Finite groups whose commuting graph is split

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Abstract

As a contribution to the study of graphs on groups, we show that the commuting graph of a finite group G is a split graph, or a threshold graph, if and only if either G is abelian or G is a generalized dihedral group

$$D(A) = \langle A, t : (\forall a \in A)(at)^2 = 1\rangle$$

where A is an abelian group of odd order.

Keywords: Commuting graph, split graph, threshold graph, generalized dihedral group

MSC: 05C25, 20D60

1 Introduction

There has been a big upsurge of research recently on graphs defined on groups so as to reflect the group structure in some way. The oldest example is the *commuting graph*, whose vertices are the group elements, two vertices joined if they commute. This was the main tool in the seminal paper of Brauer and Fowler [2] in 1955, arguably the first step towards the classification of the finite simple groups. This graph is still the subject of current research.

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The second author has suggested that the interaction between graphs and groups can benefit both areas, and that there are three main ways where this can happen:

- We learn new results about groups. The Brauer–Fowler theorem is a good example of this. A more recent result is a strengthening of the old result of Landau [8] that the order of a finite group is bounded by a function of the number of conjugacy classes; it was shown in Bhowal *et al.* [1] that the order is bounded by a function of the clique number of a graph (the *SCC graph*) whose vertices are the conjugacy classes.

- Interesting classes of groups can be defined in terms of graphs. Known examples include the minimal non-abelian, non-nilpotent, or non-solvable groups, the Dedekind groups, the 2-Engel groups, and the EPPO groups (those in which every element has prime power order): see [4].

- We may find beautiful graphs, by taking known graphs defined on groups (especially almost simple groups) and applying suitable reductions such as indentifying twin vertices.

This paper is a contribution to the second of these points. In the literature there are two methods for defining a class of groups using graphs: either restrict the graph to some well-known graph class, or take two graphs defined on the group and ask for them to be equal or complementary. We use the first method here.

Our main theorem is the classification of all finite groups for which the commuting graph is either a split graph or a threshold graph. These graph classes will be defined below before the statement of the main theorem. This answers, in part, a question of the second author [4, Question 14].

2 Split graphs and threshold graphs

Our graph theory terminology will be standard. Graphs will be simple and undirected, and the graph Γ has vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The n-vertex complete graph is denoted by K_n, the n-cycle by C_n, and the n-path by P_n. The disjoint union of m copies of Γ is denoted by $m\Gamma$.

We denote the commuting graph of a finite group G by $\Gamma(G)$. Note that, in much of the literature, vertices in the centre of G, which would be joined to all other vertices, are removed; but for our purpose it makes no difference whether this is done or not, and for convenience we will not remove these vertices.

The graph Γ is a *split graph* if $V(\Gamma)$ is a disjoint union of sets V_1 and V_2, where V_1 and V_2 induce a complete subgraph and a null subgraph in Γ respectively. Split graphs have a forbidden subgraph characterization, due to Foldes and Hammer [7].

Proposition 2.1 *A graph is split if and only if it contains no induced subgraph isomorphic to C_4, C_5, or $2K_2$.*

The graph Γ is called a *threshold graph* if there is a weight function wt on vertices and a threshold number t such that vertices x and y are joined if and only if $\mathrm{wt}(x) + \mathrm{wt}(y) > t$. The forbidden subgraph classification is due to Chvátal and Hammer [6].

Proposition 2.2 *A graph is threshold if and only if it has no induced subgraph isomorphic to P_4, C_4 or $2K_2$.*

We see that every threshold graph is split, but not conversely. However, our main theorem has the consequence that, within the class of commuting graphs of groups, these two properties coincide.

3 The main theorem

Let A be a finite abelian group. the *generalized dihedral group* $D(A)$ is defined as the semidirect product of A with a cyclic group $\langle t\rangle$ of order 2, where $t^{-1}at = a^{-1}$ for all $a \in A$. (This reduces to the usual dihedral group when A is cyclic.) It has the properties, easily checked, that every element of $D(A) \setminus A$ has order 2, and that if t' is any such element, then the centralizer of t' in A is the set of elements of order 1 or 2.

Now we can state our main theorem.

Theorem 3.1 *The following are equivalent for a finite group G:*

(a) $\Gamma(G)$ *is a split graph;*

(b) $\Gamma(G)$ *is a threshold graph;*

(c) $\Gamma(G)$ *contains no induced subgraph isomorphic to* $2K_2$*;*

(d) *either G is abelian, or it is a generalized dihedraph group $D(A)$ where A is an abelian group of odd order.*

Proof Propositions 2.1 and 2.2 show that each of (a) and (b) implies (c). Moreover it is clear that (d) implies the other three statements: the commuting graph of an abelian group is complete, while that of $D(A)$ with $|A|$ odd consists of a complete graph on A with $|A|$ pendant vertices attached to the identity.

So it remains to show that (c) implies (d). So let G be a graph whose commuting graph forbids $2K_2$. We assume that G is not abelian, and proceed in a number of steps.

Step 1: The elements of order greater than 2 in G commute pairwise, and so generate an abelian subgroup $\Omega(G)$. For let a and b be elements with order greater than 2. If a and b don't commute, then $\{a, a^{-1}, b, b^{-1}\}$ induces $2K_2$.

Step 2: $|G : \Omega(G)| \leq 2$. For clearly every element of G not in $\Omega(G)$ is an involution. So, if the claim is false, then $|G : \Omega(G)| \geq 4$, from which it follows that more than three-quarters of the elements of G satisfy $x^2 = e$. Now a folklore result shows that such a group is abelian. (Here is the proof. Let x be an involution. Then more than half the elements $g \in G$ satisfy $g^2 = (xg)^2 = e$, and so commute with x. Thus $x \in Z(G)$. Since there are at least $3|G|/4$ choices for x, we have $Z(G) = G$.)

Step 3: G is generalized dihedral $D(\Omega(G))$. For take $x \notin \Omega(G)$. For any element $a \in \Omega(G)$, we have $ax \notin \Omega(G)$, and so $x^2 = (ax)^2 = e$, whence $x^{-1}ax = a^{-1}$.

Step 4: $\Omega(G)$ has odd order. For suppose not. If every element of $\Omega(G)$ has order 2 then G is abelian. Otherwise, choose $a \in \Omega(G)$ with order $2m$, where $m > 1$, and $x \notin \Omega(G)$. Then $\{a, a^{m+1}, x, xa^m\}$ induces $2K_2$, a contradiction. □

4 Further directions

There are two obvious directions to extend this work:

- Other classes of graphs. Well-studied subgraph closed classes include perfect graphs, cographs and chordal graphs. The problem is to investigate the classes of groups whose commuting graphs belong to one of these classes. Britnell and Gill [3] determined the quasi-simple groups whose commuting graph is perfect, and the present authors with Natalia Maslova are preparing a paper on groups whose commuting graph is a cograph or a chordal graph. But the general problem is still unsolved in these cases.

- Other graphs on groups. For example, the *power graph* of a group has an edge $\{x, y\}$ whenever one of x and y is a power of the other. The second author, with Pallabi Manna and Ranjit Mehatari [5], have studied groups whose power graph is a cograph. Even in this case the complete classification is not known, and there are many other graphs which could be considered.

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