Rearrangement groups of connected spaces

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Abstract

We develop a combinatorial framework that assists in finding natural infinite "geometric" presentations for a large subclass of rearrangement groups of fractals – defined by Belk and Forrest [3], namely rearrangement groups acting on F-type topological spaces. In this framework, for a given fractal set with its group of "rearrangements", the group generators have a natural one-to-one correspondence with the standard basis of the fractal set, and the relations are all conjugacy relations.

We use this framework to produce a presentation for Richard Thompson's group F [9], [30]. This presentation has been mentioned before by Dehornoy [18], but a combinatorial method to find the length of an element in terms of the generating set of this presentation has been hitherto unknown. We provide algorithms that express an element of F in terms of our generating set and reduce a word representing the identity in F to the trivial word.

We conjecture that this framework can be used to find infinite presentations for all groups in the subclass of rearrangement groups acting on F-type topological spaces.

Acknowledgements

I would like to dedicate this thesis to my mother, Sabiha Khalid. She was a professor in pure mathematics, and my first mathematics teacher. She passed away on 1st January 2014, before I was admitted into this PhD programme. Everything I have accomplished is because of her.

I am eternally grateful to my supervisors, Collin Bleak and Martyn Quick, for their help, support and guidance at every step of this journey. I am grateful to the Commonwealth Scholarship Commission and my programme officer for their support. I am grateful to Jim Belk, Ewa Bieniecka, James Hyde and Michael Torpey for some mathematical and programming advice. I would also like to mention my lecturers and advisors from my undergraduate degree, Fazeel Anwar, Tazeen Athar and Sohail Iqbal. Without their encouragement and support, I would not have been able to apply for this PhD programme.

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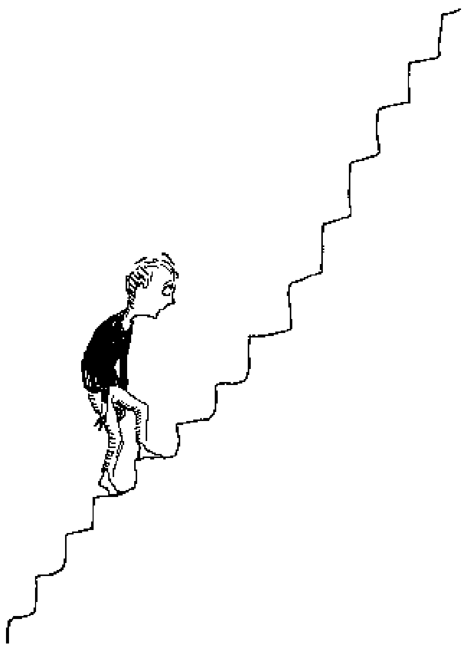
Contents

Chapter 1

Introduction



Up he went – very quickly at first – then more slowly – then in a little while even more slowly than that – and finally, after many minutes of climbing up the endless stairway, one weary foot was barely able to follow the other. Milo suddenly realized that with all his effort he was no closer to the top than when he began, and not a great deal further from the bottom. But he struggled on for a while longer, until at last, completely exhausted, he collapsed onto one of the steps.

"I should have known it," he mumbled, resting his tired legs and filling his lungs with air. "This is just like the line that goes on forever, and I'll never get there."

"You wouldn't like it much anyway," someone replied gently. "Infinity is a dreadfully poor place. They can never manage to make ends meet."

— Norton Juster, *The Phantom Tollbooth*
(Illustration by Jules Feiffer)

In this thesis, we are interested in how the topological properties of geometric spaces influence the dynamics and behaviour of the infinite groups which act on them. We have been studying groups of homeomorphisms of self-similar topological spaces – called *rearrangement groups of fractals*, introduced by James Belk and Bradley Forrest [3]. These groups include, but are not limited to, Richard Thompsons groups F, T and V (see [9], [30]).

We have developed a combinatorial framework that assists in finding natural infinite "geometric" presentations for a large subclass of rearrangement groups (rearrangement groups of F-type topological spaces, introduced in Chapter 2 and Chapter 3). In this framework, for a given fractal set with its group of "rearrangements", the group generators have a natural one-to-one correspondence with the standard basis of the fractal set, and the relations are all conjugacy relations.

As a test case for our approach, we have used the framework to produce a presentation for Richard Thompson's group F. As F is a well-studied group and our presentation is quite natural, it is unsurprising that the presentation that arose had been mentioned before in by Dehornoy in [18]. One of our key results was to provide an algorithm giving a normal form for elements of F (given as generic products in our generators) using our generating set. This was suggested as an interesting open problem in [18].

In [18], Dehornoy proved that the shortest length products in this generating set represent the shortest chain of "rotations" of rooted binary trees to get from one given tree to another, so, our algorithm may represent a new algorithm to solve this question originally posed by Thurston in [29]. Currently, there is no known algorithm solving the binary-tree rotation distance question that runs in faster than exponential time (on the size of the initial pair of trees). Our algorithm also runs in exponential time (a fact we have calculated but we have not provided a proof in this thesis). However, in all computed examples the exponential part of the algorithm always admitted easy simplifications; we have always been able to compute answers by hand with little effort. We hope in future work to decide if there is a variant of the algorithm that provably runs in polynomial time for all inputs, and to decide if our normal form really does provide shortest-length rotation sequences from one tree to another (we conjecture it does).

1.1 Richard Thompson's Groups

{1.1}

The groups F, T and V were first defined by Richard Thompson in an unpublished manuscript in 1965 [30]. They arise as subgroups of the homeomorphism group of the Cantor set. Indeed, these are groups of piecewise differentiable linear homeomorphisms of the unit interval $[0, 1]$, the unit circle and the Cantor set respectively. Thompson proved that T and V are finitely-presented infinite simple groups and F is a finitely-presented group with a simple commutator subgroup. Thompson's finite presentations have been reproduced by Cannon, Floyd and Parry [9] in their survey article. Thompson's groups have been extensively studied, and many infinite families of generalizations have been found. Some notable generalizations are:

1. The Stein-Thompson groups [28].
2. The Higman-Thompson groups $G_{n,r}$ [22], which generalize V.
3. The Röver group Γ [27], which is an amalgamation of V with the Grigorchuk group [21].
4. The Brin-Thompson group, or "higher-dimensional" nV [7].
5. Braided Thompson groups [8] [19].
6. Belk and Forrest's Basilica Thompson group [4] and rearrangement groups [3] [31].
7. The groups of piecewise-projective homeomorphisms [23], which generalize T and F.

Thompson's group F remains the most famous of these groups, both because of its connection with work on homotopy idempotents and because it is one of the most famous possible counter examples of the Von Neumann conjecture (the amenability of Thompson's group F remains an open question). It is defined as the group of orientation-preserving piecewise linear homeomorphisms of the unit interval $[0, 1]$, which are only non-differentiable at finitely many dyadic rationals, and at the periods of differentiability the derivatives are powers of 2. A finite presentation has been given in [9], and explicit combinatorial algorithms exist to compute the length of an element with respect to the finite generating set (see [2], [9]).

1.2 Rearrangement Groups of Fractals

{1.2}

In [3], Belk and Forrest defined *rearrangement groups of fractals* – groups of homeomorphisms of self-similar topological spaces. The topological spaces these groups act on are an infinite limit of finite directed graphs, constructed using edge replacement systems. Belk and Forrest have used this language of rearrangement groups to both develop presentations for some of these groups and study their finiteness properties, but they have not developed a systematic process.

We use this language of rearrangement groups to develop a combinatorial framework which we conjecture can be used to find infinite geometric presentations for the large subclass of rearrangement groups which act on F-type topological spaces. In this thesis, we apply this framework to generate an infinite geometric presentation for Thompson's group F. We have begun applying this framework to the rearrangement group of F-Basilica topological space (which is a running example in Chapter 2and Chapter 3) and have so far met no major obstructions.

1.3 An Infinite Geometric Presentation for Richard Thompson's Group F {1.3}

Our infinite geometric presentation for Thompson's group F is as follows:

$$F = \langle \mathcal{X} \mid \mathcal{R} \rangle.$$

The generating set $\mathcal{X}$ is

$$\mathcal{X} = \{ f_\alpha \mid \alpha \in \{0,1\}^* \},$$

where f_α acts as follows on points in $[0,1]$ with the prefix $\alpha = e_1 \dots e_n \in \{0,1\}^*$, and as the identity homeomorphism on the rest of the interval:

$$([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha = \begin{cases} [\alpha 0 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1}e_{n+2} = 00, \\ [\alpha 10 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1}e_{n+2} = 01, \\ [\alpha 11 e_{n+2} e_{n+3} \dots] & \text{if } e_{n+1} = 1. \end{cases}$$

This map is illustrated in the following diagram:

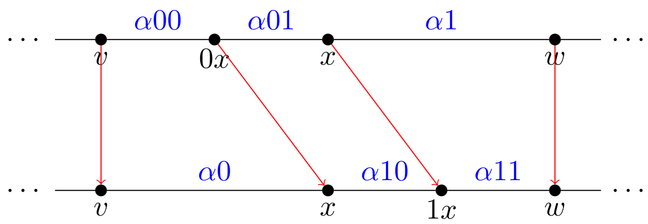


Figure 1.1: f_α

The set of relations $\mathcal{R}$ is

$$\begin{aligned} \mathcal{R} = \big\{ R1 : \quad & {f_\beta}^{f_\alpha} = f_\beta \text{ for } \alpha \perp \beta, \\ R2 : \quad & {f_{\alpha 0}}^{f_\alpha} = f_\alpha {f_{\alpha 1}}^{-1}, \\ R3 : \quad & {f_{\alpha 00\gamma}}^{f_\alpha} = f_{\alpha 0\gamma}, \\ R4 : \quad & {f_{\alpha 01\gamma}}^{f_\alpha} = f_{\alpha 10\gamma}, \\ R5 : \quad & {f_{\alpha 1\gamma}}^{f_\alpha} = f_{\alpha 11\gamma} \big\}, \end{aligned}$$

(for some $\alpha, \beta, \gamma \in \{0,1\}^*$).

{1.4}

1.4 Final Remarks

In conclusion, we would like to draw attention to two main threads in this thesis:

1. The development of a combinatorial framework which, while it was used to find an infinite geometric presentation for Thompson's group F in this thesis, can be generalised to other rearrangement groups which act on F-type topological spaces.

2. The development of a combinatorial algorithm to find the "normal form" of an element in terms of our infinite generating set, which can provide useful information regarding the rotation distance between two binary rooted trees.

Remark. We assume that the reader has some basic knowledge of combinatorial group theory, point-set topology and graph theory.

Chapter 2

The Limit Space

{basics1}

James Belk and Bradley Forrest defined a *rearrangement group* — a group of homeomorphisms of a self-similar topological space (called the limit space) — in [3]. In this chapter, we will construct this limit space by carrying out an iterative process on a sequence of graphs. We will describe the graphs in Section 2.1, we will present an extended example in Section 2.2, and we will define the limit space in Section 2.3.

2.1 Edge Replacement Systems

{2.1}

Each rearrangement group acts on its own topological space. This topological space — the *limit space* — is the limit of a sequence of graphs constructed using an iterative edge replacement process. In this section, we describe this edge replacement process in detail and construct a sequence of graphs. We will also establish our naming conventions for edges and vertices.

Definition 2.1.1 (Belk, Forrest [3], Definition 1.1 & Definition 1.4)**.** An *edge replacement system* $(G_0, \mathbf{e} \to R)$ consists of the following two things: a finite directed graph G_0 called the *base graph*, and an *edge replacement rule* $\mathbf{e} \to R$ where an edge $\mathbf{e}$ is replaced by a *replacement graph* R (where R is a finite directed graph with specified initial and terminal vertices).

In this thesis, we will be using the word "graph" synonymously with the word "digraph, as all graphs arising will be directed. We are interested in examples of edge replacement systems (and full expansion sequences constructed using them) which satisfy certain properties, outlined in the definitions below:

{dontfearthere

Definition 2.1.2 (Belk, Forrest [3], Definition 1.8)**.** An edge replacement system $(G_0, \mathbf{e} \to R)$ is *expanding* if it satisfies the following conditions:

1. Neither G_0 nor R have any isolated vertices.

2. The initial and terminal vertices of R are not connected by an edge.

3. The replacement graph R has at least three vertices and two edges.

Definition 2.1.3. An edge replacement system $(G_0, \mathbf{e} \to R)$ is *connected* if the graphs G_0 and R are both connected. (Observe that a connected graph need not be strongly connected.)

Recall that the *degree* of a vertex in a graph is the number of edges incident with it. The following definition is a specialization of the definition of "finite branching" presented in Section 4 of Belk, Forrest [3]:

Definition 2.1.4. An edge replacement system $(G_0, \mathbf{e} \to R)$ has *finite branching* if the initial and terminal vertices of R have degree 1.

Definition 2.1.5. An edge replacement system $(G_0, \mathbf{e} \to R)$ with finite branching is *oriented* if the initial vertex of R is also the initial vertex of the edge incident with it, and the terminal vertex of R is also the terminal vertex of the edge incident with it.

Recall that an automorphism of a graph G is a graph isomorphism with itself, i.e., a mapping from the vertices of G back to vertices of G such that the resulting graph is isomorphic with G. The group of all automorphisms of G is called the automorphism group of G and denoted by $\mathrm{Aut}(G)$.

{samstown}

Definition 2.1.6. We define an edge replacement system $(G_0, \mathbf{e} \to R)$ to be of *F-type* if the following hold:

1. The edge replacement system is expanding, connected, oriented and has finite branching.

2. The graph G_0 is a single edge connecting two distinct vertices.

3. The automorphism group $\mathrm{Aut}(R)$ of the graph R is trivial.

{allthatsmine}

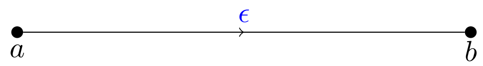
Remark 2.1.7. Let $(G_0, \mathbf{e} \to R)$ be an F-type edge replacement system. Let $e, f \in E(R)$ and $v, w \in V(R)$ such that v is the initial vertex and w is the terminal vertex of e in R and v is the initial vertex and w is the terminal vertex of f in R. Then $e = f$, otherwise there exists a non-trivial automorphism of R which transposes e and f.

For the rest of this thesis, we will assume that every edge replacement system is of F-type.

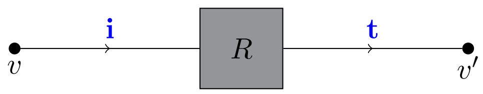
Most of the results we will prove in the following chapters can be generalized to non F-type groups, but we will leave that to the reader. Some further discussion can also be found in [3]. In the next few pages we will present a detailed exposition concerning F-type edge replacement systems and describe associated notation and constructions, which we will illustrate by an extended example in Section 2.2.

Let G be an arbitrary graph. We denote the set of edges of G by $E(G)$ and the set of vertices of G by $V(G)$. Let us fix a replacement graph R through the rest of the section and consider the edge replacement rule $\mathbf{e} \to R$. We shall write $V_{\text{int}}(R)$ for the set of vertices of R that are neither initial nor terminal.

Let us establish some naming conventions for our edge replacement system $(G_0, \mathbf{e} \to R)$. Let us label the edge in G_0 by the empty word ϵ, with the initial vertex a and terminal vertex b. Let us also label the edge incident with the initial vertex in R by $\mathbf{i}$ and the edge incident with the terminal vertex in R by $\mathbf{t}$.



G_0



R

{celestial}

Definition 2.1.8. Let α be an edge in a graph G with an initial vertex v_α and a terminal vertex w_α. Then α is a *loop* if and only if $v_\alpha = w_\alpha$.

{symbola}

Definition 2.1.9. We define a *simple expansion* G' of an arbitrary graph G by an F-type edge replacement rule $\mathbf{e} \to R$ as follows: Let G be a graph. Choose one edge of G, labeled α, and construct a new graph G' by replacing this edge by a copy of R as follows:

1. The set $V(G')$ of vertices of G' is the disjoint union of the set $V(G)$ of vertices of G and the set of new vertices $\{\alpha w \mid w \in V_{\text{int}}(R)\}$, i.e.,

$$V(G') = V(G) \cup \{\alpha w \mid w \in V_{\text{int}}(R)\}.$$

2. The set $E(G')$ of edges of G' is the disjoint union of the set $E(G)\backslash\{\alpha\}$ of the edges of G except for α and the set of new edges $\{\alpha e \mid e \in E(R)\}$, i.e.,

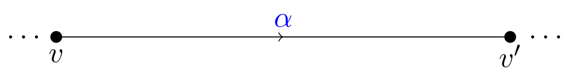
$$E(G') = E(G)\backslash\{\alpha\} \cup \{\alpha e \mid e \in E(R)\}.$$

3. Suppose the edge α leaves the vertex v and arrives to the vertex v' in G and the edge e leaves the vertex w and arrives to the vertex w' in R. The new edge αe joins vertices of G' as follows:

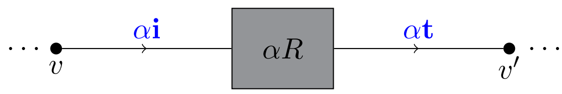
   (a) If $e = \mathbf{i}$, the initial vertex is v and the terminal vertex is $\alpha w'$.

   (b) If $e = \mathbf{t}$, the initial vertex is αw and the terminal vertex is v'.

   (c) The initial vertex is αw and the terminal vertex is $\alpha w'$ otherwise.



G



G'

This process can be repeated an arbitrary number of times using the same edge replacement rule. Any graph G constructed by performing a finite number of simple expansions to G_0 is called an *expansion* of G_0.

{deceiver}

Definition 2.1.10. We define a *full expansion* G_n of a graph G_{n-1} by an edge replacement rule $\mathbf{e} \to R$ as follows: Suppose we have defined the graph G_{n-1}. The full expansion of G_{n-1} is the graph G_n obtained by replacing each edge of G_{n-1} by the replacement graph R by the above process. The resulting graph then has the following vertices and edges:

1. The set $V(G_n)$ of vertices of G_n is the disjoint union of the set $V(G_{n-1})$ of vertices of G_{n-1} with the set of new vertices $V_n = \{\alpha w \mid \alpha \in E(G_{n-1}), w \in V_{\text{int}}(R)\}$, i.e.,

$$V(G_n) = V(G_{n-1}) \cup V_n.$$

   Formally, a new vertex in V_n can be identified with an ordered pair (α, w), where $\alpha \in E(G_{n-1})$ and $w \in V_{\text{int}}(R)$. However, to simplify notation, we follow the convention that we denote this new vertex by the symbol αw obtained by the juxtaposition of α and w.

2. The set $E(G_n)$ of edges of G_n is a one-one correspondence with the Cartesian product $E(G_{n-1}) \times E(R)$, i.e.,

$$E(G_n) = \{\alpha e \mid \alpha \in E(G_{n-1}), e \in E(R)\}.$$

   Again, an edge αe is an ordered pair (α, e), denoted via juxtaposition.

3. Suppose the edge α leaves the vertex v and arrives to the vertex v' in G_{n-1} and the edge e leaves the vertex w and arrives to the vertex w' in R. The new edge αe joins vertices of G_n as follows:

   (a) If $e = \mathbf{i}$, the initial vertex is v and the terminal vertex is $\alpha w'$.

   (b) If $e = \mathbf{t}$, the initial vertex is αw and the terminal vertex is v'.

   (c) The initial vertex is αw and the terminal vertex is $\alpha w'$ otherwise.

{hatari}

Lemma 2.1.11. *Let $(G_0, \mathbf{e} \to R)$ be an F-type edge replacement system. Let G be an expansion of G_0 by R, i.e., it is obtained from G_0 by applying the edge replacement rule $\mathbf{e} \to R$ a finite number of times. Let $e, f \in E(G)$ and $v, w \in V(G)$ such that v is the initial vertex and w is the terminal vertex of e in G and v is the initial vertex and w is the terminal vertex of f in G. Then $e = f$.*

Proof. Let $(G_0, \mathbf{e} \to R)$ be an F-type edge replacement system. Let G be an expansion, i.e., it is obtained from G_0 by applying the edge replacement rule $\mathbf{e} \to R$ a finite number of times. Let $e, f \in E(G)$ and $v, w \in V(G)$ such that v is the initial vertex and w is the terminal vertex of e in G and v is the initial vertex and w is the terminal vertex of f in G. Observe, by Definition 2.1.9 (3), that e and f share the save initial and terminal vertices if and only if one of the following is true:

1. $e, f \in E(G_0)$, in which case $e = f = \epsilon$.
2. There exist $e', f' \in E(R)$ such that $e = \alpha e'$ and $f = \alpha f'$ (where α is the edge which was replaced). In this case, there exist $v', w' \in V(R)$ such that $v = \alpha v'$ and $w = \alpha w'$. Then, by Remark 2.1.7, $e' = f'$. Hence $e = f$.

This proves the result. □

For an arbitrary edge replacement system $(G_0, \mathbf{e} \to R)$, the sequence $\{G_n\}_{n=0}^{\infty}$ is called the *full expansion sequence*, where G_0 is the base graph and each graph G_n is the full expansion of the graph G_{n-1} by the edge replacement rule $\mathbf{e} \to R$. We build our notation for the edges and vertices of the graph G_n in an arbitrary full expansion sequence as follows:

1. The set $E(G_n)$ of edges of G_n is the Cartesian product (with elements denoted via juxtaposition):

$$\begin{aligned} E(G_n) &= E(G_{n-1}) \times E(R) \\ &= E(G_0) \times E(R)^n \\ &= \{\alpha = e_1 \dots e_n \mid e_i \in E(R) \text{ for } i \geq 1\}, \end{aligned}$$

   Observe that each edge $\alpha = e_1 \dots e_n$ of G_n is an $(n+1)$-tuple, expressed as a word of length $n+1$. Observe also that $\alpha^\dagger = e_1 \dots e_{n-1}$ is the edge of G_{n-1} that was replaced by our edge replacement rule $\mathbf{e} \to R$.

2. The set V_n of new vertices introduced in the graph G_n is the set of $(n+1)$-tuples (denoted via juxtaposition, and expressed as words of length $n+1$):

$$\begin{aligned} V_n &= \{\alpha^\dagger w = e_1 \dots e_{n-1} w \mid \alpha^\dagger \in E(G_{n-1}), w \in V_{\text{int}}(R)\} \\ &= \{\alpha^\dagger w = e_1 \dots e_{n-1} w \mid e_i \in E(R) \text{ for } i \geq 1, w \in V_{\text{int}}(R)\}. \end{aligned}$$

   We define $V_0 = V(G_0)$. The complete set $V(G_n)$ of vertices of the graph G_n is the disjoint union

$$V(G_n) = \bigsqcup_{k=0}^{n} V_k.$$

Definition 2.1.12. Observe that we have defined above the *truncation function* † as the operation truncating the last letter of a word. For instance, consider a word $\alpha = e_1 \dots e_n$, then $\alpha^\dagger = e_1 \dots e_{n-1}$ and $\alpha^{(n-k)\dagger} = e_1 \dots e_k$ (for $n > k$). (Observe that we are performing iterated truncation from the right.)

{andone}

Definition 2.1.13. We define the depth of a vertex $v \in V_n$ to be the index of the set V_n (denoted by $\mathrm{depth}(v) = n$). Observe that if $v = \alpha w$ (for some $w \in V_{\mathrm{int}}(R)$) then $\mathrm{depth}(v) = |\alpha|$.

The following results characterize the adjacency of edges and vertices of a graph G_n in the full expansion sequence $\{G_n\}_{n=0}^{\infty}$ of an F-type edge replacement system $(G_0, \mathbf{e} \to R)$:

{gusgus}

Lemma 2.1.14. *Let $\{G_n\}_{n=0}^{\infty}$ be the full expansion sequence of an F-type edge replacement system $(G_0, \mathbf{e} \to R)$. Let $\alpha = e_1 \ldots e_n \in E(G_n)$ (for some $n \geq 1$). The initial and terminal vertices of α in the graph G_n are one of the following:*

1. *If $\alpha = (\mathbf{i})^n$, the initial vertex is a and the terminal vertex is $\alpha^{\dagger} w$ (where w is the terminal vertex of the edge $\mathbf{i}$ in R).*

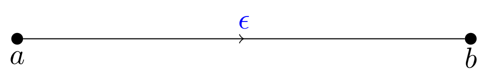
2. *If $\alpha = e_1 \ldots e_{k+1}(\mathbf{i})^{n-k-1}$ (where $e_{k+1} \neq \mathbf{i}$ for some $1 \leq k \leq n-1$), the initial vertex is $\alpha^{(n-k)\dagger} w_1$ and the terminal vertex is $\alpha^{\dagger} w_2$ (where w_1 is the initial vertex of the edge e_{k+1} in R and w_2 is the terminal vertex of the edge $\mathbf{i}$ in R).*

3. *If $\alpha = e_1 \ldots e_{k+1}(\mathbf{t})^{n-k-1}$ (where $e_{k+1} \neq \mathbf{t}$ for some $1 \leq k \leq n-1$), the initial vertex is $\alpha^{\dagger} w_1$ and the terminal vertex is $\alpha^{(n-k)\dagger} w_2$ (where w_1 is the initial vertex of the edge $\mathbf{t}$ in R and w_2 is the terminal vertex of the edge e_{k+1} in R).*

4. *If $\alpha = (\mathbf{t})^n$, the initial vertex is $\alpha^{\dagger} w$ and the terminal vertex is b (where w is the initial vertex of the edge $\mathbf{t}$ in R).*

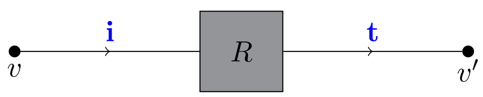
5. *The initial vertex is $\alpha^{\dagger} w_1$ and the terminal vertex is $\alpha^{\dagger} w_2$ otherwise (where w_1 is the initial vertex and w_2 is the terminal vertex of the edge e_n in R).*

Proof. Let $\{G_n\}_{n=0}^{\infty}$ be the full expansion sequence of an F-type edge replacement system $(G_0, \mathbf{e} \to R)$. Recall that, in an F-type edge replacement system, the graph G_0 is a single edge, labelled by the empty word ϵ, with an initial vertex a and a terminal vertex b.



G_0

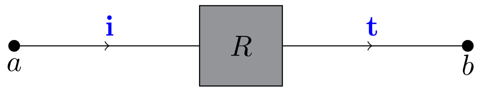
The replacement graph R is such that the automorphism group $\mathrm{Aut}(R)$ is trivial. We shall label the edge incident with the initial vertex of R by $\mathbf{i}$ and the edge incident with the terminal vertex of R by $\mathbf{t}$.



R

We will prove this result by induction on n.

In the construction of the graph G_1, the edge ϵ is replaced with a copy of R.



G_1

An edge α in the graph G_1 is of the form $\alpha = e_1$ where $e_1 \in E(R)$. Let w_1 be the initial vertex and w_2 be the terminal vertex of the edge e_1 in R. Then, by Definition 2.1.10, the initial and terminal vertices of the edge α in the graph G_1 are one of the following:

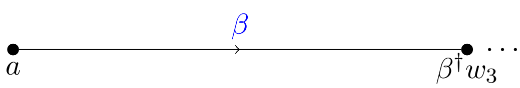
1. If $\alpha = \mathbf{i}$, the initial vertex is a and the terminal vertex is w_2, which satisfies Case 1 of the result.

2. If $\alpha = \mathbf{t}$, the initial vertex is w_1 and the terminal vertex is b, which satisfies Case 4 of the result.

3. The initial vertex is w_1 and the terminal vertex is w_2 otherwise, which satisfies Case 5 of the result.

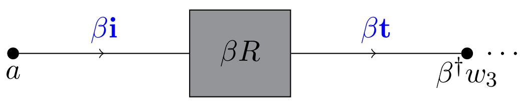
Suppose that there exists $m \geq 1$ such that the result holds for the graph G_m. Let us examine the graph G_{m+1}. Consider the edge $\beta = e_1 \dots e_m \in E(G_m)$. The initial and terminal vertices of β are as per the hypothesis. In the construction of the graph G_{m+1}, the edge β is replaced with a copy of R. Consider the edge $\alpha = \beta e_{m+1}$ in the graph G_{m+1}, where $e_{m+1} \in E(R)$. Let w_1 be the initial vertex and w_2 be the terminal vertex of the edge e_{m+1} in R. Let us examine case by case the initial and terminal vertices of the edge α in the graph G_{m+1}, given the initial and terminal vertices of the edge β in the graph G_m:

1. Let $\beta = (\mathbf{i})^m$ with initial vertex a and terminal vertex $\beta^\dagger w_3$ (where w_3 is the terminal vertex of the edge $\mathbf{i}$ in R) in G_m.

   G_m

   Then, by Definition 2.1.10, the initial and terminal vertices of α in G_{m+1} are as follows:

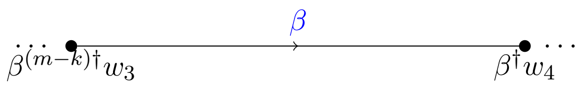
   G_{m+1}

1.1. If $\alpha = \beta\mathbf{i} = (\mathbf{i})^{m+1}$, the initial vertex is a and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$, which satisfies Case 1 of the result.

1.2. If $\alpha = \beta\mathbf{t} = (\mathbf{i})^{m}\mathbf{t}$, the initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{2\dagger}w_3 = \beta^{\dagger}w_3$, which satisfies Case 3 of the result.

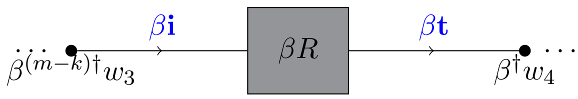
1.3. The initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$ otherwise, which satisfies Case 5 of the result.

2. Let $\beta = e_1 \dots e_{k+1}(\mathbf{i})^{m-k-1}$ (where $e_{k+1} \neq \mathbf{i}$) with initial vertex $\beta^{(m-k)\dagger}w_3$ and the terminal vertex is $\beta^{\dagger}w_4$ (where w_3 is the initial vertex of the edge e_{k+1} in R and w_4 is the terminal vertex of the edge $\mathbf{i}$ in R) in G_m.



G_m

Then, by Definition 2.1.10, the initial and terminal vertices of α in G_{m+1} are as follows:



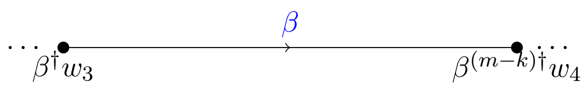
G_{m+1}

2.1. If $\alpha = \beta\mathbf{i} = e_1 \dots e_{k+1}(\mathbf{i})^{m-k}$, the initial vertex is $\alpha^{(m-k+1)\dagger}w_3 = \beta^{(m-k)\dagger}w_3$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$, which satisfies Case 2 of the result.

2.2. If $\alpha = \beta\mathbf{t} = e_1 \dots e_{k+1}(\mathbf{i})^{m-k-1}\mathbf{t}$, the initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{2\dagger}w_4 = \beta^{\dagger}w_4$, which satisfies Case 3 of the result.

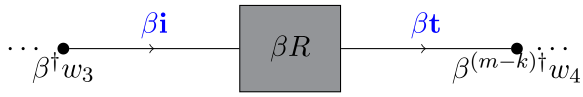
2.3. The initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$ otherwise, which satisfies Case 5 of the result.

3. Let $\beta = e_1 \dots e_{k+1}(\mathbf{t})^{m-k-1}$ (where $e_{k+1} \neq \mathbf{t}$) with initial vertex $\beta^{\dagger}w_3$ and terminal vertex $\beta^{(m-k)\dagger}w_4$ (where w_3 is the initial vertex of the edge $\mathbf{t}$ in R and w_4 is the terminal vertex of the edge e_{k+1} in R) in G_m.



G_m

Then, by Definition 2.1.10, the initial and terminal vertices of α in G_{m+1} are as follows:



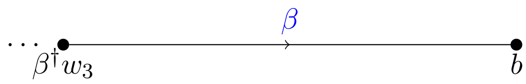
G_{m+1}

3.1. If $\alpha = \beta\mathbf{i} = e_1 \dots e_{k+1}(\mathbf{t})^{m-k-1}\mathbf{i}$, the initial vertex is $\alpha^{2\dagger}w_3 = \beta^{\dagger}w_3$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$, which satisfies Case 2 of the result.

3.2. If $\alpha = \beta\mathbf{t} = e_1 \dots e_{k+1}(\mathbf{t})^{m-k}$, the initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{(m-k+1)\dagger}w_4 = \beta^{(m-k)\dagger}w_4$, which satisfies Case 3 of the result.

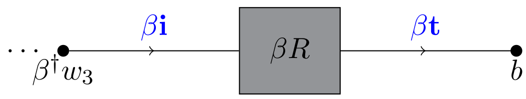
3.3. The initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$ otherwise, which satisfies Case 5 of the result.

4. Let $\beta = (\mathbf{t})^m$ with initial vertex $\beta^{\dagger}w_3$ and terminal vertex b (where w_3 is the initial vertex of the edge $\mathbf{t}$ in R) in G_m.



G_m

Then, by Definition 2.1.10, the initial and terminal vertices of α in G_{m+1} are as follows:



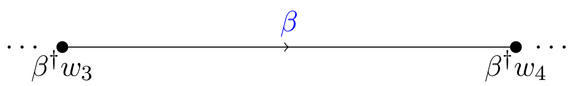
G_{m+1}

4.1. If $\alpha = \beta\mathbf{i} = (\mathbf{t})^m\mathbf{i}$, the initial vertex is $\alpha^{2\dagger}w_3 = \beta^{\dagger}w_3$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$, which satisfies Case 2 of the result.

4.2. If $\alpha = \beta\mathbf{t} = (\mathbf{t})^{m+1}$, the initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is b, which satisfies Case 4 of the result.

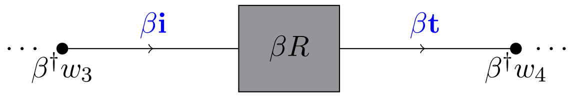
4.3. The initial vertex is $\alpha^{\dagger}w_1 = \beta w_1$ and the terminal vertex is $\alpha^{\dagger}w_2 = \beta w_2$ otherwise, which satisfies Case 5 of the result.

5. Otherwise, let β have the initial vertex $\beta^{\dagger}w_3$ and terminal vertex $\beta^{\dagger}w_4$ (where w_3 is the initial vertex and w_4 is the terminal vertex of the edge e_n in R) in G_m.



G_m

Then, by Definition 2.1.10, the initial and terminal vertices of α in G_{m+1} are as follows:



G_{m+1}

5.1. If $\alpha = \beta\mathbf{i}$, the initial vertex is $\alpha^{2\dagger}w_3 = \beta^\dagger w_3$ and the terminal vertex is $\alpha^\dagger w_2 = \beta w_2$, which satisfies Case 2 of the result.

5.2. If $\alpha = \beta\mathbf{t}$, the initial vertex is $\alpha^\dagger w_1 = \beta w_1$ and the terminal vertex is $\alpha^{2\dagger}w_4 = \beta^\dagger w_4$, which satisfies Case 3 of the result.

5.3. The initial vertex is $\alpha^\dagger w_1 = \beta w_1$ and the terminal vertex is $\alpha^\dagger w_2 = \beta w_2$ otherwise, which satisfies Case 5 of the result.

This proves the result by induction. □

{keinelust}

Lemma 2.1.15. *Let $\{G_n\}_{n=0}^\infty$ be the full expansion sequence of an F-type edge replacement system $(G_0, \mathbf{e} \to R)$. Let $\alpha = e_1 \dots e_k \in E(G_k)$, $w \in V_{\text{int}}(R)$ and $n \geq k+1$. Then*

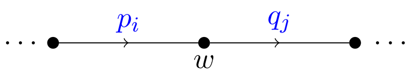
1. *The edges in G_n having αw as the initial vertex are precisely those of the form $\alpha p(\mathbf{i})^{n-k-1}$ where p is an edge of R with w as the initial vertex.*

2. *The edges in G_n having αw as the terminal vertex are precisely those of the form $\alpha q(\mathbf{t})^{n-k-1}$ where q is an edge of R with w as the terminal vertex.*

Proof. Let $\{G_n\}_{n=0}^\infty$ be the full expansion sequence of an F-type edge replacement system $(G_0, \mathbf{e} \to R)$. Let $\alpha = e_1 \dots e_k \in E(G_k)$, $w \in V_{\text{int}}(R)$ and $n \geq k+1$. Recall from Definition 2.1.10 of a full expansion that $\alpha w \in V(G_n)$.

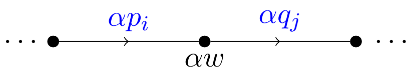
1. By Lemma 2.1.14 (2), αw is the initial vertex of edges of the form $\alpha p(\mathbf{i})^{n-k-1}$ in G_n, where p is an edge of R with w as the initial vertex.. We have to show that the edges of the form $\alpha p(\mathbf{i})^{n-k-1}$ are the only edges with αw as the initial vertex. We shall prove this by induction on n.

   Suppose w is the initial vertex of precisely the edges $p_1, \dots, p_r$ in R.

   R

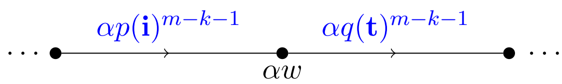
   Consider the case when $n = k+1$. The graph G_{k+1} is a full expansion of the graph G_k by Definition 2.1.10. The vertex $\alpha w \in V_{k+1}$ is a new vertex introduced in G_{k+1} when the edge α in G_k is replaced by a copy of R. It follows from the construction of G_{k+1} that αw is the initial vertex of precisely the edges $\alpha p_1, \dots, \alpha p_r$ in G_{k+1}.



G_{k+1}

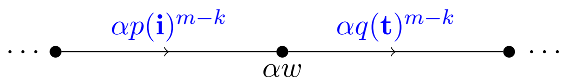
This establishes the base case of our induction argument.

Now let us assume that αw is the initial vertex of precisely the edges of the form $\alpha p(\mathbf{i})^{m-k-1}$ in the graph G_m (for some $m \geq k+1$), where p is an edge of R with w as the initial vertex.



G_m

Let us construct the full expansion G_{m+1} of G_m. Observe that, since the edge replacement system is of F-type, when an edge $\alpha p(\mathbf{i})^{m-k-1}$ gets replaced by a copy of R, the vertex αw is the initial vertex of this copy of R and hence the initial vertex of an edge $\alpha p(\mathbf{i})^{m-k}$.



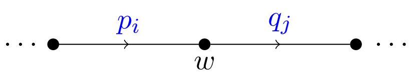
G_{m+1}

Therefore, the inductive claim holds.

2. This is proved similarly to (1). The proof is included here for completeness.

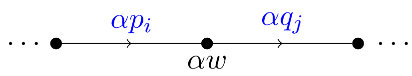
   By Lemma 2.1.14 (3), αw is the terminal vertex of edges of the form $\alpha q(\mathbf{t})^{n-k-1}$ in G_n, where q is an edge of R with w as the terminal vertex. We have to show that the edges of the form $\alpha q(\mathbf{t})^{n-k-1}$ are the only edges with αw as the terminal vertex. We shall prove this by induction on n.

   Suppose w is the terminal vertex of precisely the edges $q_1, \ldots, q_s$ in R.



R

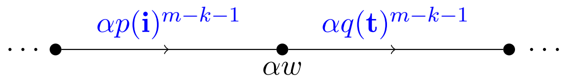
   Consider the case when $n = k+1$. The graph G_{k+1} is a full expansion of the graph G_k by Definition 2.1.10. The vertex $\alpha w \in V_{k+1}$ is a new vertex introduced in G_{k+1} when the edge α in G_k is replaced by a copy of R. It follows from the construction of G_{k+1} that αw is the terminal vertex of precisely the edges $\alpha q_1, \ldots, \alpha q_s$ in G_{k+1}.



G_{k+1}

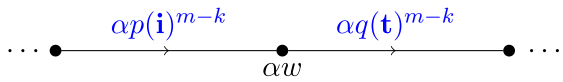
This establishes the base case of our induction argument.

Now let us assume that αw is the terminal vertex of edges of the form $\alpha q(\mathbf{t})^{m-k-1}$ in the graph G_m (for some $m \geq k+1$), where q is an edge of R with w as the terminal vertex.



G_m

Let us construct the full expansion G_{m+1} of G_m. Observe that, since the edge replacement system is of F-type, when an edge $\alpha q(\mathbf{t})^{m-k-1}$ gets replaced by a copy of R, the vertex αw is the terminal vertex of this copy of R and hence the terminal vertex of an edge $\alpha q(\mathbf{t})^{m-k}$.



G_{m+1}

Therefore, the inductive claim holds.

This completes the proof by induction. □

We can prove stronger versions of Lemma 2.1.15 and Lemma 2.1.14 for every specific rearrangement group. We will prove the analogous results for some specific rearrangement groups in later chapters.

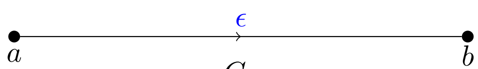
2.2 An Extended Example

{2.2}

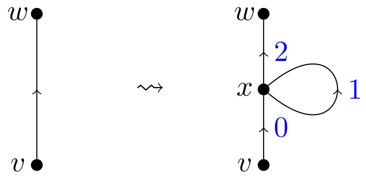
We now present an extended example which defines the edge replacement system and constructs the full expansion sequence for a particular rearrangement group, the F-*Basilica group*, and discusses the adjacency of edges and vertices for this full expansion sequence analogously to Lemma 2.1.15 and Lemma 2.1.14. We will be using this edge replacement system (and associated framework) as an example throughout this chapter and Chapter 3.

{hurtsomeone}

Example 2.2.1. We present the F-Basilica edge replacement system:



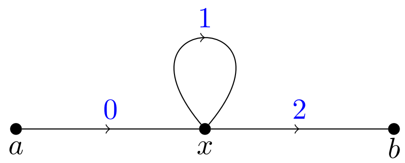
G_0



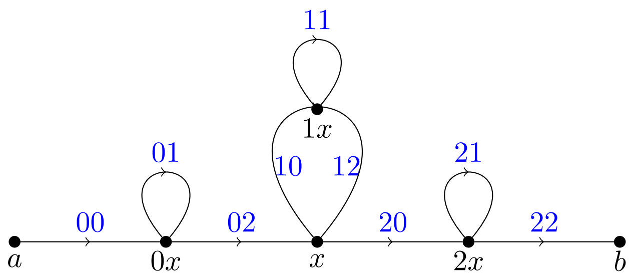
e $\quad$ R

We observe that the initial graph G_0 is comprised of one edge, labelled with the empty word ϵ, leaving a vertex a and arriving at a vertex b. Our replacement system replaces an edge $\mathbf{e}$ with the replacement graph R, which is comprised of one edge, labelled 0, leaving the specified initial vertex v and arriving to a vertex x, a second edge labelled 1 forming a loop around the vertex x, and a third edge, labelled 2, leaving the vertex x and arriving to the specified terminal vertex w. We observe that $E(G_0) = \{\epsilon\}$, $V(G_0) = \{a, b\}$, $E(R) = \{0, 1, 2\}$ and $V_{\text{int}}(R) = \{x\}$.

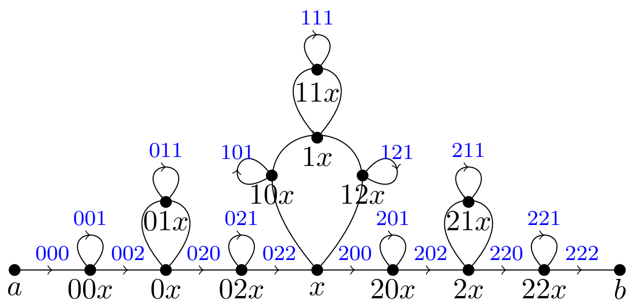
We present the first few graphs in the full expansion sequence for the F-Basilica replacement system:



G_1



G_2

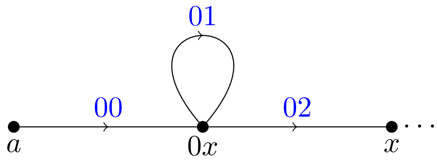


G_3

$$\vdots$$

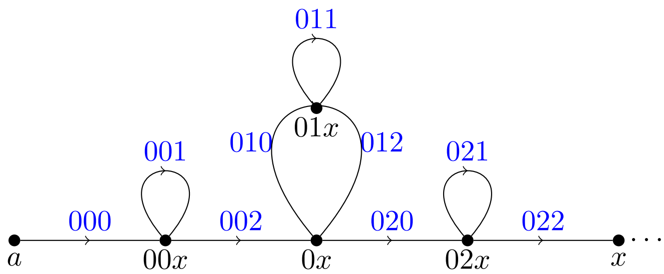
Observe that an edge of the form $\alpha 1$ in the graph G_n (for some $\alpha \in E(G_{n-1})$) is a loop for all $n \in \mathbb{N}$.

Let us discuss the adjacency of the vertex $0x$ in the first few graphs of the full expansion sequence for the F-Basilica replacement system:



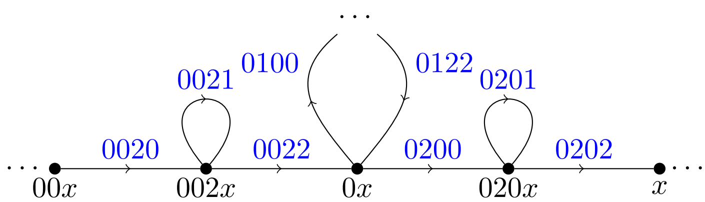
G_2

In the graph G_2, we observe that the edges departing from $0x$ are 01 and 02, and the edges arriving to $0x$ are 00 and 01.



G_3

In the graph G_3, we observe that the edges departing from $0x$ are 010 and 020, and the edges arriving to $0x$ are 002 and 012.



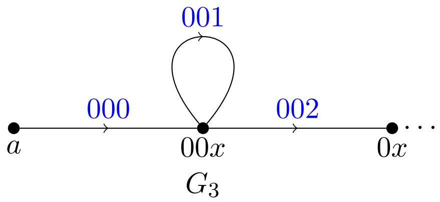
G_4

In the graph G_4, we observe that the edges departing from $0x$ are 0100 and 0200, and the edges arriving to $0x$ are 0022 and 0122. Continuing in this way we can conclude that, for the n-th full expansion graph G_n of the F-Basilica replacement system, the edges departing from $0x$ are $01(0)^{n-2}$ and $02(0)^{n-2}$, and the edges arriving to $0x$ are $00(2)^{n-2}$ and $01(2)^{n-2}$, all of which have the prefix 0.

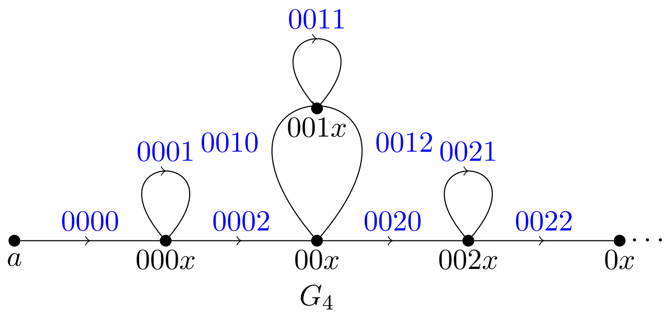
In fact we can show that, for the graph G_n of the F-Basilica replacement system, the edges departing from an arbitrary vertex αx are $\alpha 1(0)^{n-2}$ and $\alpha 2(0)^{n-2}$, and the edges arriving to αx are $\alpha 0(2)^{n-2}$ and $\alpha 1(2)^{n-2}$.

Let us now study the vertices bordering a specific edge α.

In the graph G_3, the edge 000 leaves the vertex a and arrives to the vertex $00x$.

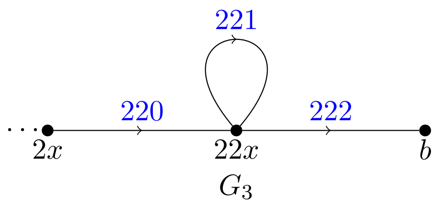


In the graph G_4, the edge 0000 leaves the vertex a and arrives to the vertex $000x$.

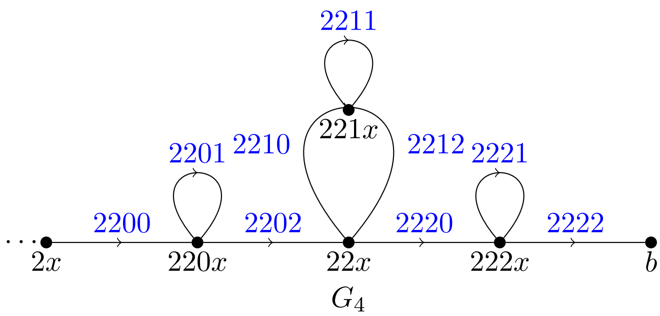


In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = (0)^n$, the initial vertex is a and the terminal vertex is $\alpha^\dagger x = (0)^{n-1}x$.

In the graph G_3, the edge 222 leaves the vertex $22x$ and arrives to the vertex b.

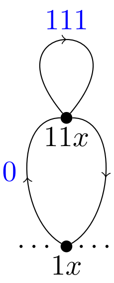


In the graph G_4, the edge 2222 leaves the vertex $222x$ and arrives to the vertex b.

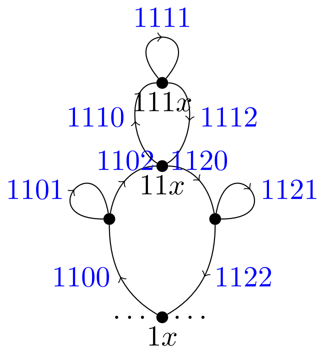


In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = (2)^n$, the initial vertex is $\alpha^\dagger x = (2)^{n-1}x$ and the terminal vertex is b.

In the graph G_3, the edge 111 leaves and arrives to the vertex $11x$.

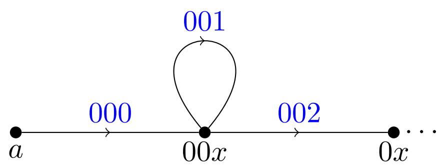


In the graph G_4, the edge 1111 leaves and arrives to the vertex $111x$.



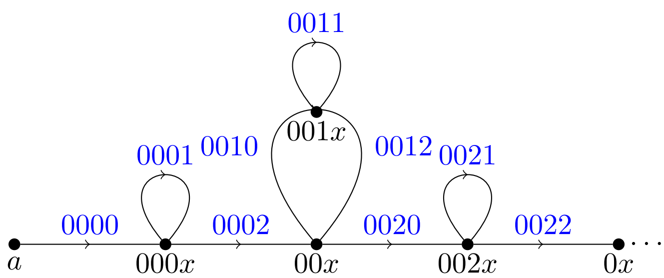
In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = e_1 \dots e_{n-1}1$, the initial and terminal vertex is $\alpha^\dagger x = e_1 \dots e_{n-1}x$.

In the graph G_3, the edge 002 leaves the vertex $00x$ and arrives to the vertex $0x$.



G_3

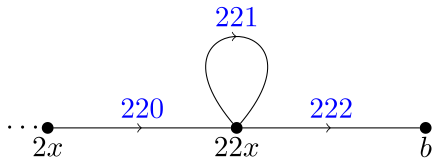
In the graph G_4, the edge 0022 leaves the vertex $002x$ and arrives to the vertex $0x$.



G_4

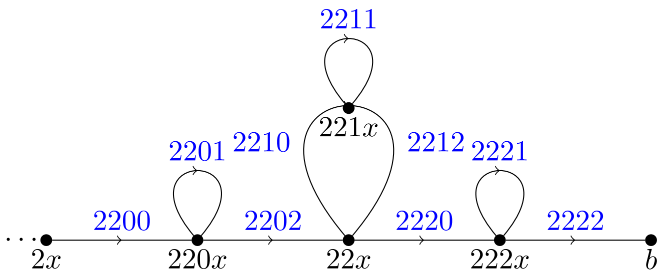
In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = e_1 \dots e_k 0(2)^{n-k-1}$ (for $n > k$), the initial vertex is $\alpha^\dagger x = e_1 \dots e_{n-1}x$ and the terminal vertex is $\alpha^{(n-k)\dagger}x = e_1 \dots e_k x$.

In the graph G_3, the edge 220 leaves the vertex $2x$ and arrives to the vertex $22x$.



G_3

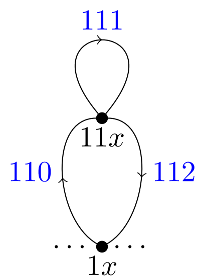
In the graph G_4, the edge 2200 leaves the vertex $2x$ and arrives to the vertex $220x$.



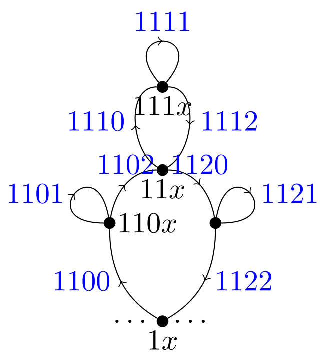
G_4

In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = e_1 \dots e_k 2(0)^{n-k-1}$ (for $n > k$), the initial vertex is $\alpha^{(n-k)\dagger}x = e_1 \dots e_k x$ and the terminal vertex is $\alpha^{\dagger}x = e_1 \dots e_{n-1}x$.

In the graph G_3, the edge 110 leaves the vertex $1x$ and arrives to the vertex $11x$.

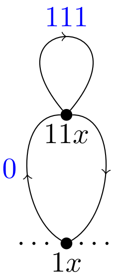


In the graph G_4, the edge 1100 leaves the vertex $1x$ and arrives to the vertex $110x$.

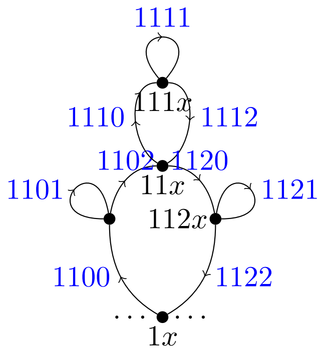


In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = e_1 \dots e_k 1(0)^{n-k-1}$ (for $n > k$), the initial vertex is $\alpha^{(n-k)\dagger}x = e_1 \dots e_k x$ and the terminal vertex is $\alpha^{\dagger}x = e_1 \dots e_{n-1}x$.

In the graph G_3, the edge 112 leaves the vertex $11x$ and arrives to the vertex $1x$.



In the graph G_4, the edge 1122 leaves the vertex $112x$ and arrives to the vertex $1x$.



In fact it follows from Lemma 2.1.14 that, in the graph G_n, when $\alpha = e_1 \dots e_k 1(2)^{n-k-1}$ (for $n > k$), the initial vertex is $\alpha^\dagger x = e_1 \dots e_{n-1} x$ and the terminal vertex is $\alpha^{(n-k)\dagger} x = e_1 \dots e_k x$.

2.3 The Limit Space

{2.3}

Let $(G_0, \mathbf{e} \to R)$ be an F-type edge replacement system. In this section, we will construct the limit space – the topological space on which our group will act.

Observe that $E(G_0) = \{\epsilon\}$ and $E(R)$ is a finite set. We define the *finite alphabet* to be the set $E(R)$.

{nocomprende}

Definition 2.3.1. We define the *set of finite words* $E(R)^*$ as follows:

$$E(R)^* = \{e_1 \dots e_n \mid e_i \in E(R) \text{ for } i = 1, \dots, n\}.$$

We denote the length of a word $\alpha \in E(R)^*$ by $|\alpha|$.

{redcow}

Definition 2.3.2 (Belk, Forrest [3])**.** We define the *symbol space* Ω to be the set $E(R)^\omega$ of all infinite sequences

$$\Omega := E(R)^\omega = \{e_1 e_2 \dots \mid e_i \in E(R) \text{ for } i = 1, 2, \dots\}.$$

Definition 2.3.3. We define a *prefix order* $\preceq$ on $E(R)^*$ as follows: For $\alpha, \beta \in E(R)^*$, α is a prefix of β, denoted by $\alpha \preceq \beta$, if there exists $\gamma \in E(R)^*$ such that $\beta = \alpha\gamma$.

Let $\alpha, \beta \in E(R)^*$. If α is a strict prefix of β, it is denoted by $\alpha \prec \beta$. If neither α or β are prefixes of the other they are said to be incomparable, denoted by $\alpha \perp \beta$. Given $\alpha \perp \beta$, there exists a *largest common prefix* $\gamma \in E(R)^*$ defined as follows: γ is the largest word such that $\gamma \prec \alpha$ and $\gamma \prec \beta$. We can extend this definition to Ω as follows: A word $\alpha \in E(R)^*$ is a prefix of a sequence $e_1e_2\ldots \in \Omega$ if and only if there exists an $n \in \mathbb{N}$ such that $\alpha = e_1 \ldots e_n$.

Observe that the prefix order on $E(R)^*$ is a partial order, since it is reflexive, transitive and anti-symmetric. A finite set $A \subset E(R)^*$ is an *antichain* if, for all $\alpha, \beta \in A$, $\alpha \perp \beta$. An antichain A is *complete* if, for all $\gamma \in E(R)^*$ such that $|\gamma| \geq N$ for some $N \in \mathbb{N}$, there exists $\alpha \in A$ such that $\alpha \preceq \gamma$. Equivalently, an antichain A is *complete* if, for all $t = e_1e_2\ldots \in \Omega$, there exists $\alpha \in A$ such that α is a prefix of t (denoted by $\alpha \prec t$). **Note that, in this thesis, we use the symbol $\subset$ to denote "contained in but not equal to" and the symbol $\subseteq$ to denote "contained in or equal to".**

Assume that we have defined a linear order $\leq$ on $E(R)$ such that $\mathbf{i}$ is the smallest and $\mathbf{t}$ is the largest. We define the *lexicographic order* $\leq_\ell$ on Ω as follows:

{bridgeburner}

Definition 2.3.4. Let $e_1e_2\ldots$ and $e_1'e_2'\ldots$ be sequences in Ω. Then $e_1e_2\cdots <_\ell e_1'e_2'\ldots$ if and only if there exists $k \in \mathbb{N}$ such that $e_1\ldots e_{k-1} = e_1'\ldots e_{k-1}'$ and $e_k < e_k'$.

We define the *partial lexicographic order* $\leq_\ell$ on $E(R)^*$ as follows:

{morningstar}

Definition 2.3.5. We define the *lexicographic order* $\leq_\ell$ on any two distinct incomparable words α and β as follows: Let $\alpha = e_1\ldots e_n$ and $\beta = e_1'\ldots e_m'$. Then there exists $k \in \mathbb{N}$ such that $e_1\ldots e_{k-1} = e_1'\ldots e_{k-1}'$ and $e_k \neq e_k'$. We define $\alpha <_\ell \beta$ if and only if $e_k < e_k'$.

While this order is a partial order on $E(R)^*$, it is a full order on a complete antichain $A \subset E(R)^*$, and is denoted by the ordered list $\mathrm{lex}(A) = (\alpha_1, \ldots, \alpha_n)$. The following results characterize some properties of complete antichains:

{blackheartedl

Lemma 2.3.6. *Let A be a complete antichain in $E(R)^*$. Let $\mathrm{lex}(A) = (\alpha_1, \ldots, \alpha_n)$. Then there does not exist $1 \leq i \leq n-1$ and $\beta \in E(R)^*$ such that*

$$\alpha_i <_\ell \beta <_\ell \alpha_{i+1}.$$

Proof. Let A be a complete antichain in $E(R)^*$. Let $\mathrm{lex}(A) = (\alpha_1, \ldots, \alpha_n)$. We will prove this by contradiction. Suppose that there exists $1 \leq i \leq n-1$ and $\beta \in E(R)^*$ such that

$$\alpha_i <_\ell \beta <_\ell \alpha_{i+1}.$$

Then $\alpha_j <_\ell \beta$ for all $j = 1, \ldots, i$ and $\beta <_\ell \alpha_k$ for all $k = i+1, \ldots, n$. This implies that $\beta \perp \alpha_1, \ldots, \alpha_n$. Then $\{\alpha_1, \ldots, \alpha_i, \beta, \alpha_{i+1}, \ldots, \alpha_n\}$ is an antichain, which contradicts the fact that A is complete. This proves the result. □

{stillsmiling}

Lemma 2.3.7. *Let A be a complete antichain in $E(R)^*$. If $\beta\gamma \in A$ (for some $\beta, \gamma \in E(R)^*$), then $D = \{\delta \mid \beta\delta \in A\}$ is a complete antichain in $E(R)^*$.*

Proof. Let A be a complete antichain in $E(R)^*$. Let $\beta\gamma \in A$ (for some $\beta, \gamma \in E(R)^*$). Let $D = \{\delta \mid \beta\delta \in A\}$.

Let us prove that D is an antichain: Let $\delta, \delta' \in D$. If $\delta \preceq \delta'$, then $\beta\delta \preceq \beta\delta'$, contrary to A being an antichain. Hence D is an antichain.

Let us prove that D is complete: Let $t \in \Omega$. Then $\beta t \in \Omega$, and there exists $\alpha \in A$ such that $\alpha \prec \beta t$. If $\alpha \preceq \beta$ then $\alpha \preceq \beta\gamma$, which contradicts the fact that A is an antichain. So $\beta \preceq \alpha$, which implies that $\alpha = \beta\delta$ for some $\delta \in D$. Then $\beta\delta \prec \beta t$. Hence $\delta \prec t$ and D is complete. □

arpdressedman}

Lemma 2.3.8. *Let $(G_0, \mathbf{e} \to R)$ be an F-type edge replacement system. Then the following hold:*

1. *If G is an expansion, i.e., it is obtained from G_0 by applying the edge replacement rule $\mathbf{e} \to R$ a finite number of times, then $E(G)$ is a complete antichain in $E(R)^*$.*

2. *If A is a complete antichain in $E(R)^*$, then there is an expansion G, obtained from G_0 by applying the edge replacement rule $\mathbf{e} \to R$ a finite number of times such that $E(G) = A$.*

Proof. Let $(G_0, \mathbf{e} \to R)$ be an F-type edge replacement system.

1. Suppose G is an expansion obtained by n simple expansions of G_0, for some $n \in \mathbb{N}$. We have to show that $E(G)$ is a complete antichain. We will prove this by induction on n.

   Let $n = 0$. Then $E(G) = \{\epsilon\}$. This is trivially an antichain, and complete because, for all $t \in \Omega$, $\epsilon \prec t$.

   Let $E(G)$ be a complete antichain for $n = m$. Let us perform a simple expansion on G to get a graph G'. Then some $\alpha \in E(G)$ is replaced by a copy of R, and hence $E(G') = (E(G)\backslash\{\alpha\}) \cup \{\alpha e \mid e \in E(R)\}$. Observe that $\alpha e \perp \beta$ for all $e \in E(R)$ and $\beta \in E(G)\backslash\{\alpha\}$ since $\alpha \perp \beta$, and $\alpha e \perp \alpha e'$ for all distinct $e, e' \in E(R)$. This proves that $E(G')$ is an antichain. Observe also that, for all $e_1 e_2 \cdots \in \Omega$, there exists an element in $E(G)$ which is a prefix of $e_1 e_2 \ldots$. If this element is $\beta \neq \alpha$, then $\beta \in E(G')$. If this element is α, observe that $\alpha = e_1 \ldots e_k$ and $e_{k+1} \in E(R)$. Then $\alpha e_{k+1} \in E(G')$. This proves that $E(G')$ is a complete antichain, where G' is an expansion obtained by $m + 1$ simple expansions of G_0.

   This completes the proof by induction.

2. Suppose A is a complete antichain in $E(R)^*$. We define a pair (m, k) as follows:

$$m = \max\{|\alpha| \mid \alpha \in A\},$$
$$k = \#\{\alpha \in A \mid |\alpha| = m\}.$$

We order these pairs lexicographically as follows: $(m,k) <_\ell (m',k')$ if and only if $m < m'$ or $m = m'$ and $k < k'$. Observe that, since both $m,k \in \mathbb{N}$, this is a well-order and there exists a lowest element. We can now prove our result by induction on (m,k).

Let $(m,k) = (1,1)$. Then $A = E(G_0)$ and hence $G = G_0$ and the inductive claim holds.

Let A be a complete antichain in $E(R)^*$ and let $m(A) = m$ and $k(A) = k$. Suppose the claim holds for all complete antichains B in $E(R)^*$ such that $(m(B),k(B)) <_\ell (m,k)$. There exists at least one $\alpha = e_1 \dots e_{m-1}$ in A of length m. Since A is an antichain, no proper prefix of α is in A. Consider $t = e_1 \dots e_{m-2}e'_{m-1} \dots \in \Omega$ where $e'_{m-1} \neq e_{m-1}$. Then there exists $\beta \in A$ such that $\beta \prec t$. Now $\beta \not\prec \alpha$ and $|\beta| \leq m$. This implies that $\beta = e_1 \dots e_{m-2}e'_{m-1}$. Since this is true for all $e'_{m-1} \in E(R)\backslash\{e_{m-1}\}$, A contains all words $e_1 \dots e_{m-2}f$ where $f \in E(R)$. Let us define the set

$$B = (A\backslash\{e_1 \dots e_{m-2}f \mid f \in E(R)\}) \cup \{e_1 \dots e_{m-2}\}.$$

Let us prove that B is a complete antichainin $E(R)^*$: Consider $\alpha, \beta \in B$. If $\alpha, \beta \in A\backslash\{e_1 \dots e_{m-2}f \mid f \in E(R)\}$, then $\alpha \perp \beta$. If $\alpha \in A\backslash\{e_1 \dots e_{m-2}f \mid f \in E(R)\}$ and $\beta = e_1 \dots e_{m-2}$, then $\alpha \perp e_1 \dots e_{m-2}$ for all $f \in E(R)$ and hence $\alpha \perp \beta$. This proves that B is an antichain. Consider $t \in \Omega$. Then there exists $\gamma \in A$ such that $\gamma \prec t$. If $\gamma \in A\backslash\{e_1 \dots e_{m-2}f \mid f \in E(R)\}$, then $\gamma \in B$. If $\gamma \in \{e_1 \dots e_{m-2}f \mid f \in E(R)\}$, then $\delta = e_1 \dots e_{m-2} \in B$ and $\delta \prec \gamma$. Hence $\delta \prec t$ and B is a complete antichain in $E(R)^*$.

Observe that $(m(B),k(B)) <_\ell (m,k)$. By our inductive claim, there exists a graph expansion G such that $B = E(G)$. Let us construct a graph expansion G' by performing an edge replacement on the edge $e_1 \dots e_{m-2} \in B = E(G)$. Then $A = E(G')$ and this completes the proof by induction.

□

Observe that the set $E(R)$ is finite. Let us endow it with the discrete topology. The symbol space Ω has a one-one correspondence with the Cartesian product

$$\prod_\omega E(R).$$

Consequently, we endow Ω with the *product topology.* Let us define the set $\Omega(\alpha) := \alpha\Omega$ (for some $\alpha \in E(R)^*$) to be the set of all infinite sequences which have the prefix α. Then the collection

$$\{\Omega(\alpha) \mid \alpha \in E(R)^*\}$$

forms a basis for the topology.

Since Ω is an infinite product of finite discrete sets, it is homeomorphic to the Cantor space. We observe that every basic open set $\Omega(\alpha)$ is also closed in Ω, and hence compact. Since Ω is a Hausdorff space, one-point sets are closed in Ω.

Recall that, in our full expansion sequence, the vertex set $V(G_n)$ of graph G_n is the disjoint union

$$V(G_n) = \bigsqcup_{k=0}^{n} V_k,$$

where V_k is the set of vertices of depth k, introduced in the graph G_k. The vertex sets of the graphs $\{G_n\}_{n=0}^{\infty}$ form a nested chain, i.e.

$$V(G_0) \subset V(G_1) \subset V(G_2) \subset \dots.$$

{silentrunners}

Definition 2.3.9. We will refer to elements of the vertex set

$$\mathcal{GV} = \bigcup_{n=0}^{\infty} V(G_n) = \bigsqcup_{n=0}^{\infty} V_n$$

as *gluing vertices* (for reasons which will become clear as we proceed).

Observe that if $v \in \mathcal{GV}$, then v is either a, b or αw (for some $\alpha \in E(R)^*$ and $w \in V_{\text{int}}(R)$). Assume that we have defined a linear order $\leq_v$ on $V_{\text{int}}(R)$. Recall from Definition 2.1.13 that the depth of a vertex $v \in \mathcal{GV}$ is the index of the set V_n such that $v \in V_n$, i.e., $\text{depth}(a) = \text{depth}(b) = 0$ and $\text{depth}(\alpha w) = |\alpha|$. Observe that, for some distinct $\alpha, \beta \in E(R)^*$, if $|\alpha| = |\beta|$, then $\alpha \perp \beta$ and they can be ordered lexicographically by Definition 2.3.5.

{mindinabox}

Definition 2.3.10. Let $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. We define an *induced vertex depth order* $\leq_d$ on $\mathcal{GV}$ as follows: for two distinct vertices z_i and z_j in $\mathcal{GV}$, $z_i <_d z_j$ if and only if one of the following holds:

1. $\text{depth}(z_i) < \text{depth}(z_j)$,
2. $\text{depth}(z_i) = \text{depth}(z_j)$ and $z_i = \alpha w$ and $z_j = \beta w'$ (for some distinct $\alpha, \beta \in E(R)^*$) and $\alpha <_\ell \beta$.
3. $\text{depth}(z_i) = \text{depth}(z_j)$ and $z_i = \alpha w$ and $z_j = \alpha w'$ (for some distinct $\alpha, \beta \in E(R)^*$) and $w <_v w'$.

{thechameleons}

Definition 2.3.11. Let Ω be the symbol space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. A sequence $e_1e_2\ldots \in \Omega$ *represents* a vertex $v \in \mathcal{GV}$ if the edge $e_1 \ldots e_n$ is incident with v in the graph G_n for all sufficiently large n.

{mortiis}

Lemma 2.3.12. *Let Ω be the symbol space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. Let $e_1e_2\ldots \in \Omega$ and $v \in \mathcal{GV}$. The sequence $e_1e_2\ldots$ represents the vertex v if and only if one of the following holds:*

1. $v = a$ and $e_1e_2\ldots = \bar{\mathbf{i}}$,

2. $v = b$ and $e_1e_2\ldots = \bar{\mathbf{t}}$,

3. $v = \alpha w$ (for some $\alpha \in E(R)^*$ and $w \in V_{\text{int}}(R)$) and $e_1e_2\ldots = \alpha p\bar{q}$, where $p \in E(R)$ is incident with w and $q = \mathbf{i}$ if w is the initial vertex of p and $q = \mathbf{t}$ if w is the terminal vertex of p.

Moreover, if it exists, the vertex v represented by the sequence $e_1e_2\ldots$ is unique.

Proof. Let Ω be the symbol space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. Let $e_1e_2\ldots \in \Omega$.

1. Let $v = a$. Suppose the sequence $e_1e_2\ldots$ represents the vertex v. Then there exists $N \in \mathbb{N}$ such that the edge $e_1\ldots e_n$ is incident with v in the graph G_n for all $n > N$. By Lemma 2.1.14 (1), we know this is only possible if $e_1\ldots e_n = (\mathbf{i})^n$ for all $n > 0$. This implies that $e_1e_2\ldots = \bar{\mathbf{i}}$ (and also that $N = 0$).

   Conversely, let $e_1e_2\ldots = \bar{\mathbf{i}}$. By Lemma 2.1.14 (1), the edge $(\mathbf{i})^n$ is incident with the vertices a and $(\mathbf{i})^{n-1}w$ in the graph G_n for some $n > 0$. It follows that a is the unique vertex such that the edge $e_1\ldots e_n$ is incident with it for all $n > 0$. And hence the sequence $e_1e_2\ldots$ represents the vertex $v = a$.

2. Let $v = b$. Suppose the sequence $e_1e_2\ldots$ represents the vertex v. Then there exists $N \in \mathbb{N}$ such that the edge $e_1\ldots e_n$ is incident with v in the graph G_n for all $n > N$. By Lemma 2.1.14 (1), we know this is only possible if $e_1\ldots e_n = (\mathbf{t})^n$ for all $n > 0$. This implies that $e_1e_2\ldots = \bar{\mathbf{t}}$ (and also that $N = 0$).

   Conversely, let $e_1e_2\ldots = \bar{\mathbf{t}}$. By Lemma 2.1.14 (1), the edge $(\mathbf{t})^n$ is incident with the vertices b and $(\mathbf{t})^{n-1}w$ in the graph G_n for some $n > 0$. It follows that b is the unique vertex such that the edge $e_1\ldots e_n$ is incident with it for all $n > 0$. And hence the sequence $e_1e_2\ldots$ represents the vertex $v = b$.

3. Let $v = \alpha w = e_1\ldots e_{k-1}w$ (for some $w \in V_{\text{int}}(R)$). Suppose the sequence $e_1e_2\ldots$ represents the vertex v. Then there exists $N \in \mathbb{N}$ such that the edge $e_1\ldots e_n$ is incident with v in the graph G_n for all $n \geq N$. Observe that v is either the initial or terminal vertex of $e_1\ldots e_n$. Let us examine both cases:

   3.1. By Lemma 2.1.14 (2), v is the initial vertex of $e_1\ldots e_n$ if $e_1\ldots e_n = \alpha p(\mathbf{i})^{n-k}$ and w is the initial vertex of p. This implies that $e_1e_2\ldots = \alpha p\bar{\mathbf{i}}$ (and $N = k$).

   3.2. By Lemma 2.1.14 (3), v is the terminal vertex of $e_1\ldots e_n$ if $e_1\ldots e_n = \alpha p(\mathbf{t})^{n-k}$ and w is the terminal vertex of p. This implies that $e_1e_2\ldots = \alpha p\bar{\mathbf{t}}$ (and $N = k$).

   Conversely, suppose $v = \alpha w$ and $e_1e_2\ldots = \alpha p\bar{q}$, where $p \in E(R)$ is incident with w and $q = \mathbf{i}$ if w is the initial vertex of p and $q = \mathbf{t}$ if w is the terminal vertex of p. Let us examine the two separate cases:

3.1. Suppose $e_1e_2\ldots = \alpha p\bar{\mathbf{i}}$ and w is the initial vertex of p. By Lemma 2.1.14 (2), the edge $e_1\ldots e_n = \alpha p(\mathbf{i})^{n-k}$ has the initial vertex αw and terminal vertex $\alpha p(\mathbf{i})^{n-k-1}w$ in the graph G_n for all $n \geq k$. This implies that $v = \alpha w$ us the unique vertex represented by $e_1e_2\ldots$.

3.2. Suppose $e_1e_2\ldots = \alpha p\bar{\mathbf{t}}$ and w is the terminal vertex of p. By Lemma 2.1.14 (3), the edge $e_1\ldots e_n = \alpha p(\mathbf{t})^{n-k}$ has the initial vertex $\alpha p(\mathbf{i})^{n-k-1}w$ and terminal vertex αw in the graph G_n for all $n \geq k$. This implies that $v = \alpha w$ us the unique vertex represented by $e_1e_2\ldots$.

This proves the result. □

{deadzone}

Definition 2.3.13. Let Ω be the symbol space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. Let $v \in \mathcal{GV}$. We define the set $Q_v \subset \Omega$ to contain all sequences $e_1e_2\ldots \in \Omega$ which represent the vertex v. It follows from Lemma 2.3.12 that

$$\begin{aligned} Q_a &= \{e_0\bar{\mathbf{i}}\} \\ Q_b &= \{e_0\bar{\mathbf{t}}\} \\ Q_{\alpha w} &= \{\, \alpha p\bar{q} \mid p \in E(R) \text{ incident with } w \\ &\qquad\qquad q = \mathbf{i} \text{ if } w \text{ is the initial vertex of } p \text{ or} \\ &\qquad\qquad q = \mathbf{t} \text{ if } w \text{ is the terminal vertex of } p\,\} \end{aligned}$$

for some $\alpha \in E(R)^*$ and $w \in V_{\text{int}}(R)$.

Observe that Q_v is a finite set for all $v \in \mathcal{GV}$ since $E(R)$ is a finite set.

{earthlings}

Definition 2.3.14 (Belk, Forrest [3])**.** Let Ω be the symbol space of an F-type edge replacement system. Two sequences from Ω

$$e_1e_2\ldots \quad \text{and} \quad e_1'e_2'\ldots$$

are said to be related to each other under the *gluing relation* $\sim$ if the edges

$$e_1\ldots e_n \quad \text{and} \quad e_1'\ldots e_n'$$

share a vertex in the graphs G_n for all $n \in \mathbb{N}$.

{newfang}

Lemma 2.3.15. *Let Ω be the symbol space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. Let $e_1e_2\ldots$ and $e_1'e_2'\ldots$ be two distinct sequences in Ω. Then*

$$e_1e_2\ldots \sim e_1'e_2'\ldots$$

if and only if there exists a vertex $\alpha w \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$ and $w \in V_{\text{int}}(R)$) such that

$$e_1e_2\ldots, e_1'e_2'\ldots \in Q_{\alpha w}.$$

Proof. Let Ω be the symbol space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system. Let $e_1e_2\dots$ and $e_1'e_2'\dots$ be two distinct sequences in Ω.

Suppose that $e_1e_2\dots \sim e_1'e_2'\dots$. Since $e_1e_2\dots$ and $e_1'e_2'\dots$ are distinct, there exists $k \in \mathbb{N}$ such that $e_1\dots e_{k-1} = e_1'\dots e_{k-1}'$ and $e_k \neq e_k'$. Let $\alpha = e_1\dots e_{k-1}$. Then αe_k and $\alpha e_k'$ are distinct edges in the graph G_k which share a vertex. By Definition 2.1.10 of the construction of the full expansion G_k, these edges arise when the edge α in G_{k-1} is replaced by a copy of R. The edges e_k and e_k' are incident with the vertex $w \in V_{\text{int}}(R)$. Then the edges αe_k and $\alpha e_k'$ are incident with the vertex $\alpha w \in V(G_k)$.

Since the edge replacement system is of F-type (in particular, expanding), αw is the unique vertex such that the edges $e_1\dots e_n$ and $e_1'\dots e_n'$ are incident with it in the graph G_n for all $n \geq k$. Then, by Lemma 2.1.15, $e_1\dots e_n = \alpha e_k(p)^{n-k}$ and $e_1'\dots e_n' = \alpha e_k'(q)^{n-k}$ (where $p = \mathbf{i}$ if w is the initial vertex of e_k in R and $p = \mathbf{t}$ if w is the terminal vertex of e_k in R, and similarly $q = \mathbf{i}$ if w is the initial vertex of e_k' in R and $q = \mathbf{t}$ if w is the terminal vertex of e_k' in R). Then, by Definition 2.3.13,

$$e_1e_2\dots, e_1'e_2'\dots \in Q_{\alpha w}.$$

Conversely, suppose there exists a vertex $\beta w \in \mathcal{GV}$ (for some $w \in V_{\text{int}}(R)$) such that $e_1e_2\dots, e_1'e_2'\dots \in Q_{\beta w}$. Let $\beta = e_1\dots e_{m-1}$. By Definition 2.3.13, $e_1e_2\dots = \beta e_m\bar{p}$ and $e_1'e_2'\dots = \beta e_m'\bar{q}$ (where $p = \mathbf{i}$ if w is the initial vertex of e_k in R and $p = \mathbf{t}$ if w is the terminal vertex of e_k in R, and similarly $q = \mathbf{i}$ if w is the initial vertex of e_k' in R and $q = \mathbf{t}$ if w is the terminal vertex of e_k' in R). Then, by Lemma 2.3.12, the sequences $e_1e_2\dots$ and $e_1'e_2'\dots$ both represent the vertex βw Hence, $e_1\dots e_n = e_1'\dots e_n'$ for all $n < m$ and the edges $e_1\dots e_n$ and $e_1'\dots e_n'$ share the vertex βw in the graphs G_n for all $n \geq m$. This implies that $\alpha = \beta$ and, by Definition 2.3.14,

$$e_1e_2\dots \sim e_1'e_2'\dots.$$

□

{deutschland}

Example 2.3.16. The symbol space for the F-Basilica group is the infinite product

$$\Omega = \{0,1,2\}^{\infty}.$$

We observed in Example 2.2.1 that the edges $00(2)^{n-2}$, $01(0)^{n-2}$, $01(2)^{n-2}$, and $02(0)^{n-2}$ in G_n share the vertex $0x$, for all $n > 1$. It follows from Definition 2.3.14 that $00\bar{2}$, $01\bar{0}$, $01\bar{2}$, and $02\bar{0}$ in Ω are all equivalent under the gluing relation.

More generally, it follows from Lemma 2.3.15 that for any two sequences $e_1e_2\dots$ and $e_1'e_2'\dots$ in Ω,

$$e_1e_2\dots \sim e_1'e_2'\dots$$

if and only if there exists a vertex $\beta x \in \mathcal{GV}$ such that

$$e_1e_2\dots, e_1'e_2'\dots \in Q_{\beta x} = \{\beta 0\bar{2}, \beta 1\bar{0}, \beta 1\bar{2}, \beta 2\bar{0}\}.$$

Belk, Forrest [3] Proposition 1.9 proved the following result for an arbitrary expanding edge replacement system. Below, we present a proof which is specific to F-type edge replacement systems.

{iamami}

Lemma 2.3.17. *Let Ω be the symbol space of an F-type edge replacement system. The gluing relation from Definition 2.3.14 is an equivalence relation.*

Proof. Let Ω be the symbol space and of an F-type edge replacement system. Let $\sim$ be the gluing relation from Definition 2.3.14. Observe that $\sim$ is always reflexive and symmetric, since sharing a vertex in a graph is always reflexive and symmetric.

To prove transitivity, suppose there exist three distinct sequences $e_1e_2\ldots$, $e_1'e_2'\ldots$ and $e_1''e_2''\ldots$ in Ω such that $e_1e_2\ldots \sim e_1'e_2'\ldots$ and $e_1'e_2'\ldots \sim e_1''e_2''\ldots$.

From Lemma 2.3.15 we know that $e_1e_2\ldots \sim e_1'e_2'\ldots$ if and only if there exists a vertex $\alpha w \in \mathcal{GV}$ such that

$$e_1e_2\ldots, e_1'e_2'\ldots \in Q_{\alpha w}.$$

Similarly, $e_1'e_2'\ldots \sim e_1''e_2''\ldots$ if and only if there exists a vertex $\beta w' \in \mathcal{GV}$ such that

$$e_1'e_2'\ldots, e_1''e_2''\ldots \in Q_{\beta w'}.$$

This implies that the sequence $e_1'e_2'\ldots$ represents the vertices αw and $\beta w'$. But we know from Lemma 2.3.12 that the vertex represented by a sequence is unique. Hence $\alpha w = \beta w'$, and therefore

$$e_1e_2\ldots, e_1'e_2'\ldots, e_1''e_2''\ldots \in Q_{\alpha w}.$$

Hence, by Lemma 2.3.15, $e_1e_2\ldots \sim e_1''e_2''\ldots$. This proves transitivity, and therefore a gluing relation of an F-type edge replacement system is an equivalence relation. □

{disappearer}

Definition 2.3.18 (Belk, Forrest [3], Definition 1.7)**.** Let Ω be the symbol space and of an F-type edge replacement system. Let $\sim$ be the gluing relation from Definition 2.3.14. We define the limit space

$$X := \Omega/{\sim}$$

to be the set of equivalence classes in Ω.

Let $x \in \Omega$. We denote the equivalence class under $\sim$ containing x by $[x]$. Since $\sim$ is an equivalence relation, the limit space X is a partition of Ω. We define the map $\phi\colon \Omega \to X$ by $\phi\colon x \mapsto [x]$. Let us assign the *quotient topology* to X, that is a set U is open in X if and only if its preimage $(U)\phi^{-1}$ is open in Ω. It follows that ϕ is the *quotient map*, and hence continuous.

We will prove in Lemma 3.1.9 that X is a Hausdorff space.

ethechemicals}

Lemma 2.3.19. *Let X be the limit space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system.*

1. *For all $v \in \mathcal{GV}$, the set $Q_v \subset \Omega$ is an equivalence class in X.*

2. *The map $\rho\colon \mathcal{GV} \to X$ defined by $(v)\rho = (y)\phi$ for all $y \in Q_v$ is injective.*

3. *Every equivalence class with more than one point has the form Q_v for some $v \in \mathcal{GV}$.*

Proof. Let X be the limit space and $\mathcal{GV}$ be the set of gluing vertices of an F-type edge replacement system.

1. By Lemma 2.3.15, we know that
$$e_1 e_2 \ldots, e_1' e_2' \ldots \in Q_v$$
for some $v \in \mathcal{GV}$ if and only if
$$e_1 e_2 \ldots \sim e_1' e_2' \ldots.$$
It follows from Lemma 2.3.17 and Definition 2.3.18 that Q_v is an equivalence class in X.

2. Observe that the map $\rho\colon \mathcal{GV} \to X$ defined by $(v)\rho = (y)\phi$ for all $y \in Q_v$ is independent of the choice of y by Lemma 2.3.15, and hence is well-defined for all $v \in \mathcal{GV}$. Consider $v_1, v_2 \in \mathcal{GV}$ such that $Q_{v_1} = Q_{v_2}$. By Definition 2.3.13, every sequence in Q_{v_1} represents the vertices v_1 and v_2. Since, by Lemma 2.3.12, the vertex represented by a sequence is unique, this implies that $v_1 = v_2$. Therefore, the map ρ is injective.

3. Let $x \in X$ be an equivalence class with at least two points. That is, there exist $y, z \in \Omega$ with $y \neq z$ such that $(y)\phi = (z)\phi = x$. Then $y \sim z$, and by Lemma 2.3.15, there exists $v \in \mathcal{GV}$ such that $y, z \in Q_v$, which is an equivalence class in X by (1).

□

To simplify notation, for all $v \in \mathcal{GV}$, we will use the vertex v as a label for the equivalence class $(v)\rho$ in X. We call the equivalence classes in X not corresponding to gluing vertices *regular points*.

Belk and Forrest [3], Theorem 1.24 proved the following result in general. We present a proof for F-type edge replacement systems:

Example 2.3.20. Let X be the limit space of the F-Basilica replacement system Recall from Example 2.3.16 that for any two sequences $e_1 e_2 \ldots$ and $e_1' e_2' \ldots$ in Ω,
$$e_1 e_2 \ldots \sim e_1' e_2' \ldots$$
if and only if there exists a vertex $\beta x \in \mathcal{GV}$ such that
$$e_1 e_2 \ldots, e_1' e_2' \ldots \in Q_{\beta x} = \{\beta 0\bar{2}, \beta 1\bar{0}, \beta 1\bar{2}, \beta 2\bar{0}\}.$$
It follows from Lemma 2.3.19 that $Q_{\beta x}$ is an equivalence class in X and corresponds to the gluing vertex βx.

Chapter 3

The Rearrangement Group of the Limit Space

{basics2}

In this chapter, let Ω be the symbol space and let X be the limit space of an F-type edge replacement system. We will define the "rearrangement group" – i.e., the group of homeomorphisms – of X. In Section 3.1, we will define a "cell" – a topological object. In Section 3.2 we will define a way to "partition" X using cells. In Section 3.3, we will define a rearrangement of X. In Section 3.4, we will prove that the set of rearrangements of X is a group.

3.1 Cells

{3.1}

In this section, we will define a "cell" – a topological object – and use it to establish our topological space in detail.

{boyharsher}

Definition 3.1.1 (Belk, Forrest [3], Definition 1.13)**.** For a word $\alpha \in E(R)^*$, a *cell* $C(\alpha)$ is the image of a basic open set $\Omega(\alpha)$ in the limit space X under the quotient map ϕ.

Observe that $C(\epsilon) = X$. For every $\alpha = e_1 \dots e_n \in E(R)^*$, the cell $C(\alpha)$ in X has a one-to-one correspondence with the edge $\alpha \in E(G_n)$.

{hallofmirrors

Definition 3.1.2. Let $\alpha = e_1 \dots e_n \in E(R)^*$. Let v_α be the initial vertex of the edge α in the graph G_n and let w_α be the terminal vertex of the edge α in the graph G_n. The equivalence classes in X corresponding to v_α and w_α are called the *boundary points* of the cell $C(\alpha)$ (with v_α being the initial boundary point and w_α being the terminal boundary point).

The complement of those boundary points in the cell $C(\alpha)$ is called the *interior* of the cell, and denoted by $\operatorname{int} C(\alpha)$. Observe that this may or may not be the same as the topological interior.

Cells and their interiors are our main topological objects, and we will use them to prove various details of the quotient topology on X. Provided X is Hausdorff, there is a simple proof that a cell is closed (hence compact). But since we require various properties of cells to prove Lemma 3.1.9 that X is a Hausdorff space, we will provide a different and slightly more technical proof in Lemma 3.1.8 that a cell is closed and the interior of a cell is open. The results which follow have been derived from Belk, Forrest [3], Section 1.5, but they have been reinterpreted for our construction.

{unlugarideal}

Lemma 3.1.3. *Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$). Then $(\alpha\bar{\mathbf{i}})\phi = v_\alpha$ and $(\alpha\bar{\mathbf{t}})\phi = w_\alpha$.*

Proof. Let $\alpha = e_1 \dots e_n \in E(R)^*$. By Lemma 2.1.14, we know that $v_\alpha \in \{a, \alpha^\dagger w, \alpha^{(n-k)\dagger} w\}$ and $w_\alpha \in \{b, \alpha^\dagger w, \alpha^{(n-k)\dagger} w\}$, for some $w \in V_{\text{int}}(R)$ and $k < n-1$. Let us examine all of these cases:

1. If $v_\alpha = a$ then $(a)\phi^{-1} = Q_a = \{\bar{\mathbf{i}}\}$. Since $a \in C(\alpha)$, then from the definition of a cell there exists a sequence $y \in \Omega(\alpha)$ such that $(y)\phi = Q_a$. Since Q_a is a singleton set, it follows that $y = \bar{\mathbf{i}}$. Also, since $\epsilon \preceq \alpha$, it follows that $y = \alpha\bar{\mathbf{i}}$.

2. If $v_\alpha = \alpha^\dagger w$ then $(\alpha^\dagger w)\phi^{-1} = Q_{\alpha^\dagger w}$. Since $\alpha^\dagger w \in C(\alpha)$, then from the definition of a cell there exists a sequence $y \in \Omega(\alpha)$ such that $(y)\phi = \alpha^\dagger w$. Then $y = \alpha^\dagger p\bar{\mathbf{i}}$ if w is the initial vertex of p or $y = \alpha^\dagger p\bar{\mathbf{t}}$ if w is the terminal vertex of p. Since v_α is the initial vertex of α, this implies that w is the initial vertex of p. And hence $y = \alpha^\dagger p\bar{\mathbf{i}}$ Since $\alpha^\dagger \prec \alpha$, we get $y = \alpha\bar{\mathbf{i}}$.

3. If $v_\alpha = \alpha^{(n-k)\dagger} w$ then $(\alpha^{(n-k)\dagger} w)\phi^{-1} = Q_{\alpha^{(n-k)\dagger} w}$. Since $\alpha^{(n-k)\dagger} w \in C(\alpha)$, then from the definition of a cell there exists a sequence $y \in \Omega(\alpha)$ such that $(y)\phi = \alpha^{(n-k)\dagger} w$. Then $y = \alpha^{(n-k)\dagger} p\bar{\mathbf{i}}$ if w is the initial vertex of p or $y = \alpha^{(n-k)\dagger} p\bar{\mathbf{t}}$ if w is the terminal vertex of p. Since v_α is the initial vertex of α, this implies that w is the initial vertex of p. And hence $y = \alpha^{(n-k)\dagger} p\bar{\mathbf{i}}$ Since $\alpha^{(n-k)\dagger} \prec \alpha$, we get $y = \alpha\bar{\mathbf{i}}$.

4. If $w_\alpha = b$ then $(b)\phi^{-1} = Q_b = \{\bar{\mathbf{t}}\}$. Since $b \in C(\alpha)$, the definition of a cell that there exists a sequence $y \in \Omega(\alpha)$ such that $(y)\phi = Q_b$. Since Q_b is a singleton set, it follows that $y = \bar{\mathbf{t}}$. Also, since $\epsilon \preceq \alpha$, it follows that $y = \alpha\bar{\mathbf{t}}$.

5. If $w_\alpha = \alpha^\dagger w$ then then $(\alpha^\dagger w)\phi^{-1} = Q_{\alpha^\dagger w}$. Since $\alpha^\dagger w \in C(\alpha)$, then from the definition of a cell there exists a sequence $y \in \Omega(\alpha)$ such that $(y)\phi = \alpha^\dagger w$. Then $y = \alpha^\dagger p\bar{\mathbf{i}}$ if w is the initial vertex of p or $y = \alpha^\dagger p\bar{\mathbf{t}}$ if w is the terminal vertex of p. Since w_α is the terminal vertex of α, this implies that w is the terminal vertex of p. And hence $y = \alpha^\dagger p\bar{\mathbf{t}}$ Since $\alpha^\dagger \prec \alpha$, we get $y = \alpha\bar{\mathbf{t}}$.

6. If $w_\alpha = \alpha^{(n-k)\dagger} w$ then $(\alpha^{(n-k)\dagger} w)\phi^{-1} = Q_{\alpha^{(n-k)\dagger} w}$. Since $\alpha^{(n-k)\dagger} w \in C(\alpha)$, then from the definition of a cell there exists a sequence $y \in \Omega(\alpha)$ such that $(y)\phi = \alpha^{(n-k)\dagger} w$. Then $y = \alpha^{(n-k)\dagger} p\bar{\mathbf{i}}$ if w is the initial vertex of p or $y = \alpha^{(n-k)\dagger} p\bar{\mathbf{t}}$ if w is the terminal vertex of p. Since w_α is the terminal vertex of α, this implies that w is the terminal vertex of p. And hence $y = \alpha^{(n-k)\dagger} p\bar{\mathbf{t}}$ Since $\alpha^{(n-k)\dagger} \prec \alpha$, we get $y = \alpha\bar{\mathbf{t}}$.

This proves the result. □

{theprice}

Remark 3.1.4. Let $C(\alpha)$ be a cell in X (for some $\alpha = e_1 \dots e_n \in E(R)^*$). Recall from Definition 2.1.8 that the edge $\alpha \in E(G_n)$ is a *loop* if and only if $v_\alpha = w_\alpha$. It follows from Lemma 3.1.3 that $C(\alpha)$ corresponds to a loop in $E(G_n)$ if and only if $\alpha\bar{\mathbf{i}} \sim \alpha\bar{\mathbf{t}}$.

Recall that our quotient map $\phi\colon \Omega \to X$ maps points in Ω to their equivalence class in X. The limit space X possesses the quotient topology, i.e., a set U is open in X if and only if $(U)\phi^{-1}$ is open in Ω.

Recall also that the basic open sets in Ω are of the form $\Omega(\alpha)$ and consist of all sequences with the prefix $\alpha = e_1 \dots e_n$. A basic open set $\Omega(\alpha)$ is both open and closed in Ω. There exist sequences in $\Omega(\alpha)$ which map to the boundary points of the $C(\alpha)$. Since one-point sets are closed in Ω, removing these sequences will still give us an open set in Ω.

The following result characterizes similar properties for the elements of the cell $C(\alpha)$:

{popwilleatits

Lemma 3.1.5. *Let $C(\alpha)$ be a cell in the limit space X (for some $\alpha \in E(R)^*$).*

1. *Every equivalence class in $C(\alpha)$ contains at least one sequence with the prefix α.*

2. *Every sequence in every equivalence class in* $\operatorname{int} C(\alpha)$ *has the prefix α.*

Proof. Let $C(\alpha)$ be a cell in the limit space X (for some $\alpha \in E(R)^*$).

1. Since $C(\alpha) = (\Omega(\alpha))\phi$, the proof follows from Definition 3.1.1.

2. Suppose there exists an equivalence class $x \in C(\alpha)$ such that there exists $y \in \Omega$ such that $(y)\phi = x$ but $y \notin \Omega(\alpha)$. We will show that x is either v_α or w_α.

   By definition of a cell and of the quotient map, there exists a $z \in \Omega(\alpha)$ such that $(z)\phi = x$. This implies that $y \sim z$. It follows from Lemma 2.3.15 that there exists $\beta w \in \mathcal{GV}$ (for some $\beta \in E(R)^*$ and $w \in V_{\text{int}}(R)$) such that $y, z \in Q_{\beta w}$. Observe that z has the prefix α but y does not. This forces $\beta \prec \alpha$, and hence $z = \alpha\bar{\mathbf{i}}$ or $z = \alpha\bar{\mathbf{t}}$. From Lemma 3.1.3, we know that $(\alpha\bar{\mathbf{i}})\phi = v_\alpha$ and $(\alpha\bar{\mathbf{t}})\phi = w_\alpha$. This proves the result.

□

{stonesfromthe

Corollary 3.1.6. *Let $C(\alpha)$ be a cell in the limit space X (for some $\alpha \in E(R)^*$). The following are true:*

1. $(C(\alpha))\phi^{-1} = \Omega(\alpha) \cup Q_{v_\alpha} \cup Q_{w_\alpha}$

2. $(\operatorname{int} C(\alpha))\phi^{-1} = \Omega(\alpha) \setminus (Q_{v_\alpha} \cup Q_{w_\alpha})$

Proof. The proof follows from Lemma 3.1.5. □

{pleasurelittl

Corollary 3.1.7. *Let $C(\alpha)$ be a cell in the limit space X (for some $\alpha \in E(R)^*$). Let $z = \beta w \in \mathcal{GV}$ (for some $\beta \in E(R)^*$ and $w \in V_{\text{int}}(R)$). Then $z \in \operatorname{int} C(\alpha)$ if and only if $\alpha \preceq \beta$.*

Proof. Let $C(\alpha)$ be a cell in the limit space X (for some $\alpha \in E(R)^*$). Let $z = \beta w \in \mathcal{GV}$ (for some $\beta \in E(R)^*$ and $w \in V_{\text{int}}(R)$). Recall from Lemma 2.3.19 that, for all $y \in \Omega$ such that $y \in (\beta w)\phi^{-1} = Q_{\beta w}$, β is a prefix of y. Recall also from Lemma 3.1.5 that, for all $y \in \Omega$ such that $y \in (x)\phi^{-1}$ for some $x \in \operatorname{int} C(\alpha)$, α is a prefix of y. Hence $\alpha \preceq \beta$. □

{grauzone}

Lemma 3.1.8 (Belk, Forrest [3] Lemma 1.25)**.** *Let X be the limit space of an F-type edge replacement system.*

1. *One-point sets are closed in X.*

2. *Each cell is closed in X and the interior of each cell is open in X.*

Proof. Let X be the limit space of an F-type edge replacement system.

1. Consider a one-point subset of X. If the point is a regular point in X, the preimage of the set under ϕ is a one-point set in Ω, which is closed in Ω. If the point is a gluing vertex, recall by Lemma 2.3.15 that the preimage of the set is $Q_v \subset \Omega$ for some $v \in \mathcal{GV}$, which is a finite set. This is also closed, as it a union of a finite number of closed sets.

2. Consider a cell $C(\alpha)$ of X. By Corollary 3.1.6 (1) the preimage of $C(\alpha)$ is as follows:

$$(C(\alpha))\phi^{-1} = \Omega(\alpha) \cup Q_{v_\alpha} \cup Q_{w_\alpha}.$$

   Recall that the basic open set $\Omega(\alpha)$ is both open and closed in Ω. The sets Q_{v_α} and Q_{w_α} are both finite sets and hence closed in Ω. It follows that $(C(\alpha))\phi^{-1}$ is closed in Ω since it is a finite union of closed sets. This proves that $C(\alpha)$ is closed in X.

   By Corollary 3.1.6 (2), the preimage of the interior of the cell is:

$$(\operatorname{int} C(\alpha))\phi^{-1} = \Omega(\alpha) \backslash \left(Q_{v_\alpha} \cup Q_{w_\alpha}\right),$$

   Recall that the basic open set $\Omega(\alpha)$ is both open and closed in Ω. The sets Q_{v_α} and Q_{w_α} are both finite sets and hence closed in Ω. Hence $(\operatorname{int} C(\alpha))\phi^{-1}$ is an open set in Ω since we have only removed a finite number of closed sets from $\Omega(\alpha)$. Therefore, the interior of each cell is open in X.

□

Note that the interior of a cell may or may not correspond to the topological interior. Each cell $C(\alpha)$ is compact, being the image of a compact set $\Omega(\alpha)$. The space X is compact since it is the quotient of a compact space.

{cubanate}

Lemma 3.1.9. *Let X be the limit space of an F-type edge replacement system. Then X is a Hausdorff space.*

Proof. Let X be the limit space of an F-type edge replacement system. Let x and y be two distinct points in X. We have to show that there exist open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$. By Definition 2.3.18 of the limit space X, $x = [e_1e_2\ldots]$ and $y = [e_1'e_2'\ldots]$ where $e_1e_2\ldots, e_1'e_2'\ldots \in \Omega$. Since $x \neq y$, $e_1e_2\ldots \neq e_1'e_2'\ldots$. We consider three cases concerning the points x and y.

1. **Suppose x and y are both regular points.**

   In this case, we can find an integer n such that $e_n \neq e_n'$. Let us define $U = \operatorname{int} C(e_1\ldots e_n)$ and $V = \operatorname{int} C(e_1'\ldots e_n')$. By Lemma 3.1.8 (2), these are open subsets of X. Observe that $x \in C(e_1\ldots e_n)$ and hence $x \in U$ since x is not a gluing vertex. Similarly $y \in V$. Let us now prove that $U \cap V = \varnothing$: Suppose there exists $z \in U \cap V$, then $z = [f_1f_2\ldots]$ for some sequence $f_1f_2\ldots \in \Omega$. Then, by Corollary 3.1.6 (2), $f_1f_2\ldots \in (U)\phi^{-1} \subset \Omega(e_1\ldots e_n)$ and $f_1f_2\ldots \in (V)\phi^{-1} \subset \Omega(e_1'\ldots e_n')$. This is impossible because $e_n \neq e_n'$. Hence $U \cap V = \varnothing$.

2. **Suppose x and y are both gluing vertices.**

   Let us define the sets U and V as follows: Consider the graph G_n such that $x, y \in V(G_n)$. Let $\alpha_1, \ldots, \alpha_k$ be the edges in the graph G_{n+1} which are incident with x and $\beta_1, \ldots, \beta_\ell$ be the edges in the graph G_{n+1} which are incident with y. By Lemma 2.1.14, for all $i = 1, \ldots, k$, the edge α_i is incident with a vertex γw where $w \in V_{\text{int}}(R)$. Similarly, for all $j = 1, \ldots, \ell$, the edge β_j is incident with a vertex δw where $w \in V_{\text{int}}(R)$. In particular, no edge α_i is incident with any edge β_j.

   Then, for all $i = 1, \ldots, k$, the cell $C(\alpha_i)$ has x as one of its boundary points. Let us define x_i to be the other boundary point. Similarly, for all $j = 1, \ldots, \ell$, the cell $C(\beta_j)$ has y as one of its boundary points. Let us define y_j to be the other boundary point. We define the sets U and V as follows:

   $$\begin{aligned} U &= C(\alpha_1) \cup \cdots \cup C(\alpha_k) \backslash \{x_1, \ldots, x_k\}, \\ V &= C(\beta_1) \cup \cdots \cup C(\beta_\ell) \backslash \{y_1, \ldots, y_\ell\}. \end{aligned}$$

   As $x \neq x_i$ for all $i = 1, \ldots, k$, the point $x \in U$. Similarly, $y \in V$.

   Let us prove that U and V are open subsets of X: Let $f_1f_2\ldots$ be a sequence in Q_x. Then $f_1f_2\ldots$ represents the vertex x and $f_1\ldots f_{n+1}$ is an edge incident with x in G_{n+1}. Hence $f_1\ldots f_{n+1} = \alpha_i$ for some $1 \leq i \leq k$, and $f_1f_2\ldots \in \Omega(\alpha_i)$. Hence

   $$Q_x \in \Omega(\alpha_1) \cup \cdots \cup \Omega(\alpha_k).$$

Therefore

$$\begin{aligned}\left(\bigcup_{i=1}^{k} C(\alpha_i)\right)\phi^{-1} &= \bigcup_{i=1}^{k} \left(C(\alpha_i)\right)\phi^{-1}\\ &= \bigcup_{i=1}^{k} \left(\Omega(\alpha_i) \cup Q_x \cup Q_{x_i}\right)\\ &= \left(\bigcup_{i=1}^{k} \Omega(\alpha_i)\right) \cup \left(\bigcup_{i=1}^{k} Q_{x_i}\right).\end{aligned}$$

Now, observe that $f_1f_2\ldots \in (U)\phi^{-1}$ if and only if $(f_1f_2\ldots)\phi \in \bigcup_{i=1}^{k} C(\alpha_i)$ and $(f_1f_2\ldots)\phi \notin \{x_1,\ldots,x_k\}$. That is, if and only if $f_1f_2\ldots \in \left(\bigcup_{i=1}^{k} C(\alpha_i)\right)\phi^{-1}$ and $f_1f_2\ldots \notin \bigcup_{i=1}^{k} Q_{x_i}$. Hence

$$(U)\phi^{-1} = \left(\bigcup_{i=1}^{k} \Omega(\alpha_i)\right) \cup \left(\bigcup_{i=1}^{k} Q_{x_i}\right).$$

This is the complement of a finite (hence closed) subset in an open set and thus $(U)\phi^{-1}$ is open in Ω. Hence U is open in X. Similarly we can prove that V is open in X.

Finally, let us prove that $U \cap V = \varnothing$: Suppose $z \in U \cap V$. Then $z = [g_1g_2\ldots]$ for some sequence $g_1g_2\ldots \in \Omega$. Then $g_1g_2\ldots \in (U)\phi^{-1}$ and hence it is contained in $\bigcup_{i=1}^{k} \Omega(\alpha_i)$ and therefore there exists an α_i for some $1 \le i \le k$ such that α_i is a prefix of $g_1g_2\ldots$. Similarly, we can find a β_j for some $1 \le j \le \ell$ such that β_j is a prefix of $g_1g_2\ldots$. However, α_i and β_j are edges in the graph G_{n+1} and hence have the same length. This forces $\alpha_i = \beta_j$, which is impossible since these edges are not even incident with the same vertices. Hence $U \cap V = \varnothing$.

3. **Suppose x is a gluing vertex and y is a regular point.**

   Let us define the sets U and V as follows: Let $y = [e_1e_2\ldots]$. Since $x \neq y$, there exists an integer n such that $e_1\ldots e_n$ is not incident with x in the graph G_n. Let $\alpha_1,\ldots,\alpha_k$ be the edges in the graph G_n which are incident with x, Then, for all $i = 1,\ldots,k$, the cell $C(\alpha_i)$ has x as one of its boundary points. Let us define x_i to be the other boundary point. We define the sets U and V as follows:

$$\begin{aligned}U &= C(\alpha_1) \cup \cdots \cup C(\alpha_k) \backslash \{x_1,\ldots,x_k\},\\ V &= \operatorname{int} C(e_1\ldots e_n).\end{aligned}$$

   We have shown in the previous cases that U and V are open subsets of X, and $x \in U$ and $y \in V$.

   It remains to prove that $U \cap V = \varnothing$: Suppose there exists $z \in U \cap V$, then $z = [f_1f_2\ldots]$ for some sequence $f_1f_2\ldots \in \Omega$. Then $f_1f_2\ldots \in (U)\phi^{-1}$ and hence it is contained in $\bigcup_{i=1}^{k} \Omega(\alpha_i)$ and therefore there exists an α_i for some $1 \leq i \leq k$ such that α_i is a prefix of $f_1 f_2 \ldots$. Also, by Corollary 3.1.6 (2), $f_1 f_2 \ldots \in (V)\phi^{-1} \subset \Omega(e_1 \ldots e_n)$. This implies that $\alpha_i = e_1 \ldots e_n$, which is a contradiction since $e_1 \ldots e_n$ is not incident with x. Hence $U \cap V = \varnothing$.

This completes the proof. □

{carbonkid}

Lemma 3.1.10. *Let $C(\alpha)$ and $C(\beta)$ be cells in X for some $\alpha, \beta \in E(R)^*$.*

1. *$C(\alpha) = C(\beta)$ if and only if $\alpha = \beta$.*
2. *$C(\alpha) \subset C(\beta)$ if and only if $\alpha \succ \beta$.*
3. *Precisely one of the following holds:*
   - *3.1. $C(\alpha) = C(\beta)$,*
   - *3.2. $C(\alpha) \supset C(\beta)$,*
   - *3.3. $C(\alpha) \subset C(\beta)$,*
   - *3.4. $C(\alpha)$ and $C(\beta)$ have disjoint interiors.*

Proof. Let $C(\alpha)$ and $C(\beta)$ be cells in X for some $\alpha, \beta \in E(R)^*$.

1. Suppose that $C(\alpha) = C(\beta)$. It follows that $(C(\alpha))\phi^{-1} = (C(\beta))\phi^{-1}$. By Corollary 3.1.6, this gives us $\Omega(\alpha) \cup Q_{v_\alpha} \cup Q_{w_\alpha} \subset \Omega(\beta) \cup Q_{v_\beta} \cup Q_{w_\beta}$. Observe that every sequence in Q_{v_β} and Q_{w_β} either ends in $\bar{\mathbf{i}}$ or $\bar{\mathbf{t}}$. Consider the sequences $\alpha\overline{\mathbf{it}}$ and $\alpha\overline{\mathbf{ti}}$ in $\Omega(\alpha)$. Then both $\alpha\overline{\mathbf{it}}$ and $\alpha\overline{\mathbf{ti}}$ are in $\Omega(\beta)$, which implies that $\beta \preceq \alpha$. Similarly consider the sequences $\beta\overline{\mathbf{it}}$ and $\beta\overline{\mathbf{ti}}$ in $\Omega(\beta)$. Then both $\beta\overline{\mathbf{it}}$ and $\beta\overline{\mathbf{ti}}$ are in $\Omega(\alpha)$, which implies that $\alpha \preceq \beta$. Hence $\alpha = \beta$.

   Conversely, let $\alpha = \beta$. Then $\Omega(\alpha) = \Omega(\beta)$, and hence $C(\alpha) = C(\beta)$.

2. Suppose that $C(\alpha) \subset C(\beta)$. Since taking preimages preserves containment, it follows that $(C(\alpha))\phi^{-1} \subset (C(\beta))\phi^{-1}$. By Corollary 3.1.6, this gives us $\Omega(\alpha) \cup Q_{v_\alpha} \cup Q_{w_\alpha} \subset \Omega(\beta) \cup Q_{v_\beta} \cup Q_{w_\beta}$. Observe that every sequence in Q_{v_β} and Q_{w_β} either ends in $\bar{\mathbf{i}}$ or $\bar{\mathbf{t}}$. Consider the sequences $\alpha\overline{\mathbf{it}}$ and $\alpha\overline{\mathbf{ti}}$ in $\Omega(\alpha)$. Then both $\alpha\overline{\mathbf{it}}$ and $\alpha\overline{\mathbf{ti}}$ are in $\Omega(\beta)$, which implies that $\beta \preceq \alpha$. Observe by (1) that if $\alpha = \beta$, $C(\alpha) = C(\beta)$. Therefore, $\alpha \succ \beta$.

   Conversely, let $\alpha \succ \beta$. Then $\Omega(\alpha) \subseteq \Omega(\beta)$, which gives us $C(\alpha) \subseteq C(\beta)$. To prove that $C(\alpha) \subset C(\beta)$, consider $p \in E(R)$ such that $\beta p \not\preceq \alpha$. Then $\beta p\bar{\mathbf{i}}\mathbf{t} \in \Omega(\beta)$ but $\beta p\bar{\mathbf{i}}\mathbf{t} \notin \Omega(\alpha)$ and $(\beta p\bar{\mathbf{i}}\mathbf{t})\phi \notin C(\alpha)$. This implies that $C(\alpha) \subset C(\beta)$.

3. Suppose that

$$\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing.$$

By Corollary 3.1.6, it follows that

$$(\Omega(\alpha)\backslash (Q_{v_\alpha} \cup Q_{w_\alpha})) \cap \left(\Omega(\beta)\backslash \left(Q_{v_\beta} \cup Q_{w_\beta}\right)\right) \neq \varnothing.$$

From this, we get

$$\Omega(\alpha) \cap \Omega(\beta) \neq \varnothing.$$

That is, there is some point y in $\Omega(\alpha)$ that is also in $\Omega(\beta)$. This means y has the prefix α and β. This is only possible if

3.1. $\alpha = \beta$, which implies that $C(\beta) = C(\alpha)$,

3.2. $\beta \prec \alpha$, which implies that $C(\beta) \supset C(\alpha)$,

3.3. $\alpha \prec \beta$, which implies that $C(\beta) \subset C(\alpha)$.

□

orgeousgeorge}

Lemma 3.1.11. *Let $C(\alpha)$ and $C(\beta)$ be cells in X (for some $\alpha, \beta \in E(R)^*$). Then $C(\alpha) = C(\beta)$ if and only if $v_\alpha = v_\beta$ and $w_\alpha = w_\beta$.*

Proof. Let $C(\alpha)$ and $C(\beta)$ be cells in X (for some $\alpha, \beta \in E(R)^*$). Suppose $C(\alpha) = C(\beta)$. By Lemma 3.1.10 (1), this gives us $\alpha = \beta$ and the result follows.

Conversely, let $v_\alpha = v_\beta$ and $w_\alpha = w_\beta$. Let $\alpha = e_1 \dots e_n$ and $\beta = e'_1 \dots e'_k$. The cell $C(\alpha)$ corresponds to the edge α in the graph G_n and the cell $C(\beta)$ corresponds to the edge β in the graph G_k. Then v_α and w_α are the initial and terminal vertices respectively of the edge α in G_n and the edge β in G_k. By Definition 2.1.2 of an expanding edge replacement system, $n = k$, and by Definition 2.1.6 and Remark 2.1.7, $\alpha = \beta$. This implies that $C(\alpha) = C(\beta)$. I'm not sure how to fix these references. □

{demondays}

Lemma 3.1.12. *Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$):*

1. *The gluing vertices of the form αw (for some $w \in V_{\text{int}}(R)$) are the gluing vertices of least depth in the interior of $C(\alpha)$.*

2. *Let v_α and w_α be the boundary points of $C(\alpha)$, then*

$$\operatorname{depth} v_\alpha < \operatorname{depth} \alpha w \quad \textit{and} \quad \operatorname{depth} w_\alpha < \operatorname{depth} \alpha w.$$

Proof. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) with boundary points v_α and w_α.

1. Let $\beta w' \in \operatorname{int} C(\alpha)$ be a gluing vertex (for some $\beta \in E(R)^*$ and $w' \in V_{\text{int}}(R)$). Then, by Corollary 3.1.7, $\alpha \preceq \beta$. It follows that, for a gluing vertex αw (for some $w \in V_{\text{int}}(R)$),
$$\operatorname{depth} \alpha w \leq \operatorname{depth} \beta w'.$$

2. Let $\alpha = e_1 \dots e_n$, then $\operatorname{depth} \alpha w = n + 1$. From Lemma 2.1.14, we know that $v_\alpha \in \{a, \alpha^\dagger w', \alpha^{(n-k)\dagger} w'\}$ and $w_\alpha \in \{b, \alpha^\dagger w', \alpha^{(n-k)\dagger} w'\}$, for some $w' \in V_{\text{int}}(R)$ and $k < n - 1$. Then $\operatorname{depth} v_\alpha \leq n$ and $\operatorname{depth} w_\alpha \leq n$ and the result follows.

□

{soylentgreen}

Definition 3.1.13. We define the *depth* of a cell $C(\alpha)$ to be

$$\operatorname{depth} C(\alpha) = |\alpha|.$$

Observe that, by Lemma 3.1.12 (1), this is the the minimum depth of a gluing vertex in $\operatorname{int} C(\alpha)$.

3.2 Cellular Partitions

{3.2}

In this section, we will define a way to "partition" X using cells.

{ruby}

Definition 3.2.1 (Belk, Forrest [3], Definition 1.14)**.** A *cellular partition* $\mathcal{P}$ of X is a cover of X by finitely many cells with disjoint interiors.

{alternativeul

Lemma 3.2.2. *The set*

$$\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$$

is a cellular partition of X if and only if A is a complete antichain in $E(R)^*$.

Proof. Suppose $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$ is a cellular partition of X. Then A is a finite subset of $E(R)^*$. We have to show that A is a complete antichain in $E(R)^*$.

Let us first prove that A is an antichain: Observe that, for all $C(\alpha), C(\beta) \in \mathcal{P}$ (for some $\alpha, \beta \in E(R)^*$), $\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) = \varnothing$. By Lemma 3.1.10 (3), this implies that $C(\alpha) \nsubseteq C(\beta)$ and $C(\alpha) \nsupseteq C(\beta)$. By Lemma 3.1.10 (1) and (2), this implies that $\alpha \npreceq \beta$ and $\alpha \nsucceq \beta$. Hence $\alpha \perp \beta$ for all $\alpha, \beta, \in A$, which proves that A is an antichain.

Let us prove that A is a complete antichain: Consider a sequence $e_1 e_2 \ldots \in \Omega$. We have to show that there exists $\alpha \in A$ such that α is a prefix of $e_1 e_2 \ldots$. Observe that $[e_1 e_2 \ldots] \in X$. Suppose that the largest length of a word α in A is n. Consider the sequence $e_1 \ldots e_n \bar{\mathbf{i}}\mathbf{t}$. Then $[e_1 \ldots e_n \bar{\mathbf{i}}\mathbf{t}]$ is not a gluing vertex by Lemma 2.3.15, and therefore there exists a cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $[e_1 \ldots e_n \bar{\mathbf{i}}\mathbf{t}] \in \operatorname{int} C(\alpha)$. By Lemma 3.1.5 (2), α is a prefix of $e_1 \ldots e_n \bar{\mathbf{i}}\mathbf{t}$, and hence α is also a prefix of $e_1 e_2 \ldots$. This proves that the antichain A is complete.

Conversely, suppose $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$ is a set of cells in X such that A is a complete antichain in $E(R)^*$. We have to show that $\mathcal{P}$ is a cellular partition of X.

Since A is an antichain, for all $\alpha, \beta \in A$, $\alpha \perp \beta$. Then by Lemma 3.1.10, $\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) = \varnothing$ for all $C(\alpha), C(\beta) \in \mathcal{P}$. Since the antichain A is complete, for all $e_1 e_2 \ldots \in \Omega$, there exists $\alpha \in A$ such that α is a prefix of $e_1 e_2 \ldots$. Then by Definition 3.1.1, $[e_1 e_2 \ldots] \in C(\alpha)$. This proves that $\mathcal{P}$ is a cellular partition of X. □

Observe that, while the cells in a cellular partition $\mathcal{P}$ of X have disjoint interiors, they may share boundary points.

{allthepeople}

Definition 3.2.3. Let $\mathcal{P}$ be a cellular partition of X. The set $\partial\mathcal{P}$ is the set of all boundary points v_α and w_α of the cells $C(\alpha)$ in $\mathcal{P}$.

{bitterdivisio

Lemma 3.2.4. *Let $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$ be a cellular partition of X. Then there exists a graph expansion $G_\mathcal{P}$ with $E(G_\mathcal{P}) = A$ and $V(G_\mathcal{P}) = \partial\mathcal{P}$.*

Proof. Let $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$ be a cellular partition of X. By Lemma 3.2.2, A is a complete antichain in $E(R)^*$. By Lemma 2.3.8, there exists a graph expansion $G_\mathcal{P}$ such that $A = E(G_\mathcal{P})$. By Definition 3.1.2 of boundary points, $\partial\mathcal{P} = V(G_\mathcal{P})$. □

We will be using graph expansions to illustrate cellular partitions of our limit space X.

Let $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$ be a cellular partition of X. Recall that, since A is a complete antichain in $E(R)^*$, there exists a lexicographic order $\leq_\ell$ on A.

{siege}

Definition 3.2.5. We define an *induced cell lexicographic order* $\leq_\ell$ on $\mathcal{P}$ as follows: for two distinct cells $C(\alpha)$ and $C(\beta)$ in $\mathcal{P}$, $C(\alpha) <_\ell C(\beta)$ if and only if $\alpha <_\ell \beta$. We denote this by that ordered list $\mathrm{lex}(\mathcal{P}) = (C(\alpha_1), \ldots C(\alpha_d))$.

Recall from Definition 2.3.10 that there exists an induced vertex depth order $\leq_d$ on $\mathcal{GV}$. We denote this by that ordered list $\mathrm{depth}(\partial\mathcal{P}) = (z_1 \ldots, z_d)$.

{astralsabbat}

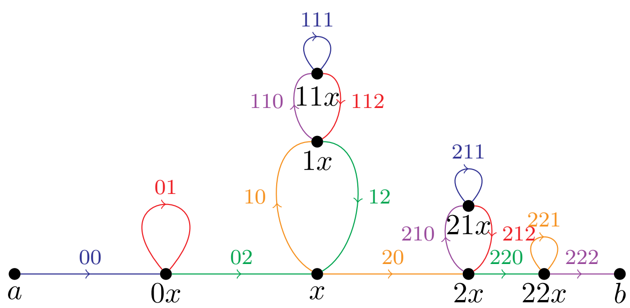
Definition 3.2.6. We define an *induced cell depth order* $\leq_d$ on $\mathcal{P}$ as follows: for two cells $C(\alpha), C(\beta) \in \mathcal{P}$, $C(\alpha) <_d C(\beta)$ if and only if $|\alpha| < |\beta|$ or $|\alpha| = |\beta|$ and $\alpha <_\ell \beta$. We denote this by the ordered list $\mathrm{depth}(\mathcal{P}) = (C(\alpha_1), \ldots C(\alpha_d))$.

christiansands}

Example 3.2.7. Let us define the set of cells $\mathcal{P}$ as follows.

$$\mathcal{P} = \big\{C(00), C(01), C(02), C(10), C(110), C(111), C(112), C(12),$$
$$C(20), C(210), C(211), C(212), C(220), C(221), C(222)\big\}.$$

Observe that the labels of the cells form a complete antichain in $E(R)^* = \{0,1,2\}^*$. Hence, by Lemma 3.2.2, the set $\mathcal{P}$ is a cellular partition of the limit space of the F-Basilica replacement system. It is illustrated by the graph expansion below:



The set of boundary points of $\mathcal{P}$ is:

$$\partial\mathcal{P} = \{0x, x, 1x, 11x, 2x, 21x, 22x\}\,.$$

The induced cell lexicographic order on $\mathcal{P}$ is

$$\begin{aligned}\operatorname{lex}(\mathcal{P}) = \big(&C(00), C(01), C(02), C(10), C(110), C(111), C(112), C(12),\\ &C(20), C(210), C(211), C(212), C(220), C(221), C(222)\big).\end{aligned}$$

The induced vertex depth order on $\partial\mathcal{P}$ is:

$$\operatorname{depth}(\partial\mathcal{P}) = (x, 0x, 1x, 2x, 11x, 21x, 22x)\,.$$

The induced cell depth order $\mathcal{P}$ is

$$\begin{aligned}\operatorname{lex}(\mathcal{P}) = \big(&C(00), C(01), C(02), C(10), C(12), C(20), C(110), C(111),\\ &C(112), C(210), C(211), C(212), C(220), C(221), C(222)\big).\end{aligned}$$

{sacred}

Lemma 3.2.8. *Let $\mathcal{P}$ be a cellular partition of X. Let $C(\alpha) \in \mathcal{P}$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$). If $C(\beta)$ is a cell in $\mathcal{P}\backslash\{C(\alpha)\}$ (for some $\beta \in E(R)^*$) then $C(\beta) \subseteq C(\gamma)$ for some*

$$\gamma \in \left\{\delta e'_k \mid \delta = e_1 \dots e_{k-1} \prec \alpha, e'_k \in E(R)\backslash\{e_k\}, k = 1, \dots, n\right\}.$$

Proof. Let $\mathcal{P}$ be a cellular partition of X. Let $C(\alpha)$ and $C(\beta)$ be distinct cells in $\mathcal{P}$ (for some $\alpha, \beta \in E(R)^*$). By Lemma 3.1.10, $\alpha \perp \beta$. Let $\alpha = e_1 \dots e_n$ and $\beta = e'_1 \dots e'_m$. Then there exists a $k \in \mathbb{N}$ such that $e_1 \dots e_{k-1} = e'_1 \dots e'_{k-1}$ and $e_k \neq e'_k$. Let $\gamma = e_1 \dots e_{k-1}\, e'_k$. Then

$$\gamma \in \left\{\delta e'_k \mid \delta = e_1 \dots e_{k-1} \prec \alpha, e'_k \in E(R)\backslash\{e_k\}, k = 1, \dots, n\right\}.$$

and $\gamma \preceq \beta$ and by Lemma 3.1.10, $C(\beta) \subseteq C(\gamma)$. ☐

{rem}

Lemma 3.2.9. *Let $\mathcal{P}$ be a cellular partition of X. Let $\alpha w' \in \partial\mathcal{P}$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$ and $w' \in V_{\text{int}}(R)$). Then*

$$\alpha w, \alpha^{\dagger} w, \alpha^{2\dagger} w, \dots, \alpha^{n\dagger} w = w \in \partial\mathcal{P}$$

for all $w \in V_{\text{int}}(R)$.

Proof. Let $\mathcal{P}$ be a cellular partition of X. Let $\alpha w' \in \partial\mathcal{P}$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$ and $w' \in V_{\text{int}}(R)$). By Lemma 3.2.4, there exists a graph expansion $G_{\mathcal{P}}$ such that $\partial\mathcal{P} = V(G_{\mathcal{P}})$. Then $\alpha w' \in V(G_{\mathcal{P}})$. We will prove this result by induction on n.

Suppose $n = 0$. Then $\alpha w' = w'$. By Definition 2.1.9 of a graph expansion, the vertex the vertex w' came into existence when the edge ϵ in the graph G_0 was replaced by a copy of the replacement graph R. This implies that the vertices $w \in V(G_{\mathcal{P}})$ for all $w \in V_{\text{int}}(R)$.

Suppose that there exists $m \in \mathbb{N}$ such that the result is true for $n = m$. Let us examine the case when $n = m + 1$. Then $\alpha w' = e_1 \dots e_{m+1} w' \in V(G_{\mathcal{P}})$. By Definition 2.1.9 of a graph expansion, the vertex $e_1 \dots e_{m+1} w'$ came into existence when the edge $e_1 \dots e_{m+1}$ was replaced by a copy of the replacement graph R. This implies that the vertices $e_1 \dots e_{m+1} w \in V(G_{\mathcal{P}})$ for all $w \in V_{\text{int}}(R)$. Observe by Lemma 2.1.14 that the edge $e_1 \dots e_{m+1}$ is bordered by a vertex $e_1 \dots e_m w''$ (for some $w'' \in V_{\text{int}}(R)$). Hence $\alpha^{\dagger} w'' \in V(G_{\mathcal{P}})$. Then, by our inductive hypothesis,

$$\alpha w, \alpha^{\dagger} w, \alpha^{2\dagger} w, \dots, \alpha^{n\dagger} w = w \in V(G_{\mathcal{P}}) = \partial \mathcal{P}$$

for all $w \in V_{\text{int}}(R)$.

This proves the result by induction. □

{kanye}

Lemma 3.2.10. *Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) and let $\mathcal{P}$ be a cellular partition of X. Then there exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that*

$$\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing.$$

Proof. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) and let $\mathcal{P}$ be a cellular partition of X.

We shall prove this result using information about the cardinalities of basic open sets in Ω and cells in X. Observe that $|\Omega| = 2^{\aleph_0}$, since there are uncountably many sequences of a finite alphabet. Then $|\Omega(\alpha)| = |\Omega| = 2^{\aleph_0}$. Observe that since, for a vertex $v \in \mathcal{GV}$, the set Q_v is finite, the quotient map ϕ identifies at most $\aleph_0$ sequences in $\Omega(\alpha)$ together. Hence $|C(\alpha)| = 2^{\aleph_0}$. It follows that $|\operatorname{int} C(\alpha)| = 2^{\aleph_0}$, since we have only removed a finite number of boundary points.

A cellular partition $\mathcal{P}$ of X is a cover of X by finitely many cells with disjoint interiors. Let $\mathcal{P} = \{C_1, \dots, C_m\}$ for some positive integer m. Recall that these cells share boundary points. Let $\partial \mathcal{P} = \{z_1, \dots, z_n\}$ for some positive integer n. Then X is the disjoint union:

$$X = \bigsqcup_{i=1}^{m} \operatorname{int} C_i \bigsqcup_{j=1}^{n} \{z_j\}.$$

Since $\operatorname{int} C(\alpha) \subseteq X$, then this disjoint union is a cover of $\operatorname{int} C(\alpha)$. Since a finite number of boundary points cannot cover an uncountable set, this implies that there exists some $C_i = C(\beta) \in \mathcal{P}$ such that

$$\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing.$$

□

{ghostman}

Lemma 3.2.11. *Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) and let $\mathcal{P}$ be a cellular partition of X. Then one of the following holds:*

1. *There exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $C(\beta) \supseteq C(\alpha)$.*

2. *There exist cells $C(\beta_1), \dots, C(\beta_m) \in \mathcal{P}$ (for some $\beta_1, \dots, \beta_m \in E(R)^*$) with*

$$C(\alpha) = \bigcup_{i=1}^{m} C(\beta_i).$$

Proof. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) and let $\mathcal{P}$ be a cellular partition of X. Suppose there is no cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $C(\beta) \supseteq C(\alpha)$. Consider all cells $C(\beta_1), \ldots, C(\beta_m) \in \mathcal{P}$ (for some $\beta_1, \ldots, \beta_m \in E(R)^*$) such that $\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta_i) \neq \varnothing$ (for $i = 1, \ldots, m$). By Lemma 3.1.10 (3),

$$C(\alpha) \supseteq \bigcup_{i=1}^{m} C(\beta_i).$$

If $x \in C(\alpha)$, then there exists a $y \in \Omega(\alpha)$ such that $(y)\phi = x$. Then, by Lemma 3.2.2, there exists a word $\gamma \in E(R)^*$ such that $C(\gamma) \in \mathcal{P}$ and either $\gamma \preceq \alpha$ or $\gamma \succeq \alpha$. But by our assumption and Lemma 3.1.10, $\alpha \not\succeq \gamma$. Hence $\alpha \preceq \gamma$, and $C(\gamma) \subseteq C(\alpha)$. Thus $\gamma = \beta_i$ for some $i = 1, \ldots, m$ and $x \in C(\beta_i)$. This gives us

$$C(\alpha) \subseteq \bigcup_{i=1}^{m} C(\beta_i).$$

This proves the result. ☐

We observe that cellular partitions have a lattice structure, detailed as follows:

{majorparkinso

Definition 3.2.12. Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X.

1. We define the *meet* $\mathcal{P} \wedge \mathcal{Q}$ as the set of cells from $\mathcal{P}$ or $\mathcal{Q}$ which do not properly contain other cells from $\mathcal{P}$ or $\mathcal{Q}$.

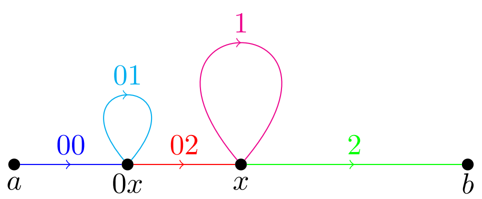
2. We define the *join* $\mathcal{P} \vee \mathcal{Q}$ as the set of cells from $\mathcal{P}$ or $\mathcal{Q}$ which are not properly contained in other cells from $\mathcal{P}$ or $\mathcal{Q}$.

Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X. Let $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in \Omega$) and let $C(\beta) \in \mathcal{Q}$ (for some $\beta \in \Omega$) such that $\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing$. Then, by Lemma 3.1.10 (3), one contains the other. Suppose without a meaningful loss of generality, $C(\beta) \subseteq C(\alpha)$. Then $C(\beta) \in \mathcal{P} \wedge \mathcal{Q}$ and $C(\alpha) \in \mathcal{P} \vee \mathcal{Q}$. We know from Lemma 3.2.11 (2) that, if $C(\alpha) \in \mathcal{P}$, then there exist cells $C(\beta_1), \ldots, C(\beta_m) \in \mathcal{P} \wedge \mathcal{Q}$ (for some $\beta_1, \ldots, \beta_m \in E(R)^*$) with

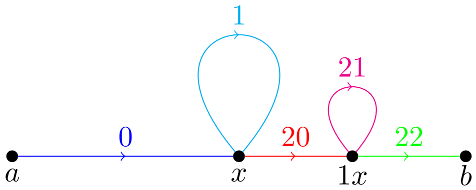
$$C(\alpha) = \bigcup_{i=1}^{m} C(\beta_i).$$

Example 3.2.13. Let X be the limit space of the F-Basilica replacement system. Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X, defined as follows:

$$\mathcal{P} = \{C(00), C(01), C(02), C(1), C(2)\}$$

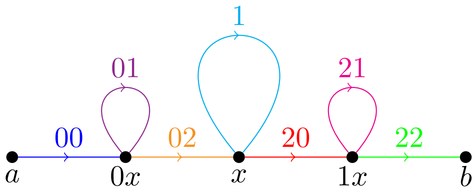


$$\mathcal{Q} = \{C(0), C(1), C(20), C(21), C(22)\},$$

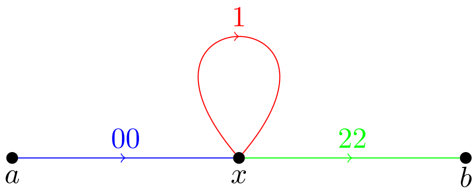


Then the refinement $\mathcal{P} \wedge \mathcal{Q}$ and corruption $\mathcal{P} \vee \mathcal{Q}$ are as follows:

$$\mathcal{P} \wedge \mathcal{Q} = \{C(00), C(01), C(02), C(1), C(20), C(21), C(22)\}$$



$$\mathcal{P} \vee \mathcal{Q} = \{C(0), C(1), C(2)\}.$$



{intruder}

Lemma 3.2.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X. Then the meet $\mathcal{P} \wedge \mathcal{Q}$ and join $\mathcal{P} \vee \mathcal{Q}$ are cellular partitions of X.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X. Observe that if $C(\alpha) \in \mathcal{P} \wedge \mathcal{Q}$ or $C(\alpha) \in \mathcal{P} \vee \mathcal{Q}$ (for some $\alpha \in E(R)^*$) then $C(\alpha) \in \mathcal{P} \cup \mathcal{Q}$. It follows that both $\mathcal{P} \wedge \mathcal{Q}$ and $\mathcal{P} \vee \mathcal{Q}$ have a finite number of cells, since $\mathcal{P}$ and $\mathcal{Q}$ have a finite number of cells.

Suppose there exist cells $C(\alpha), C(\beta) \in \mathcal{P} \wedge \mathcal{Q}$ or $C(\alpha), C(\beta) \in \mathcal{P} \vee \mathcal{Q}$ (for some $\alpha, \beta \in E(R)^*$) with $\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing$. Then, by Lemma 3.1.10, either $C(\alpha) \subseteq C(\beta)$ or $C(\alpha) \supseteq C(\beta)$. Hence, by definition of meet and join, $C(\alpha) = C(\beta)$.

It remains to show that $\mathcal{P} \wedge \mathcal{Q}$ and $\mathcal{P} \vee \mathcal{Q}$ cover X. First let us prove this for $\mathcal{P} \wedge \mathcal{Q}$: Consider a point $x \in X$, then $x \in C(\alpha)$ for some $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$). Either $C(\alpha) \in \mathcal{P} \wedge \mathcal{Q}$ or, by Lemma 3.2.11 (2), $C(\alpha) = \bigcup_{i=1}^{m} C(\beta_i)$ for some cells $C(\beta_1), \ldots, C(\beta_m) \in \mathcal{Q}$ (for some $\beta_1, \ldots, \beta_m \in E(R)^*$). By the definition of meet, $C(\beta_1), \ldots, C(\beta_m) \in \mathcal{P} \wedge \mathcal{Q}$, and at least one of the cells must contain x. This proves that $\mathcal{P} \wedge \mathcal{Q}$ is a cover of X.

Now let us prove this for $\mathcal{P} \vee \mathcal{Q}$: Consider a point $x \in X$, then $x \in C(\alpha)$ for some $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$). Either $C(\alpha) \in \mathcal{P} \vee \mathcal{Q}$ or, by Lemma 3.2.11 (1), $C(\alpha) \subset C(\beta)$ for some cell $C(\beta) \in \mathcal{Q}$ (for some $\beta \in E(R)^*$). By the definition of join, $C(\beta) \in \mathcal{P} \vee \mathcal{Q}$, and $C(\beta)$ must contain x. This proves that $\mathcal{P} \vee \mathcal{Q}$ is a cover of X. $\square$

3.3 Rearrangements

{3.3}

In this section, we will define rearrangements of the limit space X, and discuss their topological properties.

{available}

Definition 3.3.1. Let $\Omega(\alpha)$ and $\Omega(\beta)$ be basic open sets in the symbol space Ω, for some $\alpha = e_1 \dots e_n$ and $\beta = e'_1 \dots e'_k$ in $E(R)^*$. We define a *prefix replacement map*

$$\Psi \colon \Omega(\alpha) \to \Omega(\beta)$$

by

$$(\alpha\ e_{n+1}e_{n+2}\dots)\Psi = \beta\ e_{n+1}e_{n+2}\dots.$$

Lemma 3.3.2. *The map Ψ is a homeomorphism.*

Proof. Let $\Omega(\alpha)$ and $\Omega(\beta)$ be basic open sets in Ω, for some $\alpha = e_1 \dots e_n$ and $\beta = e'_1 \dots e'_k$ in $E(R)^*$. Let $\Psi \colon \Omega(\alpha) \to \Omega(\beta)$ be a prefix replacement map.

A basic open set in $\Omega(\beta)$ is of the form $\Omega(\beta\gamma)$, for some fixed $\gamma \in E(R)^*$. The preimage of $\Omega(\beta\gamma)$ under Ψ is $\Omega(\alpha\gamma)$, which is a basic open set in $\Omega(\alpha)$. This proves that Ψ is continuous.

We can define the map

$$\Psi^{-1} \colon \Omega(\beta) \to \Omega(\alpha)$$

by

$$(\beta\ e_{k+1}e_{k+2}\dots)\Psi^{-1} = \alpha\ e_{k+1}e_{k+2}\dots.$$

The composition $\Psi\Psi^{-1} = \Psi^{-1}\Psi$ is the identity homeomorphism I, which shows that Ψ^{-1} is the inverse map of Ψ. This proves that the Ψ is bijective.

We observe that Ψ^{-1} is also a prefix replacement map, and hence it is continuous by the same argument as Ψ. This proves that Ψ is a homeomorphism. □

{extremeways}

Lemma 3.3.3. *Let $\Omega(\alpha)$ and $\Omega(\beta)$ be basic open sets in Ω (for some $\alpha, \beta \in E(R)^*$). Let both α and β either be loops or non-loops. Let $\Psi \colon \Omega(\alpha) \to \Omega(\beta)$ be the prefix replacement map. Suppose y and z are two distinct sequences in $\Omega(\alpha)$. Then $y \sim z$ if and only if $(y)\Psi \sim (z)\Psi$.*

Proof. Let $\Omega(\alpha)$ and $\Omega(\beta)$ be basic open sets in Ω (for some $\alpha, \beta \in E(R)^*$). Let both α and β either be loops or non-loops. Let $\Psi \colon \Omega(\alpha) \to \Omega(\beta)$ be the prefix replacement map.

1. Consider $\alpha\bar{\mathbf{i}}, \alpha\bar{\mathbf{t}} \in \Omega(\alpha)$. If α is a loop, then by Remark 3.1.4, $\alpha\bar{\mathbf{i}} \sim \alpha\bar{\mathbf{t}}$. By our hypothesis, β is also a loop and $\beta\bar{\mathbf{i}} \sim \beta\bar{\mathbf{t}}$. Observe that $(\alpha\bar{\mathbf{i}})\Psi = \beta\bar{\mathbf{i}}$ and $(\alpha\bar{\mathbf{t}})\Psi = \beta\bar{\mathbf{t}}$. Hence $\alpha\bar{\mathbf{i}} \sim \alpha\bar{\mathbf{t}}$ if and only if $\beta\bar{\mathbf{i}} \sim \beta\bar{\mathbf{t}}$.

2. Consider two distinct sequences y and z in $\Omega(\alpha)\backslash\{\alpha\bar{\mathbf{i}}, \alpha\bar{\mathbf{t}}\}$. Observe that $(y)\phi, (z)\phi \in \operatorname{int} C(\alpha)$.

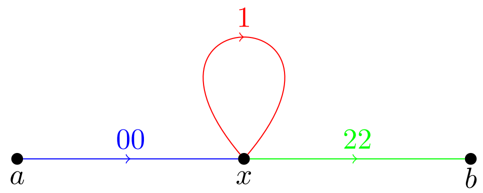
Suppose $y \sim z$. Then, by Lemma 2.3.15, there exists a gluing vertex $\gamma w \in \mathcal{GV}$ (for some $\gamma \in E(R)^*$ and $w \in V_{\text{int}}(R)$) such that $y, z \in Q_{\gamma w}$. By Corollary 3.1.7, $\alpha \preceq \gamma$. Let $\gamma = \alpha\delta$. Then, by Definition 3.3.1 of a prefix replacement map, $(Q_{\alpha\delta w})\Psi = Q_{\beta\delta w}$ and $(y)\Psi, (z)\Psi \in Q_{\beta\delta w}$. This implies that $(y)\Psi \sim (z)\Psi$.

Conversely, suppose $(y)\Psi \sim (z)\Psi$. Then, by Lemma 2.3.15, there exists a gluing vertex $\gamma w \in \mathcal{GV}$ (for some $\gamma \in E(R)^*$ and $w \in V_{\text{int}}(R)$) such that $(y)\Psi, (z)\Psi \in Q_{\gamma w}$. By Corollary 3.1.7, $\beta \preceq \gamma$. Let $\gamma = \beta\delta$. Then, by Definition 3.3.1 of a prefix replacement map, $(Q_{\beta\delta w})\Psi^{-1} = Q_{\alpha\delta w}$ and $y, z \in Q_{\alpha\delta w}$. This implies that $y \sim z$.

This proves the result. $\square$

Observe that the prefix replacement map Ψ does not preserve the equivalence relation $\sim$ if it is defined between a loop and a non-loop, as illustrated by the following example:

Example 3.3.4. Let X be the limit space of the F-Basilica replacement system.



Let us define the prefix replacement map $\Psi\colon \Omega(1) \to \Omega(0)$ Then $1\bar{0} \sim 1\bar{2}$ but $(1\bar{0})\Psi \nsim (1\bar{2})\Psi$.

Recall that the quotient map ϕ maps each sequence in Ω to its equivalence class under $\sim$ in X. Let $\Omega(\alpha)$ and $\Omega(\beta)$ be basic open sets in Ω, for some $\alpha, \beta \in E(R)^*$. Let $\Psi\colon \Omega(\alpha) \to \Omega(\beta)$ be a prefix replacement map. The map Ψ induces the map

$$\psi\colon C(\alpha) \to C(\beta),$$

where $C(\alpha)$ and $C(\beta)$ are cells in X. Observe that ψ is well-defined by Lemma 3.3.3 on $\sim$-equivalence classes. Hence the following diagram commutes:

$$\begin{tikzcd}
\Omega(\alpha) \arrow[r, "\Psi"] \arrow[d, "\phi"] & \Omega(\beta) \arrow[d, "\phi"] \\
C(\alpha) \arrow[r, "\psi"'] & C(\beta)
\end{tikzcd}$$

omorrowpeople}

Definition 3.3.5. Let $C(\alpha)$ and $C(\beta)$ be cells in X. We define the *canonical homeomorphism* of cells in X to be the map $\psi\colon C(\alpha) \to C(\beta)$.

his term is justified by the following result:

{hengebeat}

Lemma 3.3.6. *A canonical homeomorphism is a homeomorphism. Moreover, the inverse of a canonical homeomorphism is a canonical homeomorphism.*

Proof. Let $C(\alpha)$ and $C(\beta)$ be cells in X (for some $\alpha, \beta \in E(R)^*$) and let $\psi\colon C(\alpha) \to C(\beta)$ be a canonical homeomorphism. The canonical homeomorphism ψ is induced from the prefix replacement map $\Psi\colon \Omega(\alpha) \to \Omega(\beta)$, where $\Omega(\alpha)$ and $\Omega(\beta)$ are basic open sets in Ω. We have to show that ψ is a continuous map with a continuous inverse.

To prove that the map ψ is continuous, we have to show that the preimage $(U)\psi^{-1}$ is open in $C(\alpha)$ for any open set U in $C(\beta)$. By the definition of the quotient topology, the set $(U)\psi^{-1}$ is open in $C(\alpha)$ if and only if the set $(U)\psi^{-1}\phi^{-1}$ is open in $\Omega(\alpha)$. We observe that, from the definition of ψ,

$$\begin{aligned}(U)\psi^{-1}\phi^{-1} &= (U)(\phi\psi)^{-1}\\ &= (U)(\Psi\phi)^{-1}\\ &= (U)\phi^{-1}\Psi^{-1}\end{aligned}$$

The set $(U)\phi^{-1}\Psi^{-1}$ is open since $\Psi\phi$ is a continuous map (since it is the composition of continuous maps). Therefore the set $(U)\psi^{-1}$ is open in $C(\alpha)$, which proves that the map ψ is continuous.

We now prove that ψ has a continuous inverse. The homeomorphism Ψ^{-1} induces the map

$$\psi^{-1}\colon C(\beta) \to C(\alpha)$$

similarly to the map ψ. Observe that

$$\phi\psi\psi^{-1} = \Psi\phi\psi^{-1} = \Psi\Psi^{-1}\phi = \phi$$

from the definition of ψ^{-1} (since Ψ^{-1} is the inverse of Ψ). Hence, $(y)\phi\psi\psi^{-1} = (y)\phi$ for all $y \in \Omega(\alpha)$, i.e., $(x)\psi\psi^{-1} = x$ for all $x \in C(\alpha)$. Thus $\psi\psi^{-1}$ is the identity on $C(\alpha)$. Similarly, we can show that $\psi^{-1}\psi$ is the identity of $C(\beta)$. Hence ψ^{-1} is the inverse map of ψ. We observe that ψ^{-1} is continuous by the same argument which shows that ψ is continuous.

This proves that ψ is a homeomorphism. □

{youmeandmeyou

Lemma 3.3.7. *Let $C(\alpha)$ and $C(\beta)$ be cells in X (for some $\alpha, \beta \in E(R)^*$) and let $\psi\colon C(\alpha) \to C(\beta)$ be a canonical homeomorphism. Consider a cell $C(\gamma)$ in X such that $C(\gamma) \subseteq C(\beta)$. Then $\psi|_{C(\gamma)}\colon C(\gamma) \to C(\delta)$ is a canonical homeomorphism, for some $C(\delta) \subseteq C(\beta)$.*

Proof. Let $C(\alpha)$ and $C(\beta)$ be cells in X (for some $\alpha, \beta \in E(R)^*$) and let $\psi\colon C(\alpha) \to C(\beta)$ be a canonical homeomorphism. The canonical homeomorphism ψ is induced from the prefix replacement map $\Psi\colon \Omega(\alpha) \to \Omega(\beta)$, where $\Omega(\alpha)$ and $\Omega(\beta)$ are basic open sets in Ω.

Since $C(\gamma) \subseteq C(\alpha)$, then by Lemma 3.1.10, $\alpha \preceq \gamma$. Let $\alpha = e_1 \dots e_n$, $\beta = e'_1 \dots e'_k$ and $\gamma = e_1 \dots e_m$ for some $m \geq n$. Consider an element $x \in C(\gamma)$, and say $x = (e_1 \dots e_m\ e_{m+1}e_{m+2} \dots)\phi$. Then

$$\begin{aligned}(x)\psi &= (e_1 \dots e_m\ e_{m+1}e_{m+2} \dots)\phi\Psi \\ &= (e'_1 \dots e'_k\ e_{n+1} \dots e_m\ e_{m+1}e_{m+2} \dots)\phi.\end{aligned}$$

Let $\delta = e'_1 \dots e'_k\ e_{n+1} \dots e_m$. Then $\psi|_{C(\gamma)}$ is induced by the prefix replacement map $\Psi|_{\Omega(\gamma)} \colon \Omega(\gamma) \to \Omega(\delta)$. Therefore $\psi|_{C(\gamma)} \colon C(\gamma) \to C(\delta)$ is a canonical homeomorphism. $\square$

Recall that a cellular partition $\mathcal{P}$ of X is a cover of X by finitely many cells with disjoint interiors. The following definition is equivalent to Belk, Forrest [3] ????:

{omd}

Definition 3.3.8. A homeomorphism $f \colon X \to X$ is called a *rearrangement* of X if there is a cellular partition $\mathcal{P}$ of X such that

1. the set $\{(C)f \mid C \in \mathcal{P}\}$ is also a cellular partition of X,
2. the restriction $f|_C : C \to (C)f$ is a canonical homeomorphism for each cell $C \in \mathcal{P}$.

We denote the set $\{(C)f \mid C \in \mathcal{P}\}$ by $(\mathcal{P})f$. We call the pair $(\mathcal{P}, (\mathcal{P})f)$ a *cellular bipartition* for the rearrangement f.

Recall from Lemma 3.2.2 that cellular partitions of X are characterized by complete antichains. Then every rearrangement can be defined as a bijection between complete antichains of the same cardinality. This is how one would normally characterize elements of groups similar to Thompson's group V [9], with the equivalence that expanding any word in the domain antichain results in an expansion in the range antichain, which preserves the map. (Observe that we are not allowing all possible bijections, only those which respect the structure of the limit space.) This provides a natural link between rearrangement groups and Thompson theory.

Recall that each cellular partition $\mathcal{P}$ of X can be illustrated by a graph expansion $G_{\mathcal{P}}$. We present the following definition for graph-pair diagrams. This definition is equivalent to the one found in Belk, Forrest [3] Definition 1.16.

{nospillblood}

Definition 3.3.9. Let f be a rearrangement of X. Let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. A graph-pair diagram for f is the pair $(G_{\mathcal{P}}, G_{(\mathcal{P})f})$.

We will use graph-pair diagrams to illustrate rearrangements of X.

The following results characterize rearrangements of X:

{reservoir}

Lemma 3.3.10. *Let $\mathcal{P} = \{C(\alpha_1), \dots, C(\alpha_n)\}$ and $\mathcal{Q} = \{C(\beta_1), \dots, C(\beta_n)\}$ be cellular partitions of X. Let $\psi_i \colon C(\alpha_i) \to C(\beta_i)$ be a canonical homeomorphism for all $i = 1, \dots, n$, such that if z is a boundary point of both $C(\alpha_i)$ and $C(\alpha_j)$ then $(z)\psi_i = (z)\psi_j$ and if z is a boundary point of both $C(\beta_i)$ and $C(\beta_j)$ then $(z)\psi_i^{-1} = (z)\psi_j^{-1}$. Then the map $f \colon X \to X$ given by $(x)f = (x)\psi_i$ when $x \in C(\alpha_i)$ is a well-defined rearrangement of X and $(\mathcal{P}, \mathcal{Q})$ is a cellular bipartition for f.*

Proof. Let $\mathcal{P} = \{C(\alpha_1), \ldots, C(\alpha_n)\}$ and $\mathcal{Q} = \{C(\beta_1), \ldots, C(\beta_n)\}$ be cellular partitions of X. Let $\psi_i\colon C(\alpha_i) \to C(\beta_i)$ be a canonical homeomorphism for all $i = 1, \ldots, n$, such that if z is a boundary point of both $C(\alpha_i)$ and $C(\alpha_j)$ then $(z)\psi_i = (z)\psi_j$ and if z is a boundary point of both $C(\beta_i)$ and $C(\beta_j)$ then $(z)\psi_i^{-1} = (z)\psi_j^{-1}$. Let us define a map $f\colon X \to X$ by $(x)f = (x)\psi_i$ when $x \in C(\alpha_i)$. We will prove that f is a rearrangement of X by proving that f is a homeomorphism.

Let us prove that f is well-defined: Suppose that $x \in C(\alpha_i) \cap C(\alpha_j)$ for some $1 \leq i, j \leq n$ such that $i \neq j$. By Definition 3.2.1 of a cellular partition, the cells $C(\alpha_i)$ and $C(\alpha_j)$ have disjoint interiors. This implies that x is a boundary point of both cells. Then, by our hypothesis, $(x)\psi_i = (x)\psi_j$. This proves that f is well-defined.

Let us prove that f is a continuous map: Let U be an open set in X. Observe that

$$\begin{aligned} Uf^{-1}\phi^{-1} &= \left(\bigcup_{i=1}^{n} (U \cap C(\beta_i))\right) f^{-1}\phi_i^{-1} \\ &= \bigcup_{i=1}^{n} (U \cap C(\beta_i))\, f^{-1}\phi_i^{-1} \\ &= \bigcup_{i=1}^{n} (U \cap C(\beta_i))\, \psi_i^{-1}\phi_i^{-1} \\ &= \bigcup_{i=1}^{n} (U \cap C(\beta_i))\, \phi_i^{-1}\Psi_i^{-1} \end{aligned}$$

where $\Psi_i\colon \Omega(\alpha_i) \to \Omega(\beta_i)$ is the prefix replacement map which induces ψ_i for all $i = 1, \ldots, n$. By Definition 3.3.5 of a canonical homeomorphism, the following diagram commutes for all $i = 1, \ldots, n$:

$$\begin{tikzcd} \Omega(\alpha_i) \arrow[r, "\Psi_i"] \arrow[d, "\phi"] & \Omega(\beta_i) \arrow[d, "\phi"] \\ C(\alpha_i) \arrow[r, "\psi_i"'] & C(\beta_i) \end{tikzcd}$$

This implies that, since $U \cap C(\beta_i)$ is open in $C(\beta_i)$ for all $i = 1, \ldots, n$, then $(U \cap C(\beta_i))\phi_i^{-1}\Psi_i^{-1}$ is open in $\Omega(\alpha_i)$ for all $i = 1, \ldots, n$. Since $\Omega(\alpha_i)$ is open in Ω for all $i = 1, \ldots, n$, then $(U \cap C(\beta_i))\phi_i^{-1}\Psi_i^{-1}$ is open in Ω for all $i = 1, \ldots, n$. Hence $Uf^{-1}\phi^{-1}$ is open in Ω and thus $(U)f^{-1}$ is open in X. This proves that f is a continuous map.

Let us prove that the inverse map f^{-1} exists: By Definition 3.3.5 of a canonical homeomorphism, the inverse homeomorphism $\psi_i^{-1}\colon C(\beta_i) \to C(\alpha_i)$ exists for all $i = 1, \ldots, n$. By our hypothesis, if z is a boundary point of both $C(\beta_i)$ and $C(\beta_j)$ then $(z)\psi_i^{-1} = (z)\psi_j^{-1}$. Let us define a map $f^{-1}\colon X \to X$ by $(x)f = (x)\psi_i^{-1}$ when $x \in C(\beta_i)$ for some $1 \leq i \leq n$. Choose $x \in X$, then $x \in C(\beta_i)$ for some $1 \leq i \leq n$. Then $(x)ff^{-1} = (x)\psi_i\psi_i^{-1}$. Since ψ_i is a homeomorphism, $(x)\psi_i\psi_i^{-1} = x$. Since this is true for all $x \in X$, it follows that

f^{-1} is the inverse map of the map f. The map f^{-1} can be shown to be well-defined and continuous in the same way as f.

This proves that f is a continuous map with a continuous inverse, i.e., a homeomorphism. $\square$

Observe that a map might not be a rearrangement of an F-type edge replacement system unless the behaviour of the rearrangement at the boundary points is explicitly defined, as seen in the following example:

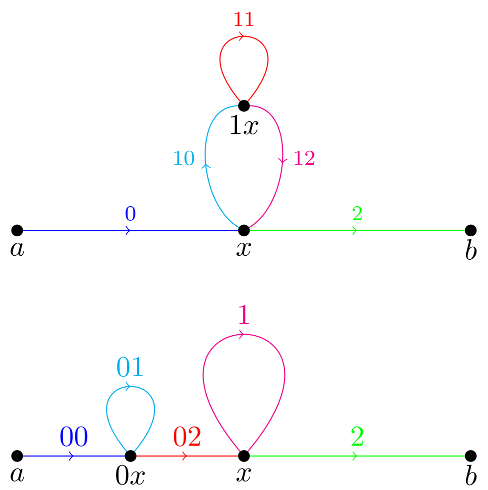
Example 3.3.11. Let X be the limit space of the F-Basilica replacement system. Let us define a map $h\colon X \to X$ as follows:

$$([e_1e_2e_3\ldots])\,h = \begin{cases} [00e_3e_4\ldots], & \text{if } e_1 = 0, \\ [01e_3e_4\ldots], & \text{if } e_1e_2 = 10, \\ [02e_3e_4\ldots], & \text{if } e_1e_2 = 11, \\ [1e_2e_3\ldots], & \text{if } e_1e_2 = 12, \\ [2e_2e_3\ldots], & \text{if } e_1 = 2. \end{cases}$$

Observe that cells get mapped as follows:

$$\begin{aligned} (C(0))\,h &= C(00), \\ (C(10))\,h &= C(01), \\ (C(11))\,h &= C(02), \\ (C(12))\,h &= C(1), \\ (C(2))\,h &= C(2). \end{aligned}$$

This map is illustrated by the following graph-pair diagram:



However, h is not a homeomorphism on the boundary points of these cells. Recall that $(x)\phi^{-1} = \{0\bar{2}, 1\bar{0}, 1\bar{2}, 2\bar{0}\}$. Then $(x)h = \{0x, x\}$. Similarly, $(1x)\phi^{-1} = \{10\bar{2}, 11\bar{0}, 11\bar{2}, 12\bar{0}\}$. Then $(1x)h = \{0x, x\}$. Hence h is not a rearrangement of X.

outhofamerica}

Lemma 3.3.12. *Let $\mathcal{P}$ be a cellular partition of X and let f be a rearrangement of X. If $f|_{C(\alpha)}$ is a canonical homeomorphism for all $C(\alpha) \in \mathcal{P}$, then $(\mathcal{P}, (\mathcal{P})f)$ is a cellular bipartition for f.*

Proof. Let $\mathcal{P}$ be a cellular partition of X and let f be a rearrangement of X. Let $f|_{C(\alpha)}$ be a canonical homeomorphism for all $C(\alpha) \in \mathcal{P}$. To prove that $(\mathcal{P}, (\mathcal{P})f)$ is a cellular bipartition for f, we have to show that $(\mathcal{P})f$ is also a cellular partition.

Observe that $(\mathcal{P})f$ contains finitely many cells since $\mathcal{P}$ contains finitely many cells. Let us prove that the cells in $(\mathcal{P})f$ have disjoint interiors: Suppose there exist cells $(C(\alpha_1))f, (C(\alpha_2))f \in (\mathcal{P})f$ (for some $\alpha_1, \alpha_2 \in E(R)^*$) such that

$$\operatorname{int}(C(\alpha_1))f \cap \operatorname{int}(C(\alpha_2))f \neq \varnothing.$$

Since f is a homeomorphism, $\operatorname{int} C(\alpha_1) \cap \operatorname{int} C(\alpha_2) \neq \varnothing$ for some cells $C(\alpha_1), C(\alpha_2) \in \mathcal{P}$. This is a contradiction, therefore

$$\operatorname{int}(C(\alpha_1))f \cap \operatorname{int}(C(\alpha_2))f = \varnothing$$

for all cells $(C(\alpha_1))f, (C(\alpha_2))f \in (\mathcal{P})f$.

Let us prove that $(\mathcal{P})f$ is a cover of X: Consider $x \in X$. Then there exists $(x)f^{-1} \in X$. Since $\mathcal{P}$ is a cover of X, there exists a cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $(x)f^{-1} \in C(\alpha)$. Then $x \in (C(\alpha))f$, which is a cell in $(\mathcal{P})f$.

This proves that $(\mathcal{P})f$ is a cellular partition of X, and hence $(\mathcal{P}, (\mathcal{P})f)$ is a cellular bipartition for f. □

{streetofdream

Example 3.3.13. Let X be the limit space of the F-Basilica replacement system. Let us define a map $g \colon X \to X$ as follows:

$$([e_1e_2e_3\ldots])\, g = \begin{cases} [0e_3e_4\ldots], & \text{if } e_1e_2 = 00 \\ [1e_3e_4\ldots], & \text{if } e_1e_2 = 01 \\ [20e_3e_4\ldots], & \text{if } e_1e_2 = 02 \\ [21e_2e_3\ldots], & \text{if } e_1 = 1 \\ [22e_2e_3\ldots], & \text{if } e_1 = 2 \end{cases}$$

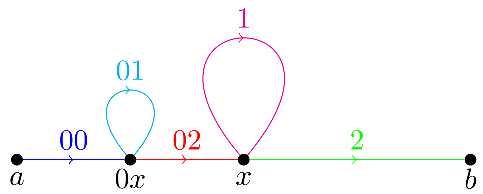
Observe that the cells and boundary boundary points get mapped as follows:

$$\begin{aligned} (C(00))\, g &= C(0) \\ (C(01))\, g &= C(1) \\ (C(02))\, g &= C(20) \\ (C(1))\, g &= C(21) \\ (C(2))\, g &= C(22) \end{aligned}$$

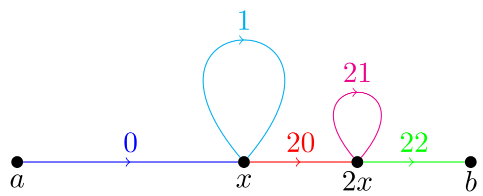
$$(0x)\,g = x$$
$$(x)\,g = 2x$$

Then the map g is a rearrangement by Lemma 3.3.10, and we can define the cellular bipartition $(\mathcal{P}, (\mathcal{P})g)$ such that g restricts to a canonical homeomorphism on each cell in $\mathcal{P}$. We illustrate this cellular bipartition using graph-pair diagrams:

$$\mathcal{P} = \{C(00), C(01), C(02), C(1), C(2)\}$$



$$(\mathcal{P})g = \{C(0), C(1), C(20), C(21), C(22)\}$$



andenburggate}

Definition 3.3.14. Let f be a rearrangement of X. The *support* of f, denoted by $\operatorname{supp} f$, is

$$\operatorname{supp} f = \{x \mid x \in X, x \neq (x)f\}.$$

Cellular bipartitions for a rearrangement f of X are not unique: there can be many pairs $(\mathcal{P}, (\mathcal{P})f)$ (where $\mathcal{P}$ is a cellular partition of X) for which a rearrangement f restricts to a canonical homeomorphism on each cell. One method to find new cellular bipartitions for f is to "refine" existing cellular bipartitions.

{giltcomplex}

Definition 3.3.15. Let f be a rearrangement of X. Let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $\mathcal{Q}$ be a cellular partition. Then the pair $(\mathcal{Q}, (\mathcal{Q})f)$ is called a *refinement* of $(\mathcal{P}, (\mathcal{P})f)$ if and only if for each cell $C(\alpha) \in \mathcal{Q}$ (for some $\alpha \in E(R)^*$), there exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) with $C(\alpha) \subseteq C(\beta)$. The pair $(\mathcal{P}, (\mathcal{P})f)$ is called a *coarsening* of $(\mathcal{Q}, (\mathcal{Q})f)$.

wonderfeeling}

Lemma 3.3.16. *Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $(\mathcal{Q}, (\mathcal{Q})f)$ be a refinement of $(\mathcal{P}, (\mathcal{P})f)$. Then $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f.*

Proof. Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $(\mathcal{Q}, (\mathcal{Q})f)$ be a refinement of $(\mathcal{P}, (\mathcal{P})f)$. Then, by Definition 3.3.15 of a refinement, for each cell $C(\alpha) \in \mathcal{Q}$ (for some $\alpha \in E(R)^*$), there exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$)

with $C(\alpha) \subseteq C(\beta)$. By Definition 3.3.8 of a rearrangement, f, $f|_{C(\beta)}$ is a canonical homeomorphism. Then, by Lemma 3.3.7, $f|_{C(\alpha)}$ is also a canonical homeomorphism. This is true for all cells $C(\alpha) \in \mathcal{Q}$, hence, by Lemma 3.3.12, $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f. $\square$

{armyofcops}

Lemma 3.3.17. *Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $(\mathcal{Q}, (\mathcal{Q})f)$ be a coarsening of $(\mathcal{P}, (\mathcal{P})f)$. Then $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f if and only if $f|_{C(\alpha)}$ is a canonical homeomorphism for all cells $C(\alpha) \in \mathcal{Q}$ (for some $\alpha \in E(R)^*$).*

Proof. Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $(\mathcal{Q}, (\mathcal{Q})f)$ be a coarsening of $(\mathcal{P}, (\mathcal{P})f)$. By Definition 3.3.15, Q is a cellular partition of X. Suppose that $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f. It follows from Definition 3.3.8 that $f|_{C(\alpha)}$ is a canonical homeomorphism for all cells $C(\alpha) \in \mathcal{Q}$ (for some $\alpha \in E(R)^*$).

Conversely, suppose that $f|_{C(\alpha)}$ is a canonical homeomorphism for all cells $C(\alpha) \in \mathcal{Q}$ (for some $\alpha \in E(R)^*$). By Lemma 3.3.12, $(\mathcal{Q})f$ is a cellular partition of X. Then by Definition 3.3.8 of a rearrangement, $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f. $\square$

{thirteen}

Corollary 3.3.18. *Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $\mathcal{Q}$ be a cellular partition of X. Then $(\mathcal{P} \wedge \mathcal{Q}, (\mathcal{P} \wedge \mathcal{Q})f)$ is a cellular bipartition for f.*

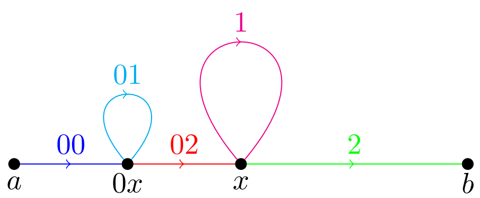
Proof. Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $\mathcal{Q}$ be a cellular partition of X. By Lemma 3.2.14, $\mathcal{P} \wedge \mathcal{Q}$ is a cellular partition of X and by Definition 3.2.12 of a meet, $(\mathcal{P} \wedge \mathcal{Q}, (\mathcal{P} \wedge \mathcal{Q})f)$ a refinement of $(\mathcal{P}, (\mathcal{P})f)$. Then, by Lemma 3.3.16, $(\mathcal{P} \wedge \mathcal{Q}, (\mathcal{P} \wedge \mathcal{Q})f)$ is a cellular bipartition for f. $\square$

{loveinacoldwo

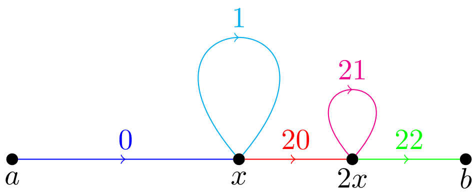
Remark 3.3.19. Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $\mathcal{Q}$ be a cellular partition of X. Then $(\mathcal{P} \vee \mathcal{Q}, (\mathcal{P} \vee \mathcal{Q})f)$ is a cellular bipartition for f if and only if $f|_{C(\alpha)}$ is a canonical homeomorphism for all cells $C(\alpha) \in \mathcal{P} \vee \mathcal{Q}$ (for some $\alpha \in E(R)^*$).

Example 3.3.20. Consider the rearrangement g of X defined in Example 3.3.13 with the following cellular bipartition (illustrated using graph-pair diagrams):

$$\mathcal{P} = \{C(00), C(01), C(02), C(1), C(2)\}$$

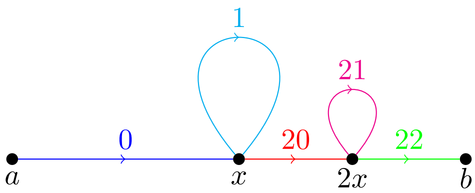


$$(\mathcal{P})g = \{C(0), C(1), C(20), C(21), C(22)\}$$



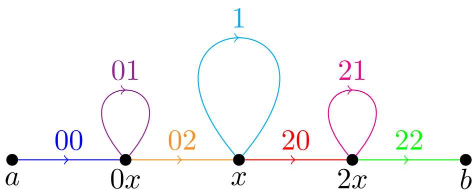
Let us define a cellular partition Q as follows:

$$\mathcal{Q} = \{C(0), C(1), C(20), C(21), C(22)\}$$



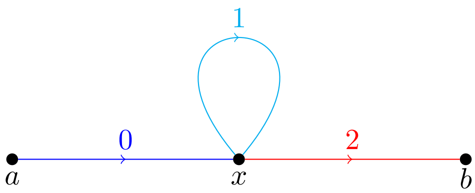
The meet $\mathcal{P} \wedge \mathcal{Q}$ is as follows:

$$\mathcal{P} \wedge \mathcal{Q} = \{C(00), C(01), C(02), C(1), C(20), C(21), C(22)\}$$



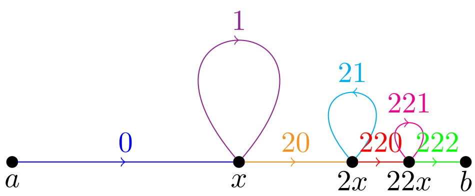
The join $\mathcal{P} \vee \mathcal{Q}$ is as follows:

$$\mathcal{P} \vee \mathcal{Q} = \{C(0), C(1), C(2)\}$$



Then, the refinement $(\mathcal{P} \wedge \mathcal{Q}, (\mathcal{P} \wedge \mathcal{Q})g)$ is a cellular bipartition for g but the coarsening $(\mathcal{P} \vee \mathcal{Q}, (\mathcal{P} \vee \mathcal{Q})g)$ is not a cellular bipartition for g, since the restriction of g on each cell of $\mathcal{P} \vee \mathcal{Q}$ is not a canonical homeomorphism. The cellular partition $(\mathcal{P} \wedge \mathcal{Q})g$ is as follows:

$$(\mathcal{P} \wedge \mathcal{Q})g = \{C(0), C(1), C(20), C(21), C(220), C(221), C(222)\}$$



{repeater}

Lemma 3.3.21. *Let f be a rearrangement of X and let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) such that $f|_{C(\alpha)}$ is a canonical homeomorphism. Then there exists a cellular bipartition $(\mathcal{P}, (\mathcal{P})f)$ for f such that $C(\alpha) \in \mathcal{P}$.*

Proof. Let f be a rearrangement of X and let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) such that $f|_{C(\alpha)}$ is a canonical homeomorphism. By Definition 3.3.8 there exists a cellular bipartition $(\mathcal{P}, (\mathcal{P})f)$ for f. Since $\mathcal{P}$ is a cellular partition then, by Lemma 3.2.10, there exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that

$$\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing.$$

Then, by Lemma 3.1.10 one of the following holds:

1. $C(\alpha) = C(\beta)$, which implies that $C(\alpha) \in \mathcal{P}$.

2. $C(\alpha) \supset C(\beta)$. Then, by Lemma 3.2.11 (2), there exist cells $C(\beta_1), \ldots, C(\beta_n) \in \mathcal{P}$ such that $C(\beta) = C(\beta_j)$ for some $1 \leq j \leq n$ and

$$C(\alpha) = \bigcup_{i=1}^{n} C(\beta_i).$$

   Let us define a set

$$\mathcal{Q} = (\mathcal{P} \backslash \{C(\beta_1), \ldots, C(\beta_n)\}) \cup \{C(\alpha)\}.$$

   Let us prove that $\mathcal{Q}$ is a cellular partition: Consider $x \in X$. Since $\mathcal{P}$ is a cover of X, either $x \in C(\beta_k)$ for some $1 \leq k \leq n$, in which case $x \in C(\alpha) \in \mathcal{Q}$. Or $x \in C(\gamma) \in \mathcal{P} \backslash \{C(\beta_1), \ldots, C(\beta_n)\}$ (for some $\gamma \in E(R)^*$), in which case $x \in C(\gamma) \in \mathcal{Q}$. Observe that there are finitely many cells in $\mathcal{Q}$, since there are finitely many cells in $\mathcal{P}$. Also observe that all cells in $\mathcal{P} \backslash \{C(\beta_1), \ldots, C(\beta_n)\}$ have disjoint interiors. Consider a cell $C(\gamma) \in \mathcal{P} \backslash \{C(\beta_1), \ldots, C(\beta_n)\}$ (for some $\gamma \in E(R)^*$). Then $\operatorname{int} C(\gamma) \cap \operatorname{int} C(\alpha) = \varnothing$, since $\operatorname{int} C(\gamma) \cap \operatorname{int} C(\beta_k) = \varnothing$ for all $k = 1, \ldots, n$. This proves that all cells in $\mathcal{Q}$ have disjoint interiors. Hence $\mathcal{Q}$ is a cellular partition of X.

   Observe that $(\mathcal{Q}, (\mathcal{Q})f)$ is a coarsening of $(\mathcal{P}, (\mathcal{P})f)$ with the rearrangement f restricting to a canonical homeomorphism on each cell in $\mathcal{Q}$. Then, by Lemma 3.3.17, $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f and $C(\alpha) \in \mathcal{Q}$.

3. $C(\alpha) \subset C(\beta)$. Let us prove that there exist cells $C(\alpha_1), \ldots C(\alpha_m) \in \mathcal{P}$ such that $C(\alpha) = C(\alpha_k)$ for some $1 \leq k \leq m$ and

$$C(\beta) = \bigcup_{i=1}^{m} C(\alpha_i).$$

   By Lemma 3.1.10 (2), $\beta \prec \alpha$. Then, $\alpha = \beta\gamma$ (for some $\gamma \in E(R)^*$). By Lemma 3.2.2, there exists a complete antichain A such that $\alpha \in A$. Then, by the properties of complete antichains, there exist $\alpha_1, \ldots, \alpha_m \in A$ such that $\alpha = \alpha_k$ for some $1 \leq k \leq m$, $\alpha_i = \beta\delta_i$ for all $i = 1, \ldots, m$ and $D = \{\delta_i \mid i = 1, \ldots m\}$ is a complete antichain.

Let us define a set

$$\mathcal{S} = (\mathcal{P} \backslash \{C(\beta)\}) \cup \{C(\alpha_1), \ldots, C(\alpha_m)\}.$$

Let us prove that $\mathcal{S}$ is a cellular partition: Consider $x \in X$. Since $\mathcal{P}$ is a cover of X, either $x \in C(\beta)$, in which case $x \in C(\alpha_j) \in \mathcal{S}$ for some $1 \leq j \leq m$. Or $x \in C(\gamma) \in \mathcal{P} \backslash \{C(\beta)\}$ (for some $\gamma \in E(R)^*$), in which case $x \in C(\gamma) \in \mathcal{S}$. Observe that there are finitely many cells in $\mathcal{S}$, since there are finitely many cells in $\mathcal{P}$. Also observe that all cells in $\mathcal{P} \backslash \{C(\beta)\}$ have disjoint interiors. Consider a cell $C(\gamma) \in \mathcal{P} \backslash \{C(\beta)\}$ (for some $\gamma \in E(R)^*$). Then $\operatorname{int} C(\gamma) \cap \operatorname{int} C(\alpha_j) = \varnothing$ for all $j = 1, \ldots, n$, since $\operatorname{int} C(\gamma) \cap \operatorname{int} C(\beta) = \varnothing$. This proves that all cells in $\mathcal{S}$ have disjoint interiors. Hence $\mathcal{S}$ is a cellular partition of X.

Observe that $(\mathcal{S}, (\mathcal{S})f)$ is a refinement of $(\mathcal{P}, (\mathcal{P})f)$. Then, by Lemma 3.3.16, $(\mathcal{S}, (\mathcal{S})f)$ is a cellular bipartition for f and $C(\alpha) \in \mathcal{S}$.

□

Recall from Definition 3.1.2 that the boundary points of a cell $C(\alpha)$ of X (for some $\alpha \in E(R)^*$) are the gluing vertices v_α and w_α such that v_α is the initial vertex of the edge α and w_α is the terminal vertex of the edge α

{dep}

Lemma 3.3.22. *Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $C(\alpha) \in \mathcal{P}$ and $C(\beta) \in (\mathcal{P})f$ (for some $\alpha, \beta \in E(R)^*$). Then $(C(\alpha))f = C(\beta)$ if and only if $(v_\alpha)f = v_\beta$ and $(w_\alpha)f = w_\beta$.*

Proof. Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $C(\alpha) \in \mathcal{P}$ and $C(\beta) \in (\mathcal{P})f$ (for some $\alpha, \beta \in E(R)^*$).

Suppose $(C(\alpha))f = C(\beta)$. Recall from Lemma 3.1.3 that $(\alpha\bar{\mathbf{i}})\phi = v_\alpha$ and $(\alpha\bar{\mathbf{t}})\phi = w_\alpha$. By Definition 3.3.8 of a rearrangement, $f|_{C(\alpha)}$ is a canonical homeomorphism. By Definition 3.3.5 of a canonical homeomorphism, the map $f|_{C(\alpha)} \colon C(\alpha) \to C(\beta)$ is induced by the prefix replacement map $\Psi \colon \Omega(\alpha) \to \Omega(\beta)$ such that the following diagram commutes:

$$\begin{tikzcd} \Omega(\alpha) \arrow[r, "\Psi"] \arrow[d, "\phi"] & \Omega(\beta) \arrow[d, "\phi"] \\ C(\alpha) \arrow[r, "\psi"'] & C(\beta) \end{tikzcd}$$

Observe that $(\alpha\bar{\mathbf{i}})\Psi = \beta\bar{\mathbf{i}}$ and $(\alpha\bar{\mathbf{t}})\Psi = \beta\bar{\mathbf{t}}$, and $(\beta\bar{\mathbf{i}})\phi = v_\beta$ and $(\beta\bar{\mathbf{i}})\phi = w_\beta$. Hence $(v_\alpha)f = v_\beta$ and $(w_\alpha)f = w_\beta$.

Conversely, suppose that $(v_\alpha)f = v_\beta$ and $(w_\alpha)f = w_\beta$. Since f is a rearrangement, by Definition 3.3.8, there exists a cell $C(\gamma) \in (\mathcal{P})f$ (for some $\gamma \in E(R)^*$) such that $(C(\alpha))f = C(\gamma)$. Then $(v_\alpha)f = v_\gamma$ and $(w_\alpha)f = w_\gamma$. This implies that $v_\beta = v_\gamma$ and $v_\beta = v_\gamma$. And hence, by Lemma 3.1.11, $C(\gamma) = C(\beta)$.

This proves the result. □

{animal}

Corollary 3.3.23. *Let f be a rearrangement of X and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$). Then $(C(\alpha))f = C(\alpha)$ if and only if $(v_\alpha)f = v_\alpha$ and $(w_\alpha)f = w_\alpha$.*

Proof. The proof follows from Lemma 3.3.22. □

We will show below that cellular bipartitions for a rearrangement f of X have a lattice structure (in terms of refinements and coarsenings). Then there exists a "coarsest" cellular bipartition, which we will call a *minimal bipartition*. This is formally defined as follows:

We use $\mathrm{cbp}(f)$ to denote the set of all possible cellular bipartitions for f, i.e.,

$$\mathrm{cbp}(f) = \{(\mathcal{P}, (\mathcal{P})f) \mid (\mathcal{P}, (\mathcal{P})f) \text{ is a cellular bipartition for } f\}.$$

We denote the number of cells in a cellular partition $\mathcal{P}$ by $|\mathcal{P}|$. We observe that $|\mathcal{P}| = |f(\mathcal{P})|$ for any pair $(\mathcal{P}, (\mathcal{P})f)$ in $\mathrm{cbp}(f)$. We can define a map $\mathfrak{N}_f \colon \mathrm{cbp}(f) \to \mathbb{N}$ by

$$\mathfrak{N}_f(\mathcal{P}, (\mathcal{P})f) = |\mathcal{P}|.$$

Definition 3.3.24. A *minimal bipartition* for f is the cellular bipartition $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ such that

$$\mathfrak{N}_f(\mathcal{P}_{f,d}, \mathcal{P}_{f,r}) = \min_{(\mathcal{P},(\mathcal{P})f)\in\mathrm{cbp}(f)} \mathfrak{N}_f(\mathcal{P}, (\mathcal{P})f).$$

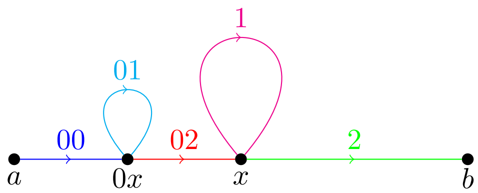
That is, a minimal bipartition for f is a cellular bipartition with the least number of cells such that f restricts to a canonical homeomorphism from each domain cell to its image cell. We will prove that it is unique.

Lemma 3.3.25. *Let f be a rearrangement for X. Then there exists a minimum bipartition $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ for f.*

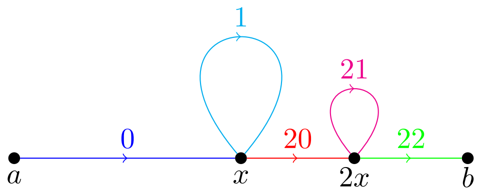
Proof. Observe that, for any rearrangement f of X, $\mathfrak{N}_f(\mathcal{P}, (\mathcal{P})f)$ is a natural number. Then there is some cellular bipartition for which $\mathfrak{N}_f(\mathcal{P}, (\mathcal{P})f)$ achieves the minimum value, and hence the result holds. □

Example 3.3.26. Consider the rearrangement g of X defined in Example 3.3.13 with the following cellular bipartition (illustrated using graph-pair diagrams):

$$\mathcal{P} = \{C(00), C(01), C(02), C(1), C(2)\}$$



$$(\mathcal{P})g = \{C(0), C(1), C(20), C(21), C(22)\}$$



The only cells properly containing the cells in $\mathcal{P}$ are $C(0)$ and $C(\epsilon)$, and $g|_{C(0)}$ and $g|_{C(\epsilon)}$ are not canonical homeomorphisms. Therefore, $(\mathcal{P}, (\mathcal{P})g)$ is the minimal bipartition for g.

The following results characterize the properties of the minimal bipartition for a rearrangement f of X.

{billytalent}

Lemma 3.3.27. *Let f be a rearrangement of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be a minimal bipartition for f. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) such that $f|_{C(\alpha)}$ is a canonical homeomorphism. Then $C(\alpha) \in \mathcal{P}_{f,d}$ if and only if there does not exist a cell $C(\beta)$ in X (for some $\beta \in E(R)^*$), such that $C(\beta) \supset C(\alpha)$ and $f|_{C(\beta)}$ is a canonical homeomorphism.*

Proof. Let f be a rearrangement of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be the minimal bipartition for f. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^*$) such that $f|_{C(\alpha)}$ is a canonical homeomorphism.

Suppose $C(\alpha) \in \mathcal{P}_{f,d}$. Let us prove the result by contradiction. Suppose there does exists a cell $C(\beta)$ in X (for some $\beta \in E(R)^*$), such that $C(\beta) \supset C(\alpha)$ and $f|_{C(\beta)}$ is a canonical homeomorphism. Then, by Lemma 3.2.11, exist cells $C(\alpha_1), \ldots, C(\alpha_m) \in \mathcal{P}_{f,d}$ (for some $\alpha_1, \ldots, \alpha_m \in E(R)^*$) such that $C(\alpha) = C(\alpha_k)$ for some $1 \leq k \leq m$ and

$$C(\beta) = \bigcup_{i=1}^{m} C(\alpha_i).$$

Let us define a set

$$\mathcal{Q} = (\mathcal{P} \backslash \{C(\alpha_1), \ldots, C(\alpha_m)\}) \cup \{C(\beta)\}.$$

Let us prove that $\mathcal{Q}$ is a cellular partition: Consider $x \in X$. Since $\mathcal{P}$ is a cover of X, either $x \in C(\alpha_j)$ for some $1 \leq j \leq m$, in which case $x \in C(\beta) \in \mathcal{Q}$. Or $x \in C(\gamma) \in \mathcal{P} \backslash \{C(\alpha_1), \ldots, C(\alpha_m)\}$ (for some $\gamma \in E(R)^*$), in which case $x \in C(\gamma) \in \mathcal{Q}$. Observe that there are finitely many cells in $\mathcal{Q}$, since there are finitely many cells in $\mathcal{P}$. Also observe that all cells in $\mathcal{P} \backslash \{C(\alpha_1), \ldots, C(\alpha_m)\}$ have disjoint interiors. Consider a cell $C(\gamma) \in \mathcal{P} \backslash \{C(\alpha_1), \ldots, C(\alpha_m)\}$ (for some $\gamma \in E(R)^*$). Then $\operatorname{int} C(\gamma) \cap \operatorname{int} C(\beta) = \varnothing$, since

$\operatorname{int} C(\gamma) \cap \operatorname{int} C(\alpha_j) = \varnothing$ for all $j = 1, \ldots, m$. This proves that all cells in $\mathcal{Q}$ have disjoint interiors. Hence $\mathcal{Q}$ is a cellular partition of X.

Observe that $(\mathcal{Q}, (\mathcal{Q})f)$ is a coarsening of $(\mathcal{P}, (\mathcal{P})f)$ with the rearrangement f restricting to a canonical homeomorphism on each cell in $\mathcal{Q}$. Then, by Lemma 3.3.17, $(\mathcal{Q}, (\mathcal{Q})f)$ is a cellular bipartition for f and $\mathfrak{N}_f(\mathcal{Q}, (\mathcal{Q})f) < \mathfrak{N}_f(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$. Then $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is not the minimal bipartition for f, which is a contradiction. Therefore, there does not exist a cell $C(\beta)$ in X (for some $\beta \in E(R)^*$), such that $C(\beta) \supset C(\alpha)$ and $f|_{C(\beta)}$ is a canonical homeomorphism.

Conversely, suppose there does not exist a cell $C(\beta)$ in X (for some $\beta \in E(R)^*$), such that $C(\beta) \supset C(\alpha)$ and $f|_{C(\beta)}$ is a canonical homeomorphism. By Lemma 3.1.10, there exists a cell $C(\gamma) \in \mathcal{P}_{f,d}$ (for some $\gamma \in E(R)^*$) such that

$$\operatorname{int} C(\alpha) \cap \operatorname{int} C(\gamma) \neq \varnothing.$$

Then only one of the following holds:

1. $C(\alpha) = C(\gamma)$, which implies that $C(\alpha) \in \mathcal{P}_{f,d}$.
2. $C(\alpha) \subset C(\gamma)$. This is not possible by our hypothesis, therefore $C(\alpha) \not\subset C(\gamma)$
3. $C(\alpha) \supset C(\gamma)$. Let us prove that there exist cells $C(\gamma_1), \ldots, C(\gamma_n) \in \mathcal{P}_{f,d}$ such that $C(\gamma) = C(\gamma_k)$ for some $1 \leq k \leq n$ and

   $$C(\alpha) = \bigcup_{i=1}^{n} C(\gamma_i).$$

   By Lemma 3.1.10 (2), $\alpha \prec \gamma$. Then, $\gamma = \alpha\delta$ (for some $\delta \in E(R)^*$). By Lemma 3.2.2, there exists a complete antichain A such that $\gamma \in A$. Then, by the properties of complete antichains, there exist $\gamma_1, \ldots, \gamma_m \in A$ such that $\gamma = \gamma_k$ for some $1 \leq k \leq n$, $\gamma_i = \alpha\delta_i$ for all $i = 1, \ldots, n$ and $D = \{\delta_i \mid i = 1, \ldots n\}$ is a complete antichain.

   Let us define a set

   $$\mathcal{S} = (\mathcal{P}_{f,d} \backslash \{C(\alpha)\}) \cup \{C(\gamma_1), \ldots, C(\gamma_n)\}.$$

   Let us prove that $\mathcal{S}$ is a cellular partition: Consider $x \in X$. Since $\mathcal{P}_{f,d}$ is a cover of X, either $x \in C(\alpha)$, in which case $x \in C(\gamma_j) \in \mathcal{S}$ for some $1 \leq j \leq n$. Or $x \in C(\delta) \in \mathcal{P}_{f,d} \backslash \{C(\alpha)\}$ (for some $\delta \in E(R)^*$), in which case $x \in C(\delta) \in \mathcal{S}$. Observe that there are finitely many cells in $\mathcal{S}$, since there are finitely many cells in $\mathcal{P}_{f,d}$. Also observe that all cells in $\mathcal{P}_{f,d} \backslash \{C(\alpha)\}$ have disjoint interiors. Consider a cell $C(\delta) \in \mathcal{P}_{f,d} \backslash \{C(\alpha)\}$ (for some $\delta \in E(R)^*$). Then $\operatorname{int} C(\delta) \cap \operatorname{int} C(\gamma_j) = \varnothing$ for all $j = 1, \ldots, n$, since $\operatorname{int} C(\delta) \cap \operatorname{int} C(\alpha) = \varnothing$. This proves that all cells in $\mathcal{S}$ have disjoint interiors. Hence $\mathcal{S}$ is a cellular partition of X.

   Observe that $(\mathcal{S}, (\mathcal{S})f)$ is a refinement of $(\mathcal{P}_{\{,\lceil}, \mathcal{P}_{f,r})$. Then, by Lemma 3.3.16, $(\mathcal{S}, (\mathcal{S})f)$ is a cellular bipartition for f and $\mathfrak{N}_f(\mathcal{S}, (\mathcal{S})f) < \mathfrak{N}_f(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$. Then

$(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is not the minimal bipartition for f, which is a contradiction. Therefore $C(\alpha) \not\supset C(\gamma)$

This proves the result. □

{catpeople}

Lemma 3.3.28. *Let f be a rearrangement of X. The minimal bipartition for f is unique.*

Proof. Let f be a rearrangement of X. Let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{Q}_{f,d}, \mathcal{Q}_{f,r})$ be minimal bipartitions for f. Consider a cell $C(\alpha) \in \mathcal{P}_{f,d}$ (for some $\alpha \in E(R)^*$). Since $\mathcal{Q}_{f,d}$ is a cellular partition of X, then, by Lemma 3.2.10, there exists a cell $C(\beta) \in \mathcal{Q}_{f,d}$ (for some $\beta \in E(R)^*$) such that

$$\operatorname{int} C(\alpha) \cap \operatorname{int} C(\beta) \neq \varnothing.$$

Then, by Lemma 3.1.10 (3), precisely one of the following holds:

1. $C(\alpha) = C(\beta)$,
2. $C(\alpha) \subset C(\beta)$,
3. $C(\alpha) \supset C(\beta)$.

Observe that f restricts to a canonical homeomorphism on both $C(\alpha)$ and $C(\beta)$. Then, by Lemma 3.3.27, $C(\alpha) \not\subset C(\beta)$ since $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is a minimal bipartition of f and $C(\alpha) \not\supset C(\beta)$ since $(\mathcal{Q}_{f,d}, \mathcal{Q}_{f,r})$ is a minimal bipartition for f. This implies that $C(\alpha) = C(\beta)$. Observe that the choice of $C(\alpha)$ is arbitrary, hence $C(\alpha) = C(\beta)$ for all $C(\alpha) \in \mathcal{P}_{f,d}$. Therefore $\mathcal{P}_{f,d} = \mathcal{Q}_{f,d}$, and it follows that $\mathcal{P}_{f,r} = \mathcal{Q}_{f,r}$. □

{tuxedomoon}

Lemma 3.3.29. *Let f be a rearrangement of X. Let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be the minimal bipartition for f and let $(\mathcal{P}, (\mathcal{P})f)$ be an arbitrary cellular bipartition for f. Then $\partial\mathcal{P}_{f,d} \subseteq \partial\mathcal{P}$ and $\partial\mathcal{P}_{f,r} \subseteq \partial(\mathcal{P})f$.*

Proof. Let f be a rearrangement of X. Let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be the minimal bipartition for f and let $(\mathcal{P}, (\mathcal{P})f)$ be an arbitrary cellular bipartition for f.

Consider a boundary point $z \in \partial\mathcal{P}_{f,d}$. By Definition 3.2.3, $z = v_\alpha$ or $z = w_\alpha$ for some cell $C(\alpha) \in \partial\mathcal{P}_{f,d}$. By Lemma 3.3.27 and Lemma 3.2.11 (2), there exist cells $C(\beta_1), \ldots, C(\beta_n) \in \mathcal{P}$ such that

$$C(\alpha) = \bigcup_{i=1}^{n} C(\beta_i).$$

Then z is a boundary point of the cell $C(\beta_j)$ for some $1 \leq j \leq n$. Hence, $z \in \partial\mathcal{P}$ and therefore $\partial\mathcal{P}_{f,d} \subseteq \partial\mathcal{P}$.

Now consider a boundary point $z \in \partial\mathcal{P}_{f,r}$. Then $(z)f^{-1} \in \partial\mathcal{P}_{f,d}$, which implies that $(z)f^{-1} \in \partial\mathcal{P}$. Therefore $z \in \partial(\mathcal{P})f$ and it follows that $\partial\mathcal{P}_{f,r} \subseteq \partial(\mathcal{P})f$. □

3.4 Rearrangement Groups

{3.4}

In this section, we will prove that the set of rearrangements of a limit space X of an F-type edge replacement system is a group.

{kmfdm}

Lemma 3.4.1. *Let f be a rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be the minimal bipartition for f. Then the inverse f^{-1} is also a rearrangement of X and $(\mathcal{P}_{f,r}, \mathcal{P}_{f,d})$ is the minimal bipartition for f^{-1}.*

Proof. Let f be a rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be the minimal bipartition for f. Since f is a homeomorphism, then there exists a homeomorphism f^{-1} such that $ff^{-1} = f^{-1}f = I$, where I is the identity homeomorphism.

Let us first prove that f^{-1} is a rearrangement: By Definition 3.3.8 of a rearrangement, $f|_{C(\alpha)}$ is a canonical homeomorphism for each cell $C(\alpha) \in \mathcal{P}_{f,d}$ (for some $\alpha \in E(R)^*$) and $(C(\alpha))f \in \mathcal{P}_{f,r}$. Then, by Lemma 3.3.6, $f^{-1}|_{(C(\alpha))f} \colon (C(\alpha))f \to C(\alpha)$ is a canonical homeomorphism for each cell $C(\alpha) \in \mathcal{P}_{f,d}$. This, by Definition 3.3.8, proves that f^{-1} is a rearrangement and $(\mathcal{P}_{f,r}, \mathcal{P}_{f,d})$ is a cellular bipartition for f^{-1}.

Let us now prove that $(\mathcal{P}_{f,r}, \mathcal{P}_{f,d})$ is the minimal bipartition for f^{-1}: We shall prove this by contradiction. Suppose that $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is not the minimal bipartition for f^{-1}. Let $(\mathcal{P}_{f^{-1},d}, \mathcal{P}_{f^{-1},r})$ be the minimal bipartition for f^{-1}. Then, by Lemma 3.3.27, there exists cells $C(\alpha) \in \mathcal{P}_{f,r}$ and $C(\beta) \in \mathcal{P}_{f^{-1},d}$ such that $C(\alpha) \subset C(\beta)$. By the definition of a canonical homeomorphism, $(C(\alpha))f^{-1} \subset (C(\beta))f^{-1}$ and $(C(\alpha))f^{-1} \in \mathcal{P}_{f,d}$ and $(C(\beta))f^{-1} \in \mathcal{P}_{f^{-1},r}$. By Definition 3.3.8 of a rearrangement, $f^{-1}|_{C(\beta)} \colon C(\beta) \to (C(\beta))f^{-1}$ is a canonical homeomorphism. Then by the definition of an inverse, $f|_{(C(\beta))f^{-1}} \colon (C(\beta))f^{-1} \to C(\beta)$ is a canonical homeomorphism. This implies that $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is not the minimal bipartition for f, which is a contradiction. Hence $(\mathcal{P}_{f,r}, \mathcal{P}_{f,d})$ is the minimal bipartition for f^{-1}. □

{tranference}

Lemma 3.4.2. *Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions for f and g respectively. Then*

1. $\big((\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}, \mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}\big)$ *is a cellular bipartition for f,*
2. $(\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}, (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})g)$ *is a cellular bipartition for g.*

Proof. Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions for f and g respectively.

1. By Lemma 3.4.1, f^{-1} is a rearrangement and $(\mathcal{P}_{f,r}, \mathcal{P}_{f,d})$ is the minimal bipartition for f^{-1}. Observe that $\mathcal{P}_{f,r} = (\mathcal{P}_{f,d})f$. It follows from Corollary 3.3.18 that $\big(\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}, (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}\big)$ is a cellular bipartition for f^{-1}, which implies that $\big((\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}, \mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}\big)$ is a cellular bipartition for f.
2. This follows immediately from Corollary 3.3.18.

□

{bringiton}

Lemma 3.4.3. *Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions for f and g respectively. Then fg is a rearrangement of X and $\big((\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}, (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})g\big)$ is a cellular bipartition for fg.*

Proof. Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions for f and g respectively. By Lemma 3.4.2, $\big((\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}, \mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}\big)$ is a cellular bipartition for f and $(\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}, (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})g)$ is a cellular bipartition for g. Consider a cell $C(\alpha) \in (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}$ (for some $\alpha \in E(R)^*$). Then $(C(\alpha))f \in \mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d}$ and hence $fg|_{C(\alpha)}$ is a canonical homeomorphism, since it is the composition of the canonical homeomorphisms $f|_{C(\alpha)}$ and $g|_{(C(\alpha))f}$. Since this is true for each cell $C(\alpha) \in (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}$, fg is a rearrangement of X and $\big((\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}, (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})g\big)$ is a cellular bipartition for fg. □

{iwantyounow}

Lemma 3.4.4. *Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions of f and g respectively. Suppose that for all cells $C(\alpha) \in \mathcal{P}_{f,r}$ (for some $\alpha \in E(R)^*$), there exists a cell $C(\beta) \in \mathcal{P}_{g,d}$ (for some $\beta \in E(R)^*$) such that $C(\alpha) \subseteq C(\beta)$. Then $(\mathcal{P}_{f,d}, (\mathcal{P}_{f,r})g)$ is a cellular bipartition for the rearrangement fg.*

Proof. Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions of f and g respectively. Suppose that for all cells $C(\alpha) \in \mathcal{P}_{f,r}$ (for some $\alpha \in E(R)^*$), there exists a cell $C(\beta) \in \mathcal{P}_{g,d}$ (for some $\beta \in E(R)^*$) such that $C(\alpha) \subseteq C(\beta)$. Then, by Definition 3.2.12 of a meet, $\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d} = \mathcal{P}_{f,r}$. By Lemma 3.4.3, we know that fg is a rearrangement and $\big((\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})f^{-1}, (\mathcal{P}_{f,r} \wedge \mathcal{P}_{g,d})g\big) = (\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ is cellular bipartition for fg. □

haveandtohold}

Lemma 3.4.5. *Let f and g be rearrangements of X and let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ and $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartitions of f and g respectively. Suppose that $\partial\mathcal{P}_{g,d} \subseteq \partial\mathcal{P}_{f,r}$. Then $\partial\mathcal{P}_{fg,d} \subseteq \partial\mathcal{P}_{f,d}$.*

Proof. The proof follows from Lemma 3.4.4 and Lemma 3.3.29. □

Belk and Forrest [3] proved the following result for rearrangement groups of all limit spaces. Here we prove it for rearrangement groups of F-type limit spaces:

{thrashme}

Lemma 3.4.6. *The rearrangements of X form a group under composition.*

Proof. Since every rearrangement of X is a homeomorphism, the set of rearrangements of X is a subset of the group of homeomorphisms of X ($\mathrm{Aut}(X)$). We will prove that this subset is a subgroup.

The identity homeomorphism I is clearly a rearrangement (and $(\{C(\epsilon)\}, \{C(\epsilon)\})$ is the minimal bipartition for I, where $C(\epsilon) = X$). We know from Lemma 3.4.1 that the inverse of a rearrangement is also a rearrangement. We know from Lemma 3.4.3 that the composition of two rearrangements is also a rearrangement. This proves that the rearrangements of X form a subgroup of $\mathrm{Aut}(X)$. □

Chapter 4

Richard Thompson's Group *F*

{f-group}

We will now use the framework we developed in Chapter 2 and Chapter 3 to define a rearrangement group of a particular limit space which is isomorphic to Richard Thompson's group F. In Section 4.1 we will present the standard definition of Thompson's group F. In Section 4.2 we will construct a particular limit space. In Section 4.3 we will show that this limit space is homeomorphic to the unit interval $[0, 1]$. And finally, in Section 4.4 we will show that the rearrangement group of this limit space is isomorphic to Thompson's group F.

4.1 Richard Thompson's Group *F*

{4.1}

The groups F, T and V were first defined by Richard Thompson in 1965. They arise as subgroups of the homeomorphism group of the Cantor set, and act on the unit interval $[0, 1]$, the unit circle S^1 and the Cantor set respectively. In this section we will present the "standard" definition of Thompson's group F. The definitions, results and terminology have been taken from the lecture notes by Cannon, Floyd and Parry [9].

{talkaboutthew

Definition 4.1.1 (Cannon, Floyd, Parry [9])**.** We define F to be the set of piecewise linear homeomorphisms from the closed unit interval $[0, 1]$ to itself that are differentiable, except at finitely many dyadic rational numbers, and such that at the intervals of differentiability the derivatives are powers of 2.

{dirtymagic}

Proposition 4.1.2 (Cannon, Floyd, Parry [9])**.** *The set F is a group under the composition of functions.*

Cannon, Floyd and Parry [9] proved this result by showing that F is a subgroup of the group of homeomorphisms of $[0, 1]$. We shall be proving this result by showing that F is a rearrangement group.

{stlouise}

Remark 4.1.3. Observe that Cannon, Floyd and Parry use left-actions in their work. Since we are using right-actions in this thesis, we shall be replacing the functions defined in [9] by their inverses.

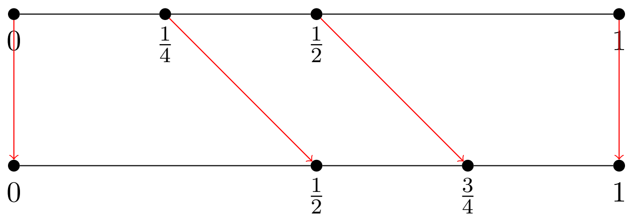
{daisycutter}

Example 4.1.4 (Cannon, Floyd, Parry [9], Example 1.1 and Example 1.2)**.** Let us define the maps A and B as follows:

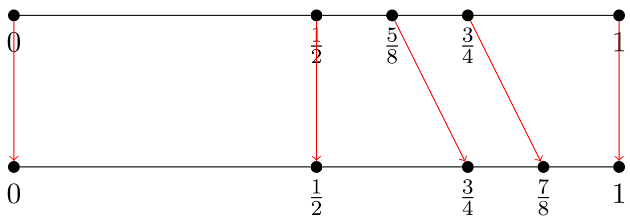
$$A(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{4} \\ x + \frac{1}{4} & \frac{1}{4} \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{2} & \frac{1}{2} \leq x \leq 1 \end{cases}$$

$$B(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 2x + \frac{1}{2} & \frac{1}{2} \leq x \leq \frac{5}{8} \\ x + \frac{1}{8} & \frac{5}{8} \leq x \leq \frac{3}{4} \\ \frac{x}{2} + \frac{1}{2} & \frac{3}{4} \leq x \leq 1 \end{cases}$$

It is easy to show that these maps are differentiable, except at finitely many dyadic rationals, and at the intervals of differentiability the derivatives are powers of 2. Hence, $A, B \in F$. We shall be illustrating these maps using the following diagrams:



A



B

Diagrams of this type are called *rectangle diagrams* or *Thurston diagrams*. They were introduced by Cannon, Floyd and Parry in [9], who in turn cite unpublished work by W. P. Thurston in 1975. In due course, we shall show that rectangle diagrams are equivalent to graph-pair diagrams defined in Definition 3.3.9.

{genius}

Definition 4.1.5 (Cannon, Floyd, Parry [9])**.** We define the maps $X_0, X_1, X_2, \dots \in F$ as follows:

- $X_0 = A$ and $X_1 = B$,
- $X_{n+1} = X_0^{-1} X_n X_0 = A^{-n} B A^n$.

The following result from [9] defines two presentations for F – one finite and one infinite. It is given here without a proof:

Proposition 4.1.6 (Cannon, Floyd, Parry [9], Theorem 3.4)**.** *The following are two presentations for Thompson's group F:*

$$F = \langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^{2}] \rangle,$$
$$F = \langle X_0, X_1, X_2, \cdots \mid X_{n+1} = X_k^{-1} X_n X_k \textit{ for some } k < n \rangle.$$

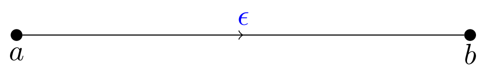
In this thesis, we shall be presenting an alternative presentation for Thompson's group F which corresponds to the basic open sets of the unit interval $[0, 1]$. We shall do this by defining F as a rearrangement group. We shall present this definition in the rest of this chapter. In Chapter 5, we will define our generating set for F. In Chapter 6, we will define our presentation for F.

4.2 A Limit Space

{4.2}

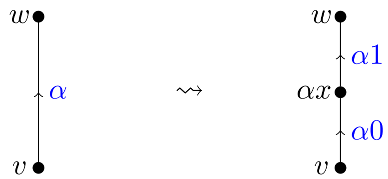
We define an edge replacement system as follows:

- Our base graph G_0 is a finite directed graph with one edge and two vertices. We denote the edge by the empty word ϵ, and the vertices by a and b.



G_0

- The graph G_n is constructed by replacing every edge $\mathbf{e}$ of the graph G_{n-1} by the replacement graph R.

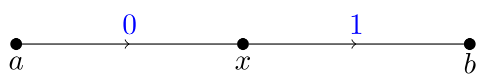


$\mathbf{e}$ R

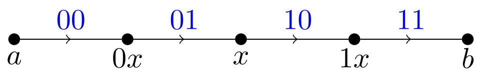
{dollsinthedar

Definition 4.2.1. We define the F *replacement system* to be the edge replacement system $(G_0, \mathbf{e} \to R)$.

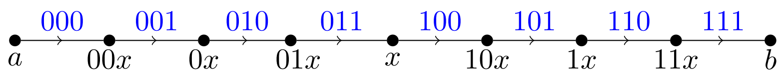
Observe that this fulfills Definition 2.1.6 of an F-type replacement system. The first few graphs of the full expansion sequence are as follows:



G_1



G_2



G_3

$$\vdots$$

Observe that $V(G_0) = \{a, b\}$, $E(G_0) = \{\epsilon\}$, $V_{\text{int}}(R) = \{x\}$ and $E(R) = \{0, 1\}$. We shall treat ϵ as the empty word. In a generic F-type edge replacement system, we used $\mathbf{i}$ to denote the edge incident with the initial vertex of R and $\mathbf{t}$ to denote the edge incident with the terminal vertex of R. Note that, in the F replacement system, $\mathbf{i} = 0$ and $\mathbf{t} = 1$. Then our notation for the edges and vertices of graph G_n is similar to the general case in Section 2.1:

1. Each edge of G_n is of the form $\alpha = e_1 \dots e_n$, where $\alpha^\dagger = e_1 \dots e_{n-1}$ is the edge of G_{n-1} that was replaced, and e_n is an edge of R. The set $E(G_n)$ of edges G_n is defined as

   $$\begin{aligned} E(G_n) &= E(R)^n \\ &= \{\alpha \mid \alpha = e_1 \dots e_n, e_i \in E(R)\}\,. \end{aligned}$$

   Each edge of G_n is a word of length n in the alphabet $E(R)$ (observe that the length of the word would be $n+1$ in a generic F-type edge replacement system).

2. Each new vertex of G_n has the form $\alpha^\dagger x = e_1 \dots e_{n-1} x$, where $\alpha^\dagger$ is the edge of G_{n-1} which was replaced. The set V_n of new vertices introduced in graph G_n (i.e., the vertices of depth n), is

   $$V_n = \left\{\alpha^\dagger x \mid \alpha^\dagger = e_1 \dots e_{n-1}, e_i \in E(R)\right\}.$$

   The set V_0 contains the vertices a and b (which have depth 0).

The complete set of vertices of G_n is the disjoint union

$$V(G_n) = \bigsqcup_{k=0}^{n} V_k.$$

{smalltownboy}

Remark 4.2.2. Observe that, in the F replacement system, the depth of a gluing vertex αx – defined in Definition 2.1.13 – is

$$\operatorname{depth}(\alpha x) = |\alpha| + 1.$$

This is because $e_0 = \epsilon$ and is treated as the empty word.

The following results are Lemma 2.1.14 and Lemma 2.1.15, restated for this setting. They detail the adjacency of edges and vertices in the graph G_n:

{theprodigy}

Lemma 4.2.3. *Let $\{G_n\}_{n=0}^{\infty}$ be the full expansion sequence of the F replacement system. Let $\alpha = e_1 \dots e_n \in E(G_n)$ (for some $n \geq 1$). The initial and terminal vertices of α are as follows:*

1. *If $\alpha = (0)^n$, the initial vertex is a and the terminal vertex is $\alpha^{\dagger}x = (0)^{n-1}x$.*
2. *If $\alpha = e_1 \dots e_k 1(0)^{n-k-1}$ (for $k < n$), the initial vertex is $\alpha^{(n-k)\dagger}x = e_1 \dots e_k x$ and the terminal vertex is $\alpha^{\dagger}x = e_1 \dots e_{n-1}x$.*
3. *If $\alpha = e_1 \dots e_k 0(1)^{n-k-1}$ (for $n > k$), the initial vertex is $\alpha^{\dagger}x = e_1 \dots e_{n-1}x$ and the terminal vertex is $\alpha^{(n-k)\dagger}x = e_1 \dots e_k x$.*
4. *If $\alpha = (1)^n$, the initial vertex is $\alpha^{\dagger}x = (1)^{n-1}x$ and the terminal vertex is b.*

{garbage}

Lemma 4.2.4. *Let $\{G_n\}_{n=0}^{\infty}$ be the full expansion sequence of the F replacement system $(G_0, \mathbf{e} \to R)$. Let $\alpha = e_1 \dots e_k \in E(G_k)$ and $n \geq k+1$. Then*

1. *The edge in G_n having αx as the initial vertex is $\alpha 1(0)^{n-k-1}$.*
2. *The edge in G_n having αx as the terminal vertex is $\alpha 0(1)^{n-k-1}$.*

We shall now recall the definitions and terminology from Chapter 2 and Chapter 3 and interpret them in the context of the F replacement system:

Recall from Definition 2.3.1 and Definition 2.3.2 that the set of finite words and the symbol space are defined as follows:

$$E(R)^* = \{e_0 \dots e_n \mid e_0 \in E(G_0) \text{ and } e_i \in E(R) \text{ for } i = 1, \dots, n\},$$

$$\Omega = \{e_0 e_1 e_2 \cdots \mid e_0 \in E(G_0) \text{ and } e_i \in E(R) \text{ for } i = 1, 2, \dots\}.$$

Observe that, in this chapter, $E(G_0) = \{\epsilon\}$, with ϵ treated as the empty word, and $E(R) = \{0, 1\}$.

Definition 4.2.5. We define the set of finite words and the symbol space of the F replacement system as follows:

$$E(R)^* = \{e_1 \dots e_n \mid e_i \in \{0,1\}\} = \{0,1\}^*,$$

$$\Omega = \{e_1 e_2 \cdots \mid e_i \in \{0,1\}\} = \{0,1\}^{\omega}.$$

We define a linear order $\leq$ on $E(R)$ such that $0 < 1$. Then recall from Definition 2.3.4 and Definition 2.3.5 that there exists a lexicographic order $\leq_{\ell}$ on Ω and every complete antichain $A \subset E(R)^*$.

As specified in the general case, we endow $\Omega = \{0,1\}^\omega$ with the product topology. Let us define $\Omega(\alpha) := \alpha\Omega$ (for some $\alpha \in E(R)^*$). Then the collection

$$\{\Omega(\alpha) \mid \alpha \in E(R)^*\}$$

form a basis for this topology. Observe that, for all $t \in \Omega(\alpha)$, $\alpha\bar{0} \leq_\ell t \leq_\ell \alpha\bar{1}$.

Recall from Definition 2.3.9 that the elements of the vertex set

$$\mathcal{GV} = \bigcup_{n=0}^{\infty} V(G_n) = \bigsqcup_{n=0}^{\infty} V_n$$

are called gluing vertices. In the F replacement system, a gluing vertex $v \in \mathcal{GV}$ is either a, b or αx (for some $\alpha \in E(R)^*$). Recall also from Definition 2.3.11 that a sequence $e_1e_2\ldots \in \Omega$ *represents* a vertex $v \in \mathcal{GV}$ if and only if the edge $e_1\ldots e_n$ is incident with v in the graph G_n for all sufficiently large n.

The following result is Lemma 2.3.12, restated for this setting:

{mothere}

Lemma 4.2.6. *Let $e_1e_2\ldots \in \Omega$ and $v \in \mathcal{GV}$. The sequence $e_1e_2\ldots$ represents the vertex v if and only if one of the following holds:*

1. *$v = a$ and $e_1e_2\ldots = \bar{0}$,*
2. *$v = b$ and $e_1e_2\ldots = \bar{1}$,*
3. *$v = \alpha x$ (for some $\alpha \in E(R)^*$) and $e_1e_2\ldots$ is either $\alpha 0\bar{1}$ or $\alpha 1\bar{0}$*

Moreover, if it exists, the vertex v represented by the sequence $e_1e_2\ldots$ is unique.

Let $v \in \mathcal{GV}$. Recall from Definition 2.3.13 that the set $Q_v \subset \Omega$ contains all sequences $e_1e_2\ldots \in \Omega$ which represent the vertex v. It follows from Lemma 4.2.6 that

$$\begin{aligned} Q_a &= \{\bar{0}\} \\ Q_b &= \{\bar{1}\} \\ Q_{\alpha x} &= \{\alpha 0\bar{1}, \alpha 1\bar{0}\} \end{aligned}$$

for each $\alpha \in E(R)^*$.

Recall from Definition 2.3.14 that two sequences from the symbol space

$$e_1e_2\ldots \quad \text{and} \quad e_1'e_2'\ldots$$

are said to be related to each other under the *gluing relation* $\sim$ if the edges

$$e_1\ldots e_n \quad \text{and} \quad e_1'\ldots e_n'$$

share a vertex in the graphs G_n for all n.

{dirtyboots}

Lemma 4.2.7. *For two distinct sequences $e_1e_2\dots$ and $e'_1e'_2\dots$ in $\Omega = \{0,1\}^\omega$,*

$$e_1e_2\dots \sim e'_1e'_2\dots$$

if and only if there exists a finite word $\beta = e_1\dots e_k$ such that

$$e_1e_2\dots = \beta 0\bar{1} \quad \text{and} \quad e'_1e'_2\dots = \beta 1\bar{0}.$$

Proof. The proof follows from Lemma 4.2.6 and Lemma 2.3.15. □

{europa}

Definition 4.2.8. Similarly to the general case in Definition 2.3.18, we define the limit space X of the F replacement system to be the quotient space:

$$X := \Omega/{\sim}.$$

It will be shown in Section 4.3 that this limit space is homeomorphic to the unit interval $[0,1]$. It follows from Lemma 2.3.15 that the sets Q_v are equivalence classes in the limit space X for all gluing vertices $v \in \mathcal{GV}$. As per the general case, will be using the gluing vertex v as a label for the equivalence class.

Recall from Definition 3.1.1 that, for $\alpha = e_1\dots e_n \in E(R)^*$, a *cell* $C(\alpha)$ is the image of a basic open set $\Omega(\alpha)$ in the limit space X under the quotient map ϕ. Recall from Definition 3.1.2 that the boundary points of $C(\alpha)$, v_α and w_α, are the gluing vertices such that v_α is the initial vertex of the edge α in G_n and w_α is the terminal vertex of the edge α in G_n. Observe from Lemma 3.1.3 that $v_\alpha, w_\alpha \in C(\alpha)$.

The characterizes cells and boundary points in the limit space X for the F replacement system:

{quicksand}

Lemma 4.2.9. *Let X be the limit space of the F replacement system. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^* = \{0,1\}^*$). Then the following hold:*

1. *$v_\alpha = (\alpha\bar{0})\phi$ and $w_\alpha = (\alpha\bar{1})\phi$.*
2. *A point $z \in C(\alpha)$ if and only if $\alpha\bar{0} \leq_\ell t \leq_\ell \alpha\bar{1}$ for some $t \in (z)\phi^{-1}$.*
3. *A point $z \in \operatorname{int} C(\alpha)$ if and only if $\alpha\bar{0} <_\ell t <_\ell \alpha\bar{1}$ for all $t \in (z)\phi^{-1}$.*

Proof. Let X be the limit space of the F replacement system. Let $C(\alpha)$ be a cell in X (for some $\alpha \in E(R)^* = \{0,1\}^*$).

1. By Lemma 3.1.3, $v_\alpha = (\alpha\bar{\mathbf{i}})\phi$ and $w_\alpha = (\alpha\bar{\mathbf{t}})\phi$. Recall that, in the F replacement system, $\mathbf{i} = 0$ and $\mathbf{t} = 1$. Hence $v_\alpha = (\alpha\bar{0})\phi$ and $w_\alpha = (\alpha\bar{1})\phi$.
2. Consider a point $z \in C(\alpha)$. Then, by Definition 3.1.1 of a cell, there exists $t \in \Omega(\alpha)$ such that $(t)\phi = z$. Hence, by Definition 2.3.4 of the lexicographic order on $\Omega(\alpha)$, $\alpha\bar{0} \leq_\ell t \leq_\ell \alpha\bar{1}$.

   Conversely, consider a sequence $t \in \Omega$ such that $\alpha\bar{0} \leq_\ell t \leq_\ell \alpha\bar{1}$. Then $t \in \Omega(\alpha)$ and, by Definition 3.1.1 of a cell, $z = (t)\phi \in C(\alpha)$.

3. Consider a point $z \in \operatorname{int} C(\alpha)$. Then, by Corollary 3.1.6 (2),

$$(z)\phi^{-1} \subseteq \Omega(\alpha) \backslash (Q_{v_\alpha} \cup Q_{w_\alpha}).$$

   That is, for all $t \in (z)\phi^{-1}$, α is a prefix of t but $t \neq \alpha\bar{0}$ and $t \neq \alpha\bar{1}$. Then, by Definition 2.3.4 of the lexicographic order on $\Omega(\alpha)$, $\alpha\bar{0} <_\ell t <_\ell \alpha\bar{1}$.

   Conversely, consider a sequence $t \in \Omega$ such that $\alpha\bar{0} <_\ell t <_\ell \alpha\bar{1}$. Then $t \in \Omega(\alpha)$ but $t \notin Q_{v_\alpha}$ and $t \notin Q_{w_\alpha}$. Hence

$$t \in \Omega(\alpha) \backslash (Q_{v_\alpha} \cup Q_{w_\alpha})$$

   and, by Corollary 3.1.6 (2), $z = (t)\phi \in \operatorname{int} C(\alpha)$.

□

Recall from Definition 3.2.1 that a cellular partition $\mathcal{P}$ of X is a cover of X by finitely many cells with disjoint interiors. Recall from Definition 3.2.3 that the set $\partial\mathcal{P}$ is the set of all boundary points of the cells in $\mathcal{P}$. The cellular partitions in X are illustrated using graph expansions.

Recall from Definition 3.2.5, Definition 2.3.10 and Definition 3.2.6 that there exist the following orders on $\mathcal{P}$ and $\partial\mathcal{P}$:

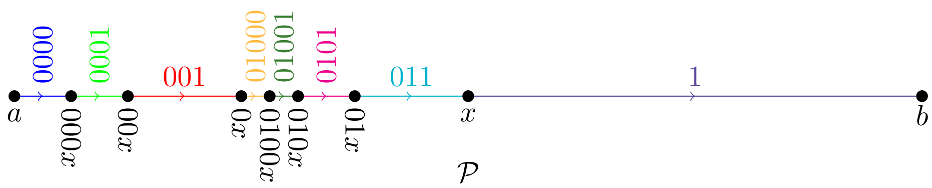
- the induced cell lexicographic order (lex($\mathcal{P}$))
- the induced vertex depth order (depth($\partial\mathcal{P}$))
- the induced cell depth order (depth($\mathcal{P}$))

ebanonhanover}

Example 4.2.10. Let X be the limit space of the F replacement system. We define a set $\mathcal{P}$ of cells in X as follows:

$$\mathcal{P} = \{C(0000), C(0001), C(001), C(01000), C(01001), C(0101), C(011), C(1)\}.$$

Observe that the edges corresponding to the cells in $\mathcal{P}$ form a complete antichain in $E(R)^*$. Hence, by Lemma 3.2.2, $\mathcal{P}$ is a cellular partition of X. It is illustrated by the following graph expansion:



$\mathcal{P}$

Then

$$\begin{aligned}
\operatorname{lex}(\mathcal{P}) &= (C(0000), C(0001), C(001), C(01000), C(01001), C(0101), C(011), C(1)) \\
\operatorname{depth}(\partial\mathcal{P}) &= (a, b, x, 0x, 00x, 01x, 000x, 010x, 0100x) \\
\operatorname{depth}(\mathcal{P}) &= (C(1), C(001), C(011), C(0000), C(0001), C(0101), C(01000), C(01001))
\end{aligned}$$

The following results characterize cellular partitions of the limit space X of the F replacement system:

{jonnyboy}

Lemma 4.2.11. *Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X. Let $z \in \partial\mathcal{P}$.*

1. *If $z = a$, then there is precisely one cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and no cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.*

2. *If $z = b$, then there is no cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and there is precisely one cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.*

3. *If $z = \gamma x$ (for some $\gamma \in E(R)^*$), then there is precisely one cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and precisely one cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.*

Proof. Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X. Let $z \in \partial\mathcal{P}$.

1. Let $z = a$. Then $(z)\phi^{-1} = Q_a = \{\bar{0}\}$.

   Let us first prove that there exists a cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$. By Lemma 3.2.2, $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$, where A is a complete antichain in $E(R)^*$. Then there exists $\alpha \in A$ such that $\alpha \prec \bar{0}$, i.e., $\alpha = (0)^n$ for some $n \in \mathbb{N}$. Then $v_\alpha = ((0)^n\bar{0})\phi = z$.

   Let us now prove that there exists only one cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$. We will prove this result by contradiction. Suppose there exist two distinct cells $C(\alpha_1), C(\alpha_2) \in \mathcal{P}$ such that $z = v_{\alpha_1}$ and $z = v_{\alpha_2}$. By Lemma 4.2.9 (1), $v_{\alpha_1} = (\alpha_1\bar{0})\phi$ and $v_{\alpha_2} = (\alpha_2\bar{0})\phi$. This implies that $\alpha_1\bar{0} \in Q_a$ and $\alpha_2\bar{0} \in Q_a$, which gives us $\alpha_1 = (0)^n$ and $\alpha_2 = (0)^m$ for some $n, m \in \mathbb{N}$. Then $C(\alpha_1) \subseteq C(\alpha_2)$ or $C(\alpha_1) \supseteq C(\alpha_2)$. Since $\mathcal{P}$ is a cellular partition, $C(\alpha_1) = C(\alpha_2)$, which is a contradiction. Hence there exists only one cell $C(\alpha) \in \mathcal{P}$ such that $z = v_\alpha$.

   Let us now prove that there does not exist a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$. By Lemma 4.2.9 (1), $w_\beta = (\beta\bar{1})\phi$. But $\beta\bar{1} \notin Q_a$ for all $\beta \in E(R)^*$. Hence there does not exist a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.

2. Let $z = b$. Then $(z)\phi^{-1} = Q_b = \{\bar{1}\}$.

   Let us first prove that there does not exist a cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$. By Lemma 4.2.9 (1), $v_\alpha = (\alpha\bar{0})\phi$. But $\alpha\bar{0} \notin Q_b$ for all $\alpha \in E(R)^*$. Hence there does not exist a cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = w_\alpha$.

   Let us now prove that there exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$. By Lemma 3.2.2, $\mathcal{P} = \{C(\beta) \mid \beta \in A\}$, where A is a complete antichain in $E(R)^*$. Then there exists $\beta \in A$ such that $\beta \prec \bar{1}$, i.e., $\beta = (1)^n$ for some $n \in \mathbb{N}$. Then $w_\beta = ((1)^n\bar{1})\phi = z$.

Let us now prove that there exists only one cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$. We will prove this result by contradiction. Suppose there exist two distinct cells $C(\beta_1), C(\beta_2) \in \mathcal{P}$ such that $z = w_{\beta_1}$ and $z = w_{\beta_2}$. By Lemma 4.2.9 (1), $w_{\beta_1} = (\beta_1\bar{1})\phi$ and $w_{\beta_2} = (\beta_2\bar{1})\phi$. This implies that $\beta_1\bar{1} \in Q_b$ and $\beta_2\bar{1} \in Q_b$, which gives us $\beta_1 = (1)^n$ and $\beta_2 = (1)^m$ for some $n, m \in \mathbb{N}$. Then $C(\beta_1) \subseteq C(\beta_2)$ or $C(\beta_1) \supseteq C(\beta_2)$. Since $\mathcal{P}$ is a cellular partition, $C(\beta_1) = C(\beta_2)$, which is a contradiction. Hence there exists only one cell $C(\beta) \in \mathcal{P}$ such that $z = w_\beta$.

3. Let $z = \gamma x$ (for some $\gamma \in E(R)^*$). Then $(z)\phi^{-1} = Q_{\gamma x} = \{\gamma 0\bar{1}, \gamma 1\bar{0}\}$.

   Let us first prove that there exists a cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and there exists a cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$. By Lemma 3.2.2, $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$, where A is a complete antichain in $E(R)^*$. Then there exists $\alpha \in A$ such that $\alpha \prec \gamma 1\bar{0}$. If $\gamma \preceq \alpha$, then by Corollary 3.1.7, $z \in \operatorname{int} C(\alpha)$ which is a contradiction since $z \in \partial\mathcal{P}$. Hence $\alpha = \gamma 1(0)^n$ for some $n \in \mathbb{N}$. Then $v_\alpha = (\gamma 1(0)^n\bar{0})\phi = z$. Similarly, there exists $\beta \in A$ such that $\beta \prec \gamma 0\bar{1}$. If $\gamma \preceq \beta$, then by Corollary 3.1.7, $z \in \operatorname{int} C(\beta)$ which is a contradiction since $z \in \partial\mathcal{P}$. Hence $\beta = \gamma 0(1)^m$ for some $m \in \mathbb{N}$. Then $w_\beta = (\gamma 0(1)^n\bar{1})\phi = z$.

   Let us now prove that there exists only one cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$. We will prove this result by contradiction. Suppose there exist two distinct cells $C(\alpha_1), C(\alpha_2) \in \mathcal{P}$ such that $z = v_{\alpha_1}$ and $z = v_{\alpha_2}$. By Lemma 4.2.9 (1), $v_{\alpha_1} = (\alpha_1\bar{0})\phi$ and $v_{\alpha_2} = (\alpha_2\bar{0})\phi$. This implies that $\alpha_1\bar{0} \in Q_{\gamma x}$ and $\alpha_2\bar{0} \in Q_{\gamma x}$, which gives us $\alpha_1 = \gamma 1(0)^n$ and $\alpha_2 = \gamma 1(0)^m$ for some $n, m \in \mathbb{N}$. Then $C(\alpha_1) \subseteq C(\alpha_2)$ or $C(\alpha_1) \supseteq C(\alpha_2)$. Since $\mathcal{P}$ is a cellular partition, $C(\alpha_1) = C(\alpha_2)$, which is a contradiction. Hence there exists only one cell $C(\alpha) \in \mathcal{P}$ such that $z = v_\alpha$.

   Let us now prove that there exists only one cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$. We will prove this result by contradiction. Suppose there exist two distinct cells $C(\beta_1), C(\beta_2) \in \mathcal{P}$ such that $z = w_{\beta_1}$ and $z = w_{\beta_2}$. By Lemma 4.2.9 (1), $w_{\beta_1} = (\beta_1\bar{1})\phi$ and $w_{\beta_2} = (\beta_2\bar{1})\phi$. This implies that $\beta_1\bar{1} \in Q_b$ and $\beta_2\bar{1} \in Q_b$, which gives us $\beta_1 = (1)^n$ and $\beta_2 = (1)^m$ for some $n, m \in \mathbb{N}$. Then $C(\beta_1) \subseteq C(\beta_2)$ or $C(\beta_1) \supseteq C(\beta_2)$. Since $\mathcal{P}$ is a cellular partition, $C(\beta_1) = C(\beta_2)$, which is a contradiction. Hence there exists only one cell $C(\beta) \in \mathcal{P}$ such that $z = w_\beta$.

This proves the result. □

Lemma 4.2.12. *Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X and let $\operatorname{lex}(P) = (C(\alpha_1), \ldots, C(\alpha_n))$. Then, for all $t \in \Omega$ such that $(t)\phi \in C(\alpha_i)$ (for some $1 \leq i \leq n$), $\alpha_{i-1}\bar{1} \leq_\ell t \leq_\ell \alpha_{i+1}\bar{0}$.*

Proof. Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X and let $\operatorname{lex}(P) = (C(\alpha_1), \dots, C(\alpha_n))$. Consider $t \in \Omega$ such that $(t)\phi \in C(\alpha_i)$ (for some $1 \leq i \leq n$). Let $(t)\phi = z$. Then we have the following choices for z:

1. Suppose $z \in \operatorname{int} C(\alpha_i)$. Then, by Lemma 4.2.9 (3), $\alpha_i\bar{0} <_\ell t <_\ell \alpha_i\bar{1}$.

2. Suppose $z = v_{\alpha_i}$. Then, by Lemma 4.2.9 (1), $z = (\alpha_i\bar{0})\phi$. Since $z \in \mathcal{GV}$, then there exists $\beta \in E(R)^*$ such that $(z)\phi^{-1} = Q_{\beta x} = \{\beta 0\bar{1}, \beta 1\bar{0}\}$. If $\beta \preceq \alpha_i$, then by Corollary 3.1.7, $z \in \operatorname{int} C(\alpha_i)$ which is a contradiction since $z \in \partial\mathcal{P}$. Hence $\alpha_i = \beta 1(0)^n$ for some $n \in \mathbb{N}$.

   Observe that $t \in Q_{\beta x}$, that is either $t = \beta 0\bar{1}$ or $t = \beta 1\bar{0} = \alpha_i\bar{0}$. To prove the result, we have to show that $\beta 0\bar{1} = \alpha_{i-1}\bar{1}$. By Lemma 3.2.2, $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$, where A is a complete antichain in $E(R)^*$. Then there exists $\alpha_j \in A$ such that $\alpha_j \prec \beta 0\bar{1}$. If $\alpha_j \preceq \beta$, then $\alpha_j \prec \alpha_i$ which is a contradiction since A is an antichain. Hence $\alpha_j = \beta 0(1)^m$ for some $m \in \mathbb{N}$. Since A is a complete antichain, $\alpha_j = \alpha_{i-1}$. Therefore $\beta 0\bar{1} = \alpha_{i-1}\bar{1}$.

3. Suppose $z = w_{\alpha_i}$. Then, by Lemma 4.2.9 (1), $z = (\alpha_i\bar{1}\phi$. Since $z \in \mathcal{GV}$, then there exists $\gamma \in E(R)^*$ such that $(z)\phi^{-1} = Q_{\gamma x} = \{\gamma 0\bar{1}, \gamma 1\bar{0}\}$. If $\gamma \preceq \alpha_i$, then by Corollary 3.1.7, $z \in \operatorname{int} C(\alpha_i)$ which is a contradiction since $z \in \partial\mathcal{P}$. Hence $\alpha_i = \gamma 0(1)^n$ for some $n \in \mathbb{N}$.

   Observe that $t \in Q_{\gamma x}$, that is either $t = \gamma 0\bar{1} = \alpha_i\bar{1}$ or $t = \gamma 1\bar{0}$. To prove the result, we have to show that $\gamma 1\bar{0} = \alpha_{i+1}\bar{1}$. By Lemma 3.2.2, $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$, where A is a complete antichain in $E(R)^*$. Then there exists $\alpha_k \in A$ such that $\alpha_k \prec \gamma 1\bar{0}$. If $\alpha_k \preceq \gamma$, then $\alpha_k \prec \alpha_i$ which is a contradiction since A is an antichain. Hence $\alpha_k = \gamma 1(0)^m$ for some $m \in \mathbb{N}$. Since A is a complete antichain, $\alpha_k = \alpha_{i+1}$. Therefore $\gamma 1\bar{0} = \alpha_{i+1}\bar{1}$.

By combining (1) – (3) above, we can conclude that

$$\alpha_{i-1}\bar{1} \leq_\ell t \leq_\ell \alpha_{i+1}\bar{0}.$$

□

{jambinai}

Lemma 4.2.13. *Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X and let* $\operatorname{lex}(P) = (C(\alpha_1), \dots, C(\alpha_n))$. *Then $C(\alpha_i) \cap C(\alpha_j) \neq \varnothing$ if and only if $|i - j| \leq 1$.*

Proof. Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X and let $\operatorname{lex}(P) = (C(\alpha_1), \dots, C(\alpha_n))$. Suppose that $C(\alpha_i) \cap C(\alpha_j) \neq \varnothing$. We shall prove that $|i - j| \leq 1$ by contradiction. Let $|i - j| \geq 2$, and suppose without a meaningful loss of generality that $j \geq i + 2$. Then $C(\alpha_i) <_\ell C(\alpha_{i+1}) <_\ell C(\alpha_j)$. Observe that $<_\ell$ is a linear order, induced from the linear lexicographic order $<_\ell$ on Ω. Then it follows that if

$t \in (C(\alpha_i))\phi^{-1}$ then $t <_\ell \alpha_{i+1}0\bar{1}$ and if $t \in (C(\alpha_{i+1}))\phi^{-1}$ then $t >_l \alpha_{i+1}\bar{0}1$. This implies that $C(\alpha_i) \cap C(\alpha_j) = \varnothing$, which is a contradiction. Hence $|i - j| \leq 1$.

Conversely, suppose that $|i - j| \leq 1$. If $i = j$, then the result follows trivially. If $i \neq j$, let us suppose without a meaningful loss of generality that $j = i + 1$. By Lemma 3.2.2, α_i and α_{i+1} belong to a complete antichain A in $E(R)^*$ and, by Definition 2.3.5, $<_\ell$ is a linear order on A. Then there exists $k \in \mathbb{N}$ such that $\alpha_i = e_1 \dots e_k 0\gamma$ and $\alpha_{i+1} = e_1 \dots e_k 1\delta$ (for some $\gamma, \delta \in E(R)^*$). Observe that, since A is a complete antichain, $\gamma = (1)^p$ and $\delta = (0)^q$ (for some $p, q \in \mathbb{N}$). Then, by Lemma 4.2.9 (1), $w_{\alpha_i} = [e_1 \dots e_k 0\bar{1}]$ and $v_{\alpha_{i+1}} = [e_1 \dots e_k 1\bar{0}]$ and, by Lemma 4.2.7, $w_{\alpha_i} = v_{\alpha_{i+1}}$. This implies that $C(\alpha_i) \cap C(\alpha_{i+1}) \neq \varnothing$. □

{emt}

Lemma 4.2.14. *Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X and let* $\operatorname{lex}(P) = (C(\alpha_1), \dots, C(\alpha_n))$. *Then*

1. $v_{\alpha_1} = a$,
2. $w_{\alpha_i} = v_{\alpha_{i+1}}$ *for all* $i = 1, \dots, n-1$,
3. $w_{\alpha_n} = b$.

Proof. Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X and let $\operatorname{lex}(P) = (C(\alpha_1), \dots, C(\alpha_n))$. By Lemma 3.2.2, $\mathcal{P} = \{C(\alpha) \mid \alpha \in A\}$, where A is a complete antichain in $E(R)^*$. Then

1. Consider the gluing vertex a. Observe that $(a)\phi^{-1} = Q_a = \{\bar{0}\}$. By Lemma 4.2.11 (1), there exists precisely one cell $C(\alpha_i) \in \mathcal{P}$ such that $v_{\alpha_i} = a$. By Lemma 4.2.9 (1), $v_{\alpha_i} = (\alpha_i\bar{0})\phi$. Hence $\alpha_i = (0)^n$ for some $n \in \mathbb{N}$. Then $\alpha_i \leq_\ell \alpha_j$ for all $j = 1, \dots, n$, and therefore $i = 1$.

2. Consider the gluing vertex γx (for some $\gamma \in E(R)^*$). Observe that $(\gamma x)\phi^{-1} = Q_{\gamma x} = \{\gamma 0\bar{1}, \gamma 1\bar{0}\}$. By Lemma 4.2.11 (3), there exists precisely one cell $C(\alpha_i) \in \mathcal{P}$ such that $v_{\alpha_i} = \gamma x$ and there exists precisely one cell $C(\alpha_j) \in \mathcal{P}$ such that $w_{\alpha_j} = \gamma x$. By Lemma 4.2.9 (1), $v_{\alpha_i} = (\alpha_i\bar{0})\phi$ and $w_{\alpha_j} = (\alpha_j\bar{1})\phi$. Hence $\alpha_i = \gamma 1(0)^n$ and $\alpha_j = \gamma 0(1)^n$ for some $n, m \in \mathbb{N}$. Since A is a complete antichain, $i = j + 1$ for all $j = 1, \dots, n-1$.

3. Consider the gluing vertex b. Observe that $(b)\phi^{-1} = Q_b = \{\bar{1}\}$. By Lemma 4.2.11 (2), there exists precisely one cell $C(\alpha_i) \in \mathcal{P}$ such that $w_{\alpha_i} = b$. By Lemma 4.2.9 (1), $w_{\alpha_i} = (\alpha_i\bar{1})\phi$. Hence $\alpha_i = (1)^n$ for some $n \in \mathbb{N}$. Then $\alpha_i \geq_\ell \alpha_j$ for all $j = 1, \dots, n$, and therefore $i = n$.

This proves the result. □

Lemma 4.2.15. *Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X with $|\mathcal{P}| = n$. Then $|\partial\mathcal{P}| = n + 1$.*

Proof. Let X be the limit space of the F replacement system. Let $\mathcal{P}$ be a cellular partition of X with $|\mathcal{P}| = n$. Let $\text{lex}(P) = (C(\alpha_1), \ldots, C(\alpha_n))$. Then, by Lemma 4.2.14

1. $v_{\alpha_1} = a$,

2. $w_{\alpha_i} = v_{\alpha_{i+1}}$ for all $i = 1, \ldots, n-1$,

3. $w_{\alpha_n} = b$.

And, by Lemma 4.2.11, for each $z \in \partial\mathcal{P}$.

1. If $z = a$, then there is precisely one cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and no cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.

2. If $z = b$, then there is no cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and there is precisely one cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.

3. If $z = \gamma x$ (for some $\gamma \in E(R)^*$), then there is precisely one cell $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) such that $z = v_\alpha$ and precisely one cell $C(\beta) \in \mathcal{P}$ (for some $\beta \in E(R)^*$) such that $z = w_\beta$.

Then a simple counting argument shows that $|\partial\mathcal{P}| = n + 1$. ☐

4.3 Our Limit Space and the Unit Interval

{4.3}

In this section, we will show that limit space X of the F replacement system, constructed in Section 4.2, is homeomorphic to the unit interval.

{nigel}

Definition 4.3.1. A *binary expansion* for a real number $x \in [0, 1]$ is a sequence $e_1 e_2 \cdots \in \{0, 1\}^\infty$ such that

$$x = \sum_{i=1}^{\infty} e_i 2^{-i}, \quad e_i \in \{0, 1\}.$$

Lemma 4.3.2. *Every real number $x \in [0, 1]$ has a binary expansion.*

Proof. Let $x \in [0, 1]$ and define a sequence $e_1 e_2 \ldots$ as follows

$$e_1 = \begin{cases} 0 \text{ if } x < \frac{1}{2} \\ 1 \text{ if } x \geq \frac{1}{2} \end{cases}$$

$$e_2 = \begin{cases} 0 \text{ if } x - \frac{e_1}{2} < \frac{1}{4} \\ 1 \text{ if } x - \frac{e_1}{2} \geq \frac{1}{4} \end{cases}$$

and so on. In general,

$$e_{k+1} = \begin{cases} 0 \text{ if } x - \sum_{i=1}^{k} \frac{e_i}{2^i} < \frac{1}{2^{k+1}} \\ 1 \text{ if } x - \sum_{i=1}^{k} \frac{e_i}{2^i} \geq \frac{1}{2^{k+1}} \end{cases}$$

We observe that $e_1 e_2 \ldots$ satisfies Definition 4.3.1. ☐

We observe that binary expansions are not unique for some elements $x \in [0,1]$.

Example 4.3.3. Observe that $1\bar{0}$ and $0\bar{1}$ are both binary expansions for $\frac{1}{2}$, since

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots .$$

Definition 4.3.4. A *dyadic rational* in the unit interval $[0,1]$ is a rational number $\frac{m}{2^n}$ such that $m, n \in \mathbb{N}$, $m \leq 2^n$ and $2 \nmid m$.

The following results describe the binary expansions of an element $x \in (0,1)$ precisely:

{helelyos}

Lemma 4.3.5. *The number $\frac{m}{2^n}$ is a dyadic rational in $(0,1)$ if and only if there exists a word $\alpha = e_1 \dots e_{n-1} \in E(R)^*$ such that $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are precisely the binary expansions for $\frac{m}{2^n}$.*

Proof. Let $\frac{m}{2^n}$ be a dyadic rational in $(0,1)$. Then $m, n \in \mathbb{N}$, $0 < m < 2^n$ and $2 \nmid m$. We can express m in base 2 as

$$m = e_1 2^{n-1} + e_2 2^{n-2} + \cdots + e_{n-1} 2 + e_n,$$

where $e_i \in \{0,1\}$ for $i = 1, \dots, n$. Since m is odd, necessarily $e_n = 1$. Hence

$$\frac{m}{2^n} = \frac{e_1}{2} + \frac{e_2}{2^2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n}.$$

We know that $\frac{1}{2^n} = \sum_{k=n+1}^{\infty} \frac{1}{2^k}$, which gives us

$$\frac{m}{2^n} = \frac{e_1}{2} + \frac{e_2}{2^2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots .$$

Setting $\alpha = e_1 \dots e_{n-1}$ gives us the binary expansions $\alpha 1\bar{0}$ and $\alpha 0\bar{1}$ for the dyadic rational $\frac{m}{2^n}$.

Conversely, consider a word $\alpha = e_1 \dots e_{n-1} \in E(R)^*$. Define

$$m = e_1 2^{n-1} + e_2 2^{n-2} + \cdots + e_{n-1} 2 + 1.$$

Observe that $0 < m < 2^n$ and $2 \nmid m$. Then

$$\begin{aligned}\frac{m}{2^n} &= \frac{e_1}{2} + \frac{e_2}{2^2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n} \\ &= \frac{e_1}{2} + \frac{e_2}{2^2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots .\end{aligned}$$

That is, $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are the binary expansions for $\frac{m}{2^n}$. This proves the result. ☐

{asp}

Corollary 4.3.6. *Let $\frac{m}{2^n}$ be a dyadic rational in $(0,1)$ with binary expansions $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ (for some $\alpha \in E(R)^*$). Then*

1. *$\alpha\bar{0}$ is a binary expansion for $\frac{m-1}{2^n}$.*
2. *$\alpha\bar{1}$ is a binary expansion for $\frac{m+1}{2^n}$.*

Proof. Let $\frac{m}{2^n}$ be a dyadic rational with binary expansions $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ (for some $\alpha \in E(R)^*$). Then

1.
$$\begin{aligned}\frac{m}{2^n} - \frac{1}{2^n} &= \left(\frac{e_1}{2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n} + \frac{0}{2^{n+1}} + \frac{0}{2^{n+2}} + \cdots\right) - \frac{1}{2^n} \\ &= \frac{e_1}{2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{0}{2^n} + \frac{0}{2^{n+1}} + \frac{0}{2^{n+2}} + \cdots .\end{aligned}$$

   That is, $\alpha\bar{0}$ is a binary expansion for $\frac{m}{2^n} - \frac{1}{2^n}$.

2.
$$\begin{aligned}\frac{m}{2^n} + \frac{1}{2^n} &= \left(\frac{e_1}{2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n} + \frac{0}{2^{n+1}} + \frac{0}{2^{n+2}} + \cdots\right) + \frac{1}{2^n} \\ &= \frac{e_1}{2} + \cdots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots .\end{aligned}$$

   That is, $\alpha\bar{1}$ is a binary expansion for $\frac{m}{2^n} + \frac{1}{2^n}$.

This proves the result. □

{truefaith}

Lemma 4.3.7. *An element $t \in (0,1)$ has a unique binary expansion if and only if it is not a dyadic rational.*

Proof. In part 1, we have proven that every dyadic rational has two binary expansions. It is sufficient to show that every element $t \in (0,1)$ with more than one binary expansion is a dyadic rational.

Consider two distinct binary expansions $e_1e_2\ldots$ and $e'_1e'_2\ldots$ for some $t \in (0,1)$. Then, by Definition 4.3.1 we have
$$\sum_{i=1}^{\infty} e_i 2^{-i} = \sum_{i=1}^{\infty} e'_i 2^{-i}.$$
We can assume without a meaningful loss of generality that there exists $n \in \mathbb{N}$ such that $e_1 \ldots e_{n-1} = e'_1 \ldots e'_{n-1}$ but $e_n = 0$ and $e'_n = 1$. Then

$$\begin{aligned}\frac{1}{2^n} &= \sum_{i=n+1}^{\infty} \frac{e_i - e'_i}{2^i} \\ &\leq \sum_{i=n+1}^{\infty} \frac{1}{2^i} = \frac{1}{2^n},\end{aligned}$$

which is only true if $e_i - e'_i = 1$ (i.e., $e_i = 1$ and $e'_i = 0$) for all $i \geq n+1$. Setting $\alpha = e_1 \ldots e_{n-1}$ gives us $e_1e_2\cdots = \alpha 0\bar{1}$ and $e'_1e'_2\cdots = \alpha 1\bar{0}$. We have proven in Lemma 4.3.5 that these are both binary expansions of the same dyadic rational $\frac{m}{2^n}$ (for some $m, n \in \mathbb{N}$ such that $0 < m < 2^n$ and $2 \nmid m$). □

{palacesofmont

Definition 4.3.8. Let us define the intervals $B_{m,n,k}$ in $(0,1)$ as follows:

$$B_{m,n,k} = \left(\frac{m}{2^n} - \frac{1}{2^k}, \frac{m}{2^n} + \frac{1}{2^k}\right),$$

for some m, n, k satisfying *Property A*, where Property A is defined as: $m, n, k \in \mathbb{N}$, $0 < m < 2^n$, $2 \nmid m$ and $k \geq n$.

{thebronx}

Lemma 4.3.9. *The collection*

$$\mathcal{B} = \{B_{m,n,k} \mid m, n, k \textit{ satisfy Property A}\}$$

forms a basis for the Euclidean topology on $(0,1)$.

Proof. Recall that open sets in the Euclidean topology on $(0,1)$ are the unions of open intervals. So it is sufficient to show that for any open interval $(x,y) \subseteq (0,1)$, there exists a collection $\mathcal{C} \subseteq \mathcal{B}$ such that

$$(x,y) = \bigcup \mathcal{C}.$$

Since dyadic rationals are dense in $(0,1)$, we can find a dyadic rational $\frac{m}{2^n}$ such that $x < \frac{m}{2^n} < y$. Consider all dyadic rationals $\frac{m}{2^n}$ contained in (x,y). Let us define our collection $\mathcal{C}$ to be the set of all intervals $B_{m,n,k}$ centred at such $\frac{m}{2^n}$ and contained in (x,y).

By the definition of $\mathcal{C}$,

$$(x,y) \supseteq \bigcup \mathcal{C}.$$

Consider a point $z \in (x,y)$. We have the following two choices for z:

1. If z is a dyadic rational, let $z = \frac{m}{2^n}$ and choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} \leq \min\{z-x, y-z\}$. Then

$$z \in B_{m,n,k} \subseteq (x,y).$$

2. If z is not a dyadic rational, choose $k \in \mathbb{N}$ such that $\frac{3}{2^k} \leq \min\{z-x, y-z\}$. Then there exist dyadic rationals of the form $\frac{\ell}{2^k}$ in (x,z) and (z,y). Choose $\ell \in \mathbb{N}$ to be as small as possible that $z < \frac{\ell}{2^k}$. It follows that

$$x < \frac{\ell-1}{2^k} < z < \frac{\ell}{2^k} < \frac{\ell+1}{2^k} < y.$$

Set $\frac{m}{2^n} = \frac{\ell}{2^k}$ (after cancellation). Then

$$z \in B_{m,n,k} \subseteq (x,y).$$

Hence

$$(x,y) = \bigcup \mathcal{C}.$$

This proves that $\mathcal{B}$ is a basis for $(0,1)$. $\square$

This basis can be extended to a basis for $[0,1]$ by including all intervals of the form $\left[0, \frac{1}{2^n}\right)$ and $\left(1-\frac{1}{2^n}, 1\right]$ for $n \geq 1$.

{daedalus}

Definition 4.3.10. Let us define a map $\vartheta\colon \Omega \to [0,1]$ by

$$(e_1e_2\ldots)\vartheta = \sum_{i=1}^{\infty} \frac{e_1}{2^i}.$$

We observe that ϑ is a surjective mapping but it is not injective, since, by Lemma 4.3.7 (1), $(\alpha 0\bar{1})\vartheta = (\alpha 1\bar{0})\vartheta$ for all $\alpha \in E(R)^*$.

The following lemma shows that ϑ preserves the lexicographic order on Ω:

{forever}

Lemma 4.3.11. *Let $e_1e_2\ldots$ and $e'_1e'_2\ldots$ be two distinct sequences in Ω such that $e_1e_2\ldots <_\ell e'_1e'_2\ldots$. Then one of the following holds:*

1. $(e_1e_2\ldots)\vartheta = (e'_1e'_2\ldots)\vartheta$ *if and only if there exists $\alpha \in E(R)^*$ such that $e_1e_2\ldots = \alpha 0\bar{1}$ and $e'_1e'_2\ldots = \alpha 1\bar{0}$.*

2. $(e_1e_2\ldots)\vartheta < (e'_1e'_2\ldots)\vartheta$ *otherwise.*

Proof. Let $e_1e_2\ldots$ and $e'_1e'_2\ldots$ be two distinct sequences in Ω such that $e_1e_2\ldots <_\ell e'_1e'_2\ldots$. Suppose that $(e_1e_2\ldots)\vartheta = (e'_1e'_2\ldots)\vartheta$. By Definition 4.3.10 of ϑ, they must be two different binary expansions of the same dyadic rational $\frac{m}{2^n}$. By Lemma 4.3.5, there exists $\alpha \in E(R)^*$ such that $e_1e_2\ldots = \alpha 0\bar{1}$ and $e'_1e'_2\ldots = \alpha 1\bar{0}$ (since they are in lexicographic order).

Conversely, suppose that there exists $\alpha \in E(R)^*$ such that $e_1e_2\ldots = \alpha 0\bar{1}$ and $e'_1e'_2\ldots = \alpha 1\bar{0}$. By Lemma 4.3.5, there exists a dyadic rational $\frac{m}{2^n}$ such that $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are the binary expansions for $\frac{m}{2^n}$. This implies that $(e_1e_2\ldots)\vartheta = (e'_1e'_2\ldots)\vartheta$. □

Example 4.3.12. Consider the interval

$$B_{1,1,2} = \left(\frac{1}{4}, \frac{3}{4}\right).$$

Observe that $(\frac{1}{4})\vartheta^{-1} = \{00\bar{1}, 01\bar{0}\}$ and $(\frac{3}{4})\vartheta^{-1} = \{10\bar{1}, 11\bar{0}\}$. Consider $e_1e_2\ldots \in \Omega$. Since, by Lemma 4.3.11, ϑ is an order-preserving map, then

$$(e_1e_2\ldots)\vartheta \in \left(\frac{1}{4}, \frac{3}{4}\right)$$

if and only if $01\bar{0} <_\ell e_1e_2\ldots < 10\bar{1}$. Hence

$$\begin{aligned}\left(\frac{1}{4}, \frac{3}{4}\right)\vartheta^{-1} &= \{e_1e_2\ldots \in \Omega \mid 01\bar{0} <_\ell e_1e_2\ldots < 10\bar{1}\}\\ &= (\Omega(01) \cup \Omega(10)) \setminus \{01\bar{0}, 10\bar{1}\}.\end{aligned}$$

This is open in Ω, since it is the union of basic open sets with the complement of a finite number of closed sets.

More generally, we have the following result:

{ceilingunlimi

Lemma 4.3.13. *Let $B_{m,n,k}$ be an interval defined in Definition 4.3.8 (for some m, n, k satisfying Property A). Let $\alpha \in E(R)^*$ such that $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are the binary expansions for the dyadic rational $\frac{m}{2^n}$.*

1. *If $k = n$, then*
$$(B_{m,n,n})\vartheta^{-1} = \Omega(\alpha)\backslash\{\alpha\bar{0}, \alpha\bar{1}\}.$$

2. *If $k > n$, then*
$$(B_{m,n,k})\vartheta^{-1} = \left(\Omega(\alpha 0(1)^{k-n-1}) \cup \Omega(\alpha 1(0)^{k-n-1})\right) \setminus \left\{\alpha 0(1)^{k-n-1}\bar{0}, \alpha 1(0)^{k-n-1}\bar{1}\right\}.$$

Proof. Let $B_{m,n,k}$ be an interval defined in Definition 4.3.8 (for some m, n, k satisfying Property A):
$$B_{m,n,k} = \left(\frac{m}{2^n} - \frac{1}{2^k}, \frac{m}{2^n} + \frac{1}{2^k}\right).$$
By Lemma 4.3.5, the dyadic rational $\frac{m}{2^n}$ has precisely two binary expansions $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ (for some $\alpha = e_1 \dots e_{n-1} \in E(R)^*$).

1. Suppose $k = n$. Then, by Corollary 4.3.6, $\alpha\bar{0}$ is a binary expansion for $\frac{m-1}{2^n}$ and $\alpha\bar{1}$ is a binary expansion for $\frac{m+1}{2^n}$.

   Consider a sequence $t \in \Omega$. By Lemma 4.3.11, ϑ is order-preserving. Then $(t)\vartheta \in B_{m,n,n}$ if and only if $\alpha\bar{0} <_\ell t <_\ell \alpha\bar{1}$. It follows that
$$(B_{m,n,n})\vartheta^{-1} = \Omega(\alpha)\backslash\{\alpha\bar{0}, \alpha\bar{1}\}.$$
   This is open in Ω, since it is the union of basic open sets with the complement of a finite number of closed sets.

2. Suppose $k > n$. Then
$$\begin{aligned}\frac{m}{2^n} - \frac{1}{2^k} &= \left(\frac{e_1}{2} + \dots + \frac{e_{n-1}}{2^{n-1}} + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots\right) - \frac{1}{2^k} \\ &= \frac{e_1}{2} + \dots + \frac{e_{n-1}}{2^{n-1}} + \frac{0}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^k} + \frac{0}{2^{k+1}} + \frac{0}{2^{k+2}} + \cdots,\end{aligned}$$
   That is, $\alpha 0(1)^{k-n-1}\bar{0}$ is a binary expansion for $\frac{m}{2^n} - \frac{1}{2^k}$. Similarly,
$$\begin{aligned}\frac{m}{2^n} + \frac{1}{2^k} &= \left(\frac{e_1}{2} + \dots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n} + \frac{0}{2^{n+1}} + \frac{0}{2^{n+2}} + \cdots\right) + \frac{1}{2^k} \\ &= \frac{e_1}{2} + \dots + \frac{e_{n-1}}{2^{n-1}} + \frac{1}{2^n} + \frac{0}{2^{n+1}} + \dots + \frac{0}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \cdots,\end{aligned}$$
   That is, $\alpha 1(0)^{k-n-1}\bar{1}$ is a binary expansion for $\frac{m}{2^n} + \frac{1}{2^k}$.

   Consider a sequence $t \in \Omega$. By Lemma 4.3.11, ϑ is order-preserving. Then $(t)\vartheta \in B_{m,n,k}$ if and only if $\alpha 0(1)^{k-n-1}\bar{0} <_\ell t <_\ell \alpha 1(0)^{k-n-1}\bar{1}$. It follows that
$$(B_{m,n,k})\vartheta^{-1} = \left(\Omega(\alpha 0(1)^{k-n-1}) \cup \Omega(\alpha 1(0)^{k-n-1})\right) \setminus \left\{\alpha 0(1)^{k-n-1}\bar{0}, \alpha 1(0)^{k-n-1}\bar{1}\right\}.$$

This is open in Ω, since it is the union of basic open sets with the complement of a finite number of closed sets.

$\square$

{holyfallout}

Lemma 4.3.14. *The function ϑ is continuous.*

Proof. Let ϑ be as defined in Definition 4.3.10. To show that ϑ is a continuous function, we have to show that $\vartheta^{-1}(U)$ is open in Ω whenever U is open in $[0,1]$. It is enough to verify this condition when U is chosen from a suitable basis for the topology on $[0,1]$ since all open sets are unions of basic open sets.

In Lemma 4.3.13, we have shown that the preimage of the intervals $B_{m,n,k}$ under ϑ is open in Ω. Observe that $\left(\frac{1}{2^n}\right)\vartheta^{-1} = (0)^n\bar{1}$ and $\left(1-\frac{1}{2^n}\right)\vartheta^{-1} = (1)^n\bar{0}$. Then, a similar argument to Lemma 4.3.13 shows that

$$\left(\left[0, \frac{1}{2^n}\right)\right)\vartheta^{-1} = \Omega((0)^n) \setminus \{(0)^n\bar{1}\}$$

and

$$\left(\left(1-\frac{1}{2^n}, 1\right]\right)\vartheta^{-1} = \Omega((1)^n) \setminus \{(1)^n\bar{0}\}.$$

These are open in Ω, since one-point sets are closed in Ω. This proves that ϑ is continuous.

$\square$

{cultofshe}

Lemma 4.3.15. *Let $e_1e_2\ldots$ and $e'_1e'_2\ldots$ be two distinct sequences in Ω. Then $(e_1e_2\ldots)\vartheta = (e'_1e'_2\ldots)\vartheta$ if and only if $e_1e_2\ldots \sim e'_1e'_2\ldots$.*

Proof. Let $e_1e_2\ldots$ and $e'_1e'_2\ldots$ be two distinct sequences in Ω. Suppose that $(e_1e_2\ldots)\vartheta = (e'_1e'_2\ldots)\vartheta$. Then, by Lemma 4.3.7, these are two distinct binary expansions of a dyadic rational. Then there exists $\alpha \in E(R)^*$ such that

$$e_1e_2\ldots, e'_1e'_2\ldots \in \{\alpha 0\bar{1}, \alpha 1\bar{0}\}.$$

Hence, by Lemma 4.2.7, $e_1e_2\ldots \sim e'_1e'_2\ldots$.

Conversely, suppose that $e_1e_2\ldots \sim e'_1e'_2\ldots$. Then, by Lemma 4.2.7, $\alpha \in E(R)^*$

$$e_1e_2\ldots, e'_1e'_2\ldots \in \{\alpha 0\bar{1}, \alpha 1\bar{0}\}.$$

It follows from Lemma 4.3.7 that are the binary expansions of the same dyadic rational. Hence $(e_1e_2\ldots)\vartheta = (e'_1e'_2\ldots)\vartheta$. $\square$

{templeoflove}

Definition 4.3.16. The function ϑ induces a unique map $\theta\colon X \to [0,1]$, where $\vartheta = \phi\theta$, i.e., the following diagram commutes:

$$\begin{tikzcd}
\Omega \arrow[r, "\phi"] \arrow[dr, "\vartheta"'] \arrow[dr, phantom, "\circlearrowleft", bend left] & X \arrow[d, "\theta"] \\
& {[0,1]}
\end{tikzcd}$$

Let $x \in X$. Then $(x)\theta = (t)\vartheta$ for all t such that $(t)\phi = x$. By Lemma 4.3.15, the map ϑ is constant on $\sim$-equivalence classes. Hence $(t)\phi\theta = (t)\vartheta$ is independent of the choice of representative t for the equivalence class $(t)\phi$. Therefore θ is well-defined.

{rotersand}

Lemma 4.3.17. *The map θ is a homeomorphism.*

Proof. Let $\theta\colon X \to [0,1]$ be the map defined in Definition 4.3.16. Observe that X is compact and $[0,1]$ is Hausdorff. Then it is sufficient to show that the map θ is continuous and bijective.

Let us prove that θ is continuous: Consider an open set U in $[0,1]$. Observe that $(U)\theta^{-1} \subseteq X$. Then, by Definition 4.3.16, $(U)\theta^{-1}\phi^{-1} = (U)\vartheta^{-1}$, which is open in Ω since ϑ is continuous by Lemma 4.3.14. Then, by definition of the quotient topology on X, $(U)\theta^{-1}$ is open in X.

Let us prove that θ is surjective: Observe, since ϑ is surjective, that for any $x \in [0,1]$ there is a $t \in \Omega$ such that $(t)\vartheta = x$. Then $(t)\phi$ is an element of X and $(y)\phi\theta = (y)\vartheta = x$.

Let us prove that θ is injective: Consider $x_1, x_2 \in X$ such that $(x_1)\theta = (x_2)\theta$. Let $t_1, t_2 \in \Omega$ such that $(t_1)\phi = x_1$ and $(t_2)\phi = x_2$. Then $\vartheta(t_1) = \vartheta(t_2)$, and, by Lemma 4.3.15, $t_1 \sim t_2$. Hence $x_1 = x_2$.

This proves that θ is a homeomorphism. $\square$

{angriff}

Lemma 4.3.18. *Let $\alpha = e_1 \dots e_{n-1} \in E(R)^*$ such that $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are the binary expansions of the dyadic rational $\frac{m}{2^n}$. Then*

$$(\alpha x)\theta = \frac{m}{2^n}.$$

Proof. Let $\alpha = e_1 \dots e_{n-1} \in E(R)^*$ such that $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are the binary expansions of the dyadic rational $\frac{m}{2^n}$. Consider the gluing vertex αx. Observe that $(\alpha x)\phi^{-1} = \{\alpha 0\bar{1}, \alpha 1\bar{0}\}$. By Lemma 4.3.5 and Definition 4.3.10, $(\alpha 0\bar{1})\vartheta = \frac{m}{2^n}$ and $(\alpha 1\bar{0})\vartheta = \frac{m}{2^n}$. Hence, by Definition 4.3.16,

$$(\alpha x)\theta = \frac{m}{2^n}.$$

$\square$

Remark 4.3.19. Observe that $(a)\theta = 0$ and $(b)\theta = 1$.

Let $\alpha = e_1 \dots e_{n-1} \in E(R)^*$ such that $\alpha 0\bar{1}$ and $\alpha 1\bar{0}$ are the binary expansions of the dyadic rational $\frac{m}{2^n}$. Let us define the depth of $\frac{m}{2^n}$ to be $\mathrm{depth}(\frac{m}{2^n}) = n$. Observe that this is equal to $\mathrm{depth}(\alpha x) = n$. There exists a natural order on dyadic rationals (which is different to the normal order on real numbers): Let $\frac{m_1}{2^{n_1}}$ and $\frac{m_2}{2^{n_2}}$ be two dyadic rationals. Then

$$\frac{m_1}{2^{n_1}} < \frac{m_2}{2^{n_2}} \quad \text{if} \quad n_1 < n_2$$

or

$$\frac{m_1}{2^{n}} < \frac{m_2}{2^{n}} \quad \text{if} \quad n_1 = n_2 \text{ and } m_1 < m_2.$$

We observe that this is the same as Definition 2.3.10 of depth order $\leq_d$ on the set $\mathcal{GV}$.

Definition 4.3.20. A *standard dyadic interval* is an interval of the form

$$\left[\frac{m-1}{2^n}, \frac{m+1}{2^n}\right],$$

where $m, n \in \mathbb{N}$, $2 \nmid m$, and $m < 2^n$.

Example 4.3.21. Consider the standard dyadic interval $\left[\frac{1}{4}, \frac{1}{2}\right]$. In this case, $\frac{m}{2^n} = \frac{3}{8}$. Observe that $\left(\frac{1}{4}\right)\vartheta = \{00\bar{1}, 01\bar{0}\}$ and $\left(\frac{1}{2}\right)\vartheta = \{0\bar{1}, 1\bar{0}\}$. Then, for $t \in \Omega$, $01\bar{0} \leq_\ell t \leq_\ell 01\bar{1}$ if and only if $frac14 \leq (t)\vartheta \leq \frac{1}{2}$. Hence

$$(\Omega(01))\vartheta = \left[\frac{1}{4}, \frac{1}{2}\right].$$

By Definition 4.3.16, it follows that

$$(C(01))\theta = \left[\frac{1}{4}, \frac{1}{2}\right].$$

More generally, we have the following result:

{dinosaurjr}

Lemma 4.3.22. *Let X be the limit space of the F replacement system. Let $C(\alpha)$ be a cell in X (for some $\alpha = e_1 \ldots e_{n-1} \in E(R)^*$). Then there exists a dyadic rational $\frac{m}{2^n}$ such that*

$$(C(\alpha))\theta = \left[\frac{m-1}{2^n}, \frac{m+1}{2^n}\right].$$

Proof. Let $\alpha = e_1 \ldots e_{n-1} \in E(R)^*$. By Lemma 4.3.5 and Corollary 4.3.6, there exists a dyadic rational $\frac{m}{2^n}$ such that

$$(\alpha\bar{0})\,\vartheta = \frac{m-1}{2^n}$$
$$(\alpha\bar{1})\,\vartheta = \frac{m+1}{2^n}.$$

Since ϑ is an order-preserving surjective map, this implies that

$$(\Omega(\alpha))\vartheta = \left[\frac{m-1}{2^n}, \frac{m+1}{2^n}\right].$$

By Definition 4.3.16, $\vartheta = \phi\theta$. Therefore

$$(C(\alpha))\theta = \left[\frac{m-1}{2^n}, \frac{m+1}{2^n}\right].$$

□

Remark 4.3.23. Observe that Definition 3.1.13 of the depth of a cell $C(\alpha)$ in X is the same as the depth of the dyadic rational $\frac{m}{2^n}$ such that

$$(C(\alpha))\theta = \left[\frac{m-1}{2^n}, \frac{m+1}{2^n}\right].$$

4.4 Thompson's Group *F* as a Rearrangement Group

{4.4}

Let X be the limit space of the F replacement system. In Section 4.3, we showed that X is homeomorphic to the unit interval $[0,1]$. Let $\mathrm{Rearr}(X)$ be the group of rearrangements of X. In this section, we will prove that $\mathrm{Rearr}(X)$ is isomorphic to Richard Thompson's group F.

Recall from Definition 4.3.16 the homeomorphism $\theta\colon X \to [0,1]$. There exists a topological conjugation $\theta^*\colon \mathrm{Aut}(X) \to \mathrm{Aut}([0,1])$, induced by θ, i.e., $\theta^*\colon f \to \theta^{-1} f \theta$ for all $f \in \mathrm{Rearr}(X)$. Recall from Lemma 3.4.6 that $\mathrm{Rearr}(X)$ is a subgroup of $\mathrm{Aut}(X)$. Let $\mathcal{G} = (\mathrm{Rearr}(X))\theta^*$. We will show that $\mathcal{G} = F$, i.e., $\theta^*|_{\mathrm{Rearr}(X)}\colon \mathrm{Rearr}(X) \to F$ is a group isomorphism. We will do this by proving the following two results:

{beastland}

Proposition 4.4.1. *The group $\mathcal{G}$ is a subgroup of Thompson's group F.*

{persephone}

Proposition 4.4.2. *Let $A, B \in F$ from Example 4.1.4. Then $A, B \in \mathcal{G}$.*

To prove these propositions, we require the following results regarding rearrangements of the limit space X of the F replacement system:

{atlasair}

Lemma 4.4.3. *Let X be the limit space of the F replacement system. Let f be a rearrangement of X. Let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) and $C(\beta) \in (\mathcal{P})f$ (for some $\beta \in E(R)^*$). Suppose one of the following holds:*

1. $(v_\alpha)f = v_\beta$
2. $(w_\alpha)f = w_\beta$

Then $(C(\alpha))f = C(\beta)$.

Proof. Let X be the limit space of the F replacement system. Let f be a rearrangement of X. Let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Let $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$) and $C(\beta) \in (\mathcal{P})f$ (for some $\beta \in E(R)^*$).

Suppose $(v_\alpha)f = v_\beta$. By the definition of a rearrangement, there exists a cell $C(\gamma) \in (\mathcal{P})f$ (for some $\gamma \in E(R)^*$) such that $(C(\alpha))f = C(\gamma)$. Then, by Lemma 3.3.22, $(v_\alpha)f = v_\gamma$, i.e., $v_\beta = v_\gamma$. By Lemma 4.2.11, there is only one cell $C(\beta) \in (\mathcal{P})f$ such that v_β is the initial vertex of $C(\beta)$. This implies that $C(\gamma) = C(\beta)$.

Suppose $(w_\alpha)f = w_\beta$. By the definition of a rearrangement, there exists a cell $C(\gamma) \in (\mathcal{P})f$ (for some $\gamma \in E(R)^*$) such that $(C(\alpha))f = C(\gamma)$. Then, by Lemma 3.3.22, $(w_\alpha)f = w_\gamma$, i.e., $w_\beta = w_\gamma$. By Lemma 4.2.11, there is only one cell $C(\beta) \in (\mathcal{P})f$ such that w_β is the terminal vertex of $C(\beta)$. This implies that $C(\gamma) = C(\beta)$. □

{paragun}

Lemma 4.4.4. *Let X be the limit space of the F replacement system. Let f be a rearrangement of X. Then $(a)f = a$ and $(b)f = b$.*

Proof. Let X be the limit space of the F replacement system. Let f be a rearrangement of X. Let $(\mathcal{P}_{f,d}, \mathcal{P}_{f,r})$ be the minimal bipartition for f. Let $\operatorname{lex}(\mathcal{P}_{f,d}) = (C(\alpha_1), \ldots, C(\alpha_n))$ and $\operatorname{lex}(\mathcal{P}_{f,r}) = (C(\beta_1), \ldots, C(\beta_n))$. We shall prove this result by contradiction.

By Lemma 4.2.14 (1), $a = v_{\alpha_1}$ and $a = v_{\beta_1}$. Suppose $(a)f \neq a$. Then, by Lemma 4.4.3, $(C(\alpha_1))f \neq C(\beta_1)$. Suppose $(C(\alpha_1))f = C(\beta_i)$ for some $2 \leq i \leq n$. Then, by Lemma 4.2.14 (2), $(a)f = w_{\beta_{i-1}}$. By the definition of a rearrangement, there exists a cell $C(\alpha_j)$ (for some $2 \leq j \leq n$) such that $(C(\beta_{i-1}))f^{-1} = C(\alpha_j)$. By Lemma 3.3.22, this implies that $((a)f)f^{-1} = w_{\alpha_j}$, i.e., $a = w_{\alpha_j}$. This contradicts Lemma 4.2.11. Therefore, $(a)f = a$.

By Lemma 4.2.14 (3), $b = w_{\alpha_n}$ and $b = w_{\beta_n}$. Suppose $(b)f \neq b$. Then, by Lemma 4.4.3, $(C(\alpha_n))f \neq C(\beta_n)$. Suppose $(C(\alpha_n))f = C(\beta_i)$ for some $1 \leq i \leq n-1$. Then, by Lemma 4.2.14 (2), $(b)f = v_{\beta_{i+1}}$. By the definition of a rearrangement, there exists a cell $C(\alpha_j)$ (for some $1 \leq j \leq n-1$) such that $(C(\beta_{i+1}))f^{-1} = C(\alpha_j)$. By Lemma 3.3.22, this implies that $((b)f)f^{-1} = v_{\alpha_j}$, i.e., $b = v_{\alpha_j}$. This contradicts Lemma 4.2.11. Therefore, $(b)f = b$. □

{cockroachking}

Lemma 4.4.5. *Let X be the limit space of the F replacement system. Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X and let* $\operatorname{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$ *and* $\operatorname{lex}(\mathcal{Q}) = (C(\beta_1), \ldots, C(\beta_n))$. *Then there exists a rearrangement f of X such that $f|_{C(\alpha_i)} \colon C(\alpha_i) \to C(\beta_i)$ is a canonical homeomorphism for all $i = 1, \ldots, n$.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ be cellular partitions of X and let $\operatorname{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$ and $\operatorname{lex}(\mathcal{Q}) = (C(\beta_1), \ldots, C(\beta_n))$. Let us define the canonical homeomorphism $\psi_i \colon C(\alpha_i) \to C(\beta_i)$ for all $i = 1, \ldots, n$. By Lemma 4.2.14,

1. $v_{\alpha_1} = a$,
2. $w_{\alpha_i} = v_{\alpha_{i+1}}$ for all $i = 1, \ldots, n-1$,
3. $w_{\alpha_n} = b$.

Observe that

1. $(v_{\alpha_1})\psi_1 = v_{\beta_1} = a$,
2. $(w_{\alpha_i})\psi_i = w_{\beta_i} = v_{\beta_{i+1}} = (v_{\alpha_{i+1}})\psi_{i+1}$ for all $i = 1, \ldots, n-1$,
3. $(v_{\alpha_n})\psi_n = v_{\beta_n} = b$.

That is, if z is a boundary point of both $C(\alpha_i)$ and $C(\alpha_j)$, then $(z)\psi_i = (z)\psi_j$. Let us define a map $f \colon X \to X$ such that $f|_{C(\alpha_i)} = \psi_i$ is a canonical homeomorphism for all $i = 1, \ldots, n$. Then, by Lemma 3.3.10, f is a rearrangement of X. □

{deathgrips}

Lemma 4.4.6. *Let X be the limit space of the F replacement system. Let f be a rearrangement of X and and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Suppose* $\operatorname{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$. *Then* $\operatorname{lex}((\mathcal{P})f) = ((C(\alpha_1))f, \ldots, (C(\alpha_n))f)$.

Proof. Let f be a rearrangement of X and and let $(\mathcal{P}, (\mathcal{P})f)$ be a cellular bipartition for f. Suppose that $\operatorname{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$ and $\operatorname{lex}((\mathcal{P})f) = (C(\beta_1), \ldots, C(\beta_n))$.

By Lemma 4.2.14 (1), $v_{\alpha_1} = a$ and $v_{\beta_1} = a$. By Lemma 4.4.4, $(a)f = a$. This gives us $(v_{\alpha_1})f = v_{\beta_1}$. Hence, by Lemma 4.4.3, $(C(\alpha_1))f = C(\beta_1)$. By Lemma 4.2.14 (2), $(w_{\alpha_i})\psi_i = w_{\beta_i} = v_{\beta_{i+1}} = (v_{\alpha_{i+1}})\psi_{i+1}$ for all $i = 1, \ldots, n-1$. And thus, by similar logic, $(C(\alpha_i))f = C(\beta_i)$ for all $i = 1, \ldots, n$. Hence, $\operatorname{lex}((\mathcal{P})f) = ((C(\alpha_1))f, \ldots, (C(\alpha_n))f)$. □

Let f be an element of Richard Thompson's group F. Recall the characterisation for f from Definition 4.1.1: $f\colon [0,1] \to [0,1]$ is a piecewise linear homeomorphism, differentiable everywhere except at finitely many dyadic rationals, and at the intervals of differentiability the derivatives are powers of 2. Let us call these dyadic rationals the *break points* of f, and denote them by the ordered list $(a_1, \ldots, a_r)$ where $a_1 = 0$ and $a_r = 1$. Then the unit interval $[0,1]$ is the overlapping union

$$[0,1] = \bigcup_{i=1}^{r-1} [a_i, a_{i+1}],$$

where the points of overlap are $a_2, \ldots, a_{r-1}$, or the disjoint union

$$[0,1] = \left(\bigsqcup_{i=1}^{r-1} (a_i, a_{i+1}) \right) \sqcup \left(\bigsqcup_{j=1}^{n} \{a_j\} \right).$$

Each interval $[a_i, a_{i+1}]$ is a finite union of standard dyadic intervals with disjoint interiors. Observe that the points of overlap of these standard dyadic intervals reveal *hidden break points*. This is illustrated in the example below:

Example 4.4.7. Let us define a function $g \in F$ as follows:

$$(x)g = \begin{cases} 2x & 0 \leq x \leq \frac{1}{8} \\ x + \frac{1}{8} & \frac{1}{8} \leq x \leq \frac{3}{4} \\ \frac{x}{2} + \frac{1}{2} & \frac{3}{4} \leq x \leq 1 \end{cases}$$

Observe that the break points of this function are: $0, \frac{1}{8}, \frac{3}{4}, 1$. This gives us the following intervals of differentiability: $\left(0, \frac{1}{8}\right), \left(\frac{1}{8}, \frac{3}{4}\right), \left(\frac{3}{4}, 1\right)$. When we convert these to standard dyadic intervals, we get: $\left(0, \frac{1}{8}\right), \left(\frac{1}{8}, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{3}{8}\right), \left(\frac{3}{8}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{5}{8}\right), \left(\frac{5}{8}, \frac{3}{4}\right), \left(\frac{3}{4}, 1\right)$. Observe that there exist *hidden break points*: $\frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}$.

Then g is a rearrangement of X, defined as follows: Let $x \in [0,1]$ with $x = [e_1 e_2 e_3 \ldots]$.

Then

$$([e_1e_2e_3\ldots])\,g = \begin{cases} [00e_3e_4\ldots], & \text{if } e_1e_2e_3 = 000 \\ [010e_3e_4\ldots], & \text{if } e_1e_2e_3 = 001 \\ [011e_3e_4\ldots], & \text{if } e_1e_2e_3 = 010 \\ [100e_3e_4\ldots], & \text{if } e_1e_2e_3 = 011 \\ [101e_3e_4\ldots], & \text{if } e_1e_2e_3 = 100 \\ [110e_3e_4\ldots], & \text{if } e_1e_2e_3 = 101 \\ [111e_2e_3\ldots], & \text{if } e_1e_2 = 11 \end{cases}$$

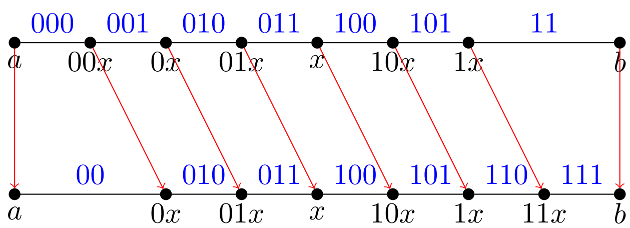
Observe that the cells and boundary points get mapped as follows:

$$(C(000))\,g = C(00)$$
$$(00x)\,g = 0x$$
$$(C(001))\,g = C(010)$$
$$(0x)\,g = 01x$$
$$(C(010))\,g = C(011)$$
$$(01x)\,g = x$$
$$(C(011))\,g = C(100)$$
$$(x)\,g = 10x$$
$$(C(100))\,g = C(101)$$
$$(10x)\,g = 1x$$
$$(C(101))\,g = C(110)$$
$$(1x)\,g = 11x$$
$$(C(11))\,g = C(111)$$

The rearrangement g has the following minimal bipartition:

$$\mathcal{P}_{g,d} = \{C(000), C(001), C(010), C(011), C(100), C(101), C(11)\}$$
$$\mathcal{P}_{g,r} = \{C(00), C(010), C(011), C(100), C(101), C(110), C(111)\}$$

The rearrangement g is illustrated by the following graph-pair diagram:



g

Let g be a rearrangement of X and let $(\mathcal{P}, \mathcal{Q})$ be a cellular bipartition for g. The following algorithm finds the minimal bipartition $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ for g:

Algorithm 4.4.8 Minimal bipartition for a rearrangement g of X. {actors}

Require: Let g be a rearrangement of X and let $(\mathcal{P}, \mathcal{Q})$ be a cellular bipartition for g, where $\mathrm{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$ and $\mathrm{lex}(\mathcal{Q}) = (C(\beta_1), \ldots, C(\beta_n))$.

function MINIMALBIPARTITION($\mathcal{P}$, $\mathcal{Q}$)
  Set `switch` $= false$.
  Set $i = 1$.
  while `switch` $= false$ **and** $i < n$ **do**
    if $C(\alpha_i) \cup C(\alpha_{i+1}) = C(\gamma)\ \gamma \in E(R)^*)$ **and**
    $C(\beta_i) \cup C(\beta_{i+1}) = C(\delta)$ (for some $\delta \in E(R)^*$) **then**
      Set `switch` $= true$.
      Set $\mathcal{P}' = \{C(\alpha_1), \ldots, C(\alpha_{i-1}), C(\gamma), C(\alpha_{i+2}), \ldots, C(\alpha_n))$
      and $\mathcal{Q}' = \{C(\beta_1), \ldots, C(\beta_{i-1}), C(\delta), C(\beta_{i+2}), \ldots, C(\beta_n)\}$.
      Set $i = i + 2$.
    else
      Set $i = i + 1$.
    end if
  end while
  if `switch` $= true$ **then**
    return MINIMALBIPARTITION($\mathcal{P}'$, $\mathcal{Q}'$).
  else
    return $(\mathcal{P}, \mathcal{Q})$.
  end if
end function

{gallowsbird}

Lemma 4.4.9. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}, \mathcal{Q})$ be a cellular bipartition for g. Let $\mathrm{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$ and $\mathrm{lex}(\mathcal{Q}) = (C(\beta_1), \ldots, C(\beta_n))$. Then* MINIMALBIPARTITION $(\mathcal{P}, \mathcal{Q})$ *is the minimal bipartition for g.*

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}, \mathcal{Q})$ be a cellular bipartition for g. Let $\mathrm{lex}(\mathcal{P}) = (C(\alpha_1), \ldots, C(\alpha_n))$ and $\mathrm{lex}(\mathcal{Q}) = (C(\beta_1), \ldots, C(\beta_n))$. We will prove this result by induction on n.

Let $n = 1$. In this case, $\mathcal{P} = \mathcal{Q} = \{C(\epsilon)\}$ and g is the identity map. In this case, implementing Algorithm 4.4.8 gives us MINIMALBIPARTITION $(\mathcal{P}, \mathcal{Q}) = (\mathcal{P}, \mathcal{Q})$ and the inductive hypothesis holds.

Suppose the inductive hypothesis is true for $n = m$. Let us examine the case when $n = m + 1$. We have the following possibilities:

1. Consider the case when $(\mathcal{P}, \mathcal{Q})$ is the minimal bipartition for g. Let us implement Algorithm 4.4.8 and calculate MinimalBipartition $(\mathcal{P}, \mathcal{Q})$. Algorithm 4.4.8 implements the while loop in lines 4-14 as follows: By Lemma 3.3.27, the conditions of the if loop in lines 5-6 are not satisfied for all $i = 1, \ldots, m+1$. Hence, Algorithm 4.4.8 skips lines 7-10 for all $i = 1, \ldots, m+1$. Since `switch` $= \mathit{false}$, Algorithm 4.4.8 skips line 16 and implements line 18, which returns $(\mathcal{P}, \mathcal{Q})$, which is the minimal bipartition for g. Therefore, the inductive hypothesis holds for $n = m+1$.

2. Consider the case when $(\mathcal{P}, \mathcal{Q})$ is not the minimal bipartition for g. Let us implement Algorithm 4.4.8 and calculate MinimalBipartition $(\mathcal{P}, \mathcal{Q})$. Algorithm 4.4.8 implements the while loop in lines 4-14 as follows: By Lemma 3.3.27, the conditions of the if loop in lines 5-6 are satisfied for some $1 \leq i \leq m+1$. Then Algorithm 4.4.8 implements lines 7-10, which gives us `switch` $= \mathit{true}$ and the sets $\mathcal{P}'$ and $\mathcal{Q}'$. Since `switch` $= \mathit{true}$, Algorithm 4.4.8 implements line 16 and calls MinimalBipartition $(\mathcal{P}', \mathcal{Q}')$. Observe that $|\mathcal{P}| = |\mathcal{Q}| \leq m$. Hence, the inductive hypothesis holds and therefore MinimalBipartition $(\mathcal{P}', \mathcal{Q}') = (\mathcal{P}'', \mathcal{Q}'')$ is the minimal bipartition for g. Therefore, the inductive hypothesis holds for $n = m+1$.

This proves the result by induction. □

We can now prove Proposition 4.4.1 and Proposition 4.4.2:

Proof. **(Proof of Proposition 4.4.1)**

Let X be the limit space of the F replacement system. Let $\mathcal{G} = (\mathrm{Rearr}(X))\theta^*$. Let F be Thompson's group F. Let $f^* \in \mathcal{G}$. Then $f^* = \theta^{-1} f \theta$ for some $f \in \mathrm{Rearr}(X)$. By Definition 3.3.8, there exists a cellular bipartition $(\mathcal{P}, (\mathcal{P})f)$ for f. Let $\partial\mathcal{P}$ be the set of boundary points of $\mathcal{P}$. Recall that $\partial\mathcal{P} \subset \mathcal{GV}$. Then, by Lemma 4.3.18, for all $z \in \partial\mathcal{P}$, $(z)\theta$ is a dyadic rational.

Let $C(\alpha) \in \mathcal{P}$ (for some $\alpha \in E(R)^*$). Then there exists $C(\beta) \in (\mathcal{P})f$ (for some $\beta \in E(R)^*$) such that $f|_{C(\alpha)} \colon C(\alpha) \to C(\beta)$ is a canonical homeomorphism. By Lemma 4.3.22, θ establishes a one-to-one correspondence between cells in X and standard dyadic intervals in $[0,1]$. We will show that $f^*|_{(C(\alpha))\theta}$ is an affine map, and its derivative is a power of 2.

Set $\alpha = e_0 \ldots e_n$ and $\beta = e'_0 \ldots e'_m$. Let $[\alpha e_{n+1} e_{n+2} \ldots] \in C(\alpha)$. Then

$$([\alpha e_{n+1} e_{n+2} \ldots])f = [\beta e_{n+1} e_{n+2} \ldots].$$

Observe that $f^*|_{(C(\alpha))\theta} = \theta^{-1} f|_{C(\alpha)} \theta$. Hence

$$([\alpha e_{n+1} e_{n+2} \ldots])\theta = \sum_{i=1}^{n} \frac{e_i}{2^i} + \sum_{i=n+1}^{\infty} \frac{e_i}{2^i}$$

and

$$([\beta e_{n+1} e_{n+2} \ldots])\theta = \sum_{i=1}^{m} \frac{e'_i}{2^i} + \sum_{i=m+1}^{\infty} \frac{e_{i+n-m}}{2^i}$$

$$= \sum_{i=1}^{m} \frac{e'_i}{2^i} + \sum_{j=n+1}^{\infty} \frac{e_j}{2^{j+m-n}}.$$

Setting $c = \sum_{i=1}^{n} \frac{e_i}{2^i}$, $d = \sum_{i=1}^{m} \frac{e'_i}{2^i}$ and $x = \sum_{i=n+1}^{\infty} \frac{e_i}{2^i}$ gives us

$$(c+x)f^*|C(\alpha) = d + \frac{x}{2^{m-n}}.$$

This proves that $f^*|_{C(\alpha)}$ is an affine map and its derivative is a power of 2. Hence

$$\mathcal{G} \leq F.$$

□

Proof. **(Proof of Proposition 4.4.2)**

Let A and B be the generators of Thompson's group F from Example 4.1.4.

Let $x \in [0,1]$ with $x = [e_1e_2e_3\ldots]$. Then we can define the maps A^* and B^* as follows:

$$([e_1e_2e_3\ldots])\,A^* = \begin{cases} [0e_3e_4\ldots], & \text{if } e_1e_2 = 00 \\ [10e_3e_4\ldots], & \text{if } e_1e_2 = 01 \\ [11e_2e_3\ldots], & \text{if } e_1 = 1 \end{cases}$$

$$([e_1e_2e_3\ldots])\,B^* = \begin{cases} [0e_2e_3\ldots], & \text{if } e_1 = 0 \\ [10e_4e_5\ldots], & \text{if } e_1e_2e_3 = 100 \\ [110e_4e_5\ldots], & \text{if } e_1e_2e_3 = 101 \\ [111e_3e_4\ldots], & \text{if } e_1e_2 = 11 \end{cases}$$

Observe that cells and boundary points get mapped as follows:

$$\begin{aligned} (C(00))\,A^* &= C(0) \\ (0x)\,A^* &= x \\ (C(01))\,A^* &= C(10) \\ (x)\,A^* &= 1x \\ (C(1))\,A^* &= C(11) \end{aligned}$$

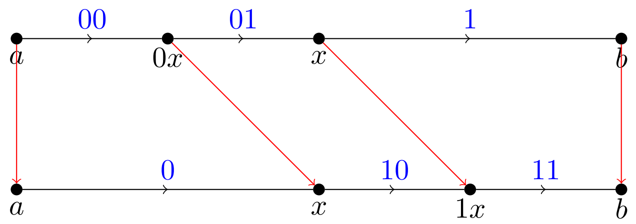
$$\begin{aligned} (C(0))\,B^* &= C(0) \\ (x)\,B^* &= x \\ (C(100))\,B^* &= C(10) \\ (10x)\,B^* &= 1x \\ (C(101))\,B^* &= C(110) \\ (1x)\,B^* &= 11x \\ (C(11))\,B^* &= C(111) \end{aligned}$$

We observe that these maps are in fact rearrangements of X, with the following minimal bipartitions:

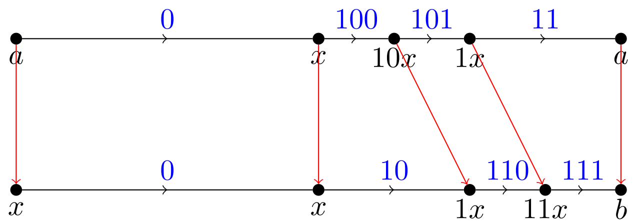
$$\mathcal{P}_{A^*,d} = \{C(00), C(01), C(1)\}$$
$$\mathcal{P}_{A^*,r} = \{C(0), C(10), C(11)\}$$

$$\mathcal{P}_{B^*,d} = \{C(0), C(100), C(101), C(11)\}$$
$$\mathcal{P}_{B^*,r} = \{C(0), C(10), C(110), C(111)\}$$

They are illustrated using the following graph-pair diagrams (which we can see are the same as rectangle diagrams):



A^*



B^*

Then $A = \theta^{-1}A^*\theta$ and $B = \theta^{-1}B^*\theta$. Hence $A, B \in \mathcal{G}$. $\square$

Cannon, Floyd, Parry [9] have shown that Thompson's group $F = \langle A, B\rangle$. By Proposition 4.4.1, $\mathcal{G} \leq F$ and, by Proposition 4.4.2, $\langle A, B\rangle \leq \mathcal{G}$. Hence $\mathcal{G} \cong F$.

From now on, we shall be referring to the group $\mathrm{Rearr}(X)$ as Thompson's group F. This equivalence induces an action of F on $\mathcal{GV}$ corresponding to the action of F on the set of dyadic rationals. Let us name this action by the homomorphism $\pi\colon F \to \mathrm{Sym}(\mathcal{GV})$ (where $\mathrm{Sym}(\mathcal{GV})$ is the symmetric group on $\mathcal{GV}$).

Chapter 5

A Generating Set for Richard Thompson's Group F

{f-generators}

In this chapter, we develop a combinatorial algorithm to define an infinite generating set for Richard Thompson's group F which follows from the structure of the topological space it acts on. The elements in this generating set correspond to small actions restricted to basic open sets of the limit space X. This enables us to build elements "locally", while previous generating sets ([9], [17]) consist of elements on the "right vine" (the right-most nodes of the binary rooted tree or the right most cells in a cellular partition).

This generating set was introduced and discussed by Patrick Dehornoy in [18]. However, there do not exist any combinatorial algorithms to find the "normal form" of an element of F in terms of this generating set (which is significantly shorter than the normal form in terms of previous generating sets – for which combinatorial methods do exist (see [2], [20])). In this chapter, we attempt to find a combinatorial algorithm.

Recall from Chapter 4 that Thompson's group F is isomorphic to the rearrangement group $\mathrm{Rearr}(X)$ of the limit space X of the F replacement system. We will be presenting results specifically for $\mathrm{Rearr}(X)$ in this chapter. Following the notation established in Chapter 4, let us fix the following in this chapter:

- The set of finite words of the F replacement system: $E(R)^* = \{0,1\}^*$.
- The symbol space of the F replacement system: $\Omega = \{0,1\}^\omega$.
- The limit space of the F replacement system: $X = [0,1]$.
- The set of gluing vertices of the F replacement system: $\mathcal{GV} = \{a, b, \alpha x \mid \alpha \in E(R)^*\}$.

5.1 The Rearrangement f_α

{5.1}

Recall from Definition 3.3.14 that the support of a rearrangement g of X is

$$\operatorname{supp} g = \{y \mid y \in X, y \neq yg\}.$$

{floorshow}

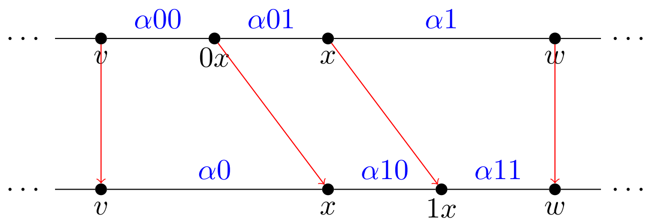
Definition 5.1.1. Let X be the limit space of the F replacement system. Let $\alpha = e_1 \dots e_n \in E(R)^* = \{0,1\}^*$. We define a map $f_\alpha \colon X \to X$ which acts as follows on points in the cell $C(\alpha)$ indexed by α, and as the identity on the rest of the interval:

$$([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha = \begin{cases} [\alpha 0 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1}e_{n+2} = 00 \\ [\alpha 10 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1}e_{n+2} = 01 \\ [\alpha 11 e_{n+2} e_{n+3} \dots], & \text{if } e_{n+1} = 1 \end{cases}$$

Observe that the cells and boundary points get mapped as follows:

$$\begin{aligned} (C(\alpha 00))\, f_\alpha &= C(\alpha 0) \\ (\alpha 0 x)\, f_\alpha &= \alpha x \\ (C(\alpha 01))\, f_\alpha &= C(\alpha 10) \\ (\alpha x)\, f_\alpha &= \alpha 1 x \\ (C(\alpha 1))\, f_\alpha &= C(\alpha 11) \end{aligned}$$

This map is illustrated by the rectangle diagram in Figure 5.1:



{silvera}

Figure 5.1: f_α

Definition 5.1.2. We define the set $\mathcal{X}$ to be as follows:

$$\mathcal{X} = \{ f_\alpha^\eta \mid \alpha \in E(R)^*, \eta \in \{\pm 1\} \}.$$

nielosullivan}

Lemma 5.1.3. *Let X be the limit space of the F replacement system. Let $f_\alpha \in \mathcal{X}$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$). Then f_α is a rearrangement of X with the minimal bipartition $(\mathcal{P}_{f_\alpha,d}, \mathcal{P}_{f_\alpha,r})$, defined as follows:*

$$\begin{aligned} \mathcal{P}_{f_\alpha,d} = \Big\{ & C(\alpha 00), C(\alpha 01), C(\alpha 1), C(\beta) \mid \beta = \alpha^{(n-i+1)\dagger} e_i' \\ & \text{where } e_i' \in E(R) \backslash \{e_i\} \text{ for all } i = 1, \dots, n \Big\}, \\ \mathcal{P}_{f_\alpha,r} = \Big\{ & C(\alpha 0), C(\alpha 10), C(\alpha 11), C(\beta) \mid \beta = \alpha^{(n-i+1)\dagger} e_i' \\ & \text{where } e_i' \in E(R) \backslash \{e_i\} \text{ for all } i = 1, \dots, n \Big\}. \end{aligned}$$

Proof. Let X be the limit space of the F replacement system. Let $f_\alpha \in \mathcal{X}$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$). Let $\mathcal{P}_{f_\alpha,d}$ and $\mathcal{P}_{f_\alpha,r}$ be the sets defined as follows:

$$\begin{aligned}\mathcal{P}_{f_\alpha,d} = \Big\{&C(\alpha 00), C(\alpha 01), C(\alpha 1), C(\beta) \mid \beta = \alpha^{(n-i+1)\dagger} e_i' \\ &\text{where } e_i' \in E(R)\backslash\{e_i\} \text{ for all } i = 1, \dots, n\Big\},\\ \mathcal{P}_{f_\alpha,r} = \Big\{&C(\alpha 0), C(\alpha 10), C(\alpha 11), C(\beta) \mid \beta = \alpha^{(n-i+1)\dagger} e_i' \\ &\text{where } e_i' \in E(R)\backslash\{e_i\} \text{ for all } i = 1, \dots, n\Big\}.\end{aligned}$$

It is easy to prove that $\mathcal{P}_{f_\alpha,d}$ and $\mathcal{P}_{f_\alpha,r}$ are cellular partitions of X. By Lemma 3.2.2, we observe that $\mathcal{P}_{f_\alpha,d}$ and $\mathcal{P}_{f_\alpha,r}$ are characterised by complete antichains in $E(R)^*$.

By Definition 3.3.8, to prove that f_α is a rearrangement of X, we have to show that $f_\alpha|_{C(\gamma)}$ is a canonical homeomorphism for all $C(\gamma) \in \mathcal{P}_{f_\alpha,d}$. Observe that

$$\begin{aligned} f_\alpha|_{C(\alpha 00)} &\colon C(\alpha 00) \to C(\alpha 0),\\ f_\alpha|_{C(\alpha 01)} &\colon C(\alpha 01) \to C(\alpha 10),\\ f_\alpha|_{C(\alpha 1)} &\colon C(\alpha 1) \to C(\alpha 11),\\ f_\alpha|_{C(\beta)} &\colon C(\beta) \to C(\beta) \quad \text{for all } \beta \text{ defined above.}\end{aligned}$$

Then $f_\alpha|_{C(\gamma)}$ is induced by the following prefix replacement maps respectively:

$$\begin{aligned} &\Psi_1 \colon \Omega(\alpha 00) \to \Omega(\alpha 0),\\ &\Psi_2 \colon \Omega(\alpha 01) \to \Omega(\alpha 10),\\ &\Psi_3 \colon \Omega(\alpha 1) \to \Omega(\alpha 11),\\ &\Psi_\beta \colon \Omega(\beta) \to \Omega(\beta) \quad \text{for all } \beta \text{ defined above.}\end{aligned}$$

Then, by Definition 3.3.5, $f_\alpha|_{C(\gamma)}$ is a canonical homeomorphism for all $C(\gamma) \in \mathcal{P}_{f_\alpha,d}$. This proves that f_α is a rearrangement of X and $(\mathcal{P}_{f_\alpha,d}, \mathcal{P}_{f_\alpha,r})$ is a cellular bipartition for f_α.

We can prove that $(\mathcal{P}_{f_\alpha,d}, \mathcal{P}_{f_\alpha,r})$ is the minimal bipartition for f_α by a straightforward application of Algorithm 4.4.8 and Lemma 4.4.9. This completes the proof. □

{juliuscaesar}

Corollary 5.1.4. *Let $f_\alpha \in \mathcal{X}$ (for some $\alpha \in E(R)^*$). Then the boundary points of the minimal bipartition $(\mathcal{P}_{f_\alpha,d}, \mathcal{P}_{f_\alpha,r})$ are as follows:*

$$\begin{aligned}\partial\mathcal{P}_{f_\alpha,d} &= \left\{\alpha 0x, \alpha x, \alpha^\dagger x, \alpha^{2\dagger} x, \dots, x\right\},\\ \partial\mathcal{P}_{f_\alpha,r} &= \left\{\alpha 1x, \alpha x, \alpha^\dagger x, \alpha^{2\dagger} x, \dots, x\right\}.\end{aligned}$$

Proof. Let $f_\alpha \in \mathcal{X}$ (for some $\alpha \in E(R)^*$). Let $(\mathcal{P}_{f_\alpha,d}, \mathcal{P}_{f_\alpha,r})$ be the minimal bipartition for f_α. The result follows from the definition of $\mathcal{P}_{f_\alpha,d}$ and $\mathcal{P}_{f_\alpha,r}$ in Lemma 5.1.3. □

Corollary 5.1.5. *Let $f_\alpha \in \mathcal{X}$ (for some $\alpha \in E(R)^*$). Then*

$$\operatorname{supp} f_\alpha = \operatorname{int} C(\alpha).$$

Proof. Let $f_\alpha \in \mathcal{X}$ (for some $\alpha \in E(R)^*$). Let $x \in X$. By Definition 5.1.1, if $x \notin \operatorname{int} C(\alpha)$ then $(x)f_\alpha = x$ and if $x \in \operatorname{int} C(\alpha)$ then $(x)f_\alpha \neq x$. The result follows. □

The following results characterise the rearrangement f_α (for some $\alpha \in E(R)^*$):

Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$). Recall from Remark 4.2.2 that $\operatorname{depth}(z) = |\alpha| + 1$. Recall from Definition 2.3.10 that there exists an induced vertex depth order on $\partial\mathcal{P}$, denoted by the ordered list $\operatorname{depth} \partial\mathcal{P} = (z_1, \ldots, z_d)$.

{control}

Lemma 5.1.6. *Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$) and let $f_\beta^\eta \in \mathcal{X}$ (for some $\beta \in E(R)^*$ and $\eta \in \{\pm 1\}$). Then $z \in \operatorname{supp} f_\beta^\eta$ if and only if $\beta \preceq \alpha$.*

Proof. Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$) and let $f_\beta^\eta \in \mathcal{X}$ (for some $\beta \in E(R)^*$ and $\eta \in \{\pm 1\}$). Suppose that f_β^η has a non-trivial action on z. Then $z \in \operatorname{int} C(\beta)$, and by Corollary 3.1.7, $\beta \preceq \alpha$. Conversely, suppose that $\beta \preceq \alpha$. Then $z \in \operatorname{int} C(\beta)$ and therefore f_β^η has a non-trivial action on z. □

{theknife}

Lemma 5.1.7. *Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$) and let $f_\beta \in \mathcal{X}$ (for some $\beta \in E(R)^*$) with $\beta \preceq \alpha$. The rearrangement f_β will act as one of the following on z:*

1. *If $z = \beta 0 x$, then $z f_\beta = \beta x$.*
2. *If $z = \beta x$, then $z f_\beta = \beta 1 x$.*
3. *If $z \in \operatorname{int} C(\beta 00)$, then $\operatorname{depth}(z f_\beta) = \operatorname{depth}(z) - 1$.*
4. *If $z \in \operatorname{int} C(\beta 01)$, then $\operatorname{depth}(z f_\beta) = \operatorname{depth}(z)$.*
5. *If $z \in \operatorname{int} C(\beta 1)$, then $\operatorname{depth}(z f_\beta) = \operatorname{depth}(z) + 1$.*

Proof. Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$) and let $f_\beta \in \mathcal{X}$ (for some $\beta \in E(R)^*$) such that $\beta \preceq \alpha$.

1. and 2. follow from Definition 5.1.1 of f_β.

3. If $z \in \operatorname{int} C(\beta 00)$, then, by Corollary 3.1.7, $\beta 00 \preceq \alpha$. Say $\alpha = \beta 00 \gamma$ for some $\gamma \in E(R)^*$. Then, by Definition 5.1.1, $z f_\beta = (\beta 00 \gamma x) f_\beta = \beta 0 \gamma x$, and hence $\operatorname{depth} z f_\beta = \operatorname{depth} z - 1$.

4. If $z \in \operatorname{int} C(\beta 01)$, then, by Corollary 3.1.7, $\beta 01 \preceq \alpha$. α. Say $\alpha = \beta 01 \gamma$ for some $\gamma \in E(R)^*$. Then, by Definition 5.1.1, $z f_\beta = (\beta 01 \gamma x) f_\beta = \beta 10 \gamma x$, and hence $\operatorname{depth} z f_\beta = \operatorname{depth} z$.

5. If $z \in \operatorname{int} C(\beta 1)$, then, by Corollary 3.1.7, $\beta 1 \preceq \alpha$. α. Say $\alpha = \beta 1 \gamma$ for some $\gamma \in E(R)^*$. Then, by Definition 5.1.1, $z f_\beta = (\beta 11 \gamma x) f_\beta = \beta 1 \gamma x$, and hence $\operatorname{depth} z f_\beta = \operatorname{depth} z + 1$.

□

{zombie}

Corollary 5.1.8. *Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha \in E(R)^*$) and let $f_\beta^{-1} \in \mathcal{X}$ (for some $\beta \in E(R)^*$) with $\beta \preceq \alpha$. The rearrangement f_β^{-1} will act as one of the following on z:*

1. *If $z = \beta 1x$, then $zf_\beta^{-1} = \beta x$.*
2. *If $z = \beta x$, then $zf_\beta^{-1} = \beta 0x$.*
3. *If $z \in \operatorname{int} C(\beta 11)$, then $\operatorname{depth}(zf_\beta^{-1}) = \operatorname{depth}(z) - 1$.*
4. *If $z \in \operatorname{int} C(\beta 10)$, then $\operatorname{depth}(zf_\beta^{-1}) = \operatorname{depth}(z)$.*
5. *If $z \in \operatorname{int} C(\beta 0)$, then $\operatorname{depth}(zf_\beta^{-1}) = \operatorname{depth}(z) + 1$.*

Proof. The proof follows from Lemma 5.1.7. □

{imonster}

Lemma 5.1.9. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X and let $z = \gamma x \in \partial\mathcal{P}_{g,d}$ (for some $\gamma \in E(R)^*$). Suppose that $yg = y$ for all $y \in \{\gamma^\dagger x, \gamma^{2\dagger}x, \ldots, x\}$. Then $zg = \alpha x$ (for some $\alpha \in E(R)^*$) with $\gamma \preceq \alpha$.*

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X and let $z = \gamma x \in \partial\mathcal{P}_{g,d}$ (for some $\gamma \in E(R)^*$). Recall, by Lemma 3.2.9 that $\gamma^\dagger x, \gamma^{2\dagger}x, \ldots, x \in \partial\mathcal{P}_{g,d}$. Suppose that $yg = y$ for all $y \in \{\gamma^\dagger x, \gamma^{2\dagger}x, \ldots, x\}$. Then, by Lemma 3.1.12, $z = \gamma x$ is the boundary point of least depth in $\operatorname{int} C(\gamma)$. Recall, by Lemma 2.1.14 that $v_\gamma, w_\gamma \in \{\gamma^\dagger x, \ldots, x, a, b\}$. Then, by our hypothesis, $v_\gamma g = v_\gamma$ and $w_\gamma g = w_\gamma$. This implies that, by Corollary 3.3.23, $(C(\gamma))g = C(\gamma)$. Since $z = \gamma x \in \operatorname{int} C(\gamma)$, then $zg = \alpha x \in \operatorname{int} C(\gamma)$ and, by Corollary 3.1.7, $\gamma \preceq \alpha$. □

{inbinary}

Corollary 5.1.10. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X and let $\operatorname{depth}(\partial\mathcal{P}_{g,d}) = (z_1, \ldots, z_d)$. Choose $z_i \in \partial\mathcal{P}_{g,d}$ such that $z_i g = z_i$ and $z_j g = z_j$ for all $j = 1, \ldots, i-1$. Let $z_i = \gamma x$ (for some $\gamma \in E(R)^*$). Then $z_j g = \alpha x$ (for some $\alpha \in E(R)^*$) with $\gamma \preceq \alpha$.*

Proof. The statement follows from Lemma 5.1.9. □

{survivalism}

Lemma 5.1.11. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X and let $z = \alpha x \in \partial\mathcal{P}_{g,r}$ (for some $\alpha \in E(R)^*$). Let $f_\beta^\eta \in \mathcal{X}$ (for some $\beta \in E(R)^*$ and $\eta \in \{\pm 1\}$) with $\beta \prec \alpha$. Then one of the following holds:*

1. *If $\beta 0 \preceq \alpha$, then $\partial\mathcal{P}_{f_\beta,d} \subseteq \partial\mathcal{P}_{g,r}$.*
2. *If $\beta 1 \preceq \alpha$, then $\partial\mathcal{P}_{f_\beta^{-1},d} \subseteq \partial\mathcal{P}_{g,r}$.*

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X and let $z = \alpha x \in \partial \mathcal{P}_{g,r}$ (for some $\alpha \in E(R)^*$). Let $f_\beta^\eta \in \mathcal{X}$ (for some $\beta \in E(R)^*$ and $\eta \in \{\pm 1\}$) with $\beta \prec \alpha$. Using Lemma 3.2.9 and the hypothesis that $z = \alpha x \in \partial \mathcal{P}_{g,r}$, we observe that $\partial \mathcal{P}_{g,r}$ contains the boundary points $\alpha^\dagger x, \ldots, x$.

1. By Corollary 5.1.4, the set of boundary points $\partial \mathcal{P}_{f_\beta,d} = \{\beta 0x, \beta x, \beta^\dagger x, \beta^{2\dagger} x, \ldots, x\}$. It follows that, if $\beta 0 \preceq \alpha$, then $\partial \mathcal{P}_{f_\beta,d} \subseteq \partial \mathcal{P}_{g,r}$.

2. Observe that $\mathcal{P}_{f_\beta^{-1},d} = \mathcal{P}_{f_\beta,r}$. Then, by Corollary 5.1.4, the set of boundary points $\partial \mathcal{P}_{f_\beta^{-1},d} = \{\beta 1x, \beta x, \beta^\dagger x, \beta^{2\dagger} x, \ldots, x\}$. It follows that, if $\beta 1 \preceq \alpha$, then $\partial \mathcal{P}_{f_\beta^{-1},d} \subseteq \partial \mathcal{P}_{g,r}$.

□

{feverray}

Lemma 5.1.12. *Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha = e_1 \ldots e_n \in E(R)^*$) and let $f_\beta^\eta \in \mathcal{X}$ (for some $\beta \in E(R)^*$ and $\eta \in \{\pm 1\}$). Then* $\operatorname{depth}(z f_\beta^\eta) = \operatorname{depth}(z) - 1$*, if and only if one of the following holds:*

1. *If $e_n = 0$, $\beta = e_1 \ldots e_{n-1}$ and $\eta = +1$.*

2. *If $e_n = 1$, $\beta = e_1 \ldots e_{n-1}$ and $\eta = -1$.*

3. *If $e_{k+1} = e_{k+2} = 0$, $\beta = e_1 \ldots e_k$ and $\eta = +1$ (for some $0 \leq k \leq n$).*

4. *If $e_{k+1} = e_{k+2} = 1$, $\beta = e_1 \ldots e_k$ and $\eta = -1$ (for some $0 \leq k \leq n$).*

Proof. Let $\mathcal{GV}$ be the set of gluing vertices of the F replacement system. Let $z = \alpha x \in \mathcal{GV}$ (for some $\alpha = e_1 \ldots e_n \in E(R)^*$) and let $f_\beta^\eta \in \mathcal{X}$ (for some $\beta \in E(R)^*$ and $\eta \in \{\pm 1\}$). Suppose that $\operatorname{depth}(z f_\beta^\eta) = \operatorname{depth}(z) - 1$. Then $z \in \operatorname{supp} f_\beta^\eta$ and, by Lemma 5.1.6, $\beta \preceq \alpha$. By Lemma 5.1.7 and Corollary 5.1.8, $\operatorname{depth}(z f_\beta^\eta) = \operatorname{depth}(z) - 1$ if

1. $\eta = +1$ and $z = \beta 0x$,

2. $\eta = +1$ and $z \in \operatorname{int} C(\beta 00)$,

3. $\eta = -1$ and $z = \beta 1x$,

4. $\eta = -1$ and $z \in \operatorname{int} C(\beta 11)$.

The result follows.

The converse is a direct consequence of Lemma 5.1.7 and Corollary 5.1.8. □

5.2 A Generating Set for Thompson's Group *F*

{5.2}

We can now present our main result for this chapter:

{drabmajesty}

Proposition 5.2.1. *Richard Thompson's group F is generated by the set $\mathcal{X}$.*

To prove Proposition 5.2.1, we will develop an algorithm to decompose an arbitrary rearrangement g of X into a word $f_{\alpha_1}^{\eta_1} \ldots f_{\alpha_N}^{\eta_N} \in W(\mathcal{X})$ (for some $\alpha_1, \ldots, \alpha_N \in E(R)^*$ and $\eta_1, \ldots, \eta_N \in \{\pm 1\}$).

Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Using Algorithm 4.4.8, we can find the minimal bipartition $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ for g. Recall from Lemma 3.3.28 that the minimal bipartition for a rearrangement is unique. Using this minimal bipartition, the following algorithm finds a word $f_{\alpha_1}^{\eta_1} \ldots f_{\alpha_N}^{\eta_N} \in W(\mathcal{X})$ (for some $\alpha_1, \ldots, \alpha_N \in E(R)^*$ and $\eta_1, \ldots, \eta_N \in \{\pm 1\}$) such that $g f_{\alpha_1}^{\eta_1} \ldots f_{\alpha_N}^{\eta_N} = I$.

Algorithm 5.2.2 The function FACTORIZATION. {killingjoke}

Require: Let g be a rearrangement of X and let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g.

```
List ∂P_{g,d}.
Define bp = depth(∂P_{g,d}) = (z_1, ..., z_d).
function FACTORIZATION(g, bp)
    Set outputword = I.
    for i = 1, ..., d do
        Set h = g outputword.
        Compute interimword = MAPBACK(h, bp, i)
        Append interimword to outputword.
    end for
    return outputword.
end function
```

Observe that Algorithm 5.2.2 calls a function MAPBACK. This function is defined in the following algorithm:

Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} = \text{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \ldots, z_d)$. Choose $z_i \in \partial \mathcal{P}_{g,d}$ such that $z_i g = z_1, \ldots, z_{i-1} g = z_{i-1}$. The function MAPBACK$(g, \mathbf{bp}, i)$ finds a word $f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} \in W(\mathcal{X})$ (for some $\beta_1, \ldots, \beta_K \in E(R)^*$ and $\eta_1, \ldots, \eta_K \in \{\pm 1\}$) such that $z_1 g f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = z_1, \ldots, z_i g f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = z_i$.

Algorithm 5.2.3 The function MAPBACK. {thetwilightsa

Require: Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} = (z_1, \ldots, z_d)$ be a list of precomputed gluing vertices ordered by depth such that $\partial \mathcal{P}_{g,d} \subseteq \mathbf{bp}$. Let $z_i g = z_1, \ldots, z_{i-1} g = z_{i-1}$.

```
function MAPBACK(g, bp = (z_1, ..., z_d), i)
    Let z_i g = αx = e_1 ... e_n x, z_i = γx = e_1 ... e_m x.
    Set outputword = I.
```

4: **if** $z_i g \neq z_i$ **then**
5: Define f_β^η such that
6: $\beta = e_1 \dots e_{n-1}$ and
7: $\eta = +1$ if $e_n = 0$ or
8: $\eta = -1$ if $e_n = 1$.
9: Append f_β^η to `outputword`.
10: Set `interimword` = MAPBACK(gf_β^η, **bp**, i).
11: Append `interimword` to `outputword`.
12: **end if**
13: **return** `outputword`.
14: **end function**

The following results and examples provide a full proof and explanation of Algorithm 5.2.3:

llyofthebeast}

Lemma 5.2.4. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let* $\text{depth}(\partial\mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. *Let $z_1 g = z_1, \dots, z_{i-1}g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma = e_1 \dots e_m \in E(R)^*$) such that $z_i g \neq z_i$. Let f_β^η be as defined in Lines 5-8 of Algorithm 5.2.3. Then*

1. $z_1 g f_\beta^\eta = z_1, \dots, z_{i-1} g f_\beta^\eta = z_{i-1}$,
2. $z_i g f_\beta^\eta = \alpha^\dagger x$.

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\text{depth}(\partial\mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. Let $z_1 g = z_1, \dots, z_{i-1}g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma \in E(R)^*$) such that $z_i g \neq z_i$. By Corollary 5.1.10, $\gamma \prec \alpha$. Suppose $\gamma = e_1 \dots e_m$ and $\alpha = e_1 \dots e_n$ for some $n, m \in \mathbb{N}$ such that $n > m$. Let f_β^η be as defined in Lines 5-8 of Algorithm 5.2.3. Then $f_\beta^\eta = f_{\alpha^\dagger}^\eta$ where $\eta = +1$ if $e_n = 0$ and $\eta = -1$ if $e_n = 1$. Then, by Lemma 5.1.7 (1) and Corollary 5.1.8 (1), $z_i g f_\beta^\eta = \alpha^\dagger x$.

Now consider the a boundary point z_j for some $1 \leq j \leq i-1$. Suppose $z_j = \delta x$. Then, by Definition 2.3.10 of the depth order, either $\delta \perp \gamma$ or $\delta \prec \gamma$. This implies that either $\delta \perp \alpha^\dagger$ or $\delta \prec \alpha^\dagger$. Hence, by Lemma 5.1.6, $z_j \notin \text{supp}\, f_{\alpha^\dagger}^\mu$ and therefore $z_j g f_\beta^\eta = z_j$. Since this is true for all $j = 1, \dots, i-1$, this proves the result. □

{seamstress}

Lemma 5.2.5. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let* $\mathbf{bp} \supseteq \text{depth}(\partial\mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. *Let $z_1 g = z_1, \dots, z_{i-1}g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma = e_1 \dots e_m \in E(R)^*$). Let* MAPBACK$(g, \mathbf{bp}, i) = f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K}$. *Then*

1. *1.1. $K = n - m$,*

1.2. $\beta_t = \alpha^{t\dagger}$ *for all* $t = 1, \ldots, K$,

1.3. $\eta_t = +1$ *if* $e_{n-t+1} = 0$ *and* $\eta_t = -1$ *if* $e_{n-t+1} = 1$ *for all* $t = 1, \ldots, K$.

2. $z_1 g f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = z_1, \ldots, z_i g f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = z_i$.

3. $z_i g f_{\beta_1}^{\eta_1} \ldots f_{\beta_s}^{\eta_s} = \alpha^{s\dagger} x$ *for all* $s = 1, \ldots, K$.

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} \subseteq \text{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \ldots, z_d)$. Let $z_1 g = z_1, \ldots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma \in E(R)^*$). By Corollary 5.1.10, $\gamma \preceq \alpha$. Suppose $\gamma = e_1 \ldots e_m$ and $\alpha = e_1 \ldots e_n$ for some $n, m \in \mathbb{N}$ such that $n \geq m$. Let $\textsc{Mapback}(g, \mathbf{bp}, i) = f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K}$.

Let us prove this result by induction on $n - m$:

Suppose $n - m = 0$. Then $\gamma = \alpha$ and $z_i g = z_i$, and Algorithm 5.2.3 omits lines 4-12 and returns I. Hence $f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = I$ and the inductive hypothesis holds.

Suppose the inductive hypothesis holds for $n - m = r$. Let us examine the case when $n - m = r + 1$. Let us implement Algorithm 5.2.3. By lines 4-9, `outputword` $= f_{\beta_1}^{\eta_1}$ where $\beta_1 = \alpha^\dagger$ and $\eta_1 = +1$ if $e_n = 0$ and $\eta_1 = -1$ if $e_n = 1$. By Lemma 5.2.4, $z_i g f_{\beta_1}^{\eta_1} = z_1, \ldots, z_{i-1} g f_{\beta_1}^{\eta_1} = z_{i-1}$ and $z_i g f_{\beta_1}^{\eta_1} = \alpha^\dagger x$. Set $h = g f_{\beta_1}^{\eta_1}$. By line 10, `interimword` $=$ $\textsc{Mapback}(h, bp, i)$. Observe that the inductive hypothesis holds for h. Therefore we have $\textsc{Mapback}(h, bp, i) = f_{\beta_2}^{\eta_2} \ldots f_{\beta_{r+1}}^{\eta_{r+1}}$ such that

1. 1.1. The length of the word $f_{\beta_2}^{\eta_2} \ldots f_{\beta_{r+1}}^{\eta_{r+1}}$ is r.

   1.2. $\beta_t = \alpha^{t\dagger}$ for all $t = 2, \ldots, r + 1$

   1.3. $\eta_t = +1$ if $e_{n-t+1} = 0$ and $\eta_t = -1$ if $e_{n-t+1} = 1$ for all $t = 2, \ldots, r + 1$

2. $z_1 h f_{\beta_2}^{\eta_2} \ldots f_{\beta_{r+1}}^{\eta_{r+1}} = z_1, \ldots, z_i h f_{\beta_2}^{\eta_2} \ldots f_{\beta_{r+1}}^{\eta_{r+1}} = z_i$.

3. $z_i h f_{\beta_2}^{\eta_2} \ldots f_{\beta_s}^{\eta_s} = \alpha^{s\dagger} x$ for all $s = 2, \ldots, K$.

By line 11, `outputword` $= f_{\beta_1}^{\eta_1} f_{\beta_2}^{\eta_2} \ldots f_{\beta_{r+1}}^{\eta_{r+1}}$. By line 13, $\textsc{Mapback}(g, bp, i) = f_{\beta_1}^{\eta_1} \ldots f_{\beta_{r+1}}^{\eta_{r+1}}$ and

1. 1.1. The length of the word $f_{\beta_1}^{\eta_1} \ldots f_{\beta_{r+1}}^{\eta_{r+1}}$ is $r + 1$.

   1.2. $\beta_t = \alpha^{t\dagger}$ for all $t = 1, \ldots, r + 1$

   1.3. $\eta_t = +1$ if $e_{n-t+1} = 0$ and $\eta_t = -1$ if $e_{n-t+1} = 1$ for all $t = 1, \ldots, r + 1$

2. $z_1 g f_{\beta_1}^{\eta_1} \ldots f_{\beta_{r+1}}^{\eta_{r+1}} = z_1, \ldots, z_i g f_{\beta_1}^{\eta_1} \ldots f_{\beta_{r+1}}^{\eta_{r+1}} = z_i$.

3. $z_i g f_{\beta_1}^{\eta_1} \ldots f_{\beta_s}^{\eta_s} = \alpha^{s\dagger} x$ for all $s = 1, \ldots, K$.

Hence, the inductive hypothesis holds for $r + 1$. This proves the result by induction. $\square$

{sistereurope}

Lemma 5.2.6. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let* $\mathbf{bp} \supseteq \mathrm{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. *Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let* $\textsc{Mapback}(g, \mathbf{bp}, i) = f^{\eta_1}_{\beta_1} \dots f^{\eta_K}_{\beta_K}$. *Then*

$$\partial \mathcal{P}_{g f^{\eta_1}_{\beta_1} \dots f^{\eta_K}_{\beta_K}, d} \subseteq \partial \mathcal{P}_{g,d}.$$

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} \supseteq \mathrm{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma \in E(R)^*$). By Corollary 5.1.10, $\gamma \preceq \alpha$. Suppose $\gamma = e_1 \dots e_m$ and $\alpha = e_1 \dots e_n$ for some $n, m \in \mathbb{N}$ such that $n \geq m$.

Let $\textsc{Mapback}(g, \mathbf{bp}, i) = f^{\eta_1}_{\beta_1} \dots f^{\eta_K}_{\beta_K}$. Consider $s \in \mathbb{N}$ such that $1 \leq s \leq K$. By Lemma 5.2.5 (3), $z_i g f^{\eta_1}_{\alpha_1} \dots f^{\eta_{s-1}}_{\alpha_{s-1}} = \alpha^{(s-1)\dagger} x$. Then, by Lemma 5.1.11,

$$\partial \mathcal{P}_{f^{\eta_s}_{\alpha_s}, d} \subseteq \partial \mathcal{P}_{g f^{\eta_1}_{\alpha_1} \dots f^{\eta_{s-1}}_{\alpha_{s-1}}, r}.$$

Hence, by Lemma 3.4.5,

$$\partial \mathcal{P}_{g f^{\eta_1}_{\alpha_1} \dots f^{\eta_s}_{\alpha_s}, d} \subseteq \partial \mathcal{P}_{g f^{\eta_1}_{\alpha_1} \dots f^{\eta_{s-1}}_{\alpha_{s-1}}, d}.$$

Since this is true for all $s = 1, \dots, K$, it follows that

$$\partial \mathcal{P}_{g f^{\eta_1}_{\beta_1} \dots f^{\eta_K}_{\beta_K}, d} \subseteq \partial \mathcal{P}_{g,d}.$$

□

ensionshifter}

Example 5.2.7. Let X be the limit space of the F replacement system. We define a rearrangement g of X which acts as follows on points in X:

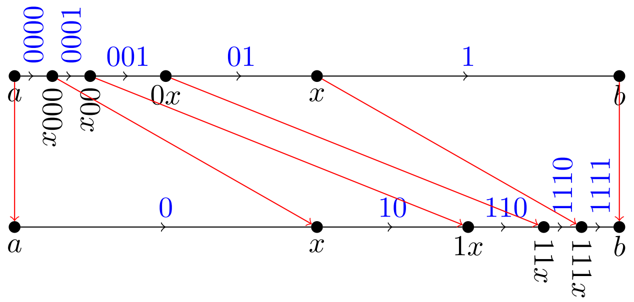
$$([e_1 e_2 e_3 \dots])\, g = \begin{cases} [0 e_5 e_6 \dots] & \text{if } e_1 e_2 e_3 e_4 = 0000, \\ [10 e_5 e_6 \dots] & \text{if } e_1 e_2 e_3 e_4 = 0001, \\ [110 e_4 e_5 \dots] & \text{if } e_1 e_2 e_3 = 001, \\ [1110 e_3 e_4 \dots] & \text{if } e_1 e_2 = 01, \\ [1111 e_2 e_3 \dots] & \text{if } e_1 = 1. \end{cases}$$

Cells and boundary points get mapped as follows:

$$(C(0000))\, g = C(0),$$
$$(000x)\, g = x,$$
$$(C(0001))\, g = C(10),$$
$$(00x)\, g = 1x,$$
$$(C(001))\, g = C(110),$$
$$(0x)\, g = 11x,$$

$$(C(01))\, g = C(1110),$$
$$(x)\, g = 111x,$$
$$(C(1))\, g = C(1111).$$

This map is illustrated by the following rectangle diagram:



We observe that $\mathbf{bp} = \mathrm{depth}(\partial\mathcal{P}_{g,d}) = (x, 0x, 00x, 000x)$. Also observe that $z_1 g = 111x \neq x = z_1$. Let us carry out MAPBACK$(g, \mathbf{bp}, 1)$:

Require: g, $\mathbf{bp}$, $i = 1$ defined above.

function MAPBACK(g, $\mathbf{bp}$, 1)

`outputword` $= I$.

$z_1 g = 111x \neq x = z_1$.

$f_\beta^\eta = f_{11}^{-1}$.

Append f_β^η to `outputword`.

`outputword` $= f_{11}^{-1}$.

`interimword` =MAPBACK(g `outputword` $= g f_{11}^{-1}$, $\mathbf{bp}$, 1).

`outputword` $= I$.

$z_1 g f_{11}^{-1} = 11x \neq x = z_1$.

$f_\beta^\eta = f_1^{-1}$.

Append f_β^η to `outputword`.

`outputword` $= f_1^{-1}$.

`interimword` =MAPBACK($g f_{11}^{-1}$ `outputword` $= g f_{11}^{-1} f_1^{-1}$, $\mathbf{bp}$, 1).

`outputword` $= I$.

$z_1 g f_{11}^{-1} f_1^{-1} = 1x \neq x = z_1$.

$f_\beta^\eta = f_\epsilon^{-1}$.

Append f_β^η to `outputword`.

`outputword` $= f_\epsilon^{-1}$.

`interimword` =MAPBACK($g f_{11}^{-1} f_1^{-1}$ `outputword` $= g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1}$, $\mathbf{bp}$, 1).

`outputword` $= I$.

$z_1 g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} = x = z_1$.

end function

`interimword` $= I$.

Append `interimword` to `outputword`.

`outputword` $= f_\epsilon^{-1}$.

end function

`interimword` $= f_\epsilon^{-1}$

Append `interimword` to `outputword`.

`outputword` $= f_1^{-1} f_\epsilon^{-1}$.

end function

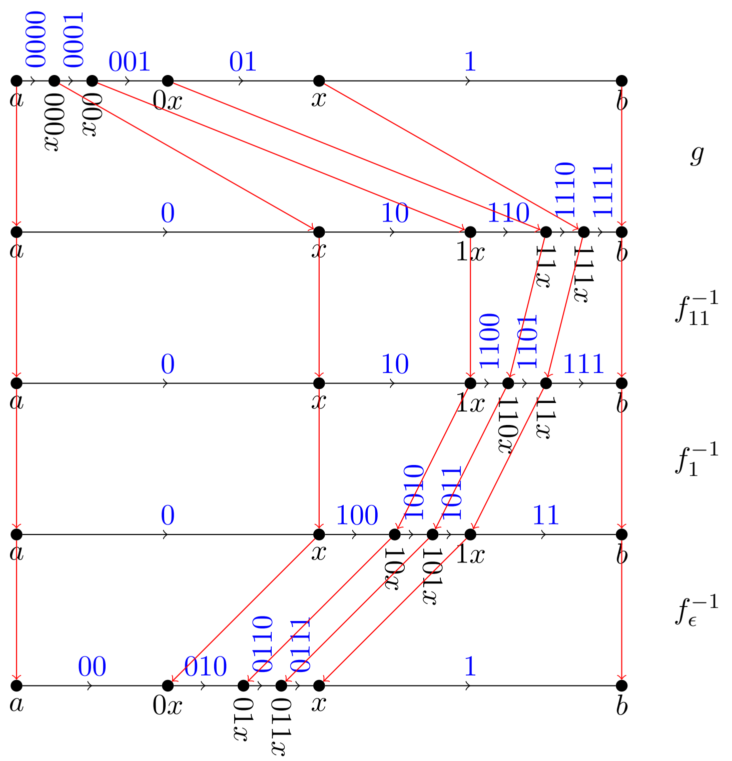
`interimword` $= f_1^{-1} f_\epsilon^{-1}$

Append `interimword` to `outputword`.

return `outputword` $= f_{11}^{-1} f_1^{-1} f_\epsilon^{-1}$.

end function

We observe that indeed $z_1 g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} = x = z_1$. We illustrate this by the following composite rectangle diagram:



hreinachliebe}

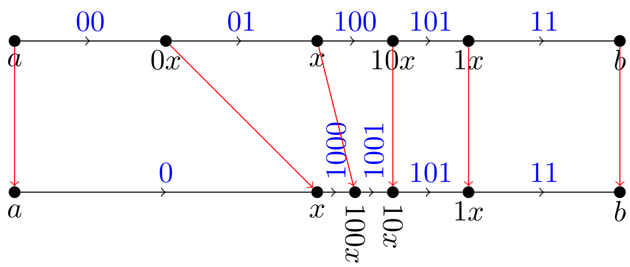
Example 5.2.8. Let X be the limit space of the F replacement system. We define a rearrangement h of X which acts as follows on points in X:

$$([e_1e_2e_3\ldots])\,h = \begin{cases} [0e_3e_4\ldots] & \text{if } e_1e_2 = 00, \\ [1000e_3e_4\ldots] & \text{if } e_1e_2 = 01, \\ [1001e_4e_5\ldots] & \text{if } e_1e_2e_3 = 100, \\ [101e_4e_5\ldots] & \text{if } e_1e_2e_3 = 101, \\ [11e_3e_4\ldots] & \text{if } e_1e_2 = 11. \end{cases}$$

Cells and boundary points get mapped as follows:

$$\begin{aligned} (C(00))\,h &= C(0), \\ (0x)\,h &= x, \\ (C(01))\,h &= C(10), \\ (x)\,h &= 100x, \\ (C(100))\,h &= C(110), \\ (10x)\,h &= 10x, \\ (C(101))\,h &= C(1110), \\ (1x)\,h &= 1x, \\ (C(11))\,h &= C(1111). \end{aligned}$$

This map is illustrated in the following rectangle diagram:



We observe that $\mathbf{bp} = \operatorname{depth}(\partial\mathcal{P}_{h,d}) = (x, 0x, 1x, 10x)$. Also observe that $z_1h = 100x \neq x = z_1$. Let us carry out MAPBACK$(h, \mathbf{bp}, 1)$:

Require: h, $\mathbf{bp}$, $i = 1$ defined above.

function MAPBACK(h, $\mathbf{bp}$, 1)

`outputword` $= I$.

$z_1h = 100x \neq x = z_1$.

$f_\beta^\eta = f_{10}$.

Append f_β^η to `outputword`.

`outputword` $= f_{10}$.

`interimword` $=$MAPBACK(h `outputword` $= hf_{10}$, $\mathbf{bp}$, 1).

`outputword` $= I$.

$z_1 h f_{10} = 10x \neq x = z_1$.

$f_\beta^\eta = f_1$.

Append f_β^η to `outputword`.

`outputword` $= f_1$.

`interimword` =MAPBACK(hf_{10} `outputword` $= hf_{10}f_1$, **bp**, 1).

`outputword` $= I$.

$z_1 h f_{10} f_1 = 1x \neq x = z_1$.

$f_\beta^\eta = f_\epsilon^{-1}$.

Append f_β^η to `outputword`.

`outputword` $= f_\epsilon^{-1}$.

`interimword` =MAPBACK($hf_{10}f_1$ `outputword` $= hf_{10}f_1f_\epsilon^{-1}$, **bp**, 1).

`outputword` $= I$.

$z_1 h f_{10} f_1 f_\epsilon^{-1} = x = z_1$.

end function

`interimword` $= I$.

Append `interimword` to `outputword`.

`outputword` $= f_\epsilon^{-1}$.

end function

`interimword` $= f_\epsilon^{-1}$

Append `interimword` to `outputword`.

`outputword` $= f_1 f_\epsilon^{-1}$.

end function

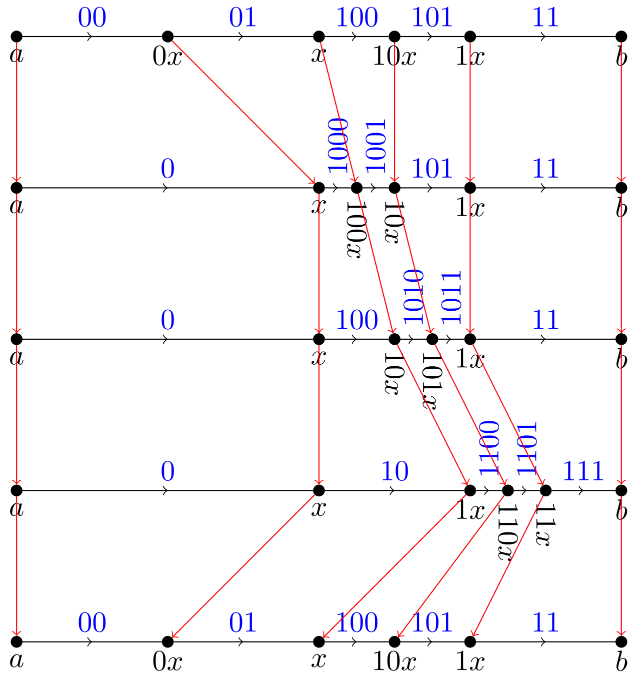
`interimword` $= f_1 f_\epsilon^{-1}$

Append `interimword` to `outputword`.

return `outputword` $= f_{10} f_1 f_\epsilon^{-1}$.

end function

We observe that indeed $z_1 h f_{10} f_1 f_\epsilon^{-1} = x = z_1$. We illustrate this in the following composite rectangle diagram:



The following results and examples provide a full proof and explanation of Algorithm 5.2.2:

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Lemma 5.2.9. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let* FACTORIZATION$(g) = f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N}$ *(for some $N \in \mathbb{N}$). Then $g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = I$.*

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let FACTORIZATION$(g) = f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N}$ (for some $N \in \mathbb{N}$). Let $\mathbf{bp} \supseteq \operatorname{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots z_d)$. Let us define the following functions:

$$
\begin{aligned}
h_{z_1} &= \textsc{Mapback}(g, \mathbf{bp}, 1), \\
h_{z_2} &= \textsc{Mapback}(g h_{z_1}, \mathbf{bp}, 2), \\
&\ \ \vdots \\
h_{z_d} &= \textsc{Mapback}(g h_{z_1} \dots h_{z_{d-1}}, \mathbf{bp}, d).
\end{aligned}
$$

We will prove that $f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = h_{z_1} \dots h_{z_d}$. Let us implement Algorithm 5.2.2. Suppose that, having completed step $i-1$ of the loop, `outputword` $= h_{z_1} \dots h_{z_{i-1}}$. Then the next step of the loop appends h_{z_i} to `outputword`. Hence, having completed step i of the loop, `outputword` $= h_{z_1} \dots h_{z_i}$. It follows that, having completed step d of the loop, `outputword` $= h_{z_1} \dots h_{z_d}$.

Then, by Lemma 5.2.5 (2),

$$z_1 g h_{z_1} = z_1$$

$$z_1 g h_{z_1} h_{z_2} = z_1, z_2 g h_{z_1} h_{z_2} = z_2$$
$$\vdots$$
$$z_1 g h_{z_1} \dots h_{z_d} = z_1, \dots, z_d g h_{z_1} \dots h_{z_{d-1}} = z_d.$$

And, by Lemma 5.2.6,

$$\partial\mathcal{P}_{gh_{z_1},d} \subseteq \partial\mathcal{P}_{g,d}$$
$$\partial\mathcal{P}_{gh_{z_1}h_{z_2},d} \subseteq \partial\mathcal{P}_{gh_{z_1},d}$$
$$\vdots$$
$$\partial\mathcal{P}_{gh_{z_1}\dots h_{z_d},d} \subseteq \partial\mathcal{P}_{gh_{z_1}\dots h_{z_{d-1}},d}.$$

Hence,

$$\partial\mathcal{P}_{gh_{z_1}\dots h_{z_d},d} \subseteq \partial\mathcal{P}_{g,d}.$$

Let $C(\alpha) \in \mathcal{P}_{gh_{z_1}\dots h_{z_d},d}$ (for some $\alpha \in E(R)^*$). Then $v_\alpha = z_i$ and $w_\alpha = z_j$. Hence $(v_\alpha)gh_{z_1} \dots h_{z_d} = v_\alpha$ and $(w_\alpha)gh_{z_1} \dots h_{z_d} = w_\alpha$. Therefore, by Lemma 4.4.3, $C(\alpha)gh_{z_1} \dots h_{z_d} = C(\alpha)$. Hence, $gh_{z_1} \dots h_{z_d}$ acts as the identity map $C(\alpha)$. This is true for all cells $C(\alpha) \in \mathcal{P}_{gh_{z_1}\dots h_{z_d},d}$ in the domain partition and hence

$$g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = g h_{z_1} \dots h_{z_d} = I.$$

□

Example 5.2.10. Recall the rearrangement g of X from Example 5.2.7. Let us now apply Algorithm 5.2.2 to compute FACTORIZATION(g):

Require: g defined above.
  $\partial\mathcal{P}_{g,d} = \{000x, 00x, 0x, x\}$.
  bp = depth $\partial\mathcal{P}_{g,d} = (z_1 = x, z_2 = 0x, z_3 = 00x, z_3 = 000x)$.
  function FACTORIZATION(g, **bp**)
    Applying for loop for $i = 1, \dots, 4$**:**
      `outputword` $= I$.
      $i = 1$.
      $h = g$ `outputword` $= g$.
      $z_1 h = 111x \neq x = z_1$.
      `interimword` =MAPBACK(h, **bp**, 1).
      return `interimword` $= f_{11}^{-1} f_1^{-1} f_\epsilon^{-1}$.
      Append `interimword` to `outputword`.
      `outputword` $= f_{11}^{-1} f_1^{-1} f_\epsilon^{-1}$.
      $i = 2$.
      $h = g$ `outputword` $= g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1}$.

$z_2h = 011x \neq 0x = z_2$.

`interimword` $=$Mapback$(h$, **bp**, $2)$.

return `interimword` $= f_{01}^{-1} f_0^{-1}$.

Append `interimword` to `outputword`.

`outputword` $= f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} f_{01}^{-1} f_0^{-1}$.

$i = 3$.

$h = g$ `outputword` $= g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} f_{01}^{-1} f_0^{-1}$.

$z_3h = 001x \neq 00x = z_3$.

`interimword` $=$Mapback$(h$, **bp**, $3)$.

return `interimword` $= f_{00}^{-1}$.

Append `interimword` to `outputword`.

`outputword` $= f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} f_{01}^{-1} f_0^{-1} f_{00}^{-1}$.

$i = 4$.

$h = g$ `outputword` $= g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} f_{01}^{-1} f_0^{-1} f_{00}^{-1}$.

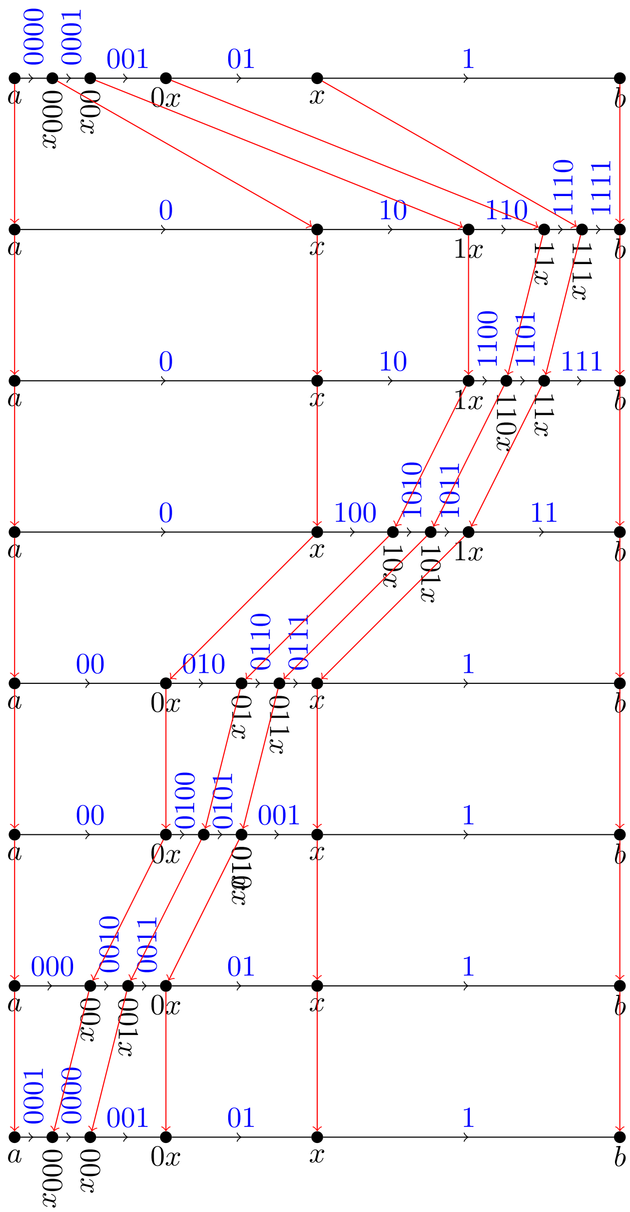
$z_4h = 000x = z_4$.

For loop ends.

return `outputword` $= f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} f_{01}^{-1} f_0^{-1} f_{00}^{-1}$.

end function

We observe that indeed $g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} f_{01}^{-1} f_0^{-1} f_{00}^{-1} = I$. We illustrate this in the following composite rectangle diagram:



Example 5.2.11. Recall the rearrangement h of X from Example 5.2.8. Let us now apply Algorithm 5.2.2 to compute FACTORIZATION(h):

Require: h defined above.

$\partial\mathcal{P}_{h,d} = \{0x, x, 10x, 1x\}$.

bp = depth $\partial\mathcal{P}_{h,d} = (z_1 = x, z_2 = 0x, z_3 = 1x, z_3 = 10x)$.

function FACTORIZATION(h, **bp**)

Applying for loop for $i = 1, \ldots, 4$**:**

`outputword` $= I$.

$i = 1$.

$\ell = h$ `outputword` $= h$.

$z_1\ell = 100x \neq x = z_1$.

`interimword` $=$MAPBACK(ℓ, **bp**, 1).

return `interimword` $= f_{10}f_1f_\epsilon^{-1}$.

Append `interimword` to `outputword`.

`outputword` $= f_{10}f_1f_\epsilon^{-1}$.

$i = 2$.

$\ell = h$ `outputword` $= hf_{10}f_1f_\epsilon^{-1}$.

$z_2h = 0x = z_2$.

`outputword` $= f_{10}f_1f_\epsilon^{-1}$.

$i = 3$.

$\ell = h$ `outputword` $= hf_{10}f_1f_\epsilon^{-1}$.

$z_3h = 1x = z_3$.

`outputword` $= f_{10}f_1f_\epsilon^{-1}$.

$i = 4$.

$h = h$ `outputword` $= hf_{10}f_1f_\epsilon^{-1}$.

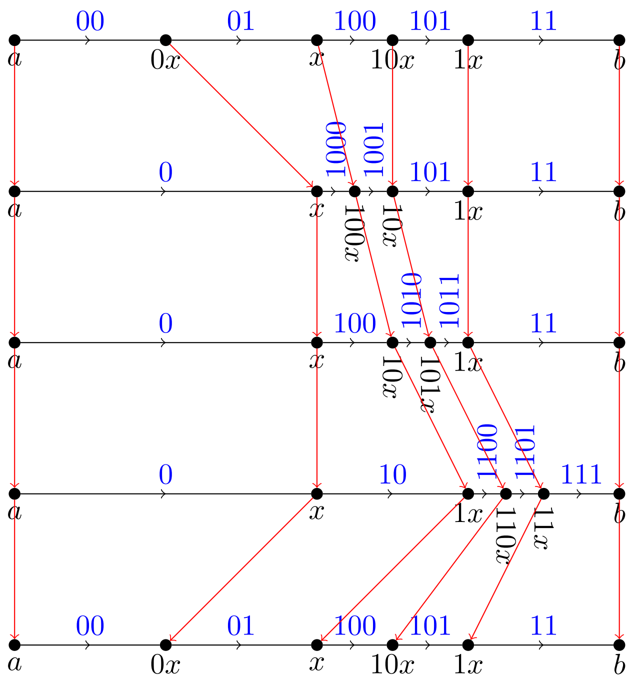
$z_4h = 10x = z_4$.

For loop ends.

return `outputword` $= f_{10}f_1f_\epsilon^{-1}$.

end function

We observe that indeed $hf_{10}f_1f_\epsilon^{-1} = I$. We illustrate this in the composite rectangle diagram below:



Proof. **(Proof of Proposition 5.2.1)**

Let X be the limit space of the F replacement system. By Proposition 4.4.1, F is the group of rearrangements of X. Let $g \in F$. Then g is a rearrangement of X. Let $\textsc{Factorization}(g) = f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N}$, where $f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} \in W(\mathcal{X})$ (for some $\alpha_1, \dots, \alpha_N \in E(R)^*$ and $\eta_1, \dots, \eta_N \in \{\pm 1\}$). By Lemma 5.2.9, $g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = I$. This implies that $g = f_{\alpha_N}^{-\eta_N} \dots f_{\alpha_1}^{-\eta_1}$. Observe that $f_{\alpha_N}^{-\eta_N} \dots f_{\alpha_1}^{-\eta_1} \in W(\mathcal{X})$. Since this is true for all $g \in F$, it follows that F is generated by $\mathcal{X}$. ☐

5.3 A Normal Form for Thompson's Group *F*

{5.3}

Let g be a rearrangement of X. Let $\textsc{Factorization}(g) = f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N}$. Observe that this is only one choice of $f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} \in W(\mathcal{X})$ (for some $\alpha_1, \dots, \alpha_N \in E(R)^*$ and $\eta_1, \dots, \eta_N \in \{\pm 1\}$) such that $g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = I$. We can find several other choices, due to conjugations in the following result. These conjugacy relations were mentioned by Dehornoy in [18]. We provide our own proof here:

erelectronics}

Proposition 5.3.1 (Dehornoy [18], Lemma 2.10)**.** *Let $\alpha, \beta, \gamma \in E(R)^*$. Then the following hold:*

1. ${f_\beta}^{f_\alpha} = f_\beta$ *if* $\alpha \perp \beta$,
2. ${f_{\alpha 0}}^{f_\alpha} = f_\alpha f_{\alpha 1}^{-1}$,
3. ${f_{\alpha 00\gamma}}^{f_\alpha} = f_{\alpha 0\gamma}$,
4. ${f_{\alpha 01\gamma}}^{f_\alpha} = f_{\alpha 10\gamma}$,
5. ${f_{\alpha 1\gamma}}^{f_\alpha} = f_{\alpha 11\gamma}$.

Proof. Let $\alpha, \beta, \gamma \in E(R)^*$. Let us prove the result case by case:

1. $\boldsymbol{{f_\beta}^{f_\alpha} = f_\beta}$ **if** $\boldsymbol{\alpha \perp \beta}$

   We examine the left-hand side of this equation, ${f_\beta}^{f_\alpha} = f_\alpha^{-1} f_\beta f_\alpha$. By Lemma 5.1.6, the rearrangements f_α and f_β have a non-identity action only on points of X with prefixes α or β. Let $\alpha = e_1 \dots e_n$ and $\beta = e'_1 \dots e'_m$. Then $f_\alpha^{-1} f_\beta f_\alpha$ acts as follows on a point $[\alpha e_{n+1} e_{n+2} \dots]$ in X:

$$([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha^{-1} f_\beta f_\alpha = \begin{cases} ([\alpha 00 e_{n+2} e_{n+3} \dots])\, f_\beta f_\alpha & \text{if } e_{n+1} = 0, \\ ([\alpha 01 e_{n+3} e_{n+4} \dots])\, f_\beta f_\alpha & \text{if } e_{n+1} e_{n+2} = 10, \\ ([\alpha 1 e_{n+3} e_{n+4} \dots])\, f_\beta f_\alpha & \text{if } e_{n+1} e_{n+2} = 11, \end{cases}$$
$$= \begin{cases} ([\alpha 00 e_{n+2} e_{n+3} \dots])\, f_\alpha & \text{if } e_{n+1} = 0, \\ ([\alpha 01 e_{n+3} e_{n+4} \dots])\, f_\alpha & \text{if } e_{n+1} e_{n+2} = 10, \\ ([\alpha 1 e_{n+3} e_{n+4} \dots])\, f_\alpha & \text{if } e_{n+1} e_{n+2} = 11, \end{cases}$$

$$= \begin{cases} [\alpha 0 e_{n+2} e_{n+3} \dots] & \text{if } e_{n+1} = 0, \\ [\alpha 10 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1} e_{n+2} = 10, \\ [\alpha 11 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1} e_{n+2} = 11. \end{cases}$$

And $f_\alpha^{-1} f_\beta f_\alpha$ acts as follows on a point $[\beta e_{m+1} e_{m+2} \dots]$ in X:

$$\begin{aligned} ([\beta e_{m+1} e_{m+2} \dots])\, f_\alpha^{-1} f_\beta f_\alpha &= ([\beta e_{m+1} e_{m+2} \dots])\, f_\beta f_\alpha \\ &= \begin{cases} ([\beta 0 e_{m+2} e_{m+3} \dots])\, f_\alpha & \text{if } e_{m+1} e_{m+2} = 00, \\ ([\beta 10 e_{m+3} e_{m+4} \dots])\, f_\alpha & \text{if } e_{m+1} e_{m+2} = 01, \\ ([\beta 11 e_{m+3} e_{m+4} \dots])\, f_\alpha & \text{if } e_{m+1} = 1, \end{cases} \\ &= \begin{cases} [\beta 0 e_{m+2} e_{m+3} \dots] & \text{if } e_{m+1} e_{m+2} = 00, \\ [\beta 10 e_{m+3} e_{m+4} \dots] & \text{if } e_{m+1} e_{m+2} = 01, \\ [\beta 11 e_{m+3} e_{m+4} \dots] & \text{if } e_{m+1} = 1. \end{cases} \end{aligned}$$

We observe that the support of $f_\alpha^{-1} f_\beta f_\alpha$ is $\operatorname{int} C(\beta)$. In this support, we observe that it acts like f_β.

2. $\boldsymbol{{f_{\alpha 0}}^{f_\alpha} = f_\alpha f_{\alpha 1}^{-1}}$

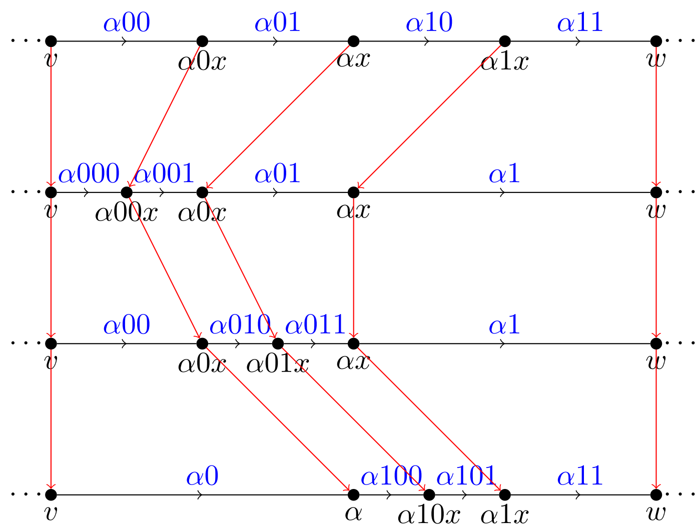
   We examine the left and right-hand sides of the equation separately and observe that they both give us the same rearrangement.

   Left-hand side: ${f_{\alpha 0}}^{f_\alpha} = f_\alpha^{-1} f_{\alpha 0} f_\alpha$

   By Lemma 5.1.6, the rearrangements f_α and $f_{\alpha 0}$ have a non-identity action only on points of X with prefix α. Let $\alpha = e_1 \dots e_n$. Then $f_\alpha^{-1} f_{\alpha 0} f_\alpha$ acts as follows on a point $[\alpha e_{n+1} e_{n+2} \dots]$ in X:

$$\begin{aligned} ([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha^{-1} f_{\alpha 0} f_\alpha &= \begin{cases} ([\alpha 00 e_{n+2} e_{n+3} \dots])\, f_{\alpha 0} f_\alpha, & \text{if } e_{n+1} = 0 \\ ([\alpha 01 e_{n+3} e_{n+4} \dots])\, f_{\alpha 0} f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\ ([\alpha 1 e_{n+3} e_{n+4} \dots])\, f_{\alpha 0} f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11 \end{cases} \\ &= \begin{cases} ([\alpha 00 e_{n+3} e_{n+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 00 \\ ([\alpha 010 e_{n+3} e_{n+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 01 \\ ([\alpha 011 e_{n+3} e_{n+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\ ([\alpha 1 e_{n+3} e_{n+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11 \end{cases} \\ &= \begin{cases} [\alpha 0 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1} e_{n+2} = 00 \\ [\alpha 100 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1} e_{n+2} = 01 \\ [\alpha 101 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1} e_{n+2} = 10 \\ [\alpha 11 e_{n+2} e_{n+3} \dots], & \text{if } e_{n+1} e_{n+2} = 11 \end{cases} \end{aligned}$$

This is illustrated in the following diagram:

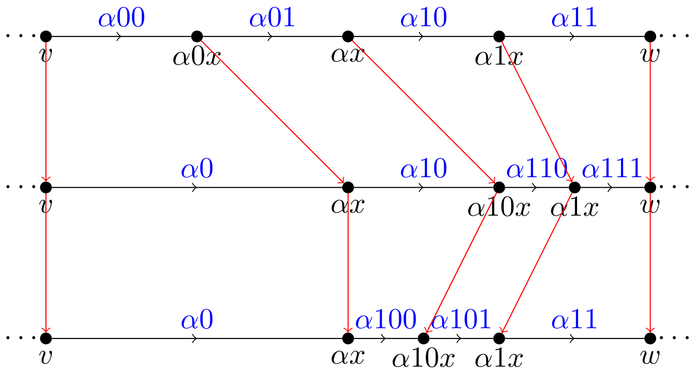


Right-hand side: $f_\alpha f_{\alpha 1}^{-1}$

By Lemma 5.1.6, the rearrangements f_α and $f_{\alpha 1}$ have a non-identity action only on points of X with prefix α. Let $\alpha = e_1 \dots e_n$. Then $f_\alpha f_{\alpha 1}^{-1}$ acts as follows on a point $[\alpha e_{n+1} e_{n+2} \dots]$ in X:

$$
\begin{aligned}
([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha f_{\alpha 1}^{-1} &= \begin{cases} ([\alpha 0 e_{n+3} e_{n+4} \dots])\, f_{\alpha 1}^{-1}, & \text{if } e_{n+1} e_{n+2} = 00 \\ ([\alpha 10 e_{n+3} e_{n+4} \dots])\, f_{\alpha 1}^{-1}, & \text{if } e_{n+1} e_{n+2} = 01 \\ ([\alpha 11 e_{n+2} e_{n+3} \dots])\, f_{\alpha 1}^{-1}, & \text{if } e_{n+1} = 1 \end{cases} \\
&= \begin{cases} [\alpha 0 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1} e_{n+2} = 00 \\ [\alpha 100 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1} e_{n+2} = 01 \\ [\alpha 101 e_{n+3} e_{n+4} \dots], & \text{if } e_{n+1} e_{n+2} = 10 \\ [\alpha 11 e_{n+2} e_{n+3} \dots], & \text{if } e_{n+1} e_{n+2} = 11 \end{cases}
\end{aligned}
$$

This is illustrated in the following diagram:

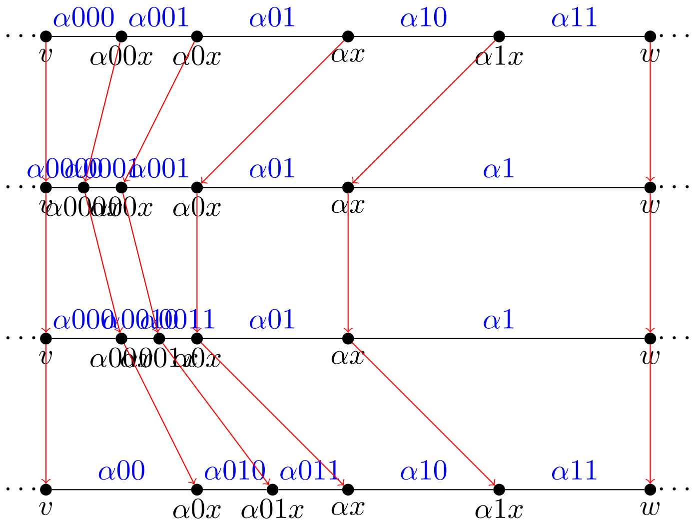


3. $\boldsymbol{{f_{\alpha00\gamma}}^{f_\alpha} = f_{\alpha0\gamma}}$

We examine the left-hand side of this equation, ${f_{\alpha00\gamma}}^{f_\alpha} = f_\alpha^{-1} f_{\alpha00\gamma} f_\alpha$. By Lemma 5.1.6, the rearrangements f_α and $f_{\alpha00\gamma}$ have a non-identity action only on points of X with prefix α. Let $\alpha = e_1 \ldots e_n$. Then $f_\alpha^{-1} f_{\alpha00\gamma} f_\alpha$ acts as follows on a point $[\alpha e_{n+1} e_{n+2} \ldots]$ in X:

$$
\begin{aligned}
&([\alpha e_{n+1} e_{n+2} \ldots])\, f_\alpha^{-1} f_{\alpha00\gamma} f_\alpha \\
&\quad = \begin{cases}
([\alpha00 e_{n+2} e_{n+3} \ldots])\, f_{\alpha00\gamma} f_\alpha, & \text{if } e_{n+1} = 0 \\
([\alpha01 e_{n+3} e_{n+4} \ldots])\, f_{\alpha00\gamma} f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\
([\alpha1 e_{n+3} e_{n+4} \ldots])\, f_{\alpha00\gamma} f_\alpha, & \text{if } e_{n+1} = 11
\end{cases} \\
&\quad = \begin{cases}
([\alpha00\gamma0 e_{m+3} e_{m+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} = 0,\ e_{n+2} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 00 \\
([\alpha00\gamma10 e_{m+3} e_{m+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} = 0,\ e_{n+2} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 01 \\
([\alpha00\gamma11 e_{m+2} e_{m+3} \ldots])\, f_\alpha, & \text{if } e_{n+1} = 0,\ e_{n+2} \ldots e_m = \gamma \text{ and } e_{m+1} = 1 \\
([\alpha00 e_{n+2} e_{n+3} \ldots])\, f_\alpha, & \text{if } e_{n+1} = 0 \text{ and } e_{n+2} \ldots e_m \neq \gamma \\
([\alpha01 e_{n+3} e_{n+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\
([\alpha1 e_{n+3} e_{n+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11
\end{cases} \\
&\quad = \begin{cases}
[\alpha0\gamma0 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} = 0,\ e_{n+2} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 00 \\
[\alpha0\gamma10 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} = 0, e_{n+2} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 01 \\
[\alpha0\gamma11 e_{m+2} e_{m+3} \ldots], & \text{if } e_{n+1} = 0, e_{n+2} \ldots em = \gamma \text{ and } e_{m+1} = 1 \\
[\alpha0 e_{n+2} e_{n+3} \ldots], & \text{if } e_{n+1} = 0 \text{ and } e_{n+2} \ldots e_m \neq \gamma \\
[\alpha10 e_{n+3} e_{n+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 10 \\
[\alpha11 e_{n+3} e_{n+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 11
\end{cases}
\end{aligned}
$$

We observe that the support of this rearrangement is $\operatorname{int} C(\alpha0\gamma)$. In this support, we observe that it acts like $f_{\alpha0\gamma}$. This is illustrated in the following diagram when $\gamma = \epsilon$:

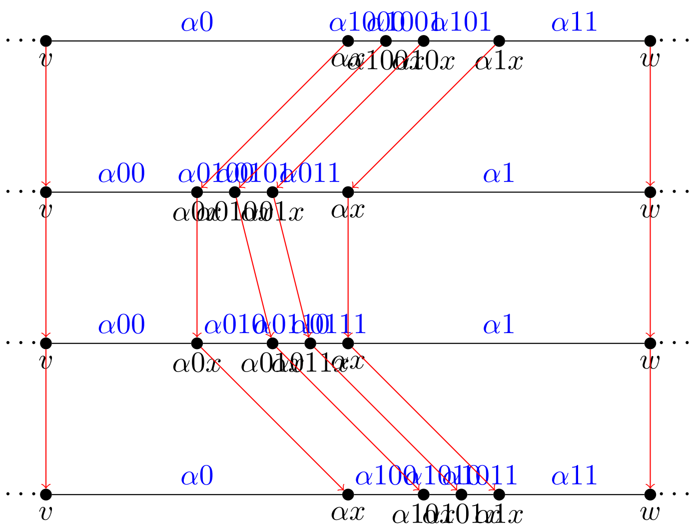


4. $\boldsymbol{f_{\alpha 01\gamma}{}^{f_\alpha} = f_{\alpha 10\gamma}}$

We examine the left-hand side of this equation, $f_{\alpha 01\gamma}{}^{f_\alpha} = f_\alpha^{-1} f_{\alpha 01\gamma} f_\alpha$. By Lemma 5.1.6, the rearrangements f_α and $f_{\alpha 01\gamma}$ have a non-identity action only on points of X with prefix α. Let $\alpha = e_1 \ldots e_n$. Then $f_\alpha^{-1} f_{\alpha 01\gamma} f_\alpha$ acts as follows on a point $[\alpha e_{n+1} e_{n+2} \ldots]$ in X:

$$
\begin{aligned}
&([\alpha e_{n+1} e_{n+2} \ldots])\, f_\alpha^{-1} f_{\alpha 01\gamma} f_\alpha \\
&\quad = \begin{cases}
([\alpha 00 e_{n+2} e_{n+3} \ldots])\, f_{\alpha 01\gamma} f_\alpha, & \text{if } e_{n+1} = 0 \\
([\alpha 01 e_{n+3} e_{n+4} \ldots])\, f_{\alpha 01\gamma} f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\
([\alpha 1 e_{n+3} e_{n+4} \ldots])\, f_{\alpha 01\gamma} f_\alpha, & \text{if } e_{n+1} = 11
\end{cases} \\
&\quad = \begin{cases}
([\alpha 00 e_{n+2} e_{n+3} \ldots])\, f_\alpha, & \text{if } e_{n+1} = 0 \\
([\alpha 01\gamma 0 e_{m+3} e_{m+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 00 \\
([\alpha 01\gamma 10 e_{m+3} e_{m+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 01 \\
([\alpha 01\gamma 11 e_{m+3} e_{m+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} = 1 \\
([\alpha 01 e_{n+3} e_{n+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m \neq \gamma \\
([\alpha 1 e_{n+3} e_{n+4} \ldots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11
\end{cases} \\
&\quad = \begin{cases}
[\alpha 0 e_{n+2} e_{n+3} \ldots], & \text{if } e_{n+1} = 0 \\
[\alpha 10\gamma 0 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 00 \\
[\alpha 10\gamma 10 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 01 \\
[\alpha 10\gamma 11 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} = 1 \\
[\alpha 10 e_{n+3} e_{n+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 10,\ e_{n+3} \ldots e_m \neq \gamma \\
[\alpha 11 e_{n+3} e_{n+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 11
\end{cases}
\end{aligned}
$$

We observe that the support of this rearrangement is $\operatorname{int} C(\alpha 10\gamma)$. In this support, we observe that it acts like $f_{\alpha 10\gamma}$. This is illustrated in the following diagram when $\gamma = \epsilon$:



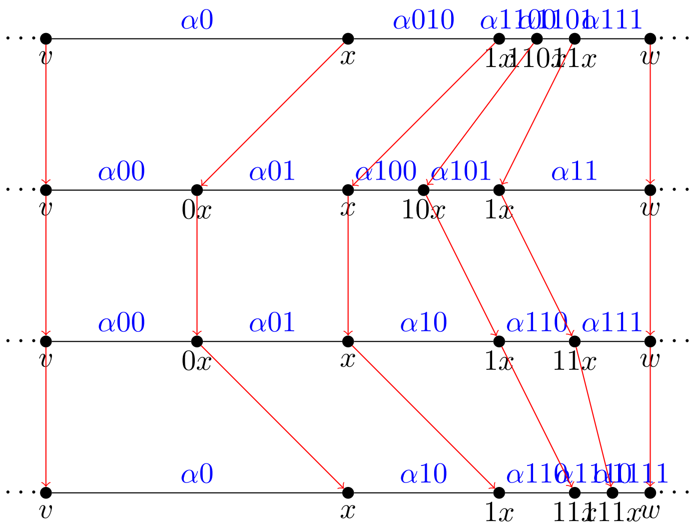
5. $\boldsymbol{{f_{\alpha 1\gamma}}^{f_\alpha} = f_{\alpha 11\gamma}}$

We examine the left-hand side of this equation, ${f_{\alpha 1\gamma}}^{f_\alpha} = f_\alpha^{-1} f_{\alpha 1\gamma} f_\alpha$. By Lemma 5.1.6, the rearrangements f_α and $f_{\alpha 1\gamma}$ have a non-identity action only on points of X with prefix α. Let $\alpha = e_1 \dots e_n$. Then $f_\alpha^{-1} f_{\alpha 1\gamma} f_\alpha$ acts as follows on a point $[\alpha e_{n+1} e_{n+2} \dots]$ in X:

$$
\begin{aligned}
&([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha^{-1} f_{\alpha 1\gamma} f_\alpha \\
&\qquad = \begin{cases}
([\alpha 00 e_{n+2} e_{n+3} \dots])\, f_{\alpha 1\gamma} f_\alpha, & \text{if } e_{n+1} = 0 \\
([\alpha 01 e_{n+3} e_{n+4} \dots])\, f_{\alpha 1\gamma} f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\
([\alpha 1 e_{n+3} e_{n+4} \dots])\, f_{\alpha 1\gamma} f_\alpha, & \text{if } e_{n+1} = 11
\end{cases} \\
&\qquad = \begin{cases}
([\alpha 00 e_{n+2} e_{n+3} \dots])\, f_\alpha, & \text{if } e_{n+1} = 0 \\
([\alpha 01 e_{n+3} e_{n+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 10 \\
([\alpha 1\gamma 0 e_{m+3} e_{m+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11, e_{n+3} \dots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 00 \\
([\alpha 1\gamma 10 e_{m+3} e_{m+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11, e_{n+3} \dots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 01 \\
([\alpha 1\gamma 11 e_{m+2} e_{m+3} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11, e_{n+3} \dots e_m = \gamma \text{ and } e_{m+1} = 1 \\
([\alpha 1 e_{n+3} e_{n+4} \dots])\, f_\alpha, & \text{if } e_{n+1} e_{n+2} = 11 \text{ and } e_{n+3} \dots e_m \neq \gamma
\end{cases}
\end{aligned}
$$

$$= \begin{cases} [\alpha 0 e_{n+2} e_{n+3} \ldots], & \text{if } e_{n+1} = 0 \\ [\alpha 10 e_{n+3} e_{n+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 10 \\ [\alpha 11 \gamma 0 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 11, e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 00 \\ [\alpha 11 \gamma 10 e_{m+3} e_{m+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 11, e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} e_{m+2} = 01 \\ [\alpha 11 \gamma 11 e_{m+2} e_{m+3} \ldots], & \text{if } e_{n+1} e_{n+2} = 11, e_{n+3} \ldots e_m = \gamma \text{ and } e_{m+1} = 1 \\ [\alpha 11 e_{n+3} e_{n+4} \ldots], & \text{if } e_{n+1} e_{n+2} = 11 \text{ and } e_{n+3} \ldots e_m \neq \gamma \end{cases}$$

We observe that the support of this rearrangement is $\operatorname{int} C(\alpha 11 \gamma)$. In this support, we observe that it acts like $f_{\alpha 11 \gamma}$. This is illustrated in the following diagram when $\gamma = \epsilon$:



This proves the result. □

We would like to draw attention to Proposition 5.3.1 (2) in particular. Dehornoy referred to it as the *pentagon relation* (however, the author misread it as the *Pentagram relation*). We find it more useful in the follwing two forms:

$$f_{\alpha 0} f_\alpha f_{\alpha 1} = f_\alpha f_\alpha$$

or

$$f_\alpha{}^{-1} f_{\alpha 0} f_\alpha f_{\alpha 1} f_\alpha{}^{-1} = I.$$

Let X be the limit space of the F replacement system. Let g be a rearrangement of X. We can now modify Algorithm 5.2.2 to develop an algorithm which outputs an optimized word $f_{\alpha_1}^{\eta_1} \ldots f_{\alpha_N}^{\eta_N}$ (where $f_{\alpha_i}^{\eta_i} \in \mathcal{X}$ (for some $\alpha_i \in E(R)^*$ and $\eta \in \{\pm 1\}$) for each $i = 1, \ldots, n$) such that $g = f_{\alpha_1}^{\eta_1} \ldots f_{\alpha_N}^{\eta_N}$. This word is often significantly shorter than $\text{Factorization}(g)$. In Section 5.3, we conjecture that it might help determine the "rotation distance" of g.

Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} \supseteq \mathrm{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \ldots, z_d)$. Choose $z_i \in \partial \mathcal{P}_{g,d}$ such that $z_i g = z_1, \ldots, z_{i-1} g = z_{i-1}$. The function MAPBACKSET$(g, \mathbf{bp}, i)$ finds a set of words $f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} \in W(\mathcal{X})$ (for some $\beta_1, \ldots, \beta_K \in E(R)^*$ and $\eta_1, \ldots, \eta_K \in \{\pm 1\}$) such that $z_1 g f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = z_1, \ldots, z_i g f_{\beta_1}^{\eta_1} \ldots f_{\beta_K}^{\eta_K} = z_i$.

Algorithm 5.3.2 The function MAPBACKSET. {desertkisses}

Require: Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} = (z_1, \ldots, z_d)$ be the list of precomputed gluing vertices ordered by depth such that $\partial \mathcal{P}_{g,d} \subseteq \mathbf{bp}$. Let $z_i g = z_1, \ldots, z_{i-1} g = z_{i-1}$.

function MAPBACKSET(g, $\mathbf{bp} = (z_1, \ldots, z_d)$, i)
  Let $z_i g = \alpha x = e_1 \ldots e_n x$, $z_i = \gamma x = e_1 \ldots e_m$ x.
  Set `outputset` = [].
  if $z_i g \neq z_i$ **then**
    Create `prelimset` consisting of all f_β^η with the following two properties:
      1. γ is a prefix of β.
      2. One of the following is true:
        $\alpha = \beta 0$, $\eta = +1$,
        $\alpha = \beta 1$, $\eta = -1$,
        $\alpha = \beta 00 \delta$, $\eta = +1$,
        $\alpha = \beta 11 \delta$, $\eta = -1$.
    for each f_β^η in `prelimset` **do**
      Set `interimset` = MAPBACKSET($g f_\beta^\eta$, $\mathbf{bp}$, i).
      if `interimset` = [] **then**
        for each h in `interimset` **do**
          Add $f_\beta^\eta h$ to `outputset`.
        end for
      else if `interimset` $\neq$ [] **then**
        Add f_β^η to `outputset`.
      end if
    end for
  end if
  Return `outputset`.
end function

The following results characterize the function MAPBACKSET, and are analogous to Lemma 5.2.4, Lemma 5.2.5 and Lemma 5.2.6 for the function MAPBACK:

{modularity}

Lemma 5.3.3. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let* $\mathrm{depth}(\partial \mathcal{P}_{g,d}) =$ $(z_1, \ldots, z_d)$*. Let $z_1 g = z_1, \ldots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha = e_1 \ldots e_n \in$*

$E(R)^*$) and $z_i = \gamma x$ (for some $\gamma = e_1 \dots e_m \in E(R)^*$) such that $z_i g \neq z_i$. Let S be the set containing all f_β^η which satisfy Lines 6-11 of Algorithm 5.3.2. Then, for each $f_\beta^\eta \in S$, the following hold:

1. $z_1 g f_\beta^\eta = z_1, \dots, z_{i-1} g f_\beta^\eta = z_{i-1}$,

2. $\mathrm{depth}(z_i g f_\beta^\eta) = \mathrm{depth}(z_i g) - 1$.

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathrm{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma \in E(R)^*$) such that $z_i g \neq z_i$. By Corollary 5.1.10, $\gamma \prec \alpha$. Suppose $\gamma = e_1 \dots e_m$ and $\alpha = e_1 \dots e_n$, for some $n, m \in \mathbb{N}$ such that $n > m$. Let us define the set S to contain all functions f_β^η which satisfy Lines 6-11 of Algorithm 5.3.2. That is, S us as follows:

$$\begin{aligned} S = \big\{ f_\beta^\eta \big| \gamma \preceq \beta \text{ and one of the following is true:} \\ \alpha = \beta 0, \eta = +1, \\ \alpha = \beta 1, \eta = -1, \\ \alpha = \beta 00 \delta, \eta = +1, \\ \alpha = \beta 11 \delta, \eta = -1 \big\}. \end{aligned}$$

Then, by Lemma 5.1.12, $\mathrm{depth}(z_i g f_\beta^\eta) = \mathrm{depth}(z_i g) - 1$, for all $f_\beta^\eta \in S$.

Now consider the a boundary point z_j for some $1 \leq j \leq i-1$. Suppose $z_j = \delta x$. Then, by Definition 2.3.10 of the depth order, either $\delta \perp \gamma$ or $\delta \prec \gamma$. By the definition of S, $\delta \perp \beta$ or $\delta \prec \beta$, for all $f_\beta^\eta \in S$. Hence, by Lemma 5.1.6, $z_j \notin \mathrm{supp}\, f_\beta^\eta$ and therefore $z_j g f_\beta^\eta = z_j$, for all $f_\beta^\eta \in S$. Since this is true for all $j = 1, \dots, i-1$, this proves the result. $\square$

{lastexit}

Lemma 5.3.4. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} \supseteq \mathrm{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma = e_1 \dots e_m \in E(R)^*$). Then* $\textsc{MapbackSet}(g, \mathbf{bp}, i)$ *is non-empty if and only if $n > m$.*

Suppose $\textsc{MapbackSet}(g, \mathbf{bp}, i)$ *is non-empty. Let* $f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K} \in \textsc{MapbackSet}(g, \mathbf{bp}, i)$. *Then the following hold:*

1. $K = n - m$.

2. $z_1 g f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K} = z_1, \dots, z_i g f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K} = z_i$.

3. $\mathrm{depth}(z_i g f_{\beta_1}^{\eta_1} \dots f_{\beta_s}^{\eta_s}) = \mathrm{depth}(z_i g) - s$ *for all* $s = 1, \dots, K$.

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} \supseteq \mathrm{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha \in E(R)^*$) and $z_i = \gamma x$ (for some

$\gamma \in E(R)^*$). By Corollary 5.1.10, $\gamma \preceq \alpha$. Suppose $\gamma = e_1 \dots e_m$ and $\alpha = e_1 \dots e_n$ for some $n, m \in \mathbb{N}$ such that $n \geq m$. Let $f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K} \in \textsc{MapbackSet}(g, \mathbf{bp}, i)$.

Let us prove this result by induction on $n - m$:

Suppose $n - m = 0$. Then $\alpha = \gamma$ and $z_i g = z_i$, and Algorithm 5.3.2 omits lines 4-22 and returns the empty set. Hence $\textsc{MapbackSet}(g, \mathbf{bp}, i) = \varnothing$ and the inductive hypothesis holds.

Suppose $n - m = 1$. Then $\text{depth}(z_i) = \text{depth}(z_i g) - 1$. Let us implement Algorithm 5.3.2. We construct `prelimset` consisting of all f_β^η which satisfy Lines 6-11. Then, by Lemma 5.3.3, for each $f_\beta^\eta \in$ `prelimset`, the following hold:

1. $z_1 g f_\beta^\eta = z_1, \dots, z_{i-1} g f_\beta^\eta = z_{i-1}$,
2. $\text{depth}(z_i g f_\beta^\eta) = \text{depth}(z_i g) - 1$.

Observe that `prelimset` $\neq \varnothing$, since there must exist f_β^η satisfying Line 8 or Line 9.

Let $f_\beta^\eta \in$ `prelimset`. Since $\gamma \preceq \beta$ and $\text{depth}(z_i g f_\beta^\eta) = \text{depth}(z_i)$, this implies that $z_i g f_\beta^\eta = z_i$. Line 13 calls `interimset` $= \textsc{MapbackSet}(g f_\beta^\eta, bp, i)$. Observe that the previous base case applies and we can conclude that `interimset` $= \textsc{MapbackSet}(g f_\beta^\eta, bp, i) = \varnothing$. Then, by Lines 18-20, $f_\beta^\eta \in$ `outputset`. Since this is true for all $f_\beta^\eta \in$ `prelimset`, then `outputset` $=$ `prelimset`. Then $\textsc{MapbackSet}(g, bp, i) =$ `outputset`. Hence every product in $\textsc{MapbackSet}(g, bp, i)$ has the form f_β^η and the inductive hypothesis holds.

Suppose the inductive hypothesis holds for $n - m = r$. Let us examine the case when $n - m = r + 1$. Let us implement Algorithm 5.3.2. By lines 5-11, we construct `prelimset` containing all $f_{\beta_1}^{\eta_1}$ such that $\gamma \preceq \beta$ and one of the following is true:

$$\alpha = \beta 0, \eta = +1,$$
$$\alpha = \beta 1, \eta = -1,$$
$$\alpha = \beta 00\delta, \eta = +1,$$
$$\alpha = \beta 11\delta, \eta = -1$$

By Lemma 5.3.3, for all $f_{\beta_1}^{\eta_1} \in$ `prelimset`, $z_i g f_{\beta_1}^{\eta_1} = z_1, \dots, z_{i-1} g f_{\beta_1}^{\eta_1} = z_{i-1}$ and $\text{depth}(z_i g f_\beta^\eta) = \text{depth}(z_i g) - 1$. Line 12 initiates a for loop on each $f_{\beta_1}^{\eta_1} \in$ `prelimset` as follows: By like 13, we construct `interimset` $= \textsc{MapbackSet}(g f_{\beta_1}^{\eta_1}, \mathbf{bp}, i)$. Observe that the inductive hypothesis holds for $g f_{\beta_1}^{\eta_1}$, for all $f_{\beta_1}^{\eta_1} \in$ `prelimset`. Therefore, for each $f_{\beta_2}^{\eta_2} \dots f_{\beta_{r+1}}^{\eta_{r+1}} \in$ `interimset` the following hold:

1. The length of the word $f_{\beta_2}^{\eta_2} \dots f_{\beta_{r+1}}^{\eta_{r+1}}$ is r.
2. $z_1 g f_{\beta_1}^{\eta_1} f_{\beta_2}^{\eta_2} \dots f_{\beta_K}^{\eta_K} = z_1, \dots, z_i g f_{\beta_1}^{\eta_1} f_{\beta_2}^{\eta_2} \dots f_{\beta_{r+1}}^{\eta_{r+1}} = z_i$.
3. $\text{depth}(z_i g f_{\beta_1}^{\eta_1} f_{\beta_2}^{\eta_2} \dots f_{\beta_s}^{\eta_s}) = \text{depth}(z_i g) - s$ for all $s = 2, \dots, r+1$.

By lines 14-16, for each $f_{\beta_2}^{\eta_2} \dots f_{\beta_{r+1}}^{\eta_{r+1}} \in$ `interimset`, we add $f_{\beta_1}^{\eta_1} f_{\beta_2}^{\eta_2} \dots f_{\beta_{r+1}}^{\eta_{r+1}}$ to `outputset`. The for loop from line 12 is terminated on line 17. And by line 19, the function $\textsc{MapbackSet}(g, bp, i)$ returns `outputset`. Hence, the following hold for each $f_{\beta_1}^{\eta_1} \dots f_{\beta_{r+1}}^{\eta_{r+1}} \in \textsc{MapbackSet}(g, bp, i)$:

1. The length of the word $f_{\beta_1}^{\eta_1} \dots f_{\beta_{r+1}}^{\eta_{r+1}}$ is r.
2. $z_1 g f_{\beta_1}^{\eta_1} \dots f_{\beta_{r+1}}^{\eta_{r+1}} = z_1, \dots, z_i g f_{\beta_1}^{\eta_1} \dots f_{\beta_{r+1}}^{\eta_{r+1}} = z_i$.
3. $\operatorname{depth}(z_i g f_{\beta_1}^{\eta_1} \dots f_{\beta_s}^{\eta_s}) = \operatorname{depth}(z_i g) - s$ for all $s = 1, \dots, r+1$.

Hence, the inductive hypothesis holds for $r+1$. This proves the result by induction.

$\square$

{theurgist}

Lemma 5.3.5. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let* $\mathbf{bp} \supseteq \operatorname{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$*. Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let* $f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K} \in \textsc{MapBackSet}(g, \mathbf{bp}, i)$*. Then*

$$\partial \mathcal{P}_{g f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K}, d} \subseteq \partial \mathcal{P}_{g,d}.$$

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\mathbf{bp} \supseteq \operatorname{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots, z_d)$. Let $z_1 g = z_1, \dots, z_{i-1} g = z_{i-1}$. Let $z_i g = \alpha x$ (for some $\alpha \in E(R)^*$) and $z_i = \gamma x$ (for some $\gamma \in E(R)^*$). By Corollary 5.1.10, $\gamma \preceq \alpha$. Suppose $\gamma = e_1 \dots e_m$ and $\alpha = e_1 \dots e_n$ for some $n, m \in \mathbb{N}$ such that $n \geq m$.

Let $f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K} \in \textsc{MapBackSet}(g, \mathbf{bp}, i)$. Observe that each $f_{\beta_i}^{\eta_i}$ satisfies the conditions of Lemma 5.1.11 in relation to αx. Then, by Lemma 5.1.11,

$$\partial \mathcal{P}_{f_{\alpha_s}^{\eta_s}, d} \subseteq \partial \mathcal{P}_{g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_{s-1}}^{\eta_{s-1}}, r}.$$

Hence, by Lemma 3.4.5,

$$\partial \mathcal{P}_{g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_s}^{\eta_s}, d} \subseteq \partial \mathcal{P}_{g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_{s-1}}^{\eta_{s-1}}, d}.$$

Since this is true for all $s = 1, \dots, K$, it follows that

$$\partial \mathcal{P}_{g f_{\beta_1}^{\eta_1} \dots f_{\beta_K}^{\eta_K}, d} \subseteq \partial \mathcal{P}_{g,d}.$$

$\square$

Example 5.3.6. Recall the rearrangement g of X from Example 5.1.17. We observe that $\mathbf{bp} = \operatorname{depth}(\partial \mathcal{P}_{g,d}) = (x, 0x, 00x, 000x)$. Also observe that $z_1 g = 111x \neq x = z_1$. Let us carry out $\textsc{MapBackSet}(g, \mathbf{bp}, 1)$:

Require: g, $\mathbf{bp}$, $i = 1$ defined above.

function $\textsc{MapBackSet}(g,\ \mathbf{bp},\ 1)$

`outputset` $= []$.

$z_1 g = 111x \neq x = z_1$.

`prelimset` $= \{f_{11}^{-1}, f_1^{-1}, f_\epsilon^{-1}\}$.

for $f_{11}^{-1} \in$ `prelimset`:

`interimset` =MAPBACKSET(gf_{11}^{-1}, **bp**, 1)
  `outputset` = [].
  $z_1 g f_{11}^{-1} = 11x \neq x = z_1$.
  `prelimset` = $\{f_1^{-1}, f_\epsilon^{-1}\}$.

  for $f_1^{-1} \in$ `prelimset`:
    `interimset` =MAPBACKSET($gf_{11}^{-1}f_1^{-1}$, **bp**, 1)
      `outputset` = [].
      $z_1 g f_{11}^{-1} f_1^{-1} = 1x \neq x = z_1$.
      `prelimset` = $\{f_\epsilon^{-1}\}$.
      for $f_\epsilon^{-1} \in$ `prelimset`:
        `interimset` =MAPBACKSET($gf_{11}^{-1}f_1^{-1}f_\epsilon^{-1}$, **bp**, 1)
          `outputset` = [].
          $z_1 g f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} = x = z_1$.
          end function
        `interimset` = [].
        Add f_ϵ^{-1} to `outputset`.
      end for
      `outputset` = $\{f_\epsilon^{-1}\}$.
      end function
    `interimset` = $\{f_\epsilon^{-1}\}$.
    Add $f_1^{-1} f_\epsilon^{-1}$ to `outputset`.

  for $f_\epsilon^{-1} \in$ `prelimset`:
    `interimset` =MAPBACKSET($gf_{11}^{-1}f_\epsilon^{-1}$, **bp**, 1)
      `outputset` = [].
      $z_1 g f_{11}^{-1} f_\epsilon^{-1} = 1x \neq x = z_1$.
      `prelimset` = $\{f_\epsilon^{-1}\}$.
      for $f_\epsilon^{-1} \in$ `prelimset`:
        `interimset` =MAPBACKSET($gf_{11}^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$, **bp**, 1)
          `outputset` = [].
          $z_1 g f_{11}^{-1} f_\epsilon^{-1} f_\epsilon^{-1} = x = z_1$.
          end function
        `interimset` = [].
        Add f_ϵ^{-1} to `outputset`.
      end for
      `outputset` = $\{f_\epsilon^{-1}\}$.
      end function
    `interimset` = $\{f_\epsilon^{-1}\}$.
    Add $f_\epsilon^{-1} f_\epsilon^{-1}$ to `outputset`.

end for

return `outputset` $= \{f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}\}$.
Add $f_{11}^{-1}f_1^{-1}f_\epsilon^{-1}$ and $f_{11}^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$ to `outputset`.

for $f_1^{-1} \in$ `prelimset`:
`interimset` =MapbackSet(gf_1^{-1}, **bp**, 1)
`outputset` = [].
$z_1gf_1^{-1} = 11x \neq x = z_1$.
`prelimset` $= \{f_1^{-1}, f_\epsilon^{-1}\}$.

for $f_1^{-1} \in$ `prelimset`:
`interimset` =MapbackSet($gf_1^{-1}f_1^{-1}$, **bp**, 1)
`outputset` = [].
$z_1gf_1^{-1}f_1^{-1} = 1x \neq x = z_1$.
`prelimset` $= \{f_\epsilon^{-1}\}$.
for $f_\epsilon^{-1} \in$ `prelimset`:
`interimset` =MapbackSet($gf_1^{-1}f_1^{-1}f_\epsilon^{-1}$, **bp**, 1)
`outputset` = [].
$z_1gf_1^{-1}f_1^{-1}f_\epsilon^{-1} = x = z_1$.
end function
`interimset` = [].
Add f_ϵ^{-1} to `outputset`.
end for
`outputset` $= \{f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_\epsilon^{-1}\}$.
Add $f_1^{-1}f_\epsilon^{-1}$ to `outputset`.

for $f_\epsilon^{-1} \in$ `prelimset`:
`interimset` =MapbackSet($gf_1^{-1}f_\epsilon^{-1}$, **bp**, 1)
`outputset` = [].
$z_1gf_1^{-1}f_\epsilon^{-1} = 1x \neq x = z_1$.
`prelimset` $= \{f_\epsilon^{-1}\}$.
for $f_\epsilon^{-1} \in$ `prelimset`:
`interimset` =MapbackSet($gf_1^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$, **bp**, 1)
`outputset` = [].
$z_1gf_1^{-1}f_\epsilon^{-1}f_\epsilon^{-1} = x = z_1$.

end function
`interimset` $= []$.
Add f_ϵ^{-1} to `outputset`.
end for
`outputset` $= \{f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_\epsilon^{-1}\}$.
Add $f_\epsilon^{-1}f_\epsilon^{-1}$ to `outputset`.
end for

return `outputset` $= \{f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}\}$.
Add $f_1^{-1}f_1^{-1}f_\epsilon^{-1}$ and $f_1^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$ to `outputset`.

for $f_\epsilon^{-1} \in$ `prelimset`:
`interimset` =MapbackSet(gf_ϵ^{-1}, **bp**, 1)
`outputset` $= []$.
$z_1gf_\epsilon^{-1} = 11x \neq x = z_1$.
`prelimset` $= \{f_1^{-1}, f_\epsilon^{-1}\}$.

for $f_1^{-1} \in$ `prelimset`:
`interimset` =MapbackSet($gf_\epsilon^{-1}f_1^{-1}$, **bp**, 1)
`outputset` $= []$.
$z_1gf_\epsilon^{-1}f_1^{-1} = 1x \neq x = z_1$.
`prelimset` $= \{f_\epsilon^{-1}\}$.
for $f_\epsilon^{-1} \in$ `prelimset`:
`interimset` =MapbackSet($gf_\epsilon^{-1}f_1^{-1}f_\epsilon^{-1}$, **bp**, 1)
`outputset` $= []$.
$z_1gf_\epsilon^{-1}f_1^{-1}f_\epsilon^{-1} = x = z_1$.
end function
`interimset` $= []$.
Add f_ϵ^{-1} to `outputset`.
end for
`outputset` $= \{f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_\epsilon^{-1}\}$.
Add $f_1^{-1}f_\epsilon^{-1}$ to `outputset`.

for $f_\epsilon^{-1} \in$ `prelimset`:

interimset $=$ MapbackSet($gf_\epsilon^{-1}f_\epsilon^{-1}$, **bp**, 1)

outputset $= []$.

$z_1 g f_\epsilon^{-1} f_\epsilon^{-1} = 1x \neq x = z_1$.

prelimset $= \{f_\epsilon^{-1}\}$.

for $f_\epsilon^{-1} \in$ prelimset:

interimset $=$ MapbackSet($gf_\epsilon^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$, **bp**, 1)

outputset $= []$.

$z_1 g f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1} = x = z_1$.

end function

interimset $= []$.

Add f_ϵ^{-1} to outputset.

end for

outputset $= \{f_\epsilon^{-1}\}$.

end function

interimset $= \{f_\epsilon^{-1}\}$.

Add $f_\epsilon^{-1}f_\epsilon^{-1}$ to outputset.

end for

return outputset $= \{f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}\}$.

end function

interimset $= \{f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}\}$.

Add $f_\epsilon^{-1}f_1^{-1}f_\epsilon^{-1}$ and $f_\epsilon^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$ to outputset.

end for

return outputset $= \{f_{11}^{-1}f_1^{-1}f_\epsilon^{-1}, f_{11}^{-1}f_\epsilon^{-1}f_\epsilon^{-1}, f_1^{-1}f_1^{-1}f_\epsilon^{-1}, f_1^{-1}f_\epsilon^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_1^{-1}f_\epsilon^{-1}, f_\epsilon^{-1}f_\epsilon^{-1}f_\epsilon^{-1}\}$.

end function

We observe that indeed $z_1 g f_{\beta_1}^{\eta_1} \dots f_{\beta_k}^{\eta_k} = x = z_1$ for all $f_{\beta_1}^{\eta_1} \dots f_{\beta_k}^{\eta_k} \in$ MapbackSet$(g, \mathbf{bp}, 1)$.

Example 5.3.7. Recall the rearrangement h of X from Example 5.1.18. We observe that $\mathbf{bp} = \mathrm{depth}(\partial\mathcal{P}_{h,d}) = (x, 0x, 1x, 10x)$. Also observe that $z_1 h = 100x \neq x = z_1$. Let us carry out MapbackSet$(h, \mathbf{bp}, 1)$:

Require: h, **bp**, $i = 1$ defined above.

function MapbackSet(h, **bp**, 1)

outputset $= []$.

$z_1 h = 100x \neq x = z_1$.

prelimset $= \{f_{10}, f_1\}$.

for $f_{10} \in$ prelimset:

interimset =MapbackSet(hf_{10}, **bp**, 1)
outputset = [].
$z_1hf_{10} = 10x \neq x = z_1$.
prelimset = $\{f_1\}$.
for $f_1 \in$ prelimset:
interimset =MapbackSet($hf_{10}f_1$, **bp**, 1)
outputset = [].
$z_1hf_{10}f_1 = 1x \neq x = z_1$.
prelimset = $\{f_\epsilon^{-1}\}$.
for $f_\epsilon^{-1} \in$ prelimset:
interimset =MapbackSet($hf_{10}f_1f_\epsilon^{-1}$, **bp**, 1)
outputset = [].
$z_1hf_{10}f_1f_\epsilon^{-1} = x = z_1$.
end function
interimset = [].
Add f_ϵ^{-1} to outputset.
end for
outputset = $\{f_\epsilon^{-1}\}$.
end function
interimset = $\{f_\epsilon^{-1}\}$.
Add $f_1f_\epsilon^{-1}$ to outputset.
return outputset = $\{f_1f_\epsilon^{-1}\}$.
end function
interimset = $\{f_1f_\epsilon^{-1}\}$.
Add $f_{10}f_1f_\epsilon^{-1}$ to outputset.

for $f_1 \in$ prelimset:
interimset =MapbackSet(hf_1, **bp**, 1)
outputset = [].
$z_1gf_1 = 10x \neq x = z_1$.
prelimset = $\{f_1\}$.
for $f_1 \in$ prelimset:
interimset =MapbackSet(hf_1f_1, **bp**, 1)
outputset = [].
$z_1hf_1f_1 = 1x \neq x = z_1$.
prelimset = $\{f_\epsilon^{-1}\}$.
for $f_\epsilon^{-1} \in$ prelimset:
interimset =MapbackSet($hf_1f_1f_\epsilon^{-1}$, **bp**, 1)
outputset = [].
$z_1hf_1f_1f_\epsilon^{-1} = x = z_1$.

end function
`interimset` $= []$.
Add f_ϵ^{-1} to `outputset`.
end for
`outputset` $= \{f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_\epsilon^{-1}\}$.
Add $f_1 f_\epsilon^{-1}$ to `outputset`.
return `outputset` $= \{f_1 f_\epsilon^{-1}\}$.
end function
`interimset` $= \{f_1 f_\epsilon^{-1}\}$.
Add $f_1 f_1 f_\epsilon^{-1}$ to `outputset`.

return `outputset` $= \{f_{10} f_1 f_\epsilon^{-1}, f_1 f_1 f_\epsilon^{-1}\}$.
end function

We observe that indeed $z_1 h f_{\beta_1}^{\eta_1} \dots f_{\beta_k}^{\eta_k} = x = z_1$ for all $f_{\beta_1}^{\eta_1} \dots f_{\beta_k}^{\eta_k} \in \textsc{MapbackSet}(h, \mathbf{bp}, 1)$.

onicgenerator}

Definition 5.3.8. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $z \in \partial \mathcal{P}_{g,d}$. Suppose $z = \alpha x$ (for some $\alpha = e_1 \dots e_n \in E(R)^*$) and $zg = \beta x$ (for some $\beta = e'_1 \dots e'_m \in E(R)^*$). Let $\gamma = e_1 \dots e_k \in E(R)^*$ be the largest common prefix of α and β. Then we define the *damage* of z associated to g as follows:

$$\text{damage}_g(z) = (n-k) + (m-k).$$

We define the *damage* of the rearrangement g as follows:

$$\text{damage}(g) = \sum_{z \in \mathcal{P}_{g,d}} \text{damage}_g(z).$$

Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Using Algorithm 5.2.2 and Definition 5.3.8, the following algorithm finds an optimized word $f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} \in W(\mathcal{X})$ (for some $\alpha_1, \dots, \alpha_N \in E(R)^*$ and $\eta_1, \dots, \eta_N \in \{\pm 1\}$) such that $g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = I$:

{godflesh} **Algorithm 5.3.9** Normal form of a rearrangement g of X.

Require: g is a rearrangement of X with minimal bipartition $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$.

```
1: List ∂P_{g,d}.
2: Define bp = depth(∂P_{g,d}) = (z_1, ..., z_d).
3: function NormalForm(g, bp)
4:     Set outputword = I.
5:     for i = 1, ..., n do
6:         Set h = g outputword.
7:         if z_i h ≠ z_i then
8:             interimset = MapbackSet(g, bp, i)
9:             Choose interimword from interimset such that interimword has the
               least damage.
10:            Append interimword to outputword.
11:        end if
12:    end for
13:    return outputword.
14: end function
```

{useless}

Lemma 5.3.10. *Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let* $\textsc{NormalForm}(g) = f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N}$ *(for some $N \in \mathbb{N}$). Then $g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = I$.*

Proof. Let X be the limit space of the F replacement system. Let g be a rearrangement of X. Let $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ be the minimal bipartition for g. Let $\textsc{Factorization}(g) = f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N}$ (for some $N \in \mathbb{N}$). Let $\mathbf{bp} \supseteq \operatorname{depth}(\partial \mathcal{P}_{g,d}) = (z_1, \dots z_d)$. Let us define the following functions:

$$
\begin{aligned}
& h_{z_1} \in \textsc{MapbackSet}(g, \mathbf{bp}, 1) \\
& \qquad \text{such that } h_{z_1} \text{ has the least damage,} \\
& h_{z_2} \in \textsc{MapbackSet}(g h_{z_1}, \mathbf{bp}, 2) \\
& \qquad \text{such that } h_{z_d} \text{ has the least damage,} \\
& \qquad \vdots \\
& h_{z_d} \in \textsc{MapbackSet}(g h_{z_1} \dots h_{z_{d-1}}, \mathbf{bp}, d) \\
& \qquad \text{such that } h_{z_d} \text{ has the least damage.}
\end{aligned}
$$

We will prove that $f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = h_{z_1} \dots h_{z_d}$. Let us implement Algorithm 5.3.9. Suppose that, having completed step $i-1$ of the for loop, `outputword` $= h_{z_1} \dots h_{z_{i-1}}$. Let us implement step i of the for loop: By Line 8, `interimset` $= \textsc{MapbackSet}(g h_{z_1} \dots h_{z_{i-1}}, \mathbf{bp}, i)$. By Line 9, we choose h_{z_i} from `interimset` such that h_{z_i} has the least damage. By

Line 10, `outputword` $= h_{z_1} \dots h_{z_i}$. It follows that, having completed step d of the loop, `outputword` $= h_{z_1} \dots h_{z_d}$.

Then, by Lemma 5.3.4 (2),

$$\begin{gathered} z_1 g h_{z_1} = z_1 \\ z_1 g h_{z_1} h_{z_2} = z_1, z_2 g h_{z_1} h_{z_2} = z_2 \\ \vdots \\ z_1 g h_{z_1} \dots h_{z_d} = z_1, \dots, z_d g h_{z_1} \dots h_{z_{d-1}} = z_d. \end{gathered}$$

And, by Lemma 5.3.5,

$$\begin{gathered} \partial \mathcal{P}_{gh_{z_1},d} \subseteq \partial \mathcal{P}_{g,d} \\ \partial \mathcal{P}_{gh_{z_1}h_{z_2},d} \subseteq \partial \mathcal{P}_{gh_{z_1},d} \\ \vdots \\ \partial \mathcal{P}_{gh_{z_1}\dots h_{z_d},d} \subseteq \partial \mathcal{P}_{gh_{z_1}\dots h_{z_{d-1}},d}. \end{gathered}$$

Hence,

$$\partial \mathcal{P}_{gh_{z_1}\dots h_{z_d},d} \subseteq \partial \mathcal{P}_{g,d}.$$

Let $C(\alpha) \in \mathcal{P}_{gh_{z_1}\dots h_{z_d},d}$ (for some $\alpha \in E(R)^*$). Then $v_\alpha = z_i$ and $w_\alpha = z_j$ (for some $i \leq i, j \leq d$). Hence $(v_\alpha) g h_{z_1} \dots h_{z_d} = v_\alpha$ and $(w_\alpha) g h_{z_1} \dots h_{z_d} = w_\alpha$. Therefore, by Lemma 4.4.3, $C(\alpha) g h_{z_1} \dots h_{z_d} = C(\alpha)$. Hence, $g h_{z_1} \dots h_{z_d}$ acts as the identity map $C(\alpha)$. Since this is true for all cells $C(\alpha) \in \mathcal{P}_{gh_{z_1}\dots h_{z_d},d}$, we can conclude that

$$g f_{\alpha_1}^{\eta_1} \dots f_{\alpha_N}^{\eta_N} = g h_{z_1} \dots h_{z_d} = I.$$

□

{moonchild}

Definition 5.3.11. Let $\mathcal{G}$ be a group. Let $\mathcal{X}$ be a generating set for $\mathcal{G}$. The *normal form* of an element $g \in \mathcal{G}$ is a uniquely determined word $w \in W(\mathcal{X})$ such that $g = w$.

The following result proves that the output of Algorithm 5.3.9 is the normal form of an element of F:

Lemma 5.3.12. *Let $w, w' \in W(\mathcal{X})$ such that $w = w'$ in F. Then* NormalForm$(w) =$ NormalForm w'.

Proof. Let $w, w' \in W(\mathcal{X})$ such that $w = w'$ in F. By Proposition 5.3.1, w and w' are equal to the same rearrangement $g \in F$. By Lemma 3.3.28, the minimal bipartition $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$ for g is unique. This implies that the input to Algorithm 5.3.9 is unique. Hence NormalForm$(w) =$ NormalForm w'. □

Example 5.3.13. Recall the rearrangement g of X from Example 5.1.17. Let us now apply Algorithm 4 to compute NormalForm(g):

Require: g defined above.
$\partial\mathcal{P}_{g,d} = \{000x, 00x, 0x, x\}$.
$\mathbf{bp} = \text{depth}\,\partial\mathcal{P}_{g,d} = (z_1 = x, z_2 = 0x, z_3 = 00x, z_4 = 000x)$.
function NormalForm(g, **bp**)
Applying for loop for $i = 1, \ldots, 4$**:**
`outputword` $= I$.

$i = 1$.
$h = g\,$`outputword` $= g$.
$z_1 h = 111x \neq x = z_1$.
`interimset` =MapbackSet(h, **bp**, 1).
return `interimset` $= \{f_{11}^{-1} f_1^{-1} f_\epsilon^{-1}, f_{11}^{-1} f_\epsilon^{-1} f_\epsilon^{-1}, f_1^{-1} f_1^{-1} f_\epsilon^{-1}, f_1^{-1} f_\epsilon^{-1} f_\epsilon^{-1}, f_\epsilon^{-1} f_1^{-1} f_\epsilon^{-1}, f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}\}$.
Check damage of all $f_{\beta_1}^{\eta_1} \ldots f_{\beta_k}^{\eta_k} \in \textsc{MapbackSet}(h, \mathbf{bp}, 1)$.
damage $f_{11}^{-1} f_1^{-1} f_\epsilon^{-1} = 6$.
damage $f_{11}^{-1} f_\epsilon^{-1} f_\epsilon^{-1} = 4$.
damage $f_1^{-1} f_1^{-1} f_\epsilon^{-1} = 6$.
damage $f_1^{-1} f_\epsilon^{-1} f_\epsilon^{-1} = 2$.
damage $f_\epsilon^{-1} f_1^{-1} f_\epsilon^{-1} = 4$.
damage $f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1} = 0$.
Choose $f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}$ and append to `outputword`.
`outputword` $= f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}$.

$i = 2$.
$h = g\,$`outputword` $= g f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}$.
$z_2 h = 0x = z_2$.

$i = 3$.
$h = g\,$`outputword` $= g f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}$.
$z_3 h = 00x = z_3$.

$i = 4$.
$h = g\,$`outputword` $= g f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}$.
$z_4 h = 000x = z_4$.

For loop ends.
return `outputword` $= f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1}$.
end function

We observe that indeed $g f_\epsilon^{-1} f_\epsilon^{-1} f_\epsilon^{-1} = I$. Observe that this word is significantly shorter than the word found in Example 5.1.20. This is illustrated in Figure 5.2.

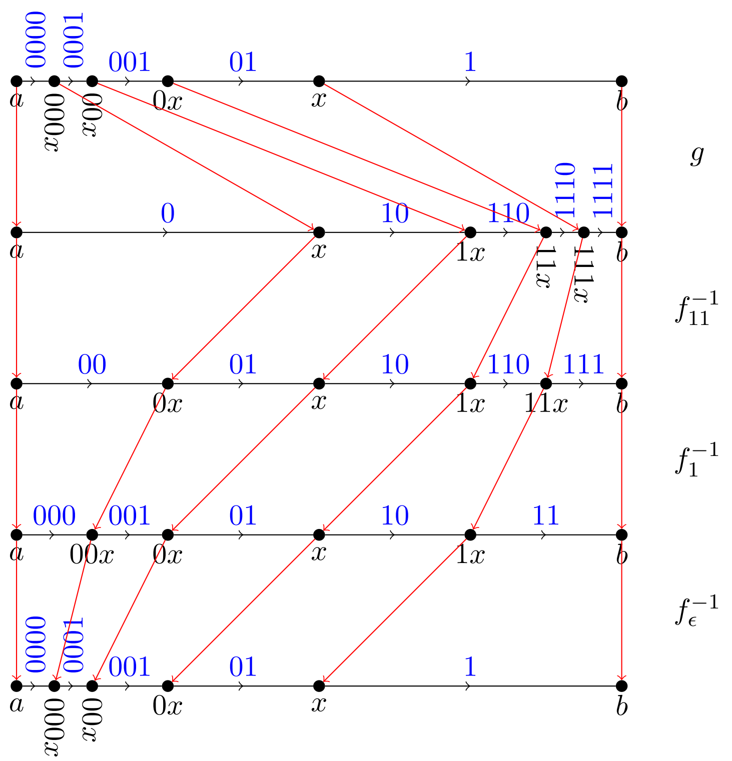


Figure 5.2: $gf_\epsilon^{-1}f_\epsilon^{-1}f_\epsilon^{-1}$ {detachment}

Example 5.3.14. Recall the rearrangement h of X from Example 5.1.18. Let us now apply Algorithm 4 to compute $\textsc{NormalForm}(h)$:

Require: h defined above.

$\partial\mathcal{P}_{h,d} = \{0x, x, 10x, 1x\}$.

$\mathbf{bp} = \text{depth}\, \partial\mathcal{P}_{h,d} = (z_1 = x, z_2 = 0x, z_3 = 1x, z_4 = 10x)$.

function $\textsc{NormalForm}(h,\ \mathbf{bp})$

Applying for loop for $i = 1, \dots, 4$**:**

`outputword` $= I$.

$i = 1$.

$h = h\,$`outputword` $= h$.

$z_1 h = 100x \neq x = z_1$.

`interimset` $= \textsc{MapbackSet}(h,\ \mathbf{bp},\ 1)$.

return `interimset` $= \{f_{10}f_1f_\epsilon^{-1}, f_1f_1f_\epsilon^{-1}\}$.

Check damage of all $f_{\beta_1}^{\eta_1} \dots f_{\beta_k}^{\eta_k} \in \textsc{MapbackSet}(h, \mathbf{bp}, 1)$.

damage $f_{10}f_1f_\epsilon^{-1} = 0$.

damage $f_1f_1f_\epsilon^{-1} = 2$.

Choose $f_{10}f_1f_\epsilon^{-1}$ and append to `outputword`.
`outputword` $= f_{10}f_1f_\epsilon^{-1}$.

$i = 2$.
$h = h\,$`outputword` $= hf_{10}f_1f_\epsilon^{-1}$.
$z_2h = 0x = z_2$.

$i = 3$.
$h = h\,$`outputword` $= hf_{10}f_1f_\epsilon^{-1}$.
$z_3h = 1x = z_3$.

$i = 4$.
$h = h\,$`outputword` $= hf_{10}f_1f_\epsilon^{-1}$.
$z_4h = 10x = z_4$.

For loop ends.
return `outputword` $= f_{10}f_1f_\epsilon^{-1}$.
end function

We observe that indeed $hf_{10}f_1f_\epsilon^{-1} = I$. Observe that this word is the same word found in Example 5.1.21. This is illustrated in Figure 5.3.

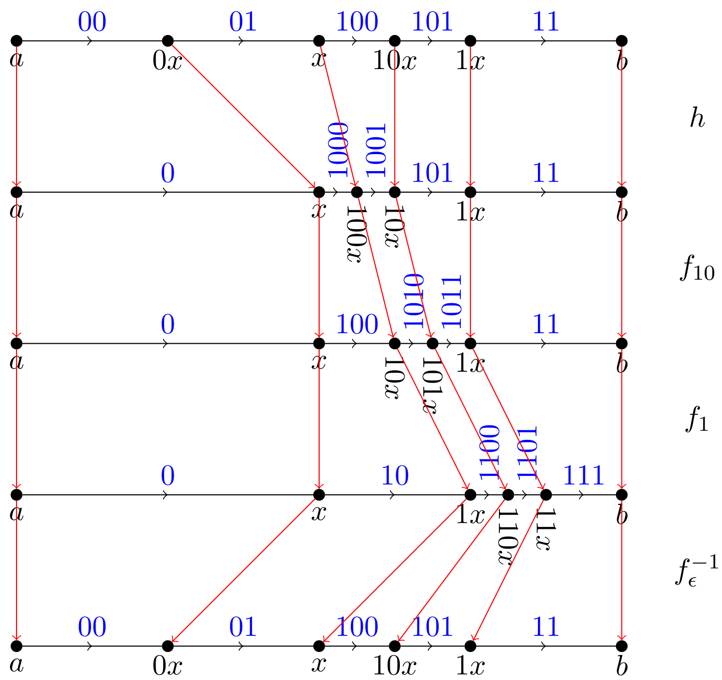


Figure 5.3: $hf_{10}f_1f_\epsilon^{-1}$

{preservationb

5.4 A Note on Rotation Distance {5.4}

Recall from Lemma 3.2.2 that cellular partitions of X are characterised by complete antichains. Then every rearrangement of X (i.e., an element of Thompson's group F) can be defined as a bijection between complete antichains of the same cardinality. Since complete antichains in $\{0,1\}^*$ can be expressed as finite rooted binary trees, a rearrangement of X defines a transformation between two binary rooted trees with the same number of nodes.

A rotation in a binary tree is a local restructuring of the tree, executed by collapsing an internal edge of the tree to a point, thereby obtaining a node with three children, and then re-expanding the node of order three in the alternative way. The rotation distance between a pair of trees with the same number of nodes is the minimum number of rotations needed to convert one tree into another. There has been a great deal of interest in the problems: what is the maximum rotation distance between any pair of n-node binary trees? Is there a polynomial time algorithm (in the number of nodes of the trees) to determine the rotation distance between a given pair of trees? See [16], [29] for bounds; [10], [11], [12], [13], [24] for results about restricted rotation distance, [14], [1], [25], [26] for approximation results; and [15] for results about tractability.

Observe that the action of an element $f_\alpha \in \mathcal{X}$ on $[0,1]$ is equivalent to a single rotation of a binary rooted tree. Using this fact, in Proposition 2.9 of [18], Dehornoy shows that the length of an element of F with respect to the generating set $\mathcal{X}$ is in fact equal to the rotation distance between two binary rooted trees. Dehornoy has left finding explicit combinatorial methods for computing this length as an open question. We conjecture that the normal form of an element of F given by Algorithm 5.3.9 provides a shortest-length sequence of rotations taking one tree to the other. Moreover, Algorithm 5.3.9 provides an explicit combinatorial method to compute this normal form. Further, while Algorithm 5.3.9 runs in exponential time, in all examples computed we have found that there have been easy reductions which enable us to carry out the calculations by hand. Thus, we wonder if the algorithm admits simplifications into a polynomial time algorithm (or at least, polynomial expected time), and answering that question represents work for the future.

5.5 An Additional Algorithm

{5.5}

We have found the following algorithm useful in our calculations. It is included here for completeness:

{digitalboy} **Algorithm 5.5.1** Factorization set of a rearrangement g

Require: g is a rearrangement of X with minimal bipartition $(\mathcal{P}_{g,d}, \mathcal{P}_{g,r})$

List $\partial\mathcal{P}_{g,d}$.

Order $\partial\mathcal{P}_{g,d}$ by depth: $\mathbf{bp} = (z_1, \ldots, z_d)$.

```
function FactorizationSet(g, bp)
    Set counter i = 1.
    while z_i g = z_i and i ≤ n do
        Set i = i + 1.
    end while
    if i > n then
        return []
    else
        prelimset = MapbackSet(g, bp, i)
        outputset = I
        for h in prelimset do
            interimset = FactorizationSet(g, bp)
            for k in interimset do
                Append kh^{-1} to outputset.
            end for
        end for
        return outputset.
    end if
end function
```

Chapter 6

A Presentation for Richard Thompson's Group F

{f-presentatio

In Chapter 5, we developed a combinatorial algorithm to express an arbitrary element of Richard Thompson's group F in *normal form* in terms of the generating set in Proposition 5.2.1. In this chapter, we will provide a combinatorial proof that the generating set in Proposition 5.2.1 and the set of relations in Proposition 5.3.1 provide a presentation for Richard Thompson's group F.

Let us recall the following definitions from the previous chapters:

- The set $\Omega = \{0, 1\}^\omega$ is the set of infinite sequences of 0 and 1, and called the *symbol space*.
- The set $E(R)^* = \{0, 1\}^*$ is the set of finite words of 0 and 1.
- The equivalence relation $\sim$ is such that $\gamma 0\bar{1} \sim \gamma 1\bar{0}$ for all $\gamma \in E(R)^*$.
- There exists a one to one correspondence between the unit interval $[0, 1]$ and the quotient $\Omega/\sim$.
- A *cell* $C(\gamma)$ is a subinterval of $[0, 1]$, consisting of the equivalence classes of all infinite sequences from Ω with the prefix γ.
- The *gluing vertex* γx is the center point of the cell $C(\gamma)$. It denotes the image of the equivalence class $\{\gamma 0\bar{1}, \gamma 1\bar{0}\}$ in $\Omega/\sim$ (and the corresponding dyadic rational in $[0, 1]$).
- The complete set of gluing vertices in $[0, 1]$ is $\mathcal{GV} = \{a = \{\bar{0}\}, b = \{\bar{1}\}, \gamma x \mid \gamma \in E(R)^*\}$.

6.1 A New Group

{6.1}

Let us define a new group combinatorially as follows:

$$\dot{F} = \langle\, \dot{\mathcal{X}} \mid \dot{\mathcal{R}}\,\rangle$$

where $\dot{\mathcal{X}}$ is a set of generators

$$\dot{\mathcal{X}} = \left\{ \dot{f}_\alpha \mid \alpha \in E(R)^* \right\}$$

and $\dot{\mathcal{R}}$ is a set of relations

$$\begin{aligned} \dot{\mathcal{R}} = \{R1: &\quad \dot{f}_\beta^{\dot{f}_\alpha} = \dot{f}_\beta \text{ if } \alpha \perp \beta, \\ R2: &\quad \dot{f}_{\alpha 0}^{\dot{f}_\alpha} = \dot{f}_\alpha \dot{f}_{\alpha 1}^{-1}, \\ R3: &\quad \dot{f}_{\alpha 00\gamma}^{\dot{f}_\alpha} = \dot{f}_{\alpha 0\gamma}, \\ R4: &\quad \dot{f}_{\alpha 01\gamma}^{\dot{f}_\alpha} = \dot{f}_{\alpha 10\gamma}, \\ R5: &\quad \dot{f}_{\alpha 1\gamma}^{\dot{f}_\alpha} = \dot{f}_{\alpha 11\gamma}\}, \end{aligned}$$

(for some $\alpha, \beta, \gamma \in E(R)^*$).

Consider the set functions $\zeta^*\colon \dot{\mathcal{X}} \to \dot{F}$ and $\bar{\zeta}^*\colon \dot{\mathcal{X}} \to F$, induced by the sets of rules $\dot{f}_\alpha \mapsto \dot{f}_\alpha$ and $\dot{f}_\alpha \mapsto f_\alpha$ respectively. These functions induce group homomorphisms $\zeta\colon \mathrm{Free}(\dot{\mathcal{X}}) \to \dot{F}$ and $\bar{\zeta}\colon \mathrm{Free}(\dot{\mathcal{X}}) \to F$ (where $\mathrm{Free}(\dot{\mathcal{X}})$ is the free group on $\dot{\mathcal{X}}$). Observe that $\ker(\zeta) \subseteq \ker(\bar{\zeta})$ and $\dot{\mathcal{X}}$ generates $\dot{F}$. Then there is an induced group homomorphism $\chi\colon \dot{F} \to F$ such that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Free}(\dot{\mathcal{X}}) \arrow[r, "\zeta"] \arrow[dr, "\bar{\zeta}"'] & \dot{F} \arrow[d, "\chi"] \arrow[dl, phantom, "\circlearrowright"] \\ & F \end{tikzcd}$$

As the maps f_α generate F, we observe that χ is surjective.

This brings us to the main result of this chapter:

{truecolours}

Theorem 6.1.1. *The group $\dot{F}$ is isomorphic to Richard Thompson's group F.*

In order to prove Theorem 6.1.1, it is sufficient to show that χ is injective. We will prove this result by developing a combinatorial algorithm to show that $\ker(\chi) = \{I\}$.

6.2 A Combinatorial Action

Let us compose χ with the "action" homomorphism $\rho\colon F \to \mathrm{Sym}(\mathcal{GV})$. This gives us an action of $\dot{F}$ on $\mathcal{GV}$ which is determined by the action of the generators:

{pinkysdream}

Lemma 6.2.1. *Let $\dot{f}_\alpha \in \dot{\mathcal{X}}$ (for some $\alpha \in E(R)^*$). Then $\dot{f}_\alpha$ acts on $z \in \mathcal{GV}$ as follows:*

1. *If $z = a$, $(a)\dot{f}_\alpha = a$,*

2. *If $z = b$, $(b)\dot{f}_\alpha = b$,*

3. *If $z = \gamma x$, (for some $\gamma \in E(R)^*$)*

$$(\gamma x)\dot{f}_\alpha = \begin{cases} \alpha 1 x & \textit{if } \gamma = \alpha, \\ \alpha x & \textit{if } \gamma = \alpha 0, \\ \alpha 0 \lambda x & \textit{if } \gamma = \alpha 00\lambda, \\ \alpha 10 \lambda x & \textit{if } \gamma = \alpha 01 \lambda, \\ \alpha 11 \lambda x & \textit{if } \gamma = \alpha 1 \lambda, \\ \gamma x & \textit{if } \alpha \not\preceq \gamma, \end{cases}$$

(for some $\lambda \in E(R)^*$).

Proof. The statement follows immediately from the action of F on $\mathcal{GV}$. □

Observe that, for an arbitrary element $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$), the action on $\gamma x \in \mathcal{GV}$ is as follows:

$$(\gamma x) g = (((\gamma x)\dot{f}_{\alpha_1}^{\eta_1})\dot{f}_{\alpha_2}^{\eta_2}) \dots \dot{f}_{\alpha_n}^{\eta_n}.$$

Observe that the function in Lemma 6.2.1 is in contrast to the partial function below:

Definition 6.2.2. Let $\dot{f}_\alpha \in \dot{\mathcal{X}}$ (for some $\alpha \in E(R)^*$), then $\dot{f}_\alpha$ induces a partial function on $E(R)^*$, defined as follows: Let $\beta \in E(R)^*$, then

$$\beta \bullet \dot{f}_\alpha = \begin{cases} \text{undefined} & \text{if } \beta \preceq \alpha, \\ \beta & \text{if } \beta \perp \alpha, \\ \alpha 0 \delta & \text{if } \beta = \alpha 00 \delta, \\ \alpha 10 \delta & \text{if } \beta = \alpha 01 \delta, \\ \alpha 11 \delta & \text{if } \beta = \alpha 1 \delta \end{cases}$$

(for some $\delta \in E(R)^*$).

This function has been shown to be a partial action by Dehornoy in [17]. This partial action has, in turn, been extended to a full action by Bleak, Matucci and Neunhöffer in [5].

Definition 6.2.3. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in E(R)^*$).

1. We define the *prefix chain* for g to be the ordered list:

$$\operatorname{prefixchain}(g) = \left(I, \dot{f}_{\alpha_1}^{\eta_1}, \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}, \dots, \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n}\right).$$

2. We define the *depth chart* of γx associated to g (denoted by $\operatorname{depthchart}_g(\gamma x)$) to be ordered list depicting the action of each element in the prefix chain for g on γx:

$$\operatorname{depthchart}_g(\gamma x) = \left(\gamma x, (\gamma x)\dot{f}_{\alpha_1}^{\eta_1}, (\gamma x)\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}, \dots, (\gamma x)\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n}\right).$$

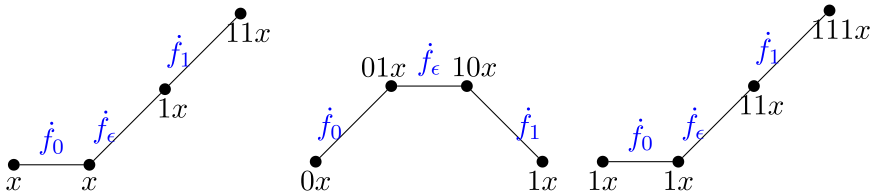
We illustrate this list by plotting the depths of each gluing vertex, as seen in the following example.

3. We say γx is *dynamic* under g if $\mathrm{depthchart}_g(\gamma x)$ is not constant.

Example 6.2.4. The depth charts for the action of the function $\dot{f}_0\dot{f}_\epsilon\dot{f}_1$ on the gluing vertices x, $0x$ and $1x$ are as follows:

$$\mathrm{depthchart}_{\dot{f}_0\dot{f}_\epsilon\dot{f}_1}(x) = (x, x, 1x, 11x)\,,$$
$$\mathrm{depthchart}_{\dot{f}_0\dot{f}_\epsilon\dot{f}_1}(0x) = (0x, 01x, 10x, 1x)\,,$$
$$\mathrm{depthchart}_{\dot{f}_0\dot{f}_\epsilon\dot{f}_1}(1x) = (1x, 1x, 11x, 111x)\,.$$

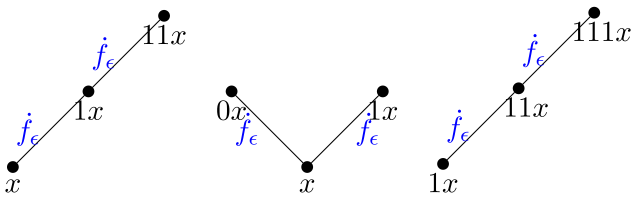
These are illustrated in the following diagram:



We observe that, while $\dot{f}_0\dot{f}_\epsilon\dot{f}_1 = \dot{f}_\epsilon\dot{f}_\epsilon$ (by the Pentagram relation $R2$), the depth charts for the action of $\dot{f}_\epsilon\dot{f}_\epsilon$ on the gluing vertices x, $0x$ and $1x$ are different:

$$\mathrm{depthchart}_{\dot{f}_\epsilon\dot{f}_\epsilon}(x) = (x, 1x, 11x)\,,$$
$$\mathrm{depthchart}_{\dot{f}_\epsilon\dot{f}_\epsilon}(0x) = (0x, x, 1x)\,,$$
$$\mathrm{depthchart}_{\dot{f}_\epsilon\dot{f}_\epsilon}(1x) = (1x, 11x, 111x)\,.$$

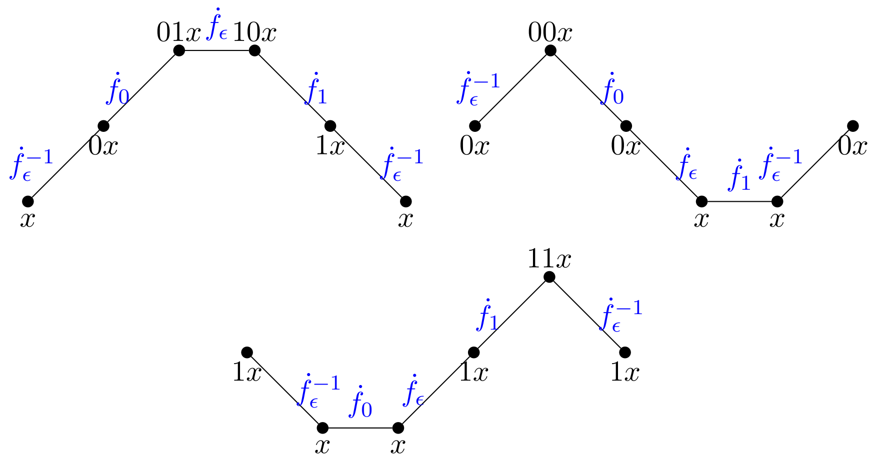
These are illustrated in the following diagram:



The depth charts for the action of the full Pentagram relation $\dot{f}_\epsilon^{-1}\dot{f}_0\dot{f}_\epsilon\dot{f}_1\dot{f}_\epsilon^{-1}$ on the gluing vertices x, $0x$ and $1x$ are as follows:

$$\mathrm{depthchart}_{\dot{f}_\epsilon^{-1}\dot{f}_0\dot{f}_\epsilon\dot{f}_1\dot{f}_\epsilon^{-1}}(x) = (x, 0x, 01x, 10x, 1x, x)\,,$$
$$\mathrm{depthchart}_{\dot{f}_\epsilon^{-1}\dot{f}_0\dot{f}_\epsilon\dot{f}_1\dot{f}_\epsilon^{-1}}(0x) = (0x, 00x, 0x, x, x, 0x)\,,$$
$$\mathrm{depthchart}_{\dot{f}_\epsilon^{-1}\dot{f}_0\dot{f}_\epsilon\dot{f}_1\dot{f}_\epsilon^{-1}}(1x) = (1x, x, x, 1x, 11x, 1x)\,.$$

These are illustrated in the following diagram:



{burundi}

Definition 6.2.5. The reverse of a depth chart is the graph obtained using left-right reflection and replacing each function by its inverse.

{parabol}

Lemma 6.2.6. *Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ and let $\gamma x \in \mathcal{GV}$. Let $\delta x = (\gamma x)g$. Then* $\operatorname{depthchart}_{g^{-1}}(\delta x)$ *is the reverse of* $\operatorname{depthchart}_g(\gamma x)$.

Proof. The statement follows from Definition 6.2.5. □

{wolfparade}

Definition 6.2.7. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). We define the set of boundary points of g to be:

$$B_g = \Big\{ (\beta x) \left(\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_{i-1}}^{\eta_{i-1}} \right)^{-1} \,\Big|\, \begin{array}{l} \beta \preceq \alpha_i 0 \text{ if } \eta_i = +1 \text{ or} \\ \beta \preceq \alpha_i 1 \text{ if } \eta_i = -1, \\ \text{for } i = 1, \dots, n \end{array} \Big\}.$$

Recall from Definition 2.3.10 that there exists a depth order on the set $\mathcal{GV}$. We shall denote the depth order on B_g by the ordered list $\operatorname{depth}(B_g) = (z_1, \dots, z_d)$.

{stripdown}

Lemma 6.2.8. *Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let B_g be the set of boundary points of g. Let $(\mathcal{P}_{g\chi,d}, \mathcal{P}_{g\chi,r})$ be the minimal bipartition for the rearrangement $g\chi \in F$. Then $B_g \supseteq \partial \mathcal{P}_{g\chi,d}$.*

Proof. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let $(\mathcal{P}_{g\chi,d}, \mathcal{P}_{g\chi,r})$ be the minimal bipartition for the rearrangement $g\chi \in F$. We will prove this result by induction on n.

If $n = 0$, $g = I$ and the result is trivially true. If $n = 1$, $g = \dot{f}_{\alpha_1}^{\eta_1}$ and

$$B_g = \Big\{ \beta x \,\Big|\, \begin{array}{l} \beta \preceq \alpha_1 0x \text{ if } \eta_1 = +1 \text{ or} \\ \beta \preceq \alpha_1 1x \text{ if } \eta_1 = -1 \end{array} \Big\}.$$

By Corollary 5.1.4, $B_g = \partial \mathcal{P}_{g\chi,d}$.

Suppose that there exists $m \in \mathbb{N}$ such that the inductive claim holds for $n = m$. We will examine the case when $n = m + 1$. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_m}^{\eta_m}$ and consider $h = g\dot{f}_{\alpha_{m+1}}^{\eta_{m+1}}$. Observe that

$$B_h = B_g \cup \left\{ (\beta x)g^{-1} \;\middle|\; \begin{array}{l} \beta \preceq \alpha_{m+1}0x \text{ if } \eta_{m+1} = +1 \text{ or} \\ \beta \preceq \alpha_{m+1}1x \text{ if } \eta_{m+1} = -1 \end{array} \right\}.$$

Observe that $h\chi = g\chi f_{\alpha_{m+1}}^{\eta_{m+1}}$. Let $(\mathcal{P}_{h\chi,d}, \mathcal{P}_{h\chi,r})$ be the minimal bipartition for the rearrangement $h\chi \in F$, then

$$\partial\mathcal{P}_{h\chi,d} \subseteq \partial\mathcal{P}_{g\chi,d} \cup (\partial\mathcal{P}_{f_{\alpha_{m+1}}^{\eta_{m+1}},d})(g\chi)^{-1}.$$

By our inductive hypothesis,

$$\partial\mathcal{P}_{g\chi,d} \subseteq B_g.$$

Observe that

$$(\partial\mathcal{P}_{f_{\alpha_{m+1}}^{\eta_{m+1}},d})(g\chi)^{-1} = \left\{ (\beta x)g^{-1} \;\middle|\; \begin{array}{l} \beta \preceq \alpha_{m+1}0x \text{ if } \eta_{m+1} = +1 \text{ or} \\ \beta \preceq \alpha_{m+1}1x \text{ if } \eta_{m+1} = -1 \end{array} \right\}.$$

Hence

$$\partial\mathcal{P}_{h\chi,d} \subseteq B_h.$$

This proves the result by induction. ☐

efatoftheland}

Corollary 6.2.9. *Let $g \in W(\dot{\mathcal{X}})$. Then there exists a cellular partition $(\mathcal{P}, (\mathcal{P})g\chi)$ for the rearrangement $g\chi \in F$ such that $B_g = \partial\mathcal{P}$.*

Proof. The statement follows from Lemma 6.2.8. ☐

{savages}

Lemma 6.2.10. *Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let B_g be the set of boundary points of g. Let $\operatorname{depth} B_g = (z_1, \dots z_d)$. Choose z_i such that $z_1, \dots z_{i-1}$ are not dynamic under g, but z_i is dynamic under g. Then*

$$\operatorname{depth}(z_i h) \geq \operatorname{depth}(z_i)$$

for all $h \in \operatorname{prefixchain}(g)$.

Proof. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let B_g be the set of boundary points of g. Let $\operatorname{depth} B_g = (z_1, \dots z_d)$. Choose z_i such that $z_1, \dots z_{i-1}$ are not dynamic under g, but z_i is dynamic under g.

Let $z_i = \gamma x$ (for some $\gamma \in E(R)^*$). By Definition 6.2.7 of B_g, $\{\gamma^\dagger x, \dots, x\} \subseteq B_g$. By our hypothesis, $\{\gamma^\dagger x, \dots, x\} \subseteq \{z_1, \dots, z_{i-1}\}$. Then, $yh = y$ for all $y \in \{\gamma^\dagger x, \dots, x\}$ and $h \in \operatorname{prefixchain}(g)$. Recall that $h\chi \in F$ for all $h \in \operatorname{prefixchain}(g)$. By Corollary 6.2.9, there exists a cellular bipartition $(\mathcal{P}, (\mathcal{P})g\chi)$ for the $g\chi$ such that $B_g = \partial\mathcal{P}$. Then $(\mathcal{P}, (\mathcal{P})h\chi)$ is a cellular bipartition for $h\chi$ for all $h \in \operatorname{prefixchain}(g)$. Then, by Lemma 5.1.9, $z_i h = \beta x$ (for some $\beta \in E(R)^*$) where $\gamma \preceq \beta$ for all $h \in \operatorname{prefixchain}(g)$. This proves the result. ☐

{pertubator}

Lemma 6.2.11. *Let* $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ *(for some* $\alpha_1, \dots, \alpha_n \in E(R)^*$ *and* $\eta_1, \dots, \eta_n \in \{\pm 1\}$*). Let* g' *be obtained from* g *by one of the following methods:*

1. *The application of an appropriate relation from* $\dot{\mathcal{R}}$ *to replace a subword* $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}$ *by either* $\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} (\dot{f}_{\alpha_i}^{\eta_i})^{\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}}$ *or* $(\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}})^{\dot{f}_{\alpha_i}^{\eta_i}} \dot{f}_{\alpha_i}^{\eta_i}$.

2. *The cancellation of* $\dot{f}_{\alpha_i}$ *by an adjacent inverse.*

Then $B_{g'} \subseteq B_g$.

Proof. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let $h = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_{i-1}}^{\eta_{i-1}}$ and $k = \dot{f}_{\alpha_{i+2}}^{\eta_{i+2}} \dots \dot{f}_{\alpha_n}^{\eta_n}$. Then $g = h \dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} k$. Observe that

$$B_g = B_h \cup (B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}}) h^{-1} \cup (B_k)(\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}})^{-1} h^{-1}.$$

Suppose $g' = h \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m} k$ (for some $m = 0, 2, 3$), where $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}$ is obtained from $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}$ by one of the following methods:

1. The application of an appropriate relation from $\dot{\mathcal{R}}$ to replace a subword $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}$ by either $\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} (\dot{f}_{\alpha_i}^{\eta_i})^{\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}}$ or $(\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}})^{\dot{f}_{\alpha_i}^{\eta_i}} \dot{f}_{\alpha_i}^{\eta_i}$.

2. The cancellation of $\dot{f}_{\alpha_i}$ by an adjacent inverse.

Then

$$B_{g'} = B_h \cup (B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}}) h^{-1} \cup (B_k)(\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m})^{-1} h^{-1}.$$

Observe that $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} = \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}$ in $\dot{F}$, and therefore $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}$ and $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}$ have the same action on the set $\mathcal{GV}$ of gluing vertices. Hence

$$(B_k)(\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}})^{-1} h^{-1} = (B_k)(\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m})^{-1} h^{-1}.$$

Hence, to show that $B_{g'} \subseteq B_g$, it sufficces to show that

$$B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} \subseteq B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}}.$$

We have the following possibilities for $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}$:

$\dot{f}_\alpha \dot{f}_\alpha^{-1}$	$\dot{f}_\alpha^{-1} \dot{f}_\alpha$	$\dot{f}_\alpha^\eta \dot{f}_\beta^\mu$	
$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 0}$	$\dot{f}_{\alpha 0} \dot{f}_\alpha$	$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 0}^{-1}$	$\dot{f}_{\alpha 0}^{-1} \dot{f}_\alpha$
$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 00\lambda}^\eta$	$\dot{f}_{\alpha 00\lambda}^\eta \dot{f}_\alpha$	$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 01\lambda}^\eta$	$\dot{f}_{\alpha 01\lambda}^\eta \dot{f}_\alpha$
	$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 1\lambda}^\eta$	$\dot{f}_{\alpha 1\lambda}^\eta \dot{f}_\alpha$	
$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 1}$	$\dot{f}_{\alpha 1} \dot{f}_\alpha$	$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 1}^{-1}$	$\dot{f}_{\alpha 1}^{-1} \dot{f}_\alpha$
$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 0\lambda}^\eta$	$\dot{f}_{\alpha 0\lambda}^\eta \dot{f}_\alpha$	$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 10\lambda}^\eta$	$\dot{f}_{\alpha 10\lambda}^\eta \dot{f}_\alpha$
	$\dot{f}_\alpha^{-1} \dot{f}_{\alpha 11\lambda}^\eta$	$\dot{f}_{\alpha 11\lambda}^\eta \dot{f}_\alpha$	

(for some $\alpha, \beta, \lambda \in E(R)^*$ such that $\alpha \perp \beta$ and $\eta, \mu \in \{\pm 1\}$).

Let us examine the following cases in detail (and leave the proof of other similar cases to the reader):

1. $\boldsymbol{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} = \dot{f}_{\alpha} \dot{f}_{\alpha}^{-1}}$

   In this case $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m} = I$. This gives us $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = \varnothing$ and the result holds trivially.

2. $\boldsymbol{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} = \dot{f}_{\alpha}^{\eta} \dot{f}_{\beta}^{\mu}}$ **(for some $\boldsymbol{\alpha, \beta \in E(R)^*}$ such that $\boldsymbol{\alpha \perp \beta}$ and $\boldsymbol{\eta, \mu \in \{\pm 1\}}$)**

   In this case, we apply relation $R1$ and $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m} = \dot{f}_{\beta}^{\mu} \dot{f}_{\alpha}^{\eta}$. This gives us $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\beta}^{\mu} \dot{f}_{\alpha}^{\eta}} = B_{\dot{f}_{\beta}^{\mu}} \cup \left(B_{\dot{f}_{\alpha}^{\eta}} \right) \dot{f}_{\beta}^{-\mu}$. Choose $\gamma x \in B_{\dot{f}_{\alpha}^{\eta}}$. Then $\gamma \preceq \alpha 0$ or $\gamma \preceq \alpha 1$. Then $\beta \npreceq \gamma$ and, by Lemma 6.2.1, $(\gamma x) \dot{f}_{\beta}^{-\mu} = \gamma x$. Since this is true for all $\gamma x \in B_{\dot{f}_{\alpha}^{\eta}}$, this implies that $\left(B_{\dot{f}_{\alpha}^{\eta}} \right) \dot{f}_{\beta}^{-\mu} = B_{\dot{f}_{\alpha}^{\eta}}$. Hence $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\beta}^{\mu}} \cup B_{\dot{f}_{\alpha}^{\eta}}$.

   Similarly, we can show that $B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}} = B_{\dot{f}_{\alpha}^{\eta}} \cup B_{\dot{f}_{\beta}^{\mu}}$.

   Hence

   $$B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}},$$

   and the result holds.

3. $\boldsymbol{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} = \dot{f}_{\alpha}^{-1} \dot{f}_{\alpha 00 \lambda}^{\eta}}$ **(for some $\boldsymbol{\alpha, \lambda \in E(R)^*}$ and $\boldsymbol{\eta \in \{\pm 1\}}$)**

   Observe that $B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}} = B_{\dot{f}_{\alpha}^{-1} \dot{f}_{\alpha 00 \lambda}^{\eta}} = B_{\dot{f}_{\alpha}^{-1}} \cup \left(B_{\dot{f}_{\alpha 00 \lambda}^{\eta}} \right) \dot{f}_{\alpha}$. Choose $\gamma x \in B_{\dot{f}_{\alpha 00 \lambda}^{\eta}}$. Then $\gamma \preceq \alpha 00 \lambda 0$ or $\gamma \preceq \alpha 00 \lambda 1$. If $\gamma \prec \alpha$, then $(\gamma x) \dot{f}_{\alpha} = \gamma x$. If $\gamma = \alpha$, then $(\gamma x) \dot{f}_{\alpha} = \alpha 1 x \in B_{\dot{f}_{\alpha}^{-1}}$. If $\gamma = \alpha 0$, then $(\gamma x) \dot{f}_{\alpha} = \alpha x \in B_{\dot{f}_{\alpha 0 \lambda}^{\eta}}$. If $\gamma = \alpha 00 \delta$ (for some $\delta \in E(R)^*$ such that $\delta \preceq \lambda 0$ or $\delta \preceq \lambda 1$), then $(\gamma x) \dot{f}_{\alpha} = \alpha 0 \delta x \in B_{\dot{f}_{\alpha 0 \lambda}^{\eta}}$. Since this is true for all $\gamma x \in B_{\dot{f}_{\alpha 00 \lambda}^{\eta}}$, this implies that $B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}} = B_{\dot{f}_{\alpha}^{-1}} \cup B_{\dot{f}_{\alpha 0 \lambda}^{\eta}}$.

   Let us apply relation $R3$ and $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m} = \dot{f}_{\alpha 0 \lambda}^{\eta} \dot{f}_{\alpha}^{-1}$. This gives us $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\alpha 0 \lambda}^{\eta} \dot{f}_{\alpha}^{-1}} = B_{\dot{f}_{\alpha 0 \lambda}^{\eta}} \cup \left(B_{\dot{f}_{\alpha}^{-1}} \right) \dot{f}_{\alpha 0 \lambda}^{-\eta}$. Choose $\gamma x \in B_{\dot{f}_{\alpha}^{-1}}$. Then $\gamma \preceq \alpha 1$. Then $\alpha 0 \lambda \npreceq \gamma$ and, by Lemma 6.2.1, $(\gamma x) \dot{f}_{\alpha 0 \lambda}^{-\eta} = \gamma x$. Since this is true for all $\gamma x \in B_{\dot{f}_{\alpha}^{-1}}$, this implies that $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\alpha 0 \lambda}^{\eta}} \cup B_{\dot{f}_{\alpha}^{-1}}$.

   Hence

   $$B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}},$$

   and the result holds.

4. $\boldsymbol{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} = \dot{f}_{\alpha}^{-1} \dot{f}_{\alpha 0}}$ **(for some $\boldsymbol{\alpha \in E(R)^*}$)**

   Observe that $B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}} = B_{\dot{f}_{\alpha}^{-1} \dot{f}_{\alpha 0}} = B_{\dot{f}_{\alpha}^{-1}} \cup \left(B_{\dot{f}_{\alpha 0}} \right) \dot{f}_{\alpha}$. Choose $\gamma x \in B_{\dot{f}_{\alpha 0}}$. Then $\gamma \preceq \alpha 00$. If $\gamma \prec \alpha$, then $(\gamma x) \dot{f}_{\alpha} = \gamma x$. If $\gamma = \alpha$, then $(\gamma x) \dot{f}_{\alpha} = \alpha 1 x \in B_{\dot{f}_{\alpha}^{-1}}$. If $\gamma = \alpha 0$, then $(\gamma x) \dot{f}_{\alpha} = \alpha x \in B_{\dot{f}_{\alpha}^{-1}}$. If $\gamma = \alpha 00$, then $(\gamma x) \dot{f}_{\alpha} = \alpha 0 x$. Since this is true for all $\gamma x \in B_{\dot{f}_{\alpha 0}}$, this implies that $B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}} = B_{\dot{f}_{\alpha}^{-1}} \cup \{\alpha 0 x\}$.

Let us apply relation $R2$ and $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m} = \dot{f}_\alpha \dot{f}_{\alpha 1}^{-1} \dot{f}_\alpha^{-1}$. This gives us $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_\alpha \dot{f}_{\alpha 1}^{-1} \dot{f}_\alpha^{-1}} = B_{\dot{f}_\alpha} \cup \left(B_{\dot{f}_{\alpha 1}^{-1}} \right) \dot{f}_\alpha^{-1} \cup \left(B_{\dot{f}_\alpha^{-1}} \right) \dot{f}_{\alpha 1} \dot{f}_\alpha^{-1}$. Choose $\gamma x \in B_{\dot{f}_{\alpha 1}^{-1}}$. Then $\gamma \preceq \alpha 11$. If $\gamma \prec \alpha$, then $(\gamma x) \dot{f}_\alpha^{-1} = \gamma x$. If $\gamma = \alpha$, then $(\gamma x) \dot{f}_\alpha^{-1} = \alpha 0 x \in B_{\dot{f}_\alpha}$. If $\gamma = \alpha 1$, then $(\gamma x) \dot{f}_\alpha^{-1} = \alpha x \in B_{\dot{f}_\alpha}$. If $\gamma = \alpha 11$, then $(\gamma x) \dot{f}_\alpha^{-1} = \alpha 1 x$. Similarly, choose $\delta x \in B_{\dot{f}_\alpha^{-1}}$. Then $\delta \preceq \alpha 1$. If $\delta \prec \alpha$, then $(\delta x) \dot{f}_{\alpha 1} \dot{f}_\alpha^{-1} = \delta x$. If $\delta = \alpha$, then $(\delta x) \dot{f}_{\alpha 1} \dot{f}_\alpha^{-1} = \alpha 0 x \in B_{\dot{f}_\alpha}$. If $\delta = \alpha 1$, then $(\delta x) \dot{f}_{\alpha 1} \dot{f}_\alpha^{-1} = \alpha 1 x$. Since this is true for all $\gamma x \in B_{\dot{f}_\alpha^{-1}}$ and $\delta x \in B_{\dot{f}_\alpha^{-1}}$, this implies that $B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_\alpha} \cup \{\alpha 1 x\}$.

Hence

$$B_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_m}^{\mu_m}} = B_{\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}},$$

and the result holds.

It follows that $B_{g'} \subseteq B_g$ in all cases, and this completes the proof. □

{hausswolff}

Lemma 6.2.12. *Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let $\operatorname{depth}(B_g) = (z_1, \dots, z_d)$. Suppose z_i is not dynamic under g for all $i = 1, \dots, d$. Then $g = I$ in $W(\dot{\mathcal{X}})$.*

Proof. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let $\operatorname{depth}(B_g) = (z_1, \dots, z_d)$.

We will prove this result by proving its contrapositive: Suppose $g \neq I$ in $W(\dot{\mathcal{X}})$. Then $n \geq 1$. Let us examine the gluing $\alpha_1 x$. By Definition 6.2.7, $\alpha_1 x \in B_g$. By Lemma 6.2.1, $\operatorname{depth}(\alpha_1 x \dot{f}_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\alpha_1 x) + 1$. Since $\dot{f}_{\alpha_1}^{\eta_1} \in \operatorname{prefixchain}(g)$, this implies that $\operatorname{depthchart}_g(\alpha_1 x)$ is not constant. Hence $\alpha_1 x$ is dynamic under g. This proves the result. □

6.3 Casework Lemmas

Let $g \in W(\dot{\mathcal{X}})$. Let B_g be the set of boundary points of g. Let $\operatorname{depth} B_g = (z_1, \dots z_d)$. Choose z_i such that $z_1, \dots z_{i-1}$ are not dynamic under g, but z_i is dynamic under g. In this section, we will present a casework argument to "reduce" the maximum depth achieved in $\operatorname{depthchart}_g(z_i)$. Since the following results are quite lengthy, we have only provided the full proof of the first result and the rest are left to the reader.

{adult}

Lemma 6.3.1. *Let $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in E(R)^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in E(R)^*$) such that the following conditions are satisfied:*

1. $\operatorname{depth}(\gamma x) = \operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2})$,
2. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in E(R)^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 0, 2, 3$ such that $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied.

3. $\operatorname{depth}(\gamma x) = \operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k})$,

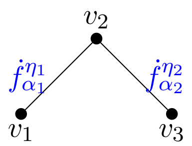
4. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_i}^{\mu_i}) = \operatorname{depth}(\gamma x) - 1$ *for* $1 \le i \le k-1$.

Proof. Let $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in E(R)^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in E(R)^*$) such that the following conditions are satisfied:

1. $\operatorname{depth}(\gamma x) = \operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2})$,

2. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1$.

Let us illustrate $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$ and $v_3 = \left(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\right) \dot{f}_{\alpha_2}^{\eta_2}$.

We will examine all cases which arise. Let us first divide into cases according to the exponents η_1 and η_2:

1. $\eta_1 = +1$ and $\eta_2 = +1$,

2. $\eta_1 = +1$ and $\eta_2 = -1$,

3. $\eta_1 = -1$ and $\eta_2 = +1$,

4. $\eta_1 = -1$ and $\eta_2 = -1$.

Observe that cases (3) and (4) are the inverses of cases (1) and (2). Then, by Lemma 6.2.6, we only need to examine cases (1) and (2).

Let us now examine these cases:

1. $\boldsymbol{\eta_1 = +1}$ **and** $\boldsymbol{\eta_2 = +1}$

   Let us set $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_{\alpha}$ (for some $\alpha \in E(R)^*$). By Lemma 6.2.1, in order for condition (2) to be satisfied

   $$\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1,$$

   we have the following choices for γx:

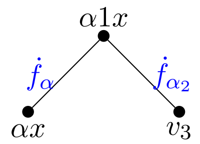
   1.1. $\gamma x = \alpha x$,

   1.2. $\gamma x = \alpha 1 \gamma x$ (for some $\gamma \in E(R)^*$).

   Let us examine each of these cases:

   1.1. $\boldsymbol{\gamma x = \alpha x}$

   We observe that $\alpha x \dot{f}_{\alpha} = \alpha 1 x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:

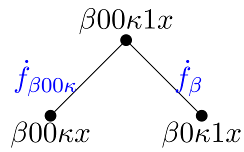


where $v_3 = \alpha 1x\dot{f}_{\alpha_2}$.

By Lemma 6.2.1, in order for condition (1) to be satisfied

$$\text{depth}(\alpha x) = \text{depth}(\alpha x\dot{f}_\alpha\dot{f}_{\alpha_2}),$$

$\alpha 1x = \beta 00\tau x$ (for some $\tau \in E(R)^*$). We can conclude that $\beta 00$ is a prefix of α. Let $\alpha = \beta 00\kappa$ for some (for some $\kappa \in E(R)^*$). Then $\dot{f}_\alpha^{\eta_1}\dot{f}_\beta^{\eta_2} = \dot{f}_{\beta 00\kappa}\dot{f}_\beta$. Observe that $\gamma x\dot{f}_{\beta 00\kappa}\dot{f}_\beta = \beta 00\kappa x\dot{f}_{\beta 00\kappa}\dot{f}_\beta = \beta 00\kappa 1x\dot{f}_\beta = \beta 0\kappa 1x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_\beta^{\eta_2}}(\gamma x)$ is then as follows:



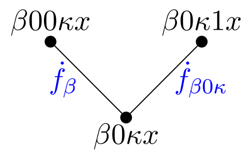
Conjugating these terms using relation $R3$ gives us $\dot{f}_{\beta 00\kappa}\dot{f}_\beta = \dot{f}_\beta\left(\dot{f}_{\beta 00\kappa}\right)^{\dot{f}_\beta} = \dot{f}_\beta\dot{f}_{\beta 0\kappa}$. Observe that $\gamma x\dot{f}_\beta\dot{f}_{\beta 0\kappa} = \beta 00\kappa x\dot{f}_\beta\dot{f}_{\beta 0\kappa} = \beta 0\kappa x\dot{f}_{\beta 0\kappa} = \beta 0\kappa 1x$, and hence conditions (3) and (4) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\beta 00\kappa x\dot{f}_\beta\dot{f}_{\beta 0\kappa}) = \text{depth}(\beta 0\kappa 1x)\\ &= \text{depth}(\beta 00\kappa x) = \text{depth}(\gamma x)\end{aligned}$$

and

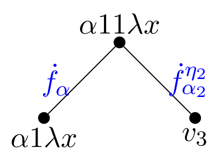
$$\begin{aligned}\text{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\beta 00\kappa x\dot{f}_\beta) = \text{depth}(\beta 0\kappa x)\\ &= \text{depth}(\beta 00\kappa x) - 1 = \text{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.2. $\boldsymbol{\gamma x = \alpha 1\lambda x}$ **(for some** $\boldsymbol{\lambda \in E(R)^*}$**)**

We observe that $\alpha 1\lambda x\dot{f}_\alpha = \alpha 11\lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



where $v_3 = \alpha 11\lambda x \dot{f}_{\alpha_2}^{\eta_2}$.

By Lemma 6.2.1, in order for condition (1) to be satisfied

$$\mathrm{depth}(\gamma x) = \mathrm{depth}(\gamma x \dot{f}_\alpha \dot{f}_{\alpha_2}^{\eta_2}),$$

we have the following possibilities for $\dot{f}_{\alpha_2}^{\eta_2}$:

1.2.1. $\alpha 11\lambda x = \alpha 11\delta 0x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}$,

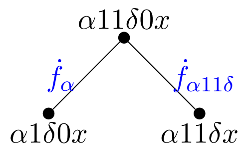
1.2.2. $\alpha 11\lambda x = \alpha 11\delta 00\tau x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}$,

1.2.3. $\alpha 11\lambda x = \beta 00\delta 11\lambda x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$,

Let us examine each of these cases separately:

1.2.1. $\boldsymbol{\alpha 11\lambda x = \alpha 11\delta 0x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}}$**.**

In this case $\dot{f}_\alpha^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 11\delta}$. Observe that $\gamma x \dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \alpha 1\delta 0x \dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \alpha 11\delta 0x \dot{f}_{\alpha 11\delta} = \alpha 11\delta x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



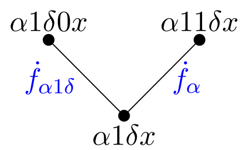
Conjugating these terms using relation $R5$ gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \left(\dot{f}_{\alpha 11\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta} \dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\alpha 1\delta} \dot{f}_\alpha = \alpha 1\delta 0x \dot{f}_{\alpha 1\delta} \dot{f}_\alpha = \alpha 1\delta x \dot{f}_\alpha = \alpha 11\delta x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\alpha 1\delta 0x \dot{f}_{\alpha 1\delta} \dot{f}_\alpha) = \mathrm{depth}(\alpha 11\delta x) \\ &= \mathrm{depth}(\alpha 1\delta 0x) = \mathrm{depth}(\gamma x)\end{aligned}$$

and

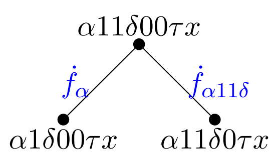
$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\alpha 1\delta 0x \dot{f}_{\alpha 1\delta}) = \mathrm{depth}(\alpha 1\delta x) \\ &= \mathrm{depth}(\alpha 1\delta 0x) - 1 = \mathrm{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.2.2. $\boldsymbol{\alpha 11\lambda x = \alpha 11\delta 00\tau x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}}$

In this case $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 11\delta}$. Observe that $\gamma x \dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \alpha 1\delta 00\tau x \dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \alpha 11\delta 00\tau x \dot{f}_{\alpha 11\delta} = \alpha 11\delta 0\tau x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



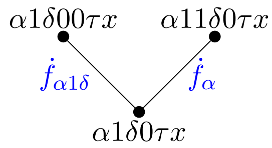
Conjugating these terms using relation $R5$ gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \left(\dot{f}_{\alpha 11\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\alpha 1\delta}\dot{f}_\alpha = \alpha 1\delta 00\tau x \dot{f}_{\alpha 1\delta}\dot{f}_\alpha = \alpha 1\delta 0\tau x \dot{f}_\alpha = \alpha 11\delta 0\tau x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\alpha 1\delta 00\tau x \dot{f}_{\alpha 1\delta}\dot{f}_\alpha) = \operatorname{depth}(\alpha 11\delta 0\tau x)\\ &= \operatorname{depth}(\alpha 1\delta 00\tau x) = \operatorname{depth}(\gamma x)\end{aligned}$$

and

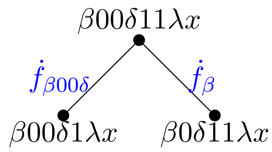
$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha 1\delta 00\tau x \dot{f}_{\alpha 1\delta}) = \operatorname{depth}(\alpha 1\delta 0\tau x)\\ &= \operatorname{depth}(\alpha 1\delta 00\tau x) - 1 = \operatorname{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.2.3. **$\alpha 11\lambda x = \beta 00\delta 11\lambda x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$**

We can conclude that $\beta 00$ is a prefix of α. Let $\alpha = \beta 00\delta$ for some $\delta \in E(R)^*$. Then $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 00\delta}\dot{f}_\beta$. Observe that $\gamma x \dot{f}_{\beta 00\delta}\dot{f}_\beta = \beta 00\delta 1\lambda x \dot{f}_{\beta 00\delta}\dot{f}_\beta = \beta 00\delta 11\lambda x \dot{f}_\beta = \beta 0\delta 11\lambda x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



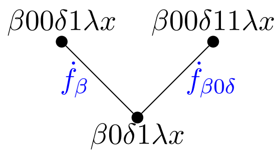
Conjugating these terms using relation $R3$ gives us $\dot{f}_{\beta 00\delta}\dot{f}_\beta = \dot{f}_\beta \left(\dot{f}_{\beta 00\delta}\right)^{\dot{f}_\beta} = \dot{f}_\beta \dot{f}_{\beta 0\delta}$. Observe that $\gamma x \dot{f}_\beta \dot{f}_{\beta 0\delta} = \beta 00\delta 1\lambda x \dot{f}_\beta \dot{f}_{\beta 0\delta} = \beta 0\delta 1\lambda x \dot{f}_{\beta 0\delta} = \beta 0\delta 11\lambda x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta 00\delta 1\lambda x \dot{f}_\beta \dot{f}_{\beta 0\delta}) = \operatorname{depth}(\beta 0\delta 11\lambda x)\\ &= \operatorname{depth}(\beta 00\delta 1\lambda x) = \operatorname{depth}(\gamma x)\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 00\delta 1\lambda x \dot{f}_\beta) = \operatorname{depth}(\beta 0\delta 1\lambda x)\\ &= \operatorname{depth}(\beta 00\delta 1\lambda x) - 1 = \operatorname{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2. $\boldsymbol{\eta_1 = +1}$ **and** $\boldsymbol{\eta_2 = -1}$

Let us set $\dot{f}_{\alpha_1}^{\eta_1} = f_\alpha$ By Lemma 6.2.1, in order for condition (2) to be satisfied

$$\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \mathrm{depth}(\gamma x) + 1,$$

we have the following choices for γx:

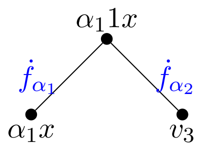
2.1. $\gamma x = \alpha x$,

2.2. $\gamma x = \alpha 1 \lambda x$ (for some $\lambda \in E(R)^*$).

Let us examine each of these cases:

2.1. $\boldsymbol{\gamma x = \alpha x}$

We observe that $\alpha x \dot{f}_\alpha = \alpha 1 x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



where $v_3 = \alpha 1 x \dot{f}_{\alpha_2}$. From Lemma 6.2.1, we observe that, in order for condition (1) of the Lemma to be satisfied

$$\mathrm{depth}(\gamma x) = \mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}),$$

we have the following possibilities for $\dot{f}_{\alpha_2}^{\eta_2}$:

2.1.1. $\alpha 1 x = \beta 1 x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$

2.1.2. $\alpha 1 x = \beta 11 \tau x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$

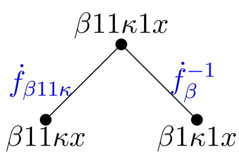
We observe that, although $\beta 0 x \dot{f}_\beta = \beta x$, this is not relevant here as $\alpha 1 x \neq \beta 0 x$. Let us examine each of these cases separately:

2.1.1. $\boldsymbol{\alpha 1 x = \beta 1 x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}}$

In this case, $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta \dot{f}_\beta^{-1} = I$, and the conditions in the Lemma are satisfied with $k = 0$.

2.1.2. $\boldsymbol{\alpha 1 x = \beta 11 \tau x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}}$

We can conclude that $\beta 11$ is a prefix of α. Let $\alpha = \beta 11 \kappa$ for some $\kappa \in E(R)^*$. Then $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 11 \kappa} \dot{f}_\beta^{-1}$. Observe that $\gamma x \dot{f}_{\beta 11 \kappa} \dot{f}_\beta^{-1} = \beta 11 \kappa x \dot{f}_{\beta 11 \kappa} \dot{f}_\beta^{-1} = \beta 11 \kappa 1 x \dot{f}_\beta^{-1} = \beta 1 \kappa 1 x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



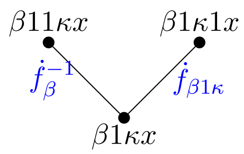
Conjugating these terms using relation $R5$ gives us $\dot{f}_{\beta 11\kappa}\dot{f}_{\beta}^{-1} = \dot{f}_{\beta}\left(\dot{f}_{\beta 11\kappa}\right)^{\dot{f}_{\beta}^{-1}} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\kappa}$. Observe that $\gamma x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\kappa} = \beta 11\kappa x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\kappa} = \beta 1\kappa x\dot{f}_{\beta 1\kappa} = \beta 1\kappa 1x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\beta 11\kappa x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\kappa}) = \mathrm{depth}(\beta 1\kappa 1x)\\ &= \mathrm{depth}(\beta 11\kappa x) = \mathrm{depth}(\gamma x)\end{aligned}$$

and

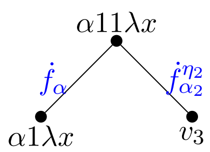
$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\beta 11\kappa x\dot{f}_{\beta}^{-1}) = \mathrm{depth}(\beta 1\kappa x)\\ &= \mathrm{depth}(\beta 11\kappa x) - 1 = \mathrm{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



2.2. $\boldsymbol{\gamma x = \alpha 1\lambda x}$ **(for some** $\boldsymbol{\lambda \in E(R)^*}$**)**

We observe that $\alpha 1\lambda x\dot{f}_{\alpha} = \alpha 11\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



where $v_3 = \alpha 11\lambda x\dot{f}_{\alpha_2}^{\eta_2}$.

Using Lemma 6.2.1, we observe that, in order for condition (1) from the Lemma to be satisfied

$$\mathrm{depth}(\gamma x) = \mathrm{depth}(\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}),$$

we have the following possibilities for $\dot{f}_{\alpha_2}^{\eta_2}$:

2.2.1. $\alpha 11\lambda x = \alpha 11\lambda x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}$,

2.2.2. $\alpha 11\lambda x = \alpha 11\delta 1x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}$,

2.2.3. $\alpha 11\lambda x = \alpha 11\delta 11\tau x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}$,

2.2.4. $\alpha 11\lambda x = \alpha 111\delta x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 1}^{-1}$,

2.2.5. $\alpha 11\lambda x = \beta 11\delta 11\lambda x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}$,

2.2.6. $\alpha 11\lambda x = \beta 111\lambda x$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}$.

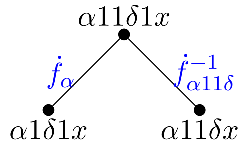
We examine each of these cases separately:

2.2.1. $\boldsymbol{\alpha 11\lambda x = \alpha 11\lambda x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}}$.

In this case, $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}\dot{f}_{\alpha}^{-1} = I$, and conditions (3) and (4) of the Lemma are satisfied with $k = 0$.

2.2.2. $\boldsymbol{\alpha 11\lambda x = \alpha 11\delta 1x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}}$.

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1}$. Observe that $\gamma x \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \alpha 1\delta 1x \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \alpha 11\delta 1x \dot{f}_{\alpha 11\delta}^{-1} = \alpha 11\delta x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



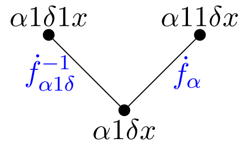
Conjugating these terms using relation $R5$ gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \left(\dot{f}_{\alpha 11\delta}^{-1}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha = \alpha 1\delta 1x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha = \alpha 1\delta x \dot{f}_\alpha = \alpha 11\delta x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

$$\begin{aligned} \text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\delta 1x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha) = \text{depth}(\alpha 11\delta x) \\ &= \text{depth}(\alpha 1\delta 1x) = \text{depth}(\gamma x) \end{aligned}$$

and

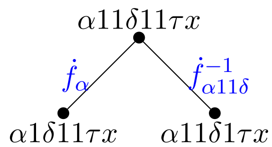
$$\begin{aligned} \text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\alpha 1\delta 1x \dot{f}_{\alpha 1\delta}^{-1}) = \text{depth}(\alpha 1\delta x) \\ &= \text{depth}(\alpha 1\delta 1x) - 1 = \text{depth}(\gamma x) - 1. \end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.2.3. $\boldsymbol{\alpha 11\lambda x = \alpha 11\delta 11\tau x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}}$.

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1}$. Observe that $\gamma x \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \alpha 1\delta 11\tau x \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \alpha 11\delta 11\tau x \dot{f}_{\alpha 11\delta}^{-1} = \alpha 11\delta 1\tau x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms using relation $R5$ gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \left(\dot{f}_{\alpha 11\delta}^{-1}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha = \alpha 1\delta 1x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha = \alpha 1\delta x \dot{f}_\alpha = \alpha 11\delta x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

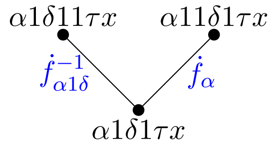
$$\begin{aligned} \text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\delta 11\tau x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha) = \text{depth}(\alpha 11\delta 1\tau x) \\ &= \text{depth}(\alpha 1\delta 11\tau x) = \text{depth}(\gamma x) \end{aligned}$$

and

$$\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) = \text{depth}(\alpha 1\delta 11\tau x \dot{f}_{\alpha 1\delta}^{-1}) = \text{depth}(\alpha 1\delta 1\tau x)$$

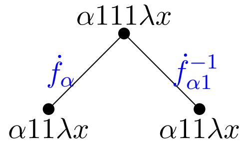
$$= \mathrm{depth}(\alpha 1\delta 11\tau x) - 1 = \mathrm{depth}(\gamma x) - 1.$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\cdots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.2.4. $\boldsymbol{\alpha 11\lambda x = \alpha 111\delta x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 1}^{-1}}$

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}\dot{f}_{\alpha 1}^{-1}$. Observe that $\gamma x\dot{f}_{\alpha}\dot{f}_{\alpha 1}^{-1} = \alpha 11\lambda x\dot{f}_{\alpha}\dot{f}_{\alpha 1}^{-1} = \alpha 111\lambda x\dot{f}_{\alpha 1}^{-1} = \alpha 11\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms using relation $R2$ gives us $\dot{f}_{\alpha}\dot{f}_{\alpha 1}^{-1} = \left(\dot{f}_{\alpha 1}^{-1}\right)^{\dot{f}_{\alpha}^{-1}}\dot{f}_{\alpha} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}\dot{f}_{\alpha}$. Observe that $\gamma x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}\dot{f}_{\alpha} = \alpha 11\lambda x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}\dot{f}_{\alpha} = \alpha 1\lambda x\dot{f}_{\alpha 0}\dot{f}_{\alpha} = \alpha 1\lambda x\dot{f}_{\alpha} = \alpha 11\lambda x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 3$:

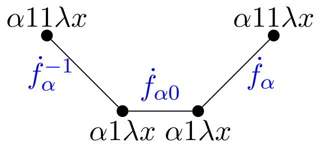
$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\alpha 11\lambda x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}\dot{f}_{\alpha}) = \mathrm{depth}(\alpha 11\lambda x)\\ &= \mathrm{depth}(\alpha 11\lambda x) = \mathrm{depth}(\gamma x),\end{aligned}$$

$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\alpha 11\lambda x\dot{f}_{\alpha}^{-1}) = \mathrm{depth}(\alpha 1\lambda x)\\ &= \mathrm{depth}(\alpha 11\lambda x) - 1 = \mathrm{depth}(\gamma x) - 1.\end{aligned}$$

and

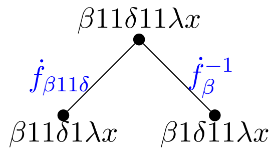
$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}\dot{f}_{\beta_2}^{\eta_2}) &= \mathrm{depth}(\alpha 11\lambda x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}) = \mathrm{depth}(\alpha 1\lambda x)\\ &= \mathrm{depth}(\alpha 11\lambda x) - 1 = \mathrm{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\cdots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.2.5. $\boldsymbol{\alpha 11\lambda x = \beta 11\delta 11\lambda x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}}$

We can conclude that $\beta 11$ is a prefix of α. Let $\alpha = \beta 11\delta$ for some $\delta \in E(R)^*$. Then $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 11\delta}\dot{f}_{\beta}^{-1}$. Observe that $\gamma x\dot{f}_{\beta 11\delta}\dot{f}_{\beta}^{-1} = \beta 11\delta 1\lambda x\dot{f}_{\beta 11\delta}\dot{f}_{\beta}^{-1} = \beta 11\delta 11\lambda x\dot{f}_{\beta}^{-1} = \beta 1\delta 11\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



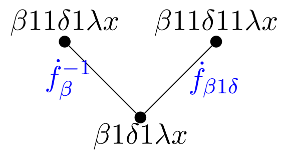
Conjugating these terms using relation $R3$ gives us $\dot{f}_{\beta 11\delta}\dot{f}_{\beta}^{-1} = \dot{f}_{\beta}^{-1}\left(\dot{f}_{\beta 11\delta}\right)^{\dot{f}_{\beta}^{-1}} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\delta}$. Observe that $\gamma x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\delta} = \beta 11\delta 1\lambda x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\delta} = \beta 1\delta 1\lambda x\dot{f}_{\beta 1\delta} = \beta 1\delta 11\lambda x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta 11\delta 1\lambda x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 1\delta}) = \operatorname{depth}(\beta 1\delta 11\lambda x)\\ &= \operatorname{depth}(\beta 11\delta 1\lambda x) = \operatorname{depth}(\gamma x)\end{aligned}$$

and

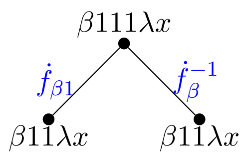
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 11\delta 1\lambda x\dot{f}_{\beta}^{-1}) = \operatorname{depth}(\beta 1\delta 1\lambda x)\\ &= \operatorname{depth}(\beta 11\delta 1\lambda x) - 1 = \operatorname{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.2.6. $\boldsymbol{\alpha 11\lambda x = \beta 111\lambda x}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}}$.

We can conclude that $\alpha = \beta 1$. In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}\dot{f}_{\beta}^{-1}$. Observe that $\gamma x\dot{f}_{\beta 1}\dot{f}_{\beta}^{-1} = \beta 11\lambda x\dot{f}_{\beta 1}\dot{f}_{\beta}^{-1} = \beta 111\lambda x\dot{f}_{\beta}^{-1} = \beta 11\lambda x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms using relation $R2$ gives us $\dot{f}_{\beta 1}\dot{f}_{\beta}^{-1} = \dot{f}_{\beta}^{-1}\left(\dot{f}_{\beta 1}\right)^{\dot{f}_{\beta}^{-1}} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}^{-1}\dot{f}_{\beta}$. Observe that $\gamma x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}^{-1}\dot{f}_{\beta} = \beta 11\lambda x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}^{-1}\dot{f}_{\beta} = \beta 1\lambda x\dot{f}_{\beta 0}^{-1}\dot{f}_{\beta} = \beta 1\lambda x\dot{f}_{\beta} = \beta 11\lambda x$, and hence conditions (3) and (4) in the Lemma are satisfied with $k = 3$:

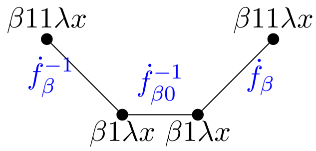
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta 11\lambda x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}^{-1}\dot{f}_{\beta}) = \operatorname{depth}(\beta 11\lambda x)\\ &= \operatorname{depth}(\beta 11\lambda x) = \operatorname{depth}(\gamma x),\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 11\lambda x\dot{f}_{\beta}^{-1}) = \operatorname{depth}(\beta 1\lambda x)\\ &= \operatorname{depth}(\beta 11\lambda x) - 1 = \operatorname{depth}(\gamma x) - 1.\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(\gamma x \dot f_{\beta_1}^{\eta_1} \dot f_{\beta_2}^{\eta_2}) &= \operatorname{depth}(\beta 11\lambda x \dot f_{\beta}^{-1} \dot f_{\beta 0}^{-1}) = \operatorname{depth}(\beta 1\lambda x)\\ &= \operatorname{depth}(\beta 11\lambda x) - 1 = \operatorname{depth}(\gamma x) - 1.\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot f_{\beta_1}^{\mu_1}\dots \dot f_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



□

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Lemma 6.3.2. *Let $\dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:*

A. $\operatorname{depth}(\gamma x \dot f_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1$,

B. $\gamma x \dot f_{\alpha_1}^{\eta_1} = \gamma x \dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2}$.

Then there exists $\dot f_{\beta_1}^{\mu_1} \dots \dot f_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 2$ such that $\dot f_{\beta_1}^{\mu_1} \dots \dot f_{\beta_k}^{\mu_k} = \dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2}$ in $\dot F$ and the following conditions are satisfied:

C. $\gamma x = \gamma x \dot f_{\beta_1}^{\mu_1}$,

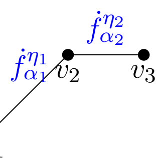
D. $\operatorname{depth}(\gamma x \dot f_{\beta_1}^{\mu_1} \dot f_{\beta_2}^{\mu_2}) = \operatorname{depth}(\gamma x) + 1$.

Proof. Let $\dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:

A. $\operatorname{depth}(\gamma x \dot f_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1$,

B. $\gamma x \dot f_{\alpha_1}^{\eta_1} = \gamma x \dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2}$.

Let us illustrate $\operatorname{depthchart}_{\dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x \dot f_{\alpha_1}^{\eta_1}$ and $v_3 = \gamma x \dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2}$.

We will examine all cases which arise. Let us first divide into cases such that Condition (A) is satisfied:

$$\operatorname{depth}(\gamma x \dot f_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1.$$

1. $\dot f_{\alpha_1}^{\eta_1} = \dot f_{\alpha}$ and $\gamma x = \alpha x$ (for some $\alpha \in E(R)^*$),

2. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha$ and $\gamma x = \alpha 1 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$),

3. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha^{-1}$ and $\gamma x = \alpha x$ (for some $\alpha \in E(R)^*$),

4. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha^{-1}$ and $\gamma x = \alpha 0 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$).

Let us now examine these cases:

1. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha}$ **and** $\boldsymbol{\gamma x = \alpha x}$ **(for some** $\boldsymbol{\alpha \in E(R)^*}$**)**

   Observe that, for Condition B to be satisfied

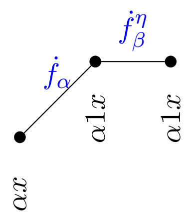
$$\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}.$$

   $\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \alpha x \dot{f}_\alpha = \alpha 1 x$ should not be in supp $\dot{f}_{\alpha_2}^{\eta_2}$. By Lemma 6.2.1, this implies that $\alpha_2 \not\preceq \alpha 1$. This gives us the following choices for $\dot{f}_{\alpha_2}^{\eta_2}$:

   1.1. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^\eta$ (for some $\beta \in E(R)^*$ such that $\beta \perp \alpha$ and $\eta \in \{\pm 1\}$),
   1.2. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 0 \delta}^\eta$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),
   1.3. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 10 \delta}^\eta$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),
   1.4. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11 \delta}^\eta$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$).

   Let us now examine these cases:

   1.1. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^\eta}$ **(for some** $\boldsymbol{\beta \in E(R)^*}$ **such that** $\boldsymbol{\beta \perp \alpha}$ **and** $\boldsymbol{\eta \in \{\pm 1\}}$**)**
   In this case $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_\beta^\eta$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \alpha x \dot{f}_\alpha \dot{f}_\beta^\eta = \alpha 1 x \dot{f}_\beta^\eta = \alpha 1 x$. The graph for depthchart$_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



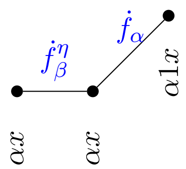
   Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_\beta^\eta = \left(\dot{f}_\beta^\eta\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_\beta^\eta \dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta^\eta \dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \alpha x \dot{f}_\beta^\eta \dot{f}_\alpha = \alpha x \dot{f}_\alpha = \alpha 1 x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\alpha x \dot{f}_\beta^\eta \dot{f}_\alpha) = \operatorname{depth}(\alpha 1 x) \\ &= \operatorname{depth}(\alpha x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

   and

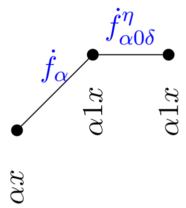
$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha x \dot{f}_\beta^\eta) = \operatorname{depth}(\alpha x) \\ &= \operatorname{depth}(\alpha x) = \operatorname{depth}(\gamma x).\end{aligned}$$

   The graph for depthchart$_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.2. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 0\delta}^{\eta}}$ **(for some** $\boldsymbol{\delta \in E(R)^*}$ **and** $\boldsymbol{\eta \in \{\pm 1\}}$**)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 0\delta}^{\eta}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha x \dot{f}_\alpha \dot{f}_{\alpha 0\delta}^{\eta} = \alpha 1x \dot{f}_{\alpha 0\delta}^{\eta} = \alpha 1x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



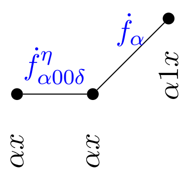
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_{\alpha 0\delta}^{\eta} = \left(\dot{f}_{\alpha 0\delta}^{\eta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \alpha x \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha = \alpha x \dot{f}_\alpha = \alpha 1x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\alpha x \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha) = \mathrm{depth}(\alpha 1x)\\ &= \mathrm{depth}(\alpha x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

and

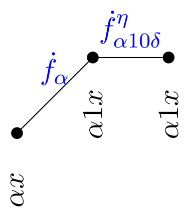
$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\alpha x \dot{f}_{\alpha 00\delta}^{\eta}) = \mathrm{depth}(\alpha x)\\ &= \mathrm{depth}(\alpha x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.3. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 10\delta}^{\eta}}$ **(for some** $\boldsymbol{\delta \in E(R)^*}$ **and** $\boldsymbol{\eta \in \{\pm 1\}}$**)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 10\delta}^{\eta}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha x \dot{f}_\alpha \dot{f}_{\alpha 10\delta}^{\eta} = \alpha 1x \dot{f}_{\alpha 10\delta}^{\eta} = \alpha 1x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



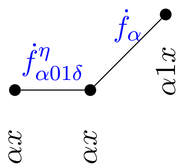
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}^\eta_{\alpha 10\delta} = \left(\dot{f}^\eta_{\alpha 10\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha$. Set $\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \alpha x \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha = \alpha x \dot{f}_\alpha = \alpha 1 x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}) &= \operatorname{depth}(\alpha x \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha) = \operatorname{depth}(\alpha 1 x)\\ &= \operatorname{depth}(\alpha x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

and

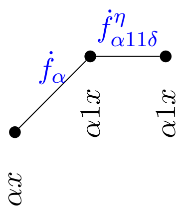
$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \operatorname{depth}(\alpha x \dot{f}^\eta_{\alpha 01\delta}) = \operatorname{depth}(\alpha x)\\ &= \operatorname{depth}(\alpha x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



1.4. $\boldsymbol{\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^\eta_{\alpha 11\delta}}$ **(for some $\boldsymbol{\delta \in E(R)^*}$ and $\boldsymbol{\eta \in \{\pm 1\}}$)**

In this case $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}_\alpha \dot{f}^\eta_{\alpha 11\delta}$. Observe that $\gamma x \dot{f}^{\eta_1}_{\alpha}\dot{f}^{\eta_2}_{\alpha_2} = \alpha x \dot{f}_\alpha \dot{f}^\eta_{\alpha 11\delta} = \alpha 1 x \dot{f}^\eta_{\alpha 11\delta} = \alpha 1 x$. The graph for $\operatorname{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



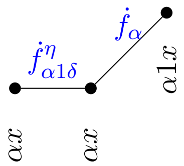
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}^\eta_{\alpha 11\delta} = \left(\dot{f}^\eta_{\alpha 11\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha$. Set $\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \alpha x \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha = \alpha x \dot{f}_\alpha = \alpha 1 x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}) &= \operatorname{depth}(\alpha x \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha) = \operatorname{depth}(\alpha 1 x)\\ &= \operatorname{depth}(\alpha x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \operatorname{depth}(\alpha x \dot{f}^\eta_{\alpha 1\delta}) = \operatorname{depth}(\alpha x)\\ &= \operatorname{depth}(\alpha x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



2. $\boldsymbol{\dot{f}^{\eta_1}_{\alpha_1} = \dot{f}_\alpha}$ **and** $\boldsymbol{\gamma x = \alpha 1\lambda x}$ **(for some** $\boldsymbol{\alpha, \lambda \in E(R)^*}$**)**

Observe that, for Condition B to be satisfied

$$\gamma x \dot{f}^{\eta_1}_{\alpha_1} = \gamma x \dot{f}^{\eta_1}_{\alpha_1} \dot{f}^{\eta_2}_{\alpha_2}.$$

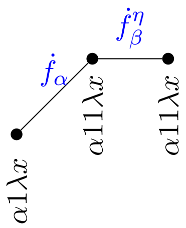
$\gamma x \dot{f}^{\eta_1}_{\alpha_1} = \alpha 1\lambda x \dot{f}_\alpha = \alpha 11\lambda x$ should not be in supp $\dot{f}^{\eta_2}_{\alpha_2}$. By Lemma 6.2.1, this implies that $\alpha_2 \not\preceq \alpha 11\lambda$. This gives us the following choices for $\dot{f}^{\eta_2}_{\alpha_2}$:

2.1. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\beta}$ (for some $\beta \in E(R)^*$ such that $\beta \perp \alpha$ and $\eta \in \{\pm 1\}$),
2.2. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 0\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),
2.3. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 10\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),
2.4. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 11\delta}$ (for some $\delta \in E(R)^*$ such that $\delta \perp \lambda$ and $\eta \in \{\pm 1\}$),
2.5. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 11\delta}$ (for some $\delta \in E(R)^*$ such that $\lambda \prec \delta$ and $\eta \in \{\pm 1\}$).

Let us now examine these cases:

2.1. $\boldsymbol{\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\beta}}$ **(for some** $\boldsymbol{\beta \in E(R)^*}$ **such that** $\boldsymbol{\beta \perp \alpha}$ **and** $\boldsymbol{\eta \in \{\pm 1\}}$**)**

In this case $\dot{f}^{\eta_1}_{\alpha_1} \dot{f}^{\eta_2}_{\alpha_2} = \dot{f}_\alpha \dot{f}^{\eta}_{\beta}$. Observe that $\gamma x \dot{f}^{\eta_1}_{\alpha} \dot{f}^{\eta_2}_{\alpha_2} = \alpha 1\lambda x \dot{f}_\alpha \dot{f}^{\eta}_{\beta} = \alpha 11\lambda x \dot{f}^{\eta}_{\beta} = \alpha 11\lambda x$. The graph for depthchart$_{\dot{f}^{\eta_1}_{\alpha_1} \dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



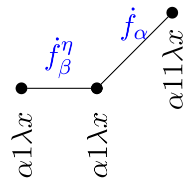
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}^{\eta}_{\beta} = \left(\dot{f}^{\eta}_{\beta}\right)^{\dot{f}^{-1}_{\alpha}} \dot{f}_\alpha = \dot{f}^{\eta}_{\beta} \dot{f}_\alpha$. Set $\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \dot{f}^{\eta}_{\beta} \dot{f}_\alpha$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \alpha 1\lambda x \dot{f}^{\eta}_{\beta} \dot{f}_\alpha = \alpha 1\lambda x \dot{f}_\alpha = \alpha 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned} \operatorname{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}) &= \operatorname{depth}(\alpha 1\lambda x \dot{f}^{\eta}_{\beta} \dot{f}_\alpha) = \operatorname{depth}(\alpha 11\lambda x) \\ &= \operatorname{depth}(\alpha 1\lambda x) + 1 = \operatorname{depth}(\gamma x) + 1, \end{aligned}$$

and

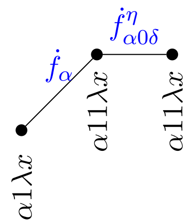
$$\begin{aligned} \operatorname{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \operatorname{depth}(\alpha 1\lambda x \dot{f}^{\eta}_{\beta}) = \operatorname{depth}(\alpha 1\lambda x) \\ &= \operatorname{depth}(\alpha 1\lambda x) = \operatorname{depth}(\gamma x). \end{aligned}$$

The graph for depthchart$_{\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



2.2. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 0\delta}^{\eta}}$ **(for some** $\boldsymbol{\delta \in E(R)^*}$ **and** $\boldsymbol{\eta \in \{\pm 1\}}$**),**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 0\delta}^{\eta}$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 1\lambda x \dot{f}_\alpha \dot{f}_{\alpha 0\delta}^{\eta} = \alpha 11\lambda x \dot{f}_{\alpha 0\delta}^{\eta} = \alpha 11\lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



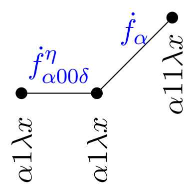
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_{\alpha 0\delta}^{\eta} = \left(\dot{f}_{\alpha 0\delta}^{\eta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \alpha 1\lambda x \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha = \alpha 1\lambda x \dot{f}_\alpha = \alpha 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\lambda x \dot{f}_{\alpha 00\delta}^{\eta}\dot{f}_\alpha) = \text{depth}(\alpha 11\lambda x)\\ &= \text{depth}(\alpha 1\lambda x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

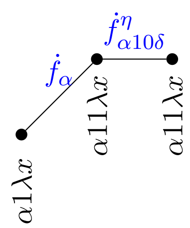
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\alpha 1\lambda x \dot{f}_{\alpha 00\delta}^{\eta}) = \text{depth}(\alpha 1\lambda x)\\ &= \text{depth}(\alpha 1\lambda x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.3. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 10\delta}^{\eta}}$ **(for some** $\boldsymbol{\delta \in E(R)^*}$ **and** $\boldsymbol{\eta \in \{\pm 1\}}$**),**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 10\delta}^{\eta}$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 1\lambda x \dot{f}_\alpha \dot{f}_{\alpha 10\delta}^{\eta} = \alpha 11\lambda x \dot{f}_{\alpha 10\delta}^{\eta} = \alpha 11\lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



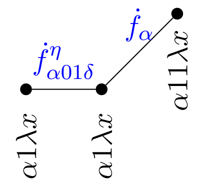
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}^\eta_{\alpha 10\delta} = \left(\dot{f}^\eta_{\alpha 10\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha$. Set $\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \alpha 1\lambda x \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha = \alpha 1 \lambda x \dot{f}_\alpha = \alpha 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}) &= \operatorname{depth}(\alpha 1\lambda x \dot{f}^\eta_{\alpha 01\delta}\dot{f}_\alpha) = \operatorname{depth}(\alpha 11\lambda x)\\ &= \operatorname{depth}(\alpha 1\lambda x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

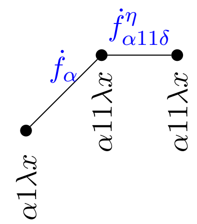
and

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \operatorname{depth}(\alpha 1\lambda x \dot{f}^\eta_{\alpha 01\delta}) = \operatorname{depth}(\alpha 1\lambda x)\\ &= \operatorname{depth}(\alpha 1\lambda x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



2.4. **$\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^\eta_{\alpha 11\delta}$ (for some $\delta \in E(R)^*$ such that $\delta \perp \lambda$ and $\eta \in \{\pm 1\}$),**
In this case $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}_\alpha \dot{f}^\eta_{\alpha 11\delta}$. Observe that $\gamma x \dot{f}^{\eta_1}_{\alpha}\dot{f}^{\eta_2}_{\alpha_2} = \alpha 1\lambda x \dot{f}_\alpha \dot{f}^\eta_{\alpha 11\delta} = \alpha 11\lambda x \dot{f}^\eta_{\alpha 11\delta} = \alpha 11\lambda x$. The graph for $\operatorname{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



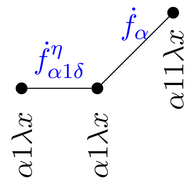
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}^\eta_{\alpha 11\delta} = \left(\dot{f}^\eta_{\alpha 11\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha$. Set $\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \alpha 1\lambda x \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha = \alpha 1 \lambda x \dot{f}_\alpha = \alpha 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}) &= \operatorname{depth}(\alpha 1\lambda x \dot{f}^\eta_{\alpha 1\delta}\dot{f}_\alpha) = \operatorname{depth}(\alpha 11\lambda x)\\ &= \operatorname{depth}(\alpha 1\lambda x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

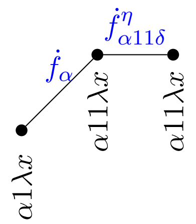
and

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \operatorname{depth}(\alpha 1\lambda x \dot{f}^\eta_{\alpha 1\delta}) = \operatorname{depth}(\alpha 1\lambda x)\\ &= \operatorname{depth}(\alpha 1\lambda x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



2.5. **$\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 11\delta}$ (for some $\delta \in E(R)^*$ such that $\lambda \prec \delta$ and $\eta \in \{\pm 1\}$).**
In this case $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}_\alpha \dot{f}^{\eta}_{\alpha 11\delta}$. Observe that $\gamma x \dot{f}^{\eta_1}_{\alpha}\dot{f}^{\eta_2}_{\alpha_2} = \alpha 1\lambda x \dot{f}_\alpha \dot{f}^{\eta}_{\alpha 11\delta} = \alpha 11\lambda x \dot{f}^{\eta}_{\alpha 11\delta} = \alpha 11\lambda x$. The graph for $\text{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



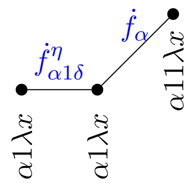
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}^{\eta}_{\alpha 11\delta} = \left(\dot{f}^{\eta}_{\alpha 11\delta}\right)^{\dot{f}^{-1}_{\alpha}} \dot{f}_\alpha = \dot{f}^{\eta}_{\alpha 1\delta}\dot{f}_\alpha$. Set $\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \dot{f}^{\eta}_{\alpha 1\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \alpha 1\lambda x \dot{f}^{\eta}_{\alpha 1\delta}\dot{f}_\alpha = \alpha 1\lambda x \dot{f}_\alpha = \alpha 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}) &= \text{depth}(\alpha 1\lambda x \dot{f}^{\eta}_{\alpha 1\delta}\dot{f}_\alpha) = \text{depth}(\alpha 11\lambda x)\\ &= \text{depth}(\alpha 1\lambda x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

$$\begin{aligned}\text{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \text{depth}(\alpha 1\lambda x \dot{f}^{\eta}_{\alpha 1\delta}) = \text{depth}(\alpha 1\lambda x)\\ &= \text{depth}(\alpha 1\lambda x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



3. **$\dot{f}^{\eta_1}_{\alpha_1} = \dot{f}^{-1}_{\alpha}$ and $\gamma x = \alpha x$ (for some $\alpha \in E(R)^*$)**
Observe that, for Condition B to be satisfied

$$\gamma x \dot{f}^{\eta_1}_{\alpha_1} = \gamma x \dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}.$$

$\gamma x \dot{f}^{\eta_1}_{\alpha_1} = \alpha x \dot{f}^{-1}_{\alpha} = \alpha 0x$ should not be in supp $\dot{f}^{\eta_2}_{\alpha_2}$. By Lemma 6.2.1, this implies that $\alpha_2 \not\preceq \alpha 0$. This gives us the following choices for $\dot{f}^{\eta_2}_{\alpha_2}$:

3.1. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\beta}$ (for some $\beta \in E(R)^*$ such that $\beta \perp \alpha$ and $\eta \in \{\pm 1\}$), 3.2. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 00\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),

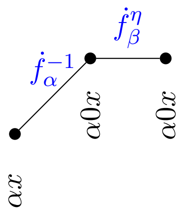
3.3. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 01\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),

3.4. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 1\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$).

Let us now examine these cases:

3.1. **$\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\beta}$ (for some $\beta \in E(R)^*$ such that $\beta \perp \alpha$ and $\eta \in \{\pm 1\}$)**

In this case $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\beta}$. Observe that $\gamma x \dot{f}^{\eta_1}_{\alpha}\dot{f}^{\eta_2}_{\alpha_2} = \alpha x \dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\beta} = \alpha 0 x \dot{f}^{\eta}_{\beta} = \alpha 0 x$. The graph for $\text{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



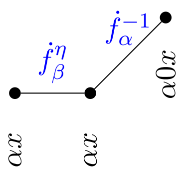
Conjugating these terms gives us $\dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\beta} = \left(\dot{f}^{\eta}_{\beta}\right)^{\dot{f}_{\alpha}} \dot{f}^{-1}_{\alpha} = \dot{f}^{\eta}_{\beta}\dot{f}^{-1}_{\alpha}$. Set $\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \dot{f}^{\eta}_{\beta}\dot{f}^{-1}_{\alpha}$. Observe that $\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k} = \alpha x \dot{f}^{\eta}_{\beta}\dot{f}^{-1}_{\alpha} = \alpha x \dot{f}^{-1}_{\alpha} = \alpha 0 x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}) &= \text{depth}(\alpha x \dot{f}^{\eta}_{\beta}\dot{f}^{-1}_{\alpha}) = \text{depth}(\alpha 0 x)\\ &= \text{depth}(\alpha x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

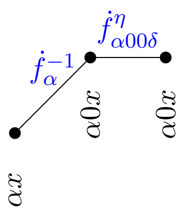
$$\begin{aligned}\text{depth}(\gamma x \dot{f}^{\eta_1}_{\beta_1}) &= \text{depth}(\alpha x \dot{f}^{\eta}_{\beta}) = \text{depth}(\alpha x)\\ &= \text{depth}(\alpha x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}^{\mu_1}_{\beta_1} \dots \dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



3.2. **$\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 00\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),**

In this case $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\alpha 00\delta}$. Observe that $\gamma x \dot{f}^{\eta_1}_{\alpha}\dot{f}^{\eta_2}_{\alpha_2} = \alpha x \dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\alpha 00\delta} = \alpha 0 x \dot{f}^{\eta}_{\alpha 00\delta} = \alpha 0 x$. The graph for $\text{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



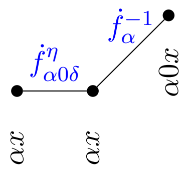
Conjugating these terms gives us $\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha00\delta}^{\eta} = \left(\dot{f}_{\alpha00\delta}^{\eta}\right)^{\dot{f}_{\alpha}} \dot{f}_{\alpha}^{-1} = \dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \alpha x\dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1} = \alpha x\dot{f}_{\alpha}^{-1} = \alpha0x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\alpha x\dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1}) = \operatorname{depth}(\alpha0x)\\ &= \operatorname{depth}(\alpha x)+1 = \operatorname{depth}(\gamma x)+1,\end{aligned}$$

and

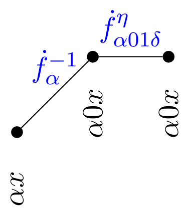
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha x\dot{f}_{\alpha0\delta}^{\eta}) = \operatorname{depth}(\alpha x)\\ &= \operatorname{depth}(\alpha x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



3.3. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha01\delta}^{\eta}}$ **(for some $\boldsymbol{\delta \in E(R)^*}$ and $\boldsymbol{\eta \in \{\pm1\}}$),**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha01\delta}^{\eta}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha01\delta}^{\eta} = \alpha0x\dot{f}_{\alpha01\delta}^{\eta} = \alpha0x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



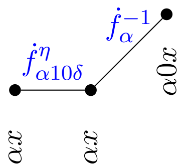
Conjugating these terms gives us $\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha01\delta}^{\eta} = \left(\dot{f}_{\alpha01\delta}^{\eta}\right)^{\dot{f}_{\alpha}} \dot{f}_{\alpha}^{-1} = \dot{f}_{\alpha10\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha10\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \alpha x\dot{f}_{\alpha10\delta}^{\eta}\dot{f}_{\alpha}^{-1} = \alpha x\dot{f}_{\alpha}^{-1} = \alpha0x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\alpha x\dot{f}_{\alpha10\delta}^{\eta}\dot{f}_{\alpha}^{-1}) = \operatorname{depth}(\alpha0x)\\ &= \operatorname{depth}(\alpha x)+1 = \operatorname{depth}(\gamma x)+1,\end{aligned}$$

and

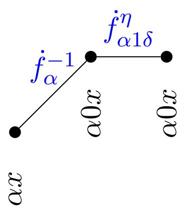
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha x\dot{f}_{\alpha10\delta}^{\eta}) = \operatorname{depth}(\alpha x)\\ &= \operatorname{depth}(\alpha x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



3.4. $\boldsymbol{\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 1\delta}}$ **(for some $\boldsymbol{\delta \in E(R)^*}$ and $\boldsymbol{\eta \in \{\pm 1\}}$).**

In this case $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\alpha 1\delta}$. Observe that $\gamma x\dot{f}^{\eta_1}_{\alpha}\dot{f}^{\eta_2}_{\alpha_2} = \alpha x\dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\alpha 1\delta} = \alpha 0x\dot{f}^{\eta}_{\alpha 1\delta} = \alpha 0x$. The graph for $\text{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}}(\gamma x)$ is then as follows:



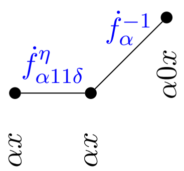
Conjugating these terms gives us $\dot{f}^{-1}_{\alpha}\dot{f}^{\eta}_{\alpha 1\delta} = \left(\dot{f}^{\eta}_{\alpha 1\delta}\right)^{\dot{f}_{\alpha}}\dot{f}^{-1}_{\alpha} = \dot{f}^{\eta}_{\alpha 11\delta}\dot{f}^{-1}_{\alpha}$. Set $\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \dot{f}^{\eta}_{\alpha 11\delta}\dot{f}^{-1}_{\alpha}$. Observe that $\gamma x\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \alpha x\dot{f}^{\eta}_{\alpha 11\delta}\dot{f}^{-1}_{\alpha} = \alpha x\dot{f}^{-1}_{\alpha} = \alpha 0x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}) &= \text{depth}(\alpha x\dot{f}^{\eta}_{\alpha 11\delta}\dot{f}^{-1}_{\alpha}) = \text{depth}(\alpha 0x)\\ &= \text{depth}(\alpha x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

$$\begin{aligned}\text{depth}(\gamma x\dot{f}^{\eta_1}_{\beta_1}) &= \text{depth}(\alpha x\dot{f}^{\eta}_{\alpha 11\delta}) = \text{depth}(\alpha x)\\ &= \text{depth}(\alpha x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



4. $\boldsymbol{\dot{f}^{\eta_1}_{\alpha_1} = \dot{f}^{-1}_{\alpha}}$ **and $\boldsymbol{\gamma x = \alpha 0\lambda x}$ (for some $\boldsymbol{\alpha, \lambda \in E(R)^*}$)**

Observe that, for Condition B to be satisfied

$$\gamma x\dot{f}^{\eta_1}_{\alpha_1} = \gamma x\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}.$$

$\gamma x\dot{f}^{\eta_1}_{\alpha_1} = \alpha 0\lambda x\dot{f}^{-1}_{\alpha} = \alpha 00\lambda x$ should not be in supp $\dot{f}^{\eta_2}_{\alpha_2}$. By Lemma 6.2.1, this implies that $\alpha_2 \not\leq \alpha 00\lambda$. This gives us the following choices for $\dot{f}^{\eta_2}_{\alpha_2}$:

4.1. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}_{\beta}$ (for some $\beta \in E(R)^*$ such that $\beta \perp \alpha$ and $\eta \in \{\pm 1\}$),

4.2. $\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{\eta}_{\alpha 1\delta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$), 4.3. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 01\delta}^{\eta}$ (for some $\delta \in E(R)^*$ and $\eta \in \{\pm 1\}$),

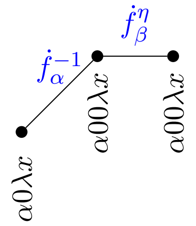
4.4. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 00\delta}^{\eta}$ (for some $\delta \in E(R)^*$ such that $\delta \perp \lambda$ and $\eta \in \{\pm 1\}$),

4.5. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 00\delta}^{\eta}$ (for some $\delta \in E(R)^*$ such that $\lambda \prec \delta$ and $\eta \in \{\pm 1\}$).

Let us now examine these cases:

4.1. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}}$ **(for some $\boldsymbol{\beta \in E(R)^*}$ such that $\boldsymbol{\beta \perp \alpha}$ and $\boldsymbol{\eta \in \{\pm 1\}}$),**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}\dot{f}_{\beta}^{\eta}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 0\lambda x \dot{f}_{\alpha}^{-1}\dot{f}_{\beta}^{\eta} = \alpha 00\lambda x \dot{f}_{\beta}^{\eta} = \alpha 00\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



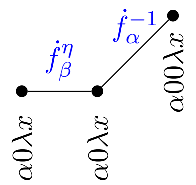
Conjugating these terms gives us $\dot{f}_{\alpha}^{-1}\dot{f}_{\beta}^{\eta} = \left(\dot{f}_{\beta}^{\eta}\right)^{\dot{f}_{\alpha}}\dot{f}_{\alpha}^{-1} = \dot{f}_{\beta}^{\eta}\dot{f}_{\alpha}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\beta}^{\eta}\dot{f}_{\alpha}^{-1}$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \alpha 0\lambda x \dot{f}_{\beta}^{\eta}\dot{f}_{\alpha}^{-1} = \alpha 0\lambda x \dot{f}_{\alpha}^{-1} = \alpha 00\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\alpha 0\lambda x \dot{f}_{\beta}^{\eta}\dot{f}_{\alpha}^{-1}) = \mathrm{depth}(\alpha 00\lambda x)\\ &= \mathrm{depth}(\alpha 0\lambda x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

and

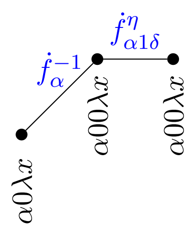
$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\alpha 0\lambda x \dot{f}_{\beta}^{\eta}) = \mathrm{depth}(\alpha 0\lambda x)\\ &= \mathrm{depth}(\alpha 0\lambda x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.2. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 1\delta}^{\eta}}$ **(for some $\boldsymbol{\delta \in E(R)^*}$ and $\boldsymbol{\eta \in \{\pm 1\}}$),**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 1\delta}^{\eta}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 0\lambda x \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 1\delta}^{\eta} = \alpha 00\lambda x \dot{f}_{\alpha 1\delta}^{\eta} = \alpha 00\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



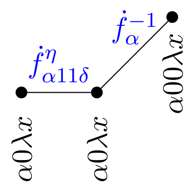
Conjugating these terms gives us $\dot{f}_\alpha^{-1}\dot{f}_{\alpha 1\delta}^{\eta} = \left(\dot{f}_{\alpha 1\delta}^{\eta}\right)^{\dot{f}_\alpha}\dot{f}_\alpha^{-1} = \dot{f}_{\alpha 11\delta}^{\eta}\dot{f}_\alpha^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 11\delta}^{\eta}\dot{f}_\alpha^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \alpha 0\lambda x\dot{f}_{\alpha 11\delta}^{\eta}\dot{f}_\alpha^{-1} = \alpha 0\lambda x\dot{f}_\alpha^{-1} = \alpha 00\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\alpha 0\lambda x\dot{f}_{\alpha 11\delta}^{\eta}\dot{f}_\alpha^{-1}) = \mathrm{depth}(\alpha 00\lambda x)\\ &= \mathrm{depth}(\alpha 0\lambda x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

and

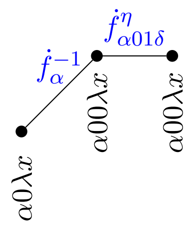
$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\alpha 0\lambda x\dot{f}_{\alpha 11\delta}^{\eta}) = \mathrm{depth}(\alpha 0\lambda x)\\ &= \mathrm{depth}(\alpha 0\lambda x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.3. $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 01\delta}^{\eta}$ **(for some** $\delta \in E(R)^*$ **and** $\eta \in \{\pm 1\}$**),**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha^{-1}\dot{f}_{\alpha 01\delta}^{\eta}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 0\lambda x\dot{f}_\alpha^{-1}\dot{f}_{\alpha 01\delta}^{\eta} = \alpha 00\lambda x\dot{f}_{\alpha 01\delta}^{\eta} = \alpha 00\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



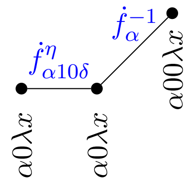
Conjugating these terms gives us $\dot{f}_\alpha^{-1}\dot{f}_{\alpha 01\delta}^{\eta} = \left(\dot{f}_{\alpha 01\delta}^{\eta}\right)^{\dot{f}_\alpha}\dot{f}_\alpha^{-1} = \dot{f}_{\alpha 10\delta}^{\eta}\dot{f}_\alpha^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 10\delta}^{\eta}\dot{f}_\alpha^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \alpha 0\lambda x\dot{f}_{\alpha 10\delta}^{\eta}\dot{f}_\alpha^{-1} = \alpha 0\lambda x\dot{f}_\alpha^{-1} = \alpha 00\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\alpha 0\lambda x\dot{f}_{\alpha 10\delta}^{\eta}\dot{f}_\alpha^{-1}) = \mathrm{depth}(\alpha 00\lambda x)\\ &= \mathrm{depth}(\alpha 0\lambda x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

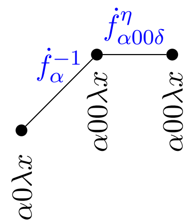
and

$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\alpha 0\lambda x\dot{f}_{\alpha 10\delta}^{\eta}) = \mathrm{depth}(\alpha 0\lambda x)\\ &= \mathrm{depth}(\alpha 0\lambda x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.4. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 00\delta}^{\eta}}$ **(for some $\boldsymbol{\delta \in E(R)^*}$ such that $\boldsymbol{\delta \perp \lambda}$ and $\boldsymbol{\eta \in \{\pm 1\}}$),**
In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 00\delta}^{\eta}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 0\lambda x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 00\delta}^{\eta} = \alpha 00\lambda x\dot{f}_{\alpha 00\delta}^{\eta} = \alpha 00\lambda x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



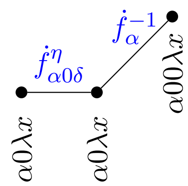
Conjugating these terms gives us $\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 00\delta}^{\eta} = \left(\dot{f}_{\alpha 00\delta}^{\eta}\right)^{\dot{f}_{\alpha}}\dot{f}_{\alpha}^{-1} = \dot{f}_{\alpha 0\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 0\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \alpha 0\lambda x\dot{f}_{\alpha 0\delta}^{\eta}\dot{f}_{\alpha}^{-1} = \alpha 0\lambda x\dot{f}_{\alpha}^{-1} = \alpha 00\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\alpha 0\lambda x\dot{f}_{\alpha 0\delta}^{\eta}\dot{f}_{\alpha}^{-1}) = \operatorname{depth}(\alpha 00\lambda x)\\ &= \operatorname{depth}(\alpha 0\lambda x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

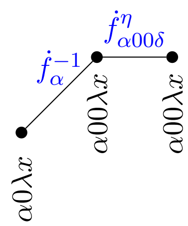
and

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha 0\lambda x\dot{f}_{\alpha 0\delta}^{\eta}) = \operatorname{depth}(\alpha 0\lambda x)\\ &= \operatorname{depth}(\alpha 0\lambda x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.5. $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 00\delta}^{\eta}}$ **(for some $\boldsymbol{\delta \in E(R)^*}$ such that $\boldsymbol{\lambda \prec \delta}$ and $\boldsymbol{\eta \in \{\pm 1\}}$).**
In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 00\delta}^{\eta}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 0\lambda x\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 00\delta}^{\eta} = \alpha 00\lambda x\dot{f}_{\alpha 00\delta}^{\eta} = \alpha 00\lambda x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



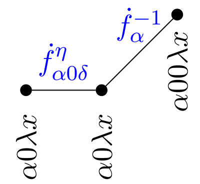
Conjugating these terms gives us $\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha00\delta}^{\eta} = \left(\dot{f}_{\alpha00\delta}^{\eta}\right)^{\dot{f}_{\alpha}} \dot{f}_{\alpha}^{-1} = \dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \alpha0\lambda x\dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1} = \alpha0\lambda x\dot{f}_{\alpha}^{-1} = \alpha00\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\alpha0\lambda x\dot{f}_{\alpha0\delta}^{\eta}\dot{f}_{\alpha}^{-1}) = \operatorname{depth}(\alpha00\lambda x)\\ &= \operatorname{depth}(\alpha0\lambda x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha0\lambda x\dot{f}_{\alpha0\delta}^{\eta}) = \operatorname{depth}(\alpha0\lambda x)\\ &= \operatorname{depth}(\alpha0\lambda x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



This completes the proof. □

{ripitup}

Lemma 6.3.3. *Let $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:*

A. $\operatorname{depth}(\gamma x\dot{f}_{\alpha_1}^{\eta_1}) = \operatorname{depth}(\gamma x) + 1$,

B. $\gamma x\dot{f}_{\alpha_1}^{\eta_1} \neq \gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ *and* $\operatorname{depth}(\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}) = \operatorname{depth}(\gamma x\dot{f}_{\alpha_1}^{\eta_1})$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm1\}$) with $k = 2, 3$ such that $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) = \operatorname{depth}(\gamma x) + 1$,

D. $\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) \leq \operatorname{depth}(\gamma x)$ *for* $i = 1, 2$.

Except for the following problem cases:

i. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta0}\dot{f}_{\beta}}$ ***and*** $\boldsymbol{\gamma x = \beta01\delta x}$ ***(for some*** $\boldsymbol{\beta, \delta \in E(R)^*}$***),***

ii. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}\dot{f}_{\beta1}^{-1}}$ ***and*** $\boldsymbol{\gamma x = \beta10\delta x}$ ***(for some*** $\boldsymbol{\beta, \delta \in E(R)^*}$***),***

iii. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta0}}$ ***and*** $\boldsymbol{\gamma x = \beta01\delta x}$ ***(for some*** $\boldsymbol{\beta, \delta \in E(R)^*}$***),***

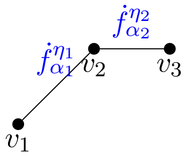
iv. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta1}^{-1}\dot{f}_{\beta}^{-1}}$ ***and*** $\boldsymbol{\gamma x = \beta10\delta x}$ ***(for some*** $\boldsymbol{\beta, \delta \in E(R)^*}$***).***

Proof. Let $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:

A. $\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \text{depth}(\gamma x) + 1$,

B. $\gamma x \dot{f}_{\alpha_1}^{\eta_1} \neq \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ and $\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}) = \text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1})$.

Let us illustrate $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$ and $v_3 = \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$.

We will examine all cases which arise. Let us first divide into cases such that Condition (A) is satisfied:

$$\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \text{depth}(\gamma x) + 1.$$

1. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha$ and $\gamma x = \alpha x$ (for some $\alpha \in E(R)^*$),
2. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha$ and $\gamma x = \alpha 1 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$),
3. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha^{-1}$ and $\gamma x = \alpha x$ (for some $\alpha \in E(R)^*$),
4. $\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha^{-1}$ and $\gamma x = \alpha 0 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$).

Let us now examine these cases:

1. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha}$ **and** $\boldsymbol{\gamma x = \alpha x}$ **(for some** $\boldsymbol{\alpha \in E(R)^*}$**)**

   Let us now divide into cases such that Condition (B) is satisfied:

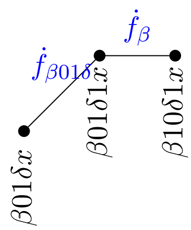
$$\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}.$$

   1.1. $\alpha = \beta 01 \delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$ (for some $\beta, \delta \in E(R)^*$),
   1.2. $\alpha = \beta 10 \delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$ (for some $\beta, \delta \in E(R)^*$),
   1.3. $\alpha = \beta 0$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$ (for some $\beta \in E(R)^*$).

   Let us now examine these cases:

   1.1. $\boldsymbol{\alpha = \beta 01 \delta}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta}$ **(for some** $\boldsymbol{\beta, \delta \in E(R)^*}$**)**

   In this case $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 01 \delta} \dot{f}_\beta$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \beta 01 \delta x \dot{f}_{\beta 01 \delta} \dot{f}_\beta = \beta 01 \delta 1 x \dot{f}_\beta = \beta 10 \delta 1 x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



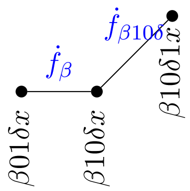
Conjugating these terms gives us $\dot{f}_{\beta 01\delta}\dot{f}_{\beta} = \dot{f}_{\beta}\left(\dot{f}_{\beta 01\delta}\right)^{\dot{f}_{\beta}} = \dot{f}_{\beta}\dot{f}_{\beta 10\delta}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\beta}\dot{f}_{\beta 10\delta}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta 01\delta x\dot{f}_{\beta}\dot{f}_{\beta 10\delta} = \beta 10\delta x\dot{f}_{\beta 10\delta} = \beta 10\delta 1x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta 01\delta x\dot{f}_{\beta}\dot{f}_{\beta 10\delta}) = \operatorname{depth}(\beta 10\delta 1x)\\ &= \operatorname{depth}(\beta 01\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

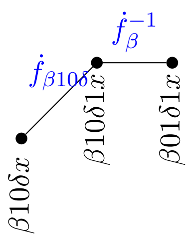
and

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 01\delta x\dot{f}_{\beta}) = \operatorname{depth}(\beta 10\delta x)\\ &= \operatorname{depth}(\beta 01\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.2. **$\alpha = \beta 10\delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}$ (for some $\beta, \delta \in E(R)^*$)**
In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 10\delta}\dot{f}_{\beta}^{-1}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta 10\delta x\dot{f}_{\beta 10\delta}\dot{f}_{\beta}^{-1} = \beta 10\delta 1x\dot{f}_{\beta}^{-1} = \beta 01\delta 1x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



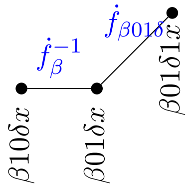
Conjugating these terms gives us $\dot{f}_{\beta 10\delta}\dot{f}_{\beta}^{-1} = \dot{f}_{\beta}^{-1}\left(\dot{f}_{\beta 10\delta}\right)^{\dot{f}_{\beta}^{-1}} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 01\delta}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 01\delta}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta 10\delta x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 01\delta} = \beta 01\delta x\dot{f}_{\beta 01\delta} = \beta 01\delta 1x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta 10\delta x\dot{f}_{\beta}^{-1}\dot{f}_{\beta 01\delta}) = \operatorname{depth}(\beta 01\delta 1x)\\ &= \operatorname{depth}(\beta 10\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

and

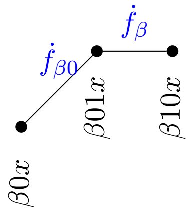
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 10\delta x\dot{f}_{\beta}^{-1}) = \operatorname{depth}(\beta 01\delta x)\\ &= \operatorname{depth}(\beta 10\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



1.3. **$\boldsymbol{\alpha = \beta 0}$ and $\boldsymbol{\dot f_{\alpha_2}^{\eta_2} = \dot f_\beta}$ (for some $\boldsymbol{\beta \in E(R)^*}$)**

In this case $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \dot f_{\beta 0}\dot f_\beta$. Observe that $\gamma x \dot f_{\alpha}^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \beta 0 x \dot f_{\beta 0}\dot f_\beta = \beta 01 x \dot f_\beta = \beta 10 x$. The graph for $\text{depthchart}_{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot f_{\beta 0}\dot f_\beta = \dot f_\beta \left(\dot f_{\beta 0}\right)^{\dot f_\beta} = \dot f_\beta \dot f_\beta \dot f_{\beta 1}^{-1}$. Set $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k} = \dot f_\beta \dot f_\beta \dot f_{\beta 1}^{-1}$. Observe that $\gamma x \dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k} = \beta 0 x \dot f_\beta \dot f_\beta \dot f_{\beta 1}^{-1} = \beta x \dot f_\beta \dot f_{\beta 1}^{-1} = \beta 1 x \dot f_{\beta 1}^{-1} = \beta 10 x$, and hence Conditions (C) and (D) are satisfied with $k = 3$:

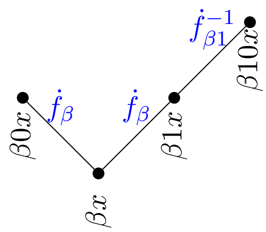
$$\begin{aligned}\text{depth}(\gamma x \dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_3}^{\mu_3}) &= \text{depth}(\beta 0 x \dot f_\beta \dot f_\beta \dot f_{\beta 1}^{-1}) = \text{depth}(\beta 10 x)\\ &= \text{depth}(\beta 0 x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\text{depth}(\gamma x \dot f_{\beta_1}^{\eta_1}) &= \text{depth}(\beta 0 x \dot f_\beta) = \text{depth}(\beta x)\\ &\leq \text{depth}(\beta 0 x) = \text{depth}(\gamma x),\end{aligned}$$

and

$$\begin{aligned}\text{depth}(\gamma x \dot f_{\beta_1}^{\mu_1}\dot f_{\beta_2}^{\mu_2}) &= \text{depth}(\beta 0 x \dot f_\beta \dot f_\beta) = \text{depth}(\beta 1 x)\\ &\leq \text{depth}(\beta 0 x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2. **$\boldsymbol{\dot f_{\alpha_1}^{\eta_1} = \dot f_\alpha}$ and $\boldsymbol{\gamma x = \alpha 1 \lambda x}$ (for some $\boldsymbol{\alpha, \lambda \in E(R)^*}$)**

Let us now divide into cases such that Condition (B) is satisfied:

$$\gamma x \dot f_{\alpha_1}^{\eta_1} = \gamma x \dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}.$$

2.1. $\alpha = \beta 01\delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$ (for some $\beta, \delta \in E(R)^*$),

2.2. $\alpha = \beta 10\delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$ (for some $\beta, \delta \in E(R)^*$),

2.3. $\alpha = \beta 0$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$ (for some $\beta \in E(R)^*$).

2.4. $\lambda = \delta 01\tau$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}$ (for some $\delta, \tau \in E(R)^*$),

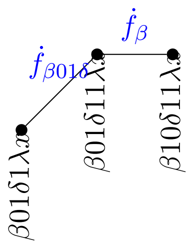
2.5. $\lambda = \delta 10\tau$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}$ (for some $\delta, \tau \in E(R)^*$),

2.6. $\lambda = 0\delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 1}^{-1}$ (for some $\delta \in E(R)^*$).

Let us now examine these cases:

2.1. **$\boldsymbol{\alpha = \beta 01\delta}$ and $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta}$ (for some $\boldsymbol{\beta, \delta \in E(R)^*}$)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 01\delta}\dot{f}_\beta$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta 01\delta 1\lambda x \dot{f}_{\beta 01\delta}\dot{f}_\beta = \beta 01\delta 11\lambda x \dot{f}_\beta = \beta 10\delta 11\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



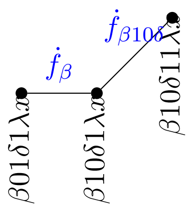
Conjugating these terms gives us $\dot{f}_{\beta 01\delta}\dot{f}_\beta = \dot{f}_\beta\left(\dot{f}_{\beta 01\delta}\right)^{\dot{f}_\beta} = \dot{f}_\beta\dot{f}_{\beta 10\delta}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta\dot{f}_{\beta 10\delta}$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \beta 01\delta 1\lambda x \dot{f}_\beta\dot{f}_{\beta 10\delta} = \beta 10\delta 1\lambda x \dot{f}_{\beta 10\delta} = \beta 10\delta 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\beta 01\delta 1\lambda x \dot{f}_\beta\dot{f}_{\beta 10\delta}) = \mathrm{depth}(\beta 10\delta 11\lambda x)\\ &= \mathrm{depth}(\beta 01\delta 1\lambda x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

and

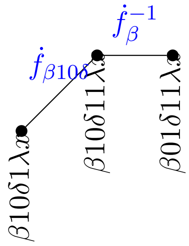
$$\begin{aligned}\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\beta 01\delta 1\lambda x \dot{f}_\beta) = \mathrm{depth}(\beta 10\delta 1\lambda x)\\ &= \mathrm{depth}(\beta 01\delta 1\lambda x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.2. **$\boldsymbol{\alpha = \beta 10\delta}$ and $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}}$ (for some $\boldsymbol{\beta, \delta \in E(R)^*}$)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 10\delta}\dot{f}_\beta^{-1}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta 10\delta 1\lambda x \dot{f}_{\beta 10\delta}\dot{f}_\beta^{-1} = \beta 10\delta 11\lambda x \dot{f}_\beta^{-1} = \beta 01\delta 11\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



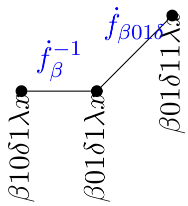
Conjugating these terms gives us $\dot{f}_{\beta 10\delta}\dot{f}_\beta^{-1} = \dot{f}_\beta^{-1}\left(\dot{f}_{\beta 10\delta}\right)^{\dot{f}_\beta^{-1}} = \dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta 10\delta 1\lambda x\dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta} = \beta 01\delta 1\lambda x\dot{f}_{\beta 01\delta} = \beta 01\delta 11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \mathrm{depth}(\beta 10\delta 1\lambda x\dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta}) = \mathrm{depth}(\beta 01\delta 11\lambda x)\\ &= \mathrm{depth}(\beta 10\delta 1\lambda x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

and

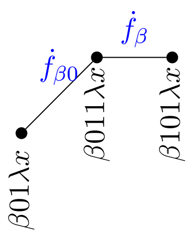
$$\begin{aligned}\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \mathrm{depth}(\beta 10\delta 1\lambda x\dot{f}_\beta^{-1}) = \mathrm{depth}(\beta 01\delta 1\lambda x)\\ &= \mathrm{depth}(\beta 10\delta 1\lambda x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.3. $\boldsymbol{\alpha = \beta 0}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta}$ **(for some** $\boldsymbol{\beta \in E(R)^*}$**)**

This is a Problem Case. In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_\beta$. Observe that $\gamma x\dot{f}_\alpha^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta 01\lambda x\dot{f}_{\beta 0}\dot{f}_\beta = \beta 011\lambda x\dot{f}_\beta = \beta 101\lambda x$. The graph for $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot{f}_{\beta 0}\dot{f}_\beta = \dot{f}_\beta\left(\dot{f}_{\beta 0}\right)^{\dot{f}_\beta} = \dot{f}_\beta\dot{f}_\beta\dot{f}_{\beta 1}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta\dot{f}_\beta\dot{f}_{\beta 1}^{-1}$. However in this case we observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta 01\lambda x\dot{f}_\beta\dot{f}_\beta\dot{f}_{\beta 1}^{-1} = \beta 10\lambda x\dot{f}_\beta\dot{f}_{\beta 1}^{-1} = \beta 110\lambda x\dot{f}_{\beta 1}^{-1} = \beta 101\lambda x$, and hence Conditions (C) and (D) are not satisfied.

$$\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_3}^{\mu_3}) = \mathrm{depth}(\beta 01\lambda x\dot{f}_\beta\dot{f}_\beta\dot{f}_{\beta 1}^{-1}) = \mathrm{depth}(\beta 101\lambda x)$$

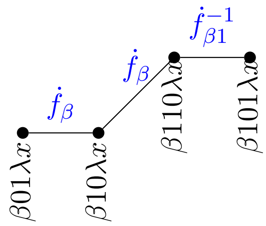
$$= \text{depth}(\beta 01\lambda x) + 1 = \text{depth}(\gamma x) + 1,$$

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\beta 01\lambda x \dot{f}_\beta) = \text{depth}(\beta 10\lambda x)\\ &\leq \text{depth}(\beta 01\lambda x) = \text{depth}(\gamma x),\end{aligned}$$

but

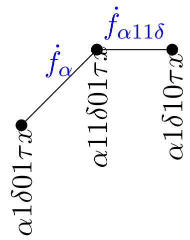
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dot{f}_{\beta_2}^{\mu_2}) &= \text{depth}(\beta 01\lambda x \dot{f}_\beta \dot{f}_\beta) = \text{depth}(\beta 110\lambda x)\\ &\geq \text{depth}(\beta 01\lambda x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.4. **$\boldsymbol{\lambda = \delta 01\tau}$ and $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}}$ (for some $\boldsymbol{\delta, \tau \in E(R)^*}$)**

In this case $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 11\delta}$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \alpha 1\delta 01\tau x \dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \alpha 11\delta 01\tau x \dot{f}_{\alpha 11\delta} = \alpha 1\delta 10\tau x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



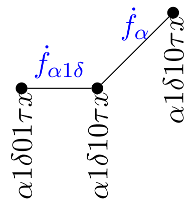
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \left(\dot{f}_{\alpha 11\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta} \dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 1\delta} \dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k} = \alpha 1\delta 01\tau x \dot{f}_{\alpha 1\delta} \dot{f}_\alpha = \alpha 1\delta 10\tau x \dot{f}_\alpha = \alpha 1\delta 10\tau x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\delta 01\tau x \dot{f}_{\alpha 1\delta} \dot{f}_\alpha) = \text{depth}(\alpha 1\delta 10\tau x)\\ &= \text{depth}(\alpha 1\delta 01\tau x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

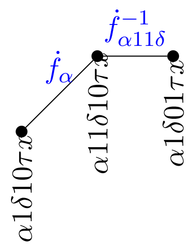
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\alpha 1\delta 01\tau x \dot{f}_{\alpha 1\delta}) = \text{depth}(\alpha 1\delta 10\tau x)\\ &= \text{depth}(\alpha 1\delta 01\tau x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \ldots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.5. $\boldsymbol{\lambda = \delta 10\tau}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}}$ **(for some** $\boldsymbol{\delta, \tau \in E(R)^*}$**)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}\dot{f}_{\alpha 11\delta}^{-1}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 1\delta 10\tau x\dot{f}_{\alpha}\dot{f}_{\alpha 11\delta}^{-1} = \alpha 11\delta 10\tau x\dot{f}_{\alpha 11\delta}^{-1} = \alpha 1\delta 01\tau x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



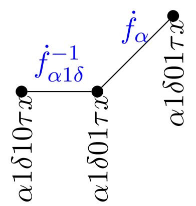
Conjugating these terms gives us $\dot{f}_{\alpha}\dot{f}_{\alpha 11\delta}^{-1} = \left(\dot{f}_{\alpha 11\delta}^{-1}\right)^{\dot{f}_{\alpha}^{-1}}\dot{f}_{\alpha} = \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_{\alpha}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_{\alpha}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \alpha 1\delta 10\tau x\dot{f}_{\alpha 1\delta}^{-1}\dot{f}_{\alpha} = \alpha 1\delta 01\tau x\dot{f}_{\alpha} = \alpha 1\delta 01\tau x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\delta 10\tau x\dot{f}_{\alpha 1\delta}^{-1}\dot{f}_{\alpha}) = \text{depth}(\alpha 1\delta 01\tau x)\\ &= \text{depth}(\alpha 1\delta 10\tau x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

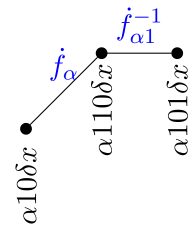
$$\begin{aligned}\text{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\alpha 1\delta 10\tau x\dot{f}_{\alpha 1\delta}^{-1}) = \text{depth}(\alpha 1\delta 01\tau x)\\ &= \text{depth}(\alpha 1\delta 10\tau x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



2.6. $\lambda = 0\delta$ **and** $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 1}^{-1}$ **(for some** $\delta \in E(R)^*$**)**

This is a Problem Case. In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha}\dot{f}_{\alpha 1}^{-1}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 10\delta x\dot{f}_{\alpha}\dot{f}_{\alpha 1}^{-1} = \alpha 110\delta x\dot{f}_{\alpha 1}^{-1} = \alpha 101\delta x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_{\alpha 1}^{-1} = \left(\dot{f}_{\alpha 1}^{-1}\right)^{\dot{f}_\alpha} \dot{f}_\alpha = \dot{f}_\alpha^{-1}\dot{f}_{\alpha 0}\dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\alpha^{-1}\dot{f}_{\alpha 0}\dot{f}_\alpha$. However in this case we observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \alpha 10\delta x \dot{f}_\alpha^{-1}\dot{f}_{\alpha 0}\dot{f}_\alpha = \alpha 01\delta x \dot{f}_{\alpha 0}\dot{f}_\alpha = \alpha 011\delta x \dot{f}_\alpha = \alpha 101\delta x$, and hence Conditions (C) and (D) are not satisfied.

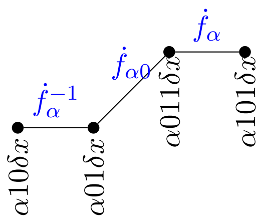
$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_3}^{\mu_3}) &= \operatorname{depth}(\alpha 10\delta x \dot{f}_\alpha^{-1}\dot{f}_{\alpha 0}\dot{f}_\alpha) = \operatorname{depth}(\alpha 101\delta x)\\ &= \operatorname{depth}(\alpha 10\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha 10\delta x \dot{f}_\alpha^{-1}) = \operatorname{depth}(\alpha 01\delta x)\\ &\leq \operatorname{depth}(\alpha 10\delta x) = \operatorname{depth}(\gamma x),\end{aligned}$$

but

$$\begin{aligned}\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dot{f}_{\beta_2}^{\mu_2}) &= \operatorname{depth}(\alpha 10\delta x \dot{f}_\alpha^{-1}\dot{f}_{\alpha 0}) = \operatorname{depth}(\alpha 011\delta x)\\ &\geq \operatorname{depth}(\alpha 10\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



3. **$\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha^{-1}$ and $\gamma x = \alpha x$ (for some $\alpha \in E(R)^*$)**

   Let us now divide into cases such that Condition (B) is satisfied:

$$\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}.$$

   3.1. $\alpha = \beta 01\delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$ (for some $\beta, \delta \in E(R)^*$),

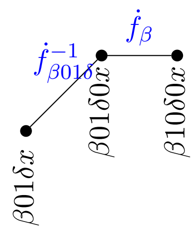
   3.2. $\alpha = \beta 10\delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$ (for some $\beta, \delta \in E(R)^*$),

   3.3. $\alpha = \beta 1$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$ (for some $\beta \in E(R)^*$).

   Let us now examine these cases:

3.1. $\boldsymbol{\alpha = \beta 01\delta}$ and $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta}$ (for some $\boldsymbol{\beta, \delta \in E(R)^*}$)

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 01\delta}^{-1}\dot{f}_\beta$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta 01\delta x \dot{f}_{\beta 01\delta}^{-1}\dot{f}_\beta = \beta 01\delta 0x \dot{f}_\beta = \beta 10\delta 0x$. The graph for depthchart$_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



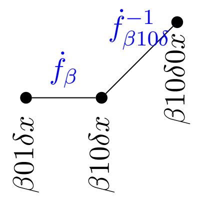
Conjugating these terms gives us $\dot{f}_{\beta 01\delta}^{-1}\dot{f}_\beta = \dot{f}_\beta\left(\dot{f}_{\beta 01\delta}^{-1}\right)^{\dot{f}_\beta} = \dot{f}_\beta\dot{f}_{\beta 10\delta}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta\dot{f}_{\beta 10\delta}^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \beta 01\delta x\dot{f}_\beta\dot{f}_{\beta 10\delta}^{-1} = \beta 10\delta x\dot{f}_{\beta 10\delta}^{-1} = \beta 10\delta 0x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta 01\delta x\dot{f}_\beta\dot{f}_{\beta 10\delta}^{-1}) = \operatorname{depth}(\beta 10\delta 0x)\\ &= \operatorname{depth}(\beta 01\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

and

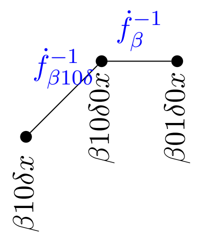
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 01\delta x\dot{f}_\beta) = \operatorname{depth}(\beta 10\delta x)\\ &= \operatorname{depth}(\beta 01\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for depthchart$_{\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



3.2. $\boldsymbol{\alpha = \beta 10\delta}$ and $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}}$ (for some $\boldsymbol{\beta, \delta \in E(R)^*}$)

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 10\delta}^{-1}\dot{f}_\beta^{-1}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta 10\delta x\dot{f}_{\beta 10\delta}^{-1}\dot{f}_\beta^{-1} = \beta 10\delta 0x\dot{f}_\beta^{-1} = \beta 01\delta 0x$. The graph for depthchart$_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot{f}_{\beta 10\delta}^{-1}\dot{f}_\beta^{-1} = \dot{f}_\beta^{-1}\left(\dot{f}_{\beta 10\delta}^{-1}\right)^{\dot{f}_\beta^{-1}} = \dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta}^{-1}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\ldots\dot{f}_{\beta_k}^{\mu_k} = \beta 10\delta x\dot{f}_\beta^{-1}\dot{f}_{\beta 01\delta}^{-1} =$

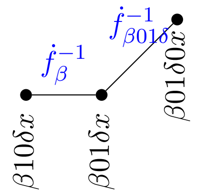
$\beta01\delta x\dot{f}_{\beta01\delta}^{-1} = \beta01\delta0x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \operatorname{depth}(\beta10\delta x\dot{f}_{\beta}^{-1}\dot{f}_{\beta01\delta}^{-1}) = \operatorname{depth}(\beta01\delta0x)\\ &= \operatorname{depth}(\beta10\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

and

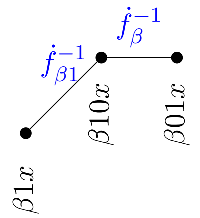
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta10\delta x\dot{f}_{\beta}^{-1}) = \operatorname{depth}(\beta01\delta x)\\ &= \operatorname{depth}(\beta10\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



3.3. $\boldsymbol{\alpha = \beta 1}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}}$ **(for some** $\boldsymbol{\beta \in E(R)^*}$**)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta1}^{-1}\dot{f}_{\beta}^{-1}$. Observe that $\gamma x\dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta1x\dot{f}_{\beta1}^{-1}\dot{f}_{\beta}^{-1} = \beta10x\dot{f}_{\beta}^{-1} = \beta01x$. The graph for $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot{f}_{\beta1}^{-1}\dot{f}_{\beta}^{-1} = \dot{f}_{\beta}^{-1}\left(\dot{f}_{\beta1}^{-1}\right)^{\dot{f}_{\beta}^{-1}} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta}^{-1}\dot{f}_{\beta0}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta}^{-1}\dot{f}_{\beta0}$. Observe that $\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta1x\dot{f}_{\beta}^{-1}\dot{f}_{\beta}^{-1}\dot{f}_{\beta0} = \beta x\dot{f}_{\beta}^{-1}\dot{f}_{\beta0} = \beta0x\dot{f}_{\beta0} = \beta01x$, and hence Conditions (C) and (D) are satisfied with $k = 3$:

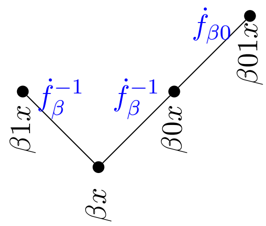
$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_3}^{\mu_3}) &= \operatorname{depth}(\beta1x\dot{f}_{\beta}^{-1}\dot{f}_{\beta}^{-1}\dot{f}_{\beta0}) = \operatorname{depth}(\beta01x)\\ &= \operatorname{depth}(\beta1x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta1x\dot{f}_{\beta}^{-1}) = \operatorname{depth}(\beta x)\\ &\le \operatorname{depth}(\beta1x) = \operatorname{depth}(\gamma x),\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dot{f}_{\beta_2}^{\mu_2}) &= \operatorname{depth}(\beta1x\dot{f}_{\beta}^{-1}\dot{f}_{\beta}^{-1}) = \operatorname{depth}(\beta0x)\\ &\le \operatorname{depth}(\beta1x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1} = \dot{f}_\alpha^{-1}}$ **and** $\boldsymbol{\gamma x = \alpha 0 \lambda x}$ **(for some** $\boldsymbol{\alpha, \lambda \in E(R)^*}$**)**

Let us now divide into cases such that Condition (B) is satisfied:

$$\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}.$$

4.1. $\alpha = \beta 01 \delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta$ (for some $\beta, \delta \in E(R)^*$),

4.2. $\alpha = \beta 10 \delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$ (for some $\beta, \delta \in E(R)^*$),

4.3. $\alpha = \beta 1$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta^{-1}$ (for some $\beta \in E(R)^*$).

4.4. $\lambda = \delta 01 \tau$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11 \delta}$ (for some $\delta, \tau \in E(R)^*$),

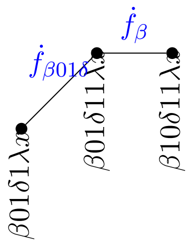
4.5. $\lambda = \delta 10 \tau$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11 \delta}^{-1}$ (for some $\delta, \tau \in E(R)^*$),

4.6. $\lambda = 1 \delta$ and $\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 0}$ (for some $\delta \in E(R)^*$).

Let us now examine these cases:

4.1. $\boldsymbol{\alpha = \beta 01 \delta}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\beta}$ **(for some** $\boldsymbol{\beta, \delta \in E(R)^*}$**)**

In this case $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 01 \delta} \dot{f}_\beta$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \beta 01 \delta 1 \lambda x \dot{f}_{\beta 01 \delta} \dot{f}_\beta = \beta 01 \delta 11 \lambda x \dot{f}_\beta = \beta 10 \delta 11 \lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



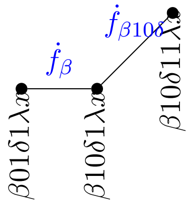
Conjugating these terms gives us $\dot{f}_{\beta 01 \delta} \dot{f}_\beta = \dot{f}_\beta \left(\dot{f}_{\beta 01 \delta} \right)^{\dot{f}_\beta} = \dot{f}_\beta \dot{f}_{\beta 10 \delta}$. Set $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_\beta \dot{f}_{\beta 10 \delta}$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \beta 01 \delta 1 \lambda x \dot{f}_\beta \dot{f}_{\beta 10 \delta} = \beta 10 \delta 1 \lambda x \dot{f}_{\beta 10 \delta} = \beta 10 \delta 11 \lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned} \text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\beta 01 \delta 1 \lambda x \dot{f}_\beta \dot{f}_{\beta 10 \delta}) = \text{depth}(\beta 10 \delta 11 \lambda x) \\ &= \text{depth}(\beta 01 \delta 1 \lambda x) + 1 = \text{depth}(\gamma x) + 1, \end{aligned}$$

and

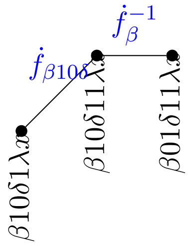
$$\begin{aligned} \text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\beta 01 \delta 1 \lambda x \dot{f}_\beta) = \text{depth}(\beta 10 \delta 1 \lambda x) \\ &= \text{depth}(\beta 01 \delta 1 \lambda x) = \text{depth}(\gamma x). \end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.2. $\boldsymbol{\alpha = \beta10\delta}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}}$ **(for some** $\boldsymbol{\beta, \delta \in E(R)^*}$**)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta10\delta}\dot{f}_{\beta}^{-1}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta10\delta1\lambda x \dot{f}_{\beta10\delta}\dot{f}_{\beta}^{-1} = \beta10\delta11\lambda x \dot{f}_{\beta}^{-1} = \beta01\delta11\lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



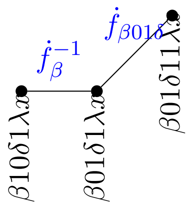
Conjugating these terms gives us $\dot{f}_{\beta10\delta}\dot{f}_{\beta}^{-1} = \dot{f}_{\beta}^{-1}\left(\dot{f}_{\beta10\delta}\right)^{\dot{f}_{\beta}^{-1}} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta01\delta}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta01\delta}$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta10\delta1\lambda x \dot{f}_{\beta}^{-1}\dot{f}_{\beta01\delta} = \beta01\delta1\lambda x \dot{f}_{\beta01\delta} = \beta01\delta11\lambda x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\beta10\delta1\lambda x \dot{f}_{\beta}^{-1}\dot{f}_{\beta01\delta}) = \text{depth}(\beta01\delta11\lambda x)\\ &= \text{depth}(\beta10\delta1\lambda x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

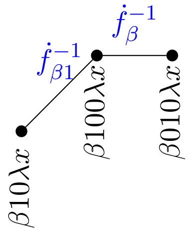
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\beta10\delta1\lambda x \dot{f}_{\beta}^{-1}) = \text{depth}(\beta01\delta1\lambda x)\\ &= \text{depth}(\beta10\delta1\lambda x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.3. $\boldsymbol{\alpha = \beta1}$ **and** $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}}$ **(for some** $\boldsymbol{\beta \in E(R)^*}$**)**

This is a Problem Case. In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta1}^{-1}\dot{f}_{\beta}^{-1}$. Observe that $\gamma x \dot{f}_{\alpha}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \beta10\lambda x \dot{f}_{\beta1}^{-1}\dot{f}_{\beta}^{-1} = \beta100\lambda x \dot{f}_{\beta}^{-1} = \beta010\lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot f_{\beta 1}^{-1}\dot f_\beta^{-1} = \dot f_\beta^{-1}\left(\dot f_{\beta 1}^{-1}\right)^{\dot f_\beta^{-1}} = \dot f_\beta^{-1}\dot f_\beta^{-1}\dot f_{\beta 0}$. Set $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k} = \dot f_\beta^{-1}\dot f_\beta^{-1}\dot f_{\beta 0}$. However in this case we observe that $\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k} = \beta 10\lambda x\dot f_\beta^{-1}\dot f_\beta^{-1}\dot f_{\beta 0} = \beta 01\lambda x\dot f_\beta^{-1}\dot f_{\beta 0} = \beta 001\lambda x\dot f_{\beta 0} = \beta 010\lambda x$, and hence Conditions (C) and (D) are not satisfied.

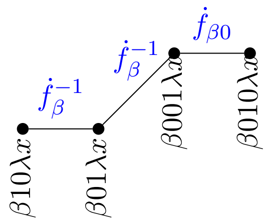
$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_3}^{\mu_3}) &= \operatorname{depth}(\beta 10\lambda x\dot f_\beta^{-1}\dot f_\beta^{-1}\dot f_{\beta 0}) = \operatorname{depth}(\beta 010\lambda x)\\ &= \operatorname{depth}(\beta 10\lambda x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\beta 10\lambda x\dot f_\beta^{-1}) = \operatorname{depth}(\beta 01\lambda x)\\ &\leq \operatorname{depth}(\beta 10\lambda x) = \operatorname{depth}(\gamma x),\end{aligned}$$

but

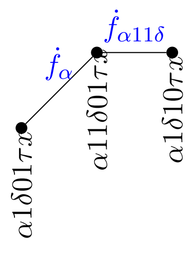
$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dot f_{\beta_2}^{\mu_2}) &= \operatorname{depth}(\beta 10\lambda x\dot f_\beta^{-1}\dot f_\beta^{-1}) = \operatorname{depth}(\beta 001\lambda x)\\ &\geq \operatorname{depth}(\beta 10\lambda x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.4. **$\lambda = \delta 01\tau$ and $\dot f_{\alpha_2}^{\eta_2} = \dot f_{\alpha 11\delta}$ (for some $\delta, \tau \in E(R)^*$)**

In this case $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \dot f_\alpha\dot f_{\alpha 11\delta}$. Observe that $\gamma x\dot f_\alpha^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \alpha 1\delta 01\tau x\dot f_\alpha\dot f_{\alpha 11\delta} = \alpha 11\delta 01\tau x\dot f_{\alpha 11\delta} = \alpha 1\delta 10\tau x$. The graph for $\operatorname{depthchart}_{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



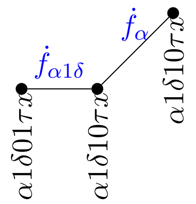
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta} = \left(\dot{f}_{\alpha 11\delta}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta}\dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 1\delta}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \alpha 1\delta 01\tau x \dot{f}_{\alpha 1\delta}\dot{f}_\alpha = \alpha 1\delta 10\tau x \dot{f}_\alpha = \alpha 1\delta 10\tau x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\delta 01\tau x \dot{f}_{\alpha 1\delta}\dot{f}_\alpha) = \text{depth}(\alpha 1\delta 10\tau x)\\ &= \text{depth}(\alpha 1\delta 01\tau x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

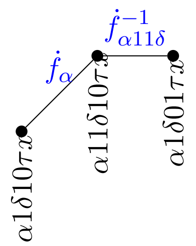
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\alpha 1\delta 01\tau x \dot{f}_{\alpha 1\delta}) = \text{depth}(\alpha 1\delta 10\tau x)\\ &= \text{depth}(\alpha 1\delta 01\tau x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.5. **$\boldsymbol{\lambda = \delta 10\tau}$ and $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\alpha 11\delta}^{-1}}$ (for some $\boldsymbol{\delta, \tau \in E(R)^*}$)**

In this case $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1}$. Observe that $\gamma x \dot{f}_\alpha^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \alpha 1\delta 10\tau x \dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \alpha 11\delta 10\tau x \dot{f}_{\alpha 11\delta}^{-1} = \alpha 1\delta 01\tau x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



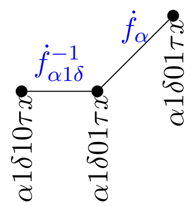
Conjugating these terms gives us $\dot{f}_\alpha \dot{f}_{\alpha 11\delta}^{-1} = \left(\dot{f}_{\alpha 11\delta}^{-1}\right)^{\dot{f}_\alpha^{-1}} \dot{f}_\alpha = \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha$. Set $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \alpha 1\delta 10\tau x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha = \alpha 1\delta 01\tau x \dot{f}_\alpha = \alpha 1\delta 01\tau x$, and hence Conditions (C) and (D) are satisfied with $k = 2$:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}) &= \text{depth}(\alpha 1\delta 10\tau x \dot{f}_{\alpha 1\delta}^{-1}\dot{f}_\alpha) = \text{depth}(\alpha 1\delta 01\tau x)\\ &= \text{depth}(\alpha 1\delta 10\tau x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

and

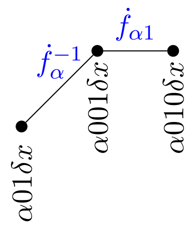
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\eta_1}) &= \text{depth}(\alpha 1\delta 10\tau x \dot{f}_{\alpha 1\delta}^{-1}) = \text{depth}(\alpha 1\delta 01\tau x)\\ &= \text{depth}(\alpha 1\delta 10\tau x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



4.6. $\boldsymbol{\lambda = 1\delta}$ **and** $\boldsymbol{\dot f_{\alpha_2}^{\eta_2} = \dot f_{\alpha 0}}$ **(for some** $\boldsymbol{\delta \in E(R)^*}$**)**

This is a Problem Case. In this case $\dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2} = \dot f_\alpha^{-1} \dot f_{\alpha 1}$. Observe that $\gamma x \dot f_\alpha^{\eta_1} \dot f_{\alpha_2}^{\eta_2} = \alpha 01\delta x \dot f_\alpha^{-1} \dot f_{\alpha 1} = \alpha 001\delta x \dot f_{\alpha 1} = \alpha 010\delta x$. The graph for depthchart$_{\dot f_{\alpha_1}^{\eta_1} \dot f_{\alpha_2}^{\eta_2}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot f_\alpha^{-1} \dot f_{\alpha 1} = \left(\dot f_{\alpha 1}\right)^{\dot f_\alpha^{-1}} \dot f_\alpha^{-1} = \dot f_\alpha \dot f_{\alpha 1}^{-1} \dot f_\alpha^{-1}$. Set $\dot f_{\beta_1}^{\mu_1} \dots \dot f_{\beta_k}^{\mu_k} = \dot f_\alpha \dot f_{\alpha 1}^{-1} \dot f_\alpha^{-1}$. However in this case we observe that $\gamma x \dot f_{\beta_1}^{\mu_1} \dots \dot f_{\beta_k}^{\mu_k} = \alpha 01\delta x \dot f_\alpha \dot f_{\alpha 1}^{-1} \dot f_\alpha^{-1} = \alpha 10\delta x \dot f_{\alpha 1}^{-1} \dot f_\alpha^{-1} = \alpha 100\delta x \dot f_\alpha^{-1} = \alpha 010\delta x$, and hence Conditions (C) and (D) are not satisfied.

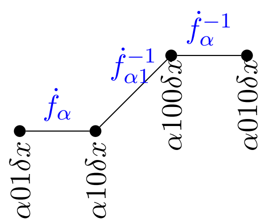
$$\begin{aligned}\operatorname{depth}(\gamma x \dot f_{\beta_1}^{\mu_1} \dots \dot f_{\beta_3}^{\mu_3}) &= \operatorname{depth}(\alpha 01\delta x \dot f_\alpha \dot f_{\alpha 1}^{-1} \dot f_\alpha^{-1}) = \operatorname{depth}(\alpha 010\delta x) \\ &= \operatorname{depth}(\alpha 01\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(\gamma x \dot f_{\beta_1}^{\eta_1}) &= \operatorname{depth}(\alpha 01\delta x \dot f_\alpha) = \operatorname{depth}(\alpha 10\delta x) \\ &\leq \operatorname{depth}(\alpha 01\delta x) = \operatorname{depth}(\gamma x),\end{aligned}$$

but

$$\begin{aligned}\operatorname{depth}(\gamma x \dot f_{\beta_1}^{\mu_1} \dot f_{\beta_2}^{\mu_2}) &= \operatorname{depth}(\alpha 01\delta x \dot f_\alpha \dot f_{\alpha 1}^{-1}) = \operatorname{depth}(\alpha 100\delta x) \\ &\geq \operatorname{depth}(\alpha 01\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for depthchart$_{\dot f_{\beta_1}^{\mu_1} \dots \dot f_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



This completes the proof. $\square$

{cantara}

Lemma 6.3.4. *Let $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:*

A. $\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \gamma x$,

B. $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}) = \mathrm{depth}(\gamma x) - 1$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 2$ such that $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}) = \mathrm{depth}(\gamma x) - 1$,

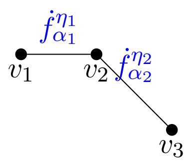
D. $\gamma x \dot{f}_{\beta_1}^{\mu_1}\dot{f}_{\beta_2}^{\mu_2} = \gamma x \dot{f}_{\beta_1}^{\mu_1}$.

Proof. Let $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:

A. $\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \gamma x$,

B. $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}) = \mathrm{depth}(\gamma x) - 1$.

Let us illustrate $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$ and $v_3 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$.

By Lemma 6.2.6, $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}}(\gamma x)$ is the reverse of $\mathrm{depthchart}_{\dot{f}_{\alpha_2}^{-\eta_2}\dot{f}_{\alpha_1}^{-\eta_1}}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2})$. All cases with this depth chart have been discussed in Lemma 6.3.2. Then there exists $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 2$ such that $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}) = \mathrm{depth}(\gamma x) - 1$,

D. $\gamma x \dot{f}_{\beta_1}^{\mu_1}\dot{f}_{\beta_2}^{\mu_2} = \gamma x \dot{f}_{\beta_1}^{\mu_1}$.

□

{usgirls}

Lemma 6.3.5. *Let $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \alpha_2 \in \Omega^*$ and $\eta_1, \eta_2 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that the following conditions are satisfied:*

A. $\gamma x \neq \gamma x \dot{f}_{\alpha_1}^{\eta_1}$ *and* $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}) = \mathrm{depth}(\gamma x)$,

B. $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}) = \mathrm{depth}(\gamma x) - 1$.

Then there exists $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}\in W(\dot{\mathcal{X}})$ *(for some* $\beta_1,\dots,\beta_k\in\Omega^*$ *and* $\mu_1,\dots,\mu_k\in\{\pm1\}$*) with* $k=2,3$ *such that* $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}=\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}$ *in* $\dot F$ *and the following conditions are satisfied:*

C. $\mathrm{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k})=\mathrm{depth}(\gamma x)-1$,

D. $\mathrm{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_i}^{\mu_i})\le\mathrm{depth}(\gamma x)-1$ *for* $i=1,2$.

Except for the following problem cases:

i. $\boldsymbol{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta}\dot f_{\beta1}}$ ***and*** $\boldsymbol{\gamma x=\beta010\delta x}$ ***(for some*** $\boldsymbol{\beta,\delta\in E(R)^*}$***),***

ii. $\boldsymbol{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta1}\dot f_{\beta}^{-1}}$ ***and*** $\boldsymbol{\gamma x=\beta101\delta x}$ ***(for some*** $\boldsymbol{\beta,\delta\in E(R)^*}$***),***

iii. $\boldsymbol{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta0}^{-1}\dot f_{\beta}}$ ***and*** $\boldsymbol{\gamma x=\beta010\delta x}$ ***(for some*** $\boldsymbol{\beta,\delta\in E(R)^*}$***),***

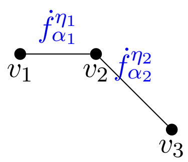
iv. $\boldsymbol{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta}^{-1}\dot f_{\beta0}^{-1}}$ ***and*** $\boldsymbol{\gamma x=\beta101\delta x}$ ***(for some*** $\boldsymbol{\beta,\delta\in E(R)^*}$***).***

Proof. Let $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}\in W(\dot{\mathcal{X}})$ (for some $\alpha_1,\alpha_2\in\Omega^*$ and $\eta_1,\eta_2\in\{\pm1\}$) and let $\gamma x\in\mathcal{GV}$ (for some $\gamma\in\Omega^*$) such that the following conditions are satisfied:

A. $\gamma x\ne\gamma x\dot f_{\alpha_1}^{\eta_1}$ and $\mathrm{depth}(\gamma x\dot f_{\alpha_1}^{\eta_1})=\mathrm{depth}(\gamma x)$,

B. $\mathrm{depth}(\gamma x\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2})=\mathrm{depth}(\gamma x)-1$.

Let us illustrate $\mathrm{depthchart}_{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}}(\gamma x)$ as follows:



where $v_1=\gamma x$, $v_2=\gamma x\dot f_{\alpha_1}^{\eta_1}$ and $v_3=\gamma x\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}$.

By Lemma 6.2.6, $\mathrm{depthchart}_{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}}(\gamma x)$ is the reverse of $\mathrm{depthchart}_{\dot f_{\alpha_2}^{-\eta_2}\dot f_{\alpha_1}^{-\eta_1}}(\gamma x\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2})$. All cases with this depth chart have been discussed in Lemma 6.3.3. Then there exists $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}\in W(\dot{\mathcal{X}})$ (for some $\beta_1,\dots,\beta_k\in\Omega^*$ and $\mu_1,\dots,\mu_k\in\{\pm1\}$) with $k=2,3$ such that $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}=\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}$ in $\dot F$ and the following conditions are satisfied:

C. $\mathrm{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k})=\mathrm{depth}(\gamma x)-1$,

D. $\mathrm{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_i}^{\mu_i})\le\mathrm{depth}(\gamma x)-1$ for $i=1,2$.

Except for the following problem cases:

i. $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta}\dot f_{\beta1}$ and $\gamma x=\beta010\delta x$ (for some $\beta,\delta\in E(R)^*$),

ii. $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta1}\dot f_{\beta}^{-1}$ and $\gamma x=\beta101\delta x$ (for some $\beta,\delta\in E(R)^*$),

iii. $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta0}^{-1}\dot f_{\beta}$ and $\gamma x=\beta010\delta x$ (for some $\beta,\delta\in E(R)^*$),

iv. $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}=\dot f_{\beta}^{-1}\dot f_{\beta0}^{-1}$ and $\gamma x=\beta101\delta x$ (for some $\beta,\delta\in E(R)^*$).

□

{sonoio}

Lemma 6.3.6. *Let $\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_3 \in \Omega^*$ and $\eta_1, \dots, \eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ is a problem case from Lemma 6.3.3 and $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3}$ is a problem case from Lemma 6.3.5, and the following conditions are satisfied:*

A. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_i}^{\eta_i}) = \operatorname{depth}(\gamma x) + 1$ *for* $i = 1, 2$,

B. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3}) = \operatorname{depth}(\gamma x)$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 4$ such that $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dot{f}_{\beta_2}^{\mu_2}) = \operatorname{depth}(\gamma x) + 1$,

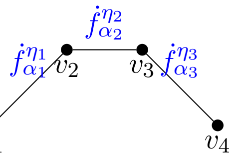
D. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_i}^{\mu_i}) = \operatorname{depth}(\gamma x)$ *for* $i = 1, 3, 4$.

Proof. Let $\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_3 \in \Omega^*$ and $\eta_1, \dots, \eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ is a problem case from Lemma 6.3.3 and $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3}$ is a problem case from Lemma 6.3.5, and the following conditions are satisfied:

A. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_i}^{\eta_i}) = \operatorname{depth}(\gamma x) + 1$ for $i = 1, 2$,

B. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3}) = \operatorname{depth}(\gamma x)$.

Let us illustrate $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \cdots \dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ as follows:



The problem cases from Lemma 6.3.3 are:

i. $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0} \dot{f}_{\beta}$ and $\gamma x = \beta 01 \delta x$ (for some $\beta, \delta \in E(R)^*$),

ii. $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta} \dot{f}_{\beta 1}^{-1}$ and $\gamma x = \beta 10 \delta x$ (for some $\beta, \delta \in E(R)^*$),

iii. $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1} \dot{f}_{\beta 0}$ and $\gamma x = \beta 01 \delta x$ (for some $\beta, \delta \in E(R)^*$),

iv. $\dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}^{-1} \dot{f}_{\beta}^{-1}$ and $\gamma x = \beta 10 \delta x$ (for some $\beta, \delta \in E(R)^*$).

The problem cases from Lemma 6.3.5 are:

i. $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha} \dot{f}_{\alpha 1}$ and $\gamma x = \alpha 010 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$),

ii. $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 1} \dot{f}_{\alpha}^{-1}$ and $\gamma x = \alpha 101 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$),

iii. $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 0}^{-1} \dot{f}_{\alpha}$ and $\gamma x = \alpha 010 \lambda x$ (for some $\alpha, \lambda \in E(R)^*$),

iv. $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}^{-1}$ and $\gamma x = \alpha 101\lambda x$ (for some $\alpha, \lambda \in E(R)^*$).

This gives us the following eight choices for $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}$:

1. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_{\beta}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}\dot{f}_{\alpha 1}$, $\gamma x = \beta 01\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 010\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
2. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_{\beta}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 1}\dot{f}_{\alpha}^{-1}$, $\gamma x = \beta 01\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
3. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 0}^{-1}\dot{f}_{\alpha}$, $\gamma x = \beta 10\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 010\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
4. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}^{-1}$, $\gamma x = \beta 10\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
5. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}\dot{f}_{\alpha 1}$, $\gamma x = \beta 01\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 010\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
6. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 1}\dot{f}_{\alpha}^{-1}$, $\gamma x = \beta 01\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
7. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta}^{-1}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 0}^{-1}\dot{f}_{\alpha}$, $\gamma x = \beta 10\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 010\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$),
8. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta}^{-1}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}^{-1}$, $\gamma x = \beta 10\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$).

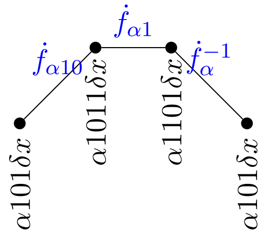
Let us examine each case in detail:

1. **$\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_{\beta}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}\dot{f}_{\alpha 1}$, $\gamma x = \beta 01\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 010\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$)**

   In this case $\beta = \alpha$. This gives us $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 0}\dot{f}_{\alpha}\dot{f}_{\alpha 1}$. Then $\gamma x = \alpha 01\delta x$. Observe that $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 01\delta x\dot{f}_{\alpha 0} = \alpha 011\delta x$. But in our hypothesis $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 010\lambda x$. There do not exist $\delta, \gamma \in E(R)^*$ such that $1\delta = 0\gamma$. **Therefore this case is not valid.**

2. **$\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_{\beta}$, $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 1}\dot{f}_{\alpha}^{-1}$, $\gamma x = \beta 01\delta x$ and $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$ (for some $\beta, \delta, \alpha, \lambda \in E(R)^*$)**

   In this case $\beta = \alpha 1$. This gives us $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 10}\dot{f}_{\alpha 1}\dot{f}_{\alpha}^{-1}$. Then $\gamma x = \alpha 101\delta x$. Observe that $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\delta x\dot{f}_{\alpha 10} = \alpha 1011\delta x$. But in our hypothesis $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$. This implies that $\lambda = 1\delta$. Hence $\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \alpha 101\delta x\dot{f}_{\alpha 10}\dot{f}_{\alpha 1}\dot{f}_{\alpha}^{-1} = \alpha 1011\delta x\dot{f}_{\alpha 1}\dot{f}_{\alpha}^{-1} = \alpha 1101\delta x\dot{f}_{\alpha}^{-1} = \alpha 101\delta x$. The graph for depthchart$_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot f_{\alpha 10}\dot f_{\alpha 1}\dot f_{\alpha}^{-1} = \dot f_{\alpha 1}\left(\dot f_{\alpha 10}\right)^{\dot f_{\alpha 1}}\dot f_{\alpha}^{-1} = \dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha 11}^{-1}\dot f_{\alpha}^{-1} = \dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha}^{-1}\left(\dot f_{\alpha 11}^{-1}\right)^{\dot f_{\alpha}^{-1}} = \dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha}^{-1}\dot f_{\alpha 1}^{-1}$. Set $\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k} = \dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha}^{-1}\dot f_{\alpha 1}^{-1}$. Observe that $\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k} = \alpha 101\delta x\dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha}^{-1}\dot f_{\alpha 1}^{-1} = \alpha 110\delta x\dot f_{\alpha 1}\dot f_{\alpha}^{-1}\dot f_{\alpha 1}^{-1} = \alpha 1110\delta x\dot f_{\alpha}^{-1}\dot f_{\alpha 1}^{-1} = \alpha 110\delta x\dot f_{\alpha 1}^{-1} = \alpha 101\delta x$, and hence Conditions (C) and (D) are satisfied:

$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}) &= \operatorname{depth}(\alpha 101\delta x\dot f_{\alpha 1}) = \operatorname{depth}(\alpha 110\delta x)\\ &= \operatorname{depth}(\alpha 101\delta x) = \operatorname{depth}(\gamma x),\end{aligned}$$

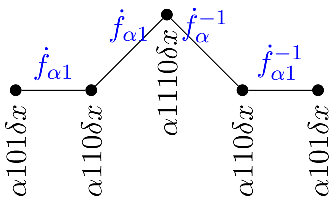
$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dot f_{\beta_2}^{\mu_2}) &= \operatorname{depth}(\alpha 101\delta x\dot f_{\alpha 1}\dot f_{\alpha 1}) = \operatorname{depth}(\alpha 1110\delta x)\\ &= \operatorname{depth}(\alpha 101\delta x) + 1 = \operatorname{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_3}^{\mu_3}) &= \operatorname{depth}(\alpha 101\delta x\dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha}^{-1}) = \operatorname{depth}(\alpha 110\delta x)\\ &= \operatorname{depth}(\alpha 101\delta x) = \operatorname{depth}(\gamma x),\end{aligned}$$

and

$$\begin{aligned}\operatorname{depth}(\gamma x\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_4}^{\mu_4}) &= \operatorname{depth}(\alpha 101\delta x\dot f_{\alpha 1}\dot f_{\alpha 1}\dot f_{\alpha}^{-1}\dot f_{\alpha 1}^{-1}) = \operatorname{depth}(\alpha 101\delta x)\\ &= \operatorname{depth}(\alpha 101\delta x) = \operatorname{depth}(\gamma x).\end{aligned}$$

The graph for $\operatorname{depthchart}_{\dot f_{\beta_1}^{\mu_1}\dots\dot f_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:

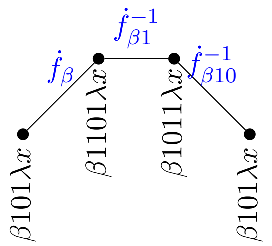


3. $\boldsymbol{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \dot f_{\beta}\dot f_{\beta 1}^{-1}}$**,** $\boldsymbol{\dot f_{\alpha_2}^{\eta_2}\dot f_{\alpha_3}^{\eta_3} = \dot f_{\alpha 0}^{-1}\dot f_{\alpha}}$**,** $\boldsymbol{\gamma x = \beta 10\delta x}$ **and** $\boldsymbol{\gamma x\dot f_{\alpha_1}^{\eta_1} = \alpha 010\lambda x}$ **(for some** $\boldsymbol{\beta, \delta, \alpha, \lambda \in E(R)^*}$**)**

   In this case $\beta 1 = \alpha 0$. But there do not exist $\alpha, \beta \in E(R)^*$ such that this is true. **Therefore this case is not valid.**

4. $\boldsymbol{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \dot f_{\beta}\dot f_{\beta 1}^{-1}}$**,** $\boldsymbol{\dot f_{\alpha_2}^{\eta_2}\dot f_{\alpha_3}^{\eta_3} = \dot f_{\alpha}^{-1}\dot f_{\alpha 0}^{-1}}$**,** $\boldsymbol{\gamma x = \beta 10\delta x}$ **and** $\boldsymbol{\gamma x\dot f_{\alpha_1}^{\eta_1} = \alpha 101\lambda x}$ **(for some** $\boldsymbol{\beta, \delta, \alpha, \lambda \in E(R)^*}$**)**

In this case $\alpha = \beta 1$. This gives us $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 10}^{-1}$. Then $\gamma x = \beta 10\delta x$. Observe that $\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \beta 10\delta x \dot{f}_{\beta} = \beta 110\delta x$. But in our hypothesis $\gamma x \dot{f}_{\alpha_1}^{\eta_1} = \beta 1101\lambda x$. This implies that $\delta = 1\lambda$. Hence $\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \beta 101\lambda x \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 10}^{-1} = \beta 1101\lambda x \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 10}^{-1} = \beta 1011\lambda x \dot{f}_{\beta 10}^{-1} = \beta 101\lambda x$. The graph for $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 10}^{-1} = \dot{f}_{\beta}\left(\dot{f}_{\beta 10}^{-1}\right)^{\dot{f}_{\beta 1}^{-1-1}}\dot{f}_{\beta 1}^{-1} = \dot{f}_{\beta}\dot{f}_{\beta 11}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1} = \left(\dot{f}_{\beta 11}\right)^{\dot{f}_{\beta}^{-1}}\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1} = \dot{f}_{\beta 1}\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1}$. Set $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\beta 1}\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1}$. Observe that $\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \beta 101\lambda x \dot{f}_{\beta 1}\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1} = \beta 110\lambda x \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1} = \beta 1110\lambda x \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1} = \beta 110\lambda x \dot{f}_{\beta 1}^{-1} = \beta 101\lambda x$, and hence Conditions (C) and (D) are satisfied:

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}) &= \text{depth}(\beta 101\lambda x \dot{f}_{\beta 1}) = \text{depth}(\beta 110\lambda x)\\ &= \text{depth}(\beta 101\lambda x) = \text{depth}(\gamma x),\end{aligned}$$

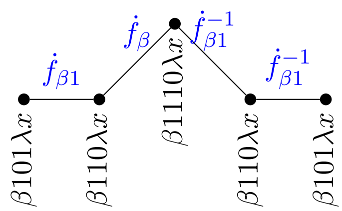
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dot{f}_{\beta_2}^{\mu_2}) &= \text{depth}(\beta 101\lambda x \dot{f}_{\beta 1}\dot{f}_{\beta}) = \text{depth}(\beta 1110\lambda x)\\ &= \text{depth}(\beta 101\lambda x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_3}^{\mu_3}) &= \text{depth}(\beta 101\lambda x \dot{f}_{\beta 1}\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}) = \text{depth}(\beta 110\lambda x)\\ &= \text{depth}(\beta 101\lambda x) = \text{depth}(\gamma x),\end{aligned}$$

and

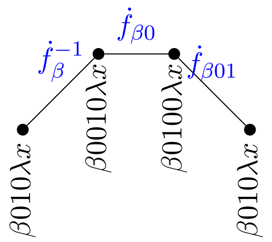
$$\begin{aligned}\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_4}^{\mu_4}) &= \text{depth}(\beta 101\lambda x \dot{f}_{\beta 1}\dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}\dot{f}_{\beta 1}^{-1}) = \text{depth}(\beta 101\lambda x)\\ &= \text{depth}(\beta 101\lambda x) = \text{depth}(\gamma x).\end{aligned}$$

The graph for $\text{depthchart}_{\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:



5. **$\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2} = \dot f_\beta^{-1}\dot f_{\beta0}$, $\dot f_{\alpha_2}^{\eta_2}\dot f_{\alpha_3}^{\eta_3} = \dot f_\alpha \dot f_{\alpha1}$, $\gamma x = \beta01\delta x$ and $\gamma x \dot f_{\alpha_1}^{\eta_1} = \alpha010\lambda x$ (for some $\beta,\delta,\alpha,\lambda \in E(R)^*$)**

In this case $\alpha = \beta0$. This gives us $\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}\dot f_{\alpha_3}^{\eta_3} = \dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta01}$. Then $\gamma x = \beta01\delta x$. Observe that $\gamma x \dot f_{\alpha_1}^{\eta_1} = \beta01\delta x \dot f_\beta^{-1} = \beta001\delta x$. But in our hypothesis $\gamma x \dot f_{\alpha_1}^{\eta_1} = \beta0010\lambda x$. This implies that $\delta = 0\lambda$. Hence $\gamma x \dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}\dot f_{\alpha_3}^{\eta_3} = \beta010\lambda x \dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta01} = \beta0010\lambda x \dot f_{\beta0}\dot f_{\beta01} = \beta0100\lambda x \dot f_{\beta01} = \beta010\lambda x$. The graph for $\mathrm{depthchart}_{\dot f_{\alpha_1}^{\eta_1}\dot f_{\alpha_2}^{\eta_2}\dot f_{\alpha_3}^{\eta_3}}(\gamma x)$ is then as follows:



Conjugating these terms gives us $\dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta01} = \dot f_\beta^{-1}\left(\dot f_{\beta01}\right)^{\dot f_{\beta0}^{-1}}\dot f_{\beta0} = \dot f_\beta^{-1}\dot f_{\beta00}^{-1}\dot f_{\beta0}\dot f_{\beta0} = \left(\dot f_{\beta00}^{-1}\right)^{\dot f_\beta^{-1-1}}\dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta0} = \dot f_{\beta0}^{-1}\dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta0}$. Set $\dot f_{\beta_1}^{\mu_1}\ldots\dot f_{\beta_k}^{\mu_k} = \dot f_{\beta0}^{-1}\dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta0}$. Observe that $\gamma x \dot f_{\beta_1}^{\mu_1}\ldots\dot f_{\beta_k}^{\mu_k} = \beta010\lambda x \dot f_{\beta0}^{-1}\dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta0} = \beta001\lambda x \dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta0} = \beta0001\lambda x \dot f_{\beta0}\dot f_{\beta0} = \beta001\lambda x \dot f_{\beta0} = \beta010\lambda x$, and hence Conditions (C) and (D) are satisfied:

$$\begin{aligned}\mathrm{depth}(\gamma x \dot f_{\beta_1}^{\mu_1}) &= \mathrm{depth}(\beta010\lambda x \dot f_{\beta0}^{-1}) = \mathrm{depth}(\beta001\lambda x)\\ &= \mathrm{depth}(\beta010\lambda x) = \mathrm{depth}(\gamma x),\end{aligned}$$

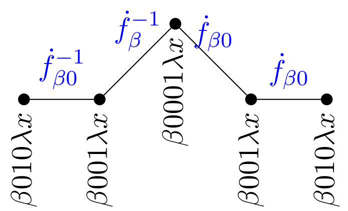
$$\begin{aligned}\mathrm{depth}(\gamma x \dot f_{\beta_1}^{\mu_1}\dot f_{\beta_2}^{\mu_2}) &= \mathrm{depth}(\beta010\lambda x \dot f_{\beta0}^{-1}\dot f_\beta^{-1}) = \mathrm{depth}(\beta0001\lambda x)\\ &= \mathrm{depth}(\beta010\lambda x) + 1 = \mathrm{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\mathrm{depth}(\gamma x \dot f_{\beta_1}^{\mu_1}\ldots\dot f_{\beta_3}^{\mu_3}) &= \mathrm{depth}(\beta010\lambda x \dot f_{\beta0}^{-1}\dot f_\beta^{-1}\dot f_{\beta0}) = \mathrm{depth}(\beta001\lambda x)\\ &= \mathrm{depth}(\beta010\lambda x) = \mathrm{depth}(\gamma x),\end{aligned}$$

and

$$\begin{aligned}\mathrm{depth}(\gamma x \dot f_{\beta_1}^{\mu_1}\ldots\dot f_{\beta_4}^{\mu_4}) &= \mathrm{depth}(\beta010\lambda x \dot f_{\beta0}^{-1}\dot f_\beta^{-1}\dot f_{\beta0}\dot f_{\beta0}) = \mathrm{depth}(\beta010\lambda x)\\ &= \mathrm{depth}(\beta010\lambda x) = \mathrm{depth}(\gamma x).\end{aligned}$$

The graph for $\mathrm{depthchart}_{\dot f_{\beta_1}^{\mu_1}\ldots\dot f_{\beta_k}^{\mu_k}}(\gamma x)$ is then as follows:

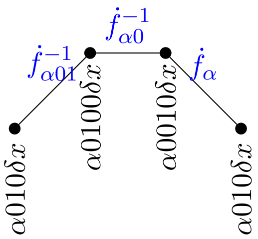


6. $\boldsymbol{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{-1}_{\beta}\dot{f}_{\beta 0}}$, $\boldsymbol{\dot{f}^{\eta_2}_{\alpha_2}\dot{f}^{\eta_3}_{\alpha_3} = \dot{f}_{\alpha 1}\dot{f}^{-1}_{\alpha}}$, $\boldsymbol{\gamma x = \beta 01\delta x}$ **and** $\boldsymbol{\gamma x\dot{f}^{\eta_1}_{\alpha_1} = \alpha 101\lambda x}$ **(for some** $\boldsymbol{\beta, \delta, \alpha, \lambda \in E(R)^*}$**)**

   In this case $\beta 0 = \alpha 1$. But there do not exist $\alpha, \beta \in E(R)^*$ such that this is true. **Therefore this case is not valid.**

7. $\boldsymbol{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2} = \dot{f}^{-1}_{\beta 1}\dot{f}^{-1}_{\beta}}$, $\boldsymbol{\dot{f}^{\eta_2}_{\alpha_2}\dot{f}^{\eta_3}_{\alpha_3} = \dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}}$, $\boldsymbol{\gamma x = \beta 10\delta x}$ **and** $\boldsymbol{\gamma x\dot{f}^{\eta_1}_{\alpha_1} = \alpha 010\lambda x}$ **(for some** $\boldsymbol{\beta, \delta, \alpha, \lambda \in E(R)^*}$**)**

   In this case $\beta = \alpha 0$. This gives us $\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}\dot{f}^{\eta_3}_{\alpha_3} = \dot{f}^{-1}_{\alpha 01}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}$. Then $\gamma x = \alpha 010\delta x$. Observe that $\gamma x\dot{f}^{\eta_1}_{\alpha_1} = \alpha 010\delta x\dot{f}^{-1}_{\alpha 01} = \alpha 0100\delta x$. But in our hypothesis $\gamma x\dot{f}^{\eta_1}_{\alpha_1} = \alpha 010\lambda x$. This implies that $\lambda = 0\delta$. Hence $\gamma x\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}\dot{f}^{\eta_3}_{\alpha_3} = \alpha 010\delta x\dot{f}^{-1}_{\alpha 01}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha} = \alpha 0100\delta x\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha} = \alpha 0010\delta x\dot{f}_{\alpha} = \alpha 010\delta x$. The graph for $\text{depthchart}_{\dot{f}^{\eta_1}_{\alpha_1}\dot{f}^{\eta_2}_{\alpha_2}\dot{f}^{\eta_3}_{\alpha_3}}(\gamma x)$ is then as follows:



   Conjugating these terms gives us $\dot{f}^{-1}_{\alpha 01}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha} = \dot{f}^{-1}_{\alpha 0}\left(\dot{f}^{-1}_{\alpha 01}\right)^{\dot{f}^{-1}_{\alpha 0}}\dot{f}_{\alpha} = \dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha 00}\dot{f}_{\alpha} = \dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}\left(\dot{f}_{\alpha 00}\right)^{\dot{f}_{\alpha}} = \dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}\dot{f}_{\alpha 0}$. Set $\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}\dot{f}_{\alpha 0}$. Observe that $\gamma x\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k} = \alpha 010\delta x\dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}\dot{f}_{\alpha 0} = \alpha 001\delta x\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}\dot{f}_{\alpha 0} = \alpha 0001\delta x\dot{f}_{\alpha}\dot{f}_{\alpha 0} = \alpha 001\delta x\dot{f}_{\alpha 0} = \alpha 010\delta x$, and hence Conditions (C) and (D) are satisfied:

$$\begin{aligned}\text{depth}(\gamma x\dot{f}^{\mu_1}_{\beta_1}) &= \text{depth}(\alpha 010\delta x\dot{f}^{-1}_{\alpha 0}) = \text{depth}(\alpha 001\delta x)\\ &= \text{depth}(\alpha 010\delta x) = \text{depth}(\gamma x),\end{aligned}$$

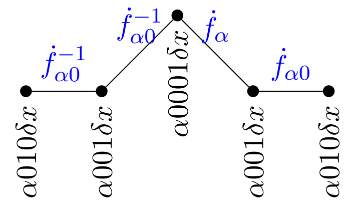
$$\begin{aligned}\text{depth}(\gamma x\dot{f}^{\mu_1}_{\beta_1}\dot{f}^{\mu_2}_{\beta_2}) &= \text{depth}(\alpha 010\delta x\dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}) = \text{depth}(\alpha 0001\delta x)\\ &= \text{depth}(\alpha 010\delta x) + 1 = \text{depth}(\gamma x) + 1,\end{aligned}$$

$$\begin{aligned}\text{depth}(\gamma x\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_3}_{\beta_3}) &= \text{depth}(\alpha 010\delta x\dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}) = \text{depth}(\alpha 001\delta x)\\ &= \text{depth}(\alpha 010\delta x) = \text{depth}(\gamma x),\end{aligned}$$

   and

$$\begin{aligned}\text{depth}(\gamma x\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_4}_{\beta_4}) &= \text{depth}(\alpha 010\delta x\dot{f}^{-1}_{\alpha 0}\dot{f}^{-1}_{\alpha 0}\dot{f}_{\alpha}\dot{f}_{\alpha 0}) = \text{depth}(\alpha 010\delta x)\\ &= \text{depth}(\alpha 010\delta x) = \text{depth}(\gamma x).\end{aligned}$$

   The graph for $\text{depthchart}_{\dot{f}^{\mu_1}_{\beta_1}\dots\dot{f}^{\mu_k}_{\beta_k}}(\gamma x)$ is then as follows:



8. $\boldsymbol{\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta}^{-1}}$, $\boldsymbol{\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}^{-1}}$, $\boldsymbol{\gamma x = \beta 10\delta x}$ **and** $\boldsymbol{\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x}$ **(for some** $\boldsymbol{\beta, \delta, \alpha, \lambda \in E(R)^*}$**)**

   In this case $\beta = \alpha$. This gives us $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3} = \dot{f}_{\alpha 1}^{-1}\dot{f}_{\alpha}^{-1}\dot{f}_{\alpha 0}^{-1}$. Then $\gamma x = \alpha 10\delta x$. Observe that $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 10\delta x\dot{f}_{\alpha 1}^{-1} = \alpha 100\delta x$. But in our hypothesis $\gamma x\dot{f}_{\alpha_1}^{\eta_1} = \alpha 101\lambda x$. There do not exist $\delta, \gamma \in E(R)^*$ such that $0\delta = 1\gamma$. **Therefore this case is not valid.**

This completes the proof. □

{strawberryswi

Lemma 6.3.7. *Let $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1,\dots,\alpha_3 \in \Omega^*$ and $\eta_1,\dots,\eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ is a problem case from Lemma 6.3.3, and the following conditions are satisfied:*

A. $\mathrm{depth}(\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_i}^{\eta_i}) = \mathrm{depth}(\gamma x) + 1$ *for* $i = 1, 2, 3$,

B. $\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} = \gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1,\dots,\beta_k \in \Omega^*$ and $\mu_1,\dots,\mu_k \in \{\pm 1\}$) with $k = 3$ such that $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \mathrm{depth}(\gamma x)$ *for* $i = 1$,

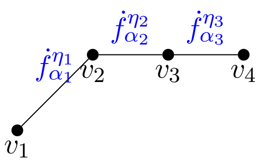
D. $\mathrm{depth}(\gamma x\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \mathrm{depth}(\gamma x) + 1$ *for* $i = 2, 3$.

Proof. Let $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1,\dots,\alpha_3 \in \Omega^*$ and $\eta_1,\dots,\eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ is a problem case from Lemma 6.3.3, and the following conditions are satisfied:

A. $\mathrm{depth}(\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_i}^{\eta_i}) = \mathrm{depth}(\gamma x) + 1$ for $i = 1, 2, 3$,

B. $\gamma x\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} = \gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$.

Let us illustrate $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x\dot{f}_{\alpha_1}^{\eta_1}$ and $v_3 = v_4 = \gamma x\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$. The problem cases from Lemma 6.3.3 are:

1. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_{\beta}$ and $\gamma x = \beta 01\delta x$ (for some $\beta, \delta \in E(R)^*$),
2. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}$ and $\gamma x = \beta 10\delta x$ (for some $\beta, \delta \in E(R)^*$),
3. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}$ and $\gamma x = \beta 01\delta x$ (for some $\beta, \delta \in E(R)^*$),
4. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta}^{-1}$ and $\gamma x = \beta 10\delta x$ (for some $\beta, \delta \in E(R)^*$).

The rest of this proof proceeds similarly to Lemma 6.3.6.

□

toutofmyhouse}

Lemma 6.3.8. *Let $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_3 \in \Omega^*$ and $\eta_1, \dots, \eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ is a problem case from Lemma 6.3.3, and the following conditions are satisfied:*

A. $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_i}^{\eta_i}) = \mathrm{depth}(\gamma x) + 1$ *for* $i = 1, 2, 3$,

B. $\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \neq \gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ *and* $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}) = \mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2})$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 2, 3$ such that $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \mathrm{depth}(\gamma x)$ *for* $i = 1$,

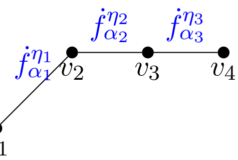
D. $\mathrm{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \mathrm{depth}(\gamma x) + 1$ *for* $i = 2, 3$.

Proof. Let $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_3 \in \Omega^*$ and $\eta_1, \dots, \eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ is a problem case from Lemma 6.3.3, and the following conditions are satisfied:

A. $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_i}^{\eta_i}) = \mathrm{depth}(\gamma x) + 1$ for $i = 1, 2, 3$,

B. $\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \neq \gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ and $\mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}) = \mathrm{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2})$.

Let us illustrate $\mathrm{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\cdots\dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$, $v_3 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ and $v_4 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3}$. The problem cases from Lemma 6.3.3 are:

i. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 0}\dot{f}_{\beta}$ and $\gamma x = \beta 01\delta x$ (for some $\beta, \delta \in E(R)^*$),

ii. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}\dot{f}_{\beta 1}^{-1}$ and $\gamma x = \beta 10\delta x$ (for some $\beta, \delta \in E(R)^*$),

iii. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta}^{-1}\dot{f}_{\beta 0}$ and $\gamma x = \beta 01\delta x$ (for some $\beta, \delta \in E(R)^*$),

iv. $\dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2} = \dot{f}_{\beta 1}^{-1}\dot{f}_{\beta}^{-1}$ and $\gamma x = \beta 10\delta x$ (for some $\beta, \delta \in E(R)^*$).

The rest of this proof proceeds similarly to Lemma 6.3.6. □

{newnoise}

Lemma 6.3.9. *Let $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1,\dots,\alpha_3 \in \Omega^*$ and $\eta_1,\dots,\eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3}$ is a problem case from Lemma 6.3.5, and the following conditions are satisfied:*

A. $\gamma x = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$,

B. $\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}) = \text{depth}(\gamma x)$,

C. $\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}) = \text{depth}(\gamma x) - 1$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1,\dots,\beta_k \in \Omega^*$ and $\mu_1,\dots,\mu_k \in \{\pm 1\}$) with $k = 3$ such that $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

D. $\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \text{depth}(\gamma x)$ *for* $i = 1$,

E. $\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \text{depth}(\gamma x) - 1$ *for* $i = 2, 3$.

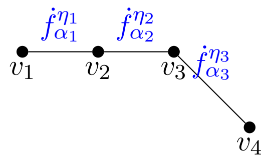
Proof. Let $\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1,\dots,\alpha_3 \in \Omega^*$ and $\eta_1,\dots,\eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3}$ is a problem case from Lemma 6.3.5, and the following conditions are satisfied:

A. $\gamma x = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$,

B. $\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}) = \text{depth}(\gamma x)$,

C. $\text{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}) = \text{depth}(\gamma x) - 1$.

Let us illustrate $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ as follows:



where $v_1 = v_2 = \gamma x$, $v_3 = \gamma x \dot{f}_{\alpha_2}^{\eta_2}$ and $v_4 = \gamma x \dot{f}_{\alpha_2}^{\eta_2}\dot{f}_{\alpha_3}^{\eta_3}$. By Lemma 6.2.6, $\text{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ is the reverse of $\text{depthchart}_{\dot{f}_{\alpha_3}^{-\eta_3}\dots\dot{f}_{\alpha_1}^{-\eta_1}}(\gamma x \dot{f}_{\alpha_1}^{\eta_1}\dots\dot{f}_{\alpha_3}^{\eta_3})$. All cases with this depth chart have been discussed in Lemma 6.3.7. Then there exists $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1,\dots,\beta_k \in \Omega^*$ and $\mu_1,\dots,\mu_k \in \{\pm 1\}$) with $k = 3$ such that $\dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1}\dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \text{depth}(\gamma x)$ for $i = 1$,

D. $\text{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1}\dots\dot{f}_{\beta_i}^{\mu_i}) = \text{depth}(\gamma x) - 1$ for $i = 2, 3$.

□

{blackcelebrat

Lemma 6.3.10. *Let $\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_3 \in \Omega^*$ and $\eta_1, \dots, \eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3}$ is a problem case from Lemma 6.3.5, and the following conditions are satisfied:*

A. $\gamma x \neq \gamma x \dot{f}_{\alpha_1}^{\eta_1}$,

B. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_i}^{\eta_i}) = \operatorname{depth}(\gamma x)$ *for* $i = 1, 2$,

C. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3}) = \operatorname{depth}(\gamma x) - 1$.

Then there exists $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 2, 3$ such that $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

D. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_i}^{\mu_i}) = \operatorname{depth}(\gamma x)$ *for* $i = 1$,

E. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_i}^{\mu_i}) = \operatorname{depth}(\gamma x) + 1$ *for* $i = 2, 3$.

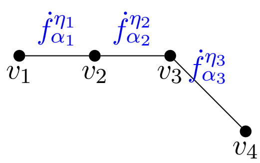
Proof. Let $\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_3 \in \Omega^*$ and $\eta_1, \dots, \eta_3 \in \{\pm 1\}$) and let $\gamma x \in \mathcal{GV}$ (for some $\gamma \in \Omega^*$) such that $\dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3}$ is a problem case from Lemma 6.3.5, and the following conditions are satisfied:

A. $\gamma x \neq \gamma x \dot{f}_{\alpha_1}^{\eta_1}$,

B. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_i}^{\eta_i}) = \operatorname{depth}(\gamma x)$ for $i = 1, 2$,

C. $\operatorname{depth}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3}) = \operatorname{depth}(\gamma x) - 1$.

Let us illustrate $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ as follows:



where $v_1 = \gamma x$, $v_2 = \gamma x \dot{f}_{\alpha_1}^{\eta_1}$, $v_3 = \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ and $v_4 = \gamma x \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2} \dot{f}_{\alpha_3}^{\eta_3}$. By Lemma 6.2.6, $\operatorname{depthchart}_{\dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3}}(\gamma x)$ is the reverse of $\operatorname{depthchart}_{\dot{f}_{\alpha_3}^{-\eta_3} \dots \dot{f}_{\alpha_1}^{-\eta_1}}(\gamma x \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_3}^{\eta_3})$. All cases with this depth chart have been discussed in Lemma 6.3.8. Then there exists $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} \in W(\dot{\mathcal{X}})$ (for some $\beta_1, \dots, \beta_k \in \Omega^*$ and $\mu_1, \dots, \mu_k \in \{\pm 1\}$) with $k = 2, 3$ such that $\dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_k}^{\mu_k} = \dot{f}_{\alpha_1}^{\eta_1} \dot{f}_{\alpha_2}^{\eta_2}$ in $\dot{F}$ and the following conditions are satisfied:

C. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_i}^{\mu_i}) = \operatorname{depth}(\gamma x)$ for $i = 1$,

D. $\operatorname{depth}(\gamma x \dot{f}_{\beta_1}^{\mu_1} \dots \dot{f}_{\beta_i}^{\mu_i}) = \operatorname{depth}(\gamma x) + 1$ for $i = 2, 3$.

$\square$

6.4 A Presentation for Thompson's Group $\boldsymbol{F}$

{laibach}

Proposition 6.4.1. *Let* $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ *(for some* $\alpha_1, \dots, \alpha_n \in E(R)^*$ *and* $\eta_1, \dots, \eta_n \in \{\pm 1\}$*). Let* B_g *be the set of boundary points of* g*. Let* $\mathrm{depth}(B_g) = (z_1, \dots, z_d)$*. Suppose* $(z_i)g = z_i$ *for all* $i = 1, \dots, d$*. Then* $g = I$ *in* $\dot{F}$*.*

Proof. Let $g = \dot{f}_{\alpha_1}^{\eta_1} \dots \dot{f}_{\alpha_n}^{\eta_n} \in W(\dot{\mathcal{X}})$ (for some $\alpha_1, \dots, \alpha_n \in E(R)^*$ and $\eta_1, \dots, \eta_n \in \{\pm 1\}$). Let B_g be the set of boundary points of g. Let $\mathrm{depth}(B_g) = (z_1, \dots, z_d)$. Suppose $(z_i)g = z_i$ for all $i = 1, \dots, d$.

If $n = 0$, then $g = I$ and the result is trivially true.

Suppose $n > 0$. Then B_g is a non-empty finite set. By Lemma 6.2.12, there exist some dynamic boundary points in B_g. Choose $z_i \in B_g$ such that $z_1, \dots z_{i-1}$ are not dynamic under g, but z_i is dynamic under g. By Lemma 6.2.10, $\mathrm{depth}(z_i h) \geq \mathrm{depth}(z_i)$ for all $h \in \mathrm{prefixchain}(g)$.

We define a pair $(M_g(z_i), C_g(z_i))$ as follows:

$$\begin{aligned} M_g(z_i) &= \max \{\mathrm{depth}(z_i h) - \mathrm{depth}(z_i) \mid h \in \mathrm{prefixchain}(g)\}, \\ C_g(z_i) &= \# \{h \mid \mathrm{depth}(z_i h) - \mathrm{depth}(z_i) = M_g(z_i)\}. \end{aligned}$$

Recall that, since $M_g(z_i), C_g(z_i) \in \mathbb{N}$, there exists a lexicographic order on the pair $(M_g(z_i), C_g(z_i))$. The lexicographic order is a well-order, with a least element.

We construct a word $g' \in W(\dot{\mathcal{X}})$ by one of the following methods:

1. The application of an appropriate relation from $\dot{\mathcal{R}}$ to replace a subword $\dot{f}_{\alpha_i}^{\eta_i} \dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}$ by either $\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}} (\dot{f}_{\alpha_i}^{\eta_i})^{\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}}}$ or $(\dot{f}_{\alpha_{i+1}}^{\eta_{i+1}})^{\dot{f}_{\alpha_i}^{\eta_i}} \dot{f}_{\alpha_i}^{\eta_i}$.

2. The cancellation of a generator $\dot{f}_{\alpha_i}$ by an adjacent inverse.

Then $g = g'$ in $\dot{F}$. By Lemma 6.2.11, $B_{g'} \subseteq B_g$. By Lemma 6.3.1, Lemma 6.3.2, Lemma 6.3.4, Lemma 6.3.3, Lemma 6.3.5, Lemma 6.3.6, Lemma 6.3.7, Lemma 6.3.8, Lemma 6.3.9 and Lemma 6.3.10, $(M_{g'}(z_i), C_{g'}(z_i)) < (M_g(z_i), C_g(z_i))$.

We can only repeat this process finitely many times until we achieve a word such that z_i is no longer dynamic under it. Repeating this process for all dynamic points in B_g gives us a word k which has no dynamic boundary points. By Lemma 6.2.12, $k = I$. Then $g = k = I$ in $\dot{F}$. □

We can now prove our main result:

Proof. **(Proof of Theorem 6.1.1)**

Recall the homomorphism $\chi \colon \dot{F} \to F$, induced by the map $\dot{f}_\alpha \mapsto f_\alpha$, where $f_\alpha \in \mathcal{X}$ in Definition 5.1.1. We have already observed that χ is surjective. We will prove that it is injective.

Let $g \in \dot{F}$ such that $g\chi = I$. Let $h \in \mathrm{prefixchain}(g)$. By Lemma 6.2.1, the action of h on B_g is the same as the action of $h\chi$ on B_g. Since this is true for all $h \in \mathrm{prefixchain}(g)$, it follows that, for all $z \in B_g$, $zg = z$. Then, by Proposition 6.4.1, $g = I$ in $\dot{F}$. This proves that $\ker(\chi) = \{I\}$. Hence

$$\dot{F} \cong F.$$

□

Chapter 7

The F-Basilica Group F_B

{f-basilica}

In this Chapter, we conjecture a presentation for the F-Basilica group F_B, i.e., the group of rearrangements of the F-Basilica replacement system defined in Example 2.2.1. This replacement system and its rearrangements were used as an example in Chapter 2 and Chapter 3.

The F-Basilica group F_B is the group of homeomorphisms of one edge of the Basilica Julia set. This group was briefly mentioned in [3], and shown to not be finitely generated in Remark 4.7 (since it can be constructed as an infinite wreath product of Thompson's group F).

Our infinite presentation for F_B is very similar to our infinite presentation for Thompson's group F, and follows similarly from the geometric structure of the topological space.

Our conjecture for an infinite presentation for F_B is as follows:

$$F = \langle \mathcal{X} \mid \mathcal{R} \rangle$$

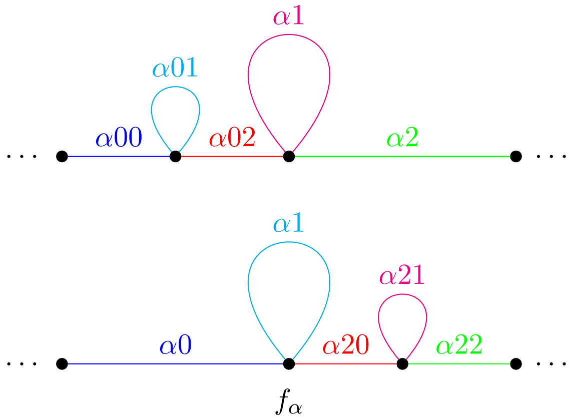
where the generating set $\mathcal{X}$ is

$$\mathcal{X} = \{ f_\alpha \mid \alpha \in \{0,1,2\}^* \}$$

with f_α acting as follows on points in the topological space with the prefix $\alpha = e_1 \dots e_n$ with each $e_i \in \{0,1,2\}$ for $i = 1, \dots, n$, and as the identity homeomorphism on the rest of the space:

$$([\alpha e_{n+1} e_{n+2} \dots])\, f_\alpha = \begin{cases} [\alpha 0 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1}e_{n+2} = 00, \\ [\alpha 1 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1}e_{n+2} = 01, \\ [\alpha 20 e_{n+3} e_{n+4} \dots] & \text{if } e_{n+1}e_{n+2} = 02, \\ [\alpha 21 e_{n+2} e_{n+3} \dots] & \text{if } e_{n+1} = 1, \\ [\alpha 22 e_{n+2} e_{n+3} \dots] & \text{if } e_{n+1} = 2. \end{cases}$$

This map is illustrated in the following diagrams:



and the set of relations $\mathcal{R}$ is

$$\begin{aligned}
\mathcal{R} = \big\{ R1 : \quad & {f_\beta}^{f_\alpha} = f_\beta \text{ for } \alpha \perp \beta, \\
R2 : \quad & {f_{\alpha 0}}^{f_\alpha} = f_\alpha {f_{\alpha 2}}^{-1}, \\
R3 : \quad & {f_{\alpha 00\gamma}}^{f_\alpha} = f_{\alpha 0\gamma}, \\
R4 : \quad & {f_{\alpha 01\gamma}}^{f_\alpha} = f_{\alpha 1\gamma}, \\
R5 : \quad & {f_{\alpha 02\gamma}}^{f_\alpha} = f_{\alpha 20\gamma}, \\
R6 : \quad & {f_{\alpha 1\gamma}}^{f_\alpha} = f_{\alpha 21\gamma}, \\
R7 : \quad & {f_{\alpha 2\gamma}}^{f_\alpha} = f_{\alpha 22\gamma} \big\},
\end{aligned}$$

where $\alpha, \beta, \gamma \in \{0, 1, 2\}^*$.

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