TOPOLOGICAL EMBEDDINGS INTO TRANSFORMATION MONOIDS

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Abstract. In this paper we consider the questions of which topological semigroups embed topologically into the full transformation monoid $\mathbb{N}^\mathbb{N}$ or the symmetric inverse monoid $I_\mathbb{N}$ with their respective canonical Polish semigroup topologies. We characterise those topological semigroups that embed topologically into $\mathbb{N}^\mathbb{N}$ and belong to any of the following classes: commutative semigroups; compact semigroups; groups; and certain Clifford semigroups. We prove analogous characterisations for topological inverse semigroups and $I_\mathbb{N}$. We construct several examples of countable Polish topological semigroups that do not embed into $\mathbb{N}^\mathbb{N}$, which answer, in the negative, a recent open problem of Elliott et al. Additionally, we obtain two sufficient conditions for a topological Clifford semigroup to be metrizable, and prove that inversion is automatically continuous in every Clifford subsemigroup of $\mathbb{N}^\mathbb{N}$. The former complements recent works of Banakh et al.

1. Introduction

As is well-known, Cayley's Theorem states that every group is isomorphic to a subgroup of a symmetric group S_X on some set X. Analogous statements hold for semigroups and inverse semigroups. More specifically, every semigroup is isomorphic to a subsemigroup of the monoid X^X consisting of all transformations of some set X with operation the usual composition of functions; for more details see [20, Theorem 1.1.2]. Similarly, every inverse semigroup is isomorphic to an inverse subsemigroup of a symmetric inverse monoid I_X, consisting of all partial permutations of the set X with operation the usual composition of binary relations; see [20, Theorem 5.1.7]. Monoids of transformations and partial permutations have been extensively studied in the literature; of particular relevance to this paper are [8, 13, 15, 17, 22, 23, 24, 25, 26, 27, 28, 32, 33, 34, 36].

It seems natural enough to ask if there is an analogue of Cayley's Theorem for semigroups endowed with topologies that are compatible with their algebraic structures. The monoids $\mathbb{N}^\mathbb{N}$, $S_\mathbb{N}$, and $I_\mathbb{N}$ each possess a canonical topology with respect to which their operations are continuous. As a topological space, $\mathbb{N}^\mathbb{N}$ with the Tychonoff product topology arising from the discrete topology on $\mathbb{N}$, is the well-known *Baire space*. The space $\mathbb{N}^\mathbb{N}$ is Polish, *i.e.* completely metrizable and separable; for further details see [21, Section 3]. Multiplication as a function from $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ (with the product topology) to $\mathbb{N}^\mathbb{N}$ is continuous, and as such $\mathbb{N}^\mathbb{N}$ is also a *topological semigroup*. It was shown in [14, Theorem 5.4(b)] that the unique Polish semigroup topology on $\mathbb{N}^\mathbb{N}$ is the topology of the Baire space. We refer to this topology as the canonical topology for $\mathbb{N}^\mathbb{N}$.

Topological groups and inverse semigroups are defined analogously, where both multiplication and inversion are continuous. Since $S_\mathbb{N}$ is a G_δ subset of the Baire space $\mathbb{N}^\mathbb{N}$, the subspace topology on $S_\mathbb{N}$ is Polish. Gaughan [18] showed that this topology is contained in every Hausdorff group topology on $S_\mathbb{N}$. On the other hand, every Borel measurable bijection between Polish groups is a homeomorphism ([14, Propositions 3.2 and 3.3]). Hence if G is a Polish topological group with respect to topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ where $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then the identity function from $(G, \mathcal{T}_2)$ to $(G, \mathcal{T}_1)$ is continuous, and hence

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Borel measurable, which implies that $\mathcal{T}_1 = \mathcal{T}_2$. It follows that the subspace topology induced by the canonical topology on $\mathbb{N}^\mathbb{N}$ is the unique Polish group topology on $S_\mathbb{N}$. As such we refer to the subspace topology on $S_\mathbb{N}$ induced by the canonical topology on $\mathbb{N}^\mathbb{N}$ as the canonical topology for $S_\mathbb{N}$. Although not a subspace of $\mathbb{N}^\mathbb{N}$, the symmetric inverse monoid $I_\mathbb{N}$ also possesses a natural unique Polish inverse semigroup topology [14, Theorem 5.15(ix)]; a subbasis for this topology is given in (2). The symmetric group $S_\mathbb{N}$ is a subgroup of both $I_\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ and the canonical topologies on these semigroups both induce the canonical topology on $S_\mathbb{N}$. The canonical topologies on $S_\mathbb{N}$, $\mathbb{N}^\mathbb{N}$ and $I_\mathbb{N}$ are zero-dimensional, i.e. they each possess a basis consisting of clopen sets (see (1) and (2) for further details). Since all three spaces are Polish, zero-dimensional, and their compact subsets have empty interior, the Alexandrov-Urysohn Theorem ([21, Theorem 7.7]) implies that $S_\mathbb{N}, I_\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ are homeomorphic, although they are clearly not isomorphic monoids.

Throughout the remainder of this paper, unless explicitly stated otherwise, we write $S_\mathbb{N}$, $\mathbb{N}^\mathbb{N}$, and $I_\mathbb{N}$ to mean the corresponding topological group, monoid, and inverse monoid endowed with their canonical topologies.

In this paper we consider the problems of which topological semigroups can be topologically embedded into $\mathbb{N}^\mathbb{N}$, and which topological inverse semigroups embed topologically into $I_\mathbb{N}$. It is well-known that a Hausdorff topological group G embeds topologically into the symmetric group $S_\mathbb{N}$ if and only if G is second-countable and possesses a neighbourhood basis of the identity consisting of open subgroups; see, for example, [8, Theorem 5.1]. This characterisation can be readily applied to an arbitrary topological group G to determine whether or not G embeds topologically into the symmetric group. For example, the additive group $\mathbb{Q}$ of rational numbers endowed with the subspace topology inherited from the real line, despite being second-countable and zero-dimensional, cannot be embedded into $S_\mathbb{N}$, because $(-1,1)\cap\mathbb{Q}$ is an open neighborhood of 0 which contains no open subgroup of $\mathbb{Q}$.

There are natural analogues, in the contexts of semigroups and inverse semigroups, of the aforementioned characterisation for groups, in terms of right congruences; see Propositions 3.1 and 3.2. Right congruences of topological semigroups are, in general, much more complicated, and harder to work with than subgroups of topological groups. As such, unlike in the case of groups, Propositions 3.1 and 3.2 do not often simplify the process of determining whether or not a topological semigroup is topologically isomorphic to a subsemigroup of $\mathbb{N}^\mathbb{N}$ or $I_\mathbb{N}$. Every countable Polish group is discrete, and so the trivial group is a neighbourhood basis of the identity; that is, every countable Polish group embeds topologically into $S_\mathbb{N}$, and hence into $\mathbb{N}^\mathbb{N}$ and $I_\mathbb{N}$ also. On the other hand, the situation for countable Polish semigroups that are not groups is unclear; the following question was posed in [14].

Question 1.1 (cf. Question 5.6 in [14])**.** *Does every countable Polish semigroup embed topologically into* $\mathbb{N}^\mathbb{N}$*?*

In this paper, we provide several alternate characterisations of particular types of topological semigroup that embed into $\mathbb{N}^\mathbb{N}$ or $I_\mathbb{N}$.

The paper is organised as follows. In Section 2 we state the main results of the paper; in Section 3 we prove the main theorems; and in Section 4 we provide a number of examples demonstrating the sharpness of our main results and we show that the answer to Question 1.1 is negative.

2. Statement of main results

We start by giving some preliminary material required to state the main results of this paper. Recall that, a semigroup S is called an *inverse semigroup* if for each element $x\in S$ there exists a unique inverse element $x^{-1}\in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. An inverse semigroup S is called *Clifford* if $xx^{-1} = x^{-1}x$ for all $x\in S$; or, equivalently, S is a strong semilattice of groups. Recall that a *semilattice* is a commutative semigroup of idempotents. For every inverse semigroup S the set $E(S) = \{e\in S : e^2 = e\}$ is a semilattice. For a partial function f on X we denote the domain and image of f by $\operatorname{dom}(f)$ and $\operatorname{im}(f)$, respectively. Throughout this paper we will write partial functions to the right of their arguments and compose from left to right; it will also be convenient to identify partial functions $f : X \to X$ with their graphs $\{(x, (x)f) : x \in \mathrm{dom}(f)\} \subseteq X \times X$. Elements of I_X are referred to as *partial permutations of the set* X.

A subbases for the canonical topologies on $\mathbb{N}^{\mathbb{N}}$ consists of the sets:

$$U_{x,y} = \left\{ f \in \mathbb{N}^{\mathbb{N}} : (x, y) \in f \right\}. \tag{1}$$

The complement of $U_{x,y}$ is $\bigcup_{z \neq y} U_{x,z}$, which shows that the subbasic open sets are clopen, and hence $\mathbb{N}^{\mathbb{N}}$ with its canonical topology is zero-dimensional. The following sets form a subbasis for the canonical topology on $I_{\mathbb{N}}$:

$$U_{x,y} = \{h \in I_{\mathbb{N}} : (x, y) \in h\}, \ W_x = \{h \in I_{\mathbb{N}} : x \notin \mathrm{dom}(h)\}, \ W_x^{-1} = \{h \in I_{\mathbb{N}} : x \notin \mathrm{im}(h)\}, \tag{2}$$

where $x, y \in \mathbb{N}$.

An *embedding* of a semigroup S into a semigroup T is an injective homomorphism from S to T. For topological semigroups S and T an embedding $\phi : S \to T$ is called *topological* if both of the maps ϕ and $\phi^{-1} : (S)\phi \to S$ are continuous. The term *topological isomorphism* is defined analogously.

In this paper we characterize the commutative topological subsemigroups of $\mathbb{N}^{\mathbb{N}}$ and $I_{\mathbb{N}}$ as follows.

Theorem 2.1. *A commutative topological semigroup S embeds topologically into $\mathbb{N}^{\mathbb{N}}$ if and only if there exists a countable family $\{S_n : n \in \mathbb{N}\}$ of countable discrete semigroups such that S embeds topologically into the Tychonoff product $\prod_{n\in\mathbb{N}} S_n$.*

A semigroup S with adjoined external zero is denoted by S^0.

Theorem 2.2. *A commutative topological inverse semigroup S embeds topologically into $I_{\mathbb{N}}$ if and only if there exists a countable family $\{G_n : n \in \mathbb{N}\}$ of countable groups such that S embeds topologically into the Tychonoff product $\prod_{n\in\mathbb{N}} G_n^0$, where each factor is discrete.*

A topological space X is called *totally disconnected* if the only connected subsets of X are singletons. It is well-known that every Tychonoff zero-dimensional space is totally disconnected, but there exists a Polish totally disconnected topological group which is not zero-dimensional [12, Proposition 4.3]. However these two notions coincide for subspaces of the real line. It is also well-known, see [21, Theorem 7.8] for example, that a topological space X is homeomorphic to a subspace of $\mathbb{N}^{\mathbb{N}}$ if and only if X is metrizable, zero-dimensional, and second-countable. In this paper we obtain the following characterization of the compact subsemigroups of $\mathbb{N}^{\mathbb{N}}$.

Theorem 2.3. *Let S be a compact topological semigroup. Then the following are equivalent:*

(i) *S is homeomorphic to a subspace of $\mathbb{N}^{\mathbb{N}}$ (and $I_{\mathbb{N}}$);*
(ii) *S embeds topologically into $\mathbb{N}^{\mathbb{N}}$;*
(iii) *S is metrizable and totally disconnected.*

The compact inverse subsemigroups of $I_{\mathbb{N}}$ are characterized as follows.

Theorem 2.4. *Let S be a compact inverse topological semigroup. Then the following are equivalent:*

(i) *S is homeomorphic to a subspace of $\mathbb{N}^{\mathbb{N}}$ (and $I_{\mathbb{N}}$);*
(ii) *S embeds topologically into $I_{\mathbb{N}}$;*
(iii) *S embeds topologically into $\mathbb{N}^{\mathbb{N}}$;*
(iv) *S is metrizable and totally disconnected.*

Turning to topological groups, we obtain the following characterisation.

Theorem 2.5. *Let G be a Hausdorff topological group. Then the following conditions are equivalent:*

(i) *G embeds topologically into $S_{\mathbb{N}}$;*
(ii) *G embeds topologically into $I_{\mathbb{N}}$;*

(iii) *G embeds topologically into $\mathbb{N}^{\mathbb{N}}$;*
(iv) *G is second-countable and has a neighbourhood basis of the identity consisting of open subgroups.*

Perhaps the next most natural objective is to characterize those countable Polish Clifford semigroups that topologically embed into $\mathbb{N}^{\mathbb{N}}$ or $I_{\mathbb{N}}$. The following notion introduced by Banakh and Pastukhova [6] is crucial for this purpose.

Definition 2.6. An inverse semigroup X endowed with a semigroup topology is called *ditopological* if inversion is continuous; and for any point $x \in X$ and any neighborhood O of x there are neighborhoods U and W of x and xx^{-1}, respectively, such that

$$\{s \in S : \exists b \in U,\ \exists e \in W \cap E(S) \text{ such that } b = es\} \cap \{s \in S : ss^{-1} \in W\} \subseteq O.$$

The countable Polish Clifford subsemigroups of $I_{\mathbb{N}}$ are characterised as follows.

Theorem 2.7. *A countable Polish Clifford semigroup S embeds topologically into $I_{\mathbb{N}}$ if and only if S is ditopological and the semilattice $E(S)$ embeds topologically into $I_{\mathbb{N}}$.*

There exist countable commutative Polish Clifford semigroups with compact semilattice of idempotents that are not ditopological; see Proposition 4.6.

Automatic continuity of inversion in paratopological groups, or more general, inverse topological semigroups was investigated by many authors in [4, 7, 9, 16, 19, 29, 35, 38, 39]. The following theorem implies that inversion is automatically continuous in every Clifford subsemigroups of $\mathbb{N}^{\mathbb{N}}$.

Theorem 2.8. *Each Clifford subsemigroup of $\mathbb{N}^{\mathbb{N}}$ is ditopological.*

Also, Theorem 2.8 allows us to characterize countable Polish Clifford subsemigroups of $\mathbb{N}^{\mathbb{N}}$.

Theorem 2.9. *A countable Polish Clifford semigroup S embeds topologically into $\mathbb{N}^{\mathbb{N}}$ if and only if S is ditopological and the semilattice $E(S)$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$.*

Given Theorem 2.7 and Theorem 2.9 it is natural to ask if the semilattice $E(S)$ embeds into $\mathbb{N}^{\mathbb{N}}$ if and only if it embeds into $I_{\mathbb{N}}$ when S is any countable Polish Clifford semigroup. We will show in Proposition 3.9 that every subsemilattice of $I_{\mathbb{N}}$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. The converse implication, however, fails, see Proposition 4.2 for a counter-example. Furthermore, in Proposition 4.7 we give an example of a countable Polish linear semilattice which is not topologically isomorphic to a subsemigroup of either $\mathbb{N}^{\mathbb{N}}$ or $I_{\mathbb{N}}$.

For an element x of a semilattice X let ${\uparrow}x = \{y \in X : x \leq y\}$.

Definition 2.10. A semilattice X endowed with a topology is called a:
(i) *U-semilattice*, if for every open set U and every $x \in U$ there exist $y \in U$ and an open neighborhood V of x such that $V \subseteq {\uparrow}y$;
(ii) *U_2-semilattice*, if for every open set U and every $x \in U$ there exist $y \in U$ and a clopen ideal $I \subseteq X$ such that $x \in X \setminus I \subseteq {\uparrow}y$.

Clearly, any U_2-semilattice is a U-semilattice, but the converse implication fails (see [6] and [10] for more details).

Theorems 2.7 and 2.9 are derived from the following more general theorems.

Theorem 2.11. *Let S be a Clifford topological semigroup whose set of idempotents $E(S)$ is a U-semilattice. Then S embeds topologically into $I_{\mathbb{N}}$ if and only if S is Hausdorff, ditopological and every maximal subgroup of S, as well as the semilattice $E(S)$, embed topologically into $I_{\mathbb{N}}$.*

Theorem 2.12. *Let S be a Clifford topological semigroup whose set of idempotents $E(S)$ is a U_2-semilattice. Then S embeds topologically into $\mathbb{N}^{\mathbb{N}}$ if and only if S is Hausdorff, ditopological and every maximal subgroup of S, as well as the semilattice $E(S)$, embed topologically into $\mathbb{N}^{\mathbb{N}}$.*

Theorems 2.11 and 2.12 imply the following corollary which complements results of Banakh et al. about the metrizability of Clifford topological inverse semigroups [1, 3, 5].

Corollary 2.13. *A topological Clifford semigroup S is metrizable and zero-dimensional if one of the following conditions holds:*

(i) *S is Hausdorff, ditopological, $E(S)$ is a U_2-semilattice which embeds topologically into $\mathbb{N}^{\mathbb{N}}$ and each maximal subgroup of S embeds topologically into $\mathbb{N}^{\mathbb{N}}$;*

(ii) *S is Hausdorff, ditopological, $E(S)$ is a U-semilattice which embeds topologically into $I_{\mathbb{N}}$ and each maximal subgroup of S embeds topologically into $I_{\mathbb{N}}$.*

3. Proofs of the main theorems

In this section we prove the main results of this paper each in its own subsection. We begin by recording some results that are useful throughout this section.

An equivalence relation ρ on a semigroup S is called a *right congruence* if for any $x, y, z \in S$, $(x, y) \in \rho$ implies $(xz, yz) \in \rho$. If ρ is a right congruence on a semigroup S and $x \in S$, then the set $\{y \in S : (x, y) \in \rho\}$ is denoted by $[x]_\rho$.

Proposition 3.1 (cf. Theorem 5.5 [14])**.** *A Hausdorff topological semigroup S is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$ if and only if there exists a countable family $\{\rho_i : i \in \mathbb{N}\}$ of right congruences of S, each having countably many equivalence classes, such that the family $\{[x]_{\rho_i} : x \in S, i \in \mathbb{N}\}$ is a basis for the topology on S.*

A right congruence ρ on an inverse monoid S is called *Vagner-Preston* if for every $s \in S$ either: $t \in [s]_\rho$ implies that $1 \in [tt^{-1}]_\rho$; or $[st]_\rho = [s]_\rho$ for all $t \in S$. The following analogue of Proposition 3.1 for the symmetric inverse monoid $I_{\mathbb{N}}$ was proven in [14, Theorem 5.21].

Proposition 3.2. *A Hausdorff topological inverse monoid S is topologically isomorphic to an inverse subsemigroup of $I_{\mathbb{N}}$ if and only if there exists a sequence $\{\rho_i : i \in \mathbb{N}\}$ of Vagner-Preston right congruences, each having countably many equivalence classes, such that the family $\{[s]_{\rho_i}, [s]_{\rho_i}^{-1} : s \in S, i \in \mathbb{N}\}$ is a subbasis*[1] *of the topology on S.*

As we mentioned in the introduction, for any countable semigroup S the diagonal congruence $\Delta_S = \{(x, x) : x \in S\}$ has countably many equivalence classes and the family $\{[x]_{\Delta_S} : x \in S\}$ forms a basis for the discrete topology on S. Combined with Proposition 3.1 this observation yields the following corollary.

Corollary 3.3. *Each countable discrete semigroup embeds topologically into $\mathbb{N}^{\mathbb{N}}$.*

The following lemma will be useful for detecting semigroups topologically embeddable into $\mathbb{N}^{\mathbb{N}}$.

Lemma 3.4. *Suppose that S_n is a topological semigroup for every $n \in \mathbb{N}$. Then the following hold:*

(i) *If S_n is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$ for every $n \in \mathbb{N}$, then the Tychonoff product $\prod_{n\in\mathbb{N}} S_n$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$.*

(ii) *If S_n is topologically isomorphic to a subsemigroup of $I_{\mathbb{N}}$ for every $n \in \mathbb{N}$, then the Tychonoff product $\prod_{n\in\mathbb{N}} S_n$ embeds topologically into $I_{\mathbb{N}}$.*

Proof. We only prove part (ii), the proof of item (i) is similar. We will show that the Tychonoff power $I_{\mathbb{N}}^{\mathbb{N}}$ embeds topologically into $I_{\mathbb{N}}$. Fix any partition of $\mathbb{N}$ into countably many infinite subsets A_i for $i \in \mathbb{N}$. Consider the subsemigroup $Y = \{f \in I_{\mathbb{N}} : (A_i)f \subseteq A_i \text{ for all } i \in \mathbb{N}\}$ of $I_{\mathbb{N}}$. For each $f \in Y$ let $f_n = f{\restriction}_{A_n}$. Then it is straightforward to check that the map $\phi : Y \to \prod_{i\in\mathbb{N}} I_{A_i} \cong I_{\mathbb{N}}^{\mathbb{N}}$ defined by $(f)\phi = (f_n)_{n\in\mathbb{N}}$ is the desired topological isomorphism. □

[1] The term "subbasis" cannot be replaced with "basis" here, unlike in Proposition 3.1.

A semigroup S is called *countably prodiscrete* if S can be embedded into the Tychonoff product of countably many countable discrete semigroups. Corollary 3.3 and Lemma 3.4 imply the following.

Corollary 3.5. *Every countably prodiscrete semigroup can be topologically embedded into* $\mathbb{N}^{\mathbb{N}}$.

3.1. **Proof of Theorem 2.1.** We need to show that a commutative topological semigroup S embeds topologically into $\mathbb{N}^{\mathbb{N}}$ if and only if S is countably prodiscrete.

Proof. ($\Leftarrow$) By Corollary 3.5 each countably prodiscrete semigroup embeds topologically into $\mathbb{N}^{\mathbb{N}}$.

($\Rightarrow$) Fix any commutative subsemigroup S of $\mathbb{N}^{\mathbb{N}}$. By Proposition 3.1, the semigroup S possesses a countable family $\{\rho_n : n \in \mathbb{N}\}$ of right congruences of S, each having countably many classes, such that the family $\{[x]_{\rho_n} : x \in S, n \in \mathbb{N}\}$ is a basis of the subspace topology on S inherited from $\mathbb{N}^{\mathbb{N}}$. Since S is commutative, for each $n \in \mathbb{N}$, ρ_n is a two-sided congruence. Clearly, for every $n \in \mathbb{N}$ the quotient semigroup $S_n = S/\rho_n$ is countable. We will show that S embeds topologically into the Tychonoff product $\prod_{n\in\mathbb{N}} S_n$, where each factor is endowed with the discrete topology. For every $n \in \mathbb{N}$ let $f_n : S \to S_n$ be the homomorphism associated with the congruence ρ_n. Since each equivalence class $[x]_{\rho_n}$ is clopen, the homomorphism f_n (onto the discrete semigroup S_n) is continuous. It follows that the diagonal map $\Phi : S \to \prod_{n\in\mathbb{N}} S_n$, $(x)\Phi = ((x)f_n)_{n\in\mathbb{N}}$ is continuous as well. Since $\mathbb{N}^{\mathbb{N}}$ is Hausdorff, S is Hausdorff and so $\bigcap_{n\in\mathbb{N}}[x]_{\rho_n} = \{x\}$ for every $x \in S$. Thus, Φ is injective. To show that Φ is a topological embedding, fix any open basic set $[x]_{\rho_n}$ of S. The image

$$([x]_{\rho_n})\Phi = \{(z_i)_{i\in\mathbb{N}} \in (S)\Phi : z_n = (x)f_n\}$$

is open in $(S)\Phi$. Hence the map Φ is open onto its image, which implies that Φ is a topological embedding. □

Recall that for a semigroup S by S^1 and S^0 we denote the semigroup S with adjoined external identity and zero, respectively. If S is a topological semigroup, then, unless otherwise stated explicitly, we write S^1 to mean the topological semigroup obtained from S by extending the topology of S and where the adjoined identity is isolated. An analogous statement holds for S^0. In order to prove Theorem 2.2 we need the following lemmas.

Lemma 3.6. *For a topological semigroup S the following assertions are equivalent:*

(i) *S embeds topologically into $\mathbb{N}^{\mathbb{N}}$;*
(ii) *S^1 embeds topologically into $\mathbb{N}^{\mathbb{N}}$;*
(iii) *S^0 embeds topologically into $\mathbb{N}^{\mathbb{N}}$.*

Proof. The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) are trivial. Let S be a topological semigroup which embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Put $2\mathbb{N} = \{2n : n \in \mathbb{N}\}$. Since $2\mathbb{N}^{2\mathbb{N}}$ is topologically isomorphic to $\mathbb{N}^{\mathbb{N}}$ we obtain that S is topologically isomorphic to a subsemigroup T of $2\mathbb{N}^{2\mathbb{N}}$. For each element $t \in T$ let

$$t' = t \cup \{(1, 1)\} \cup \{(2n + 1, 3) : n \in \mathbb{N} \setminus \{0\}\}.$$

Clearly, $T' = \{t' : t \in T\}$ is a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. It is easy to check that S^1 is topologically isomorphic to a subsemigroup $T' \cup \{\mathrm{id}_{\mathbb{N}}\}$ of $\mathbb{N}^{\mathbb{N}}$, where by $\mathrm{id}_{\mathbb{N}}$ we denote the identity permutation of $\mathbb{N}$. Hence the implication (i) $\Rightarrow$ (ii) holds. Also, it is easy to check that S^0 is topologically isomorphic to a subsemigroup $T' \cup \{z\}$ of $\mathbb{N}^{\mathbb{N}}$, where $z = \{(n, 1) : n \in \mathbb{N}\}$. Hence the implication (i) $\Rightarrow$ (iii) holds. □

Similarly one can prove the following lemma.

Lemma 3.7. *For a topological semigroup S the following assertions are equivalent:*

(i) *S embeds topologically into $I_{\mathbb{N}}$;*
(ii) *S^1 embeds topologically into $I_{\mathbb{N}}$;*
(iii) *S^0 embeds topologically into $I_{\mathbb{N}}$.*

Lemma 3.8. *Every quotient of a commutative inverse monoid by a Vagner-Preston congruence is a group or a group with zero adjoined.*

Proof. Let S be a commutative inverse monoid and ρ be a Vagner-Preston congruence on S. Clearly, S is a Clifford semigroup and so is the quotient semigroup S/ρ. If for each $x \in S$, $xx^{-1} \in [1]_\rho$, then the Clifford semigroup S/ρ contains the unique idempotent $[1]_\rho$, which implies that S/ρ is a group. Assume that there exists an element $s \in S$ such that $ss^{-1} \notin [1]_\rho$. Fix any element $t \in S$ such that $tt^{-1} \notin [1]_\rho$. Since ρ is a Vagner-Preston congruence and the semigroup S is commutative, we get that $[s]_\rho$ and $[t]_\rho$ are two-sided ideals in S. Then $[s]_\rho \cap [t]_\rho \neq \emptyset$, which shows that $[t]_\rho = [s]_\rho$. Thus, for each $x \in S$ either $xx^{-1} \in [1]_\rho$ or $x \in [s]_\rho$. At this point it is easy to see that the quotient semigroup S/ρ contains only two idempotents $[1]_\rho$ and $[s]_\rho$. Moreover, $[1]_\rho$ is an identity of S/ρ and $[s]_\rho$ is zero of S/ρ. Hence S/ρ is a group with adjoined zero. □

3.2. **Proof of Theorem 2.2.** We need to show that a commutative topological inverse semigroup S is topologically isomorphic to a subsemigroup of $I_\mathbb{N}$ if and only if S embeds into a Tychonoff product $\prod_{n\in\mathbb{N}} G_n^0$, where G_n is a discrete countable group for every $n \in \mathbb{N}$.

Proof. ($\Leftarrow$) Clearly, for each group G the diagonal congruence on the monoid G^0 is a Vagner-Preston congruence. Proposition 3.2 implies that for every countable group G the discrete monoid G^0 embeds topologically into $I_\mathbb{N}$. Lemma 3.4(ii) implies that for every countable family $\{G_n : n \in \mathbb{N}\}$ of countable discrete groups the Tychonoff product $\prod_{n\in\mathbb{N}} G_n^0$ embeds topologically into $I_\mathbb{N}$. It follows that each topological subsemigroup S of $\prod_{n\in\mathbb{N}} G_n^0$ embeds topologically into $I_\mathbb{N}$.

($\Rightarrow$) Consider a commutative inverse subsemigroup S of $I_\mathbb{N}$ and assume that S carries the subspace topology inherited from $I_\mathbb{N}$. By Lemma 3.7, S^1 embeds topologically into $I_\mathbb{N}$. Proposition 3.2 implies that the monoid S^1 possesses a countable family $\{\rho_n : n \in \mathbb{N}\}$ of Vagner-Preston right congruences, each having countably many classes, such that the family $\{[x]_{\rho_n}, [x]_{\rho_n}^{-1} : x \in S^1, n \in \mathbb{N}\}$ is a subbasis of the topology on S^1. By the commutativity of S^1, each ρ_n is a congruence on S^1. It follows that $[x]_{\rho_n}^{-1} = [x^{-1}]_{\rho_n}$ for every $x \in S^1$ and $n \in \mathbb{N}$. Hence the family $\{[x]_{\rho_n} : x \in S, n \in \mathbb{N}\}$ is a subbasis of the topology on S^1. For each $n \in \mathbb{N}$ let $\mu_n = \bigcap_{i\leq n} \rho_i$. It is straightforward to check that $\{[x]_{\mu_n} : x \in S, n \in \mathbb{N}\}$ is a basis of the topology on S^1. By Lemma 3.8, for each $n \in \mathbb{N}$ the quotient semigroup S^1/μ_n is isomorphic either to G_n or G_n^0 for some group G_n. By the definition of μ_n the group G_n is countable. For every $n \in \mathbb{N}$ let $f_n : S^1 \to G_n^0$ be the homomorphism associated with the congruence μ_n. Since each equivalence class $[x]_{\mu_n}$ is clopen, we get that the homomorphism f_n (into a discrete monoid G_n^0) is continuous. Similarly as in the proof of Theorem 2.1 it can be checked that the diagonal map $\Phi : S^1 \to \prod_{n\in\mathbb{N}} G_n^0$, $(x)\Phi = ((x)f_n)_{n\in\mathbb{N}}$ is a topological embedding of the monoid S^1 into the Tychonoff product $\prod_{n\in\mathbb{N}} G_n^0$, where each factor is discrete. It follows that the topological semigroup S is topologically isomorphic to a subsemigroup of $\prod_{n\in\mathbb{N}} G_n^0$. □

3.3. **Proof of Theorem 2.3.** We will show that for a compact topological semigroup S, the following conditions are equivalent:

(i) S is homeomorphic to a subspace of $\mathbb{N}^\mathbb{N}$ (and $I_\mathbb{N}$);
(ii) S embeds topologically into $\mathbb{N}^\mathbb{N}$;
(iii) S is metrizable and totally disconnected.

Proof. Let S be a compact topological semigroup. The implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (ii). The celebrated result of Numakura [30, Theorem 1] states that each Hausdorff compact totally disconnected topological semigroup S is profinite, i.e. can be embedded into a Tychonoff product of finite discrete semigroups. Moreover, from the proof of [30, Theorem 1] follows that if the diagonal $\Delta = \{(x, x) : x \in S\}$ is a G_δ subset of $S\times S$, then S can be embedded into the Tychonoff product of countably many finite discrete semigroups. Since the diagonal of any metrizable space X is a G_δ subset of $X\times X$, the result of Numakura implies that each compact metrizable totally disconnected topological semigroup S is countably prodiscrete. By Corollary 3.5, S embeds topologically into $\mathbb{N}^\mathbb{N}$. □

3.4. **Proof of Theorem 2.4.** We need to prove that for any compact topological inverse semigroup S the following conditions are equivalent:

(i) S is homeomorphic to a subspace of $\mathbb{N}^\mathbb{N}$ (and $I_\mathbb{N}$);
(ii) S embeds topologically into $I_\mathbb{N}$;
(iii) S embeds topologically into $\mathbb{N}^\mathbb{N}$;
(iv) S is metrizable and totally disconnected.

Proof. The equivalences (i) ⇔ (iii) and (iii) ⇔ (iv) are established in Theorem 2.3.

The implication (ii) ⇒ (iv) follows from the fact that $I_\mathbb{N}$ is metrizable and totally disconnected.

(iv) ⇒ (ii). As we already showed in the proof of Theorem 2.3, each totally disconnected compact metrizable topological semigroup S embeds into a Tychonoff product $\prod_{n\in\mathbb{N}} Y_n$ of finite discrete semigroups Y_n, $n\in\mathbb{N}$. For each $n\in\mathbb{N}$ consider the projection $(S)\pi_n\subseteq Y_n$ of S on the n-th coordinate. Clearly, the semigroup $(S)\pi_n$ is inverse. By the Wagner-Preston Theorem, each countable inverse semigroup embeds into $I_\mathbb{N}$. Since for each $n\in\mathbb{N}$ the semigroup $(S)\pi_n$ is finite, the discrete semigroup $(S)\pi_n$ embeds topologically into $I_\mathbb{N}$. Taking into account that $S\subseteq\prod_{n\in\mathbb{N}}(S)\pi_n$, Lemma 3.4(ii) implies that S is topologically isomorphic to a subsemigroup of $I_\mathbb{N}$. □

Besides the canonical topology, $I_\mathbb{N}$ can be endowed with a Polish semigroup topology $\mathcal{T}$ which is generated by the subbasis consisting of the sets $U_{x,y}$ and W_x (defined in the introduction), where $x,y\in\mathbb{N}$. The topology $\mathcal{T}$ was investigated in [14] under the name $\mathcal{I}_2$, and is used in the proof of the next proposition.

Proposition 3.9. *If a topological Clifford semigroup S embeds topologically into $I_\mathbb{N}$, then S is topologically isomorphic to a subsemigroup of $\mathbb{N}^\mathbb{N}$.*

Proof. Observe that each Clifford subsemigroup S of $I_\mathbb{N}$ consists of partial permutations of subsets of $\mathbb{N}$, i.e. for any $x\in S$, $\operatorname{dom}(x)=\operatorname{im}(x)$. Thus, for any Clifford subsemigroup S of $I_\mathbb{N}$ the subspace topology on S inherited from the canonical topology on $I_\mathbb{N}$ coincides with the subspace topology inherited from $\mathcal{T}$ (as defined above). Hence it remains to check that $(I_\mathbb{N},\mathcal{T})$ embeds topologically into $\mathbb{N}^\mathbb{N}$. A routine verification shows that the map $\phi: I_\mathbb{N}\to\mathbb{N}^\mathbb{N}$ defined by

$$(g)\phi=\{(x+1,y+1):(x,y)\in g\}\cup\{(x+1,0):x\in\mathbb{N}\setminus\operatorname{dom}(g)\}\cup\{(0,0)\}$$

is a topological embedding of $(I_\mathbb{N},\mathcal{T})$ into $\mathbb{N}^\mathbb{N}$. □

Lemma 3.10. *Let G be a subgroup of $\mathbb{N}^\mathbb{N}$ and e_G be the identity of G. Then the following conditions hold:*

(i) *$(x)e_G=x$ for every $x\in\operatorname{im}(e_G)$;*
(ii) *$\operatorname{im}(g)=\operatorname{im}(f)$ for any $f,g\in G$;*
(iii) *for any $g\in G$ the restriction $g\restriction_{\operatorname{im}(g)}$ is a permutation of $\operatorname{im}(g)$;*
(iv) *for any $g\in G$ the restriction $g^{-1}\restriction_{\operatorname{im}(g)}$ is equal to the inverse permutation of $g\restriction_{\operatorname{im}(g)}$ in $S_{\operatorname{im}(g)}$;*
(v) *for any $f,g\in G$, $f=g$ if and only if $f\restriction_{\operatorname{im}(f)}=g\restriction_{\operatorname{im}(g)}$;*
(vi) *if $f\in G$, $x\in\mathbb{N}$ and $x'\in\operatorname{im}(f)$ be such that $(x)f=(x')f$, then for every $g\in G$, $(x)f=(x)g$ if and only if $(x')f=(x')g$.*

Proof. (i) Since e_G is the identity of G we obtain that $((x)e_G)e_G=(x)e_G$ for every $x\in\mathbb{N}$. Then $(y)e_G=y$ for each $y\in\operatorname{im}(e_G)$.

(ii) Since $f=fg^{-1}g$ we obtain that $\operatorname{im}(f)\subseteq\operatorname{im}(g)$. Similarly, the equality $g=gf^{-1}f$ implies that $\operatorname{im}(g)\subseteq\operatorname{im}(f)$. Hence $\operatorname{im}(g)=\operatorname{im}(f)$ for each $f,g\in G$.

(iii) Fix $g \in G$ and $x, y \in \operatorname{im}(g)$ such that $(x)g = (y)g$. By item (ii), $x, y \in \operatorname{im}(e_G)$. By item (i),

$$x = (x)e_G = ((x)g)g^{-1} = ((y)g)g^{-1} = (y)e_G = y,$$

and so the map $g\restriction_{\operatorname{im}(g)}$ is injective. Item (i) implies that $\operatorname{im}(gg) = \operatorname{im}(g)$. It follows that $g\restriction_{\operatorname{im}(g)}$ is a permutation.

(iv) Follows from items (i), (ii) and (iii).

(v) Assume that $f\restriction_{\operatorname{im}(f)} = g\restriction_{\operatorname{im}(g)}$ for some $f, g \in G$. Then $(gf^{-1})\restriction_{\operatorname{im}(g)}$ is the identity permutation of $\operatorname{im}(g)$. It is straightforward to check that gf^{-1} is an idempotent. Since $E(\mathbb{N}^{\mathbb{N}}) \cap G = \{e_G\}$, we get that $gf^{-1} = e_G$, and so $f = g$.

(vi) First assume that $(x')f = (x)f = (x)g$ for some $f, g \in G$, $x \in \mathbb{N}$ and $x' \in \operatorname{im}(f)$. Since $x' \in \operatorname{im}(g)$ we obtain the following:

$$(x')g = ((x')e_G)g = ((x')f)f^{-1}g = ((x)ff^{-1})g = ((x)e_G)g = ((x)g)e_G = (x)g = (x')f.$$

Assume that $(x)f = (x')f = (x')g$ for some $f, g \in G$, $x \in \mathbb{N}$ and $x' \in \operatorname{im}(f)$. Then

$$(x)g = ((x)e_G)g = ((x)f)f^{-1}g = ((x')ff^{-1})g = ((x')e_G)g = (x')g = (x)f. \qquad \square$$

3.5. **Proof of Theorem 2.5.** We need to prove that for a topological group G the following conditions are equivalent:

(i) G embeds topologically into $S_{\mathbb{N}}$;
(ii) G embeds topologically into $I_{\mathbb{N}}$;
(iii) G embeds topologically into $\mathbb{N}^{\mathbb{N}}$;
(iv) G is second-countable and has a neighbourhood basis of the identity consisting of open subgroups.

Proof. The implication (i) $\Rightarrow$ (ii) is trivial; the implication (ii) $\Rightarrow$ (iii) follows from Proposition 3.9; and the equivalence (i) $\Leftrightarrow$ (iv) is well-known and follows from [14, Theorem 5.5].

(iii) $\Rightarrow$ (i). Let G be a subgroup of $\mathbb{N}^{\mathbb{N}}$. Let $X = \operatorname{im}(g)$ for some $g \in G$. Lemma 3.10(ii) implies that the set X does not depend on the choice of g. By Lemma 3.10(v), G acts faithfully by permutations on the set X. This gives a natural algebraic embedding $\phi : G \to S_X$ defined by $(g)\phi = g\restriction_{\operatorname{im}(g)}$. It remains to show that this embedding is topological. A subbasic open set in the topology on $(G)\phi$ has the form

$$\{g \in (G)\phi : (x, y) \in g\}$$

for some $x, y \in X$. By the definition of ϕ,

$$(\{g \in (G)\phi : (x, y) \in g\})\phi^{-1} = \{g \in G : (x, y) \in g\},$$

which is open in the topology on G. Hence the map ϕ is continuous. Conversely, a subbasic open set in the topology on G has the form

$$\{g \in G : (x, y) \in g\}$$

for some $x \in \mathbb{N}$ and $y \in X$. By Lemma 3.10(iii), there exists $x' \in X$ such that $(x')g = y$. Lemma 3.10(vi) implies that

$$(\{g \in G : (x, y) \in g\})\phi = \left\{g \in (G)\phi : (x', y) \in g\right\}$$

and hence the map ϕ is a topological embedding. $\square$

Definition 3.11. A topological inverse semigroup X is called *weakly ditopological* if for any point $x \in X$ and a neighborhood O of x there are neighborhoods U, V and W of the points x, $x^{-1}x$ and xx^{-1}, respectively, such that

$$\{s \in S : \exists b \in U,\ \exists e \in W \cap E(S) \text{ such that } b = es\} \cap \left\{s \in S : ss^{-1} \in W\right\} \cap \left\{s \in S : s^{-1}s \in V\right\} \subseteq O.$$

Clearly, each ditopological inverse semigroup is weakly ditopological. However, the converse is not true (see [31, Example 3.4]). Since $xx^{-1} = x^{-1}x$ in Clifford semigroups, we get the following.

Proposition 3.12. *A Clifford semigroup S is ditopological if and only if S is weakly ditopological.*

Proposition 3.13. *The topological inverse semigroup $I_{\mathbb{N}}$ is weakly ditopological.*

Proof. Fix a partial injection $f \in I_{\mathbb{N}}$ and a basic open neighborhood O of f. Then there exist finite subsets A_1, A_2, A_3 of $\mathbb{N}$ such that $A_1 \subseteq \operatorname{dom}(f)$, $A_2 \cap \operatorname{dom}(f) = \emptyset$, $A_3 \cap \operatorname{im}(f) = \emptyset$ and

$$O = \left\{g \in I_{\mathbb{N}} : g{\restriction}_{A_1} = f{\restriction}_{A_1},\ \operatorname{dom}(g) \cap A_2 = \emptyset \text{ and } \operatorname{im}(g) \cap A_3 = \emptyset\right\}.$$

By id_A we denote the identity function on a subset $A \subseteq \mathbb{N}$. Consider the open neighborhoods

$$W = \left\{g \in I_{\mathbb{N}} : g{\restriction}_{A_1} = \mathrm{id}_{A_1} \text{ and } \operatorname{dom}(g) \cap A_2 = \emptyset\right\}$$

and

$$V = \left\{g \in I_{\mathbb{N}} : g{\restriction}_{(A_1)f} = \mathrm{id}_{(A_1)f} \text{ and } \operatorname{im}(g) \cap A_3 = \emptyset\right\}$$

of ff^{-1} and $f^{-1}f$, respectively. Put $U = O$ and

$$D = \{s \in I_{\mathbb{N}} : \exists b \in U,\ \exists e \in W \cap E(S) \text{ such that } b = es\} \cap \left\{s \in I_{\mathbb{N}} : ss^{-1} \in W\right\} \cap \left\{s \in I_{\mathbb{N}} : s^{-1}s \in V\right\}.$$

In order to show that $D \subseteq O$ fix any $y \in D$. There exist $b \in U$ and $e \in W$ such that $b = ey$. Consequently, $b{\restriction}_{\operatorname{dom}(e)} = y{\restriction}_{\operatorname{dom}(e)}$. Since $A_1 \subseteq \operatorname{dom}(e)$ and $f{\restriction}_{A_1} = b{\restriction}_{A_1}$ we deduce that $y{\restriction}_{A_1} = f{\restriction}_{A_1}$. By the definition of W, for each element $z \in \left\{s \in I_{\mathbb{N}} : ss^{-1} \in W\right\}$ we have that $\operatorname{dom}(z) \cap A_2 = \emptyset$. Consequently, $\operatorname{dom}(y) \cap A_2 = \emptyset$. Analogously, for each $z \in \left\{s \in I_{\mathbb{N}} : s^{-1}s \in V\right\}$ we have that $\operatorname{im}(z) \cap A_3 = \emptyset$. It follows that $\operatorname{im}(y) \cap A_3 = \emptyset$. Hence $y \in O$, and so the semigroup $I_{\mathbb{N}}$ is weakly ditopological. □

By $2^{\mathbb{N}}$ we denote the Cantor set endowed with the semilattice operation of taking coordinate-wise minimum.

Lemma 3.14. *The semilattice of idempotents of $I_{\mathbb{N}}$ is topologically isomorphic to $2^{\mathbb{N}}$.*

Proof. Note that any idempotent e of $I_{\mathbb{N}}$ is the identity map on the set $\operatorname{dom}(e)$. A routine verification shows that the map $\phi : E(I_{\mathbb{N}}) \to 2^{\mathbb{N}}$ which assigns to each element $e \in E(I_{\mathbb{N}})$ the characteristic function of $\operatorname{dom}(e)$ is a topological isomorphism. □

For an element x of a semilattice X let

$$\Uparrow\! x = \{z \in X : \text{there exists an open neighborhood } V \text{ of } z \text{ such that } V \subseteq \uparrow\! x\}.$$

A subset A of a topological semilattice X is called *U-dense* if for each point $x \in X$ and a neighborhood U of x in X there exists a point $y \in U \cap A$ such that $x \in \Uparrow\! y$. A semilattice X is called *U-separable* if it possesses a countable U-dense subset.

The following result was proved in [6, Proposition 2.5]

Proposition 3.15. *Each second-countable U-semilattice is U-separable.*

Recall that if X is a topological semigroup, then X^0 carries the unique topology $\mathcal{T}$ such that the point 0 is isolated in $(X^0, \mathcal{T})$ and the subspace topology on X inherited from $(X^0, \mathcal{T})$ coincides with the original topology on X. Let X be an inverse semigroup and $e \in E(X)$. Then the maximal subgroup $\left\{x \in X : xx^{-1} = e = x^{-1}x\right\}$ is denoted by H_e. The following nontrivial result proved by Banakh and Pastukhova in [6, Theorem 3.2] is crucial for this paper.

Theorem 3.16. *Let S be a Hausdorff ditopological Clifford semigroup whose set of idempotents $E(S)$ is a U_2-semilattice, and A be any U-dense subset of S. Then S can be topologically embedded into the Tychonoff product*

$$E(S) \times \prod_{e \in A} (H_e^0)^{A \cap \Uparrow e},$$

where H_e is endowed with the subspace topology inherited from S.

3.6. **Proof of Theorem 2.11.** We need to show that a topological Clifford semigroup S whose set of idempotents $E(S)$ is a U-semilattice embeds topologically into $I_{\mathbb{N}}$ if and only if S is Hausdorff, ditopological and every maximal subgroup of S, as well as the semilattice $E(S)$, embed topologically into $I_{\mathbb{N}}$.

Proof. ($\Rightarrow$) If a Clifford topological semigroup S embeds topologically into $I_{\mathbb{N}}$, then S is Hausdorff. Also the maximal subgroups of S and the semilattice $E(S)$ embed topologically into $I_{\mathbb{N}}$. According to [31] each inverse subsemigroup of a weakly ditopological inverse semigroup is weakly ditopological. Then Proposition 3.13 implies that the Clifford semigroup S is weakly-ditopological. Proposition 3.12 yields that S is ditopological.

($\Leftarrow$) Let S be a Hausdorff ditopological Clifford semigroup whose set of idempotents $E(S)$ satisfies the following properties:

(1) $E(S)$ is a U-semilattice which embeds topologically into $I_{\mathbb{N}}$;
(2) for every $e \in E(S)$ the maximal subgroup $H_e = \{x \in S : xx^{-1} = e = x^{-1}x\}$ embeds topologically into $I_{\mathbb{N}}$.

By [6, Proposition 2.4(6)], each U-semilattice which embeds topologically into $2^{\mathbb{N}}$ is a U_2-semilattice. Lemma 3.14 implies that $E(S)$ is a U_2-semilattice. Hence S satisfies conditions of Theorem 3.16. Therefore, for any U-dense subset $A \subseteq S$, S can be topologically embedded into the Tychonoff product

$$E(S)\times \prod_{e\in A} (H_e^0)^{A\cap\Uparrow e}.$$

Since the space $I_{\mathbb{N}}$ is Polish, the semilattice $E(S)$ is second-countable. Proposition 3.15 implies that we lose no generality assuming that the set A is countable.

By Lemma 3.7, for each idempotent $e \in S$ the topological monoid H_e^0 embeds topologically into $I_{\mathbb{N}}$. Since the set A is countable, Lemma 3.4(ii) implies that for every $e \in E(S)$ the topological semigroup $(H_e^0)^{A\cap\Uparrow e}$ embeds topologically into $I_{\mathbb{N}}$. Using one more time Lemma 3.4(ii) we get that $\prod_{e\in A}(H_e^0)^{A\cap\Uparrow e}$ embeds topologically into $I_{\mathbb{N}}$.

By the assumption, $E(S)$ is topologically isomorphic to a subsemigroup of $E(I_{\mathbb{N}})$. Lemma 3.4(ii) ensures that $E(S)\times\prod_{e\in A}(H_e^0)^{A\cap\Uparrow e}$ embeds topologically into $I_{\mathbb{N}}$. Hence S is topologically isomorphic to a subsemigroup of $I_{\mathbb{N}}$. □

Proposition 3.17. *Let S be a Clifford subsemigroup of $\mathbb{N}^{\mathbb{N}}$. Then S endowed with the subspace topology is a topological inverse semigroup.*

Proof. Suppose that S is any Clifford subsemigroup of $\mathbb{N}^{\mathbb{N}}$. Throughout this proof it will be convenient to denote $s\restriction_{\operatorname{im}(s)}$ by ϕ_s for every $s \in S$. Since every Clifford semigroup is a union of groups, if $s \in S$, then s belongs to a subgroup of $\mathbb{N}^{\mathbb{N}}$. In particular, by Lemma 3.10, for each $s \in S$, ϕ_s is a permutation of $\operatorname{im}(s)$ and $\phi_s^{-1} = \phi_{s^{-1}}$, where by ϕ_s^{-1} we mean the inverse permutation of ϕ_s. Although $\phi_s \notin \mathbb{N}^{\mathbb{N}}$ unless s is a permutation itself, if $t \in \mathbb{N}^{\mathbb{N}}$ is any transformation such that $\operatorname{im}(t) \subseteq \operatorname{im}(s)$, then the compositions $t \circ \phi_s$ and $t \circ \phi_s^{-1}$ (as binary relations) belong to $\mathbb{N}^{\mathbb{N}}$. For the sake of brevity, we will denote them as $t\phi_s$ and $t\phi_s^{-1}$, respectively. Taking into account that $(x)\phi_s = (x)s$ for all $x \in \operatorname{im}(s)$, Lemma 3.10 implies the following: $s\phi_s^{-1}s = s$ and $s\phi_s^{-1} = ss^{-1} = s^{-1}s = s^{-1}\phi_s$ for any $s \in S$.

Clearly, it is enough to show that inversion is continuous in S. For a finite partial function f on $\mathbb{N}$ consider a nonempty basic open set $U = \{u \in S : f \subseteq u\}$. Fix an arbitrary $s \in U^{-1}$. In order to show that the set U^{-1} is open, we need to find an open neighbourhood V of s such that $V \subseteq U^{-1}$. Let

$$T = (\operatorname{dom}(f))s\phi_s^{-1} = \{x \in \operatorname{im}(s) : (x)s \in (\operatorname{dom}(f))s\},$$

and

$$Z = \operatorname{dom}(f) \cup T \cup (T)\phi_s^{-1}.$$

Observe that the set Z is finite. We define

$$W' = \left\{t \in S : \forall x \in Z, (x)s\phi_s^{-1} \in \operatorname{im}(t) \text{ and } (x)t = (x)s\phi_s^{-1}t\right\}$$

and

$$W = \{t \in S : \operatorname{im}(s) \cap Z \subseteq \operatorname{im}(t)\} \cap W'.$$

Taking into account that for each finite subset $A \subseteq \mathbb{N}$ the set $\left\{g \in \mathbb{N}^{\mathbb{N}} : A \subseteq \operatorname{im}(g)\right\}$ is open in $\mathbb{N}^{\mathbb{N}}$, it is routine to verify that the set W' is open and contains s. It follows that the set W is an open neighborhood of s. We will show that

$$W \subseteq \{t \in S : \operatorname{im}(t) \cap Z = \operatorname{im}(s) \cap Z\}.$$

It suffices to check that for every $t \in W$ if $x \in Z \setminus \operatorname{im}(s)$, then $x \in Z \setminus \operatorname{im}(t)$. If $x \in Z \setminus \operatorname{im}(s)$, then $x \in \operatorname{dom}(f)$ and, consequently, $(x)s\phi_s^{-1} \in T \subseteq \operatorname{im}(s) \cap Z$. By the definition of W', $(x)s\phi_s^{-1} \in \operatorname{im}(t)$ and $(x)t = (x)s\phi_s^{-1}t$. But $x \notin \operatorname{im}(s)$ and $(x)s\phi_s^{-1} \in \operatorname{im}(s)$ and so $x \neq (x)s\phi_s^{-1}$. On the other hand, $(x)s\phi_s^{-1} \in \operatorname{im}(t)$ and t is a permutation of $\operatorname{im}(t)$, and so $x \notin \operatorname{im}(t)$. Thus, $W \subseteq \{t \in S : \operatorname{im}(t) \cap Z = \operatorname{im}(s) \cap Z\}$.

We define $V = W \cap \{t \in S : t\restriction_Z = s\restriction_Z\}$. Clearly, V is an open neighborhood of s. It remains to check that $V \subseteq U^{-1}$. Let $t \in V$ be arbitrary. We need to show that $f \subseteq t^{-1}$. As

$$t \in V = W \cap \{k \in S : k\restriction_Z = s\restriction_Z\} \subseteq \{k \in S : \operatorname{im}(k) \cap Z = \operatorname{im}(s) \cap Z\} \cap \{k \in S : k\restriction_Z = s\restriction_Z\},$$

we know that $t\restriction_Z = s\restriction_Z$ and $\operatorname{im}(t) \cap Z = \operatorname{im}(s) \cap Z$.

Let $x \in \operatorname{dom}(f)$ be arbitrary. We will show that $(x)t^{-1} = (x)f$. Since $tt^{-1} = t^{-1}t$, Lemma 3.10(iii) and (iv) imply the following:

$$(x)t^{-1} = (x)t^{-1}tt^{-1} = (x)tt^{-1}t^{-1} = (x)t\phi_t^{-1}\phi_t^{-1}. \tag{3}$$

Since $s \in U^{-1} = \{u \in S : f \subseteq u\}^{-1} = \left\{u^{-1} \in S : f \subseteq u\right\} = \left\{u \in S : f \subseteq u^{-1}\right\}$ we get that

$$(x)f = (x)s^{-1} = (x)s^{-1}ss^{-1} = (x)ss^{-1}s^{-1} = (x)s\phi_s^{-1}\phi_s^{-1}. \tag{4}$$

By assumption $x \in Z$ and so $(x)t = (x)s$. Since $(x)s\phi_s^{-1} \in T \subseteq Z$ and $s\restriction_Z = t\restriction_Z$, we get that

$$(x)s\phi_s^{-1}t = (x)s\phi_s^{-1}s = (x)s. \tag{5}$$

Clearly, $(x)s\phi_s^{-1} \in \operatorname{im}(s) \cap Z = \operatorname{im}(t) \cap Z$, and so $(x)s\phi_s^{-1} \in \operatorname{im}(t)$. Hence

$$(x)s\phi_s^{-1} = (x)s\phi_s^{-1}\phi_t\phi_t^{-1} = (x)s\phi_s^{-1}t\phi_t^{-1} = (x)s\phi_t^{-1} \tag{6}$$

(the last equality holds by (5)). Similarly, $(x)s\phi_s^{-1}\phi_s^{-1} \in (T)\phi_s^{-1} \subseteq Z \cap \operatorname{im}(s)$ implies $(x)s\phi_s^{-1}\phi_s^{-1} \in \operatorname{im}(t)$. Since $s\restriction_Z = t\restriction_Z$ we have that $(x)s\phi_s^{-1}\phi_s^{-1}t = (x)s\phi_s^{-1}\phi_s^{-1}s = (x)s\phi_s^{-1}$. So

$$(x)s\phi_s^{-1}\phi_s^{-1} = (x)s\phi_s^{-1}\phi_s^{-1}t\phi_t^{-1} = (x)s\phi_s^{-1}\phi_t^{-1}. \tag{7}$$

Hence

$$\begin{aligned}
(x)t^{-1} &= (x)t\phi_t^{-1}\phi_t^{-1} && \text{by (3)}\\
&= (x)s\phi_t^{-1}\phi_t^{-1} && \text{as } x \in \operatorname{dom}(f) \subseteq Z \text{ and } s\restriction_Z = t\restriction_Z\\
&= (x)s\phi_s^{-1}\phi_t^{-1} && \text{by (6)}\\
&= (x)s\phi_s^{-1}\phi_s^{-1} && \text{by (7)}\\
&= (x)f && \text{by (4).}
\end{aligned}$$

Since $x \in \operatorname{dom}(f)$ was arbitrary, we obtain that $f \subseteq t^{-1}$, and so $t \in U^{-1}$. It follows that $V \subseteq U^{-1}$. Thus, the set U^{-1} is open, implying the continuity of inversion in S. □

By the proof of Proposition 3.9, the inverse topological semigroup $(I_{\mathbb{N}}, \mathcal{T})$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. However the inversion is not continuous in $(I_{\mathbb{N}}, \mathcal{T})$. Hence Proposition 3.17 doesn't hold for an arbitrary inverse subsemigroup of $\mathbb{N}^{\mathbb{N}}$.

Proposition 3.18. *Each topological inverse subsemigroup of $\mathbb{N}^{\mathbb{N}}$ is ditopological.*

Proof. Let S be a topological inverse subsemigroup of $\mathbb{N}^{\mathbb{N}}$. By $\mathcal{T}$ we denote the subspace topology on S inherited from $\mathbb{N}^{\mathbb{N}}$. By Proposition 3.1, there exists a family $\{\rho_n : n \in \mathbb{N}\}$ of right congruences on S, each having countably many equivalence classes, such that the family $\{[x]_{\rho_n} : x \in S,\ n \in \mathbb{N}\}$ is a basis of $\mathcal{T}$.

To show that S is ditopological fix any $x \in S$ and an open neighborhood O of x. Then there exists $n \in \mathbb{N}$ such that $[x]_{\rho_n} \subseteq O$. Let $U = [x]_{\rho_n}$ and $W = [xx^{-1}]_{\rho_n}$. It remains to check that

$$D = \{y \in S : \exists b \in U,\ \exists e \in W \cap E(S) \text{ such that } b = ey\} \cap \left\{y \in S : yy^{-1} \in W\right\} \subseteq O.$$

Fix any $y \in D$. Then there exist $b \in U$ and $e \in W \cap E(S)$ such that $b = ey$. The choice of the sets U and W implies that $(b, x) \in \rho_n$ and $(e, xx^{-1}) \in \rho_n$. Since $yy^{-1} \in W$ we get that $(xx^{-1}, yy^{-1}) \in \rho_n$ and, consequently, $(e, yy^{-1}) \in \rho_n$. Then

$$[x]_{\rho_n} = [b]_{\rho_n} = [ey]_{\rho_n} = [yy^{-1}y]_{\rho_n} = [y]_{\rho_n}.$$

Hence $y \in U$, and so S is a ditopological inverse semigroup. □

3.7. **Proof of Theorem 2.8.** We need to show that each Clifford subsemigroup of $\mathbb{N}^{\mathbb{N}}$ is ditopological.

Proof. By Proposition 3.17, each Clifford subsemigroup of $\mathbb{N}^{\mathbb{N}}$ is a topological inverse semigroup. Proposition 3.18 implies that Clifford subsemigroups of $\mathbb{N}^{\mathbb{N}}$ are ditopological. □

3.8. **Proof of Theorem 2.12.** We need to show that a Clifford topological semigroup S whose set of idempotents $E(S)$ is a U_2-semilattice embeds topologically into $\mathbb{N}^{\mathbb{N}}$ if and only if S is Hausdorff, ditopological and every maximal subgroup of S, as well as the semilattice $E(S)$, embed topologically into $\mathbb{N}^{\mathbb{N}}$.

Proof. ($\Rightarrow$) Let S be a Clifford topological subsemigroup of $\mathbb{N}^{\mathbb{N}}$. Then S is Hausdorff and every maximal subgroup of S, as well as the semilattice $E(S)$, embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Theorem 2.8 implies that S is ditopological.

($\Leftarrow$) Let S be a Hausdorff ditopological Clifford semigroup whose set of idempotents $E(S)$ is a U_2-semilattice which embeds topologically into $\mathbb{N}^{\mathbb{N}}$ and for every $e \in E(S)$ the maximal subgroup $H_e = \left\{x \in S : xx^{-1} = e = x^{-1}x\right\}$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Then S satisfies conditions of Theorem 3.16. Therefore, for any U-dense subset $A \subseteq S$, S can be topologically embedded into the Tychonoff product

$$E(S)\times \prod_{e\in A} (H_e^0)^{A\cap{\Uparrow}e}.$$

Since the space $\mathbb{N}^{\mathbb{N}}$ is Polish, the semilattice $E(S)$ is second-countable. By Proposition 3.15 we can assume that the set A is countable.

By Lemma 3.6, for each idempotent $e \in S$ the topological monoid H_e^0 embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Since the set A is countable, Lemma 3.4(i) implies that for every $e \in E(S)$ the topological semigroup $(H_e^0)^{A\cap{\Uparrow}e}$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Using one more time Lemma 3.4(i) we get that $\prod_{e\in A}(H_e^0)^{A\cap{\Uparrow}e}$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. By the assumption, $E(S)$ is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. Lemma 3.4(i) ensures that $E(S)\times\prod_{e\in A}(H_e^0)^{A\cap{\Uparrow}e}$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Hence S is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. □

A space X is called *scattered* if every subset A of X contains an isolated (in the subspace topology) point. Recall that Cantor-Bendixson derivatives of a scattered space X are defined by transfinite induction as follows, where X' is the set of all accumulation points of X:

(i) $X^0 = X$;
(ii) $X^{\alpha+1} = \left(X^\alpha\right)'$;
(iii) $X^\alpha = \bigcap_{\beta<\alpha} X^\beta$, if α is a limit ordinal.

The set $X^\alpha \setminus X^{\alpha+1}$ is called the α-th *Cantor-Bendixson level* of X and is denoted by $X^{(\alpha)}$ (this notation is used in the proof of Lemma 3.19). The *height* of a scattered space X is the smallest ordinal $ht(X)$ such that $X^{ht(X)} = \emptyset$.

A semilattice X endowed with a topology $\mathcal{T}$ is called:

(i) *semitopological* if for every $a \in X$ the shift $l_a : X \to X$, $(x)l_a = xa$ is continuous;
(ii) *U_2-semilattice at a point x* if for every open neighborhood V of x there exist a point $y \in V$ and a clopen ideal $I \subseteq E$ such that $x \in X \setminus I \subseteq \uparrow y$.

Lemma 3.19. *Each scattered T_1 semitopological semilattice is a U_2-semilattice.*

Proof. It is easy to check that for each open subset U of X the upper set $\uparrow U = \bigcup_{x \in U} \uparrow x$ is open. Since each singleton is closed in X and the semilattice X is semitopological, the set $\uparrow x$ is closed for every $x \in X$. Hence for every $x \in X^{(0)}$ (the 0-th Cantor-Bendixson level of X) the set $\uparrow x$ is clopen. Then for each $x \in X^{(0)}$ the clopen ideal $I = X \setminus \uparrow x$ together with the point x implying that X has the U_2 property at $x \in X^{(0)}$. Assume that for some ordinal $\alpha < ht(X)$, X has the U_2 property at every $x \in \bigcup_{\xi \in \alpha} X^{(\xi)}$. Fix any $x \in X^{(\alpha)}$ and open neighborhood U of x. Since the space X is scattered, we lose no generality assuming that $U \subseteq \bigcup_{\xi \leq \alpha} X^{(\xi)}$ and $U \cap X^\alpha = \{x\}$. The continuity of shifts in X yields the existence of an open neighborhood V of x such that $xV \subseteq U$. There are two cases to consider:

(1) $xz = x$ for all $z \in V$;
(2) there exists $z \in V$ such that $xz = y \in U \setminus \{x\}$.

In case 1 we have that $V \subseteq \uparrow x$. Clearly, the set $\uparrow x$ is closed. Let us show that the upper cone $\uparrow x$ is open. Pick any $a \in \uparrow x$. The continuity of shifts in X yields an open neighborhood W of a such that $Wx \subseteq V \subseteq \uparrow x$, establishing that the element a belongs to the interior of $\uparrow x$. Hence the set $\uparrow x$ is clopen. Thus, the clopen ideal $I = X \setminus \uparrow x$ together with the point x show that X is a U_2-semilattice at x.

Assume that case 2 holds. The choice of U implies that $y \in \bigcup_{\xi \in \alpha} X^{(\xi)}$. By the inductive assumption, there exist $p \in U$ and a clopen ideal I such that $y \in X \setminus I \subseteq \uparrow p$. Note that

$$xp = x(yp) = (xy)p = xxzp = xzp = yp = p,$$

implying that $x \in \uparrow p$. Since $xy = xxz = xz = y$ we get that $x \notin I$, because otherwise $y = xy \in I$, contradicting the choice of I. Thus, the point $p \in U$ and the clopen ideal I prove that the semilattice X has the U_2 property at the point x.

Hence X is a U_2-semilattice at each point $x \in X$, implying that X is a U_2-semilattice. □

3.9. **Proof of Theorem 2.7.** We need to show that a countable Polish Clifford semigroup S embeds topologically into $I_{\mathbb{N}}$ if and only if S is ditopological and the semilattice $E(S)$ embeds topologically into $I_{\mathbb{N}}$.

Proof. ($\Rightarrow$) According to [31] each inverse subsemigroup of a weakly ditopological inverse semigroup is weakly ditopological. Then Proposition 3.13 implies that each Clifford subsemigroup S of $I_{\mathbb{N}}$ is weakly ditopological. Proposition 3.12 yields that S is ditopological. Clearly, since the entire semigroup S embeds into $I_{\mathbb{N}}$, so too does its semilattice of idempotents.

($\Leftarrow$) Let S be a ditopological countable Polish Clifford semigroup such that the semilattice $E(S)$ embeds topologically into $I_{\mathbb{N}}$. The continuity of the inversion in S implies that maximal subgroups of S are closed and hence Polish. Since non-discrete Polish topological groups are of cardinality continuum, we deduce that each maximal subgroup of S is discrete and, thus, embeds topologically into $I_{\mathbb{N}}$ by Theorem 2.5. Clearly, every countable Polish space is scattered. By Lemma 3.19, the semilattice $E(S)$ is a U_2-semilattice and, consequently, a U-semilattice. Theorem 2.11 implies that X embeds topologically into $I_{\mathbb{N}}$. □

3.10. **Proof of Theorem 2.9.** We need to show that a countable Polish Clifford semigroup S embeds topologically into $\mathbb{N}^{\mathbb{N}}$ if and only if S is ditopological and the semilattice $E(S)$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$.

Proof. ($\Rightarrow$) By Theorem 2.8 each Clifford subsemigroup S of $\mathbb{N}^{\mathbb{N}}$ is ditopological. Clearly the semilattice of idempotents of S embeds in $\mathbb{N}^{\mathbb{N}}$, since S embeds into $\mathbb{N}^{\mathbb{N}}$.

($\Leftarrow$) Let S be a countable Polish ditopological Clifford semigroup such that the semilattice $E(S)$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Similarly as in the proof of Theorem 2.7 it can be checked that each maximal subgroup of S embeds topologically into $\mathbb{N}^{\mathbb{N}}$ and the semilattice $E(S)$ is a U_2-semilattice. Theorem 2.12 implies that X embeds topologically into $\mathbb{N}^{\mathbb{N}}$. □

4. Counterexamples

In this section we collect counterexamples to Question 1.1 as well as other examples which show the sharpness of the results proved in the previous section.

Given Theorem 2.4 and Proposition 3.9, it might be tempting to think that $I_{\mathbb{N}}$ embeds topologically in $\mathbb{N}^{\mathbb{N}}$. The following lemma shows that this is not the case.

Proposition 4.1. *The topological inverse semigroup $I_{\mathbb{N}}$ cannot be topologically embedded into $\mathbb{N}^{\mathbb{N}}$.*

Proof. Seeking a contradiction, we suppose that there is such an embedding. It follows from Proposition 3.1, that there is a countable family $\{\rho_i : i \in \mathbb{N}\}$ of right congruences on $I_{\mathbb{N}}$ such that the equivalence classes of these congruences form a basis for the canonical topology on $I_{\mathbb{N}}$.

Clearly, the set $U = \{f \in I_{\mathbb{N}} : 0 \notin \operatorname{im}(f)\}$ is an open neighborhood of $\emptyset$ in $I_{\mathbb{N}}$. Then there is $k \in \mathbb{N}$ such that $[\emptyset]_{\rho_k} \subseteq U$. Since the set $[\emptyset]_{\rho_k}$ is open in $I_{\mathbb{N}}$, there are finite subsets $X, Y \subseteq \mathbb{N}$ such that

$$V = \{g \in I_{\mathbb{N}} : \operatorname{dom}(g) \cap X = \emptyset, \operatorname{im}(g) \cap Y = \emptyset\}$$

satisfies

$$\emptyset \in V \subseteq [\emptyset]_{\rho_k} \subseteq U.$$

Fix an arbitrary $n \in \mathbb{N} \setminus (X \cup Y)$. We have that $\{(n, n)\} \in V \subseteq [\emptyset]_{\rho_k}$. Since ρ_k is a right congruence we get that

$$\{(n, 0)\} = \{(n, n)\} \circ \{(n, 0)\} \in [\emptyset \circ \{(n, 0)\}]_{\rho_k} = [\emptyset]_{\rho_k} \subseteq U = \{f \in I_{\mathbb{N}} : 0 \notin \operatorname{im}(f)\},$$

which is a contradiction. □

Each semilattice X carries a natural partial order $\leq$ defined by $e \leq f$ if $ef = e$ for any $e, f \in X$. A semilattice X is called *chain-finite* if every linearly ordered subset in $(X, \leq)$ is finite. The following proposition shows that Proposition 3.9 cannot be reversed.

Proposition 4.2. *Each countable infinite discrete chain-finite semilattice X embeds topologically into $\mathbb{N}^{\mathbb{N}}$, but not into $I_{\mathbb{N}}$.*

Proof. Since the semilattice X is countable and discrete, Corollary 3.3 implies that X embeds topologically into $\mathbb{N}^{\mathbb{N}}$. Lemma 4.1 from [2] implies that for each topological embedding ϕ of X into a zero-dimensional Hausdorff topological semigroup Y the image $(X)\phi$ is closed in Y. Consequently, for each topological embedding $\phi : X \to I_{\mathbb{N}}$ the image $(X)\phi \subseteq E(I_{\mathbb{N}})$ is closed. Since the semilattice X is noncompact, Lemma 3.14 yields that X cannot be embedded topologically into $I_{\mathbb{N}}$. □

Looking at Theorems 2.11 and 2.12 it is natural to ask whether each subsemilattice of $I_{\mathbb{N}}$ or $\mathbb{N}^{\mathbb{N}}$ is a U-semilattice. The following lemma gives a negative answer to this question.

Proposition 4.3. *There exists a topological semilattice X which is not a U-semilattice at any of its points, but embeds topologically into $I_{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let $\mathbb{R}$ be the real line endowed with the usual topology and the semilattice operation of taking the minimum. Consider the discrete subsemilattice $Z = \left\{\frac{1}{n+1} : n \in \mathbb{N}\right\} \cup \left\{2 - \frac{1}{n+1} : n \in \mathbb{N}\right\}$ of $\mathbb{R}$. Note that $\overline{Z} = Z \cup \{0, 2\}$ is a compact zero-dimensional topological semilattice. Let

$$X = \left\{(x_n)_{n\in\mathbb{N}} \in Z^{\mathbb{N}} : x_n = 1 \text{ for all but finitely many } n \in \mathbb{N}\right\}$$

be a subsemilattice of the Tychonoff product $Z^{\mathbb{N}}$. It is easy to see that $Y = \overline{Z}^{\mathbb{N}}$ is a compact metrizable zero-dimensional topological semilattice which contains X. By Theorem 2.4, Y embeds topologically into $I_{\mathbb{N}}$. It follows that X embeds topologically into $I_{\mathbb{N}}$ and hence into $\mathbb{N}^{\mathbb{N}}$ as well, by Proposition 3.9.

Fix a point $(x_n)_{n\in\mathbb{N}} \in X$, an open neighborhood U of x and a point $(y_n)_{n\in\mathbb{N}} \in U$. In order to show that X is not a U-semilattice at x consider a basic open neighborhood $V = \{(z_n)_{n\in\mathbb{N}} \in X : z_n = x_n$ for all $n \leq k\}$ of x. Let $t \in Z$ be such that $t < y_{k+1}$. It is clear that $(x_0, \ldots, x_n, t, 1, \ldots, 1, \ldots) \in V \setminus {\uparrow}y$ implying that x doesn't belong to the interior of ${\uparrow}y$. Since y was chosen arbitrarily, X is not a U-semilattice at the point x. □

Let X be a non-empty topological space. The *strong Choquet game* on X is defined as follows: Player I chooses a pair (x_0, U_0) where U_0 is an open subset of X and $x_0 \in U_0$. Player II responds with an open subset V_0 such that $x_0 \in V_0 \subseteq U_0$. At stage n Player I chooses a pair (x_n, U_n) such that U_n is an open subset of X and $x_n \in U_n \subseteq V_{n-1}$. Player II responds with an open set $V_n \subseteq U_n$ which contains x_n. If $\bigcap_{n\in\omega} U_n = \emptyset$, then Player I wins. Otherwise, Player II wins. The following result was proved in [11].

Theorem 4.4. *A metrizable space X is completely metrizable if and only if Player II has a winning strategy in strong Choquet game on X.*

The following lemma is helpful in detecting scattered Polish spaces.

Lemma 4.5. *A scattered space X is Polish if and only if X is regular and second-countable.*

Proof. ($\Rightarrow$) Clearly, each Polish space is regular and second-countable.

($\Leftarrow$) By the Urysohn Metrization Theorem, each regular second-countable space X is metrizable and separable. By Theorem 4.4, to prove that X is Polish it suffices to show that Player II has a winning strategy in the strong Choquet game on X. Assume that we are at stage n of the strong Choquet game and Player I chose a corresponding pair (U_n, x_n). Find an ordinal $\alpha \in ht(X)$ such that x_n belongs to the Cantor-Bendixson level $X^{(\alpha)}$. Player II can respond with any open neighborhood $V_n \subseteq U_n$ of x_n such that $V_n \cap X^{(\alpha)} = V_n \cap X^{\alpha} = \{x_n\}$. Then, using the fact that ordinals do not possess infinite decreasing sequences, it is straightforward to check that this is a winning strategy for Player II. □

In the following four propositions we construct counterexamples to Question 1.1.

Proposition 4.6. *There exists a countable commutative (and hence Clifford) Hausdorff topological inverse semigroup S such that*

(i) *S is locally compact and Polish;*
(ii) *the semilattice $E(S)$ is compact;*
(iii) *S cannot be topologically embedded into $\mathbb{N}^{\mathbb{N}}$.*

Proof. By T we denote the set $\{0\} \cup \{x_n : n \in \mathbb{N}\}$ endowed with the semilattice operation

$$ab = \begin{cases} a, & \text{if } a = b; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{1,-1\}$ be the two-element multiplicative group. Then the direct product $S=T\times\{1,-1\}$ is a countable commutative inverse semigroup whose semilattice $E(S)$ coincides with the set $T\times\{1\}$. We endow S with the topology $\mathcal{T}$ defined as follows:

(1) each element $x\in S\setminus\{(0,1)\}$ is isolated;
(2) the sets $U_n=\{(0,1)\}\cup\{(x_i,1):i\geq n\}$, $n\in\mathbb{N}$ form an open neighborhood basis at $(0,1)$.

It is easy to check that $(S,\mathcal{T})$ is a locally compact regular scattered second-countable topological inverse semigroup and the semilattice $E(S)=T\times\{1\}$ is compact. By Lemma 4.5, the space $(S,\mathcal{T})$ is Polish. To derive a contradiction, assume that $(S,\mathcal{T})$ can be embedded topologically into $\mathbb{N}^{\mathbb{N}}$. By Proposition 3.1, there exists a family $\{\rho_n:n\in\mathbb{N}\}$ of right congruences on S such that the collection $\{[x]_{\rho_n}:x\in S,n\in\mathbb{N}\}$ is a base of the topology $\mathcal{T}$. Since S is commutative, ρ_n is a two-sided congruence for every $n\in\mathbb{N}$. It follows that for any $n,m\in\mathbb{N}$, $((x_n,1),(0,1))\in\rho_m$ if and only if $((x_n,-1),(0,-1))\in\rho_m$. By the definition of $\mathcal{T}$, for each $m\in\mathbb{N}$ there exists $q_m\in\mathbb{N}$ such that $((x_n,1),(0,1))\in\rho_m$ for every $n\geq q_m$. It follows that for each $m\in\mathbb{N}$, $((x_n,-1),(0,-1))\in\rho_m$ for every $n\geq q_m$, and so the point $(0,-1)$ is not isolated in $(S,\mathcal{T})$. The obtained contradiction implies that $(S,\mathcal{T})$ cannot be topologically embedded into $\mathbb{N}^{\mathbb{N}}$. □

Proposition 4.7. *There exists a countable linearly ordered locally compact Polish topological semilattice X which cannot be topologically embedded into $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let X be the semilattice $(\left\{\frac{1}{n+1}:n\in\mathbb{N}\right\}\cup\{0\},\min)$ endowed with the topology $\mathcal{T}$ which is defined as follows: each non-zero element of X is isolated and an open neighborhood basis at 0 consists of the sets $U_m=\left\{\frac{1}{2n+1}:n\geq m\right\}\cup\{0\}$, $m\in\mathbb{N}$. One can easily check that X is a locally compact regular scattered second-countable linearly ordered topological semilattice. By Lemma 4.5, the space $(X,\mathcal{T})$ is Polish. To derive a contradiction, assume that X is topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. Taking into account the commutativity of X, Proposition 3.1 yields the existence of a family $\{\rho_n:n\in\mathbb{N}\}$ of congruences on X such that the collection $\{[x]_{\rho_n}:x\in X,n\in\mathbb{N}\}$ is a basis of the topology $\mathcal{T}$. Then there exists $n\in\mathbb{N}$ such that $[0]_{\rho_n}\subseteq U_1=\left\{\frac{1}{2n+1}:n\geq 1\right\}\cup\{0\}$. Observe that if there exists $m\in\mathbb{N}$ such that $\frac{1}{2m+1}\in[0]_{\rho_n}$, then $\frac{1}{2m+2}\in[0]_{\rho_n}\setminus U_1$, which contradicts our assumption. Otherwise, $[0]_{\rho_n}=\{0\}$, and so 0 is an isolated point in $(X,\mathcal{T})$, which contradicts the definition of $\mathcal{T}$. The obtained contradictions imply that X is not topologically isomorphic to a subsemigroup of $\mathbb{N}^{\mathbb{N}}$. □

Proposition 4.8. *Let S be a right simple semigroup. Then there exists no topology $\mathcal{T}$ on S^0 such that 0 is not isolated and $(S^0,\mathcal{T})$ embeds topologically into $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let S be a right simple semigroup. Fix a right congruence ρ on S^0 such that $[0]_\rho$ is not singleton. Then there exists $a\in X$ such that $(0,a)\in\rho$. By the right simplicity of S, $aS=S$. It follows that for every $b\in S$ there exists $c\in S$ such that $b=ac$. Then $[b]_\rho=[ac]_\rho=[0c]_\rho=[0]_\rho$. Hence for each right congruence ρ on S^0 the equivalence class $[0]_\rho$ is either singleton or coincides with S^0. By Proposition 3.1, if 0 is not isolated in $(S^0,\mathcal{T})$, then $(S^0,\mathcal{T})$ doesn't embed topologically into $\mathbb{N}^{\mathbb{N}}$. □

A semigroup S is called *congruence-free* if S admits only trivial (diagonal and universal) two-sided congruences.

Proposition 4.9. *There exists a countable congruence-free Hausdorff locally compact Polish topological inverse semigroup S with a compact semilattice of idempotents which cannot be topologically embedded into $\mathbb{N}^{\mathbb{N}}$.*

Proof. Let S be the subsemigroup of $I_{\mathbb{N}}$ which consists of all partial bijections of cardinality ≤ 1, and $\mathcal{T}$ be the topology on S which satisfies the following conditions:

(1) each nonempty partial bijection is isolated in $(S, \mathcal{T})$;
(2) the sets $U_k = \{\emptyset\} \cup \{\{(n,n)\} : n \geq k\}$, $k \in \mathbb{N}$ form an open neighborhood basis at $\emptyset$.

Clearly the semigroup S is isomorphic to the countable Brandt semigroup over the trivial group. By [37, Theorem 2], S is congruence free. One can check that $(S, \mathcal{T})$ is a regular locally compact second-countable topological inverse semigroup, and the semilattice $E(S) = \{(n,n) : n \in \omega\} \cup \{\emptyset\}$ is compact. Lemma 4.5 implies that the space $(S, \mathcal{T})$ is Polish.

Assume that $(\{(n,n)\}, \emptyset) \in \rho$ for some right congruence ρ on S. Then for any $m \in \mathbb{N}$ we have that

$$(\{(n,m)\}, \emptyset) = (\{(n,n)\} \circ \{(n,m)\}, \emptyset \circ \{(n,m)\}) \in \rho.$$

Consequently, $\{\{(n,m)\} : m \in \mathbb{N}\} \subseteq [0]_\rho$. Hence for each right congruence ρ on S the inclusion $[\emptyset]_\rho \subseteq \{\{(n,n)\} : n \in \mathbb{N}\} \cup \{\emptyset\}$ implies that $[\emptyset]_\rho = \{\emptyset\}$. Proposition 3.1 yields that the topological inverse semigroup $(S, \mathcal{T})$ cannot be topologically embedded into $\mathbb{N}^{\mathbb{N}}$. □

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