

# GENUS AND CROSSCAP OF SOLVABLE CONJUGACY CLASS GRAPHS OF FINITE GROUPS

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ABSTRACT. The solvable conjugacy class graph of a finite group  $G$ , denoted by  $\Gamma_{sc}(G)$ , is a simple undirected graph whose vertices are the non-trivial conjugacy classes of  $G$  and two distinct conjugacy classes  $C, D$  are adjacent if there exist  $x \in C$  and  $y \in D$  such that  $\langle x, y \rangle$  is solvable. In this paper, we discuss certain properties of genus and crosscap of  $\Gamma_{sc}(G)$  for the groups  $D_{2n}$ ,  $Q_{4n}$ ,  $S_n$ ,  $A_n$ , and  $\text{PSL}(2, 2^d)$ . In particular, we determine all positive integers  $n$  such that their solvable conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal. We shall also obtain a lower bound for the genus of  $\Gamma_{sc}(G)$  in terms of order of the center and number of conjugacy classes for certain groups. As a consequence, we shall derive a relation between the genus of  $\Gamma_{sc}(G)$  and the commuting probability of certain finite non-solvable group.

## 1. INTRODUCTION

The genus of a graph  $\Gamma$ , denoted by  $\gamma(\Gamma)$ , is the smallest non-negative integer  $n$  such that the graph can be embedded on the surface obtained by attaching  $n$  handles to a sphere. Graphs having genus zero are called planar, while those having genus one are called toroidal. Graphs having genus two and three are called double-toroidal triple-toroidal respectively. Let  $N_k$  be the connected sum of  $k$  projective planes. A simple graph which can be embedded in  $N_k$  but not in  $N_{k-1}$ , is called a graph of crosscap  $k$ . The crosscap of a graph  $\Gamma$  is denoted by  $\bar{\gamma}(\Gamma)$ . A graph  $\Gamma$  is called projective if  $\bar{\gamma}(\Gamma) = 1$ . It is well-known that  $\gamma(\Gamma) \geq \gamma(\tilde{\Gamma})$  and  $\bar{\gamma}(\Gamma) \geq \bar{\gamma}(\tilde{\Gamma})$ , if  $\tilde{\Gamma}$  is a subgraph of  $\Gamma$ . Also, for  $n \geq 3$  and  $r, s \geq 2$  we have

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \quad (1)$$

$$\gamma(K_{r,s}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil \quad (2)$$

and

$$\bar{\gamma}(K_n) = \begin{cases} \lceil \frac{1}{6}(n-3)(n-4) \rceil, & \text{if } n \neq 7 \\ 3, & \text{if } n = 7, \end{cases} \quad (3)$$

where  $K_n$  is the complete graph on  $n$  vertices and  $K_{r,s}$  is the complete bipartite graph with two parts of sizes  $r$  and  $s$ .

The solvable conjugacy class graph (SCC-graph) of a finite group  $G$ , denoted by  $\Gamma_{sc}(G)$ , is a simple undirected graph whose vertices are the non-trivial conjugacy

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classes of  $G$  and two distinct conjugacy classes  $C, D$  are adjacent if there exist  $x \in C$  and  $y \in D$  such that  $\langle x, y \rangle$  is solvable. The SCC-graph of  $G$  is introduced and studied in [3] extending the notions of commuting conjugacy class graph [13] and nilpotent conjugacy class graph [14].

It was shown in [3, Theorem 3.6] that there are only finitely many finite groups  $G$  whose SCC-graph has given genus. Therefore, if some bounds for  $|G|$  are known in terms of genus of  $\Gamma_{sc}(G)$  then one may have characterizations of finite groups such that their SCC-graphs are planar, toroidal, double-toroidal or triple-toroidal. It is worth mentioning that the genus of the commuting graph and commuting conjugacy class graph of various classes of finite groups have been computed in [1, 9, 4] and characterized finite non-abelian groups such that their commuting/commuting conjugacy class graphs are planar, toroidal, double-toroidal or triple-toroidal. In [8], all finite non-nilpotent groups are characterized such that their nilpotent graphs are planar or toroidal. Further, in [5, 6], it was shown that solvable and non-solvable graphs of finite non-solvable groups are neither planar, toroidal, double-toroidal or triple-toroidal.

In this paper, we discuss certain properties of genus and crosscap of  $\Gamma_{sc}(G)$  for the groups  $D_{2n}$ ,  $Q_{4n}$ ,  $S_n$ ,  $A_n$  and  $\text{PSL}(2, 2^d)$ . In particular, we determine all cases for which their SCC-graphs are planar, toroidal, double-toroidal or triple-toroidal. We shall also obtain a lower bound for  $\gamma(\Gamma_{sc}(G))$  in terms of order of the center and number of conjugacy classes for certain groups. As a consequence, we shall derive a relation between  $\gamma(\Gamma_{sc}(G))$  and commuting probability of certain finite non-solvable groups.

## 2. GENERALITIES ON SCC-GRAPHS

As noted earlier, it was shown in the earlier paper [3] that the order of a group  $G$  is bounded in terms of the clique number of its SCC-graph. We mentioned in that paper the problem of finding an explicit bound.

We noted in the preceding section that the genus and crosscap of the complete graph  $K_n$  are known explicitly, and are functions which tend to infinity with  $n$ . It follows immediately that

**Proposition 2.1.** *Given  $k$ , there are only finitely many finite groups  $G$  such that the genus or crosscap of  $\Gamma_{sc}(G)$  is equal to  $k$ .*

We pose the analogous question:

**Problem 2.2.** *Find an explicit bound for the order of a finite group  $G$  for which  $\gamma(\Gamma_{sc}(G)) = k$ . Same problem for  $G$  for which  $\bar{\gamma}(\Gamma_{sc}(G)) = k$ .*

In particular, if  $G$  is a solvable group, then  $\Gamma_{sc}(G)$  is a complete graph  $K_k$ , where  $k$  is one less than the number of conjugacy classes of  $G$ ; so the genus and crosscap of  $\Gamma_{sc}(G)$  are determined by the results in the preceding section.

The analysis is complicated by the fact that, if  $G$  and  $H$  are groups with  $G \leq H$ , it is not necessarily true that  $\Gamma_{sc}(G)$  is a subgraph of  $\Gamma_{sc}(H)$ . For example, the cyclic group of order 5 has five conjugacy classes but the dihedral group of order 10 only has four.

This example also shows that a natural monotonic graph parameter (the clique number) does not induce a monotonic parameter on groups.

However, for one important class of groups, things are better.

**Proposition 2.3.** *For any  $n$ , there is a natural embedding  $f_n$  from the vertex set of  $\Gamma_{sc}(S_n)$  to that of  $\Gamma_{sc}(S_{n+1})$ , which embeds the first graph as a subgraph of the second.*

*Proof.* Under the natural embedding of  $S_n$  into  $S_{n+1}$  as the stabilizer of  $n+1$ , two elements of  $S_n$  are conjugate in  $S_{n+1}$  if and only if they are conjugate in  $S_n$ . (This is because elements of the symmetric group are conjugate if they have the same cycle structure; and the natural embedding simply adds one cycle of length 1 to each element of  $S_n$ .) So the set of conjugacy classes of  $S_n$  embeds naturally into that for  $S_{n+1}$ .

If two conjugacy classes in  $S_n$  contain elements which generate a solvable group, then this is still true in  $S_{n+1}$ . So the embedding above maps edges to edges. (Note that non-edges are not necessarily mapped to non-edges; the classes of a 3-cycle and a 5-cycle are non-adjacent in  $S_n$  for  $n = 5, 6, 7$  but are adjacent for  $n \geq 8$ . So the embedding is not as induced subgraph in general.)  $\square$

It follows that both  $\gamma(\Gamma_{sc}(S_n))$  and  $\bar{\gamma}(\Gamma_{sc}(S_n))$  are non-decreasing functions of  $n$ . We can observe further that, if we embed  $S_n$  in  $S_{n+k}$  by composing the maps  $f_n, f_{n+1}, \dots, f_{n+k-1}$ , then for sufficiently large  $k$ , the image of the composite map is a complete graph. (It suffices to take  $k = n$ ; for, given any two elements of  $S_n$ , we can find conjugates of them in  $S_{2n}$  with disjoint support, and hence commuting.) In particular, we get the following fractional exponential lower bounds, using the known results on the genus and crosscap of the complete graph. Here  $p(n)$  denotes the number of partitions of the integer  $n$  (which is the number of conjugacy classes of the symmetric group  $S_n$ ). Thus, if  $k = p(\lfloor n/2 \rfloor) - 1$  then  $\Gamma_{sc}(S_n)$  has a subgraph isomorphic to a complete graph having  $k$  vertices. Hence, using (1) and (3) we get the following result.

**Theorem 2.4.** *Given  $n \geq 10$ , let  $k = p(\lfloor n/2 \rfloor) - 1$ . Then*

$$\gamma(\Gamma_{sc}(S_n)) \geq \left\lceil \frac{(k-3)(k-4)}{12} \right\rceil \quad \text{and} \quad \bar{\gamma}(\Gamma_{sc}(S_n)) \geq \left\lceil \frac{(k-3)(k-4)}{6} \right\rceil.$$

Inspection of the proof shows that the same bound holds for the commuting and nilpotent conjugacy class graphs of the symmetric group [13, 14].

Recall that  $k(G)$  denotes the number of conjugacy classes of  $G$ . The following lemma is useful in obtaining a lower bound for  $\gamma(\Gamma_{sc}(G))$  as mentioned above.

**Lemma 2.5.** *Let  $G$  be a finite non-solvable group with non-trivial center  $Z(G)$ . Then  $\Gamma_{sc}(G)$  has a subgraph isomorphic to  $K_{|Z(G)|-1, k(G)-|Z(G)|}$ .*

*Proof.* Let  $S = \{x^G : x \in Z(G) \setminus \{1\}\}$  and  $T = \{y^G : y \in G \setminus Z(G)\}$ . We consider the subgraph  $S_\Gamma$  of  $\Gamma_{sc}(G)$  by removing edges between the vertices of  $S$  as well as removing edges between the vertices of  $T$ . Then the subgraph thus obtained is isomorphic to  $K_{|Z(G)|-1, k(G)-|Z(G)|}$ .  $\square$

**Theorem 2.6.** *Let  $G$  be a finite non-solvable group with non-trivial center  $Z(G)$ . Then*

$$4\gamma(\Gamma_{sc}(G)) \geq (|Z(G)| - 3)(k(G) - |Z(G)| - 2).$$

*Proof.* By Lemma 2.5, it follows that  $\Gamma_{sc}(G)$  has a subgraph isomorphic to  $K_{|Z(G)|-1, k(G)-|Z(G)|}$ . We have

$$\gamma(\Gamma_{sc}(G)) \geq \gamma(K_{|Z(G)|-1, k(G)-|Z(G)|}).$$

Therefore, by (2), we have

$$\begin{aligned}\gamma(\Gamma_{sc}(G)) &\geq \left\lceil \frac{(|Z(G)| - 3)(k(G) - |Z(G)| - 2)}{4} \right\rceil \\ &\geq \frac{(|Z(G)| - 3)(k(G) - |Z(G)| - 2)}{4}.\end{aligned}$$

Hence, the result follows on simplification.  $\square$

We conclude this section with the following relation between commuting probability (which is the probability that a randomly chosen pair of elements of  $G$  commute) and genus of SCC-graph of finite non-solvable group with non-trivial center.

**Corollary 2.7.** *Let  $G$  be a finite non-solvable group and  $|Z(G)| > 3$ . If  $\Pr(G)$  is the commuting probability of  $G$  then*

$$\Pr(G) \leq \frac{4\gamma(\Gamma_{sc}(G)) + (|Z(G)| - 3)(|Z(G)| + 2)}{|G|(|Z(G)| - 3)}.$$

*Proof.* The result follows from Theorem 2.6 and the fact that  $\Pr(G) = \frac{k(G)}{|G|}$  as noted in [12].  $\square$

It is worth mentioning that many bounds for  $\Pr(G)$  have been obtained using various group theoretic notions over the years (see [11, 15]). However, the bound for  $\Pr(G)$  obtained in Corollary 2.7 is the first of its kind involving genus of certain graph defined on groups though it is difficult to compute genus of  $\Gamma_{sc}(G)$  in general.

### 3. GENUS AND CROSSCAP OF SCC-GRAPHS

We begin with the characterizations of dihedral groups ( $D_{2n} = \langle a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$ ) and quaternion groups ( $Q_{4n} = \langle a, b : a^{2n} = 1, b^2 = x^n, bab^{-1} = a^{-1} \rangle$ ) such that their SCC-graphs are planar, toroidal, double-toroidal or triple-toroidal.

Since the dihedral and quaternion groups are solvable, their SCC graphs are complete graphs  $K_k$  where  $k$  is one less than the number of conjugacy classes.

**Theorem 3.1.** (a)  $\Gamma_{sc}(D_{2n})$  is planar if and only if  $n = 2, 3, 4, 5$  and 7.

(b)  $\Gamma_{sc}(D_{2n})$  is toroidal if and only if  $n = 6, 8, 9, 10, 11$  and 13.

(c)  $\Gamma_{sc}(D_{2n})$  is double-toroidal if and only if  $n = 12$  and 15.

(d)  $\Gamma_{sc}(D_{2n})$  is triple-toroidal if and only if  $n = 14$  and 17.

(e)  $\gamma(\Gamma_{sc}(D_{2n})) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{48} \right\rceil, & \text{when } n \geq 19 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{48} \right\rceil, & \text{when } n \geq 16 \text{ and } n \text{ is even.} \end{cases}$

(f)  $\bar{\gamma}(\Gamma_{sc}(D_{2n})) = 0$  if and only if  $n = 2, 3, 4, 5$  and 7.

(g)  $\Gamma_{sc}(D_{2n})$  is projective if and only if  $n = 6, 8, 9$  and 11.

(h)  $\bar{\gamma}(\Gamma_{sc}(D_{2n})) = \begin{cases} \left\lceil \frac{(n-5)(n-7)}{24} \right\rceil, & \text{when } n \geq 13 \text{ and } n \text{ is odd} \\ \left\lceil \frac{(n-2)(n-4)}{24} \right\rceil, & \text{when } n \geq 10 \text{ and } n \text{ is even.} \end{cases}$

*Proof.* We consider the following cases.

**Case 1.** If  $n$  is odd.

The non-trivial conjugacy classes in  $D_{2n}$  are  $a^{D_{2n}}, (a^2)^{D_{2n}}, \dots, (a^{\frac{n-1}{2}})^{D_{2n}}, b^{D_{2n}}$ . There are  $\frac{n+1}{2}$  such conjugacy classes which give  $\Gamma_{sc}(D_{2n}) = K_{\frac{n+1}{2}}$ .

For  $n = 3, 5$  and  $7$  we have  $\Gamma_{sc}(D_{2n}) = K_2, K_3$  and  $K_4$  respectively; and so  $\gamma(\Gamma_{sc}(D_{2n})) = 0$  as well as  $\bar{\gamma}(\Gamma_{sc}(D_{2n})) = 0$ . For  $n = 9, 11$  and  $13$  we have  $\Gamma_{sc}(D_{2n}) = K_5, K_6$  and  $K_7$  respectively; so  $\gamma(\Gamma_{sc}(D_{2n})) = 1$  and  $\bar{\gamma}(\Gamma_{sc}(D_{2 \times 9})) = \bar{\gamma}(\Gamma_{sc}(D_{2 \times 11})) = 1$ . For  $n = 15$  we have  $\Gamma_{sc}(D_{2n}) = K_8$  and so  $\gamma(\Gamma_{sc}(D_{2n})) = 2$ . For  $n = 17$  we have  $\Gamma_{sc}(D_{2n}) = K_9$  and so  $\gamma(\Gamma_{sc}(D_{2n})) = 3$ . For  $n \geq 19$ , by (1), we have

$$\gamma(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-5)(n-7)}{48} \right\rceil \geq 4.$$

For  $n \geq 13$ , by (3), we have

$$\bar{\gamma}(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-5)(n-7)}{24} \right\rceil \geq 2.$$

**Case 2.** If  $n$  is even.

The non-trivial conjugacy classes in  $D_{2n}$  are  $a^{D_{2n}}, (a^2)^{D_{2n}}, \dots, (a^{\frac{n}{2}})^{D_{2n}}, b^{D_{2n}}, (ab)^{D_{2n}}$ . There are  $\frac{n+4}{2}$  such conjugacy classes which give  $\Gamma_{sc}(D_{2n}) = K_{\frac{n+4}{2}}$ .

For  $n = 2$  and  $4$  we have  $\Gamma_{sc}(D_{2n}) = K_3$  and  $K_4$  respectively; and so  $\gamma(\Gamma_{sc}(D_{2n})) = 0$  as well as  $\bar{\gamma}(\Gamma_{sc}(D_{2n})) = 0$ . For  $n = 6, 8$  and  $10$  we have  $\Gamma_{sc}(D_{2n}) = K_5, K_6$  and  $K_7$  respectively; so  $\gamma(\Gamma_{sc}(D_{2n})) = 1$  and  $\bar{\gamma}(\Gamma_{sc}(D_{2 \times 6})) = \bar{\gamma}(\Gamma_{sc}(D_{2 \times 8})) = 1$ . For  $n = 12$  we have  $\Gamma_{sc}(D_{2n}) = K_8$  and so  $\gamma(\Gamma_{sc}(D_{2n})) = 2$ . For  $n = 14$  we have  $\Gamma_{sc}(D_{2n}) = K_9$  and so  $\gamma(\Gamma_{sc}(D_{2n})) = 3$ . For  $n \geq 16$ , by (1), we have

$$\gamma(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-2)(n-4)}{48} \right\rceil \geq 4.$$

For  $n \geq 10$ , by (3), we have

$$\bar{\gamma}(\Gamma_{sc}(D_{2n})) = \left\lceil \frac{(n-2)(n-4)}{24} \right\rceil \geq 2.$$

Hence, the result follows.  $\square$

**Theorem 3.2.** (a)  $\Gamma_{sc}(Q_{4n})$  is planar if and only if  $n = 1$  and  $2$ .

(b)  $\Gamma_{sc}(Q_{4n})$  is toroidal if and only if  $n = 3, 4$  and  $5$ .

(c)  $\Gamma_{sc}(Q_{4n})$  is double-toroidal if and only if  $n = 6$ .

(d)  $\Gamma_{sc}(Q_{4n})$  is triple-toroidal if and only if  $n = 7$ .

(e)  $\gamma(\Gamma_{sc}(Q_{4n})) = \left\lceil \frac{(n-1)(n-2)}{12} \right\rceil$  for  $n \geq 8$ .

(f)  $\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = 0$  if and only if  $n = 1$  and  $2$ .

(g)  $\Gamma_{sc}(Q_{4n})$  is projective if and only if  $n = 3$  and  $4$ .

(h)  $\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = \begin{cases} 3, & \text{when } n = 5 \\ \left\lceil \frac{(n-1)(n-2)}{6} \right\rceil, & \text{when } n \geq 6. \end{cases}$

*Proof.* The non-trivial conjugacy classes in  $Q_{4n}$  are  $a^{Q_{4n}}, (a^2)^{Q_{4n}}, \dots, (a^n)^{Q_{4n}}, b^{Q_{4n}}$  and  $(ab)^{Q_{4n}}$ . There are  $n+2$  such conjugacy classes which give  $\Gamma_{sc}(Q_{4n}) = K_{n+2}$ .

For  $n = 1$  and  $2$  we have  $\Gamma_{sc}(Q_{4n}) = K_3$  and  $K_4$  respectively; and so  $\gamma(\Gamma_{sc}(Q_{4n})) = 0$  as well as  $\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = 0$ . For  $n = 3, 4$  and  $5$  we have  $\Gamma_{sc}(D_{2n}) = K_5, K_6$  and  $K_7$  respectively; so  $\gamma(\Gamma_{sc}(Q_{4n})) = 1$  and  $\bar{\gamma}(\Gamma_{sc}(Q_{4 \times 3})) = \bar{\gamma}(\Gamma_{sc}(Q_{4 \times 4})) = 1$ .

For  $n = 6$  we have  $\Gamma_{sc}(D_{2n}) = K_8$  and so  $\gamma(\Gamma_{sc}(Q_{4n})) = 2$ . For  $n = 7$  we have  $\Gamma_{sc}(Q_{4n}) = K_9$  and so  $\gamma(\Gamma_{sc}(Q_{4n})) = 3$ . For  $n \geq 8$ , by (1), we have

$$\gamma(\Gamma_{sc}(Q_{4n})) = \left\lceil \frac{(n-1)(n-2)}{12} \right\rceil \geq 4.$$

By (3), we have  $\bar{\gamma}(\Gamma_{sc}(Q_{4 \times 5})) = 3$  and for  $n \geq 6$ ,

$$\bar{\gamma}(\Gamma_{sc}(Q_{4n})) = \left\lceil \frac{(n-1)(n-2)}{6} \right\rceil \geq 4.$$

Hence, the result follows.  $\square$

In general, we have the following result for the SCC-graph of a finite solvable group.

**Theorem 3.3.** *Let  $G$  be a finite solvable group with  $k(G)$  conjugacy classes. Then*

- (a)  $\Gamma_{sc}(G)$  is planar if and only if  $k(G) = 1, 2, 3, 4$  and 5.
- (b)  $\Gamma_{sc}(G)$  is toroidal if and only if  $k(G) = 6, 7$  and 8.
- (c)  $\Gamma_{sc}(G)$  is double-toroidal if and only if  $k(G) = 9$ .
- (d)  $\Gamma_{sc}(G)$  is triple-toroidal if and only if  $k(G) = 10$ .
- (e)  $\gamma(\Gamma_{sc}(G)) = \left\lceil \frac{(k(G)-4)(k(G)-5)}{12} \right\rceil$  for  $k(G) \geq 11$ .
- (f)  $\bar{\gamma}(\Gamma_{sc}(G)) = 0$  if and only if  $k(G) = 1, 2, 3, 4$  and 5.
- (g)  $\Gamma_{sc}(G)$  is projective if and only if  $k(G) = 6$  and 7.
- (h)  $\bar{\gamma}(\Gamma_{sc}(G)) = \begin{cases} 3, & \text{when } k(G) = 8 \\ \left\lceil \frac{(k(G)-4)(k(G)-5)}{6} \right\rceil, & \text{when } k(G) \geq 9. \end{cases}$

*Proof.* We have  $\Gamma_{sc}(G) = K_{k(G)-1}$ . If  $k(G) = 1$  then  $\Gamma_{sc}(G)$  is the null graph. For  $k(G) = 2, 3, 4$  and 5 we have  $\Gamma_{sc}(G) = K_1, K_2, K_3$  and  $K_4$  respectively; and so  $\gamma(\Gamma_{sc}(G)) = 0$  as well as  $\bar{\gamma}(\Gamma_{sc}(G)) = 0$ . For  $k(G) = 6, 7$  and 8 we have  $\Gamma_{sc}(G) = K_5, K_6$  and  $K_7$  respectively; and so  $\gamma(\Gamma_{sc}(G)) = 1$ . If  $k(G) = 9$  then we have  $\Gamma_{sc}(G) = K_8$  and so  $\gamma(\Gamma_{sc}(G)) = 2$ . If  $k(G) = 10$  then  $\Gamma_{sc}(G) = K_9$  and so  $\gamma(\Gamma_{sc}(G)) = 3$ . For  $k(G) \geq 11$ , by (1), we have

$$\gamma(\Gamma_{sc}(G)) = \left\lceil \frac{(k(G)-4)(k(G)-5)}{12} \right\rceil \geq 4.$$

For  $k(G) = 6, 7$ , by (3), we have  $\bar{\gamma}(\Gamma_{sc}(G)) = 1$ . If  $k(G) = 8$  then  $\bar{\gamma}(\Gamma_{sc}(G)) = 3$ . For  $k(G) \geq 9$ , by (3), we have

$$\bar{\gamma}(\Gamma_{sc}(G)) = \left\lceil \frac{1}{6}(k(G)-4)(k(G)-5) \right\rceil \geq 4.$$

Hence, the result follows.  $\square$

The groups  $S_3, S_4, A_3$  and  $A_4$  are solvable, with respectively 3, 5, 3 and 4 conjugacy classes; so their SCC-graphs are complete graphs on 2, 4, 2 and 3 vertices respectively. All these graphs are planar. Also,  $\bar{\gamma}(\Gamma_{sc}(G)) = 0$ , if  $G$  is one of the groups  $S_3, S_4, A_3$  and  $A_4$ . The SCC-graphs of other small symmetric and alternating groups are shown in the following figures, where a vertex is labelled with a representative of its conjugacy class.

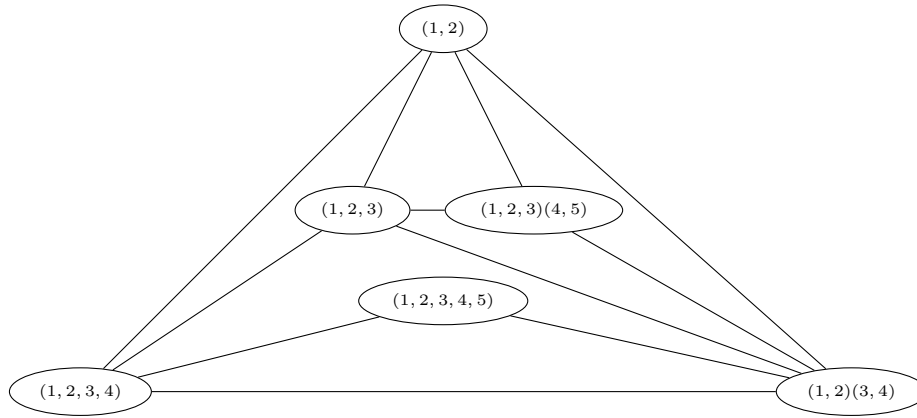


Figure 1.  $\Gamma_{sc}(S_5)$

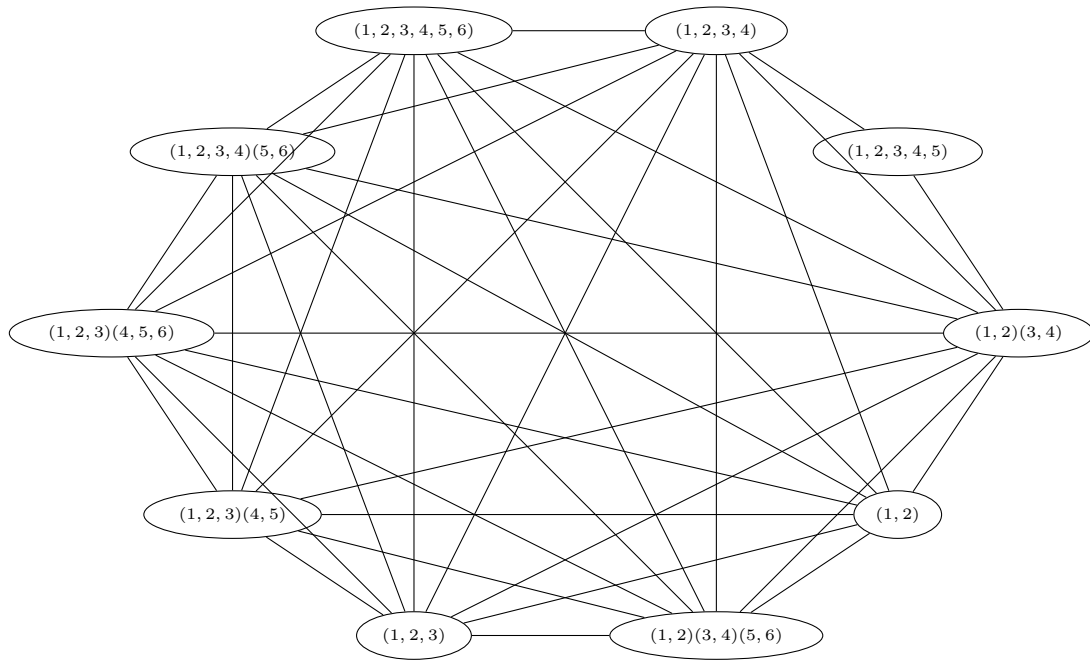
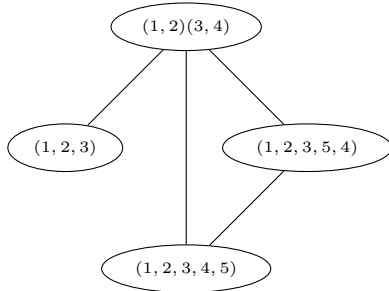
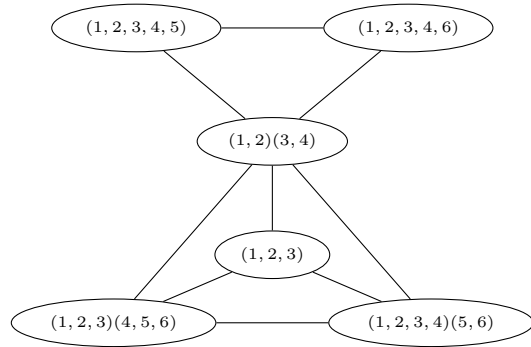


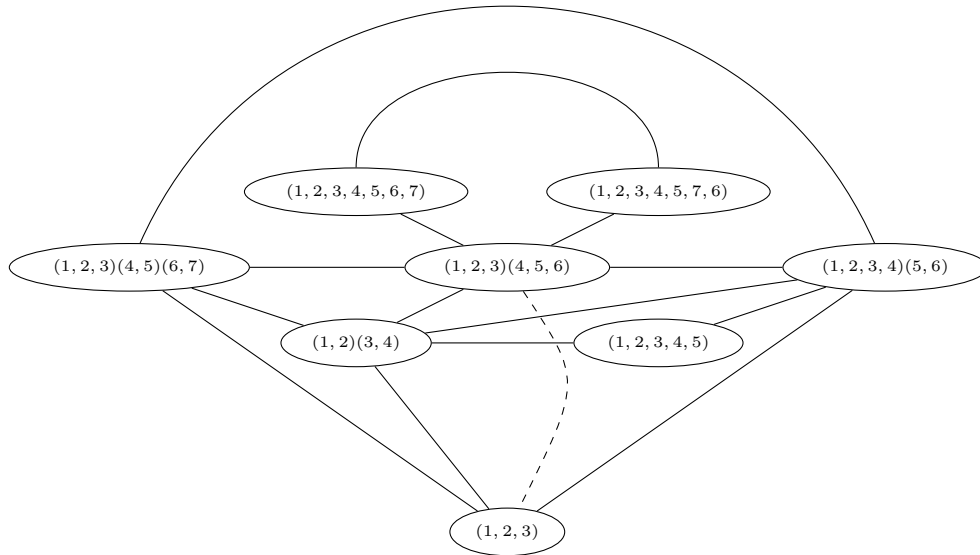
Figure 2.  $\Gamma_{sc}(S_6)$



**Figure 3.**  $\Gamma_{sc}(A_5)$



**Figure 4.**  $\Gamma_{sc}(A_6)$

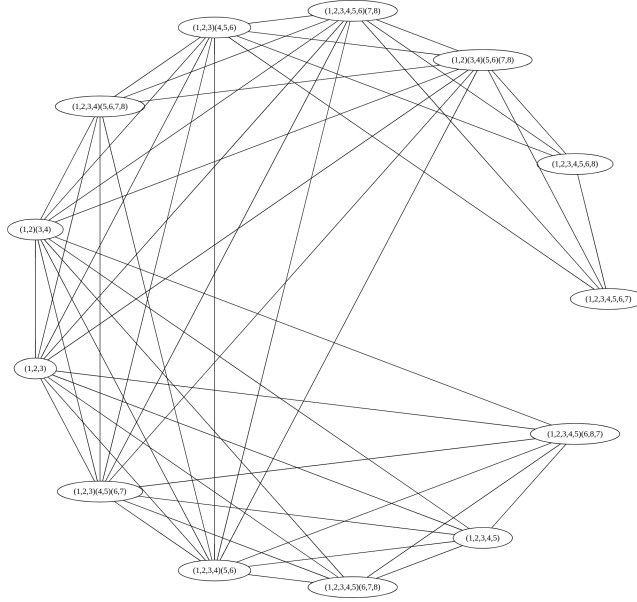


**Figure 5.**  $\Gamma_{sc}(A_7)$

The symmetric and alternating groups whose SCC-graphs have small genus or are projective are given in the following results. First, the symmetric groups.

**Theorem 3.4.** (a)  $\Gamma_{sc}(S_n)$  is planar if and only if  $n \leq 5$ .





**Figure 6.**  $\Gamma_{sc}(A_8)$

- (b) If  $n \geq 7$  then  $\Gamma_{sc}(S_n)$  is neither planar, toroidal, double-toroidal nor triple-toroidal.
- (c)  $\Gamma_{sc}(S_6)$  is not toroidal.
- (d) If  $n \geq 6$  then  $\Gamma_{sc}(S_n)$  is not projective.

*Proof.* (a) If  $n \leq 5$  then, from our earlier remarks and Figure 1, it follows that  $\Gamma_{sc}(S_n)$  is planar. If  $n \geq 6$  then it is easy to show that the elements  $(1, 2)$ ,  $(1, 2, 3)$ ,  $(1, 2)(3, 4)$ ,  $(1, 2, 3, 4)$ ,  $(1, 2, 3)(4, 5)$  induce a clique in  $\Gamma_{sc}(S_n)$ . Hence,

$$\gamma(\Gamma_{sc}(S_n)) \geq \gamma(K_5) = 1$$

and so  $\Gamma_{sc}(S_n)$  is not planar.

- (b) One can show that the ten elements

$$(1, 2), (1, 2, 3), (1, 2)(3, 4), (1, 2, 3, 4), (1, 2, 3)(4, 5), (1, 2)(3, 4)(5, 6), \\ (1, 2, 3, 4)(5, 6), (1, 2, 3)(4, 5, 6), (1, 2, 3)(4, 5)(6, 7), (1, 2, 3, 4)(5, 6, 7)$$

induce a clique in  $\Gamma_{sc}(S_n)$ . Hence,

$$\gamma(\Gamma_{sc}(S_n)) \geq \gamma(K_{10}) = 4$$

and so  $\Gamma_{sc}(S_n)$  is neither planar, toroidal, double-toroidal nor triple-toroidal.

- (c) From Figure 2, it follows that  $\Gamma_{sc}(S_6)$  contains a subgraph isomorphic to  $K_9$  (which is induced by  $V(\Gamma_{sc}(S_6)) \setminus \{(1, 2, 3, 4, 5)^{S_6}\}$ ). Therefore,

$$\gamma(\Gamma_{sc}(S_6)) \geq \gamma(K_9) = 3.$$

Hence, the result follows from (a) and (b).

(d) In addition to the five permutations listed in the proof of (a), also the elements

$$(1, 2), (1, 2)(3, 4)(5, 6), (1, 2, 3, 4)(5, 6), (1, 2, 3)(4, 5, 6), (1, 2, 3, 4, 5, 6)$$

induce a clique. Consequently,  $\Gamma_{sc}(S_n)$  contains two copies of  $K_5$  which share a single vertex. This subgraph is isomorphic to the graph denoted by  $A_1$  in [10]. Therefore,  $\Gamma_{sc}(S_n)$  is not projective.  $\square$

Here is the analogous results for alternating groups.

- Theorem 3.5.** (a)  $\Gamma_{sc}(A_n)$  is planar if and only if  $n \leq 6$ .  
 (b) If  $n \geq 9$  then  $\Gamma_{sc}(A_n)$  is neither planar, toroidal, double-toroidal nor triple-toroidal.  
 (c)  $\Gamma_{sc}(A_n)$  is toroidal if and only if  $n = 7$ .  
 (d) If  $n \geq 8$  then  $\Gamma_{sc}(A_n)$  is not projective.

*Proof.* (a) If  $n \leq 6$  then, as shown in Figures 3 and 4, it follows that  $\Gamma_{sc}(A_n)$  is planar.

If  $n \geq 7$  then the permutations

$$(1, 2, 3), (1, 2)(3, 4), (1, 2, 3, 4)(5, 6), (1, 2, 3)(4, 5, 6), (1, 2, 3)(4, 5)(6, 7)$$

induce a clique in  $\Gamma_{sc}(A_n)$  (note that the elements have pairwise distinct cycle types). Therefore,

$$\gamma(\Gamma_{sc}(A_n)) \geq \gamma(K_5) = 1.$$

and so  $\Gamma_{sc}(A_n)$  is not planar.

(b) The ten elements

$$(1, 2, 3), (1, 2)(3, 4), (1, 2, 3, 4)(5, 6), (1, 2, 3)(4, 5, 6), (1, 2, 3)(4, 5)(6, 7), (1, 2)(3, 4)(5, 6)(7, 8), \\ (1, 2, 3, 4, 5, 6)(7, 8), (1, 2, 3, 4)(5, 6, 7)(8, 9), (1, 2, 3)(4, 5, 6)(7, 8, 9), (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

induce a clique in  $\Gamma_{sc}(A_n)$ . Thus, the result follows as in Theorem 3.4(b).

(c) The fact that  $\Gamma_{sc}(A_7)$  is toroidal follows from Figure 5 and part (a).

It is easy to see in Figure 6, that the subgraph induced by the permutations

$$(1, 2, 3)(4, 5, 6), (1, 2, 3, 4, 5, 6, 7), (1, 2, 3, 4, 5, 6, 8), (1, 2)(3, 4)(5, 6)(7, 8), (1, 2, 3, 4, 5, 6)(7, 8)$$

and

$$(1, 2, 3, 4, 5), (1, 2, 3, 4)(5, 6), (1, 2, 3)(4, 5)(6, 7), (1, 2, 3, 4, 5)(6, 7, 8), (1, 2, 3, 4, 5)(6, 8, 7)$$

contains a subgraph isomorphic to  $K_5 \amalg K_5$ , where  $\amalg$  stands for disjoint union. Therefore,

$$\gamma(\Gamma_{sc}(A_8)) \geq \gamma(K_5 \amalg K_5) = 2.$$

Hence, the result follows from parts (a) and (b).

(d) There are two 5-cliques induced by

$$(1, 2, 3), (1, 2)(3, 4), (1, 2, 3, 4, 5), (1, 2, 3, 4)(5, 6), (1, 2, 3)(4, 5)(6, 7), \\ (1, 2, 3), (1, 2, 3)(4, 5, 6), (1, 2)(3, 4)(5, 6)(7, 8), (1, 2, 3, 4, 5, 6)(7, 8), (1, 2, 3, 4)(5, 6, 7, 8),$$

which share a single vertex. Thus, the claim follows as in Theorem 3.4(d).  $\square$

We conclude with one more family of groups to show that the results can be extended, namely the groups  $\text{PSL}(2, q)$ , where  $q$  is a power of 2. This group is isomorphic to  $S_3$  if  $q = 2$ , and to  $A_5$  if  $q = 4$ ; these cases have already been considered. Relevant group-theoretic information can be found in the ATLAS of finite groups [7].

The group  $G = \text{PSL}(2, q)$ , for  $q = 2^d$ , has a unique conjugacy class of involutions. The remaining elements lie in cyclic groups of orders  $q \pm 1$ , whose normalisers are dihedral of orders  $2(q \pm 1)$ ; any element of odd order is conjugate to one in a fixed dihedral group of order  $2(q \pm 1)$ , and two such elements are conjugate if and only if they are conjugate in this dihedral group. From our analysis of dihedral groups, we see that  $\Gamma_{sc}(\text{PSL}(2, q))$  is the union of complete graphs of orders  $(q + 2)/2$  and  $q/2$  with one vertex from each (corresponding to the class of involutions) identified. (The remaining solvable subgroups are the normalizers of Sylow 2-subgroups, of order  $q(q - 1)$ , and give rise to the same edges as the dihedral group of order  $2(q - 1)$ .) For  $q = 8$ , the graph is shown in Figure 7.

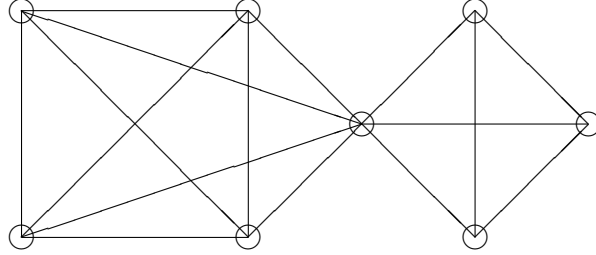


Figure 7.  $\Gamma_{sc}(\text{PSL}(2, 8))$

**Theorem 3.6.** For  $q = 2^d$ , with  $d \geq 3$ , the genus of  $\Gamma_{sc}(\text{PSL}(2, q))$  is either  $\gamma(K_{q/2}) + \gamma(K_{q/2+1})$  or  $\gamma(K_{q/2}) + \gamma(K_{q/2+1}) - 1$ .

*Proof.* The upper bound comes from the observation that we can embed each of the two complete graphs in a surface of minimal genus, and then identify one vertex (by deleting a small disc containing the vertex on each surface and putting a cylinder connecting the boundaries of the two discs with one new vertex on it).

The value of this sum is  $\lceil (m-3)(m-4)/12 \rceil + \lceil (m-2)(m-3)/12 \rceil$ , where  $m = q/2$ . Now  $m \equiv 4$  or  $8 \pmod{12}$ . So  $(m-3)(m-4) \equiv 0$  or  $8$ , and  $(m-2)(m-3) \equiv 2$  or  $6 \pmod{12}$ . Thus, taking the ceiling means adding 10 to the numerator, giving

$$g \leq \frac{m^2 - 7m + 12 + m^2 - 5m + 6 + 10}{12} = \frac{m^2 - 6m + 14}{6}.$$

For the lower bound, we note that the numbers of vertices and edges are given by  $V = 2m$ ,  $E = m^2$ , where  $m = q/2$ . Since a face has at least three edges, the number  $F$  of faces satisfies  $3F \leq 2E$ , so  $F \leq \frac{2}{3}m^2$ . Now Euler's formula gives

$$V - E + F = 2 - 2g,$$

so  $g \geq (m^2 - 6m + 6)/6$ . Since  $g$  is integral, we can round up. Now again  $m \equiv 2$  or  $4 \pmod{6}$ , so we add 2 to the numerator to reach a multiple of 6. Thus

$$g \geq \frac{m^2 - 6m + 8}{6}.$$

The lower and upper bounds differ by 1, giving the result.  $\square$

**Problem 3.7.** *Which of the two values is correct?*

In particular, we see that for  $q \leq 4$  the graph is planar; for  $q = 8$  it is toroidal, and for  $q \geq 16$  it has genus at least 4.

We note in passing that these groups have the property that the unique class of involutions is a dominant vertex in the SCC graph.

**Problem 3.8.** *Which non-solvable groups  $G$  have the property that the SCC graph contains a dominant vertex?*

We note that the Janko group  $J_1$  also has this property.

A similar analysis (with some differences in detail) could be carried out for the groups  $\text{PSL}(2, q)$  and  $\text{PGL}(2, q)$ , where  $q$  is an odd prime power; we leave this as an exercise for readers, after working one case, the group  $\text{PSL}(2, 7)$ . In this case, the two dihedral subgroups of orders 6 and 8 are contained in a subgroup isomorphic to  $S_4$ . Since all involutions are conjugate, this gives a triangle on the classes of elements of orders 2, 3 and 4. The elements of order 7 are contained in subgroups of order 21, each meeting both conjugacy classes. So the two classes of order 7 and the class of order 3 form another triangle, with one vertex in common with the first; so  $\Gamma_{sc}(\text{PSL}(2, 7))$  is the “bowtie graph”, and is planar.

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