$
\left.\begin{array}{r}
c_{0} q_{0}^{(2 n)}+c_{1} q_{1}^{(2 n)}+c_{2} q_{2}^{(2 n)}+\ldots+c_{2 n-1} q_{2 n-1}^{(2 n)}+c_{2 n} q_{2 n}^{(2 n)}=\gamma_{1}^{(2 n)} ; \\
c_{0} q_{1}^{(2 n)}+c_{1} q_{2}^{(2 n)}+\ldots+c_{2 n-2} q_{2 n-1}^{(2 n)}+c_{2 n-1} q_{2 n}^{(2 n)}=p_{0}^{(2 n),} \\
c_{0} q_{2}^{(2 n)}+\ldots+c_{2 n-3} q_{2 n-1}^{(2 n)}+c_{2 n-2} q_{2 n}^{(2 n)}=p_{1}^{(2 n)},  \tag{3.2.22}\\
c_{0} q_{2 n-1}^{(2 n)}+c_{1} q_{2 n}^{(2 n)}=p_{2 n-2}^{(2 n)}, \\
c_{0} q_{2 n}^{(2 n)}=p_{2 n-1}^{(2 n)},
\end{array}\right\}
$$

$$
\begin{gather*}
c_{1} q_{0}^{(2 n+1)}+c_{2} q_{1}^{(2 n+1)}+\ldots+c_{2 n+1} q_{2 n}^{(2 n+1)}+c_{2 n+2} q_{2 n+1}^{(2 n+1)}=\gamma_{1}^{(2 n+1)}, \\
c_{0} q_{0}^{(2 n+1)}+c_{1} q_{1}^{(2 n+1)}+\ldots+c_{2 n+1} q_{2 n}^{(2 n+1)}+c_{2 n+2} q_{2 n+1}^{(2 n+1)}=0, \\
\quad c_{0} q_{1}^{(2 n+1)}+\ldots+c_{2 n-1} q_{2 n}^{(2 \dot{n}+1)}+c_{2 n} q_{2 n+1}^{(2 n+1)}=p_{0}^{(2 n+1)}, \tag{3.2.23}
\end{gather*}
$$

$$
\begin{aligned}
& c_{0} q_{2 n}^{(2 n+1)}+c_{1} q_{2 n+1}^{(2 n+1)}=p_{2 n-1}^{(2 n+1)}, \\
& c_{0} q_{2 n+1}^{(2 n+1)}=p_{2 n}^{(2 n+1)},
\end{aligned}
$$

$$
\begin{align*}
& { }^{-c_{-2 n-1}} q_{0}^{(2 n)}-c_{-2 n} q_{1}^{(2 n)}-\ldots-c_{-3} q_{2 n-2}^{(2 n)}-c_{-2} q_{2 n-1}^{(2 n)}-c_{-1} q_{2 n}^{(2 n)}=\rho_{1}^{(2 n)}, \\
& { }^{-c_{-2 n}} q_{0}^{(2 n)}-c_{-2 n+1} q_{1}^{(2 n)}-\ldots-c_{-2} q_{2 n-2}^{(2 n)}-c_{-1} q_{2 n-1}^{(2 n)}=p_{2 n-1}^{(2 n)}, \\
& { }^{-c_{-2 n+1}} q_{0}^{(2 n)}-c_{-2 n+2} q_{1}^{(2 n)}-\ldots-c_{-1} q_{2 n-2}^{(2 n)} \quad=p_{2 n-2}^{(2 n)}, \tag{3.2.24}
\end{align*}
$$

$$
\begin{aligned}
& { }_{-} c_{-2} q_{0}^{(2 n)}-c_{-1} q_{1}^{(2 n)} \quad=p_{1}^{(2 n)}, \\
& { }^{-c}{ }_{-1} q_{0}^{(2 n)} \\
& =p_{0}^{(2 n)} \text {, }
\end{aligned}
$$

and

$$
\begin{align*}
& c_{-2 n-1} q_{0}^{(2 n+1)}-c_{-2 n} q_{1}^{(2 n+1)}-\ldots-c_{-2} q_{2 n-1}^{(2 n+1)}-c_{-1} q_{2 n}^{(2 n+1)} \\
& =\rho_{1}^{(2 n+1)}+p_{2 n}^{(2 n+1)} \text {, } \\
& { }^{-c_{-2 n} q_{0}^{(2 n+1)}-c_{-2 n+1} q_{1}^{(2 n+1)}-\ldots-c_{-1} q_{2 n-1}^{(2 n+1)}=p_{2 n-1}^{(2 n+1)}, ~}  \tag{3.2.25}\\
& -c_{-2} q_{0}^{(2 n+1)}-c_{-1} q_{1}^{(2 n+1)} \\
& =p_{1}^{(2 n+1)} \text {, } \\
& { }_{-c_{-1}} q^{(2 n+1)} \\
& =\mathrm{p}_{0}^{(2 \mathrm{n}+1)} .
\end{align*}
$$

Now, subtracting each equation of (3.2.24) from the corresponding equation of (3.2.22), gives the system of equations

$$
\begin{align*}
& c_{-2 n-1} q_{0}^{(2 n)}+c_{-2 n} q_{1}^{(2 n)}+\ldots+c_{-2} q_{2 n-1}^{(2 n)}+c_{-1} q_{2 n}^{(2 n)}=\rho_{1}^{(2 n)}, \\
& c_{-2 n} q_{0}^{(2 n)}+c_{-2 n+1} q_{1}^{(2 n)}+\ldots+c_{-1} q_{2 n-1}^{(2 n)}+c_{0} q_{2 n}^{(2 n)}=0, \\
& c_{-2 n+1} q_{0}^{(2 n)}+c_{-2 n+2} q_{1}^{(2 n)}+\ldots+c_{0} q_{2 n-1}^{(2 n)}+c_{1} q_{2 n}^{(2 n)}=0,  \tag{3.2.26}\\
& c_{-1} q_{0}^{(2 n)}+c_{0} q_{1}^{(2 n)}+\ldots+c_{2 n-2}^{q_{2 n-1}^{(2 n)}+c_{2 n-1} q_{2 n}^{(2 n)}=0,} \\
& c_{0} q_{0}^{(2 n)}+c_{1} q_{1}^{(2 n)}+\ldots+c_{2 n-1} q_{2 n-1}^{(2 n)}+c_{2 n} q_{2 n}^{(2 n)}=\gamma_{1}^{(2 n)},
\end{align*}
$$

Hence, by remembering $q_{2 n}^{(2 n)}=1$, and applying Cramer's rule to the last $2 n+1$ equations, we obtain

$$
\begin{equation*}
Y_{1}^{(2 n)}=H_{2 n+1}^{(-2 n)} / H_{2 n}^{(-2 n)}, \quad n \geqslant 0 \tag{3.2.27}
\end{equation*}
$$

while applying Cramer's rule to the first $2 n+1$ equations we find

$$
\begin{equation*}
\rho_{1}^{(2 n)}=H_{2 n+1}^{(-(2 n+1))} / H_{2 n}^{(-2 n)}, \quad n \geqslant 0 \tag{3.2.28}
\end{equation*}
$$

Also from (3.2.26) taking the last $2 \mathrm{n}+1$ equations and replacing the last of these $2 n+1$ equations by

$$
1 \cdot q_{0}^{(2 n)}+z q_{1}^{(2 n)}+\ldots+z^{2 n-1} q_{2 n-1}^{(2 n)}+z^{2 n} q_{2 n}^{(2 n)}=Q_{2 n}(z)
$$

we obtain

$$
Q_{2 n}(z)=\frac{1}{H_{2 n}^{(-2 n)}}\left|\begin{array}{lllll}
c_{2 n} & c_{-2 n+1} & \cdots & c_{0} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
c_{-1} & c_{0} & \cdots \cdots \cdots & c_{2 n-1} \\
1 & z & \ldots \ldots . & z^{2 n}
\end{array}\right|, \quad n \geqslant 1
$$

From this it follows that

$$
\begin{equation*}
Q_{2 n}(0)=\sum_{r=1}^{n}\left(D_{2 r-1} D_{2 r}-N_{2 r}\right)=H_{2 n}^{(-(2 n-1))} / H_{2 n}^{(-2 n)} \tag{3.2.29}
\end{equation*}
$$

for $n \geqslant 1$.
Now, subtracting each equation of (3.2.25) from the corresponding equation of $(3.2 .23)$, provides the system

$$
\begin{aligned}
& c_{-2 n-1} q_{0}^{(2 n+1)}+c_{-2 n} q_{1}^{(2 n+1)}+\ldots+c_{-1} q_{2 n}^{(2 n+1)}+c_{0} q_{2 n+1}^{(2 n+1)}=-p_{1}^{(2 n+1)}, \\
& c_{-2 n} q_{0}^{(2 n+1)}+c_{-2 n+1} q_{1}^{(2 n+1)}+\ldots+c_{0} q_{2 n}^{(2 n+1)}+c_{1} q_{2 n+1}^{(2 n+1)}=0, \\
& c_{-2 n+1} q_{0}^{(2 n+1)}+c_{-2 n+2} q_{1}^{(2 n+1)}+\ldots+c_{1} q_{2 n}^{(2 n+1)}+c_{2} q_{2 n+1}^{(2 n+1)}=0, \\
& c_{0} q_{0}^{(2 n+1)}+c_{1} q_{1}^{(2 n+1)}+\ldots+c_{2 n} q_{2 n}^{(2 n+1)}+c_{2 n+1} q_{2 n+1}^{(2 n+1)}=0, \\
& c_{1} q_{0}^{(2 n+1)}+c_{2} q_{1}^{(2 n+1)}+\ldots+c_{2 n+1} q_{2 n}^{(2 n+1)}+c_{2 n+2} q_{2 n+1}^{(2 n+1)}=r_{1}^{(2 n+1)},
\end{aligned}
$$

Consequently from this we find that

$$
\begin{gather*}
\gamma_{1}^{(2 n+1)}=H_{2 n+2}^{(-2 n)} / H_{2 n+1}^{(-2 n)}, \quad n \geqslant 0,  \tag{3.2.30}\\
\rho_{1}^{(2 n+1)}=H_{2 n+2}^{(-(2 n+1))} / H_{2 n+1}^{(-2 n)}, \quad n \geqslant 0, \tag{3.2.31}
\end{gather*}
$$

and

$$
Q_{2 n+1}(z)=\frac{1}{H_{2 n+1}^{(-2 n)}}\left|\begin{array}{lllll}
c_{-2 n} & c_{-2 n+1} & \cdots & c_{1} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
c_{0} & c_{1} & \cdots \cdots & c_{2 n+1} \\
1 & z & \cdots \cdots & z^{2 n+1}
\end{array}\right|, \quad n \geqslant 0
$$

from which

$$
\begin{equation*}
Q_{2 n+1}(0)=D_{2 n+1} \sum_{r=1}^{n}\left(D_{2 r-1} D_{2 r}-N_{2 r}\right)=-H_{2 n+1}^{(-(2 n-1))} / H_{2 n+1}^{(-2 n)}, \quad n \geqslant 0 \tag{3.2.32}
\end{equation*}
$$

for all $n \geqslant 0$.
The only assumption we made to arrive at these results was that $Q_{2 n}(0) \neq 0$ for all $\mathrm{n} \geqslant 1$. Hence, from (3.2.29) we must have

$$
\begin{equation*}
H_{2 n}^{(-(2 n-1))} \neq 0, \text { for all } n \geqslant 1 \tag{3,2.33}
\end{equation*}
$$

Now, substituting for $\gamma_{l}^{(n)}, \rho_{1}^{(n)}$ and $Q_{n}(0)$ the previous expressions in terms of the coefficients $N_{r}$ and $D_{r}$ we immediately arrive at the relations (3.2.11) for these coefficients. Thus by summarising the above results, we obtain

Theorem 3.2.1 : If the condition (3.2.12) holds, then there exists a $\hat{J}$-fraction of the form (3.2.8) with coefficients $\mathrm{N}_{\mathrm{r}}$ and $\mathrm{D}_{\mathrm{r}}$ given by (3.2.11), such that its 2 n -th convergent corresponds to exactly 2 n terms of the series $\mathrm{f}_{0}(\mathrm{z})$ and at least 2 n terms of the series $g_{0}(z)$, while its $(2 \mathrm{n}+1)$-th convergent corresponds to at least $2 \mathrm{n}+2$ terms of the series $f_{0}(z)$.

The correspondence of the $(2 n+1)$-th convergent of this $\hat{J}-$ fraction to the series $g_{0}(z)$ has not yet been considered. From (3.2.20) it follows that
$g_{0}(z)-L_{2 n+1}(z)=\frac{1}{Q_{2 n+1}(z)}\left\{\rho_{1}^{(2 n+1)} z^{2 n}+\right.$ higher order terms $\}, \quad n \geqslant 0$.

Now let us suppose that for a given $m, Q_{2 m+1}(0)$ is not equal to zero. Thus, we can write

$$
g_{0}(z)-L_{2 m+1}(z)=\mu_{1}^{(2 m+1)} 2 m+\text { higher order terms } .
$$

Using (3.2.32) we find that the condition required for this to be the case is $H_{2 m+1}^{(-(2 m-1))} \neq 0$.

But for any given $m$, if $Q_{2 m+1}(0)=0$ and $Q_{2 m+1}^{\prime}(0) \neq 0$, that is from (3.2.14) $1+N_{2 m+1} \neq 0$, then we have

$$
g_{0}(z)-L_{2 m+1}(z)=\mu_{1,1}^{(2 m+1)} z^{2 m-1}+\text { higher order terms }
$$

and the additional conditions required for this to be the case is found to be $H_{2 m+1}^{(-(2 m-1))}=0$ and $H_{2 m}^{(-(2 m-2))} \neq 0$.

In particular if the condition

$$
H_{2 n}^{(-2 n)}>0 \text { and } H_{2 n+1}^{(-2 n)}>0, \quad n \geqslant 0
$$

is satisfied then using the Jacobi identity (1.2.11) we also find that

$$
\mathrm{H}_{2 \mathrm{n}}^{(-(2 n-2))} \neq 0, \quad \mathrm{n} \geqslant 0
$$

Thus, summarising these results, we obtain

Theorem 3.2.2 : If the condition

$$
H_{2 n+1}^{(-2 n)}>0, \quad H_{2 n}^{(-2 n)}>0 \quad \text { and } H_{2 n}^{(-(2 n-1))} \neq 0, \quad n \geqslant 0,(3.2 .34)
$$

holds then there exists a $\hat{J}$-fraction of the form (3.2.8) with coefficients $\mathrm{N}_{\mathrm{r}}$ and $\mathrm{D}_{\mathrm{r}}$ given by (3.2.11), such that its 2n-th convergent corresponds to exactly $2 n$ terms of the series $f_{0}(z)$ and at least 2 n terms of the series $\mathrm{g}_{0}(z)$, while its $(2 n+1)$-th convergent corresponds to exactly $(2 n+2)$ terms of the series $f_{0}(z)$ and at least $(2 n-1)$ terms of the series $g_{0}(z)$.

If we also consider a $\hat{J}$-fraction of the form

$$
\begin{equation*}
\frac{\left\{c_{-1}\right\}^{2} / c_{-2}}{z+D_{1}^{*}}-\frac{N_{2}^{*} z^{2}}{\left(1+N_{2}^{*}\right) z+D_{2}^{*}}-\frac{N_{3}^{*}}{z+D_{3}^{*}}-\frac{N_{4}^{*} z^{2}}{\left(1+N_{4}^{*}\right) z+D_{4}^{*}}-\frac{N_{5}^{*}}{z+D_{5}^{*}}-\ldots \tag{3.2.35a}
\end{equation*}
$$

where

$$
\begin{align*}
& N_{2 r}^{*}=\frac{H_{2 r}^{(-2 r)} \cdot H_{2 r-2}^{(-(2 r-2))}}{\left\{H_{2 r-1}^{(-(2 r-1))}\right\}^{2}}, \quad r \geqslant 1, \\
& D_{2 r}^{*}=\frac{-H_{2 r-1}^{(-2 r)} \cdot H_{2 r}^{(-(2 r-1))}}{H_{2 r-1}^{(-(2 r-1))} \cdot H_{2 r}^{(-2 r)}}, \quad r \geqslant 1,  \tag{3.2.35b}\\
& N_{2 r+1}^{*}=\left\{\frac{H_{2 r+1}^{(-(2 r+1))}}{H_{2 r}^{(-2 r)}}\right\}^{2} \frac{H_{2 r-1}^{(-2 r)}}{H_{2 r+1}^{(-(2 r+2))}}, \quad r \geqslant 1, \\
& D_{2 r+1}^{*}=\frac{-H_{2 r+1}^{(-(2 r+1))} \cdot H_{2 r}^{(-(2 r+1))}}{H_{2 r}^{(-2 r)} \cdot H_{2 r+1}^{(-(2 r+2))}}, \quad r \geqslant 0,
\end{align*}
$$

then we also have

Theorem 3.2.3 : If the condition

$$
\begin{equation*}
H_{2 n+1}^{(-2 n)}>0, \quad H_{2 n}^{(-2 n)}>0 \quad \text { and } \quad H_{2 n+1}^{(-(2 n+1))} \neq 0, \quad n \geqslant 0 \tag{3,2.36}
\end{equation*}
$$

holds then there exists the $\hat{J}$-fraction (3.2.35) for which the 2 n -th convergent corresponds to exactly 2 n terms of the series $\mathrm{f}_{0}(\mathrm{z})$ and at least $2 \mathrm{n}-1$ terms of the series $\mathrm{g}_{0}(\mathrm{z})$, while the $(2 \mathrm{n}+1)$-th convergent corresponds to exatly 2 n terms of the series $\mathrm{f}_{0}(z)$ and at Least $2 \mathrm{n}+2$ terms of the series $\mathrm{g}_{0}(\mathrm{z})$.

### 3.3 METHODS OF DERIVATION

Given the power series expansions $f_{0}(z)$ and $g_{0}(z)$ then the coefficients $n_{r}$ and $d_{r}$ of the corresponding $M^{(0)}$-fraction, when it exists, can be obtained for example by the q-d algorithm (1.2.8). These coefficients in turn can be used in the relations of (3.2.5) to arrive at the coefficients of the $\hat{J}$-fraction (3.2.8). Likewise using a similar relation the coefficients of the $\hat{J}$-fraction (3.2.35) can also be obtained.

A draw back in this method of derivation is that the $M^{(0)}$ fraction might not exist, even then the $\hat{J}$-fraction exists. This is because the required condition for the existence of a $\hat{J}$-fraction is only part of the condition required for the existence of the $M^{(0)}$ fraction. An interesting method of obtaining a $\hat{J}$-fraction, whenever it exists, is the corresponding sequence algorithm of Murphy and O'Donohoe [1977].

Let us consider the $\hat{J}$-fraction (3.2.8). This fraction can be generated by the recurrence relations
$h_{1}(z)=c_{0}-\left(z+D_{1}\right) h_{0}(z)$,
$h_{2 n}(z)=-N_{2 n} h_{2 n-2}(z)-\left(z+D_{2 n}\right) h_{2 n-1}(z), \quad n \geqslant 1$,
$h_{2 n+1}(z)=-N_{2 n+1} z^{2} h_{2 n-1}(z)-\left\{\left(1+N_{2 n+1}\right) z+D_{2 n+1}\right\} h_{2 n}(z), n \geqslant 1$,
such that it follows that
$h_{0}(z)=\frac{c_{0}}{z+D_{1}}-\frac{N_{2}}{z+D_{2}}-\frac{N_{3} z^{2}}{\left(1+N_{3}\right) z+D_{3}}-\ldots-\frac{N_{2 n}}{z+D_{2 n}}+\frac{h_{2 n}(z)}{h_{2 n-1}(z)}, n \geqslant 1$,
and that

$$
\begin{gathered}
h_{0}(z)=\frac{c_{0}}{z+D_{1}}-\frac{N_{2}}{z+D_{2}}-\frac{N_{3} z^{2}}{\left(1+N_{3}\right) z+D_{3}}-\ldots-\frac{N_{2 n+1} z^{2}}{\left(1+N_{2 n+1}\right) z+D_{2 n+1}}+\frac{h_{2 n+1}(z)}{h_{2 n}(z)} \\
n \geqslant 1 .
\end{gathered}
$$

Furthermore, from (3.3.1) and the three term relations (3.2.13), it also follows that

$$
\begin{equation*}
Q_{n}(z) \cdot h_{0}(z)-P_{n}(z)=(-1)^{n_{h}}(z), \quad n \geqslant 0 \tag{3.3.2}
\end{equation*}
$$

Hence, if we denote the power series expansions of $h_{n}(z)$, about the origin and about infinity, by $g_{n}(z)$ and $f_{n}(z)$ respectively, then for the continued fraction to satisfy the correspondence properties given by (3.2.18) and (3.2.20), we must have

$$
\left.\begin{array}{l}
f_{2 n}(z)=f_{0}^{(2 n)} z^{-1}+f_{1}^{(2 n)} z^{-2}+f_{2}^{(2 n)} z^{-3}+\ldots, n \geqslant 0, \\
g_{2 n}(z)=g_{0}^{(2 n)} z^{2 n}+g_{1}^{(2 n)} z^{2 n+1}+g_{2}^{(2 n)} z^{2 n+2}+\ldots, n \geqslant 0, \\
f_{2 n+1}(z)=f_{0}^{(2 n+1)} z^{-2}+f_{1}^{(2 n+1)} z^{-3}+f_{2}^{(2 n+1)} z^{-4}+\ldots, n \geqslant 0,  \tag{3.3.3}\\
g_{2 n+1}(z)=g_{0}^{(2 n+1)} z^{2 n}+g_{1}^{(2 n+1)} z^{2 n+1}+g_{2}^{(2 n+1)} z^{2 n+2}+\ldots, n \geqslant 0,
\end{array}\right\}
$$

where

$$
f_{r}^{(0)}=c_{r} \text { and } g_{r}^{(0)}=-c_{-r-1}, \text { for } r \geqslant 0
$$

From (3.3.1) we find, by taking expansions about infinity and the origin, that

$$
\begin{aligned}
& f_{1}(z)=c_{0}-\left(z+D_{1}\right) f_{0}(z) \\
& g_{0}(z)=c_{0}-\left(z+D_{1}\right) g_{0}(z), \\
& f_{2 n}(z)=-N_{2 n} f_{2 n-2}(z)-\left(z+D_{2 n}\right) f_{2 n-1}(z), \\
& g_{2 n}(z)=-N_{2 n} g_{2 n-2}(z)-\left(z+D_{2 n}\right) g_{2 n-1}(z), \\
& f_{2 n+1}(z)=-N_{2 n+1} z^{2} f_{2 n-1}(z)-\left\{\left(1+N_{2 n+1}\right) z+D_{2 n+1}\right\} f_{2 n}(z), \\
& g_{2 n+1}(z)=-N_{2 n+1} z^{2} g_{2 n-1}(z)-\left\{\left(1+N_{2 n+1}\right) z+D_{2 n+1}\right\} g_{2 n}(z), n \geqslant 1 .
\end{aligned}
$$

Therefore, if we use (3.3.3) in the above equations, then by equating the corresponding coefficients the following (corresponding sequence) algorithm is obtained.

$$
\begin{aligned}
& D_{1}=-f_{1}^{(0)} / f_{0}^{(0)}, \\
& f_{r}^{(1)}=-f_{r+2}^{(0)}-D_{1} f_{r+1}^{(0)}, \quad r \geqslant 0, \\
& g_{0}^{(1)}=f_{0}^{(0)}-D_{1} g_{0}^{(0)}, \\
& g_{r}^{(1)}=-g_{r-1}^{(0)}-D_{1} g_{r}^{(0)}, \quad r \geqslant 1, \\
& N_{2 n}=\left\{g_{0}^{(2 n-1)}\right\}^{2} /\left\{g_{0}^{(2 n-2)} g_{1}^{(2 n-1)}-g_{1}^{(2 n-2)} g_{0}^{(2 n-1)}\right\},
\end{aligned}
$$

$$
\begin{aligned}
& D_{2 n}=-N_{2 n} g_{0}^{(2 n-2)} / g_{0}^{(2 n-1)}, \\
& f_{0}^{(2 n)}=-N_{2 n} f_{0}^{(2 n-2)}-f_{0}^{(2 n-1)}, \\
& f_{r}^{(2 n)}=-N_{2 n} f_{r}^{(2 n-2)}-f_{r}^{(2 n-1)}-D_{2 n^{2} r-1}^{(2 n-1)}, \quad r \geqslant 1, \\
& g_{r}^{(2 n)}=-N_{2 n^{\prime}} g_{r+2}^{(2 n-2)}-D_{2 n} g_{r+2}^{(2 n-1)}-g_{r+1}^{(2 n-1)}, \quad r \geqslant 0, \\
& N_{2 n+1}=-f_{0}^{(2 n)} /\left\{f_{0}^{(2 n)}+f_{0}^{(2 n-1)}\right\}, \\
& D_{2 n+1}=-N_{2 n+1} f_{l}^{(2 n-1)}+\left(1+N_{2 n+1}\right) f_{1}^{(2 n)} \cdot / f_{0}^{(2 n)}, \\
& f_{r}^{(2 n+1)}=-N_{2 n+1} f_{r+2}^{(2 n-1)}-\left(1+N_{2 n+1}\right) f_{r+2}^{(2 n)}-D_{2 n+1} f_{r+1}^{(2 n)}, \quad r \geqslant 0, \\
& g_{0}^{(2 n+1)}=-N_{2 n+1} g_{0}^{(2 n-1)}-D_{2 n+1} g_{0}^{(2 n)}, \\
& g_{r}^{(2 n+1)}=-N_{2 n+1} g_{r}^{(2 n-1)}-D_{2 n+1} g_{r}^{(2 n)}-\left(1+N_{2 n+1}\right) g_{r-1}^{(2 n)}, \quad r \geqslant 1,
\end{aligned}
$$

for $n \geqslant 1$.

As an example consider the power series expansions

$$
\begin{gathered}
f_{0}(z)=\frac{c_{1}}{z}+\frac{c_{2}}{z^{2}}+\frac{c_{3}}{z^{3}}+\cdots, \\
g_{0}(z)=-c_{-1}-c_{-2} z-c_{-3} z^{2}-\ldots,
\end{gathered}
$$

where the coefficients $c_{r}$ are given by

$$
\begin{equation*}
c_{r}=\frac{e}{\sqrt{2 \pi}} \int_{-\infty}^{\infty} t^{r} e^{-\frac{1}{2}\left(t^{2}+1 / t^{2}\right)} d t, \quad r=\ldots-2,-1,0,1,2, \ldots . \tag{3.3.4}
\end{equation*}
$$

Using the method of integration by parts, these coefficients $c_{r}$ can be shown to satisfy

$$
\begin{align*}
& c_{0}=1, \\
& c_{2 s+1}=0, \\
& c_{2 s+2}=(2 s+1) c_{2 s}+c_{2 s-2},  \tag{3.3.5}\\
& c_{-s-2}=c_{s},
\end{align*}
$$

for all $s \geqslant 0$.

For these coefficients, the corresponding M-fraction does not exist. Therefore, the method of using the relation (3.2.5) to obtain the coefficients of the $\hat{J}$-fraction (3.2.8) is not applicable. On the other hand using the above corresponding sequence algorithm, we get

$$
\begin{aligned}
& \mathrm{D}_{\mathrm{n}}=0, \quad \mathrm{n} \geqslant 1 \\
& \mathrm{~N}_{2 \mathrm{n}}=1, \quad \mathrm{~N}_{2 \mathrm{n}+1}=\mathrm{n}, \quad \mathrm{n} \geqslant 1,
\end{aligned}
$$

to give the $\hat{J}$-fraction

$$
\begin{equation*}
\frac{1}{z}-\frac{1}{z}-\frac{z^{2}}{2 z}-\frac{1}{z}-\frac{2 z^{2}}{3 z}-\frac{1}{z}-\frac{3 z^{2}}{4 z}-\frac{1}{z}-\ldots \tag{2.3.6}
\end{equation*}
$$

Here, since $D_{n}=0$ for all $n$, we must have from the relation (3.2.11) and the condition (3.2.34)

$$
H_{2 n+1}^{(-(2 n+1))}=0 \text { and } H_{2 n+1}^{(-(2 n-1))}=0
$$

for all $n \geqslant 0$. Hence, it can be seen from theorem 3.2.3 that in this case the corresponding $\hat{J}$-fraction (3.2.35) does not exist.
$3.4 \hat{J}$-Fractions and the strong hamburger moment Problem

The problem in study, "the strong Hamburger moment problem", can be stated as follows.

Given a double sequence of finite valued real numbers $\left\{c_{n}\right\}_{n=-\infty}^{\infty}$ find conditions to ensure the existence of a bounded non-decreasing function $\psi(t)$ in the interval $(-\infty, \infty)$ such that

$$
\begin{equation*}
\int_{-\infty}^{\infty} t^{n} d \psi(t)=c_{n}, \quad n=\ldots-2,-1,0,1,2, \ldots . \tag{3.4.1}
\end{equation*}
$$

If we first assume that there exists such a bounded nondecreasing function $\psi(t)$, which is sometimes called a solution, then we must have for all $\mathrm{n} \geqslant 0$

$$
\begin{equation*}
\int_{-\infty}^{\infty}\left\{\sum_{r=-n}^{n} \xi_{r} t^{r}\right\}^{2} d \psi(t)>0, \tag{3.4.2}
\end{equation*}
$$

whenever

$$
\sum_{r=-n}^{n} \xi_{r}^{2}>0 .
$$

Hence, using the relation (3.4.1) in (3.4.2), we obtain the quadratic form

$$
\left[\begin{array}{llll}
\xi_{-n} \xi_{-n+1} & \cdots & \xi_{n}
\end{array}\right]\left[\begin{array}{llll}
c_{-2 n} & c_{-2 n+1} & \cdots & c_{0} \\
c_{-2 n+1} & c_{-2 n+2} & \cdots & c_{1} \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
c_{0} & c_{1} & \cdots & c_{2 n}
\end{array}\right]\left[\begin{array}{l}
\xi_{-n} \\
\xi_{-n+1} \\
\vdots \\
\vdots \\
\xi_{n}
\end{array}\right]>0, \quad n \geqslant 0
$$

Hence for all $n \geqslant 0$, the above square matrix must be positive definite. The positive definiteness of this matrix is in fact equivalent to the condition (see Wall [1948])

$$
\begin{equation*}
H_{2 n+1-r}^{(-2 n)}>0, \quad r=0,1,2, \ldots, 2 n+1 \tag{3.4.3}
\end{equation*}
$$

Thus it follows for all $n \geqslant 0$ that the above condition is necessary for the strong Hamburger moment problem to have a solution. Let us now consider the $\hat{J}$-fraction (3.2.8). For this fraction we have from the three term relations (3.2.13) that

$$
\begin{aligned}
& \left\{P_{2 n}(z) Q_{2 n-1}(z)-P_{2 n-1}(z) Q_{2 n}(z)\right\}=N_{2 n} N_{2 n-1} \cdots N_{2} c_{0} z^{2 n-2}, \quad n \geqslant 1 \\
& \left\{P_{2 n+1}(z) Q_{2 n}(z)-P_{2 n}(z) Q_{2 n+1}(z)\right\}=N_{2 n+1} N_{2 n} \cdots N_{2} c_{0} z^{2 n}, \quad n \geqslant 1.4
\end{aligned}
$$

In addition if

$$
K_{n}(z)=\left\{Q_{n}^{\prime}(z) Q_{n-1}(z)-Q_{n-1}^{\prime}(z) Q_{n}(z)\right\}, \quad n \geqslant 1
$$

where $Q_{n}^{\prime}(z)$ is the derivative of $Q_{n}(z)$, then

$$
\begin{equation*}
K_{2 n+2}(z)=\left\{Q_{2 n+1}(z)\right\}^{2}+N_{2 n+2} K_{2 n+1}(z) \tag{3.4.5a}
\end{equation*}
$$

and

$$
\begin{align*}
K_{2 n+1}(z)= & \left\{Q_{2 n}(z)\right\}^{2}+N_{2 n+1}\left\{Q_{2 n}(z)-z Q_{2 n-1}(z)\right\}^{2}+N_{2 n+1} N_{2 n} z^{2}\left\{Q_{2 n-2}(z)\right\}^{2} \\
& +N_{2 n+1} N_{2 n} N_{2 n-1} z^{2}\left\{Q_{2 n-2}(z)-z Q_{2 n-3}(z)\right\}^{2}+\ldots \\
& \ldots+N_{2 n+1} N_{2 n} \cdots N_{3} N_{2} z^{2 n_{\{ }}\left\{Q_{0}(z)\right\}^{2} \tag{3.4.5b}
\end{align*}
$$

In the $\hat{J}$-fraction ( 3.2 .8 ), suppose we have

$$
\begin{equation*}
c_{0} \neq 0, \quad N_{r+1}>0, \quad Q_{2 r}(0) \neq 0 \text { and } D_{r} \text { real } \tag{3.4.6}
\end{equation*}
$$

for all $\mathrm{r} \geqslant 1$. Then we see from the three term relation (3.2.13) and from (3.4.5) that $K_{n}(z)>0$ for all $z$ real and for all $n \geqslant 1$. An immediate consequence of this result is that all the roots of $Q_{n}(z)$ are real, distinct and different from those of $Q_{n-1}(z)$. The proof is similar to that of the proof in the case of any set of orthogonal polynomials, and can be found for example in Szegö [1959] (see also theorem 2.4.6).

Now using the relation (3.4.4) it is seen that if $z_{r}^{(n)}$, a root of the polynomial $Q_{n}(z)$, is non-zero then it is not a root of $P_{n}(z)$, while if it is zero then it is also a root of $P_{n}(z)$.

Since all the roots $z_{r}^{(n)}, r=1,2, \ldots, n$, of the polynomial
$Q_{n}(z)$ are real and distinct and since the non-zero ones are not roots of $P_{n}(z)$ the quotient $P_{n}(z) / Q_{n}(z)$ has a partial decomposition of the form

$$
\begin{equation*}
\frac{P_{n}(z)}{Q_{n}(z)}=\sum_{r=1}^{n} \frac{e_{r}^{(n)}}{z-z_{r}^{(n)}}, \quad n \geqslant 1 \tag{3.4.7}
\end{equation*}
$$

where

$$
\begin{equation*}
\ell_{r}^{(n)}=\frac{P_{n}\left(z_{r}^{(n)}\right)}{Q_{n}^{\prime}\left(z_{r}^{(n)}\right)}, \quad r=1,2, \ldots, n \tag{3.4.8}
\end{equation*}
$$

If we rewrite $\ell_{r}^{(n)}$ in the form

$$
\ell_{r}^{(n)}=\frac{P_{n}\left(z_{r}^{(n)}\right) Q_{n-1}\left(z_{r}^{(n)}\right)-P_{n-1}\left(z_{r}^{(n)}\right) Q_{n}\left(z_{r}^{(n)}\right)}{Q_{n}^{\prime}\left(z_{r}^{(n)}\right) Q_{n-1}\left(z_{r}^{(n)}\right)-Q_{n-1}^{\prime}\left(z_{r}^{(n)}\right) Q_{n}\left(z_{r}^{(n)}\right)}, \quad r=1,2, \ldots, n,
$$

and then use the equations (3.4.4) and (3.4.5), we obtain

$$
e_{r}^{(2 m)}=\frac{N_{2 m} N_{2 m-1} \cdots N_{2} c_{0}\left\{z_{r}^{(2 m)}\right\}^{2 m-2}}{K_{2 m .}\left(z_{r}^{(2 m)}\right)}, \quad r=1,2, \ldots, 2 m
$$

$$
\begin{equation*}
\ell_{r}^{(2 m+1)}=\frac{N_{2 m+1} N_{2 m} \cdots N_{2} c_{0}\left\{z_{r}^{(2 m+1)}\right\}^{2 m}}{K_{2 m+1}\left(z_{r}^{(2 m+1)}\right)}, \quad r=1,2, \ldots, 2 m+1 \tag{3.4.9}
\end{equation*}
$$

Consequently we note from (3.4.9), for the $\hat{J}$-fraction (3.2.8), if the condition (3.4.6) holds and further $c_{0}$ is also greater than zero, then

$$
e_{r}^{(n)} \geqslant 0, \quad r=1,2, \ldots, n, \quad n \geqslant 1
$$

and

$$
e_{r}^{(n)}>0, \text { whenever } z_{r}^{(n)} \neq 0
$$

Furthermore, we have

$$
\sum_{r=1}^{n} e_{r}^{(n)}=\operatorname{Lt}_{z \rightarrow \infty} \sum_{r=1}^{n} \frac{z e_{r}^{(n)}}{z-z_{r}^{(n)}}=\operatorname{Lt}_{z \rightarrow \infty} \frac{z P_{n}(z)}{Q_{n}(z)}, \quad n \geqslant 1
$$

As $P_{n}(z)$ is a polynomial of the form

$$
\mathrm{c}_{0} \mathrm{z}^{\mathrm{n}-1}+\text { lower order terms }
$$

then from this and from (3.2.14), we obtain

$$
\sum_{r=1}^{n} e_{r}^{(n)}=c_{0}, \text { for } n \geqslant 1
$$

Now, let us define a sequence of step functions $\left\{\psi_{n}(t)\right\}$ by

$$
\psi_{n}(t)= \begin{cases}0, & \text { for }-\infty<t \leqslant z_{1}^{(n)}  \tag{3.4.10}\\ \sum_{s=1}^{r} \ell_{s}^{(n)}, & \text { for } z_{r}^{(n)}<t \leqslant z_{r+1}^{(n)}, \quad r=1,2, \ldots, n-1 \\ c_{0}, & \text { for } z_{n}^{(n)}<t<\infty\end{cases}
$$

for $n \geqslant 1$. Thus, it can be noted from (3.4.7) that the quotients $P_{n}(z) / Q_{n}(z)$ can be given as

$$
\begin{equation*}
\frac{p_{n}(z)}{Q_{n}(z)}=\int_{-\infty}^{\infty} \frac{1}{z-t} d \psi_{n}(t), \quad n \geqslant 1 \tag{3.4.11}
\end{equation*}
$$

This result immediately follows from the definition of the Stielties integral (see for example Widder [1952])

To proceed any further we require the following result which is due to Helly.

Theorem : Let $f(t)$ be a continuous, complex valued function of the real variable $t$ such that

$$
\operatorname{Lt}_{t \rightarrow \pm \infty} f(t)=0
$$

and let $\left\{\phi_{n}(t)\right\}$ be a sequence of real valued non-decreasing functions defined on $(-\infty, \infty)$, such that

$$
\lambda \leqslant \phi_{n}(t) \leqslant \mu \text { for }-\infty<t<\infty \text { and } n=1,2, \ldots
$$

Then there exists a subsequence $\left\{n_{k}\right\}$ of positive integers such that

$$
\operatorname{Lt}_{k} \rightarrow \infty{ }_{n_{k}}(t)=\phi(t), \text { for }-\infty<t<\infty \text {, }
$$

and

$$
\operatorname{Lt}_{k \rightarrow \infty} \int_{-\infty}^{\infty} f(t) d \phi_{n_{k}}(t)=\int_{-\infty}^{\infty} f(t) d \phi(t)
$$

where $\phi(t)$ is also a real valued, non-decreasing function defined on $(-\infty, \infty)$ such that $\lambda \leqslant \phi(t) \leqslant \mu$.

The proof of this theorem can be found in wall [1948] and Natanson [1955].

Using the Helly theorem we immediately see that for the sequence of functions $\left\{\psi_{n}(t)\right\}$ defined by (3.4.10), there exist a subsequence $\left\{n_{r}\right\}$ of positive integers such that

$$
\operatorname{Lt}_{r} \rightarrow \infty \psi_{n_{r}}(t)=\psi(t), \quad-\infty<t<\infty
$$

and

$$
\begin{equation*}
\underset{n_{r} \rightarrow \infty}{\operatorname{Lt}} \int_{-\infty}^{\infty} \frac{1}{z-t} d \psi_{n_{r}}(t)=\int_{-\infty}^{\infty} \frac{1}{z-t} d \psi(t) \tag{3.4.12}
\end{equation*}
$$

where $\psi(t)$ is a real valued non-decreasing function such that $0 \leqslant \psi(t) \leqslant c$, for $-\infty<t<\infty$.

Now, given the double sequence of real numbers $\left\{c_{n}\right\}_{n=-\infty}^{\infty}$ we define the $\hat{J}$-fraction (3.2.8) such that its coefficients are given by (3.2.11). Then we see from theorem 3.2.2 that under the condition (3.2.34) this $\hat{J}$-fraction corresponds to the power series expansions

$$
f_{0}(z)=\frac{c_{0}}{z}+\frac{c_{1}}{z^{2}}+\frac{c_{2}}{z^{3}}+\cdots,
$$

and

$$
g_{0}(z)=-c_{-1}-c_{-2} z-c_{-3} z^{2}-\cdots .
$$

From (3.2.11) and (3.2.29), we also see that this $\hat{J}$-fraction satisfies the condition (3.4.6) with $c_{0}>0$. Therefore, it follows from (3.4.11) and (3.4.12) that there exists a real valued non-
decreasing function $\psi(t)$ such that $0 \leqslant \psi(t) \leqslant c_{0}$ for $-\infty<t<\infty$ and

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{1}{z-t} d \psi(t) & =\frac{c_{1}}{z}+\frac{c_{2}}{z^{2}}+\frac{c_{3}}{z^{3}}+\ldots, \\
& =-c_{-1}-c_{-2} z-c_{-3} z^{2}-\ldots,
\end{aligned}
$$

for $z$ large and small respectively.
Hence, expanding the left hand side of the above equation about the origin and about infinity, and comparing the corresponding coefficients, we find the required results (3.4.1).

The only requirement on the real numbers $c_{n}$ for arriving at this result was that the condition (3.2.34) is satisfied. In other words, that the condition (3.2.34) is sufficient for the existence of a solution $\psi(t)$ to the Hamburger moment problem.

Similarly by considering the corresponding $\hat{J}$-fraction (3.2.35), we can also arrive at condition (3.2.36) as another sufficient condition for the existence of a solution to the strong Hamburger moment problem.

We further note from the Jacobi identity (1.2.11) that if

$$
\begin{equation*}
H_{2 n+1}^{(-2 n)}>0 \quad \text { and } \quad H_{2 n}^{(-2 n)}>0, \quad n \geqslant 0 \tag{3.4.13}
\end{equation*}
$$

then for any value of $r>1$

$$
H_{2 r-1}^{(-(2 r-1))} \neq 0 \text { and } H_{2 r+1}^{(-(2 r+1))} \neq 0
$$

if $\mathrm{H}_{2 \mathrm{r}}^{(-(2 r-1))}=0$, while

$$
\mathrm{H}_{2 \mathrm{r}}^{(-(2 \mathrm{r}-1))} \neq 0 \text { and } \mathrm{H}_{2 \mathrm{r}+2}^{(-(2 r+1))} \neq 0
$$

if $H_{2 r+1}^{(-(2 r+1))}=0$.
Using these results we find it is always possible to construct
a corresponding $\hat{J}$-fraction with condition (3.4.13) alone. The convergents of these new $\hat{J}$-fractions are chosen from the convergents of the regular $\hat{\jmath}$-fractions (3.2.8) and (3.2.35). There may be more than one such corresponding $\hat{J}$-fraction which can be constructed. To understand this, let us look at a possible construction.

Suppose we have $H_{2 r}^{(-(2 r-1))}=0$ and $H_{2 s}^{(-(2 s-1))} \neq 0$ for $s=0,1, \ldots, r-1$. Then it follows that $H_{2 r-1}^{(-(2 r-1))} \neq 0$ and $H_{2 r+1}^{(-(2 r+1))} \neq 0$. Hence, we start with the $\hat{j}$-fraction (3.2.8) and at the ( $2 r-1$ )-th stage we switch to the $\hat{J}$-fraction (3.2.35). This can be done as follows
$\ldots-\frac{N_{2 r-3} z^{2}}{\left(1+N_{2 r-3}\right) z+D_{2 r-3}}-\frac{N_{2 r-2}}{z+D_{2 r-2}}-\frac{N_{2 r-1}^{(-)} z^{2}}{\left(1+N_{2 r-1}^{(-)}\right) z+D_{2 r-1}^{(-)}}$

$$
-\frac{N_{2 r}^{*} z^{2}}{\left(1+N_{2 r}^{*}\right) z+D_{2 r}^{*}}-\frac{N_{2 r+1}^{*}}{z+D_{2 r+1}^{*}}-\cdots
$$

Further if $H_{2 r+2 m+1}^{(-(2 r+2 m+1))}=0$ for some $m>1$ then we switch back to the $\hat{J}$-fraction (3.2.8) as follows

$$
\begin{gathered}
-\frac{N_{2 r+2 m-1}^{*}}{z+D_{2 r+2 m-1}^{*}}-\frac{N_{2 r+2 m^{*}}^{*}-}{\left(1+N_{2 r+2 m}^{*}\right) z+D_{2 r+2 m}^{*}}-\frac{N_{2 r+2 m+1}^{(+)}}{z+D_{2 r+2 m+1}^{(+)}} \\
\\
-\frac{N_{2 r+2 m+2}}{z+D_{2 r+2 m+2}}-\cdots
\end{gathered}
$$

Such change over is made whenever necessary. The coefficients $N_{2 s+1}^{(-)}, D_{2 s+1}^{(-)}, N_{2 s+1}^{(+)}$and $D_{2 s+1}^{(+)}$which are used for these switches can be proved to satisfy:

$$
\begin{align*}
& N_{2 s+1}^{(-)}=\left\{\begin{array}{l}
H^{(-(2 s+1))} \\
H_{2 s+1}^{(-(2 s-1))}
\end{array}\right\}^{2} \frac{\mathrm{H}^{(-(2 s-2))}}{\mathrm{H}_{2 \mathrm{~s}+1}^{(-(2 s+2))}}, \\
& \mathrm{D}_{2 S+1}^{(-)}=-\frac{H_{2 S}^{(-2 s)} H_{2 S+1}^{(-(2 s+1))}}{H_{2 S}^{(-(2 S-1))} H_{2 S+1}^{(-(2 S+2))}},  \tag{3.4,14}\\
& N_{2 s+1}^{(+)}=\frac{H_{2 s-1}^{(-2 s)} H_{2 s+1}^{(-2 s)}}{\left\{H_{2 s}^{(-2 s)}\right\}^{2}}, \\
& D_{2 s+1}^{(+)}=-\frac{H_{2 s+1}^{(-2 s)} H_{2 s}^{(-(2 s-1))}}{H_{2 s+1}^{(-(2 s+1))} H_{2 s}^{(-2 s)}}-\frac{H_{2 s+2}^{(-(2 s+1))} H_{2 s}^{(-2 s)}}{H_{2 s+1}^{(-2 s)} H_{2 s+1}^{(-(2 s+1))}},
\end{align*}
$$

From (3.2.11), (3.2.35) and (3.4.14) we note that the partial numerators of this new corresponding $\hat{J}$-fraction are positive. Hence, as for the regular $\hat{J}$-fractions (3.2.8) and (3.2.35), it is also possible to find integral representations for the convergents of this
$\hat{J}$-fraction. This in turn implies that the condition (3.4.13) alone is sufficient for the existence of a solution to the strong Hamburger moment problem. Consequently, looking also at (3.4.3) we can conclude that the necessary and sufficient condition for the existence of a solution to the strong Hamburger moment problem is that the real numbers $c_{n}$ satisfy (3.4.13).

> CHAPTER FOUR

CONVERGENCE BEHAVIOUROF A CLASS OF $\hat{J}-F R A C T I O N S$

### 4.1 POSITIVE DEFINITE $\hat{J}$-FRACTIONS

As defined in Chapter 3, a $\hat{J}$-fraction is a continued fraction of the form

$$
\begin{equation*}
\frac{1}{z+b_{1}}-\frac{\left\{a_{2}(z)\right\}^{2}}{z+b_{2}}-\frac{\left\{a_{3}(z)\right\}^{2}}{z+b_{3}}-\frac{\left\{a_{4}(z)\right\}^{2}}{z+b_{4}}-\ldots, \tag{4.1.1}
\end{equation*}
$$

in which all the $b_{n}$ are complex constants and each $\left\{a_{n}(z)\right\}$ is either a complex constant ( $l_{n}+i m_{n}$ ) or a complex variable of the form $\left(\ell_{n}+i m_{n}\right) z$. Here, all the $\ell_{n}$ and $m_{n}$ are real.

If all the $b_{n}$ are real and all the $m_{n}$ are equal to zero then the continued fraction is referred to as a real $\hat{J}$-fraction. Following the definition of a positive definite J-fraction by Wall [1948], the fraction (4.1.1) will be referred to as a "positive definite $\hat{\jmath}$ fraction" if the coefficients $\left\{a_{n}(z)\right\}$ and $b_{n}$ satisfy the following property:

For all $\mathrm{y}>0$ and for all $\mathrm{n} \geqslant 1$

$$
\begin{equation*}
\sum_{r=1}^{n}\left(y+\beta_{r}\right) \xi_{r}^{2}-2 \sum_{r=1}^{n-1} \alpha_{r+1}(z) \xi_{r} \xi_{r+1}>0 \tag{4.1.2}
\end{equation*}
$$

whenever

$$
\sum_{r=1}^{n} \xi_{r}^{2}>0
$$

where

$$
\mathrm{y}=I \mathrm{~m}(\mathrm{z}), \quad \beta_{\mathrm{r}}=I \mathrm{~m}\left(\mathrm{~b}_{\mathrm{r}}\right) \quad \text { and } \quad \alpha_{\mathrm{r}+1}(\mathrm{z})=I \mathrm{~m}\left\{\mathrm{a}_{\mathrm{r}+1}(\mathrm{z})\right\}
$$

The term positive definite comes from the fact that the relation (4.1.2) is equivalent to the positive definiteness of the tri-diagonal matrix

for $\mathrm{y}>0$ and for $\mathrm{n} \geqslant 1$.
We know, from the theory of continued fractions, that the numerator $P_{n}(z)$ and the denominator $Q_{n}(z)$ of the $n$-th approximant of the $\hat{J}$-fraction (4.1.1) satisfy

$$
\begin{aligned}
& P_{n}(z)=\left(z+b_{n}\right) P_{n-1}(z)-\left\{a_{n}(z)\right\}^{2} P_{n-2}(z), \\
& \\
& n \geqslant 1, \quad \text { (4.1.3) }
\end{aligned}
$$

$$
Q_{n}(z)=\left(z+b_{n}\right) Q_{n-1}(z)-\left\{a_{n}(z)\right\}^{2} Q_{n-2}(z),
$$

with

$$
a_{1}(z)=1, P_{-1}(z)=-1, Q_{-1}(z)=0, P_{0}(z)=0 \text { and } Q_{0}(z)=1
$$

Using (4.1.3) it can be seen that the denominator polynomials $Q_{n}(z)$ can also be given in terms of the following determinant formula

|  | $\left(z+b_{1}\right)$ | $-a_{2}(z)$ | 0 | 0 |  | 0 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $-a_{2}(z)$ | $\left(z+b_{2}\right)$ | $-a_{3}(z)$ | 0 | . ......... | 0 |
| . | 0 | $-a_{3}(z)$ | $\left(z+b_{3}\right)$ | $-a_{4}(z)$ | -•......... | 0 |
| $Q_{n}(z)=$ |  |  |  |  |  |  |
|  | 0 |  |  | $-a_{n-1}(z)$ | $\left(z+b_{n-1}\right)$ | $-a_{n}(z)$ |
|  | 0 |  | . . . . ${ }^{\text {a }}$ | 0 | $-a_{n}(z)$ | $\left(z+b_{n}\right)$ |

for $n \geqslant 1$.
Considering now the following system of linear homogeneous equations in the complex variable $U_{r}$,

$$
\begin{aligned}
\left(z+b_{1}\right) U_{1}-a_{2}(z) U_{2} & =0 \\
-a_{2}(z) U_{1}+\left(z+b_{2}\right) U_{2}-a_{3}(z) U_{3} & =0 \\
-a_{n-1}(z) U_{n-2}+\left(z+b_{n-1}\right) U_{n-1}-a_{n}(z) U_{n} & =0 \\
-a_{n}(z) U_{n-1}+\left(z+b_{n}\right) U_{n} & =0
\end{aligned}
$$

we note that this system has $Q_{n}(z)$ as its determinant. Hence, this system has no solution other than the trivial solution if, and only if, $Q_{n}(z)$ is non-zero.

Multiplying the equations of (4.1.4) by $\bar{U}_{1}, \bar{U}_{2}, \ldots, \bar{U}_{n}$ respectively and adding the resulting equations, we obtain

$$
\begin{equation*}
\sum_{r=1}^{n}\left(z+b_{r}\right)\left|U_{r}\right|^{2}-\sum_{r=1}^{n-1} a_{r+1}(z)\left(U_{r} \bar{U}_{r+1}+\bar{U}_{r} U_{r+1}\right)=0 \tag{4.1.5}
\end{equation*}
$$

Now, if the fraction (4.1.1) is a positive definite $\hat{J}$-fraction then from (4.1.2) we also have

$$
\begin{equation*}
\sum_{r=1}^{n}\left(y+\beta_{r}\right)\left|U_{r}\right|^{2}-\sum_{r=1}^{n-1} \alpha_{r+1}(z)\left(U_{r} \bar{U}_{r+1}+\bar{U}_{r} U_{r+1}\right)>0 \tag{4.1.6}
\end{equation*}
$$

for all $y>0$ and for all $n \geqslant 1$ provided $\sum_{r=1}^{n}\left|U_{r}\right|^{2}>0$.
Since the left hand side of (4.1.6) is the imaginary part of the left hand side of $(4.1 .5)$, we see that if (4.1.1) is positive definite then, for $y>0$, (4.1.5) is true if and only if, $\mathrm{U}_{1}=\mathrm{U}_{2}=\ldots=\mathrm{U}_{\mathrm{n}}=0$, or equivalently, when the determinant $Q_{n}(z) \neq 0$. Hence, the following result is established.

Theoren 4.1.1 : If the $\hat{J}$-fraction (4.1.1) is positive definite then the denominators $Q_{n}(z)$ of all its convergents are non-zero for all $z$ for which the imaginary part is positive.

It is very difficult to see when a $\hat{J}$-fraction is positive definite using (4.1.2). Hence, we require another way of identifying a positive definite $\hat{J}$-fraction. There is in fact such a method and it can be given as:

Theorem 4.1.2: The $\hat{J}$-fraction (4.1.1) is positive definite if, and only if

I

$$
\begin{equation*}
B_{n}=I m\left(b_{n}\right) \geqslant 0, \quad n=1,2, \ldots, \tag{4.1.7}
\end{equation*}
$$

II There exist numbers $g_{1}, g_{2}, g_{3}, \ldots$, satisfying $g_{1}=0,0 \leqslant g_{n} \leqslant 1$, $(\mathrm{n} \geqslant 2)$ such that for all $\mathrm{n} \geqslant 1$

$$
\begin{equation*}
m_{n+1}^{2}=\beta_{n} \beta_{n+1}\left(1-g_{n}\right) g_{n+1} \tag{4.1.8a}
\end{equation*}
$$

if $a_{n+1}(z)=\ell_{n+1}+i m_{n+1}$, but

$$
\begin{equation*}
m_{n+1}=0 \text { and } l_{n+1}^{2}=\left(1-g_{n}\right) g_{n+1} \tag{4.1.8b}
\end{equation*}
$$

if $a_{n+1}(z)=\left(e_{n+1}+i m_{n+1}\right) z$.
In (4.1.8a) $g_{n+1}$ is taken to be zero whenever $m_{n+1}$ is zero and in (4.1.8b) $g_{n+1}$ must be less than unity if $\beta_{r+1}=0$ for $r=0,1,2,3, \ldots, n$.

Proof : Let us consider the definition (4.1.2) of a positive definite $\hat{J}$-fraction. Here, if we let $\xi_{r}=0$ for all r except $r=p \leqslant n$ then it becomes

$$
\left(y+\beta_{p}\right)>0 .
$$

Consequently, since $y>0$, the necessity of the relation (4.1.7) follows immediately. Now in (4.1.2), letting $\xi_{3}=\xi_{4}=\ldots=0$, gives

$$
\left(y+\beta_{1}\right) \xi_{1}^{2}-2 \alpha_{2}(z) \xi_{1} \xi_{2}-\left(y+\beta_{2}\right) \xi_{2}^{2}>0,
$$

and here completing the square then leads to

$$
\left\{\left(y+\beta_{1}\right)^{\frac{3}{2}} \xi_{1}-\frac{\alpha_{2}(z)}{\left(y+\beta_{1}\right)^{\frac{3}{2}}} \xi_{2}\right\}^{2}+\left\{\frac{\left(y+\beta_{1}\right)\left(y+\beta_{2}\right)-\left(\alpha_{2}(z)\right)^{2}}{\left(y+\beta_{1}\right)^{\frac{3}{2}}}\right\} \xi_{2}>0 .
$$

Therefore we note, it is necessary that

$$
\left\{\alpha_{2}(z)\right\}^{2}<\left(y+\beta_{1}\right)\left(y+\beta_{2}\right)
$$

There are two different cases to look at, namely $a_{2}(z)=\ell_{2}+i m_{2}$ and $a_{2}(z)=\left(\ell_{2}+i m_{2}\right) z$. In the first case we have $\alpha_{2}(z)=m_{2}$ and thus $m_{2}{ }^{2}<\left(y+\beta_{1}\right)\left(y+\beta_{2}\right)$. Since this relation must hold for all $y>0$ it follows that

$$
m_{2}^{2} \leqslant \beta_{1} \beta_{2} .
$$

Hence, we can choose, $\mathrm{g}_{1}=0$ and $0 \leqslant \mathrm{~g}_{2} \leqslant 1$, with $\mathrm{g}_{2}=0$ whenever $m_{2}=0$, such that

$$
\begin{equation*}
\left\{\alpha_{2}(z)\right\}^{2}=m_{2}^{2}=\beta_{1} \beta_{2}\left(1-g_{1}\right) g_{2} . \tag{4.1.9}
\end{equation*}
$$

In the second case, when $a_{2}(z)=\left(\ell_{2}+i m_{2}\right) z$, we have $\alpha_{2}(z)=\ell_{2} y+m_{2} x$, where $z=x+i y$. Hence, for the condition

$$
\left(l_{2} y+m_{2} x\right)^{2}<\left(y+\beta_{1}\right)\left(y+\beta_{2}\right)
$$

to hold for all $y>0$ and for all $x$, we must have

$$
m_{2}=0 \text { and } \ell_{2}^{2} \leqslant 1 \text {, (with } \ell_{2}^{2}<1 \text { for } \beta_{1}=\beta_{2}=0 \text { ). }
$$

Thus, we can choose $g_{1}=0$ and $0 \leqslant g_{2} \leqslant 1,\left(g_{2}<1\right.$, if $\left.\beta_{1}=\beta_{2}=0\right)$, such that

$$
m_{2}=0 \quad \text { and } \ell_{2}^{2}=\left(1-g_{1}\right) g_{2} .
$$

From this we also have

$$
\begin{equation*}
\left\{\alpha_{2}(z)\right\}^{2}=\ell_{2}{ }^{2} y^{2}=y^{2}\left(1-g_{1}\right) g_{2} . \tag{4.1.10}
\end{equation*}
$$

Now, taking $\xi_{4}=\xi_{5}=\ldots=0$, we obtain from the definition (4.1.2),

$$
\begin{equation*}
\left(y+\beta_{1}\right) \xi_{1}^{2}-2 \alpha_{2}(z) \xi_{1} \xi_{2}+\left(y+\beta_{2}\right) \xi_{2}^{2}-2 \alpha_{3}(z) \xi_{2} \xi_{3}+\left(y+\beta_{3}\right) \xi_{3}^{2}>0 \tag{4.1.11}
\end{equation*}
$$

Suppose $a_{2}(z)=\ell_{2}+i m_{2}$. Then using (4.1.9) for $\alpha_{2}(z)$, the relation (4.1.11) can be written as

$$
\begin{aligned}
& y \xi_{1}^{2}+\left\{\left(1-g_{1}\right)^{\frac{3}{2}} \beta_{1}^{\frac{1}{2}} \xi_{1}-\mathrm{g}_{2}^{\frac{1}{2}} \beta_{2}^{\frac{1}{2}} \xi_{2}\right\}^{2}+g_{2} y \xi_{2}^{2} \\
& \\
& \quad+\left(1-g_{2}\right)\left(y+\beta_{2}\right) \xi_{2}^{2}-2 \alpha_{3}(z) \xi_{2} \xi_{3}+\left(y+\beta_{3}\right) \xi_{3}^{2}>0
\end{aligned}
$$

On the other hand, suppose $a_{2}(z)=\left(\ell_{2}+i m_{2}\right) z$. Then, using (4.1.10), the relation (4.1.11) can be written as

$$
\begin{aligned}
& \beta_{1} \xi_{1}^{2}+y\left\{\left(1-g_{1}\right)^{\frac{3}{2}} \xi_{1}-g_{2}^{\frac{3}{2}} \xi_{2}\right\}^{2}+g_{2} \beta_{2} \xi_{2}^{2} \\
& \\
& \quad+\left(1-g_{2}\right)\left(y+\beta_{2}\right) \xi_{2}^{2}-2 \alpha_{3}(z) \xi_{2} \xi_{3}+\left(y+\beta_{3}\right) \xi_{3}^{2}>0
\end{aligned}
$$

Therefore, it is necessary that

$$
\left(1-g_{2}\right)\left(y+\beta_{2}\right) \xi_{2}^{2}-2 \alpha_{3}(z) \xi_{2} \xi_{3}+\left(y+\beta_{3}\right) \xi_{3}^{2} \geqslant 0
$$

Here, only the inequality is assumed when $\beta_{1}=\beta_{2}=0$. From this we obtain

$$
\left\{\alpha_{3}(z)\right\}^{2} \leqslant\left(1-g_{2}\right)\left(y+\beta_{2}\right)\left(y+\beta_{3}\right)
$$

Once again considering the two possible cases, we find that for $a_{3}(z)=\ell_{3}+i m_{3}$,

$$
\left\{\alpha_{3}(z)\right\}^{2}=m_{3}^{2}=\beta_{2} \beta_{3}\left(1-g_{2}\right) g_{3}
$$

where $0 \leqslant g_{3} \leqslant 1$, with $g_{3}=0$ when $m_{3}=0$.

For $a_{3}(z)=\left(\ell_{3}+i m_{3}\right) z$,

$$
m_{3}=0 \text { and } \ell_{3}^{2}=\left(1-g_{2}\right) g_{3},
$$

where $0 \leqslant g_{3} \leqslant 1$, with $g_{3}<1$ if $\beta_{1}=\beta_{2}=\beta_{3}=0$.
Continuing this manner, it can be seen that the condition (4.1.8) is also necessary for (4.1.2) to hold.

To prove the sufficiency of the relations (4.1.7) and (4.1.8), let us consider these relations. From (4.1.8) we have

$$
\begin{array}{r}
\left\{\alpha_{n+1}(z)\right\}^{2}=m_{n+1}^{2} \leqslant\left(y+\beta_{n}\right)\left(y+\beta_{n+1}\right)\left(1-g_{n}\right) g_{n+1}, \quad \text { (4.1.12a) } \\
\text { if } a_{n+1}(z)=\ell_{n+1}+i m_{n+1}, \text { while if } a_{n+1}(z)=\left(\ell_{n+1}+i m_{n+1}\right) z \text { then }
\end{array}
$$

$$
\begin{equation*}
\left\{\alpha_{n+1}(z)\right\}^{2}=\ell_{n+1}^{2} y^{2} \leqslant\left(y+\beta_{n}\right)\left(y+\beta_{n+1}\right)\left(1-g_{n}\right) g_{n+1} \tag{4.1.12b}
\end{equation*}
$$

In (4.1.12a) equality holds only when $g_{n+1}=0$ and in (4.1.12b) equality holds only when $\beta_{n}=\beta_{n+1}=0$. An important point to note here is that when equality holds in (4.1.12) for $n=1,2, \ldots, m$, then we have

$$
\begin{equation*}
g_{m+1}<1 \tag{4.1,13}
\end{equation*}
$$

Using (4.1.12) we have

$$
\begin{align*}
& \sum_{r=1}^{m}\left(y+\beta_{r}\right) \xi_{r}^{2}-2 \sum_{r=1}^{m-1} \alpha_{r+1}(z) \xi_{r} \xi_{r+1} \\
&= \sum_{r=1}^{m}\left\{\left(1-g_{r}\right)^{\frac{3}{2}}\left(y+\beta_{r}\right)^{\frac{1}{2}} \xi_{r}\right.
\end{aligned} \begin{aligned}
& \left.-g_{r+1}^{\frac{1}{2}}\left(y+\beta_{r+1}\right)^{\frac{1}{2}} \xi_{r+1}\right\}^{2} \\
& +\left(1-g_{m}\right)\left(y+\beta_{m}\right) \xi_{m}^{2} \tag{4.1.14a}
\end{align*}
$$

whenever equality holds in (4.1.12) for $n=1,2, \ldots, m$, and otherwise

$$
\begin{align*}
& \sum_{r=1}^{m}\left(y+\beta_{r}\right) \xi_{r}^{2}-2 \sum_{r=1}^{m-1} \alpha_{r+1}(z) \xi_{r} \xi_{r+1} \\
& \quad>\sum_{r=1}^{m}\left\{\left(1-g_{r}\right)^{\frac{1}{2}}\left(y+\beta_{r}\right)^{\frac{1}{2}}\left|\xi_{r}\right|-g_{r+1}^{\frac{1}{2}}\left(y+\beta_{r+1}\right)^{\frac{1}{2}}\left|\xi_{r+1}\right|\right\}^{2} \\
& \tag{4.1.14b}
\end{align*}
$$

Thus using (4.1.7) and (4.1.13) we see that the left hand side of (4.1.14) always takes a positive value, under (4.1.7) and (4.1.8). This concludes the proof of the sufficiency of (4.1.7) and (4.1.8) for the positive definite relation (4.1.2) to be true and hence the theorem is proved.

If the $\hat{J}$-fraction (4.1.1) is real then it follows that $\beta_{n}$ is zero for all $n$ and, for any $n, a_{n+1}(z)$ is either $\ell_{n+1}$ or $\ell_{n+1} z$. Thus from theorem (4.1.2), we have the result

Corollary 4.1.2a: If the $\hat{J}$-fraction (4.1.1) is real then it is positive definite if and only if, there exist numbers $g_{1}, g_{2}, g_{3}, \ldots$ satisfying $\mathrm{g}_{1}=0,0 \leqslant \mathrm{~g}_{\mathrm{n}}<1(\mathrm{n} \geqslant 2)$ and such that, for $\mathrm{n} \geqslant 1$

$$
g_{n+1}=0
$$

if $a_{n+1}(z)=\ell_{n+1}$ and

$$
g_{n+1}=\ell_{n+1}^{2} /\left(1-g_{n}\right)
$$

if $a_{n+1}(z)=\ell_{n+1} z$.

Let us now consider a special case of a real $\hat{J}$-fraction given by

$$
\begin{equation*}
\frac{1}{z+b_{1}}-\frac{\ell_{2}^{2}}{z+b_{2}}-\frac{\ell_{3}^{2} z^{2}}{z+b_{3}}-\frac{\ell_{4}^{2}}{z+b_{4}}-\frac{\ell_{5}^{2} z^{2}}{z+b_{5}}-\frac{\ell_{6}^{2}}{z+b_{6}}-\ldots \tag{4.1.15}
\end{equation*}
$$

where all the $b_{n}$ and $\ell_{n}$ are real.
The partial numerators of (4.1.15) alternately take constant and variable values as indicated. If this regular real $\hat{J}$-fraction is positive definite then in theorem 4.1 .2 we have that all the $g_{2 n}$ are equal to zero and all the $g_{2 n+1}$ are less than unity. Hence, we have

Corollary 4.1 .2 b : The regutar reat $\hat{J}$-fraction (4.1.15) is positive definite if and only if

$$
e_{2 n+1}^{2}<1 \text { for } n=1,2,3, \ldots
$$

It can be noted that the $\hat{J}$-fraction (3.2.8) with all the $N_{r}$ positive and all the $D_{r}$ real is equivalent to a positive definite $\hat{J}$-fraction of the form (4.1.15).

### 4.2 CONVERGENCE CIRCLE

The $\hat{J}$-fraction (4.1.1) can be generated by the following linear fractional transformations ( $\ell . f . t s$ ).

$$
\begin{align*}
& t_{0}(z, w)=1 / w, \quad t_{p}(z, w)=z+b_{p}-\left\{a_{p+1}(z)\right\}^{2} / w, \quad p \geqslant 1  \tag{4.2.1}\\
& T_{0}(z, w)=t_{0}(z, w), \quad T_{p}(z, w)=T_{p-1}\left(z, t_{p}(z, w)\right), \quad p \geqslant 1 \tag{4.2.2}
\end{align*}
$$

Thus,

$$
\begin{equation*}
T_{p}(z, w)=\frac{1}{z+b_{1}}-\frac{\left\{a_{2}(z)\right\}^{2}}{z+b_{2}}-\ldots-\frac{\left\{a_{p}(z)\right\}^{2}}{z+b_{p}}-\frac{\left\{a_{p+1}(z)\right\}^{2}}{w} \tag{4.2.3}
\end{equation*}
$$

From (4.2.3) it follows that the $n$-th convergent of the $\hat{J}$ fraction (4.1.1), is

$$
\begin{equation*}
\frac{P_{n}(z)}{Q_{n}(z)}=T_{n}(z, \infty) \tag{4.2.4}
\end{equation*}
$$

Now, by assuming that the $\hat{J}$-fraction (4.1.1) is positive definite, let us consider for the l.f.ts $t_{p}(z, w)$ the range of values of w given by

$$
\begin{equation*}
W_{p+1}(z) \equiv\left\{w: \operatorname{Im}(w) \geqslant\left(y+\beta_{p+1}\right) g_{p+1}, y=\operatorname{Im}(z)\right\}, \quad p \geqslant 1 \tag{4.2.5}
\end{equation*}
$$

where the numbers $g_{p}$ are defined according to theorem 4.1.2.
We see that for any $\varepsilon>0$, the values of $w$ lying in the halfplane region $W_{p+1}(z)$ also satisfy, for $y>0$

$$
\operatorname{Im}(w) \geqslant \frac{\left(y+\beta_{p}\right)\left(y+\beta_{p+1}\right)\left(1-g_{p}\right) g_{p+i}}{\left(y+\beta_{p}\right)\left(1-g_{p}\right)+\varepsilon}, \quad p \geqslant 1
$$

Hence using (4.1.12) in this relation then gives, for $y>0$,

$$
\begin{equation*}
\operatorname{Im}(w) \geqslant \frac{\left\{\alpha_{p+1}(z)\right\}^{2}}{\left(y+\beta_{p}\right)\left(1-g_{p}\right)+\varepsilon}, \quad \varepsilon>0, \quad p \geqslant 1 \tag{4.2.6}
\end{equation*}
$$

We now make an interesting observation that all the values of w satisfying (4.2.6) also satisfy the following relation.

$$
\begin{equation*}
\left|w+\frac{i\left\{a_{p+1}(z)\right\}^{2}}{2\left\{\left(y+\beta_{p}\right)\left(1-g_{p}\right)+\varepsilon\right\}}\right| \geqslant\left|\frac{\left|\left\{a_{p+1}(z)\right\}^{2}\right|}{2\left\{\left(y+\beta_{p}\right)\left(1-g_{p}\right)+\varepsilon\right\}}\right|, \quad \varepsilon \geqslant 0, p \geqslant 1 \tag{4.2.7}
\end{equation*}
$$

This can be seen pictorially in the following diagram, in which the relation (4.2.6) describes the region on or above the dotted line, and the relation (4.2.7) describes the region on or outside the circle.


It immediately follows from relation (4.2.7) that

$$
\left|2\left\{\left(y+\beta_{p}\right)\left(1-g_{p}\right)+\varepsilon\right\}-\frac{i\left\{a_{p+1}(z)\right\}^{2}}{w}\right| \geqslant\left|\frac{\left\{a_{p+1}(z)\right\}^{2}}{w}\right|
$$

Thus, squaring both sides and then expressing each side in terms of their real and imaginary parts, we find that

$$
\left(y+\beta_{p}\right)\left(1-g_{p}\right)+\varepsilon-\operatorname{Im}\left[\frac{\left\{a_{p+1}(z)\right\}^{2}}{w}\right] \geqslant 0
$$

Here, however small the value of $\varepsilon$, this condition is true whenever $\operatorname{Im}(z)=y>0$ and $w$ lies in $W_{p+1}(z)$. Hence, for all such values of $w$ the above condition must also hold for $\varepsilon=0$. Thus, we obtain

$$
\left(y+\beta_{p}\right)-\operatorname{Im}\left[\frac{\left\{a_{p+1}(z)\right\}^{2}}{w}\right] \geqslant\left(y+\beta_{p}\right) g_{p}
$$

Now, since the left hand side of this inequality can be identified as the imaginary part of $t_{p}(z, w)$, we arrive at the following result.

For any $z$ for which $\operatorname{Im}(z)=y>0$,

$$
\operatorname{Im}\left\{t_{p}(z, w)\right\} \geqslant\left(y+\beta_{p}\right) g_{p}, \quad p \geqslant 1
$$

whenever

$$
\operatorname{Im}(w) \geqslant\left(y+\beta_{p+1}\right) g_{p+1}, \quad p \geqslant 1
$$

By a similar argument we can also show (see also Wall [1948]) that for any $z$ for which $\operatorname{Im}(z)=y>0$

$$
\operatorname{Im}\left\{t_{p}(z, w)\right\} \geqslant y+\beta_{p} g_{p}, \quad p \geqslant 1
$$

whenever

$$
\operatorname{Im}(w) \geqslant\left(y+\beta_{p+1}\right) g_{p+1} \text { and } a_{p+1}(z)=\ell_{p+1}+i m_{p+1}, \quad p \geqslant 1 .
$$

Summarising these results we have

Theorem 4.2.1 : If the $\hat{J}$-fraction (4.1.1) is positive definite then for any $z$ lying above the real axis the corresponding l.f.ts $t_{p}(z, w), p \geqslant 1$, defined by (4.2.1) satisfy:

$$
t_{p}(z, w) \in w_{p}(z),
$$

whenever $w$ lies in $W_{p+1}(z)$. In particular, for any $p \geqslant 1$, we also have

$$
\operatorname{Im}\left\{t_{p}(z, w)\right\} \geqslant \dot{y}+\beta_{p} g_{p}
$$

whenever $w$ lies in $W_{p^{+1}}(z)$ and $a_{p+1}(z)$ is a constant. Here, the real numbers $g_{p}, p \geqslant 1$ are defined according to theorem 4.1.2.

We can now use this result to study the image of the half-plane region $W_{p+1}(z)$, under the linear fractional transformation $T_{p}(z, w)$. Let us first denote this image by $K_{p}(z)$. We can easily see from the equation (4.2.2) and the theorem 4.2.1 that the region $K_{p+1}(z)$ is contained in $K_{p}(z)$, for any $z$ having a positive imaginary part.

These regions $K_{p}(z), p=1,2,3, \ldots$, may be either circular regions or half-plane regions. But, if $K_{q}(z)$ is a circular region for any $q \geqslant 1$, then for all $r \geqslant 1$ the regions given by $K_{q+r}(z)$ are also circular regions.

Let us now look at the region $K_{1}(z)$, which is the image of the half-plane region $W_{2}(z)$, under the l.f.t. $T_{1}(z, w)$. That is

$$
K_{1}(z)=T_{1}\left(z, W_{2}(z)\right) .
$$

Since $T_{1}\left(z, W_{2}(z)\right)=T_{0}\left(z, t_{1}\left(z, W_{2}(z)\right)\right)$ then from theorem 4.2.1
we note that when $a_{2}(z)=\ell_{2}+i m_{2}$

$$
K_{1}(z)=T_{1}\left(z, W_{2}(z)\right) \subseteq T_{0}\left(z, W_{1}^{\prime}(z)\right), \text { for } y>0,
$$

where $W_{1}^{\prime}(z)$ is the half-plane region given by

$$
\left\{w: \operatorname{Im}(w) \geqslant y+\beta_{1} g_{1}\right\}
$$

But $g_{1}=0$, and hence

$$
W_{1}^{\prime}(y) \equiv\{w: \operatorname{Im}(w) \geqslant y\} .
$$

Thus from $T_{0}(z, w)=1 / w$, we obtain that the region $T_{0}\left(z, W_{1}^{\prime}(z)\right)$ is the circular region given by

$$
\begin{equation*}
\left|w+\frac{i}{2 y}\right| \leqslant \frac{i}{2 y}, \tag{4.2.8}
\end{equation*}
$$

for any $z$ for which $\operatorname{Im}(z)=y>0$. Hence, we have the following:

Theorem 4.2.2: If the $\hat{J}$-fraction (4.1.1) is positive definite then for the corresponding l.f.t. $\mathrm{T}_{\mathrm{n}}(\mathrm{z}, \mathrm{w})$, given by (4.2.2), the image $K_{n}(z)$ of the half-plane region $W_{n+1}(z)$ satisfies

$$
K_{n}(z) \supseteq K_{n+1}(z), \quad n \geqslant 1,
$$

for any $z$ with a positive imaginary part. In pariticular, if $a_{2}(z)$ is a constant, then all these regions $K_{n}(z), n \geqslant 1$, are circular regions and satisfy

$$
T_{0}\left(z, W_{1}^{\prime}(y)\right) \supseteq K_{1}(z) \supseteq K_{2}(z) \supseteq K_{3}(z) \supseteq \ldots
$$

for ony such $z$.

The point $\infty$ lies on the boundary of the half-plane region $W_{n+1}(z)$. Therefore, the $n$-th convergent $T_{n}(z, \infty)$ of the positive definite $\hat{J}$-fraction of the form (4.1.1) must lie on the boundary of $K_{n}(z)$. In particular if $a_{2}(z)=\ell_{2}+i m_{2}$ then $K_{n}(z)$ lies inside the circle $T_{0}\left(z, W_{1}^{\prime}(z)\right)$. Consequently, we have, for $T_{n}(z, \infty)=P_{n}(z) / Q_{n}(z)$,

$$
\begin{align*}
& \left|P_{n}(z) / Q_{n}(z)\right| \leqslant 1 / y, \quad n \geqslant 1,  \tag{4.2.9a}\\
& \operatorname{Im}\left\{P_{n}(z) / Q_{n}(z)\right\} \leqslant 0, \quad n \geqslant 1, \tag{4.2.9b}
\end{align*}
$$

provided that $\operatorname{Im}(z)=y>0$.

### 4.3 LIMIT POINT CASE

If the sequence of circular regions $\left\{\mathrm{K}_{\mathrm{n}}(\mathrm{z})\right.$ \} in theorem 4.2.2 converges to a limit point for any $z$ for which $y>0$ then it is said that the positive $\hat{\mathcal{J}}$-fraction with $\mathrm{a}_{2}(z)$ a constant satisfies the limit point case for that $z$. Otherwise the sequence of circular regions converges to a limit circle, and hence, the fraction is said to satisfy the limit circle case, for that $z$.

Let us denote the $\hat{J}$-fraction with $a_{2}(z)=\ell_{2}+i m_{2}$ as a $\hat{J}_{*}$ fraction. Hence, if the positive definite $\hat{J}_{*}$-fraction satisfies the limit point case for any $z$, then for this value of $z$ the convergents of this fraction also converge to this limit point. On the other hand if this fraction satisfies the limit circle case for any $z$ then the value of the $n$-th convergent of the fraction will be on the boundary of $K_{n}(z)$, but as $n$ increases these values do not necessarily converge to a single point in the boundary of the limit circle. But if it does so then we have, for this particular value of $z$, convergence in the limit circle case for the positive definite $\hat{J}_{*}$ fraction.

To be certain of whether the limit point case or the limit circle case holds, some understanding of the radius of the circular regions $K_{n}(z)$ is required. We shall denote this radius by $r_{n}(z)$.

Using (4.2.3), the l.f.t. $T_{n}(z, w)$ can be given in terms of $P_{n}(z)$ and $Q_{n}(z)$ as follows

$$
\begin{equation*}
T_{n}(z, w)=\frac{w P_{n}(z)-\left\{a_{n+1}(z)\right\}^{2} P_{n-1}(z)}{w Q_{n}(z)-\left\{a_{n+1}(z)\right\}^{2} Q_{n-1}(z)}, \quad n \geqslant 1 . \tag{4.3.1}
\end{equation*}
$$

Let us now for convenience define two new sequences of functions $\left\{X_{n}(z)\right\}$ and $\left\{Y_{n}(z)\right\}$ by

$$
\begin{aligned}
& X_{0}(z)=-1, \quad Y_{0}(z)=0, X_{1}(z)=0, Y_{1}(z)=1, \\
& X_{n+1}(z)=\frac{P_{n}(z)}{a_{2}(z) a_{3}(z) \ldots a_{n+1}(z)}, \quad n \geqslant 1, \\
& Y_{n+1}(z)=\frac{Q_{n}(z)}{a_{2}(z) a_{3}(z) \cdots a_{n+1}(z)}, \quad n \geqslant 1 .
\end{aligned}
$$

Then it follows that

$$
T_{n}(z, w)=\frac{w X_{n+1}(z)-a_{n+1}(z) X_{n}(z)}{w Y_{n+1}(z)-a_{n+1}(z) Y_{n}(z)}, \quad n \geqslant 1
$$

Using (4.1.3), the functions $X_{n}(z)$ and $Y_{n}(z)$ can easily be shown to satisfy the three term relations

$$
-a_{n}(z) x_{n-1}(z)+\left(z+b_{n}\right) x_{n}(z)-a_{n+1}(z) x_{n+1}(z)=0,
$$

$$
n \geqslant 1 . \quad(4.3 .4)
$$

$-a_{n}(z) Y_{n-1}(z)+\left(z+b_{n}\right) Y_{n}(z)-a_{n+1}(z) Y_{n+1}(z)=0$,

Here, $a_{1}(z)$ is taken to be equal to unity.

Furthermore these functions can also be shown to satisfy the determinant formulas

$$
\begin{equation*}
X_{n+1}(z) Y_{n}(z)-X_{n}(z) Y_{n+1}(z)=1 / a_{n+1}(z) \tag{4.3.5}
\end{equation*}
$$

and

$$
\begin{equation*}
X_{n+2}(z) Y_{n}(z)-X_{n}(z) Y_{n+2}(z)=\left(z+b_{n+1}\right) / a_{n+2}(z) a_{n+1}(z) \tag{4.3.6}
\end{equation*}
$$

for $n \geqslant 1$.
Since $K_{n}(z)$ is the image of the half-plane region $W_{n+1}(z)$ under the l.f.t. $T_{n}(z, w)$, it follows from (4.3.3) that the point which has the centre of $K_{n}(z)$ as its image point is (see for example Wall [1948])

$$
w_{0}^{(n)}(z)=2 i\left(y+\beta_{n+1}\right) g_{n+1}+\bar{a}_{n+1}(z) \bar{Y}_{n}(z) / \bar{Y}_{n+1}(z)
$$

Thus from the fact $T_{n}(z, \infty)$ is on the boundary of $K_{n}(z)$, we find that the radius $\mathrm{r}_{\mathrm{n}}(\mathrm{z})$ can be given as

$$
r_{n}(z)=\left|T_{n}\left(z, w_{0}^{(n)}(z)\right)-T_{n}(z, \infty)\right| .
$$

Consequently from the relation (4.3.3) we obtain

$$
r_{n}(z)=\frac{1}{\left|Y_{n+1}(z)\left\{w_{0}^{(n)}(z) Y_{n+1}(z)-a_{n+1}(z) Y_{n}(z)\right\}\right|}
$$

When substituting the value of $w_{0}^{(n)}(z)$ we find, after some simple manipulation,

In order to change (4.3.7) into a more convenient form we take from (4.3.4) the equation

$$
-a_{r}(z) Y_{r-1}(z)+\left(z+b_{r}\right) Y_{r}(z)-a_{r+1}(z) Y_{r+1}(z)=0
$$

Multiplying this equation by $\bar{Y}_{\mathrm{r}}(z)$ and summing over $r=1,2, \ldots, n$, we arrive at

$$
\begin{aligned}
& -a_{n+1}(z) Y_{n}(z) \bar{Y}_{n+1}(z)=\sum_{r=1}^{n}\left(z+b_{r}\right)\left|Y_{r}(z)\right|^{2} \\
& \quad-\sum_{r=1}^{n} a_{r+1}(z)\left\{Y_{r+1}(z) \bar{Y}_{r}(z)+Y_{r}(z) \bar{Y}_{r+1}(z)\right\}
\end{aligned}
$$

Hence, using the imaginary part of this equation in (4.3.7), gives

$$
\begin{aligned}
2 r_{n}(z)= & \frac{1}{\left.\left|\sum_{r=1}^{n+1}\left(y+\beta_{r}\right)\right| Y_{r}(z)\right|^{2}-\sum_{r=1}^{n} \alpha_{r+1}(z)\left\{Y_{r+1}(z) \bar{Y}_{r}(z)+Y_{r}(z) \bar{Y}_{r+1}(z)\right\}}
\end{aligned}
$$

Now, applying the result (4.1.14a) in this equation, yields for the positive definite and real $\hat{J}_{*}$-fraction
$2 r_{n}(z)=\frac{1}{\sum_{r=1}^{n+1}\left|\left(y+\beta_{r}\right)^{\frac{1}{2}}\left(1-g_{r}\right)^{\frac{1}{2}} Y_{r}(z)-\left(y+\beta_{r+1}\right)^{\frac{1}{2}} g_{r+1}^{\frac{1}{2}} Y_{r+1}(z)\right|^{2}}$.

Relation (4.3.8) suggests, if the series given by

$$
\begin{equation*}
\sum_{\mathrm{r}=1}^{\infty}\left|\left(\mathrm{y}+\beta_{\mathrm{r}}\right)^{\frac{3}{2}}\left(1-\mathrm{g}_{\mathrm{r}}\right)^{\frac{3}{2}} \mathrm{Y}_{\mathrm{r}}(z)-\left(y+\beta_{\mathrm{r}+1}\right)^{\frac{3}{2}} \mathrm{~g}_{\mathrm{r}+1}^{\frac{1}{2}} \mathrm{Y}_{\mathrm{r}+1}(z)\right|^{2} \tag{4.3.9}
\end{equation*}
$$

is divergent for any $z$, then $r_{n}(z) \rightarrow 0$, giving the limit point case for the positive definite and real $\hat{J}_{*}$-fraction for that $z$.

We also note from corollary 4.1.2b, that for the positive definite and real $\hat{J}$-fraction $\wedge^{(4 \cdot 11}$ all the $g_{2 r}$ are equal to zero and all the $g_{2 r+1}$ are equal to $l_{2 r+1}^{2}$. Hence for this continued fraction the series becomes

$$
y \sum_{r=1}^{\infty}\left|Y_{2 r}(z)-\ell_{2 r+1} Y_{2 r+1}(z)\right|^{2}+y \sum_{r=1}^{\infty}\left(1-\ell_{2 r+1}^{2}\right)\left|Y_{2 r+1}(z)\right|^{2} .
$$

As a consequence, we have the following:

Theorem 4.3.1 : For the positive definite and real $\hat{J}_{*}$-fraction of the form (4.1.15) the limit point case holds for any $z$ such that $\operatorname{Im}(z)>0$ if, and only $i f$, one or both of the following series

$$
\begin{equation*}
\sum_{\mathrm{r}=1}^{\infty}\left(1-\ell_{2 \mathrm{r}+1}^{2}\right)\left|Y_{2 \mathrm{r}+1}(z)\right|^{2} \tag{4.3.10}
\end{equation*}
$$

$$
\begin{equation*}
\sum_{r=1}^{\infty}\left|Y_{2 r}(z)-\ell_{2 r+1} Y_{2 r+1}(z)\right|^{2} \tag{4.3.11}
\end{equation*}
$$

diverges.

From the determinant equation (4.3.6), we obtain for the real $\hat{J}_{*}$-fraction (4.1.15)

$$
\frac{z+b_{2 r}}{l_{2 r+1} l_{2 r} z}=X_{2 r+1}(z) Y_{2 r-1}(z)-X_{2 r-1}(z) Y_{2 r+1}(z), \quad r \geqslant 1
$$

Hence, multiplying this by $\left(1-l_{2 r-1}^{2}\right)^{\frac{\pi}{2}}\left(1-l_{2 r+1}^{2}\right)^{\frac{1}{2}}$ and summing over $r=2,3, \ldots, n$, we find

$$
\begin{aligned}
& \sum_{r=2}^{n}\left(1-l_{2 r-1}^{2}\right)^{\frac{3}{2}}\left(1-l_{2 r+1}^{2}\right)^{\frac{3}{2}}\left\{\frac{z+b_{2 r}}{l_{2 r+1} l_{2 r}^{2}}\right\} \\
& \quad=\sum_{r=2}^{n}\left(1-l_{2 r-1}^{2}\right)^{\frac{3}{2}}\left(1-l_{2 r+1}^{2}\right)^{\frac{3}{2}}\left\{x_{2 r+1}(z) Y_{2 r-1}(z)-x_{2 r-1}(z) Y_{2 r+1}(z)\right\}
\end{aligned}
$$

NOTE : Since the fraction is positive definite $\ell_{2 \mathrm{r}+1}^{2}$ must be less than unity for all $r \geqslant 1$.

Thus, if we apply to the right hand side of the above equation the Schwarz inequality

$$
\left|\sum_{p=1}^{n} U_{p} V_{p}\right|^{2} \leqslant \sum_{p=1}^{n}\left|U_{p}\right|^{2} \cdot \sum_{p=1}^{n}\left|V_{p}\right|^{2}, \quad n \geqslant 1,
$$

where $U_{p}, V_{p}, p \geqslant 1$ are any complex numbers, we then obtain

$$
\begin{align*}
& \left|\sum_{r=2}^{n}\left(1-\ell_{2 r-1}\right)^{\frac{1}{2}}\left(1+\ell_{2 r+1}\right)^{\frac{1}{2}}\left\{\frac{z+b_{2 r}}{\ell_{2 r+1} \ell_{2 r}^{z}}\right\}\right| \\
& \quad \leqslant\left\{2 \cdot \sum_{r=1}^{n}\left(1-\ell_{2 r-1}^{2}\right)\left|X_{2 r+1}(z)\right|^{2} \cdot \sum_{r=1}^{n}\left(1-\ell_{2 r+1}^{2}\right)\left|Y_{2 r+1}(z)\right|^{2}\right\} . \tag{4.3.13}
\end{align*}
$$

Suppose now that the left hand side of (4.3.13) diverges as $n \rightarrow \infty$. Then from the right hand side it follows that one of the series

$$
\left.\begin{array}{l}
\sum_{r=1}^{\infty}\left(1-l_{2 r+1}^{2}\right)\left|X_{2 r+1}(z)\right|^{2}  \tag{4.3.14}\\
\sum_{r=1}^{\infty}\left(1-l_{2 r+1}^{2}\right)\left|Y_{2 r+1}(z)\right|^{2}
\end{array}\right\}
$$

must also be divergent. But from (4.2.9) and (4.3.2), it follows that if (4.1.15) is positive definite then

$$
\frac{\left|X_{2 n+1}(z)\right|}{\left|Y_{2 n+1}(z)\right|} \leqslant 1 / y,
$$

for all $y=\operatorname{Im}(z)>0$ and for all $n \geqslant 1$.
Hence we see that if the first of the series of (4.3.14) diverges then so does the other. Consequently we arrive at the following:

Theorem 4.3.2: A sufficient condition for the positive definite and real $\hat{J}$-fraction of the form (4.1.15) to satisfy the limit point case for all $z$ for which $\operatorname{Im}(z)>0$, is that one or both of the following hold.

$$
\begin{align*}
& \sum_{r=2}^{\infty}\left(1-\ell_{2 r-1}^{2}\right)^{\frac{3}{2}}\left(1-\ell_{2 r+1}^{2}\right)^{\frac{3}{2}} \frac{1}{\ell_{2 r+1} \ell_{2 r}}=\infty,  \tag{4.3.15}\\
& \sum_{r=2}^{\infty}\left(1-\ell_{2 r-1}^{2}\right)^{\frac{3}{2}}\left(1-\ell_{2 r+1}^{2}\right)^{\frac{3}{2}} \frac{b_{2 r}}{\ell_{2 r+1} l_{2 r}}=\infty . \tag{4.3.16}
\end{align*}
$$

Let us now consider the series (4.3.11), and if we use the three term relation (4.3.4) on this we find that this series is equivalent to

$$
\sum_{r=1}^{\infty}\left|\ell_{2 r} Y_{2 r-1}(z)-b_{2 r} Y_{2 r}(z)\right|^{2}
$$

Thus, suppose $b_{2 r}=0$ for all $r \geqslant 1$. Then a sufficient condition for the positive definite $\hat{J}_{*}$-fraction (4.1.15) to satisfy the limit point case is

$$
\sum_{r=1}^{\infty} \ell_{2 r}^{2}\left|Y_{2 r-1}(z)\right|^{2}=\infty
$$

Hence, as before using the formula (4.3.12) and the Schwarz inequality, we arrive at the following result.

Theorem 4.3.3: A sufficient condition for the positive definite and real $\hat{\mathrm{J}}$-fraction of the form (4.1.15), with all the $\mathrm{b}_{2 \mathrm{r}}=0$, to satisfy the limit point case for all $z$ for which $\operatorname{Im}(z)>0$, is that

$$
\begin{equation*}
\sum_{r=1}^{\infty}\left|\left\{\ell_{2 r+2} / \ell_{2 r+1}\right\}\right|=\infty . \tag{4.3.17}
\end{equation*}
$$

### 4.4 LIMIT CIRCLE CASE

Restricting $z$ to values which satisfy the conditions $z=i y$ and $y>0$, the positive definite $\hat{J}$-fraction of the form (4.1.1) can be given, after a suitable equivalence transformation, as

$$
\begin{equation*}
\frac{\lambda_{1}}{i+\mu_{1}}+\frac{\lambda_{2}}{i+\mu_{2}}+\frac{\lambda_{3}}{i+\mu_{3}}+\frac{\lambda_{4}}{i+\mu_{4}}+\ldots, \tag{4.4.1}
\end{equation*}
$$

where

$$
\begin{aligned}
& \mu_{n}=b_{n} / y, \quad n \geqslant 1, \quad \lambda_{1}=1 / y \\
& \lambda_{n+1}=-\left(\ell_{n+1}+i m_{n+1}\right)^{2} / y^{2}
\end{aligned}
$$

if $a_{n+1}(z)=\ell_{n+1}+i m_{n+1}$ and while if $a_{n+1}(z)=\ell_{n+1} z$

$$
\lambda_{n+1}=\ell_{n+1}^{2}, \text { for all } n \geqslant 1
$$

The continued fraction (4.4.1) can be generated by the l.f.ts

$$
\begin{equation*}
s_{n}(y, w)=\lambda_{n} /\left(i+\mu_{n}+w\right), \quad n \geqslant 1 \tag{4.4.2}
\end{equation*}
$$

and

$$
s_{1}(y, w)=s(y, w), \quad s_{n}(y, w)=s_{n-1}\left(y, s_{n}(y, w)\right), \quad n \geqslant 1
$$

Hence, from (4.4.3) we note that the $n$-th convergent of (4.4.1) is given by $S_{n}(y, 0)$.

It is quite easy to verify that the l.f.t. $T_{n}(z, w)$ of Section 4.2 and the l.f.t. $S_{n}(y, w)$ are related by

$$
\begin{equation*}
S_{n}(y, w)=T_{n}\left(i y,-\left\{a_{n+1}(i y)\right\}^{2} /(w y)\right), \quad n \geqslant 1 \tag{4.4.4a}
\end{equation*}
$$

$$
\begin{equation*}
S_{n}\left(y,-\left\{a_{n+1}(i y)\right\}^{2} /(w y)\right)=T_{n}(i y, w), \quad n \geqslant 1 \tag{4.4.4b}
\end{equation*}
$$

It has been shown in Section 4.2 that for the positive definite $\hat{J}_{*}$-fraction the l.f.ts $T_{n}(z, w)$ satisif

$$
\begin{gathered}
K_{n}(z)=T_{n}\left(z, W_{n+1}(z)\right), \quad n \geqslant 1, \\
T_{0}\left(z, W_{1}^{\prime}(z)\right) \supseteq T_{1}\left(z, W_{2}(z)\right) \supseteq T_{2}\left(z, W_{3}(z)\right) \supseteq T_{3}\left(z, W_{4}(z)\right) \supseteq \ldots,
\end{gathered}
$$

where $T_{0}\left(z, W_{1}^{\prime}(z)\right)$ is the circular region given by (4.2.8) and $W_{n+1}(z)$, $n \geqslant 1$, are the half-plane regions given by (4.2.5). Hence, for the regions $V_{n}(y), n \geqslant 1$ defined by

$$
\begin{equation*}
V_{n}(y)=-\left\{a_{n+1}(i y)\right\}^{2} /\left(y W_{n+1}(i y)\right), \quad n \geqslant 1 \tag{4.4.5}
\end{equation*}
$$

it follows from (4.4.4) that

$$
\begin{equation*}
T_{0}\left(i y, W_{1}^{\prime}(i y)\right) \supseteq S_{1}\left(y, V_{1}(y)\right) \supseteq S_{2}\left(y, V_{2}(y)\right) \supseteq S_{3}\left(y, V_{3}(y)\right) \supseteq \ldots . \tag{4.4.6}
\end{equation*}
$$

This implies that for any $y>0$, if the positive definite $\hat{J}_{*}$ fraction satisfies the limit circle case then the sequence of circular regions $S_{n}\left(y, V_{n}(y)\right)$ must converge to a limit circle.

Now, considering the positive definite and real $\hat{\mathrm{J}}_{*}$-fraction of the form (4.1.15), it follows, since $a_{2 \pi}(z)=\ell_{2 n}, g_{2 \pi}=0$, $\mathrm{a}_{2 \mathrm{n}+1}(\mathrm{z})=\ell_{2 \mathrm{n}+1} \mathrm{z}$ and $\mathrm{g}_{2 \mathrm{n}+1}=\ell_{2 \mathrm{n}+1}^{2}<1$, that

$$
\begin{align*}
& \mathrm{V}_{2 \mathrm{n}-1}(y) \equiv \mathrm{V}_{1}^{*} \equiv\{\mathrm{w}: \operatorname{Im}(\mathrm{w}) \geqslant 0\}, \quad \text { (a half-plane) } \\
& \mathrm{V}_{2 \mathrm{n}}(y) \equiv \mathrm{V}_{2}^{*} \equiv\left\{\mathrm{w}:\left|\mathrm{w}+\frac{1}{2}\right| \leqslant \frac{1}{2}\right\}, \quad \text { (a circle) } \tag{4.4.7}
\end{align*}
$$

for $n \geqslant 1$.
Such regions $V_{1}^{*}$ and $V_{2}^{*}$, which are called "twin regions", have been studied extensively by Thron [1944, 1949] and by Jones and Thron [1970].

For a given value of $y>0$, let us now define for convenience a new sequence of $\ell . f . t s\left\{H_{n}(w)\right\}$ by

$$
\begin{align*}
H_{2 n-1}(w) & =S_{2 n-1}\left(y, v_{1}^{-1}(w)\right), \quad n \geqslant 1  \tag{4.4.8}\\
H_{2 n}(w) & =S_{2 n}\left(y, v_{2}^{-1}(w)\right), \quad n \geqslant 1
\end{align*}
$$

where $v_{1}^{-1}(w)=(w-i) /(i w-1)$ and $v_{2}^{-1}(w)=(w-i) / 2$ are the $2 . f . t s$ which map the unit circle $U$ onto, respectively, $V_{1}^{*}$ and $V_{2}^{*}$.

Hence from (4.4.6) we have

$$
\begin{equation*}
\mathrm{T}_{0}\left(\mathrm{iy}, \mathrm{~W}_{1}^{\prime}(\mathrm{i} y)\right) \supseteq \mathrm{H}_{1}(\mathrm{U}) \supseteq \mathrm{H}_{2}(\mathrm{U}) \supseteq \mathrm{H}_{3}(\mathrm{U}) \supseteq \ldots \tag{4.4.9}
\end{equation*}
$$

Thron [1963] has shown that any sequence of l.f.ts $\left\{H_{n}(w)\right\}$ satisfying (4.4.9) can be given as

$$
H_{n}(w)=C_{n}+R_{n}\left\{\left(w+\bar{F}_{n}\right) /\left(w F_{n}+1\right)\right\}, \quad n \geqslant 1,
$$

in which

$$
\left|F_{n}\right|=f_{n}<1 \text { and }\left|C_{n}-C_{m}\right| \leqslant r_{m}-r_{n} \text { for all } m \leqslant n
$$

where $C_{n}$ is the centre of $H_{n}(U)$ and $r_{n}=\left|R_{n}\right|$ is the radius of $H_{n}(U)$. Thus, for the limit circle case to occur for the given $z=i y$ the radius $r_{n}$ of $H_{n}(U)$ must satisfy

$$
r_{n} \not r>0 \text { as } n \rightarrow \infty
$$

For the positive definite and real $\hat{J}_{*}$-fraction of the form (4.1.15) to converge under the limit circle case for the given $z=i y$, it is required that its convergents $\mathrm{T}_{\mathrm{n}}(\mathrm{iy}, \infty)$ converge to a single point in the limit circle. In other words, the sequence $\left\{T_{n}(i y, \infty)=\right.$ $\left.S_{n}(y, 0)=H_{n}(i)\right\}$ is a Cauchy sequence. To find out when this is so, we need to know some properties of this sequence.

We have from (4.4.1) and (4.4.3) that for the real $\hat{\mathrm{J}}_{*}$-fraction (4.1.15)

$$
\begin{aligned}
& S_{2 n+1}\left(y,-i-b_{2 n+1} / y\right)=s_{2 n}(y, \infty), \quad n \geqslant 1, \\
& s_{2 n}(y, \infty)=s_{2 n-1}(y, 0), \quad n \geqslant 1, \\
& s_{2 n-1}(y, 0)=s_{2 n-2}\left(y, l_{2 n-1}^{2} /\left(i+b_{2 n-1} / y\right)\right), \quad n \geqslant 2,
\end{aligned}
$$

and

$$
\begin{aligned}
& S_{2 n}\left(y,-i-b_{2 n} / y\right)=S_{2 n-1}(y, \infty), \quad n \geqslant 1, \\
& S_{2 n}(y, \infty)=S_{2 n-2}(y, 0), \quad n \geqslant 2, \\
& S_{2 n-2}(y, 0)=S_{2 n-3}\left(y,\left(-l_{2 n-2}^{2} / y^{2}\right) /\left(i+b_{2 n-2} / y\right)\right), \quad n \geqslant 2 .
\end{aligned}
$$

Thus, for the particular value of $z=i y$, using (4.4.8) we immediately obtain the following relations.

$$
\begin{align*}
& H_{2 n+1}\left\{\frac{2-i b_{2 n+1} / y}{b_{2 n+1} / y}\right\}=H_{2 n}(\infty), \quad n \geqslant 1,  \tag{4.4.11a}\\
& H_{2 n}(\infty)=H_{2 n-1}(i), \quad n \geqslant 1,  \tag{4.4.1.1b}\\
& H_{2 n-1}(i)=H_{2 n-2}\left\{\frac{\left(2 l_{2 n-1}^{2}-1\right)+i b_{2 n-1} / y}{i+b_{2 n-1} / y}\right\}, \quad n \geqslant 2,(4.4 .11 c) \\
& H_{2 n}\left(-i-2 b_{2 n} / y\right)=H_{2 n-1}(-i), \quad n \geqslant 1, \\
& H_{2 n-1}(-i)=H_{2 n-2}(i), \quad n \geqslant 2,  \tag{4.4.12b}\\
& H_{2 n-2}(i)=H_{2 n-3}\left\{\frac{\left(1-l_{2 n-2}^{2}\right) i+b_{2 n-2} / y}{\left(1+l_{2 n-2}^{2}\right)-i b_{2 n-2} / y}\right\}, \tag{4.4.12c}
\end{align*}
$$

Hence, from (4.4.10) and (4.4.11b) we find the relation

$$
C_{2 n}+R_{2 n} \cdot 1 / F_{2 n}=C_{2 n-1}+R_{2 n-1} \cdot \sigma_{2 n-1}, n \geqslant 1,
$$

where

$$
\sigma_{2 n-1}=\frac{i+\bar{F}_{2 n-1}}{i F_{2 n-1}+1}
$$

Rearranging the terms and then taking the modules of both sides gives

$$
\frac{r_{2 n}}{f_{2 n}} \leqslant\left|c_{2 n-1}-c_{2 n}\right|+r_{2 n-1}\left|\sigma_{2 n-1}\right|, \quad n \geqslant 1
$$

Thus, using the fact $\left|\sigma_{2 n-1}\right|=1$ and also the fact that

$$
\left|C_{2 n-1}-C_{2 n}\right| \leqslant r_{2 n-1}-r_{2 n},
$$

we find

$$
\frac{r_{2 n}}{r_{2 n-1}}\left\{1-\frac{1-f_{2 n}}{1+f_{2 n}}\right\}, \quad n \geqslant 1
$$

Since $f_{2 n}<1$ for all $n \geqslant 1$, the right hand side of this inequality is bounded by zero and one. Furthermore, we have from (4.4.9) and (4.4.10) that $r_{2 n-1} / r_{2 n-2} \leqslant 1$ for all $n$. Hence, by taking the product of these quotients, we obtain

$$
\begin{equation*}
r_{2 n} \leqslant A \prod_{k=1}^{n}\left\{1-\frac{1-f_{2 k}}{1+f_{2 k}}\right\}, \quad n \geqslant 1 \tag{4.4.13}
\end{equation*}
$$

where $A$ is a constant independent of $n$.
Since, we must have $r_{2 n} \nmid r>0$ for the limit circle case, it follows from (4.4.13) that

$$
\sum_{k=1}^{\infty}\left\{\frac{1-f_{2 k}}{1+f_{2 k}}\right\}<\infty
$$

From above it also follows that

$$
\sum_{k=1}^{\infty}\left(1-f_{2 k}\right)<\infty .
$$

Now, let us consider the relations (4.4.11a) and (4.4.11b). From these we have

$$
H_{2 n+1}\left\{\frac{2-i b_{2 n+1} / y}{b_{2 n+1} / y}\right\}=H_{2 n-1}(i), \quad n \geqslant 1 .
$$

If we use (4.4.10) on this we find

$$
C_{2 n+1}+R_{2 n+1} \sigma_{2 n+1}^{\prime}=C_{2 n-1}+R_{2 n-1} \sigma_{2 n-1}, \quad n \geqslant 1,
$$

where

$$
\sigma_{2 n+1}^{\prime}=\frac{w^{\prime}+\bar{F}_{2 n+1}}{w^{\prime} F_{2 n+1}+1}, \quad w^{\prime}=\frac{2-i b_{2 n+1} / y}{b_{2 n+1} / y}
$$

and

$$
\sigma_{2 n+1}=\frac{i+\bar{F}_{2 n-1}}{i F_{2 n+1}+1}
$$

Therefore as before, rearranging the terms and taking the modulus of both sides we find

$$
\frac{r_{2 n+1}}{r_{2 n-1}} \leqslant \frac{2}{1+\left|\sigma_{2 n+1}^{\prime}\right|}, \quad n \geqslant 1
$$

Under the limit circle case the left hand side of this inequality is bounded from below by some number greater than zero, and therefore, $\left|\sigma_{2 n+1}^{\prime}\right|$ must be bounded from above. Further, since $\left|w^{\prime}\right|>1$ we also must have that $\left|\sigma_{2 n+1}^{\prime}\right|>1$. Hence, by rewriting the above inequality as

$$
\frac{r_{2 n+1}}{r_{2 n-1}} \leqslant 1-\frac{\left|\sigma_{2 n+1}^{\prime}\right|-1}{\left|\sigma_{2 n+1}^{\prime}\right|+1}, \quad n \geqslant 1
$$

we arrive at

$$
\sum_{k=1}^{\infty}\left\{1-1 /\left|\sigma_{2 k+1}^{\prime}\right|\right\}<\infty
$$

This result enables us to establish (see Jones and Thron [1970]) that

$$
\sum_{k=1}^{\infty}\left(1-f_{2 k+1}\right)<\infty
$$

Sumnarising these results yields the following:

Theorem 4.4.1 : If the positive definite and real $\hat{J}$-fraction of the form (4.1.15) satisfies the Iimit circle case for any $z=i y, y>0$, then for the corresponding 2.f.t. $\mathrm{H}_{\mathrm{n}}(\mathrm{w})$ given by (4.4.10), the following hold

$$
\begin{align*}
& \sum_{n=1}^{\infty}\left(1-f_{2 n}\right)<\infty  \tag{4.4.14}\\
& \sum_{n=1}^{\infty}\left(1-f_{2 n+1}\right)<\infty \tag{4.4.15}
\end{align*}
$$

Let us now consider the following three cases respectively, coming from (4.4.11a), (4.4.12b) and (4.4.11c).

$$
\begin{aligned}
& K_{n}=-R_{2 n+1}\left\{\frac{1-f_{2 n+1}^{2}}{F_{2 n+1}+1 / n_{2 n+1}}\right\}+R_{2 n}\left\{\frac{1-f_{2 n}^{2}}{F_{2 n}}\right\}, \quad n \geqslant 1, \\
& K_{n}=-R_{2 n+1}\left\{\frac{1-f_{2 n+1}^{2}}{F_{2 n+1}+i}\right\}+R_{2 n}\left\{\frac{1-f_{2 n}^{2}}{F_{2 n}-i}\right\}, \quad n \geqslant 1, \\
& K_{n}=-R_{2 n+1}\left\{\frac{1-f_{2 n+1}^{2}}{F_{2 n+1}-i}\right\}+R_{2 n}\left\{\frac{1-f_{2 n}^{2}}{F_{2 n}+1 / \delta}\right\}, \quad n \geqslant 1,
\end{aligned}
$$

where

$$
\begin{gather*}
K_{n}=R_{2 n+1} \bar{F}_{2 n+1}-R_{2 n} \bar{F}_{2 n}-C_{2 n}+C_{2 n+1}, \quad n \geqslant 1, \\
n_{2 n+1}=\frac{2-i b_{2 n+1} / y}{b_{2 n+1}}, \quad n \geqslant 1, \tag{4.4.16a}
\end{gather*}
$$

and

$$
\begin{equation*}
\delta_{2 n}=\frac{\left(2 l_{2 n+1}^{2}-1\right)+i b_{2 n+1} / y}{i+b_{2 n+1} / y}, \quad n \geqslant 1 \tag{4.4.16b}
\end{equation*}
$$

Here, subtracting one equation from the other and then taking the quotients of the resulting equations, we get

$$
\begin{align*}
& \frac{\left(i-1 / n_{2 n+1}\right)\left\{F_{2 n+1}-i\right\}}{\left(i+1 / n_{2 n+1}\right)\left\{F_{2 n+1}+i\right\}}=\frac{i\left\{F_{2 n}+1 / \delta_{2 n}\right\}}{\left(1 / \delta_{2 n}\right)\left\{F_{2 n}-i\right\}},  \tag{4.4.17}\\
& \frac{\left(i-1 / n_{2 n+1}\right)\left\{F_{2 n+1}-i\right\}}{2 i\left\{F_{2 n+1}+1 / n_{2 n+1}\right\}}=\frac{i\left\{F_{2 n}+1 / \delta_{2 n}\right\}}{\left(i+1 / \delta_{2 n}\right) F_{2 n}},
\end{align*}
$$

for $n \geqslant 1$.
Similarly considering the three cases which arise from (4.4.11b), (4.4.12a) and (4.4.12c) we also obtain

$$
\begin{align*}
& \frac{\left(1 / \rho_{2 n}\right)\left\{F_{2 n}-i\right\}}{i\left\{F_{2 n}+1 / \rho_{2 n}\right\}}=\frac{2 i\left\{F_{2 n-1}+1 / \gamma_{2 n-1}\right\}}{\left(i+1 / \gamma_{2 n-1}\right)\left\{F_{2 n-1}+i\right\}}, \\
& \frac{i\left\{F_{2 n}+1 / \rho_{2 n}\right\}}{\left(i+1 / \rho_{2 n}\right) F_{2 n}}=\frac{\left(i+1 / \gamma_{2 n-1}\right)\left\{F_{2 n-1}+i\right\}}{\left(i-1 / \gamma_{2 n-1}\right)\left\{F_{2 n-1}-i\right\}}, \tag{4.4.18}
\end{align*}
$$

for $\mathrm{n} \geqslant 1$, where

$$
\begin{equation*}
\rho_{2 n}=-i-2 b_{2 n} / y, \quad n \geqslant 1, \tag{4.4.19a}
\end{equation*}
$$

and

$$
\begin{equation*}
\gamma_{2 n-1}=\frac{\left(1-\ell_{2 n}^{2}\right) i+b_{2 n} / y}{\left(1+\ell_{2 n}^{2}\right)-i b_{2 n} / y}, \quad n \geqslant 1 \tag{4.4.19b}
\end{equation*}
$$

Now, suppose that $\ell_{2 n}^{2} \geqslant \varepsilon>0$ for all $n \geqslant 1$. Then from (4.4.19b) we have $\left|\gamma_{2 n-1}\right| \leqslant 1-\varepsilon_{1}$, where $\varepsilon_{1}>0$. Hence, under the limit circle case, from theorem 4.4.1 and from equations (4.4.17) and $(4,4,18)$ we obtain

$$
\begin{equation*}
\mathrm{F}_{2 \mathrm{n}-1} \rightarrow \mathrm{i} \text { and } \mathrm{F}_{2 \mathrm{n}} \nrightarrow \mathrm{i} \text { as } \mathrm{n} \rightarrow \infty \tag{4.4.20}
\end{equation*}
$$

Consider again the following cases which arise from (4.4.1.1a) and (4.4.11b):
$R_{2 n+1} \bar{F}_{2 n+1}-R_{2 n} \bar{F}_{2 n}=\left(C_{2 n}-C_{2 n+1}\right)-R_{2 n+1} \frac{1-f_{2 n+1}^{2}}{F_{2 n+1}+1 / n} 2 n+1 \quad R_{2 n} \cdot \frac{1-f_{2 n}^{2}}{F_{2 n}}$,
and

$$
R_{2 n} \bar{F}_{2 n}-R_{2 n-1} \bar{F}_{2 n-1}=\left(C_{2 n-1}-C_{2 n}\right)-R_{2 n} \cdot \frac{1-f_{2 n}^{2}}{F_{2 n}}+R_{2 n-1} \cdot \frac{1-f_{2 n-1}^{2}}{F_{2 n-1}-i},
$$

for all $n \geqslant 1$.
Here, using the telescopic effect of these equations together with the properties in theorem 4.4 .1 and equation (4.4.20), we immediately establish that $\left\{\mathrm{R}_{\mathrm{n}} \overline{\mathrm{F}}_{\mathrm{n}}\right\}$ is a Cauchy sequence.

Therefore, realising that $H_{n}(w)$ can also be given in the form

$$
H_{n}(w)=C_{n}+\frac{R_{n} \bar{F}_{n}}{f_{n}^{2}}\left\{1-\frac{1-f_{n}^{2}}{F_{n} w+1}\right\}
$$

we establish that the sequence $\left\{H_{n}(i)\right\}$ is also a Cauchy sequence. Consequently the result:

Theorem 4.4.2 : If the positive definite and real $\hat{J}$-fraction of the form (4.1.15) satisfies the limit circle case for ony $z=i y,(y>0)$, then for this $z$ the fraction converges to a single point in the limit circle, if there exists an $\varepsilon>0$ such that

$$
\begin{equation*}
e_{2 n}^{2} \geqslant \varepsilon \text { for all } n \geqslant 1 \text {. } \tag{4.4.21}
\end{equation*}
$$

### 4.5 UNIFORM CONVERGENCE

We see from the theorems 4.3 .2 and 4.3 .3 that when the coefficients of the positive definite and real $\hat{J}$-fraction of the form (4.1.15) satisfy any of the conditions $(4.3 .15),(4.3 .16)$ and (4.3.17) then for any $z$ for which $\operatorname{Im}(z)>0$, the fraction converges to a finite value.

Similarly we also note from theorem 4.4.2 that when the coefficients $\ell_{2 n},(n \geqslant 1)$ satisfy the condition (4.4.21) then the positive definite and real $\hat{J}$-fraction of the form (4.1.15) converges to a finite value for any $z=i y$ for which $y>0$.

So far we have only considered the convergence of the fraction at any particular point $z$ lying inside some domain. The question of uniform convergence in any domain still remains to be answered.

Let us now consider the following theorem known as the "convergence continuation theorem".

Theorem : Let $\left\{\mathrm{F}_{\mathrm{p}}(z)\right\}$ be an infinite sequence of functions, analytic over a simply connected open domain $S$, which is uniformiy bounded over every finite closed domain $S^{\prime}$ entirely within $S$. Let the sequence converge over an infinite set of points hoving at least one Zimit-point interior to $S$. Then, the sequence converges uniformly over every finite closed domain entirely within $S$ to a function of $z$ which is analytic in $S$.

The proof of this theorem can be found for example in Wall
[1948]. We now consider the sequence $F_{n}(z)=P_{n}(z) / Q_{n}(z)$, which is the $n$-th convergent of the positive derinite and real $\hat{J}$-fraction of
the form (4.1.15). Since all the zeros of $Q_{n}(z)$ lie outside the region $\{z: \operatorname{Im}(z)>0\}$ it follows that $F_{n}(z)$ is analytic for all $n$ in the domain $S$, which is the half-plane given by $\{z: \operatorname{Im}(z)>0\}$. We also have from (4.2.9b) that $F_{n}(z)$ is uniformly bounded over every finite closed domain $S^{\prime}$ entirely within $S$.

Thus using these results together with the results of the theorems $4.3 .2,4.3 .3$ and 4.4 .2 on the convergence continuation theorem we immediately obtain the following.

Theorem 4.5.1: For all z lying entirely on the top half of the complew plane, the positive definite and real $\hat{J}$-fraction of the form (4.1.15) converges uniformiy to a function analytic in this region. if the coefficients of the fraction satisfy any of the folzowing:

$$
\begin{aligned}
& \text { I } \sum_{r=2}^{\infty}\left\{\left(1-\ell_{2 r-1}^{2}\right)^{\frac{3}{2}}\left(1-\ell_{2 r+1}^{2}\right)^{\frac{3}{2}} \frac{1}{\ell_{2 r^{\ell}}^{2 r+1}}\right\}=\infty, \\
& \text { II } \sum_{r=2}^{\infty}\left\{\left(1-l_{2 r-1}^{2}\right)^{\frac{1}{2}}\left(1-\ell_{2 r+1}^{2}\right)^{\frac{1}{2}} \frac{b_{2 r}}{\ell_{2 r^{l}}{ }_{2 r+1}}\right\}=\infty, \\
& \text { III } b_{2 r}=0, r \geqslant 1 \text { and } \sum_{r=1}^{\infty}\left\{l_{2 r+2} / \ell_{2 r+1}\right\}=\infty, \\
& \text { IV } \ell_{2 r}^{2} \geqslant \varepsilon>0 \text { for all } r \geqslant 1 .
\end{aligned}
$$

## REFERENCES

1. Akhieser, N.I. (1965), "The Classical Moment Prob1ems and Some Related Questions", Hafner, New York.
2. Baker, George A. Jr. and Graves-Morris, P.R. (1981a), "Padé Approximants, Part I : Basic Theory", Addison-Wesley, Reading, Massachusetts.
3. Baker, George A. Jr. and Graves-Morris, P.R. (1981b), "Padé Approximants, Part II : Extensions and Applications', Addison-Wesley, Reading, Massachusetts.
4. Boas, R.P. (1939), "The Stieltjes Moment Problem for Functions of Bounded Variation", Bull. Amer. Math. Soc. 45, 399-404.
5. Brezinski, C. (1980), "Padé-Type Approximation and General Orthogonal Polynomials", Birkhäuser, Boston Inc.
6. Carleman, T. (1922), "Sur les problème des moments", Comptes Rendus, 174, 1680-1682.
7. Carleman, T. (1923), "Sur les équations intégrales singulières a noyau réel et symétrique", Uppsala Universitets Årsskrift, 228 pp.
8. Carleman, T. (1926), "Les fonctions quasi-analytiques", Gauthier-Villars, Paris, 115 pp .
9. Chihara, T.S. (1978), "An Introduction to Orthogonal Polynomials", Mathematics and its Applications Ser., Gordon and Breach.
10. Drew, D.M. and Murphy, J.A. (1977), "Branch Points, M-Fractions and Rationai Approximant Generated by Linear Equations", J. Inst. Maths. Applics. 19, 169-185.
11. Erdelyi, A. (1953), "Higher Transcendental Functions", Vo1.2, McGraw-Hill, New York.
12. Frobenius, G. (1881), "Über Relationen zwischen den Näherungsbruchen von Potenzreihen', J. Reine Angew. Math. 90, 1-17.
13. Gauss, C.F. (1814), "Methodus Nova Integralium Valores per Approximationem Inveniendi", Werke, Vol.3, Göttingen, 1876, 165-196.
14. Grundy, R.E. (1977), "Laplace Transform Inversion Using TwoPoint Rational Approximants', J. Inst. Maths. Applics. 20, 299-306.
15. Grundy, R.E. (1978a), 'The Solution of Volterra Integral Equations of the Convolution Type Using Two-Point Rational Approximants", J. Inst. Maths. Applics. 22, 147-158.
16. Grundy, R.E. (1978b), "On the Solution of Nonlinear Volterra Integral Equations Using Two-Point Padé Approximants", J. Inst. Maths. Applics. 22, 317-320.
17. Hamburger, H. (1920), "Über eine Erweiterung des Stieltjesschen Momentenproblems', Part I, Math. Ann. 81, 235-319.
18. Hamburger, H. (1921), "Über eine Erweiterung des Stieltjesschen Momentenproblems", Part II, III, Math. Ann. 82, 120-164, 168-187.
19. Hausdorff, F. (1923), "Momentprobleme für ein endliches Intervall", Mathematische Zeitschrift, 16, 220-248.
20. Heilermann, J.B.H. (1846), "Über die Verwandlung der Reihen in Kettenbrüche', J. Reine Angew. Math. 33.
21. Heine, E. (1878), "Handbuch der Kugelfunktionen", Vol.1, 2nd Ed., Berlin.
22. Heine, E. (1881), "Handbuch der Kugelfunktionen", Vol.2, 2nd Ed., Berlin.
23. Hellinger, E. (1922), "Zur Stieltjes'schen Kettenbruchtheorie", Mathematische Annalen, 86, 18-29.
24. Henrici, P. (1974), "Applied and Computational Complex Analysis Vol.1, Power Series, Integration, Conformal Mapping and Location of Zeros", Wiley, New York.
25. Henrici, P. (1977), "Applied and Computational Complex Analysis Vol.2, Special Functions, Integral Transforms, Asymptotics and Continued Fractions", Wiley, New York.
26. Jones, W.B. and Magnus, Arne (1980), "Computation of Poles of Two-Point Padé Approximants and their Limits", Journal CAM, 105-119.
27. Jones, W:B. and Thron, W.J. (1970), "Twin-Convergence Regions for Continued Fractions $K\left(a_{n} / 1\right)^{\prime \prime}$, Trans. Aner. Math. Soc. 150, 93-119.
28. Jones, W.B. and Thron, W.J. (1977), "Two-Point Padé Tables and T-Fractions", Bull. Amer. Math. Soc. 83, 388-390.
29. Jones, W.B. and Thron, W.J. (1980), "Continued Fractions :

Analytic Theory and Applications", Addison-Wesley, Reading, Massachusetts.
30. Jones, W.B. and Thron, W.J. (1981), "Families of Orthogonal Laurent Polynomials and Related Quadrature Formulas", Preprint 49 pp.
31. Jones, W.B., Thron, W.J. and Njåstad, Olav (1983a), "Orthogonal Laurent Polynomials and the Strong Hamburger Moment Problem", J. Math. Anal. Appl. (to appear).
32. Jones, W.B., Thron, W.J. and Njåstad, Olav (1983b), "Continued Fractions and Strong Hamburger Moment Problems", (to appear).
33. Jones, W.B., Thron, W.J. and Waadeland, H. (1980), "A Strong Stieltjes Moment Problem", Trans. Amer. Math. Soc.
34. Khovanskii, A.N. (1963), 'The Application of Continued Fractions and Their Generalisations to Problems in Approximation Theory (translated by P. Wynn)", P. Noordhoff, Groningen, The Netherlands.
35. McCabe, J.H. (1974), "A Continued Fraction Expansion, with a Truncation Error Estimate for Dawson's Integral", Math. Comp. 28, No.127, 811-816.
36. McCabe, J.H. (1975), "A Formal Extension of the Pade Table to Include Two Point Padé Quotients", J. Inst. Math. Applics. 15, 363-372.
37. McCabe, J.H. (1981), "Perron fractions : an algorithm for computing the Padé Table", J. Comp. Appl. Math., Vol.15, 363-372.
38. McCabe, J.H. (1983), "The Quotient-Difference Algorithm and the Padé Table : An Alternative Form and a General Continued Fraction', Math. Comp. 41, No.163, 183-197.
39. McCabe, J.H. and Murphy, J.A., "Continued Fractions Which Correspond to Power Series Expansions of Two Points", J. Inst. Math. Applics. 17, 233-247.
40. Murphy, J.A. (1971), "Certain Rational Function Approximations to $\left(1+x^{2}\right)^{-\frac{7}{2}} \eta$, J. Inst. Math. Applics. 7, 138-150.
41. Murphy, J.A. and O'Donohoe, M.R. (1977), "A Class of Algorithm for Obtaining Rational Approximants to Functions Which are Defined by Power Series", J. Appl. Math. and Phys. (ZAMP) 28, 1121-1131.
42. Natanson, I.P. (1955), 'Theory of functions of a real variable", I, Unger, New York.
43. Nevañlinna, R. (1922), "Asymptotische Entwickelungen beschränkter Funktionen and des Stieltjessche Momentenproblem", Annales Academiae Scientiarum Fennicae, (A) 18 (5), 52 pp.
44. Padé, H. (1892), "Sur la représentation approchée d'une fonctions par des fractions rationelles", Annales Scientifiques de
l'Ecole Normale Supérieure, (3) 9 (Supplement), S3-93.
45. Phillips, G.M. and Taylor, P.J. (1973), 'Theory and Applications of Numerical Analysis", Academic Press, London.
46. Riesz, M. (1921), "Sur le problème des moments. Premiè̀re Note", Arkiv för matematik astronomi och fysik, 16 (12), 23 pp.
47. Riesz, M. (1922), "Sur le problème des moments. Deuxième Note", Arkiv för matematik, astronomi och fysik, 16 (19), 21 pp .
48. Riesz, M. (1923), "Sur le problème des moments. Troisième Note", Arkiv för matematik; astronomi och fysik, 17 (16), 52 pp.
49. Rudin, W. (1976), "Principles of Mathematical Analysis", McGraw-Hill.
50. Rutishauser, H. (1954), "Der Quetienten-Differenzen-Algorithmus", Z. Angew. Math. Phys. 5, 233-251.
51. Shohat, J.A. and Tamarkin, J.D. (1943), "The Problem of Moments", Mathematical Surveys No.1, Amer. Math. Soc.
52. Stern, M.A. (1832), "Theorie der Kettenbrüche und ihre

Anwendung", J. Reine Angew. Math. 10, 11.
53. Stieltjes, T.J. (1889), "Sur la réduction en fraction continue d'une série précédant suivant les puissances descendents d'une variable", Ann. Fac. Sci. Toulouse; 3H, 1-17.
54. Stieltjes, T.J. (1894-95), "Recherches sur les fractions continues", Ann. Fac. Sci. Toulouse, 8J, 1-122; 9A, 1-47.
55. Szegö, G. (1959), "Orthogonal Polynomials", Colloquium Publications, Vol. 23 , Amer. Math. Soc. New York.
56. Thron, W.J. (1944), "Twin Convergence Regions for Continued Fractions $\mathrm{b}_{0}+\mathrm{K}\left(1 / \mathrm{b}_{\mathrm{n}}\right)^{\prime \prime}$, Amer. J. Math. 66, 428-438.
57. Thron, W.J. (1948), "Some properties of continued fractions $1+\mathrm{d}_{0} \mathrm{z}+\mathrm{K}\left(\mathrm{z} / 1+\mathrm{d}_{\mathrm{n}} \mathrm{z}\right)^{\prime \prime}$, Bull. Amer. Math. Soc. 54, 206-218.
58. Thron, W.J. (1949), "Twin Convergence Regions for Continued Fractions $b_{0}+K\left(1 / b_{n}\right)$, II', Amer. J. Math. 71, 112-120.
59. Thron, W.J. (1963), "Convergence of Sequencés of Linear Fractional Transformations and of Continued Fractions", J. Indian Math. Soc. 27, 103-127.
60. Thron, W.J. (1977), "Two-point Padé Tables, T-Fractions and Sequence of Schur", Saff and Vargo (Eds.), Padé and Rational Approximation, Academic Press, New York, 215-226.
61. Wall, H.S. (1948), "Analytic Theory of Continued Fractions", Van Nostrand, New York.
62. Widder, D.V. (1948), "The Laplace transform", Princeton Univ. Press, Princeton, N.J.
63. Wynn, P. (1960), "The Rational Approximation of Functions Which are Formally Defined by a Power Series Expansion", Math. Comp. 14, 147-186.

