# Indeterminacy and the law of the excluded middle

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# Abstract

This thesis is an investigation into indeterminacy in the foundations of mathematics and its possible consequences for the applicability of the law of the excluded middle (LEM). It characterises different ways in which the natural numbers as well as the sets may be understood to be indeterminate, and asks in what sense this would cease to support applicability of LEM to reasoning with them. The first part of the thesis reviews the indeterminacy phenomena on which the argument is based and argues for a distinction between two notions of indeterminacy: a) indeterminacy as applied to domains and b) indefiniteness as applied to concepts. It then addresses possible attempts to secure determinacy in both cases. The second part of the thesis discusses the advantages that an argument from indeterminacy has over traditional intuitionistic arguments against LEM, and it provides the framework in which conditions for the applicability of LEM can be explicated in the setting of indeterminacy. The final part of the thesis then applies these findings to concrete cases of indeterminacy. With respect to indeterminacy of domains, I note some problems for establishing a rejection of LEM based on the indeterminacy of the height of the set theoretic hierarchy. I show that a coherent argument can be made for the rejection of LEM based on the indeterminacy of its width, and assess its philosophical commitments. A final chapter addresses the notion of indefiniteness of our concepts of set and number and asks how this might affect the applicability of LEM.

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# Contents

# 0. Introduction

This thesis is an investigation into indeterminacy in the foundations of mathematics and its possible consequences for the applicability of the law of the excluded middle (LEM). It characterises different ways in which the natural numbers, the sets in general, and the ordinals in particular, may be understood to be indeterminate, either a) with respect to their *domain*, or b) with respect to their *concept* itself. It then asks, in what sense this would cease to support applicability of LEM to reasoning about them. The determinacy of the natural numbers as well as the sets has been questioned on several occasions; similarly, there is a long history of challenges to LEM, indeed, it is the defining trait of the intuitionistic tradition. But a connection between the two has only been observed (rather) recently and a systematic investigation—or so I shall argue—is largely missing. This thesis aims to fill that gap.

## 0.1. On the Connection Between Indeterminacy and LEM

The natural numbers as well as the sets can be conceived of as indeterminate in broadly two ways. One way is connected to the paradoxes and captured in Dummett's notions of indefinite extensibility (Dummett 1973, Shapiro and Wright 2006). This concerns a), the natural numbers, the sets in general, and the ordinals in particular, as they are conceived as a domain of objects. For any finite collection of natural numbers, there is a natural number bigger than all of them; for any collection of sets, one can find a corresponding Russell set; and for any ordinal, one can find a successor. For these reasons it is not possible to circumscribe the collection of all natural numbers in a finite way, and the collection of all sets or all ordinals by standard comprehension. In this way, indefinite extensibility is also closely connected to the problem of absolute generality (cf. Rayo et al. 2006). Indeterminacy as it results from indefinite extensibility of those various sorts will be called **indeterminacy of height**. An analysis of this phenomenon will be given in chapter 1.

In addition to that, indefinite extensibility has also been mentioned in connection to the incompleteness theorems with respect to Peano Arithmetic ($\mathsf{PA}$) (Dummett 1978).[1] As far as this phenomenon is concerned, the indeterminacy that it is taken to characterise arises not with respect to the domain of objects in question, but with respect to b), the concept that the respective axiom system is supposed to explicate. Throughout the thesis, 'indeterminacy' is sometimes used as an umbrella term for both phenomena, but is will mostly denote the indeterminacy of domains. To keep the two apart, I will use the term 'indefiniteness' when talking about concepts and their axiomatisations. The distinction between indeterminacy of domains and indefiniteness of concepts is substantial. While it is prima facie questionable what the extension of an indefinite concept is, the converse need not hold. With respect to the ordinals, for instance, we will see that we have a definite concept with an indeterminate domain.

Another notion of indeterminacy, which applies exclusively to the sets is related to the independence of the continuum hypothesis ($\mathsf{CH}$) from the axioms of $\mathsf{ZFC}$.[2] This form of indeterminacy arises because there is (as of yet) no formal (axiomatic) or informal characterisation of the notion of set to determine the truth value of sentences such as $\mathsf{CH}$. Not only that, there are mutually incompatible (but initially plausible) axiomatic extensions of $\mathsf{ZFC}$, which testify that there is more than one way of specifying the notion. In this respect, we have little claim to whether $\mathsf{CH}$ or $\neg\mathsf{CH}$ holds for 'the sets'. This sort of indeterminacy, which will be called **indeterminacy of width**, can be understood in both ways, with respect to the domain of sets as well as with respect to the concept of set: Is the domain of sets sufficiently determinate to settle the truth value of $\mathsf{CH}$ and/or is the concept of set sufficiently definite in order to motivate the adoption of additional

[1] A list of the axioms of $\mathsf{PA}$ can be found in any standard textbook, such as Button and Walsh (2018, p. 29) and van Dalen (2013, p. 82).

[2] A list of the axioms of $\mathsf{ZFC}$ can be found in any standard textbook, such as Button and Walsh (2018, p. 31) and Jech (2002, p. 3).

axioms to settle it? Indeterminacy of width, will be discussed in chapter 2.

Approaching the matter from the other side, scepticism regarding LEM has a long tradition. It has risen to prominence within modern logic and mathematics through the work of L.E.J. Brouwer, who challenges its applicability by combining an ontology of mathematics as given via mental constructions with a notion of human finitude (cf. Brouwer 1907, Brouwer 1923, Brouwer 1928, Brouwer 1981). Brouwer's argument is largely based on the fact that we are not able to perform infinitely many mental operations and for this reason, LEM on infinite domains possesses no legitimacy. Similar arguments, now clad in broadly epistemic terms (and thus eschewing such a heavy dependency on mathematical ontology) have been given by Dummett (Dummett 1983, Dummett 2000), Martin-Löf (Martin Löf 1996), and Bishop (Bishop 1973). Other arguments against LEM (or rather for intuitionistic logic) are motivated by the desire to track provability or by a connection to epistemic mathematics (cf. Kolmogorov 1932, Melikhov 2015, Melikhov 2017, Melikhov 2018, Shapiro 1985). Finally, there are arguments against LEM based on considerations of (proof theoretic) harmony (Dummett 1991b, Tennant 2002).

These arguments, however, are not of (primary) concern in this thesis. They differ in kind from the arguments that are being developed here, because they don't deal directly with the indeterminacy phenomena mentioned above. This will turn out to be a major advantage. The arguments developed in this thesis don't rely on standard intuitionistic premises and the problems that they bring with them (cf. Raatikainen 2004). They don't rest on the (highly problematic) equation of truth and provability, and neither do they require any epistemic constraints (like the arguments given by Dummett or Bishop), nor a heavy constructive ontology like the one espoused by Brouwer. Furthermore, unlike traditional intuitionistic arguments, they are compatible with a classical meta-logic. All these points make the connection between indeterminacy and LEM a genuine unique and worthwhile object of inquiry.

The question is thus, how is indeterminacy related to a rejection of LEM?

## 0.2. Current Views Regarding Indeterminacy and LEM

Dummett has claimed on several occasions (cf. Dummett 1978, Dummett 1973, Dummett 1991a, Dummett 1994) that indefinite extensibility should lead to a rejection of LEM. In *Frege. Philosophy of Mathematics* for instance, he writes:

> Quantification over the objects falling under an indefinitely extensible concept obviously does not yield statements with determinate truth conditions ... and the logic governing such statements is not classical, but intuitionistic. (Dummett 1991a, p. 321)

This may be taken to apply to the natural numbers (insofar as they are taken to be indefinitely extensible) as well as to the ordinals.

As a generalisation of this idea, Santos (2013) argues that with respect to absolute generality (which suffers from problems that are similar or even indeed instances of indefinite extensibility) intuitionistic logic will be more appropriate. The reason he gives is that the intuitionistic universal quantifier (unlike its classical counterpart) does not require a completed domain of objects to be available.

Without any concrete use of the notion of indefinite extensibility, similar claims have been made with respect to the set theoretic hierarchy, where conceptions of large cardinals are still being developed and their existence debated. An earliest argument by recourse to this open endedness of the set concept has been given by Lear (1977). Lear points out that "we are living in a period in which we are learning about sets, and this historical, sociological, and epistemic fact must affect our ability to talk about sets"[3](Lear 1977, p. 92) As a consequence, he argues that the extension of the set concept should be open ended, and for that reason, the semantics of set theory should be given by a Kripke model with its typical interpretation of conditional and negation, as well as the non-duality of the quantifiers. Similarly, William Tait claims that

> When we affirm the truth of a propositions about the universe of sets, we must speak of it on the basis of those operations for constructing sets that have been accepted, while admitting that further principles might also be admitted. For this reason, the logic that applies to arbitrary formulas of set theory ... should be constructive, not classical logic. (Tait 1998, p. 478)

[3] As we will see in chapters 2 and 3, even though this claim was made almost 50 years ago, it is still an adequate description of contemporary set theoretic practice.

Apart from indeterminacy of height, Lear points out that the width of the set theoretic universe is affected as well, because it is also the case that "our understanding of *arbitrary subset* or *combination in all possible ways* develops over time"(Lear 1977, p. 99). This idea has been revisited (independently to my knowledge) by Solomon Feferman. In his unpublished, but widely circulated EFI-talk on the issue, he takes the independence of CH from ZFC to show that the notion of the powerset of the natural numbers is not a definite notion with a determinate extension (Feferman 2011). He then suggests a hybrid version of classical and intuitionist logic, so-called semi-intuitionistic logic, to be appropriate for dealing with such indeterminacy.

> a natural conception of the set theoretic universe is as an indefinite (or "potential") totality, to which intuitionistic logic is more appropriately applied, while each set is taken to be a definite (or "completed") totality, for which classical logic is appropriate. (Feferman 2010)

Feferman then characterises a system called SCS to reflect what is determinate and what is not. This idea has been developed further in Scambler (2020) and Rathjen (2019), based explicitly on the principle: "What's determinate is the domain of classical logic, what's not that of intuitionistic logic"(Scambler 2020, p. 562). Rathjen (2019) contains a proof that CH is indeed independent of SCS + $\mathscr{P}(\omega)$ exists, and Scambler (2020) states a slight extension of SCS, so-called $\mathsf{SCS}^+$ about which he argues that it reflects an indeterminate set theoretic ontology.

All these claims—Dummett's explicitly and the others at least implicitly—are based on an indeterminacy of the respective *domain* of reasoning. In an early paper by Dummett there is another idea mentioned which can be developed into an argument according to which the rejection of LEM follows from the indefiniteness of the *concept* of inquiry. Dummett (1978) connects the notion of indefinite extensibility to the incompleteness of Peano Arithmetic (PA). This he takes to show that the notion of an arithmetic truth is in a similar vein indeterminate. Unlike the arguments mentioned before, however, these arguments apply indefinite extensibility to *theories*, i.e. collections of formulas, intended to characterise the natural numbers. Consequently, the underlying indeterminacy is not connected to the domain of objects, but to the *concept* (of natural number) itself which those theories are taken to express. This idea I call *the argument from con-*

*ceptual incompleteness.* It represents an approach that differs to a larger extend from the indeterminacy of height and indeterminacy of width considerations with respect to domains of objects. As it turns out, this argument can be applied to the concept of set with considerable more success than to the concept of natural number. The reason for this, and possible consequences of it will be discussed in detail in chapter 8.

## 0.3. Criticisms of Current Views

There are responses to these claims in the literature (except for the argument from conceptual completeness, which has not been sufficiently developed yet). Many scholars have rejected the notion of indefinite extensibility on grounds of it being obscure (cf. Boolos 1998, Tait 1998, Burgess 2004, Rumfitt 2015), and indeed, one well known problem with the notion is that it relies on an unspecified notion of determinacy with respect to which domains are considered to be extensible. This problem, too, will be discussed when the notion is introduced in chapter 1. Similarly, Peter Koellner has given a long discussion in rejection of Feferman's claims that CH is an indefinite mathematical problem and that the notion of an arbitrary subset of the natural numbers is an indefinite mathematical concept (cf. Koellner 2017). These types of criticism seem to be primarily focused on rejecting the underlying notion of indeterminacy. Of course, it has been disputed that indefinite extensibility as well as the independence of CH from ZFC are indeed to be understood to testify the indeterminacy of our fundamental mathematical domains or concepts. Categoricity theorems as well as broad metaphysical assumptions of realism have been proclaimed to secure determinacy. The arguments to this effect will be discussed and criticised in chapter 3.

But even if said indeterminacy is acknowledged, that does not mean that its consequences are clear. There are other objections against the above claims that do not reject the indeterminacy in play right away, but rather focus on the way it is dealt with. Lear's arguments in particular have been discussed in Paseau (2003) who objects against the alleged non-duality of the quantifiers when modelling the open-endedness of the extension of the set concept. Any extension of the range of the universal quantifier beyond what is 'presently available', according to Paseau, warrants an extension of the range of the existential quantifier in the same manner. And conversely, any reason to restrict the existential quantifier similarly is a reason to restrict the universal one.

Incurvati (2008a) points out that different ways of conceiving the indeterminacy of the set theoretic universe may lead to different logics between intuitionistic- and classical logic. He further notes that such a restriction might also be incompatible with certain axioms of ZFC such as the Axiom of Choice (AC) and the axiom of Foundation. Furthermore, we should note that in the case of set theory and ZFC, there is more than one way to obtain a corresponding intuitionistic system once a notion of indeterminacy is recognised. SCS, as mentioned before, is only one such option. There are other possibilities, for instance, intuitionistic Kripke-Platek set theory (IKP), SCS + $\mathscr{P}(\omega)$, or even Intuitionistic Zermelo Fraenkel set theory (IZF) and Constructive Zermelo-Fraenkel set theory (CZF).[4] It is an open question which of those should be chosen in the face of indeterminacy.

However, all these criticisms not withstanding, the above positions have a more fundamental shortcoming. Even if the underlying notion of indeterminacy is accepted with all its accompanying commitments and the need for a revision of the semantics or even the logic is acknowledged, the above claims appear to be essentially that: *claims*, and not complete arguments! As the matter stands right now, the road from indeterminacy to a restriction of LEM is simply not obvious. Why is it that indeterminacy should yield intuitionistic logic? Intuitively, the matter might seem compelling: one might argue that LEM does not reflect a logic of constructive reasoning, and that whatever may count as secure in the face of indeterminacy has to be (explicitly) constructed. However, on second thought, this leaves the connection to the indeterminacy phenomena and all their individual differences out of the story. Why does indefinite extensibility, the independence of CH, or the incompleteness results *in particular* lead to a rejection of *precisely* this logical law?

To illustrate how this question is largely left open, consider the most detailed analysis of Dummett's claim that we have to this day, which is given in Linnebo (2018a). To reconstruct Dummett's idea, Linnebo distinguishes between the intensional and exten-

[4]For different kinds of constructive set theories, see Crosilla (2020).

sional determinacy of a domain.[5] A domain is said to be **intensionally determinate** iff it is decidable, i.e.,

$$ID(X) \qquad \text{iff} \qquad \forall x(Xx \vee \neg Xx)$$

a domain is said to be **extensionally determinate** iff quantification over it preserves intensional determinacy:

$$ED(X) \qquad \text{iff} \qquad \forall Y(ID(Y) \rightarrow ID(\lambda u(\forall x(Xx \rightarrow Yxu))).$$

The idea behind this definition is that $X$ is extensionally determinate just as long as quantification over it is determinate. If $Yxy$ is decidable, then the concept $\lambda u(\forall x(Xx \rightarrow Yxu))$ also remains decidable.

Now, Linnebo shows that the assumption that all instances of LEM are valid is inconsistent with the assumption that every extensionally determinate domain according to the above criterion defines a set (and that the principle of extensionality holds for sets). The reason for this is that due to LEM holding ubiquitously, every domain would become intensionally determinate and hence, the definition of extensional determinateness would apply to all of them—including concepts like $\lambda x.x \notin x$, which leads to a contradiction. Looking at this in closer detail, for $\lambda x.x \notin x$ to fail to characterise an extensionally determinate domain, there would have to be an intensionally determinate $Yxy$ and object $u$ such that the following does *not* hold:

$$\forall x(x \notin x \rightarrow Yxu) \vee \neg\forall x(x \notin x \rightarrow Yxu) \tag{0.1}$$

But why should that be the case? Any why is this criterion a good way to capture whether something is extensionally determinate in the first place? In order to answer these questions, we need to give an account of how the quantified expression works. Of course, (0.1) does not come out valid if the quantifiers are understood intuitionistically,

[5]Linnebo uses the terms 'definite' and 'concept'. I have changed the vocabulary accordingly in order to not cause any conflation with the notions of indefiniteness of concepts that this thesis investigates as well.

but this is what we were interested in showing in the first place.

In this respect, Linnebo rightly notes that intuitionistic logic, this particular definition of extensional determinacy, and the assumption that every extensionally determinate domain according to the above criterion defines a set is a "package deal". But in order to decide whether to accept the package we need to decide whether the definition of $ED(\cdot)$ and thus (0.1) captures the notion of extensional determinacy in the first place. And in order to do so for the domain of an indefinitely extensible concept, we need an account of what it means to quantify over it. Should it turn out that quantification over it contrary to Dummett's claim licenses classical logic, then the above can no longer work as a criterion for its extensional determinacy (if that in turn is taken to entail set existence). In other words, Linnebo's analysis of Dummett's claims makes it even more apparent that Dummett is lacking a clear argument as to why indefinite extensibility should lead to a rejection of LEM.

But this neither shows that we should automatically assume that classical logic is the more adequate choice. For the analysis above has unveiled that an actual account of quantification over indefinitely extensible concepts is still missing—and it is this account, which determines whether (0.1) holds and whether the definition of $ED(\cdot)$ is adequate. Linnebo further points out that there is such an account available that uses modal resources to model quantification over indefinitely extensible concepts. This account and its connection to the intuitionistic argument will be discussed in chapter 6.

A similar argument can be made with respect to the claims by the other authors as well. For instance, Feferman (2010) and Rathjen (2019) defend the system SCS as giving an account of what is to count as determinate and what as indeterminate, but the question remains why the semi-intuitionistic logic on which it is based should actually correspond to the indeterminacy that was introduced by the independence results. Feferman motivates the use of intuitionistic logic on account of Kripke semantics, which, he emphasizes, relate closely to "the question of dealing with conceptions of structures involving possibly indefinite notions and domains"(Feferman 2014, p. 82). But this requires the Kripke model to actually reflect the indeterminacy of CH, and in doing so to actually provide counterexamples to LEM.[6]

[6]Furthermore, as Incurvati (2008a) notes, even then, there are scenarios in which the acceptance of

The situation is even more precarious when focusing on the argument from conceptual incompleteness. While a Kripke model with its growing domains may be intuitively close to representing indefinite extensibility and, as I argue in this thesis, indeed the indeterminacy of width as well, the question is entirely open for the indefiniteness of the concepts of natural number and of set. Furthermore, it is not obvious, and in fact also not the case that all notions of indeterminacy have this same effect.

The main question this thesis is concerned with is thus: How are we to connect the notions of indeterminacy mentioned above to a rejection of LEM? And how do we determine the logical laws that are nonetheless available when reasoning with them?

## 0.4. The Structure of the Argument and of the Thesis

I will follow a two step procedure inspired by Ian Rumfitt.[7] The first (and main part) of the argument is to construct a semantics for expressions whose meaning involves said independence phenomena.

> **Step 1.** Given an indeterminate concept $C$, construct a model $\mathcal{M}_C$ which is argued to reflect its extension.

Then I will ask whether the semantics validates LEM.

> **Step 2.** Inquire whether $\mathcal{M}_C \vDash \mathsf{LEM}$.

Finally, what logical consequences does this semantics license:

> **Step 3.** Does $\mathcal{M}_C \nvDash \mathsf{LEM}$ translate to $C \nvdash \mathsf{LEM}_C$?

The transition from step 2 to step 3 is rather straightforward when we are dealing with indeterminacy of domains. The matter is more complicated regarding indefiniteness of concepts. This latter point will be left open to some extent at the end of the thesis.

The thesis contains three parts. The first part is aimed at the indeterminacy phenomena themselves, the second part is an investigation into the wherewithal of LEM, and

AC might lead to classical logic.

[7]The argument made in Rumfitt (2015, ch.9) purports an intuitionistic semantic, but actually argues for its compatibility with classical logic. It will be discussed in chapter 7.1.

the third part brings the two together to give a semantics of the respective notions of indeterminacy in which LEM may or may not be validated.

**The first part** is made up of chapters 1-3. They discuss, as already mentioned, indeterminacy of height, indeterminacy of width, and possible ways of attaining determinacy via categoricity and assumptions of realism. At the end of part one, the indeterminacy phenomena, as well as the ontological assumptions that are required to draw any desired consequences from them should be sufficiently clear.

**The second part** consists of chapters four and five.

Chapter four discusses the traditional intuitionistic arguments against LEM in order to distinguish them from the ones developed in this thesis. It also seeks to establish a semantics that does not prefigure the meaning of the logical constants in either direction, and in which the question regarding LEM can be discussed on neutral grounds. The chosen semantics is based on the notion of a *warrant*, which itself is neutral between a truth and a proof theoretic understanding of the logical constants. It will be formulated via topological models which are taken to exhibit the right mixture of abstraction and specification between Kripke frames and algebraic models. These models will then be used to give an analysis of some intuitionistic non-validities in a neutral setting.

Chapter five introduces the notion of generic generality (cf. Weyl 1921, Linnebo 2022a), which is a form of generality that consists in the ascription of essential properties and as such does not require the availability for any instances of quantification. It thus presents itself as the best way of understanding quantification over indeterminate domains. In his analysis of this type of generality, Linnebo shows that intuitionistic logic is always available for generic generalisations, but classical logic might not be. I provide a generalisation of this notion which allows for cases in which generic generality is also paired with classical logic. The availability of classical logic on a generic generalisation is shown to depend on one of two possible ways of understanding corresponding existential statements.

Furthermore, the chapter contains a discussion of two ways in which the negation of a generic universal generalisation is to be understood. That something is not an essential property of something can be understood as to say that it is either *not entailed* by it, or in a stronger sense, that it is *excluded* by it. This distinction corresponds to the two types of indeterminacy; while the stronger form of negation is the proper choice for reasoning with indeterminate domains, the weaker notion of negation is the one best applicable in cases of conceptual indefiniteness. This distinction will set the structure for the final part of the thesis.

**The third part** of the thesis applies the findings from chapters 4 and 5 to actual models of different instances indeterminacy as they were introduced in part one. The stronger form of negation and indeterminacy of domains will be investigated in chapters 6 and 7, while the weaker form of negation in connection to the argument from conceptual incompleteness will be investigated in chapter 8. As promising as the distinction between the argument from indeterminacy and traditional intuitionistic arguments against LEM are, as difficult it turns out to actually find ways to model indeterminacy to that effect. For both cases, indeterminacy of height as well as indeterminacy of width, a rejection of LEM comes with considerable concessions.

Chapter six focuses on indeterminacy of height. It contains an analysis of why indefinite extensibility no longer allows for a classical understanding of quantification and gives an answer how it is alternatively to be understood. To this end, modal resources will be employed and the standpoint of potentialism will be discussed. I argue that indefinite extensibility can best be understood via a frame based model and the corresponding logic can then be obtained by a suitable translation from modal to non-modal vocabulary. In this respect also an account of the modality will be given and it will be shown that this understanding rules out possible revenge scenarios where the modality is understood to be indefinitely extensible as well. Regarding the question of LEM it will turn out to depend on the type of potentialism that is taken with respect to the indefinitely extensible domain. I discuss liberal potentialism, strict potentialism, and branching potentialism. Liberal potentialism is known to license LEM, while strict potentialism and branching potentialism do not. However, the common examples of indefinite extensibility like the natural numbers and the ordinals do not exhibit branching behaviour when understood potentially, and there are other obstacles to understanding them in a strict potentialist manner.

Chapter seven is concerned with indeterminacy of width. I will discuss three possible ways of modelling the indeterminncy that is due to the independence of CH from ZFC. The first one is due to Rumfitt (2015). It will be argued that it does not achieve its goal because it relies on the notion of categoricity, which ultimately prohibits the indeterminacy that is of concern here to be adequately represented in the model. The second proposal is due to Väänänen (2014). It proposes a supervaluationist understanding of the set theoretic multiverse and licenses classical logic. We will see that this understanding is actually an instance of generic generality with a special way of understanding the existential quantifier. This leaves us, as of now, with no argument against LEM based on the indeterminacy of CH. We can obtain such an argument if we introduce the requirement that set theoretic statements have a unique truthmaker. This can be modelled by applying the Gödel translation to the set theoretic multiverse. The Gödel translation does not work to model indeterminacy of height, but a case can be made for its application to indeterminacy of width. This will turn out to *almost* lead to the system of $\mathsf{SCS}^+$ as defended in Scambler (2020)—with the important distinction that $\Delta_0$-Markov's Principle will no longer be valid.

Chapter eight finally deals with conceptual indefiniteness. It introduces a notion of potentialism for theories, which can be understood as modelling refinements of concepts as they are represented by axiomatic systems. This approach now deals with the weaker notion of negation as it was introduced in chapter five. The question whether LEM holds for this kind of approach is the question whether one can infer from the fact that something is not essentially entailed by a concept that it must be excluded by it. This will be presented as a measure of the concepts completeness. An analysis to this effect will be applied to PA and to ZFC. It will turn out that, given the requirements introduced by Dummett, this argument fares at least technically better than the one based on indefinite extensibility of domains. I will then inquire into the difference between its applications with respect to PA and ZFC and raise the question to what extend the whole argument may actually lead towards revising the logic of arithmetic and set theory.

# Part I.

# Indeterminacy

# 1. Indeterminacy of Height

Before considering possible routes to a rejection of LEM it is important to give a precise characterisation of the indeterminacy that an argument to that effect depends on. The notion of indeterminacy that will be looked at in this chapter is a generalisation of traditional forms of scepticism regarding the infinite. Scepticism regarding the infinite is nothing new. Already Aristotle distinguished between an actual and a potential infinite and opted for the latter. Up until the 19th century a rejection of reasoning with the infinite (especially) in mathematics was the common view. Kant flat out asserts: "Synthesis of an infinite series is impossible"(A 426/B 454). Similar reservations were expressed, often in explicit acknowledgement of Kant, around the turn of the 20th century by figures such as Poincaré, Brouwer, and Weyl, but also by non-revisionists like Hilbert (cf. Hilbert 1925). Characteristic for all these approaches was to pose some form of connection between the computational power of the mind and the mathematical objects it can process. A domain that is infinite was then taken to be indeterminate, because its infinity essentially made it untraversable for a finite (human) mind. The reactions to this were diverse, and included, amongst others, intuitionism with its rejection of LEM.

But the computational power of the mind is not the only measure of determinacy. The account can be generalised. With the rise of the set theoretic paradoxes, it has also been questioned that certain notions which are prominently involved in our reasoning with the infinite are in a more general sense *conceptually* determinate. While something that is epistemically accessible is in this regard *a fortiori* conceptually determinate, the converse need not be the case. To analyse this notion of conceptual determinacy, Michael Dummett has introduced the notion of *indefinite extensibility*[1], which goes back

[1] To my knowledge, the first mentioning of the term occurs in Dummett (1978). The concept features prominently in both Dummett (1973) and Dummett (1991a), and later in Dummett (1994). A

to Russell's *Vicious Circle Principle* (Russell 1908, Russell 1992). This notion forms the basis of the investigation conducted in this chapter. The peculiarity of this notion, which is brought out by the analysis in Shapiro and Wright (2006), is that it essentially leaves open the ground with respect to which other concepts then count as *in*determinate. When being finite (or epistemically accessible for that matter) is seen as a measure of determinacy, indefinite extensibility becomes a representation of earlier scepticism regarding the infinite. But other bases of indeterminacy are also conceivable.

In this chapter, I will look at the natural numbers and the ordinals as indefinitely extensible concepts and discern which notions of indeterminacy they involve. There are different ways in which the natural numbers and the ordinals can be considered indeterminate, which are, in turn, connected to indefinite extensibility in different ways. The natural numbers as well as the ordinals can be understood as

a) a *domain* of objects which is to be (informally) circumscribed.

This characterisation puts them in close proximity to the paradoxes and to Russell's Vicious Circle Principle, such that the notion of impredicativity will play a central role in characterising their indeterminacy. However, it will turn out that there is a sense of impredicativity ascribable to both, the naturals numbers and the ordinals, and we will see that there is a striking difference between them.

In addition to that, the natural numbers in particular can also be investigated as

b) a *concept* that is to be characterised by axiomatisation.

In this respect, it is not their domain that is considered as indefinitely extensible, but the collection of arithmetic truths which are supposed to characterise it.

I will start with a formalisation of the notion of indefinite extensibility which brings out the components that are relevant to the discussion in this chapter in a perspicuous way (section 1). Afterwards, I inquire into its connection to impredicativity where both the natural numbers as well as the ordinals are investigated as domains of objects (sections 2 and 3). Finally, I will discuss the connection of indefinite extensibility to conceptual indefiniteness (section 4).

formalisation using higher order logic was given in Shapiro and Wright (2006), using bi-modal logic in Studd (2019), and using intuitionistic plural logic in Linnebo (2018a).

## 1.1. Indefinite Extensibility

To point out the moving pieces, start with an informal characterisation. Given a finite set of natural numbers, say, an initial segment, adding the next bigger natural number to it produces another finite set. Finiteness is thus a property of each of those sets of initial segments of natural numbers, and the process of adding the next bigger number can be repeated indefinitely but finitely many times such that the outcome still yields a finite set. Similarly, we observe that adding a successor to any initial segment of the ordinals (of arbitrary length) yields an ordinal again. Hence, being an ordinal is a property of any set of ordinals.[2] This can be stated more formally:

**Definition 1.1.** (Shapiro and Wright 2006) A concept $P$ is **indefinitely extensible with respect to a property of concepts** $\Pi$ iff there is a function $\delta$ from concepts to objects, defined on each determinate subconcept $X \subset P$ with $\Pi(X)$ s.t.

$$\neg X(\delta(X))$$

$$P(\delta(X))$$

$$\Pi(\lambda y.Xy \vee y = \delta(X))$$

$P$ is called **reflexively** indefinitely extensible if additionally $\Pi(P)$.

Call a **totality** indefinitely extensible if it is given as the extension of an indefinitely extensible concept.

One important aspect of the definition is the property $\Pi$. It provides the *basis* upon which a concept (i.e. its extension) is considered to be indefinitely extensible. Concepts

[2] Ordinals are here understood the von Neumann way and thereby directly identified with certain sets. Alternatively, ordinals can be understood as order-types $ord(R)$ of well-orders $R$ that get introduced by abstraction, i.e.

$$\forall R_1, R_2(ord(R_1) = ord(R_2) \leftrightarrow R_1 \cong R_2)$$

Since these order types can themselves be ordered, we can form a well ordering $<$ of order types. Burali-Forti's paradox arises, because each order-type can be shown to be the order-type of an initial segment of $<$. This yields a contradiction when we consider the order-type of $<$ itself. In this respect $\Pi$ would indicate the property of having an order type for well-orderings. The reason the von Neumann ordinals are used here, is that the abstractionist understanding does not directly take into account how new ordinals are generated upon a given set of ordinals, which will be the prime focus of the discussion in section 1.3.

are indefinitely extensible with respect to a certain basis, but may cease to be so given another basis. The underlying notion of determinacy is therefore left open and can be filled in different ways. The ordinals as well as the natural numbers already provide two examples.

1. The initial segments of the natural numbers are indefinitely extensible with respect to being finite, where $\Pi(X) =$ 'X is finite' and $\delta(X) = n + 1$ for $X = \{0, \ldots, n\}$. Then naturally $X \cup \{\delta(X)\}$ is finite, too. Of course, the natural numbers are only extensible up to $\omega$, for which we then have $\neg\Pi(\omega)$.

2. Let $\Pi(x) =$ '$x$ is an ordinal' and set $\delta(x) = x \cup \{x\}$, which is already the extended concept, i.e. $x \cup \delta(x) = \delta(x)$.[3] Since we also have $\Pi(x \cup \{x\})$ each time, we have the peculiar case of the ordinals being indefinitely extensible *with respect to being ordinals*. Similar constructions can be made for Russell's Paradox and for the notion of a cardinal number.

There are also intermediate cases between the two (cf. Shapiro and Wright 2006):

3. If $\kappa$ is a regular cardinal[4] then the ordinals smaller than $\kappa$ are indefinitely extensible with respect to being smaller than $\kappa$. Define $\Pi(x) = x < \kappa$ and $\delta(x) = x \cup \{x\}$.

4. The real numbers are indefinitely extensible with respect to being countable. Let $\Pi X =$ '$X$ is countable' and $\delta(X)$ be the element obtained from the Cantor diagonal construction of $X$. Then $\delta(X)$ is a real number not in $X$ and $X \cup \{\delta(X)\}$ is countable.

These examples illustrate how the notion of $\Pi$ changes, and how some concepts (like the natural numbers) are indefinitely extensible with respect to one basis (e.g. being finite), but not with respect to another one (e.g. being an ordinal). It then remains to be seen if we can motivate independent reasons for such a basis to be determinate. This addresses the question in what way indefinitely extensible concepts are considered

[3] Note that we don't need the extra set formation here, because it is already integrated in the definition of $\delta$ (which is due to the fact that ordinals are transitive sets).

[4] A cardinal $\kappa$ is regular, if its cofinality $cf(\kappa) = \kappa$, i.e. if it has no unbounded subset of size smaller than $\kappa$.

to be indeterminate or yield indeterminate domains, which becomes especially acute for reflexively extensible concepts.

Reflexively indefinitely extensible concepts are especially problematic due to their connection to the paradoxes. Each reflexively extensible concept yields and inclosure scheme in the sense of Priest (1994) and Priest (1995, ch.9)).

**Definition 1.2.** An *inclosure schema* is a triple $I = \langle \Omega, \Theta, \delta \rangle$, where

$$\begin{aligned} &\Omega \text{ is a set of objects} \\ &\Theta \subseteq \mathscr{P}(\Omega), \text{ s.t. } \Omega \in \Theta \\ &\delta : \Theta \to \Omega \text{ s.t. for all } x \in \Theta, \delta(x) \notin x \end{aligned}$$

For indefinitely extensible concept $P$ we can define an inclosure schema $I_P$ (and vice versa) by setting $\Omega_P$ to be the extension of $P$, $\Theta_P$ the extension of the corresponding higher order property $\Pi$, and $\delta_P(x) = \delta(X)$ for $x \in \Theta_P$ corresponding to $X \subset P$. Finally, the requirement that $\Omega_P \in \Theta_P$ corresponds to $P$ being reflexively indefinitely extensible. In this case, a contradiction arises for $\delta_P(\Omega_P) \in \Omega_P$, because the range of $\delta_P$ is equal to $\Omega_P$ and $\delta_P(\Omega_P) \notin \Omega_P$ by the third condition. As a concrete example, consider the Burali-Forti paradox. If $\Omega$ is the set of all ordinals, then $\delta(\Omega)$ would be (on the von Neumann conception) $\Omega \cup \{\Omega\}$. But then this would have to be an element of $\Omega$ (on account of $\Omega$ being the set of all ordinals) which leads to a contradiction.[5]

As Priest's analysis is closely connected to Russell's own analysis of the technical structure of the paradoxes (cf. Russell 1905), one might think that the philosophical consequences Russell drew also transfer to our case. In particular, then, one might think that at the heart of the indefinite extensibility phenomena there is indeed a phenomenon of circularity as referred to in Russell's *Vicious Circle Principle* (VCP). The VCP has been formulated by Russell in several ways, in 1908 he wrote:

> "[N]o totality can contain members defined in terms of itself." This principle, in

[5]The structural properties characterising the paradoxes can in fact be further generalised to incorporate not only an analysis of paradox but also of a wide range of mathematical arguments pertaining to diagonalisation cf. Lawvere (2006), Yanofsky (2003), Karimi et al. (2013). This also incorporates the distinction between reflexively and non-reflexively indefinitely extensible concepts.

> our technical language, becomes: "Whatever contains an apparent variable must not be a possible value of that variable."(Russell 1908, p. 237)

and later on, in *Principia*, it is put as:

> "Whatever involves all of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total."(Whitehead et al. 1927, p. 37)

In our case, the member of the collection in question would be the one obtained by applying the diagonal function $\delta$ to $\Omega$ itself. This and the additional requirement that $\delta(\Omega) \in \Omega$ is responsible for the above contradiction—and at least one of the two is indeed excluded by the VCP. Thus, at first glance, indeterminacy seems to be given by reflexively indefinitely extensible concepts which are characterised by the above analysis to yield a circular definition of $\Pi$ and $P$.

Both formulations of the VCP, however, are not precise enough to demarcate the type of indeterminacy that we are interested in. In fact, there exist counterexamples to both formulations, i.e. cases which would be considered determinate but which would nonetheless be excluded by it. Ramsey's example "the tallest man in the room"(Ramsey 1990) (if it is accepted as a correct definition) stands in direct contradiction to the first formulation, and as the next section will show, the second formulation does not fare better. The reason for this is that the circularity that the VCP alludes to, which is nowadays better characterised as impredicativity, is an oversimplification. In the following, we are going to see that even non-reflexively indefinitely extensible concepts can have an impredicative component, which leads us to locate the difference between reflexivity and non-reflexivity in another aspect of the way that $\Pi$ is given. This difference can be observed in a detailed discussion of the natural numbers and the ordinals.

## 1.2. The Natural Numbers and the Impredicativity of Induction

A counterexample to the second formulation of the VCP is the impredicativity of the natural number concept. This is a special case of the impredicativity of generalised inductive definitions.[6] Starting from the natural numbers, we will see that there is a correspondence between generalised inductive definitions and indefinitely extensible concepts, such that technical results regarding the former can be applied to the latter. This helps in understanding how indefinitely extensible concepts characterise domains of reasoning, and how a certain form of circularity is, contrary to the VCP, unproblematic in this respect.

The impredicativity of the natural numbers comes about in the following way. One can characterise the natural number predicate $N$ by defining it as the *smallest* predicate closed under the rules

$$\frac{}{N(0)} \qquad \frac{N(n)}{N(n+1)}\, Succ$$

This definition is an instance of a more general notion of inductive definition[7], where an inductive definition of a predicate $P$ is considered as the *smallest* predicate that is closed under a given set of rules, such as:

$$\frac{}{P(c)} \qquad \frac{P(n)}{P(f(n))}\, f$$

where $c$ is a starting constant and $f$ is a function on an underlying domain $A$. The domain $A$ thus already contains the set that we are trying to define as a subset and the inductive definition seeks to carve it out.

How do we explicate the idea that what we have defined here is indeed the *smallest* set closed under the respective operations? There are roughly two approaches to this, which can be called 'top down' and 'bottom up'.[8] I will start with the former, which is impredicative in a quite obvious way. But we will later see that even the latter one does not quite escape that mark.

[6] This point was made in Parsons (1992), based on an remark in Dummett (1978), and in Nelson (1986), cf. also Crosilla (2018).

[7] For discussion of inductive definitions, see Aczel (1977), esp. ch.1.3 and ch.4. Much of the material on inductive definitions here is drawn from a course by Andre Popescu at MGS 2021.

[8] For this distinction with regards to the natural numbers, see in particular George and Velleman (1998). The systems EFS and its extensions EFSC and EFSC* introduced in Feferman and Hellman (1995) can be understood as a middle ground between the two approaches discussed here. (But see footnote 10.) Feferman and Hellman (2000) prove existence and uniqueness of the natural number structure in a system based on the notion of finite set and containing a weak form of second order comprehension. They argue that the derivation of the natural number structure and induction is predicative, given the previously accepted notions. Parsons (2008) objects that there are still impredicatively defined sets used in the proof of induction. I will focus on the difference between bottom up and top down, since their connection to indefinitely extensible concepts is closer at hand.

### 1.2.1. The 'top down' approach

The 'top down' approach is the paradigm case for the discussion of the impredicativity of induction. In general, given a set of rules $\chi$ on the domain $A$, we can define the (extension of the) smallest predicate simply as

$$P_\chi = \bigcap \{X \subseteq A \mid X \text{ is closed under } \chi\} \tag{1.1}$$

i.e. the intersection of all predicates closed under the rules in $\chi$. In particular, for the natural numbers, we get

$$N = \bigcap \{X \subseteq A \mid X \text{ is closed under } Succ\} \tag{1.2}$$

Note that since for any predicate $Q$ closed under $Succ$, we have that $N \subseteq Q$. Thus defining the set of natural numbers in this way immediately entails that the principle of induction

$$\frac{Nm \qquad Q(0) \qquad \forall n(Qn \to Q(n+1))}{Q(m)}$$

holds for them.

The above definitions, however, are clearly impredicative, since a subset/subconcept of $A$ is defined by quantifying over a collection of subconcepts of $A$ which includes the one that we are defining. Thus, this definition contradicts the first formulation of the VCP. This is not a problem, though, since the collection of subsets of $A$ is considered as given (independently) and (1.1) and (1.2) simply serve to pick out the smallest of these (which happens to be by reference to all the others). More problematically, perhaps, it also contradicts the second formulation in the sense that $N$ is defined as a member of the collection of all sets closed under $Succ$. Accordingly, then, this collection would not exist. This, however, seems to be a widely unwanted and unjustified consequence. Why should it follow that a collection defined in this way does not exist?

But before we get into that, could we have a definition of the natural numbers even without this type of impredicativity? What about the 'bottom up' approach? The answer is No, even in this case there is no way around impredicativity.

### 1.2.2. The 'bottom up' approach

Let a predicate $P$ be thought of as a function from a domain $A$ into the set 2, e.g. $P : A \to 2$. On basis of the domain $A$ one can define an order on predicates by inclusion, such that for two such predicates $P_1$ and $P_2$ it holds that

$$P_1 \subseteq P_2 \qquad \text{iff} \qquad \forall x \in A : P_1(x) \to P_2(x)$$

and use this to define the lattice $2^A = \langle A \to 2, \subseteq \rangle$ of predicates. For a set of predicates $X \subseteq (A \to 2)$ its infimum and supremum is given by

$$\bigwedge X = \lambda a.\forall P \in X.Pa \qquad \bigvee X = \lambda a.\exists P \in X.Pa$$

where the first expression collects all objects that fall under every predicate in $X$ whereas the second expression collects all objects that fall under at least one predicate in $X$.

Corresponding to each predicate $P_f$ to be defined inductively using a rule $f$ we define on $2^A$ a higher order operation $\mathcal{F} : (A \to 2) \to (A \to 2)$ that takes predicates to predicates. For a predicate $Q \in 2^A$, then, we define:

$$\mathcal{F}Q \equiv \lambda m.m = 0 \vee Qm \vee (\exists n : m = f(n) \wedge Qn)$$

Thus $\mathcal{F}$ returns the predicate that contains all elements of $Q$ plus (possibly) the element obtained by one further application of the function/rule $f$ on the elements of $Q$. The operation $\mathcal{F}$, then, is a monotonically increasing function on predicates.

How does the operation $\mathcal{F}$ serve to specify the predicate $P_f$ as the smallest one closed under the application of $f$? This depends on the following important result:

**Theorem 1.1.** (Knaster-Tarski) *Given a complete*[9] *lattice $\mathcal{L} = \langle L, \subseteq \rangle$, and a function $\mathcal{F} : L \to L$ which is monotonically increasing, let $I_{\mathcal{F}} = \bigwedge\{P \in \mathcal{L} \mid \mathcal{F}P \subseteq P\}$ be the infimum of the set of pre-fixpoints. Then $I_{\mathcal{F}}$ is the least fixpoint of $\mathcal{F}$.*

*Proof.* Let $X = \{P \mid \mathcal{F}P \subseteq P\}$ and show that $\mathcal{F}I_{\mathcal{F}}$ is a lower bound of $X$. Indeed,

[9] A lattice $\mathcal{L} = \langle L, \subseteq \rangle$ is complete iff it contains the supremum and the infimum of every set $X \subseteq L$.

- given a $P \in X$ we have $I_{\mathcal{F}} \subseteq P$, which implies by monotonicity $\mathcal{F}I_{\mathcal{F}} \subseteq \mathcal{F}P$.
- We have $\mathcal{F}P \subseteq P$, which combined with the abvoe implies $\mathcal{F}I_{\mathcal{F}} \subseteq P$.

And because we know that $I_{\mathcal{F}}$ is a pre-fixpoint, i.e. $\mathcal{F}I_{\mathcal{F}} \subseteq I_{\mathcal{F}}$, $I_{\mathcal{F}} \in X$ and hence $I_{\mathcal{F}}$ is the least pre-fixpoint.

But also, since $\mathcal{F}\mathcal{F}I_{\mathcal{F}} \subseteq \mathcal{F}I_{\mathcal{F}}$, i.e. $\mathcal{F}I_{\mathcal{F}} \in X$, we have $I_{\mathcal{F}} \subseteq \mathcal{F}I_{\mathcal{F}}$.

Both imply that $I_{\mathcal{F}} = \mathcal{F}I_{\mathcal{F}}$ □

Using this theorem one can define the predicate $P_f$ as the fixpoint $I_{\mathcal{F}}$ on the lattice of predicates. All that is required for this is that the corresponding lattice $\mathcal{L}_{P_f}$ has the required infimum. For instance, we are now able to say that $N = I_{\mathcal{S}}$, where $\mathcal{S}$ corresponds to the successor function *Succ*. Furthermore, using this construction, we can derive the induction principle for the respective inductively defined predicate, e,g. for $N = I\mathcal{S}$. This follows from the fact that $I_{\mathcal{S}}$ is the *least* fixpoint of the operation $\mathcal{S}$, which implies that for any predicate $Q$ closed under *Succ* it is the case that $N \leq Q$.

This construction bears some strong resemblances to the notion of an indefinitely extensible concept. $\mathcal{F}$ and $f$ correspond to the working of the diagonal function $\delta$ and the subsequent construction of the predicate with extension $X \cup \{\delta(X)\}$. But how is the higher order property $\Pi$ to be incorporated into the above schema? It seems natural to think of $\Pi$ as

$$\Pi(P) \qquad \text{iff} \qquad P \in \{P \mid \exists Q \; s.t. \; \mathcal{F}Q = P \text{ and } P \subseteq I_{\mathcal{F}}\}$$

The left-to-right direction in this respect is no problem. If we assume that $\Pi(P)$, then, because $P$ is finite, we know that there is a $Q$ s.t. $P \equiv \mathcal{F}Q = \lambda x.Q(x) \vee x = \delta Q$ and $P \subseteq I_{\mathcal{F}}$. But deducing from the fact that $\mathcal{F}Q = P \subseteq I_{\mathcal{F}}$ that $\Pi(P)$ is not directly possible. Take *Succ*, for instance, and consider the set of even numbers $E$. If $E$ is in $\mathcal{L}_N$, then by the above two suggestions $E$ would be in $\Pi$ and hence misclassified as finite. To exclude a case like that, we could demand that the diagonal element is not bounded by any element of $P$ in addition to its being non-identical with any of them. This can be achieved by defining the operation $\mathcal{G} : (A \to 2) \to (A \to 2)$

$$\mathcal{G}P \equiv \lambda m.m = 0 \vee Pm \vee (\exists n : m = f(n) \wedge Pn \wedge \forall x P(x) : m > x).$$

But if we choose to use $\mathcal{G}$ instead, it follows that for predicates with infinite extension like *even*, we'd obtain that $\mathcal{G}even = even$. Hence we would need to exclude such predicates from the lattice. We should be able to do so by requiring that each $P$ has to be bounded in $I_{\mathcal{G}}$. (This should not interfere with the completeness of the lattice, since the set of even numbers has as its supremum simply $N$.) Now we should be able to derive induction based on Knaster-Tarski for this lattice just the same. Furthermore, we have

$$I_{\mathcal{F}} = I_{\mathcal{G}} = \bigvee \Pi$$

This leaves us with two options. Either we restrict the lattice to sets that are bounded in $I_{\mathcal{G}}$ and obtain a close representation of indefinite extensibility, or we allow other subsets as well (like in the common case of inductive definitions). The result is just the same. This should also not be surprising, since our notion of natural number (including induction), or of any domain of reasoning for that matter, should be considered determinate independently of what we'd consider its subconcepts to be in addition to the ones we'd obtain through successive application of the defining rules.

Thus, indefinitely extensible concepts have an intimate relationship with inductive definitions. The notion of determinacy that is involved in each indefinitely extensible concept can be characterised in terms of the fixpoint on the lattice of subsets that is induced by the respective diagonal function. But have we made any progress in excluding the notion of impredicativity in this? First note that in our characterisation of $\Pi$ by the set $\{P \mid \exists Q \; s.t. \; \mathcal{G}Q = P \text{ and } P \subseteq I_{\mathcal{G}}\}$ we make use of a notion of finitude from the outset—the lattice itself is defined by setting that each of its elements $Q$ is bounded in $I_{\mathcal{G}}$. In addition to that, in both cases, $\mathcal{F}$ and $\mathcal{G}$, lingers impredicativity. The reason for this lies in one of the assumptions required for the proof of the Knaster-Tarski theorem, namely that the lattice is complete. This guarantees that there is an infimum of all the pre-fixpoints, namely $I_{\mathcal{G}} = \bigwedge\{P \subseteq \mathcal{L}_n \mid \mathcal{G}P \subseteq P\}$, which is itself a pre-fixpoint, and hence bears witness to impredicativity.[10]

Both approaches then, 'bottom up' as well as 'top down', rely on some sort of impredicativity. But the impredicativity involved here is not vicious. It does not lead to an inclosure paradox. This shows that the second formulation of the VCP is implausible as well. But it also raises the question how the problematic cases regarding reflexively indefinitely extensible concepts, where there *is* an inclosure paradox, are appropriately distinguished from the above. In the next section we will see that the above analysis *cannot* be extended to reflexively indefinitely extensible concepts. The ordinals will provide an example of another form of circularity that does not conform to the style of an inductive definition and that, consequently, requires different treatment.

## 1.3. The Ordinals

Covertly, the ordinals were already used in the previous section. They served as an indicator of the length of the operation whose fixpoint was determined. In the 'top down' approach this was evident because sets that contained numbers beyond the naturals were used. In the 'bottom up' approach at least the initial section of the ordinals (in terms of the finite numbers) was explicitly assumed. On the one hand, the ordinals present themselves now as a natural extension of the natural numbers. On the other hand, there are some decisive differences between the two. These differences concern the notion of $\Pi$ and involve a form of circularity which is to be distinguished from the foundational impredicativity considered above.

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[10] The definition of the natural number structure in George (1987) and Feferman and Hellman (2000) (with a modification by Aczel) can be read as

$$x \in \mathcal{N} \quad \text{iff} \quad \forall A[x \in A \wedge Clos^{-}(A) \rightarrow 0 \in A] \wedge \exists A[x \in A \wedge Clos^{-}(A)],$$

where $Clos^{-}$ means closed under predecessor, and where it is specified that the range of the second order variables consists only of all the finite sets. I think this corresponds to my construction of the lattice of predicates using the $\mathcal{G}$ operation. If so, then the objection by Parsons (2008) that this notion is impredicative as well because the proof of induction makes use of predicates that may be defined impredicatively can be resisted. That induction follows for any predicate is due to the position of $\mathcal{N}$ in the lattice, namely that for any other predicate $Q$ closed under successor it is the case that $\mathcal{N} \leq Q$. Whether or not $Q$ is defined predicatively or impredicatively seems to be accidental to the whole affair. The only "impredicative" part here is thus the range of the quantifiers over finite sets which it seems corresponds to my assumption that the fixpoint $I_{\mathcal{S}}$ exists.

Even though it requires a notion of finite number, or an impredicative conception of the natural numbers, the higher order property Π—once it is determined—does no longer yield a finite extension. In other words, the natural numbers in their totality do not have the property of being finite. This is the reason that the impredicativity in their case is manageable. Even though it can only be circularly determined, there *is* a point at which the iteration stops. In case of the ordinals, however, there is an *additional* difficulty which is due to the fact that the ordinals themselves (in their entirety) may be said to have the property of being an ordinal, in other words, the extensible property $P$ and not just its subproperties have the property Π. But in this case, there is no independent ground to determine what $P$ is supposed to be. This suggests that it is not necessarily impredicativity *per se* that causes the indeterminacy in question, but the lack of an ambient conception that serves to delimit the higher order property Π.[11]

In what sense does this show that the ordinals are indeterminate? It appears that with the number of limit stages unspecified there is no ground to answer the question of *how far* the ordinals go. The problem is that any such specification already involves a notion of 'how far' that itself is conceived as given by the ordinals. Usually, the length of a series is given by an isomorphism to an initial segment of the ordinals, but if we were to give the length of the ordinals themselves we can only say they are longer than any initial segment and thus as long as the ordinals itself. In other words, the ordinals themselves are usually seen as the ultimate length indicator.[12]

[11]If we want to relate this observation to the solutions of the Burali-Forti paradox, we would say that on the von Neumann conception of the ordinals we don't have absolute generality regarding the object domain. The lack of an ambient concept illustrates this idea insofar as it would serve to circumscribe a domain for the extension of Π. If we consider the ordinals as obtained by abstraction, i.e. by $\forall R_1, R_2(ord(R_1) = ord(R_2) \leftrightarrow R_1 \cong R_2)$ then we get additional options for blocking the paradox. These are a) restricting second order comprehension, b) rejecting second order absolute generality, and of course, c) rejecting ordinal abstraction itself. More recently Florio and Leach-Krouse (2017) have presented a solution according to which ordinals are not considered to be objects but to be of a different sort. An ordering $<$ of the ordinals on this analysis is different from any ordering of objects $R$ such that $ord(x)$ only takes $R$ but not $<$ as input. They show how talk of ordinals in this way can be expressed as theorems of third order logic. In this, they managed to allow for absolute generality in the first order domain, but with that, too, they forego any determination of the size of the domain. Their proposal goes well with their aim to relegate ordinal talk as "purely logical talk", but it does not address the problem that I am concerned with here.

[12]This is a general concern in the set theoretic depiction of the ordinals that does not only apply to the iterative understanding of sets, where the length of the iteration has to be provided, but also to the limitation of size approach, where one excludes some collections of being sets due to having a size equal to that of the universe. I this approach the size of the universe can only be measured by

One possible characterisation of this is to suggest that the indeterminacy is due to a particular form of *circularity* in its defining conditions. This explanation is offered in Wright (2018). Defining conditions are circular in this respect not if the characterisation of the object involves an ambient space *of* which it is part, but the other way round, if the ambient space is itself taken to be part (or indeed identical) to the object. This also seems to be what is captured with the inclosure schema rather than with Russell's formulation of the VCP. While the definition of natural number was considered to be irreducibly impredicative, but still sufficiently demarcated, the current predicament is thus due to a conflation of the definition itself and the ambient space in which it occurs.

This remains the case even when we introduce new conceptual resources. This is not to say, however, that the insight that these new concepts introduce is generally vacuous. Many of them establish new meaningful connections between concepts, but they are uninformative with respect to a determination of the length of the ordinals. The central one of these is the notion of a plurality (cf. Boolos 1984, Linnebo 2022b). A plurality $xx$ is something that is referred to by expressions like "some things" or "the things that $A$". One can express that $u$ is a member of the plurality $xx$ by a specific membership predicate $u \prec xx$ and introduce quantification over pluralities, written as $\forall xx$ and $\exists xx$ and read as "for all things" and "there are some things". Even though pluralities are, in this sense, understandable as collections of elements, they are not identical to sets. Unlike a set, a plurality is here taken to be prima facie nothing 'above' its elements. We don't have the elements and additionally the plurality that contains them, like we do in case of sets.

Regarding the interaction between pluralities and sets, two principles are of particular relevance. One is *Plural Collapse*:

$$\forall xx \exists y \forall u(u \in y \leftrightarrow u \prec xx)$$

which says that every plurality can be collected into a set, and *Plural Comprehension*:

$$\exists xx \forall u(u \in xx \leftrightarrow \varphi(u))$$

comparison with itself. For an extended discussion, see Incurvati (2020).

which says that every condition $\varphi$ defines a plurality.

Now, however, if we let $\varphi$ express the condition of being an ordinal, these two principles (in conjunction with unrestricted range of the objectual quantifiers) imply that there is a set of all ordinals—and hence the Burali-Forti paradox. One of these assumptions thus has to go, and an argument has to be supplied as to why. If we choose to restrict *Plural Comprehension* and argue that the ordinals don't form a plurality we need to give some criterion as to what constitutes a plurality in the first place, but ultimately, this cannot be done by reference to length or size (which requires the ordinals) (cf. Linnebo 2010).

Alternatively, one could admit that the ordinals do form a plurality, but reject versions of collapse. To express this distinction further, one can categorically distinguish between **exhaustive** and **distributive** readings of pluralities (cf. Oliver et al. 2013). An application of this idea to the ordinals was suggested in Rumfitt (2018a). On the distributive reading, a generalisation over a plurality expresses properties that each element has, on the exhaustive reading the attribution of a property concerns the plurality *as one* collection. To say "the four horses draw the carriage" is an instance of an exhaustive reading, while to say "the four horses drink water" is an instance of a distributive reading. In case of the ordinals to say that each ordinal has a successor is an instance of a distributive understanding (since each ordinal individually has one), but to say that the ordinals form a collection to be treated as an object itself is an exhaustive reading. This proposal is a good way to make the point that the ordinals do not form a collection in the sense of constituting a set but still give them some status,[13] but the question remains as to what makes a plurality empty on the exhaustive reading, and, again, using the length of the ordinals is no longer a permissible indicator.

All of the above is not to say that identifying the ordinals as a non-plurality, or as a non-exhaustive one, is not a meaningful conception to be established, or that the

[13] This distinction in the understanding of a plurality characterises a difference between the approaches by Rumfitt and Linnebo. Yet I am not sure if this particular disagreement between Rumfitt and Linnebo is really a fundamental one, or rather about the way the notion of plurality is used. Surely, insofar as comprehension is ubiquitously accepted, not every instance of it can yield an exhaustive non-empty plurality, and insofar as any plurality permits an exhaustive reading, not every condition may yield one. I suspect that the cases in which Linnebo allows for plural comprehension are precisely those in which Rumfitt allows for them to form exhaustively non-empty pluralities. Cf. their discussion in Linnebo (2018c) and Rumfitt (2018b).

interaction between pluralities and sets is not highly illuminating, but it puts some constraints on the possibility of such an identification as determining their length. When it comes to determining an ambient space in which the ordinals can be seen as indefinitely extensible, these suggestions fail to elevate themselves above the ordinals in the relevant sense. Introducing new concepts like that of a plurality helps elucidate the matter at hand, but the explanatory force lies with the notion of the ordinals themselves. This suggests that understanding what an ordinal is, simply is understanding this fundamental indeterminacy.[14]

This point will be of critical importance in chapter 6, where the question is discussed whether one can express generalisations over all ordinals and use them as a domain of quantification. Usually, a domain of quantification needs to be specified in a suitable manner for the quantified statement to be endowed with a desired meaning. The standard account of the quantifiers understands them as ranging over all the individuals in a domain, such that at least *some* circumscription of what these objects are is required. But the above shows that this cannot be done with the ordinals in the same way in which it is done with other domains of quantification. *The* ordinals cannot be treated in the same way as *an* ordinal (i.e. any of their initial segments) can be treated—for this would lead to the Burali-Forti paradox. But even though this shows that the ordinals cannot (without vicious(!) circularity) be construed as a domain of quantification in the ways suggested above, this does not mean that generalisations over the ordinals are impossible. From chapter 5 onwards, a different understanding of quantification will be developed which not only incorporates, but relies on these features of the reflexively indefinitely extensible concepts.

And only with respect to such an understanding of quantification, we can formulate the question of whether $\mathsf{LEM}$ holds in a precise manner.

[14]Interestingly, the characterisation of the ordinals as indefinitely extensible does not mention any limit stage. Thus, if we set the number of limit stages to 0 we again obtain the natural numbers, and the problem transfers. In this case, we have to change the above account of $N$ into being reflexively indefinitely extensible with respect to being a natural number. This takes us back to the position of the early skeptics of the infinite.

## 1.4. Indefinite Contractability

The previous discussion took the natural numbers as well as the ordinals to be domains of objects. But the notion of indefinite extensibility can also be applied to collections of formulas. This idea was mentioned in Dummett (1978) and analysed in Shapiro and Wright (2006). It will turn out to play a central role in the discussion whether the concept of natural number is indefinite.

**Definition 1.3.** A property of the natural numbers $P$ is **arithmetic** iff there is a formula $\chi_P(x)$ in the language of $\mathsf{PA}$, such that

$$\mathcal{N} \models \chi_P(n) \qquad \text{iff} \qquad n \in P.$$

where $\mathcal{N}$ is the standard model of arithmetic.

By Tarski's theorem, there is no formula $T$ in the language of $\mathsf{PA}$ such that

$$\mathcal{N} \models T(n) \qquad \text{iff} \qquad \text{there is some } \varphi \text{ s.t. } n = \ulcorner \varphi \urcorner \text{ and } \mathcal{N} \models \varphi$$

i.e. such that $T(n)$ holds for all Gödelnumbers of truths of arithmetic. One consequence of this is that the set of (Gödelnumbers of) truths of arithmetic is not recursively enumerable. However, take a set $C_0$ of (Gödelnumbers of) truths of arithmetic that is recursively enumerable (like all the theorems of $\mathsf{PA}$). This set is itself arithmetic since all recursively enumerable sets are $\Sigma_1$ definable and thus arithmetic (cf. van Dalen 2013, 245f.). We can now define a diagonal function $\delta$ on $C_0$ such that

$$\delta(C_0) = \ulcorner \varphi \urcorner \quad \text{for} \quad \varphi = \neg\chi_{C_0}(\ulcorner \varphi \urcorner).$$

Then $\varphi$ is a truth of arithmetic not in $C_0$. A similar result can be obtained by setting

$$\delta(C_0) = \ulcorner Con(C_0) \urcorner$$

Furthermore, we can go on to define $C_1 := C_0 \cup \{\delta(C_0)\}$, $C_2 := C_1 \cup \{\delta(C_1)\}$, etc. where $C_1$ and $C_2$ are still recursively enumerable. What this shows is that 'begin a truth of arithmetic' is indefinitely extensible with respect to being recursively enumerable. This process of extending $C_0$ can be iterated into the transfinite. Take $C_\omega$ to be the union of $C_0, C_1, C_2, \ldots$. Then $C_\omega$ is recursively enumerable as well, and we can define $C_{\omega+1}, C_{\omega+2}, C_{\omega+3}, \ldots$ all the way up to $C_{2\omega}$. However, due to the fact that there are uncountably many countable ordinals and that the set of formulas is only countable, there is a countable ordinal $\lambda$ such that the construction runs out before that. Furthermore, recursively iterating this construction does not give us the complete set of arithmetic truths.

How could this be taken to affect the determinacy of the natural number concept? Dummett writes the following:

> The only way to explain the meanings of the quantifiers whose variables range over the natural numbers is to state principles for recognising as true a statement which involves it; Gödel's discovery amounted to the demonstration that the class of these principles cannot be specified exactly once and for all, but must be acknowledged to be an indefinitely extensible class. (Dummett 1978, p. 199)

In this quote Dummett seems to understand that "stating principles for recognizing" true statements about the natural numbers at least requires them to be recursively enumerable, like, for instance, PA and all its theorems. PA seems to be the best candidate to start with, since it consists of recursive definitions of addition and multiplication plus induction for every predicate in the language of arithmetic. However, since there are truths of arithmetic not contained amongst the theorems of PA and there is no recursively enumerable extension of it that encompasses everything that we do recognize to be an arithmetical truth, these principles cannot be stated "one and for all". Adding such truths to our collection of formulas or principles enriches our explicit characterisation of the concept, but by staying within recursive enumerability we never achieve any form of completeness where every truth of arithmetic will be listed. In this sense, Dummett seems to take the concept of the natural numbers to be "inherently vague"(Dummett 1978, p. 197).

The same idea can be approached from another angle. Hartry Field suggests that:

> It would be natural to argue that if the concept of property of natural numbers is the concept of an indefinitely extensible totality, then the concept of natural

> numbers is the concept of an indefinitely contractible totality (since an increase in the supply of properties can lead to a decrease in the things closed under these properties).(Field 1994, p. 411)

But this raises a question. As noted above, the notion of indefinite extensibility at play here was related to collections of (Gödelnumbers of) formulas, whereas Field's reference to indefinite contractability was related to possible extension of the number concept, and thus to objects. How do the two relate?

Given a Gödel sentence $\theta_{\mathsf{PA}}$ of $\mathsf{PA}$, there is a non-standard model $\mathcal{M}$ of $\mathsf{PA}$ such that $\mathcal{M} \vDash \mathsf{PA} + \neg\theta_{\mathsf{PA}}$. This means that $\mathcal{M} \vDash \exists x Prov_{\mathsf{PA}}(x, \ulcorner\theta\urcorner_{\mathsf{PA}})$, where $Prov_{\mathsf{PA}}(x, y)$ says that $x$ is the Gödelnumber of a $\mathsf{PA}$ proof of $y$. But we know that $x$ cannot be a standard number since then it would encode an actual proof of $\theta_{\mathsf{PA}}$ which cannot be the case. Let this non-standard number be called $e$. Now, of course $\mathcal{M}$ cannot be a model of $\mathsf{PA} + \theta_{\mathsf{PA}}$. But even so, since we have no criterion of identity of non-standard numbers, we cannot give any precise meaning to the claim that *the* non-standard number $e$ from $\mathcal{M}$ is no longer an element of another model $\mathcal{M}^*$ of $\mathsf{PA} + \theta_{\mathsf{PA}}$. It is not the case that a model of $\mathsf{PA} + \theta_{\mathsf{PA}}$ can be obtained by scrutinizing the elements of a non-standard model of $\mathsf{PA}$. It is also not the case that the non-standard models of $\mathsf{PA} + \theta_{\mathsf{PA}}$ have in any sense a smaller cardinality that the non-standard models of ordinary $\mathsf{PA}$. In chapter 8, when possible consequences of conceptual indefiniteness will be discussed, I will make a suggestion on how the indefinite extensibility of theories could be connected to a change in the extension of the natural number concept.

## 1.5. Conclusion

In this chapter we have seen a general characterisation of the types of indeterminacy that are connected to indefinite extensibility. It was observed that this notion of indeterminacy is intimately connected to impredicativity or circularity in the concepts defining conditions, but it was noted that there is a difference between harmless and potentially harmful ways such circularity can unfold. In case of the natural numbers, we have seen that when we take them to be given inductively, impredicativity is an irreducible part of their definition. But this is benign insofar as there is an ambient space in which they can be seen as properly demarcated. In case of the ordinal numbers we have seen that such an ambient demarcation is missing and indeed impossible insofar as the ordinals are understood as the ultimate length indicator. This accounts for their paradoxical nature. In this respect, some attempts were discussed to determine the length of the ordinals, which, however, were all shown to depend on the ordinals themselves. Their other merits notwithstanding, they didn't carry any real information on the length of the ordinals.

In addition to that, another type of indeterminacy was discussed in connection to the concept of natural number. On this approach, the concept of natural number was understood to be characterised by a collection of formulas, for instance by PA and its theorems. This was also observed to be connected to indefinite extensibility, this time with respect to possible (recursively enumerable) extensions of theories like PA.

The overall question of the thesis is in which way indeterminacy might lead to a rejection of LEM. In this chapter we have encountered two notions of indeterminacy in relation of indefinite extensibility. This leads to two specifications of our question:

1. Can the indeterminacy in characterisation of the ordinals as a domain of quantification lead to a rejection of LEM? (cf. Tait 1998, Dummett 1994)

2. Can the indefinite extensibility of theories characterising our concept of natural number (by suitably extending PA) lead to a rejection of LEM. (cf. Dummett 1978)

These questions will receive an answer in chapters 6 and 8, respectively. But before getting to that, the fuller extent of the indeterminacy phenomena has to be explored. In the next chapter, I will discuss another form of indeterminacy, namely with-indeterminacy.

# 2. Indeterminacy of Width

This chapter addresses the second relevant notion of indeterminacy, so-called indeterminacy of width. Width-indeterminacy refers to the extent of the powerset operation (not the limits of its iterability, which was discussed in chapter 1). The question whether to accept or restrict the full powerset operation arises on several levels. The most prominent of those is given by the independence of the Continuum Hypothesis (CH) from the axioms of ZFC, on which Feferman's case against LEM in set theory rests. Since it is the main point of Feferman's argument, the case of CH will be treated in the most detail in this chapter. But there is already indeterminacy on levels 'below' CH, which will be briefly addressed as well.

These cases of indeterminacy can be understood with respect to both, the concept of set as well as its domain, the set theoretic universe $V$. Whether or not determinacy is achieved regarding CH or other instances, however, is in both cases judged upon whether an axiomatic extension of ZFC exists that settles the matter appropriately and that is plausibly justified. I will give a brief overview of certain axiom candidates and their status amongst mathematicians.

Section 1 introduces the constructible universe and presents the proof of the consistency of CH. Section 2 is concerned with the proof of the independence of CH from ZFC and will introduce the technique of forcing to the extent to which it is required for the discussion in subsequent chapters. Section 3 will situate the case of CH in a broader context by (i) comparing forcing to indefinite extensibility, (ii) presenting related cases of indeterminacy and suggestions for axioms to settle it, and (iii) arguing that indeterminacy of height and indeterminacy of width are independent from each other.

## 2.1. The consistency of CH

If one accepts the results from the previous chapter and takes the natural numbers plus induction as sufficiently determinate (which almost all mathematicians do), the next question would be to look for impredicativity on the next higher level, i.e. regarding the notion of a property of the natural numbers. Weyl, for instance, in *Das Kontinuum* (Weyl 1994) takes the natural numbers and the rationals to be extensionally determinate, but the concept of a property of a natural number as extensionally indefinite. According to him, we have a definite conception for the construction of the natural numbers and of induction which is grounded in intuition, but there are certain properties of natural numbers which can only be defined using an impredicative and, in his eyes, maliciously circular construction.To introduce definiteness regarding our notion of a subset or arbitrary property of the natural numbers, Weyl therefore restricts concept formation to be predicative.[1]

Moving up one level further, to properties of properties or subsets of subsets of the natural numbers, we reach the level at which CH will be decided. CH is the statement that $2^{\aleph_0} = \aleph_1$, which can alternatively expressed as $2^\omega = \mathscr{P}(\omega)$. Written out, this formula says that:

$$\forall x \subseteq \mathscr{P}(\omega) : (x \neq \emptyset \rightarrow (\exists g : \omega \twoheadrightarrow x \vee \exists f : x \twoheadrightarrow \mathscr{P}(\omega)))$$

where $\exists g : \omega \twoheadrightarrow x$ is shorthand for $\exists g(g : \omega \twoheadrightarrow x)$ and the double headed arrow $\twoheadrightarrow$ indicates that $g$ is surjective. The statement uses quantification over subsets of the real numbers, which means over subsets of the collection subsets of the natural numbers ($\mathscr{P}(\omega)$).

Gödel has shown that this statement is consistent with ZFC. His proof can be seen as a generalisation of the predicative construction to such higher levels (cf. Feferman 2005, Crosilla 2018). The proof rests on the introduction of the constructible universe $L$, which is a model of ZFC that validates not only CH, but its extension, the generalised

[1]The idea of predicativity given the natural numbers was thoroughly investigated independently by Feferman and Schütte (cf. Feferman 1964). They have shown that if we let the above process of constructing predicatively defined collections of properties iterate into the transfinite, we arrive at a certain limit of predicativity given by the ordinal $\Gamma_0$.

continuum hypothesis GCH according to which the equation $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ holds for all cardinals $\aleph_\alpha$ and their successors $\aleph_{\alpha+1}$. The constructible universe $L$ is specified in the following way:

**Definition 2.1.** A set $y$ is **constructible over** $\langle M, \in\rangle$, if there exists a formula $\varphi$ such that $y = \{x \in M \mid \langle M, \in\rangle \models \varphi(x)\}$.

The set $L(M) := \{y \subseteq M \mid y \text{ is constructible over } \langle M, \in\rangle\}$ is the set of **constructible subsets** of a set $M$.

Now one can obtain the constructible hierarchy by restricting the powerset operation applied to each stage to its definable subsets:

**Definition 2.2.** The **constructible hierarchy L**:

$$L_0 = \emptyset \qquad L_{\alpha+1} = L(L_\alpha) \qquad L_\lambda = \bigcup_{\beta<\lambda} L_\beta, \quad \lambda \text{ is a limit ordinal}$$

$$L = \bigcup_{\alpha \in Ord} L_\alpha$$

This generalises the idea of restricting the notion of an arbitrary property or subset of the natural numbers to the definable ones into the transfinite.[2] With respect to ZFC, each such $L_\alpha$ is considered a set, but $L$ itself is a class in ZFC. This is due to the fact that $L$ contains all ordinals, for which ZFC proves (as a consequence of the Burali-Forti paradox) that they are not a set.

By the axioms of ZFC, it is not excluded that the powerset may be understood as predicative in this way, i.e. that the powerset of a set $u$ is interpreted as $L(u)$. The assertion that every set is constructible, which is called the axiom $V = L$, is consistent with ZFC, and the constructible universe $L$ is a model of it. Gödel showed that adding

[2]It should be pointed out, however, that the definition of $L$ is not predicative in the strictest sense. The ordinals themselves along which the construction iterates, are given by impredicative reasoning, since their definition involves the notion of a well-ordering, which says that a set $u$ is well-ordered iff all of its non-empty subsets have a least element. If the ordinals are defined in the von Neumann way, this can be expressed by

$$\forall x \subseteq u(x \neq \emptyset \to \exists y \forall z(y \in z \to z \notin y))$$

where the outer quantifier $\forall x$ ranges over a domain of which $u$ is a member.

$V = L$ to $\mathsf{ZFC}$ settles $\mathsf{CH}$ and $\mathsf{GCH}$ to the positive. In the following, I will write $\lambda$, and $\lambda^+$ for cardinals $\aleph_\alpha$ and their successor $\aleph_{\alpha+1}$.

**Theorem 2.1.** (Gödel, Consistency of $\mathsf{GCH}$). $\mathsf{ZFC} + V = L \vdash 2^\lambda = \lambda^+$ *for all cardinals* $\lambda$ *and their successors* $\lambda^+$.

In general, since $2^\lambda \geq \lambda^+$ by Cantor's theorem, in order to show that $2^\lambda = \lambda^+$ we have to show that $\mathscr{P}(L_\lambda) \subset L_{\lambda^+}$ holds in $L$. This means that we have to show that for each $m \subset L_\lambda$, $m \in L_{\lambda^+}$. This follows from a series of lemmas.[3]

In order to make proper use of constructibility we need to make sure to be working with transitive models:

**Lemma 2.1.** (Mostowski-Collapse) *Every extensional and well founded class* $\langle M, \in\rangle$ *ordered by* $\in$ *is isomorphic to a unique transitive class* $\langle N, \in\rangle$. *If* $C \subset M$ *is transitive, then the respective part of the isomorphism from* $M$ *to* $N$ *is the identity on* $C$.

Transitive models become relevant when combined with the definition of absoluteness:

**Definition 2.3.** A formula $\varphi(x_1, \ldots, x_n)$ is **absolute** with respect to two models $\mathcal{M} = \langle M, \in\rangle$ and $\mathcal{N} = \langle N, \in\rangle$ if for all $a_1, \ldots, a_n \in M \cap N$

$$\mathcal{M} \vDash \varphi(a_1, \ldots, a_n)^M \qquad \text{iff} \qquad \mathcal{N} \vDash \varphi(a_1, \ldots, a_n)^N$$

The following lemma states that transitive models have special properties with respect to absoluteness. Let $\mathsf{ZF} - \mathscr{P}$ be $\mathsf{ZFC}$ minus the powerset axiom.

**Lemma 2.2.** *The property of being constructible,* $y = L_\alpha$, *and* $x \in L_\alpha$ *are absolute for all transitive models* $\mathcal{M}$ *of* $\mathsf{ZF} - \mathscr{P}$.

Finally, the proof of the theorem requires some constraints on the size of the structures that we are concerned with.

**Lemma 2.3.** (Downward-Löwenheim-Skolem-Tarski (DLST) Theorem, cf. Kunen (2013), Theorem I.15.10, p.88) *Let* $\mathcal{M} = \langle M, \in\rangle$ *be any structure for the language of set theory.*

[3] A full account of these matters can be found in Jech (2002, ch.13), Kunen (1980, ch.VI) and Kunen (2013, ch.II.6). I found Smullyan et al. (2010, ch.10-15) particularly accessible.

*Let $\kappa$ be a cardinal such that $|\omega| \leq \kappa \leq |M|$ and fix an $S \subset M$ with $|S| \leq \kappa$. Then there is a model $\mathcal{N} = \langle N, \in \rangle$ such that $S \subset N$, $\mathcal{N} \preceq \mathcal{M}$, and $|N| = \kappa$.*[4]

All this can be combined for the proof of the theorem

*Proof of Theorem (2.1).* As a reminder, we have to show that for each $m \subset L_\lambda$, $m \in L_{\lambda^+}$, which means that we have to show that there is an ordinal $\alpha$ such that $L_\alpha \subseteq L_{\lambda^+}$ and $m \in L_\alpha$.

If $m \subset L_\lambda$ then it is the case that $|L_\lambda \cup \{m\}| = \lambda$. This follows from the fact that $|L_\lambda| = |L|$, which essentially is the case, because there are only countably many formulas and each formula only contains finitely many parameters (cf. Kunen 2013, Lemma II.6.9, p.136).

Now, if $\theta$ is a regular uncountable cardinal, then $L(\theta) \vDash \mathsf{ZF} - \mathscr{P} + V = L$ (cf. Kunen (2013), Lemma II.6.22, p.140). Pick such a $\theta$ for which $L_\lambda \cup \{m\} \subset L(\theta)$. We can then apply the DLST Theorem to $L(\theta)$ to obtain an $\mathcal{M} = \langle M, \in \rangle$ (of which $L_\lambda \cup \{m\}$ is a part) with $|M| \leq \lambda$ and $\mathcal{M} \vDash \mathsf{ZF} - \mathscr{P} + V = L$.

The next step is to apply the Mostowski-Collapse lemma to $\mathcal{M}$ and obtain a transitive model $\mathcal{N}$. Since $m \subset L_\lambda$ is transitive, the collapsing function is the identity function on its part. Hence $m \in \mathcal{N}$.

Finally, $m$ is constructible is in $\mathcal{N}$ (because it was constructible in $L(\theta)$ and the subsequent modifications did not alter its truth value). This means that there is an ordinal $\alpha \in \mathcal{N}$ such that $\mathcal{N} \vDash m \in L_\alpha$. By absoluteness, $\alpha$ is a genuine ordinal. Since we have $|M| = |N| \leq \lambda$, we have that $|\alpha| \leq \lambda$, and hence $|\alpha| \leq \lambda^+$. Therefore we have, what we wanted to show, namely that $m \in L_{\lambda^+}$. □

This has established the consistency of $\mathsf{CH}$. A natural question to ask at this point is: Why not just accept $V = L$ as an axiom and settle the matter? But $V = L$ has some unwanted consequences. It is inconsistent with other axioms that are deemed desirable such as the assertion of the existence of a measurable cardinal. This suggests that $V = L$ is not a good candidate to determinately settle the question of $\mathsf{CH}$ to the positive. But if $V = L$ doesn't hold, Cohen showed by using the method of forcing that there is a

[4] $\mathcal{N} \preceq \mathcal{M}$ here means that there is an elementary embedding from $\mathcal{N}$ into $\mathcal{M}$ (Jech 2002, 156f.).

model of ZFC in which the negation of CH is true, and, furthermore, that according to ZFC (and many of its extensions) the continuum can have almost any cardinality.

## 2.2. The independence of CH

In this section I will sketch the proof of the following:

**Theorem 2.2.** (Cohen, Independence of CH) ZFC *is consistent with* $\neg$CH*. More precisely, there is a model* $\mathcal{M}$ *of* ZFC *such that* $\mathcal{M} \vDash 2^{\aleph_0} = \aleph_\alpha$ *for an ordinal* $\alpha$ *for which* $cf(\aleph_\alpha) > \omega$*.*[5]

This is proved by the method of forcing. A textbook account of this procedure usually runs over one or two full chapters. My presentation here only serves to express the general idea. I aim to give enough detail for the forcing construction to be relatable to the arguments from categoricity introduced in the next chapter and for the discussion in chapter 7.3. My presentation will follow a template provided in Bell (1986), and the constructions are largely based on Bell (2005).[6] The general idea is that in order to show that a statement is independent, we change the underlying logic that is used for reasoning with the axioms to allow us to create a model in which a translation of CH into this logic fails. If there is a respective correspondence between the usual two-valued logic and the new model, we know that CH cannot be provable in that logic—on pain of contradiction in the new system.

The logic I choose is that of a complete boolean algebra $\mathbb{B}$.[7] The first step is thus to transition from a two valued model $\mathcal{M}$ to a boolean valued model $\mathcal{M}^{\mathbb{B}}$.

$$\mathcal{M} \longrightarrow \mathcal{M}^{\mathbb{B}}$$

Under certain restrictions, then, one can even obtain another transition to a two valued

[5] This requirement is due to König's Lemma, which states that $cf(2^\kappa) > \kappa$ for any infinite cardinal $\kappa$.

[6] For a more extensive introductory explanation, see Chow (2007). For the details the reader may also consult Jech (2002, ch. 14).

[7] Other approaches make use of intuitionistic logic (Fitting 1969), and modal logic (Smullyan et al. 2010).

model $\mathcal{M}^*$ that differs in important respects from $\mathcal{M}$, i.e.,

$$\mathcal{M} \longrightarrow \mathcal{M}^{\mathbb{B}} \longrightarrow \mathcal{M}^*,$$

where we enrich the starting system and then collapse it back to obtain the desired differences.

**Forcing with boolean valued models**

If we are only interested in the first step, $\mathcal{M} \longrightarrow \mathcal{M}^{\mathbb{B}}$, we can use the full set theoretic universe $V$ and create a boolean valued model $V^{\mathbb{B}}$ from it. The definition is analogous to the usual one:

$$V_0^{\mathbb{B}} = \emptyset \qquad V_{\alpha+1}^{\mathbb{B}} = \text{ set of all functions } x \text{ with } dom(x) \subset V_\alpha^{\mathbb{B}} \text{ and values in } \mathbb{B}$$
$$V_\lambda^{\mathbb{B}} = \bigcup_{\alpha<\lambda} V_\alpha^{\mathbb{B}} \text{ for limit ordinal } \lambda$$
$$V^{\mathbb{B}} = \bigcup_{\alpha \in Ord} V_\alpha^{\mathbb{B}}$$

Note that $V^{\mathbf{2}}$ for the two-valued boolean algebra $\mathbf{2}$ is isomorphic to $V$.

The logical connectives and the quantifiers are then interpreted in the usual way as boolen operations, and one can define co-inductively the interpretation of primitive expressions:

$$[\![x \in y]\!] = \bigvee_{t \in dom(y)} ([\![x = y]\!] \wedge y(t))$$
$$[\![x = y]\!] = \bigwedge_{w \in dom(x)} (x(w) \Rightarrow [\![w \in y]\!]) \wedge \bigwedge_{w \in dom(y)} (y(w) \Rightarrow [\![w \in x]\!])$$

which guarantee that the principle of extensionality is satisfied. The (interpretation of) the powerset $\mathscr{P}(u)$ is given by setting $dom(\mathscr{P}(u)) = \mathbb{B}^{dom(u)}$ and for each $x \in dom(\mathscr{P}(u))$, $\mathscr{P}(u)(x) = [\![x \subset u]\!]$. Note that $\mathscr{P}(u)$ is a function from 'elements' of $V^{\mathbb{B}}$ into $\mathbb{B}$.

It should also be noted, however, that this definition cannot be given by the means of

$\mathsf{ZFC}$ alone, but needs to be given 'meta linguistically'. This problem can be circumvented by the second approach on forcing that I will highlight in due course. There are, however, considerable costs to this, too.

With $V^{\mathbb{B}}$ defined like this, one can show that $V^{\mathbb{B}} \models \mathsf{ZFC}$ and, with some additional constraints on $\mathbb{B}$, one can be certain that the transition from $V$ to $V^{\mathbb{B}}$ preserves cardinality. Then, to violate $\mathsf{CH}$ in $V^{\mathbb{B}}$ one chooses $\mathbb{B}$ in such a way that $\mathscr{P}(\omega)$ ends up being large in $V^{\mathbb{B}}$.

To this end, consider the set $C(x, y)$ of mappings from finite subsets of $x$ to $y$. In particular, we might think of $C(\omega \times \omega_\alpha, 2)$ in case we want to add $\omega_\alpha$ new subsets to $\omega$. This goes as follows: Any $p \in C(x, y)$ gives rise to the set

$$N(p) = \{f \in y^x : p \subset f\}$$

i.e. the set of all possible extensions of $p$. Each such $p$ can thereby be understood as a finite piece of information of adding a new subset to $w$, and $N(p)$ as the set of its possible continuations. Those sets then form the basis of a boolean algebra $\mathbb{B}$ which can be understood to adjoin $\omega_\alpha$ new subsets of $\omega$ in $V^{\mathbb{B}}$. One can then show that $V^{\mathbb{B}} \models 2^{\aleph_0} = \aleph_{\hat{\alpha}}$, where $\hat{\alpha}$ is an image of the original $\alpha \in V$. The argument to this effect exploits that $|dom(\widehat{\mathscr{P}(\omega)})| = |\mathbb{B}^{dom(\hat{\omega})}|$ which is, given the size of $\mathbb{B}$, smaller or equal to $\aleph_{\hat{\alpha}}$.

Similarly, given infinite cardinals $\kappa$ and $\lambda$ with $\kappa < \lambda$ we can construct a $\mathbb{B}$ such that $V^{\mathbb{B}} \models |\hat{\kappa}| = |\hat{\lambda}|$. For this we may use the set $K(x, y)$ of partial functions $f$ from $\kappa$ into $\lambda$ with $dom(f) < \kappa$. Analogously, we can define for $p \in K(x, y)$

$$N(p) = \{f \in \lambda^\kappa : p \subset f\}$$

Each $p$ is then a smaller than $\kappa$ piece of information to add a new function from $\kappa$ into $\lambda$. Now, if we consider the sets $b_{m,\eta} = \{f \in \lambda^\kappa : f(m) = \eta\}$, i.e. the set of all functions that have the same value $\eta$ at input $m$, we have $b_{m,\eta} \wedge b_{m,\theta} = 0$ in the corresponding boolean algebra. This follows, because no function can have two values for the same input. Additionally, we have that $\bigvee\limits_{m<\kappa} b_{m,\eta} = 1$ for each $\eta < \lambda$. Hence all these sets encode a total function from $\hat{\kappa}$ onto $\hat{\lambda}$. This argument analogously exploits the fact that $|dom(\kappa)| = |\lambda^\kappa| = |dom(\lambda)|$. For the special case where $\kappa = \omega$ we can say that the (uncountable) cardinal $\lambda$ is made countable by forcing (cf. Bell 2005, 109f. Jech 2002, 237f.).

**Forcing with countable transitive models**

This is already enough to show the independence of CH via the transition $V \to V^{\mathbb{B}}$. But there is one further step that we can take to obtain a regular two-valued model again. For this, however, we can no longer use $V$ as a starting point. Instead, we start with a countable transitive model $M$ and perform the whole construction inside $M$, i.e. we define $M^{\mathbb{B}}$ and the map $M \to M^{\mathbb{B}}$ all inside $M$.[8] Because $M$ is countable, the conditions are fulfilled to specify a certain set $G$ on the basis of $M$ for which $G \notin M$. We can then adjoin $G$ to $M$ to obtain a new model $M[G]$ with $M \subset M[G]$.

The properties that $G$ needs to have, in order for this to work are the following. If we consider again that each $p \in \mathbb{B}$ encodes a finite piece of knowledge, we may further think of two pieces of knowledge $p, q \in \mathbb{B}$ to be compatible, if there is an $r \in \mathbb{B}$ such that $p \leq r$ and $q \leq r$, i.e. if they are both refinements of a common 'previous' piece of information. We obtain a body of knowledge if we close under compatibility, i.e. under going 'upwards'. Closure under compatibility is fulfilled, if $G$ is an ultrafilter on $\mathbb{B}$.[9] In this case $G$ is also called *generic*. We can then define

$$\emptyset^G = \emptyset \qquad x^G = \{y^G : x(y) \in G\}$$

i.e. $x^G$ as containing the elements as recognised by $G$.[10] This gives us a new model

[8] Throughout this thesis, models are usually denoted by calligraphic letters $\mathcal{M}$, but for elements of the forcing multiverse which are discussed in this section and in chapter 7 I follow the standard convention of using normal lettering for better readability.

[9] Given any poset $P$. a set $F \subset P$ is a *filter* on $P$ iff

(i) $0 \notin F$

(ii) If $x, y \in F$ then $x \wedge y \in F$

(iii) If $x \in F$ and $x < y$ then $y \in F$

$F$ is an *ultrafilter* if additionally for all $x \in P$, either $x \in F$ or $P \setminus \{x\} \in F$.

[10] This is actually a combination of forming the quotient-model $M^{\mathbb{B}}/U$ followed by an application of

$M[G] = \{x^G : x \in M^{\mathbb{B}}\}$, for which it holds that $G \in M[G]$, and

$$M[G] \vDash \varphi(x_1^G, \ldots, x_n^G) \quad \text{iff} \quad [\![\varphi(x_1^G, \ldots, x_n^G)]\!] \in G$$

which is the so-called forcing theorem (cf. Jech 2002, p. 216, Bell 2005, ch.4).

Now, if $\mathbb{B}$ is based on $C(x, y)$ then $\bigcup G$ is a function from $x$ to $y$, e.g. from $\omega \times \omega_\alpha \to 2$. So $\bigcup G$ is coding a sequence of functions from $\omega \to 2$, and since $\bigcup G$ has domain $\omega$ (because each $p \in G$ is finite), $\omega$ has $\omega_\alpha$ many subsets in $M[G]$. Similarly, if $\mathbb{B}$ is based on $K(\kappa, \lambda)$, then $\bigcup G$ is a function from $\kappa$ into $\lambda$. That is because each $p < \kappa$, we have that $dom(\bigcup G) = \kappa$ and because $G$ intersects with every set $\{p \in K(\kappa, \lambda) : \alpha \in ran(p)\}$ for every $\alpha < \lambda$, we have that $ran(\bigcup G) = \lambda$.

But there is one important detail that has been left unmentioned till now. For this whole construction to have the desired effect, it needs to be the case that $G$ is not part of the ground model, i.e. $G \notin M$. That this indeed holds in the relevant cases can be explained (roughly) by the observation that there must be a $p$ in the respective poset $P$, such that for some $q \in G$, $p \wedge q = 0$, and hence that $p \notin G$. However, due to the nature of the poset, we have that there is an $r$ such that $q \leq r$ and $p \leq r$, and hence, by closure under compatibility, $p \in G$, which leads to a contradiction.

## 2.3. Further points

This section will situate the case of CH in a broader context by (i) comparing forcing to indefinite extensibility, (ii) presenting related cases of indeterminacy and suggestions for axioms to settle it, and (iii) arguing that indeterminacy of height and indeterminacy of width are independent from each other.

### 2.3.1. Forcing and indefinite extensibility

Forcing can be understood as a technique to create *new models* (with alternating truth values for certain statements). It can also be understood as a technique to create *new elements*. It was suggested that forcing, when understood in the latter way, comparable

the Mostowski Collapse lemma to it.

to indefinite extensibility, where the generic extension $M[G]$ of a model $M$ is comparable to adding a diagonal element to a determinate sub-concept of an indefinite extensible expression. This can be formulated in the following way (cf. Meadows 2014):

1. Obtain $G$ on basis of $M$.
2. Define a set $E = \{p \in P \mid p \notin G\}$. Now, it is the case that for each $p \in P$ there is a $q \in E$ such that $q \leq p$.
3. This means that there is an $r \in G$ such that $q \leq r$, and hence by closure under compatibility, $q \in G$.
4. But then we obtain $q \notin G \leftrightarrow q \in E \leftrightarrow q \in G \leftrightarrow q \notin E$.
5. Therefore $G \notin M$

Thus on this analogy each $M$, $M[G]$, etc., corresponds to a determinate sub-concept or subset and each forcing extension is an application of the diagonal function. But there are elements that disrupt this analogy. First, it is not clear in what sense each forcing extension is understood as (the extension of) a sub-concept. What is the concept that it is a sub-concept of? After all, each model created by the forcing extensions satisfies all the axioms of ZFC. This lends some legitimacy to the claim that each of them may be considered as a full, but *separate* extension of the set concept (as it is, for instance, defended in Hamkins 2012). This is not the case for any of the indefinitely extensible concepts that were discussed in the previous chapter. Each (finite) sub-concept of the natural numbers, for instance, does not have the properties of the natural number structure. Rather, the natural numbers are understood as the limit of this process. From this perspective, then, the structural similarity that the above shows should not be overestimated. We might say that the notion of being a forcing extension is indefinitely extensible with respect to being a countable transitive model of ZFC—but it is not the notion of a forcing extension, but that of a model that we are interested in. This is not to say, that the other perspective, which emphasises forcing primarily as a technique to introduce new elements, does not have its own philosophical legitimacy. In fact, in chapter 7, we will see that both perspectives will lend themselves to an approach towards addressing indeterminacy of width—but only one of them will ultimately lead to a rejection of LEM.

This discussion also points to the question whether the independence of CH should be understood to affect the *domain of sets* (i.e. the set theoretic universe $V$) or our *concept of set* as it is explicated by the ZFC axioms. What seems to be clear, however, is that once an axiom is formulated and generally accepted to settle CH, then this would constitute a case of determinacy. In this respect, and in preparation for the discussion in the next chapter, it will prove to be enlightening to find similar statements like CH and inquire into their (possible) 'resolution'. The question is thus: How do we situate the independence of CH in the broader context of independence phenomena?

### 2.3.2. $V_{\omega+1}$ and $V_{\omega+2}$

In some way, the independence of CH can be seen in line with the independence of the parallel postulate from the axioms of euclidean geometry, and in the realm of set theory with that of Borel Determinacy, which can't be settled in Zermelo's original system Z, but which requires addition of the axiom of replacement, hence the system ZF. If we consider ZF or ZFC itself, there is independence at the levels of $V$ corresponding to the different levels of predicativity discussed in section 2.1. Corresponding to the indeterminacy regarding the notion of an arbitrary subset of the natural numbers, we have indeterminacy with respect to the level $V_{\omega+1}$ of the set theoretic universe; and corresponding to the indeterminacy regarding the notion of an arbitrary subset of the collection of arbitrary subsets of the natural numbers, we have indeterminacy with respect to the level $V_{\omega+2}$.[11] CH, since it contains quantification over subsets of $\mathscr{P}(\omega)$, will be decided on $V_{\omega+2}$. Thus, there is already indeterminacy below the level on which CH is dealt with. Furthermore, following the reasoning in Koellner (2009), to these levels also correspond different interpretability degrees for axiomatic extensions of ZFC. Two axiom systems belong to the same interpretability degree, if they are mutually interpretable. Importantly, however, there are axiom systems which belong to the same interpretability

[11] The difference between the two can also be understood as the difference between second and third order arithmetic, see Koellner (2014) and Koellner (2013).

degree which are inconsistent with each other.[12] There is thus a range of prima facie competing candidates to introduce determinacy on these levels.

The indeterminacy on $V_{\omega+1}$ centres around the question of projective determinacy in descriptive set theory. While it is not accepted as an axiom by all mathematicians, there is at least a statement that serves as a candidate for an axiom and that introduces determinacy regarding $V_{\omega+1}$. This is the statement $\mathsf{AD}^{L(\mathbb{R})}$ in the presence of a proper class of Woodin cardinals[13]. $\mathsf{AD}^{L(\mathbb{R})}$ is robust in terms of forcing: It determines the theory of $V_{\omega+1}$ in a way that it cannot be altered via forcing—and it is the only axiom with that property (cf. Koellner (2013), Koellner (2006)). Additionally, it is the only axiom that yields intuitively compelling consequences for the structure theory of the definable reals, and it also naturally features in the large cardinal hierarchy.[14] According to Koellner, these features make it a suitable addition to $\mathsf{ZFC}$. However, not everyone agrees to grant $\mathsf{AD}^{L(\mathbb{R})}$ such a special status. Saharon Shelah, for instance, referring to axioms like $\mathsf{AD}^{L(\mathbb{R})}$ as semi-axioms, writes:

> The judgements of certain semi-axioms as best is based on the groups of problems you are interested in. For the California school, descriptive set theory problems are central. While I agree that they are important and worth investigating, for me they are not "the center". Other groups of problems suggest different semi-axioms as best; other universes may be the nicest from a different perspective.(Shelah 2003, p. 212)

Thus, for the level of $V_{\omega+1}$ there is, although not ubiquitously accepted, a way to introduce determinacy. But regarding indeterminacy on the level of $V_{\omega+2}$ and hence regarding $\mathsf{CH}$, there is some important discontinuity to all of the examples given above. As of now, there is no axiom which is known to settle $\mathsf{CH}$ (cf. Koellner 2006, Koellner 2009, Koellner 2013), and, furthermore, as Levy et al. (1976) have known, such an axiom cannot be one of the currently available large cardinal notions (cf. Jech (2002), theorem 21.2). The independence of $\mathsf{CH}$ is thus the paradigm case of width-indeterminacy that as of now remains unsettled. For this reason, it will stay the focus in the remainder of the thesis. There are two main research programs that aim to settle $\mathsf{CH}$. One is

[12]See, for instance, the examples given in Koellner (2009, p. 100).
[13]For the notion of a Woodin cardinal, see Kanamori (2009, 360f.).
[14]For the case why $\mathsf{AD}^{L(\mathbb{R})}$ is compelling in this respect, see Koellner (2006), Koellner (2009).

concerned with finding strong enough forcing axioms to express the fact that certain statements cannot be altered via forcing any more. Whether or not there are axioms that settle $\mathsf{CH}$ on this level depends on whether the so-called strong $\Omega$-conjecture holds. This conjecture is about the consequence relation in $\Omega$-logic, which is concerned with statements that are invariant to forcing. The technical success of this development, which would lead to $\neg\mathsf{CH}$, and its bearing on the philosophical issues is not decided yet. Competing with this approach is the so-called Ultimate-$L$ program which aims to create an $L$-like model that is (unlike plain $L$) consistent with all large cardinals (cf. Woodin 2017), and that would entail $\mathsf{CH}$. Both, in any case, entail $\mathsf{AD}^{L(\mathbb{R})}$.[15] However, even if these programs establish their technical goals, it is another question if they possess compelling philosophical motivations to be counted as discoveries of the truth or falsity of $\mathsf{CH}$ rather than as ways to merely express general (artificial) consensus about it (for an extended discussion of this issue, see Lingamneni (2020)).

### 2.3.3. Dependencies between width and height indeterminacy?

Combining the previous discussion with the results of chapter 1, $\mathsf{ZFC}$ and thereby also the universe of sets it is taken to characterise exhibits both indeterminacy in height (the ordinals) and in width (interpretation of $\mathscr{P}$). Now, before asking what logical consequences width and height indeterminacy may have, we should ask if there are any synergy effects between them. We should be aware of how both relate to each other. One idea where dependencies might arise is when there are further mutually incompatible axiomatic extensions of $\mathsf{ZFC}$ that narrow down possible interpretations of the powerset operation. One such example is the existence of a measurable cardinal, which contradicts the axiom of constructibility.[16]

**Definition 2.4.** A cardinal $\alpha$ is **measurable** if it contains a non-principal $\alpha$-complete ultrafilter.[17]

[15]There is also an axiom candidate called the *Inner Model Hypothesis*, which is motivated by another research program called the *Hyperuniverse Program*, and which implies that $\mathsf{AD}^{L(\mathbb{R})}$ is false (cf. Barton and Friedman 2020, Antos, Barton, et al. 2021).

[16]To my knowledge, the smallest cardinal whose existence is inconsistent with $V = L$ is $0^\sharp$. I will use the notion of a measurable cardinal, because it seems to be more well known.

[17]An ultrafilter $U$ is **$\kappa$-complete** if and only if any subset $E \subset U$ with $|E| < \kappa$, $\bigwedge E \in U$; $U$ is

**Theorem 2.3.** (Scott, cf. Chang et al. (1990, Theorem 4.2.18)) *If there exists a measurable cardinal bigger than $\omega$, then $V \neq L$, i.e. the axiom of constructibility fails.*

But does this have any bearing on the length of the ordinals under the different axiomatisations? The answer is no. We know that (in the presence of the axiom of Foundation) an ordinal can be defined as being transitive and totally well ordered by $\in$. Both properties are expressible via $\Delta_0$-formulas, which are formulas that only contain bounded quantification. It is a well known fact that $\Delta_0$-definable properties are absolute with respect to transitive models (which includes $V$ and $L$) (cf. Jech 2002, p. 163). Thus any ordinal in $L$ is thereby also an ordinal of $V$. Furthermore, $L$ is defined by ordinal recursion in $V$ and hence it must share its height. In general, it can be noted that if $\mathcal{M}$ is a model of set theory with the axiom that there exists an uncountable measurable cardinal $\kappa$, we can form a submodel $\mathcal{M}_L$, which has the same length as $\mathcal{M}$ but which satisfies the axiom of constructibility (and hence doesn't contain $\kappa$) (cf. Bell and Slomson 1969, p. 306). Thus the existence of a measurable cardinal only depends on the existence of an ultrafilter, and it is the existence of that, which is ultimately in conflict with $V = L$. Any absolute length is not affected, only the question whether there is a particular set that has a "horizontal" property.

## 2.4. Conclusion

In this chapter we have seen the paradigm case of width indeterminacy, namely the independence of CH from the ZFC axioms, and observed that at least insofar as its axiomatic extensions are concerned, it is independent from height indeterminacy. This leads to another sharpening of our question, which is the following:

3. Does indeterminacy of width in the guise of the independence of CH lead to a rejection of LEM in set theory? (Feferman 2011)

Feferman is presumably thinking about the *domain of sets*, but since the question of ensuring determinacy is so entangled with finding a new axiom, we can equally ask

**principal** if $U = \{y \mid a \leq y\}$ for some $a \in U$.

whether the *concept of set* that is characterised by ZFC might be indeterminate, too (especially since there are competing ways to sharpen it that are inconsistent with each other). But before getting to that, we need to ensure that there is at least a certain kind of robustness to the claims of indeterminacy that have been put forward in the first two chapters. We need to be sure that any investigation into their consequences has its proper place. For that reason, the next chapter is dedicated to a discussion of attempts to argue for and re-gain determinacy in light of the technical landscape that has been sketched.

# 3. Arguments for Determinacy

In the last two chapters the notion of indeterminacy was sharpened in different ways. The discussion of indefinite extensibility from chapter 1 resulted in two possible charges against determinacy. It was observed that the ordinals, due to being reflexively indefinitely extensible, were subject to a sort of indeterminacy that arose due to the lack of an ambient notion in which any demarcation of their length could be situated. And there was a charge against the determinacy of the natural number concept because the first order theories intended to characterise them were considered indefinitely extensible. In the previous chapter, the indeterminacy of the powerset operation explicated through the method of forcing was discussed. Before considering possible consequences of these instances of indeterminacy for the respective applicability of LEM, we need to make sure that these notions of indeterminacy are indeed substantive and to a certain extend robust.

To this extent, I will address some common arguments that the natural numbers, as well as the sets, are determinate after all. I will first discuss the relationship between categoricity and determinacy. Categoricity may be taken to countenance the indeterminacy that arises through the first order limitative results and through forcing, but it has no bearing on the charge of indeterminacy due to the circularity that the ordinals face. However, even with the first two points, it will turn out that categoricity is not up to the task, because its semantics will be argued to be indeterminate. Afterwards I will briefly address the arguments based on internal categoricity results which eschew any semantic dependency. They, however, will also turn out to be insufficient to establish determinacy. Finally, I will discuss the notion of realism as it arises in relation to the indeterminacy phenomena. This serves to specify the possible position against determinacy and $LEM$ from an ontological/metaphysical perspective to be (in a certain sense) anti-realist. To this end, categoricity will have a reprise not as a sufficient, but as a necessary condition for determinacy when it is coupled with an assumption of realism.

Section 1 states the categoricity theorems. Section 2 discusses their philosophical significance, while section 3 addresses internal categoricity. Section 4, discusses the notion of realism and its role in the debate on determinacy.

## 3.1. The Categoricity theorems

In this section, I will state the relevant categoricity theorems and afterwards discuss their philosophical interpretation. The central definition that we will be working with in this section is that of an isomorphism between structures.

**Definition 3.1.** Let $\mathcal{M} = \langle M, R^{\mathcal{M}}, f^{\mathcal{M}}, c^{\mathcal{M}} \rangle$ and $\mathcal{N} = \langle N, R^{\mathcal{N}}, f^{\mathcal{N}}, c^{\mathcal{N}} \rangle$ be two structures for the same signature. A bijection $h : M \to N$ is an **isomorphism from $\mathcal{M}$ to $\mathcal{N}$** if and only if for each constant symbol $c$, relation symbol $R$, and function symbol $f$, and for all $a_1, \ldots, a_n \in M$:

$$
\begin{aligned}
& h(c^{\mathcal{M}}) = c^{\mathcal{N}} \\
& R^{\mathcal{M}}(a_1, \ldots, a_n) \text{ iff } R^{\mathcal{N}}(h(a_1), \ldots, h(a_n)) \\
& h(f^{\mathcal{M}}(a_1, \ldots, a_n)) = f^{\mathcal{N}}(h(a_1), \ldots, h(a_n))
\end{aligned}
$$

Write $\mathcal{M} \cong \mathcal{N}$ to express that there is an isomorphism between $\mathcal{M}$ and $\mathcal{N}$.

A theory $\mathsf{T}$ is said to be **categorical** if and only if all its models are isomorphic. In this sense, then $\mathsf{T}$ is said to pick out a unique structure. A paradigmatic example of this is Second order Peano Arithmetic $\mathsf{PA2}$.

### The categoricity of $\mathsf{PA2}$

$\mathsf{PA2}$ consists of the usual first order axioms plus a second order Comprehension Schema:

$$\exists X \forall v (\varphi(v) \leftrightarrow X(v))$$

for every formula $\varphi$ which does not contain $X$ free, and the Induction axiom

$$\forall X([X(0) \land \forall x(X(x) \to X(Sx))] \to \forall x X(x))$$

The Categoricity of PA2 was proved in Dedekind (1888).

**Theorem 3.1** (Categoricity of PA2)**.** *Let* $\mathcal{M} = \langle M, S_1, +_1, \cdot_1, 0_1 \rangle$ *and* $\mathcal{N} = \langle M, S_2, +_2, \cdot_2, 0_2 \rangle$ *be two full models of* PA2*. Then* $\mathcal{M} \cong \mathcal{N}$*.*

The proof requires all the listed second order principles. See, for instance, Shapiro (1991, p. 82) and Button and Walsh (2018, p. 155).

This theorem may be evoked to countenance the charge of indefinite extensibility issued in chapter 1.4. But note that it distinctly leaves the realm of recursive enumerability. Insofar as this requirement is viciously upheld (as it is the case for the argument analysed in chapter 8), the categoricity theorem is not available.

A more challenging case to assess is the purported effect of categoricity on the determinacy of CH. For ZFC2, the second order variant of ZFC, we can prove a quasi-categoricity theorem which asserts that any two of its models are either isomorphic or one is an initial segment of the other. This issue warrants a closer reading of its proof.

### The quasi-categoricity of ZFC2

ZFC2 consists of the axioms of Extensionality, Pairing, Union, Powerset, Foundation, the full second order Comprehension Schema (like above) and the axioms of Separation

$$\forall x \forall Y \exists y \forall w (w \in y \Leftrightarrow [w \in x \land Yw]),$$

and Replacement

$$\forall G \forall w (\forall x \in w \exists! y G(x, y) \to \exists z (\forall x \in w)(\exists y \in z) G(x, y))$$

Strictly speaking, Separation is redundant in the presence of second order Replacement. I still state both explicitly to make their individual contributions to the quasi-categoricity result salient.

To prove the quasi-categoricity theorem, we will use the definition of the set theoretic universe $V$:

$$V_0 = \emptyset \qquad V_{\alpha+1} = \mathscr{P}(V_\alpha) \qquad V_\lambda = \bigcup_{\beta<\lambda} V_\beta, \quad \lambda \text{ is a limit ordinal}$$
$$V = \bigcup_{\alpha \in Ord} V_\alpha.$$

Here $\mathscr{P}(\cdot)$ is understood as the 'actual' powerset operation. With this understanding, we can now show:

**Theorem 3.2** (Zermelo 1930)**.** $\langle V_\kappa, \in\rangle \vDash \mathsf{ZFC2}$ *iff* $\kappa$ *is an inaccessible cardinal.*

The point where the second order perspective becomes relevant is to establish that the models in question are well-founded, closed under subsets, and verify every instance of Replacement. In the following I will only look at the Powerset axiom and the axiom of Replacement, which are of importance for the purpose of this chapter. The proofs are largely taken from Button and Walsh (2018, 186f.). For a complete account of these proofs, the reader is referred to their book.

*Proof.* For the right to left direction, assume that $\kappa$ is inaccessible.

For the Powerset axiom, it is enough to assume that $\kappa$ is a limit ordinal, and we can show that if $a \in V_k$, then $\mathscr{P}(a) \in V_\kappa$. If $a \in V_\kappa$ then $a \in V_\gamma$ for some $\gamma < \kappa$. This means that for any $b \subset a$, $b \in V_{\gamma+1}$ and hence $\mathscr{P}(a) \in V_{\gamma+2}$. Furthermore, by absoluteness of subsethood, it is the case that if $V_\kappa \vDash b \subset a$, then $b \in \mathscr{P}(a)$, and conversely, if $b \in \mathscr{P}^{V_\kappa}(a)$, then $b \subset a$ and hence $b \in \mathscr{P}(a)$. Thus $V_\kappa$ is (unsurprisingly) right about all subsets of $a$.

Regarding Replacement, let $G$ be a function whose domain is $a \in V_\kappa$. Then $|a| < \kappa$. Now, let $b \in V_\kappa$ be the image of $G$, since $|b| \leq |a| < \kappa$ it is the case that $b \in V_\kappa$.

The left to right direction is the crucial one. If $V_\kappa \vDash \mathsf{ZFC2}$, then we know that $\kappa$ is a cardinal larger than $\omega$, since $V_\kappa$ must contain an infinite ordinal.

That $\kappa$ is regular follows from Replacement. Consider any map $F : \beta \to \kappa$ for some $\beta < \kappa$. Then $F(\beta) = a \in V_\kappa$. Since $\kappa$ is a limit ordinal, we have that $a \in V_\gamma$ for some $\gamma < \kappa$ and thus $a \subset \gamma$, which shows that $F$ is bounded in $V_\kappa$.

Now, pick any $a \in V_\kappa$, then $\mathscr{P}^{V_\kappa}(a) \subset \mathscr{P}(a)$ follows by transitivity and absoluteness and $\mathscr{P}(a) \subset \mathscr{P}^{V_\kappa}(a)$ follows by second order Separation. For suppose that $X \subset a$, then $X \subset a \subset V_\kappa$. By Separation, it follows that $X \in V_\kappa$.

□

Especially the last argument in the proof shows how ZFC2 captures the full powerset using second order Separation. Zermelo's quasi categoricity theorem is now a direct consequence of the following:

**Theorem 3.3** (Quasi-categoricity of ZFC2). *If $\mathcal{M} \vDash$ ZFC2 then $\mathcal{M} \cong V_\alpha$ for some (limit) ordinal $\alpha$.*

*Proof.* By second order separation we know that $\mathcal{M}$ is well-founded. Hence, by the Mostowski Collapse lemma, $\mathcal{M}$ is isomorphic to a transitive model $\mathcal{N}$.

We thus need to show that $\mathcal{N} = V_\alpha$ for some (limit) ordinal $\alpha$. This is shown by induction on the ordinals in $\mathcal{N}$. The crucial bit is, again, the powerset axiom.

It is trivially the case that $\emptyset^{\mathcal{N}} = \emptyset = V_0$. Thus assume that $V_\gamma^{\mathcal{N}} = V_\gamma$ for some ordinal $\gamma$. Then for any $b \in V_\gamma^{\mathcal{N}}$, we need to show that $\mathscr{P}^{\mathcal{N}}(b) = \mathscr{P}(b)$. That $\mathscr{P}^{\mathcal{N}}(b) \subset \mathscr{P}(b)$ follows by transitivity and absoluteness. To show the converse, suppose that $X \subset b$. Then by second order separation $X$ exists in $\mathcal{N}$. Hence $\mathscr{P}^{\mathcal{N}}(V_\gamma^{\mathcal{N}}) = \mathscr{P}(V_\gamma)$. Now, if $\lambda$ is a limit ordinal in $\mathcal{N}$ and $V_\gamma^{\mathcal{N}} = V_\gamma$ for all $\gamma < \lambda$, then $\bigcup_{\gamma<\lambda} V_\gamma^{\mathcal{N}} = \bigcup_{\gamma<\lambda} V_\gamma$.

Finally, let $\alpha$ be the least ordinal not in $\mathcal{N}$. Then $B \subset V_\alpha$, because if $a \in B$ then $a \in V_\gamma^{\mathcal{N}} = V_\gamma$ for some $\gamma < \alpha$. But also $V_\alpha \subset N$, because if $b \in V_\alpha$, then $b \in V_\gamma = V_\gamma^{\mathcal{N}}$ for some $\gamma < \alpha$. Thus $\mathcal{N} = \langle V_\alpha, \in \rangle$ and from this it follows that $\mathcal{M} \cong \langle V_\alpha, \in \rangle$. □

Again, it is second order Separation that makes sure that any models get the subsets right.

Unlike for PA2, we only get this quasi-categoricity result and not full categoricity. We can force full categoricity by adding axioms of the form

'There exists no inaccessible cardinal'
'There exists exactly one inaccessible cardinal $\kappa$'
'There exist exactly two inaccessible cardinals $\kappa$'
and so forth...

to ZFC2. Each corresponding categorical model, however, has the size of the next bigger inaccessible cardinal whose existence is excluded by the respective axioms (cf. Hamkins and Solberg 2020). But such a limitation seems artificial with respect to the discussion in chapter 1.3. If we can acknowledge the size of each of these models to be the next inaccessible, it seems uncalled for to assert their non-existence (which is also Zermelo's own interpretation of his theorem). In this respect, then, categoricity does not extend to indeterminacy of height. Furthermore, quasi categoricity has little to do with the fact that ZFC is a very powerful system. As the presentation of the proofs should have made salient, the fact that the respective models are all inaccessible ranks is mostly due to the presence of Replacement. The absence of replacement in the last proof shows that without it, the same arguments could have been given for $V_\alpha$ with $\alpha$ being a limit ordinal. For instance, let Z2 = ZFC2 - Replacement, then it is already the case that $V_{\omega+\omega} \models$ Z2.

## 3.2. Philosophical Significance of Categoricity Results

A first pass on the philosophical significance of categoricity results can be gotten by observing what distinguishes categorical from non-categorical theories. The axioms of group theory, for instance, are not categorical. This, however, is not seen as a defect of any sort. To explain why, we may say that the group axioms characterise a *general* structure while the axioms of PA(2) and ZFC(2) characterise a *particular* one. The group axioms are general in that they characterise similarities between mathematical objects or structures that are given *prior* and *independently* of axiom systems like PA(2) and ZFC(2) are more closely connected to our conception of the objects or structures that they characterise. In a way, the whole debate regarding the relationship between categoricity and determinacy that now follows can be understood as an attempt to clarify this connection.

Thus, given a theory T that is intended to characterise a certain structure or collection of objects. A categoricity theorem for T can be understood as (involved in) guaranteeing that T is indeed able to do so. From the outset, this guarantee can be more or less extensive. It could be understood (cf. Martin 2001, Meadows 2013)

a) to guarantee the *existence* of the structure in question, or

b) to guarantee its *uniqueness*, as far as its existence is not disputed.

There is a general doubt that categoricity can serve to do a) (cf. Walmsley 2002, Meadows 2013). Martin (2001) admits that without any existence assumption one cannot really get far in postulating any model, except noting that set collection goes as far as there are sets (cf. p. 8). In a similar vein, Isaacson (2011) acknowledges that "[t]here is no way that the existence of any particular structure can be proved"(p. 37). Thus, the commonplace is to go for b). In the following, I will consider positive arguments to that effect and afterwards objections to them. At the end of this section, we will see that categoricity is not up to the task of guaranteeing uniqueness or determinacy, either. In the final section of the chapter, however, it will make a reprise as a necessary condition of determinacy when paired with a suitable notion of realism.

### 3.2.1. Using categoricity as a notion of determinacy

Perhaps the most elaborated standpoint on the use of the categoricity theorems to secure uniqueness and determinacy is the structuralist viewpoint advocated in Isaacson (2011). According to this proposal, we start out with an informal understanding of a mathematical structure like the natural numbers or the sets. A first step is then to characterise them by axiomatisation using second order resources. If the axiomatisation turns out to be categorical, then we have succeeded in grasping the particular structure in question. In Isaacson's words: "We need second order categorical considerations such that we can say that there is clarity in the structure we are dealing with"(Isaacson 2011, p. 39).[1]

Isaacson's argument makes heavy use of Georg Kreisel's notion of "informal rigour"(Kreisel 1967a). Isaacson writes: "Categoricity gives us the particularity of the particular structure. Informal rigour is how we know what we are talking about."(Isaacson 2011, p. 32) Unfortunately, Kreisel does not offer a sharp definition of the term. But looking at

[1]See also Rumfitt (2015) who similarly argues that the idea of having grasped (a conception that yields) a determinate domain is *made explicit* by categoricity (p. 265-7). A similar point has been made in Martin (2001).

some of his remarks we can determine some essential characteristics and their bearing on the question of determinacy.[2] Informal rigour refers to the idea that our *informal* notions of 'set' and 'number' are not only considered to be significant, but even more representative of the concept than their subsequent axiomatisations (whose categoricity merely *confirms* (but does not establish) that we are indeed dealing with something determinate). Antecedently to axiomatisation, informal rigour thus sets two aims: to "eliminate doubtful properties of intuitive notions" and "not to leave undecided questions which can be decided by full use of evident properties of these intuitive notions"(Kreisel 1967a, p. 138)

One such example may be the informal notion of natural number as the smallest set closed under the successor function, whose determinacy is reflected in the categoricity of PA2. We have an informally clear notion of natural number, so the story goes, and because of that, the second order perspective delivers a categoricity result. Rumfitt (2015) in this respect speaks of a "coherent story"(p.286) that someone defending determinacy is able to tell.

For Kreisel, another example of a clear and significant informal notion is that of the cumulative hierarchy that has emerged from the discussion of the paradoxes in Zermelo's 1930 formalisation. According to Kreisel, it achieves the separation of the notions of 'abstract property' and 'set of something' (nowadays often called the logical and the combinatorical concept of set), whose conflation (in the naive concept of set) may be understood as a source of the paradoxes. Once separated from the notion of 'abstract property', the notion of 'set of something' becomes "marvellously clear and comprehensive"(Kreisel 1967a, p. 143). In particular, this notion of 'set of something' in combination with the idea of the cumulative hierarchy "provides a coherent source of axioms", which, however, only "played an auxiliary role"(Kreisel 1967a, p. 146) when it comes to understanding and articulating the informal notion.

Now, this informally rigorous notion of the cumulative hierarchy licenses the use of second order logic and at the same time is reflected in it.

> A moments reflection shows that the evidence of the first order axiom schema de-

[2] See also the study by Dean and Kurokawa (2021) that gives a (unifying) formalisation of this notion by pooling together most centrally the works Kreisel (1965), Kreisel (1967a), Kreisel (1967b).

> rives from the second order schema: the difference is that when one puts down the first order schema one is supposed to have convinced oneself that the specific formulae used (in particular, the logical operations) are well defined in any structure that one considers.(Kreisel 1967a, p. 148)

In other words, our notion of "all subsets" is clear *enough* by virtue of our conception of the cumulative hierarchy, which is expressed by its determinacy in the second order formulation.

But in what sense is it clear enough? Kreisel explicitly acknowledges that whoever is sceptical about the powerset operation will likewise be sceptical about the determinacy of the second order quantifiers (cf. Kreisel 1969, endnote 2). Furthermore, highlighting the difference between the first and the second order, Kreisel suggests on several occasions that "any decision [of CH] would involve considerations of a quite different character from those which have led to existing axioms"(Kreisel 1969, p. 108, cf. Kreisel 1965, Kreisel 1967b). Nonetheless, Kreisel claims that the categoricity results deliver some assurance that this is indeed plausible. This leads to a number of questions:

(i) What kind of evidence do Kreisel and the others have that suggests that second order logic *reflects* if not *secures* determinacy to this end?

(ii) If we follow Kreisel in his claim that first order understanding already requires second order understanding (as illustrated above), but grant at the same time that there is a distinction between first and second order, and that the method of forcing exploits this, how do we interpret (this possibility of) the forcing constructions?

(iii) Kreisel notes that we need to go beyond ZFC to settle the question of CH, perhaps even beyond the iterative conception. What does this look like and, importantly, why does a categoricity theorem show that there is only one way of doing so?

Regarding (i), the main argument by Isaacson and Kreisel to the claim that one can rely on (quasi)categoricity to testify that CH has a determinate truth value is rejecting the analogy of CH to phenomena like the unsolvability of Euclid's parallel postulate or the independence of the axiom of Replacement from the system Z (see chapter 2). This analogy is commonly used by opponents of the view that CH has a determinate truth value (e.g. in Hamkins 2012). The parallel postulate is still independent from the second order axioms of geometry, and so is the axiom of Replacement from Z2.[3] CH, on the other hand, is decided even in Z2.[4] While this difference is certainly striking, it remains to be assessed if the second order resources can be understood to be itself determinate enough to make this difference relevant. This point will be revisited further below.

(ii) What are we to make of the forcing constructions? Isaacson does not discuss the forcing constructions in much detail, other than noting that forcing models are usually treated as non-standard models. Somewhat more detail we find in Martin (2001) (which is also echoed in Koellner (2013) and Koellner (2017)). As noted in chapter 2, forcing can be done within ZFC using countable transitive models or 'from the outside' using boolean valued models. Martin rejects forcing on countable transitive models as a proof to indeterminacy because these constructions are done *within* ZFC and because they are not 'the real thing'. Even if these models are well-founded (which they are usually specified to be), judging from an external perspective, they are countable and hence don't contain all possible subsets of certain sets. Nonetheless these models can be taken to reflect essential properties of $V$. This idea is the so-called toy model perspective is not an isolated phenomenon that is only connected to interpreting forcing constructions, but occurs on a wider scale as well (for instance, in Hamkins (2012) and Hamkins (2014)).[5] On this account, each universe may be understood to be a separate instantiation of the set concept (or reflect essential properties of it), and in this respect, it may be taken to be one way in which the notion of set theoretic indeterminacy can be explicated. The toy-model perspective will be revisited throughout the thesis.

Regarding $V^{\mathbb{B}}$, the immediate criticism is indeed that it is boolean valued and thereby differs from standard $V$. To this, however, Martin adds: "What does seem plausible is the assertion that Boolean-valued extensions are epistemically indistinguishable from $V$"(Martin 2001, p. 15). He does not specify what he means by "epistemically indistinguishable", but it seems clear that since $V$ and $V^{\mathbb{B}}$ are both models of ZFC, they cannot

[3]This is due to the fact that Z2 is satisfied by $V_{\omega+\omega}$ while the (second order) axiom of Replacement is not.

[4]This is because CH is expressed via quantification over subsets of $\mathscr{P}(\omega)$, which are all collected at $V_{\omega+2}$.

[5]The 'naturalistic account of forcing' of Hamkins (2012) falls under this perspective as well.

be distinguished from its perspective. In other words, ZFC cannot exclude the possibility that we assign degrees of belief or certainty to our set theoretical statements. Then, of course, a cardinality claim may not only count complete convictions, but could take into account all the degrees of belief or finite pieces of information as well. In other words, we cannot distinguish exactly what to count (by the formal means given so far).

Now, it might be the case that we will not succeed in sharpening this conception (without artificially reducing it) to be able to tell $V$ and $V^{\mathbb{B}}$ apart (or at least the cardinality claims in it). This could explain why Martin writes that "[i]f my feeling is right, then it is epistemically possible that not all sentences of set theory have truth values."(Martin 2001, p. 15) Presumably, Martin's use of epistemic possibility refers to the impossibility of distinguishing $V$ and $V^{\mathbb{B}}$ from 'within' ZFC. But this need not mean that there is no more to say on the matter given resources that go *beyond* ZFC but still pertain to our informally rigorous conception of the cumulative hierarchy.

This leads to point (iii). As mentioned in the previous chapter, any sharpening or explication of our informal notion of set goes by way of introducing another axiom to that effect. But in light of what has been said in chapter 2.3.2, how do we know that there is only one way of doing so? Furthermore, if axiomatisation is a necessary requirement of concept formation in this respect, then corresponding to different possible axiomatic extensions of ZFC there could exist different notions of set that would need to be separated (just like the notions of 'abstract property' and 'set of something'). According to Kreisel's position, however, this cannot be the case, since our second order categoricity theorem tells us that our concept is informally rigorous to begin with and does not allow for such differentiation.

Thus, the categoricicist's position with respect to set theory can be characterised as the claim that there is a unique universe and some unique extensions of ZFC that a) is informally motivatable, and b) that this is reflected in the categoricity results. But can the second order logic that is used in the categoricity theorem really guarantee this much? When we now look at critics of the categoricity position, this is what we need to find out.

### 3.2.2. Problems with categoricity as determinacy

The merit of categoricity theorems has been criticised extensively in the literature. But it remains to be seen which of the objections actually work against the case at hand, i.e. the overall claim that our notion of 'set of something' is informally rigorous enough to equip CH with a truth value, and that this is reflected in its second order determinacy.

One common objection is that simply knowing that two models of a certain system are isomorphic does not give us a conception of the model itself. This can be put in conjunction with the observation that second order logic has an incomplete proof system, which means that even if a statement of second order logic were (semantically) determinate, it does not entail that *we* can determine them (cf. Field 2003, Hamkins and Solberg 2020). These points, however, do not cause much trouble to the above position, since it was readily admitted that we need some extension of ZFC (and thus to the proof system) to settle CH. It was 'just' claimed that these extensions are informally rigorously motivatable.

A better objection to the claim that categoricity explicates informal rigour is that this whole business of fixing determinacy via second order logic is hopelessly circular, for in order to determine the extension of our second order systems we need to require what we aim to show. This claim was put forward in different ways by Meadows (2013), Hamkins (2021), and Koellner (2017). The problem arises for PA2 as well as for ZFC2. For ZFC2 it can be elucidated in the following way. In the characterisation of $V$ given in section 3.1 we made free use of the 'actual powerset', and the proof of the categoricity theorem amounted to the fact that second order separation managed to capture all the subsets thus introduced. But without an independent characterisation of the 'actual powerset' operation, this amounts to the simple claim that there are as many subsets (say of $\omega$) as there are. Now, it was already acknowledged by Kreisel, that doubt about the determinacy of the 'actual' powerset operation may encompass doubt about the determinacy of the second order quantifiers, such that the above may be understood as a package deal: determinacy of the 'actual' powerset operation is tantamount to determinacy of the second order quantifiers. But we might doubt that the informal notion of 'arbitrary subset of the natural numbers' as it is part of the notion of the cumulative hierarchy, for instance, is rich enough, and for this, critics of categoricity may demand an argument that our semantic understanding of second order logic is in fact up to the task.

This point becomes even more acute due to the following observation made in Koellner (2010). Define $\mathcal{V}_{SO} = \{\ulcorner\varphi\urcorner \mid \emptyset \vDash \varphi\}$ where $\vDash$ is the consequence relation with respect to full second order structures. By a theorem of Väänänen (2001) we know that, under the assumption of ZFC, any $\Pi_2$ definable set of integers is Turing reducible to $\mathcal{V}_{SO}$. This follows from the fact that $\mathcal{V}_{SO}$ is $\Pi_2$-definable over $V$, and if $X \subset \omega$ is $\Pi_2$ definable over $V$, then it is Turing reducible to $\mathcal{V}_{SO}$. Thus, we get the following equivalence:

$$Th_{\Pi_2}(V) \equiv_T \mathcal{V}_{SO}$$

where $Th_{\Pi_2}(V)$ is the set of $\Pi_2$-definable truths over $V$. However, given a forcing extension $V[G]$, we similarly have

$$Th_{\Pi_2}(V[G]) \equiv_T (\mathcal{V}_{SO})^{V[G]}$$

This shows that what we take to be expressible in second order logic is not invariant to forcing. In other words, since the validities of second order logic may be expressed via $\Pi_2$ definability over the universe, and what we can say given those $\Pi_2$ resources may vary with forcing, second order logic itself may vary alongside it.

Is there a reprise for the categoricicist to this? They may respond by rejecting the eligibility of forcing constructions for this in the way previously suggested, and consider sets like $Th_{\Pi_2}(V[G])$ to be defective in some kind—after all, all this follows from the assumption of only ZFC, and there should be a way to fix the domain over which the $\Pi_2$ sets are defined. At this point, the dialectic between the two starts to become a matter of who shoots first. Categoricicists undermine forcing as testifying to a mere epistemic indeterminacy, while their critics undermine the categoricity theorems on the basis of variability along forcing. Yet, it does become difficult to see how second order logic alone manages to secure and indeed even *reflect*(!) determinacy in light of the observation that it too can vary according to a background theory. So, even though second order logic highlights a difference between cases like CH and the Parallel Postulate or Replacement, this does not mean that there is no other form of indeterminacy which it fails to detect.

But there is a problem now: If our informally rigorous first order notion happens to be not so rigorous with respect to the interpretation of the powerset operation (as the above argument illustrates), then the use of second order resources becomes doubtful. It is hard to see what bearing a categoricity theorem has on on whether there are different plausible axiomatic extensions of ZFC or just one. Furthermore, insofar as we take each universe (of a suitably specified collection of universes) to be a separate instantiation of the set concept, then we might wonder if using second order logic in this manner might also *obscure* different ways of sharpening the notion, i.e. it might fulfil a similar role as our naive conception of sets, in which the notions of 'abstract property' and 'set of something' were not properly separated. Hence, the appeal to categoricity theorems to secure uniqueness and determinacy is not up to the task.

## 3.3. Internal Categoricity

Thus, the claim that categoricity secures determinacy can be resisted. But that doesn't mean that there are no other ways to secure determinacy. One might argue that the fault of the foregoing approach lay in the illegitimate reliance on second order *semantics* in any kind of way. But perhaps from a deductive perspective alone, determinacy can be secured. This is the approach of *internal categoricity.*

Barring any kind of semantics, but using second order logic, one can express that there is a set which is **Dedekind infinite**, i.e. that has an injective but not surjective function into itself. We can further define the successor relation $S$, a predicate $N$ that is the smallest one closed under it, and an element $z$ that has no successor in $N$. Then, let $PA(N, S, z)$ be the conjunction of the Peano Axioms relativised to $N, S, z$, where e.g. uniqueness of successor is expressed as

$$N(z) \wedge (\forall x(N(x) \rightarrow \exists! y(N(y) \wedge S(x) = y))).$$

Call such a $PA(N, S, z)$ a **Peano Structure**. Now, from the assumption that there exists a Dedekind infinite set, one can derive a second order sentence which says that that there are such $N, S, z$ which form a Peano Structure. Furthermore, there exists a formula $ISO(F, U, V)$ in second order logic saying that $F$ is an isomorphism between $U$ and $V$. With all this in place one can prove the internal categoricity of $\mathsf{PA}$, which is the following statement.

**Theorem 3.4.** (Väänänen and Wang 2015)

$$\vdash_2 \forall N_1, S_1, z_1, N_2, S_2, z_2[(PA(N_1, S_1, z_1) \wedge PA(N_2, S_2, z_2)) \rightarrow \exists F\, ISO(F, N_1, N_2)]$$

The proof of this theorem makes essential use of second order comprehension, but since it is a deductive theorem, it holds in all models of second order logic, including Henkin Models. The theorem also implies Dedekind's categoricity theorem for full semantics. However, structures that satisfy it need not be isomorphic to each other (it is only required that each of these structures individually satisfies that there is an isomorphism between its instantiations of $N_1$ and $N_2$). Furthermore (insofar as Henkin Semantics is used), we also don't avoid (Gödelian) incompleteness, and $PA(N, S, z)$ for fixed $N, S, z$ does not prove $Con(PA(N, S, z))$.

We do, however, get what Button and Walsh (2018) have called *intolerance.*

**Theorem 3.5.** (Button and Walsh 2018, p. 245) *Let $\varphi(N, S, z)$ be a formula with quantifiers restricted to $N$ and in which $N, S, z$ are the only free variables. Then it holds that*

$$\vdash_2 \forall N, S, z[(PA(N, S, z) \rightarrow \varphi(N, S, z))] \vee \forall N, S, z[(PA(N, S, z) \rightarrow \neg\varphi(N, S, z))]$$

The upshot of this is the following. We can fix any such $PA(N, S, z)$ and (since they are all isomorphic) use it to reason with all structures simultaneously, since whatever is true of one such structure has to be true of all of them. This is different to the internalised version of the group axioms and of the group structures in general, where intolerance fails.

Even though we don't use any semantics for second order logic, one might still object that the proofs of these theorems make essential use of the impredicative second order comprehension schema. As a possible response to this, there is a first order version of these theorems available as well. Thus consider two first order theories $PA_1(N_1)$ and $PA_2(N_2)$ with their own vocabularies $\{0_1, 1_1, +_1, \cdot_1\}$ and $\{0_2, 1_2, +_2, \cdot_2\}$ defined on $N_1$ and $N_2$, respectively. To prove a version of internal categoricity for these two, we would need to define a function $\psi$ which acts as an isomorphism. In order to do this, we need each theory to allow the vocabulary of the other in their respective induction schema, and, since $\psi$ relies on coding, we require an additional version of PA to prove that it has the required properties. We then get the following theorem:

**Theorem 3.6.** (Väänänen 2021)

$$PA \cup PA_1(N_1) \cup PA_2(N_2) \vdash ISO_\psi(N_1, N_2)$$

Here the formulas stand to the left of the turnstile, because they are given via schemas and are thus infinitely many.

Before going into a philosophical discussion of these results, let us note that there is an analogue to each of these theorems for set theory. Let $\varphi^{(X_1)}(E_1)$ be a formula with quantifiers relativised to $X_1$ and its non-logical symbol (for elementhood) expressed by $E_1$. Then we can express the ZF axioms in this language by

$$ZF^{2(X_1)}(E_1) = \bigwedge\{\varphi^{(X_1)}(E_1) : \varphi(E_1) \in ZF(E_1)\}$$

and similarly for $ZF^{2(X_2)}(E_2)$. We also need another sentence $IA$ expressed in the vocabulary $\{X_1, E_1, X_2, E_2\}$ which says that the class of inaccessible cardinals of $(X_1, E_1)$ and $(X_2, E_2)$ is the same (this is to exclude the quasi-categoricity). Then one can prove the theorem:

**Theorem 3.7.** (Väänänen 2019)

$$\vdash_2 ZF^{2(X_1)}(E_1) \wedge ZF^{2(X_2)}(E_2) \wedge IA \to ISO((X_1, E_1), (X_2, E_2))$$

which also gives us a version of intolerance, such as

**Theorem 3.8.** (Button and Walsh 2018) *Let* $\Gamma = \bigwedge\{ZF^2 \cup \neg\exists \text{ inacc. } \kappa\}$, *then*

$$\vdash_2 \forall X \forall E(\Gamma^{(X)}(E) \to \varphi(X)(E)) \vee \forall X \forall E(\Gamma^{(X)}(E) \to \varphi(X)(E))$$

Similarly, there is a first order version of the result, i.e. there is a first order formula $\varphi(x, y)$ such that the following holds:

**Theorem 3.9.** (Väänänen 2019) *Let* $IO_\pi$ *be the sentence expressing that there is an isomorphism between the ordinals of* $(X_1, E_1)$ *and* $(X_2, E_2)$, *then*

$$ZF \cup ZF^{(X_1)}(E_1) \cup ZF^{(X_2)}(E_2) \cup IO_\pi \vdash ISO\varphi((X_1, E_1), (X_2, E_2))$$

Again, for this theorem it is not impredicative comprehension, but letting the vocabulary of one theory in the axiom schema of the other that makes for the desired result. Thus, these theorems show that there is a version of categoricity that can be explicated without the need of full second order semantics, and given some additional assumptions, even without second order logic at all.

But have we made any progress in securing determinacy? Against the charge of indefinite extensibility of theories, internal categoricity arguments don't work for the simple reason that they don't avoid Gödelian incompleteness. Regarding the first order versions of these results, we encounter a revenge problem since we require an additional theory apart from the two which we are trying to prove isomorphic—and thus the same question can be raised about that one. Regarding the second order versions, Maddy et al. (2022) note that since it is exclusively deductive it becomes hard to make a connection to any sort of *semantic* determinacy. In particular, they point out that intolerance asserts a disjunction without specifying any of the disjuncts. Since, in particular second order logic is deductively incomplete we have yet no way of determining which disjunct is true. A further axiomatic extension of ZF is required to settle it. But this is exactly a reformulation of the trouble we had to begin with.

## 3.4. Realism

In order to characterise the phenomenon of indeterminacy further, a remark on ontology should be made at this point. This section addresses the question of how the distinction between realism and anti-realism affects the handling of indeterminacy of the natural numbers and the sets. A first step is to note that there are two ways in which the notion of realism has been used: (i) as an explanation or feature regarding the subject matter of certain truths, henceforth called '*subject matter realism*', or (ii) as a particular stance on which logical principles are valid, the most prominent of which is the principle of bivalence, i.e. that any sentence has at most one of two truth values and, in some versions of it, only one truthmaker. Call this '*truth value realism*'.[6]

To give an example outside the realm of mathematics: A subject matter anti-realist might hold that the world only exists in the mind of a perceiver (or of god) instead of attributing its physical structure independent existence. Generally, however, a stance on this matter does not entail which logical principles in particular are to hold of the world. For instance, one could be a subject matter anti-realist regarding the existence of the world and still think the classical physical laws apply (in which case it just needs to be a particular strong mind to assure the sufficient determinacy) and one could be a subject matter realist about it and argue that the laws of quantum mechanics are in fact the correct ones.

The notion of realism that is incompatible with the argument from indeterminacy developed in this thesis is truth value realism. Accepting the principle of bivalence predetermines the acceptance of the law of the excluded middle and would thus undermine the whole project. As a consequence, any position regarding the subject matter of mathematics that is compatible with the argument from indeterminacy should not entail truth value realism. One position on the ontology of mathematics that does not do so is Feferman's own, according to which "the basic objects of mathematical thought exist only as mental conceptions"(Feferman 2014, p. 74) and mathematical objectivity "is a special case of intersubjective objectivity that is ubiquitous in social reality"(Feferman

[6]This is the way in which, for instance, realism is defined in Dummett (1991b). Note that this notion of realism differs from how others use term in the philosophy of set theory, e.g. in Venturi (2016) and in Antos, Friedman, et al. (2015) which is along the lines of (i).

2014, p. 75). This allows for the fact that mathematical reality including the extension or reference of certain mathematical concepts (like sets) is not predetermined entirely and may permit competing further developments. This can be combined with a view such as the one espoused in Hewitt (2015), where it is argued that what sets there are can only be judged upon what sets mathematicians consider there to be. The primary way of measuring this seems to be by what additional axioms they accept, or at least what candidates for axioms are seriously considered by the mathematical community.[7]

It should also be noted that truth-value realism (and consequently truth-value anti-realism) comes in degrees. The strongest realistic assumption in this respect is that of Universism, which is the position that there is one single universe of sets $V$ in which any set theoretical question can be settled (cf. Woodin 2011b). The universist offers an explanation of the plethora of different model constructions and universe extensions as all being interpretable with respect to and thus taking place in $V$. Arguments for this position can be claims of naturality, of categoricity, as well as considerations related to $\Omega$-logic and arguments against the coherence of the multiverse conception (cf. Woodin 2011a). But since there is, as of yet, no (axiomatic) characterisation of such a universe that settles all these questions, this position goes beyond what we are currently able to determine and assumes that our talk of the universe of sets is indeed determinate. Consequently, there is no restriction in the logical machinery.

The assumption of determinacy that the universist's position embodies can also be applied more gradually, for instance, with respect to the levels of $V_{\omega+1}$, or $V_{\omega+2}$ as discussed in chapter 2.3.2. For each of the two levels, truth value realism will turn into a significant claim if it is being held (i) that there are axiomatic extensions corresponding to the respective interpretability degree which have a similarly appropriate level of theoretical legitimisation but which are mutually inconsistent, and (ii) that nonetheless one of them is *the* true one. Having an appropriate level of theoretical legitimisation (e.g. being supported by the iterative conception of set (like certain reflection principles) or having it has intuitively compelling consequences for related areas of mathematics) seems to exclude unwanted statements like $\neg Con(\mathsf{ZFC})$ form the pool of candidates.

[7]Hewitt's point is epistemic in nature, but coupled with an ontology such as Feferman's, it can easily be adjusted to support the present approach.

In this respect, the notion of categoricity becomes useful. As Barton (2022) argues, since there is no categoricity theorem for $\mathsf{ZFC2} + \neg Con(\mathsf{ZFC2})$, such an axiomatic extension can be excluded by requiring categoricity as a *necessary* condition. This allocates the appropriate task to the categoricity theorems which which this chapter has started. Barton (2022) points out that there are categoricity theorems for these intermediate positions that are given in a logic weaker than full second order.[8] Someone who is in doubt of determinacy already at $V_{\omega+1}$ might express their views in a **weak second order logic**, which is second order logic minus function variables, where the second order quantifier ranges only over finite relations, and where comprehension is only available for formulas with finite extension. ZFC2 in weak second order logic yields categoricity up to the rational numbers, but not including real analysis. Someone who thinks that $V_{\omega+1}$ is indeed determinate (e.g. through the acceptance of $\mathsf{AD}^{L(\mathbb{R})}$) can express their view using **quasi weak second order logic**, which extends weak second order logic in assigning countable relations to the second order variables. ZFC2 in quasi weak second order logic yields categoricity and well-foundedness up to real analysis, but not on the level of $V_{\omega+2}$.

Thus, since truth value realism may be expressed using (weaker forms) of second order logic, what about truth value anti-realism? The indeterminacy of the set concept may be understood to be reflected in *different* possible axiomatic extensions of ZFC or it might be understood to be given by a multiverse of equally legitimate instantiations of the set concept. With respect to the latter, there are different ways the position can be spelled out (cf. Barton 2021), turning on what exactly the multiverse is allowed to contain. Examples of such are Hamkins (2012), Steel (2014), Woodin (2011a), and Väänänen (2014).[9] These and their relevant differences will be discussed in more detail in chapter 7. All these, however, agree in that there are certain statements that do not have a "global" truthmaker and not even a global truth-value and that that makes them indeterminate.

For the argument developed in this thesis, of course, only those anti-realist positions are of interest, for only they have room for the possibility of deviation from classical

[8]For technical details, see Barton (2022) and Shapiro (2001).
[9]These versions may also differ with respect subject matter realism (cf. Antos, Friedman, et al. 2015).

logic. However, just because set theoretic truths, for instance, might not be bivalent or might not have unique truth-makers in the multiverse doesn't automatically entail that LEM has to fail with respect to reasoning with them. It is not directly observable which semantics and which logic these conceptions licence, especially since we have not yet considered what truth and generality for such cases of indeterminacy amount to. Only once such a conception is provided, the question of LEM can be addressed.

## 3.5. Conclusion

In this chapter I have reviewed attempts to secure determinacy in light of the technical results discussed in chapters 1 and 2. I have considered determinacy by reliance on categoricity, internal categoricity, and the broader notion of determinacy as a realist assumption. The discussion of categoricity and related attempts has shown that—due to the indeterminacy of the higher order resources that it relies on—their attempt turns out to be circular in an unacceptable way. A similar conclusion was reached with respect to the use of internal categoricity results. Finally different realistic assumptions and their bearing on determinacy were discussed. This was observed to be compatible with taking categoricity as a necessary condition of determinacy. This served the purpose of classifying the argument against LEM as based on an anti-realisitc premise, in the particular sense that bivalence and uniqueness of truthmakers was not explicitly assumed. Whether or not the lack of bivalence in these respective cases entails a rejection of LEM, however, is a question that can only be answered by inquiring into the particular semantics of indeterminacy.

# Part II.

# LEM

# 4. Traditional Intuitionism and Truth-conditional Semantics without LEM

Now that the conceptions of indeterminacy and their broader connection to questions of ontology are sufficiently clear, it is time to look more closely to LEM. There already exists a variety of arguments against the applicability of LEM, and it is reasonable to ask in what way these arguments differ from any arguments that can be put forward based on the notion of indeterminacy.

The first section of the chapter is devoted to traditional arguments against LEM with the aim of demarcating these from the argument against LEM based on indeterminacy. The crucial difference between traditional arguments against LEM and the argument from indeterminacy is the fact that traditional arguments rely on incorporating (broadly understood) epistemic notions into the interpretation of the logical constants, while the argument from indeterminacy is still based on an interpretation of the logical constants with respect to truth conditions. This will turn out to avoid some major problems that traditional intuitionism faces, but it also leaves us with the following dilemma. The semantics for traditional intuitionism manages to articulate a failure of LEM, but in doing so, it relies on an interpretation of the logical constants that is not suitable for our case. On the other hand, given the usual bivalent semantics for classical logic, it is not possible to analyse LEM.

The challenge is thus to find an independent framework in which LEM can be evaluated such that neither the intuitionistic nor the classical bivalent meaning of the logical constants is presupposed. This will be presented in the second section of the chapter. To this end, I will introduce the notion of a *warrant* which permits a classical as well as an intuitionistic understanding and I will formalise the interpretation that it suggests by using topological models, which have the right degree of specification to pinpoint what is exactly at issue with the validity of LEM.

## 4.1. Traditional Intuitionism

The term 'traditional intuitionism' will be used as an umbrella term for all those positions that reject LEM by incorporating a notion of proof or construction into the evaluation of logical formulas. This section will review some of the arguments to this effect and characterise the common semantics that are used to explicate the position. It will also characterise some important non-validities and two major problems that these accounts face.

### 4.1.1. Epistemic arguments against LEM

When it comes to motivating intuitionsitc logic, we can distinguish between a non-revisionist way where intuitionistic logic is considered as a logic of provability *alongside* the classical logic of truth (cf. Kolmogorov 1932, Melikhov 2017), and a revisionist way, according to which intuitionistic reasoning *replaces* a faulty classical one. Obviously, only the latter one is of interest here. Within this camp, recent literature (Sundholm 1994, van Atten 2004, Sundholm and van Atten 2008, van Atten 2018) suggests a division between a (Brouwerian) ontological conception and a meaning theoretic (better: epistemic) one (Dummett 1983, Dummett 2000, and Martin Löf 1996). Both of these conceptions, we will now see, do not match with the setting of indeterminacy.

Brouwer's intuitionism rests on an ontological thesis about the nature of mathematical objects. For Brouwer, mathematics is a languageless creation of the mind (cf. Brouwer 1928, Brouwer 1981), and proofs are mental entities (cf. Brouwer 1967). Language in itself only serves as an imperfect means of communication and should not be hypostasised as providing an adequate reflection of mathematical activity. This set-up naturally leans towards a rejection of LEM, especially when it comes to quantified statements over infinite domains. On the assumption that there is a refutation of a universal claim for such an infinite domain, there is no guarantee that this can be turned into the construction of a counterexample, and hence the inference from $\neg\forall x\varphi(x)$ to $\exists x\neg\varphi(x)$ fails on Brouwer's terms.[1] However, it should be clear that this ontological premise is too strict and completely overpowers the subtleties of the instances of indeterminacy that we are concerned with.

Moving on to the epistemic arguments, the backbone of Dummett's criticism of classical logic in Dummett (1983) and Dummett (2000) aims to develop a so-called *molecular* theory of meaning for the logical constants. Contrary to its opposite, the *holistic* approach according to which the meaning of any linguistic expressions is exclusively given by the way they interact with *all* the other expressions, a molecular theory of meaning entertains primitive or molecular expressions whose meaning is previously fixed and rather *constitutes* the way they interact with other expressions. To fix the meaning of those molecular expressions, Dummett claims that we require an explanation of what it means for someone to use them correctly. The ability to correctly use an expression, the argument goes, is to be understood as the ability to recognise when a sentence containing this expression can be asserted truthfully. Thus, one needs to account for how someone could have the knowledge of the truth that the sentence containing the relevant expression can be used to assert. The truth condition of such a sentence therefore must be "one which [the speaker] is capable of recognizing as obtaining whenever it in fact obtains"(Dummett 2000, p. 259). This is the case when truth is coupled with a notion of proof, which is by its very nature finitistic and as such recognisable. In this way, truth is intimately connected to proof on the molecular account.

Per Martin-Löf arrives via a different route at essentially the same position. In Martin Löf (1996) we find the distinction between judgements and propositions. Judgements are classified as evident and not evident, which automatically involves a connection to a judging subject *for which* they are evident or fail to be so. There are two types of

[1] Brouwer has also provided so called strong conterexamples to LEM (Brouwer 1923, Brouwer 1928, van Dalen 1999, Posy 2020), in which he defines a number that cannot be determinately said to be greater, smaller, or equal to zero.

judgement: That something is a proposition and that a proposition is true or that it is false. A proposition, then, is understood as something that can be (i.e. for which it makes sense to say that it is) either true or false. It is taken to be true if it can be verified and false if it cannot (for non-arbitrary reasons) be verified. Thus in order to make a correct judgement that something is a proposition, it needs to be evident (to a judging subject) that it makes sense to ask for a verification of it, and in order to make the correct judgement that the proposition is true, it needs to be equally evident that the process of its verification can be concluded to the positive. As Martin-Löf points out, this set-up actually purports a change in the notion of proposition. A proposition is no longer taken to be something that *is true or false*, but essentially something that *can be verified*.

For this changed notion of proposition and the intimate connection between truth and proof conditions the standard truth conditional semantics are no longer adequate. The BHK interpretation provides an alternative in explicating the meaning of complex propositions via proof conditions: (cf. Heyting 1931, Troelstra and van Dalen 1988):

> A proof of $A \wedge B$ is given by a proof of $A$ and a proof of $B$.
>
> A proof of $A \vee B$ is given by an explicit choice between $A$ and $B$ and a proof of the chosen proposition.
>
> A proof of $A \rightarrow B$ is a *general method* of transforming any proof of $A$ into a proof of $B$.
>
> There is no proof of $\perp$, the absurd proposition.
>
> A proof of $\forall x A(x)$ is a *general method* of proving $A(a)$ for arbitrary $a$.
>
> A proof of $\exists x A(x)$ is given by providing an object $a$ and a proof of $A(a)$.

Now, what about LEM on this account? For any atomic proposition $A$, it is of course not the case that the lack of a proof of $A$ comes down to a refutation of $A$, i.e. a general method of transforming any proof of $A$ into absurdity. The same holds for quantified propositions as well. Dummett explains that if the truth of a universal generalisation on an infinite domain is coupled with proof, we lack the required guarantee to assert LEM when it (already) comes to countably infinite domains.

> For an arithmetical statement involving the universal quantifier, there will be no guarantee that, if it is true, we shall be able to recognise its truth-conditions as fulfilled: not only do we not have any means to bring ourselves into a position to be able to recognize this, but, for all we know, the condition may not be one which any human being will ever be capable of recognizing as obtaining.(Dummett 2000, p. 261)

Thus, since there is no guarantee that we might or can find either a general proof or a counterexample, LEM and the duality of the quantifier fails on precisely those epistemic grounds that Dummett incorporates in order to secure the meaning of our molecular expressions.

Similarly, on Martin-Löf's account, adherence to LEM with respect to infinite totalities would be tantamount to the firm belief that these (possibly infinite) verifications are in principle possible. Since this is by no means obvious, LEM does not seem to enjoy ubiquitous justification.

> The trouble arises when you come to the laws for forming quantified propositions, the quantifiers not being restricted to finite domains [...] To my mind, at least, they simply fail to be evident. [...] Others must make up their minds whether these laws are really evident to them when they conceive of propositions simply as truth values.(Martin Löf 1996)

Of course, the last part of the quote needs to be understood appropriately, that is with respect to the criterion that subject-related evidence is required in much the same sense as Dummett.

Thus, both arguments can be analysed in a two step matter. First, the notion of proof is incorporated in the notion of truth or in the notion of proposition, and secondly, by reference to proof conditions LEM is rejected. But, without making any claim regarding the plausibility of these arguments from a meaning theoretic perspective, this somewhat changes the rules of the game. Dummett's and Martin-Löf's arguments introduce epistemic points which cut off the discourse in a way that it no longer reflects the notion of indeterminacy we are concerned with. We are not interested in whether LEM holds based on a notion of provability, or whether we should accept provability based on a general theory of meaning, but, given an otherwise classical understanding of proposition as something that is true, we want to know if our inability to circumscribe a certain domain of objects may lead to a rejection of LEM with respect to *this* notion of proposition. Accepting the arguments from Dummett and Martin-Löf will not get us an investigation of that.

However, this difference is also a decisive advantage because it avoids two central problems that the traditional intuitionist faces. One of them will be mentioned now, the other at the end of section 4.1.2. The first problem is that taking provability to be an essential part of truth conflicts with our common understanding of mathematical necessity. If 'provable' means 'having an actual proof', then truth values of mathematical sentences will change over time. The same is true if 'provable' is understood as 'being provable with the means available today (even though an actual proof has not yet been found)'. But a more liberalised notion of 'provable in principle (with whatever techniques may be developed in the future)' seems to be too metaphysically loaded for the intuitionist to accept. These claims obviously require a more thorough discussion than that (and for some intriguing details, see Raatikainen (2004)), but mentioning them should already make us appreciate that we don't have to think about them as far as the argument from indeterminacy is concerned.

### 4.1.2. Principles of traditional intuitionism

But because the argument from indeterminacy does not concern itself with provability conditions, we may ask how it relates to other intuitionistic non-validities that are usually rejected based on this understanding. Examples of such are the following:

- The **decidability of atomic sentences**

$$A \vee \neg A$$

  It is not guaranteed that we have a proof or a refutation of each such statement.

- The following inference of the **de Morgan law**:

$$\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$$

That we have a proof that two propositions are mutually inconsistent does not mean that we have a proof that one of them is inconsistent.

- The **principle of omniscience**:

$$\forall x(A(x) \vee \neg A(x)) \rightarrow \exists x A(x) \vee \neg \exists x A(x)$$

According to BHK, it is not only the proof of a particular proposition $A(a)$ that we need to be concerned with, but also the availability/construction of the object $a$ that is relevant. The statement can be weakened to the **bounded principle of omniscience** (BOM):

$$\forall x(A(x) \vee \neg A(x)) \rightarrow [\forall x(\varphi(x) \rightarrow A(x)) \vee \exists x(\varphi(x) \wedge \neg A(x))]$$

where the range of quantification is restricted to a domain characterised by $\varphi$. But this might also not enjoy intuitionistic validity in case this domain is infinite.

- A bit more controversial is **Markov's Principle** for decidable formulas $A(x)$

$$\neg\neg\exists x A(x) \rightarrow \exists x A(x)$$

Here is a case in which it might be accepted: Considering the underlying domain to be the natural numbers, then the antecedent says that it is absurd that there are no counterexamples to $A(x)$. In that case, one just needs to go through the natural numbers one by one and check of each $n$ whether $A(n)$. Eventually, then, the procedure will terminate and we end with the required witness.

Finally, there is also a feature that intuitionistic logic has that classical logic lacks, which is the following:

- Because we are interested in modelling our epistemic prowess, the logic should (and does) satisfy the **disjunction property**, i.e.

$$\vdash A \vee B \quad \vdash A \text{ or } \vdash B$$

$$\vdash \exists x A(x) \quad \vdash A(a) \text{ for some } a$$

where $\vdash A$ can be loosely understood as 'there is a proof of $A$'. Half a proof of $\varphi$ and half a proof of $\psi$ does not amount to a complete proof of $\varphi \vee \psi$, and half a construction of one object and half a construction of another object does not amount to one full object.[2]

However, even though the BHK interpretation and the foregoing explanation of these principles captures our convictions regarding traditional intuitionistic reasoning, the BHK interpretation is just too imprecise and too informal to allow for a thorough explanation of the intuitionistic constants. For this reason they are usually explicated via Kripke models.[3]

**Definition 4.1.** A **Kripke model** is a triple $\mathcal{K} = \langle W, \leq, \vDash \rangle$, where $W$ is a set of worlds or nodes, $\langle W, \leq \rangle$ is a poset and $\vDash$ is a mapping from $W$ and the set of formulas to truth values. For atomic formulas $A$ and elements $w \in W$ we have the following monotonicity condition:

$$w \Vdash A \quad \text{imples} \quad w' \Vdash A \text{ for all } w' \geq w$$

For arbitrary formulas $\varphi$ and $\psi$

$$\begin{array}{lll} w \Vdash \varphi \wedge \psi & \text{iff} & w \Vdash \varphi \text{ and } w \Vdash \psi \\ w \Vdash \varphi \vee \psi & \text{iff} & w \Vdash \varphi \text{ or } w \Vdash \psi \\ w \Vdash \neg \varphi & \text{iff} & \text{for all } w' \in W \text{ such that } w' \geq w\text{: } w' \nVdash \varphi \\ w \Vdash \varphi \to \psi & \text{iff} & \text{for all } w' \in W \text{ such that } w' \geq w\text{: if } w' \Vdash \varphi \text{ then } w' \Vdash \psi. \end{array}$$

These models can be used to give a precise account to the above counterexamples. Furthermore, the use of these models allows for an explication of the logical constants

[2] This strict reading of the disjunction might also be problematic, for, as Rumfitt (2012) has pointed out, there are possible scenarios in which one has enough evidence to affirm a disjunction without being able to confirm either of its disjunctions.

[3] Much of what is to follow can also be applied to Beth models, such that they will not be mentioned specifically. For an extended discussion of Beth models and traditional intuitionistic semantics, see Dummett (2000, ch.5.7)

in a more general way. The nodes in each of them may be understood as states of information such that moving along the branches models growth of information. There are, however, information related aspects that these frames cannot account for. One is the difference between global absence of verification and local falsification (cf. van Benthem 2009). An intuitionistic model based on a Kripke frame already fails to verify $\varphi$ when there is a node that just lacks the verification of $\varphi$ (like the bottom node in any case). In the next section, we will see how this can be generalised.

But before that, there is one more important point to note about traditional intuitionism and its explication via Kripke models. Kripke models are chosen over the BHK interpretation, because they manage to pin down the meaning of the logical constants in a way that makes them susceptible to a completeness proof. Such a proof is required to show that they indeed capture all and only those intuitionistically valid principles. But for the proof to go through we require the use of classical meta-logic—to which the traditional intuitionist with their ubiquitous rejection of LEM does not seem to be entitled to. This is a second major problem for traditional intuitionism. If provability is an essential part of truth, then this should extent to the meta-level. This leaves the position in a certain state of tension (cf. Dummett 2000, ch.5.6).

In the following chapters, when we deal with working out the details of the arguments based on indeterminacy, we will revisit these issues. As it will turn out, the rejection of LEM based on indeterminacy does not adhere to all those restrictions that traditional intuitionism (based on the notion of proof) contains, and also manages to circumvent the problem of using classical meta-logic.

## 4.2. Truth-conditional Semantics of LEM

The rejection of traditional intuitionist arguments for our endeavour, however, raises a different question. On the one hand, the argument from indeterminacy relies on a truth conditional understanding of the logical constants, which is most commonly given by a two-valued semantics that allows no room for a negotiation of LEM. Kripke/Beth models, on the other hand, can make the failure of LEM salient, but, it can be argued, their purported information theoretic rendering of the logical constants also changes their meaning (cf. Rumfitt 2012). Hence, we need to provide a general perspective from which the status of LEM can be evaluated *with respect to a truth conditional understanding of the logical constants.* In this section, such a perspective will be given via the use of topological models.

### 4.2.1. A general semantics for negation

The main task in finding such a general perspective is to give an account of negation that incorporates both its classical as well as its intuitionistic use. Such a general notion of negation plus an accompanying semantics has emerged in the discussion of logical pluralism as it was introduced in Beall et al. (2006). Building on that, Berto (2015) and Berto and Restall (2019) address negation explicitly. According to their proposal, a unified notion of negation is to be explicated in terms of incompatibility (cf. also Wright 1993).

> For any statement $A$, the negation of $A$ is the weakest proposition that is incompatible with $A$.

This understanding of negation allows for different precisifications that validate either classical or intuitionistic negation. Berto (2015) argues that, due to its connection to the notion of incompatibility, negation gains a distinctly modal character and should therefore be explicated using possible worlds. However, the possible world model he uses is considerably fine grained (for instance, it allows for the distinction between LEM and the double negation elimination rule (DN) which are intuitionistically equivalent). Furthermore, the precise contribution that LEM makes (and whatever is responsible for its absence) is either expressed in distinctly classical or intuitionistic terms. If LEM holds, then propositions are understood in a classical sense as sets of possible worlds; if it doesn't hold, then worlds are rather considered as information states, which is the epistemic notion that was rejected in the previous section. What I am interested in here is a notion that reflects the distinction between the acceptance and rejection of LEM in terms of determinacy, i.e. in terms of truth conditions.

The algebraic approach that is briefly mentioned in Berto and Restall (2019) comes closer to this aim. Consider the complete algebra $\mathcal{G} = \langle G, \leq, \curlywedge, \bigvee \rangle$ where $G$ is a set whose elements are ordered by the preorder $\leq$, $\curlywedge$ is the incompatibility relation which says that $a \curlywedge b$ iff $a \wedge b = 0$ and $\bigvee$ is a join operation. A proposition $A$ is interpreted as an element $a \in G$. To interpret negation, let $E_a = \{b \mid a \curlywedge b\}$ be the set of (interpretations of) propositions that are incompatible with $a$. Then the negation of $A$ can be interpreted as

$$\neg a = \bigvee E_a$$

Note that, technically, since $a$ is an element of the algebra $\mathcal{G}$, $\neg$ is an operation on $A$ rather than a piece of syntax. I abuse notation in writing $\neg a$ for the interpretation of the proposition $\neg A$ in $\mathcal{G}$. That $\neg a$ is (the interpretation of) the weakest proposition incompatible with $a$ can be expressed by

$$x \leq \neg a \qquad \text{iff} \qquad x \curlywedge a.$$

This follows from the definition of $\neg a$ and says that anything that entails $\neg a$ is incompatible with $a$ and anything that is incompatible with $a$ must entail $\neg a$.

First, note that $\mathcal{S}$ no longer encodes a two-valued (i.e. bivalent) scenario. Consequently $A \vee \neg A$ can be true if $a \neq 1$ and $\neg a \neq 1$. Now, whether $\mathcal{G}$ verifies $A \vee \neg A$, i.e. whether it holds that $a \vee \neg a = 1$ in $\mathcal{G}$ depends of course on its structure (namely whether $\mathcal{G}$ is a boolean or a heyting algebra). The algebraic approach manages to express both the intuitionistic as well as the classical notion of negation via the expression $\bigvee E_a$. But what does this say about the reasons for and against LEM? How can we connect the distinction between determinacy and indeterminacy to the features of $\mathcal{G}$ on which LEM hinges?

To make these points more precise, I will work with topological models.[4] Topological models are based on the notion of an open set, which can be used to characterise a notion of determinacy. In fact, the collection of all open sets of a topology form a Heyting algebra. However, what gets lost in the transition from topology to Heyting algebra is the following. Due to their underlying structure, using topologies also gives us

[4]The standard reference for Topology is Munkres (2014). For the connection of topologies to logic, see van Dalen (2002), and more advanced Melikhov (2015).

the notion of a boundary (of an open set), which will be shown to provide the adequate measure of indeterminacy. Topologies, unlike possible world models who still incorporate a notion of growing information, and unlike algebraic models, who only express what is determinate, get the focus just right.

**Definition 4.2.** A **topology** $\mathcal{T} = \langle X, \mathcal{O}_X \rangle$ on a set $X$ is given by a collection of subsets $\mathcal{O}_X$ of $X$ (called *open* sets) which fulfil the following requirements:

$$\emptyset, X \in \mathcal{O}_X$$

$$\text{If } U_1, \dots \in \mathcal{O}_X \text{ then } \bigcup_i U_i \in \mathcal{O}_X$$

$$\text{If } U_1, \dots, U_n \in \mathcal{O}_X \text{ then } \bigcap_{i=1}^{n} U_i \in \mathcal{O}_X$$

In other words, $\mathcal{O}_X$ is closed under arbitrary unions and finite intersections.

Furthermore, a set $P \subset X$ is said to be *closed*, if it is the complement of an open set, i.e. $P = U^c$ for $U \in \mathcal{O}_X$.

The **interior** of a set $Y$, denoted by $Y^\circ$ is the largest open set contained in $Y$. Of course—and most importantly for what's to come—if $Y$ is open, then $Y = Y^\circ$.

Conversely, the **closure** of a set $Z$, denoted by $\overline{Z}$ or $cl(Z)$, is the smallest closed set which contains $Z$. If $Z$ is closed, then $\overline{Z} = Z$.

Finally, the **boundary** $\partial Z$ of a set $Z$, which is also called the set of limit points of $Z$, is its closure minus its interior, i.e. $\partial Z = \overline{Z} \setminus Z^\circ$.

**Example 4.1.** The prime example for a topological space is the real line with the open intervals $(a, b)$ on it as open sets. Closures of those are the closed intervals $[a, b]$. The boundary of $(a, b)$ is $\partial(a, b) = [a, b] \setminus (a, b) = \{a, b\}$.

There is, of course, a connection between topologies and Kripke models. For this, we need the notion of a basis of a topology.

**Definition 4.3.** A collection $\mathcal{B}$ of subsets of a set $X$ is a **basis** for a topology $\mathcal{T} = \langle X, \mathcal{O}_X \rangle$ on $X$ if each $U \subset X$, $U \in \mathcal{O}_X$ if and only if it is a union of sets in $\mathcal{B}$.

**Example 4.2.** A Kripke frame $\langle W, \leq \rangle$ can be turned into a topology by taking for each point $x \in W$ its upset $R^{\uparrow}(x) = \{y \mid x \leq y\}$ to be a basic open set.[5] The closure of a set $R^{\uparrow}(x)$ can be obtained by adding elements 'walking backwards' from $x$ to the first branching point below it. Consequently, the boundary of $R^{\uparrow}(x)$ is just the collection of points from the first branching point up to but not including $x$.

Traditionally, the space $X$ is understood as a space of proofs. A formula $\varphi$ is allocated a certain section of a space, i.e. an open set $U \in \mathcal{O}_X$, which is traditionally understood as the collection of its proofs. But as already noted in the first section, we are not necessarily interested in modelling epistemic prowess or in incorporating any notion of proof *per se*. For this reason, we may understand $U$ more abstractly as containing the *warrants* for $\varphi$. A warrant may be understood as something that is a ground of truth of $\varphi$, or something in virtue of which $\varphi$ obtains. This notion is deliberately left broad. The traditional intuitionist may only allow a proof of $\varphi$ to be counted as one of its warrants, but the notion can also be expanded to include 'classical' truth conditions.[6] If $\varphi$ is a tautology then it is warranted by whole space; conversely, $\perp$ doesn't have any warrant and is thus interpreted as the empty set. This is the basic idea of how a topological space may be used as a model for a logic that generalises upon the proof theoretic notion of the traditional intuitionist as well as the bivalent notion of classical two-valued models. The details are as follows:

**Definition 4.4.** A **topological model** $\mathcal{T} = \langle \mathcal{O}_X, [\![\cdot]\!] \rangle$ is given by a topology of open subsets $\mathcal{O}_X$ of a space $X$, and an interpretation function $[\![\cdot]\!]$ from formulas into $\mathcal{O}_X$.

$$\begin{array}{ll} [\![A]\!] \in \mathcal{O}_X \text{ for atomic formulas } A & \\ [\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!] & [\![\varphi \to \psi]\!] = ([\![\varphi]\!]^c \cup [\![\psi]\!])^\circ \\ [\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] & [\![\perp]\!] = \emptyset \end{array}$$

[5]Even though Kripke frames can be viewed as topologies, the general class of topological models is a bit more versatile than the ones corresponding to Kripke frames. The corresponding topology can be independently characterised as an Alexandroff topology, which has the special property that *arbitrary* (not just finite) intersections of open sets yield an open set as well.

[6]In the sense that snow being white is a warrant of the proposition "snow is white" without that being accompanied by any actual observation/explanation that (pure) snow is white.

This gives in particular $[\![\neg\varphi]\!] = [\![\varphi \rightarrow\perp]\!] = ([\![\varphi]\!]^c)^\circ$. A formula $\varphi$ is valid in $\mathcal{T}$, if $[\![\varphi]\!] = X$.

Note that just like in non-rooted Kripke models, a conjunction $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$ may be true in a model even if neither of its disjuncts are, i.e. if $[\![\varphi]\!] \neq X$ and $[\![\psi]\!] \neq X$. In this sense, these models don't reflect the standard intuitiontist interpretation of the logical constants according to which a proof of a disjunction amounts to an explicit choice between one of the disjuncts and a proof of the chosen formula.[7] To have this variability for our purpose seems to be desirable: it may be the case that it is not determinate which one of two options obtains, but nonetheless determinate that one of them has to obtain.

The case of implication and subsequently negation stands out for its use of the interior operator. The traditional intuitionist understands $\varphi \rightarrow \psi$ as a procedure that takes a proof of $\varphi$ and transforms it into a proof of $\psi$. Thus if we take the space $X$ as a space of proofs, and if we take an *open* set in $[\![\varphi]\!]^c \cup [\![\psi]\!]$ and intersect it with $[\![\varphi]\!]$ we get a set of proofs that prove $\varphi$ and $\psi$. The largest such open set which contains all such proofs that go from $\varphi$ to $\psi$ is thus the interior of $[\![\varphi]\!]^c \cup [\![\psi]\!]$, hence $[\![\varphi \rightarrow \psi]\!] = ([\![\varphi]\!]^c \cup [\![\psi]\!])^\circ$.

Given our more general understanding we might interpret the conditional as saying that the warrants of $\varphi$ are included in those of $\psi$. In this sense, the interior of $[\![\varphi]\!]^c \cup [\![\psi]\!]$ is understood as the set of those warrants that ground $\varphi$ if they also ground $\psi$. Note that we also have the classical interpretation of implication by $[\![\varphi \rightarrow \psi]\!] = [\![\varphi]\!]^c \cup [\![\psi]\!]$ contained in the above. The crucial difference, then, between intuitionistic models and classical ones lies solely in the use of the interior operator. If every set is open, and thereby identical with its interior we would have a model of classical logic. This will become a crucial point for the interpretation of LEM.

### 4.2.2. Interpreting intuitionistic non-validities

How does the model behave with respect to the typical intuitionistic invalidities? In this section I will look at the most basic propositional non-validities: LEM, DN, and deMorgan. The subsequent chapter will be focused on quantification and assess in what

[7] Of course, this leaves untouched the completeness theorem according to which we can find a countermodel for any formula that is not provable in intuitionistic logic.

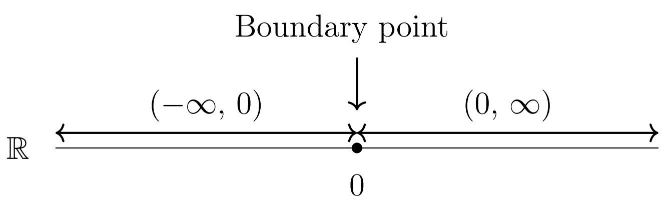
way the availability of objects may impact the validity of the other principles on p.87.

### LEM and DN in topological models

Regarding LEM, note that

$$\mathcal{T} \nvDash A \vee \neg A \qquad \text{iff} \qquad [\![A \vee \neg A]\!] = [\![A]\!] \cup ([\![A]\!]^c)^\circ \neq X$$

which is the case if $([\![A]\!]^c) \neq ([\![A]\!]^c)^\circ$. But in this case the boundary of $A$, $\partial[\![\neg A]\!] = cl(([\![A]\!]^c)^\circ) \setminus ([\![A]\!]^c)^\circ \neq \emptyset$ and similarly, $\partial[\![A]\!] = cl(([\![A]\!]) \setminus [\![A]\!] \neq \emptyset$. This can be interpreted as saying that LEM doesn't hold with respect to $A$ if there are warrants that are neither warrants for $A$ nor $\neg A$. This can be illustrated by taking $\mathcal{T}$ to be the real line with open intervals.




$[\![A]\!] = (-\infty, 0)$

$[\![\neg A]\!] = (0, \infty)$

$0 \notin [\![A]\!] \cup [\![\neg A]\!]$

$[\![A]\!] \cup [\![\neg A]\!] \neq \mathbb{R}$

It is for this exact reason that DN fails as well.

$$\mathcal{T} \nvDash \neg\neg A \rightarrow A \qquad \text{iff} \qquad ([\![\neg\neg A]\!]^c \cup [\![A]\!])^\circ \neq X$$

i.e. if $[\![\neg\neg A]\!] \neq [\![A]\!]$. Since $[\![A]\!] \subset [\![\neg\neg A]\!]$, the inequality holds if

$$((([\![A]\!]^c)^\circ)^c)^\circ \subseteq [\![A]\!].$$

It is worthwhile to spend some time in understanding these containments. That $[\![A]\!] \subseteq ((([\![A]\!]^c)^\circ)^c)^\circ$ means that the warrants for $A$ are contained in the warrants for $\neg\neg A$, but not necessarily every warrant of $\neg\neg A$ might establish $A$. Thus if the converse does not hold, this means that whatever establishes $\neg\neg A$ may simply not suffice to establish $A$. In this case, again, $\partial[\![A]\!] = cl(([\![A]\!]) \setminus [\![A]\!] \neq \emptyset$.

In general, the above discussion illustrates two ways of understanding negation. These one may call *negation as mere absurdity* and *negation as affirmation of the opposite*. Insofar as $[\![A]\!]^c \neq ([\![A]\!]^c)^\circ$, we know that $[\![A]\!]^c$ contains everything that is not a warrant for $A$, but it also contains warrants that are insufficient to *deny* $A$. In this case, the negation expressed by the complement operator alone is merely one of limitation. We have sufficient warrant to reject $A$ but no sufficient warrant to affirm $\neg A$. On the other hand, if $[\![A]\!]^c = ([\![A]\!]^c)^\circ$, every warrant to deny $A$ is also a warrant to affirm $\neg A$.

Now, it is well known that the open sets corresponding to a topological space form an algebra ordered by inclusion. In the algebra as well as in the topology we can express incompatibility between $A$ and $\neg A$ by setting $a \wedge \neg a = 0$ and $[\![A]\!] \cap [\![\neg A]\!] = \emptyset$, respectively. Using the language of topologies, however, we can give a precise technical description of what is at issue in the distinction. If the topology satisfies $U^c = (U^c)^\circ$ for an open set $U$, its corresponding element $u$ in the algebra satisfies $u \vee \neg u = 1$. Using the notion of a closure we are not able to express this issue by saying that in this case, in addition to $[\![A]\!] \cap [\![\neg A]\!] = \emptyset$ it is also the case that $cl([\![A]\!]) \cap cl([\![\neg A]\!]) = \emptyset$. This expresses that there is no boundary between $[\![A]\!]$ and $[\![\neg A]\!]$, and hence that there is no room for indeterminacy.

A topological space that is the union of two disjoint open sets is called *disconnected*. Insofar as these closed and open sets are taken to be the warrants of a formula $\varphi$, we may say that a space that is disconnected at $[\![\varphi]\!]$ is *determinate with respect to its warrants* of $\varphi$. A connected space, on the other hand, is one that cannot be described as the union of two disjoint open sets. Such a space may be considered ***in****determinate with respect to its warrants* of $\varphi$. Insofar as we are admitting LEM with respect to a formula $\varphi$, which means that we understand its negation as confirmation of the opposite, we equivalently have a notion of the underlying space as disconnected, and thereby a conception of its determinacy implanted in the interpretation of $\varphi \vee \neg \varphi$. Furthermore, a topological space may be disconnected in just one, or in many places, which means that it may license LEM with respect to one or many formulas. The class of topological spaces thus corresponds to a class of logics intermediate between intuitionistic and fully classical logic.

### deMorgan in topological models

While the topological semantics gives the same diagnosis to DN and LEM, the deMorgan law requires a slightly different explanation, which, importantly, does not primarily focus on negation.

$$\mathcal{T} \not\models \neg(A \wedge B) \to (\neg A \vee \neg B) \qquad \text{iff} \qquad [\![\neg(A \wedge B)]\!] \neq [\![\neg A \vee \neg B]\!]$$

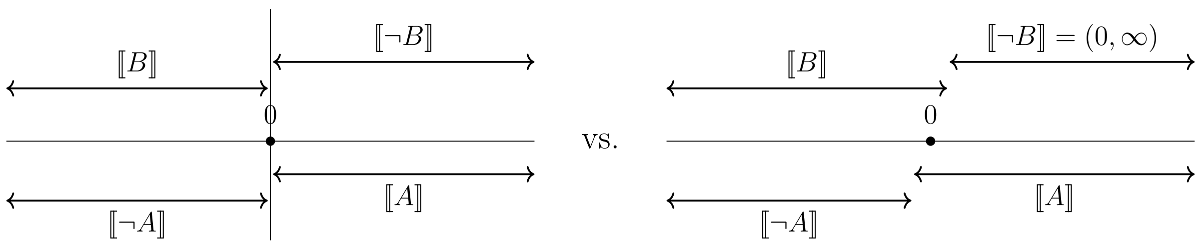
The left term evaluates to

$$[\![\neg(A \wedge B)]\!] = (([\![A]\!] \cap [\![B]\!])^c)^\circ = ([\![A]\!]^c \cup [\![B]\!]^c)$$

and the right term evaluates to

$$[\![\neg A \vee \neg B]\!] = ([\![A]\!]^c)^\circ \cup ([\![B]\!]^c)^\circ$$

These two are not equal when $[\![A]\!]$ and $[\![B]\!]$ share a boundary, which can be seen in the following example:




For the left side it is the case that $F = \partial[\![A]\!] \cap \partial[\![B]\!] \neq \emptyset$. Then $F \subset [\![A]\!]^c \cup [\![B]\!]^c$ but $F \nsubseteq ([\![A]\!]^c)^\circ$ and neither $F \nsubseteq ([\![B]\!]^c)^\circ$. Thus, deMorgan is not validated if there are warrants which are not warrants for either of the disjuncts. This is somewhat different than the clauses for LEM and DN, since the reason for its failure does not go through negation *per se*. This is quite a significant departure form the traditional BHK related explanation of why deMorgan fails. Furthermore, it signals that the two ideas may come apart in our approach.

### 4.2.3. Connections to logical pluralism

Finally, since the analysis of negation was introduced with reference to logical pluralism, one might wonder how it, and indeed the whole argument from indeterminacy, relates to that position. A full analysis of this cannot be given here (and it is also more important to actually develop the argument in the first place) but a brief remark seems to be warranted. Logical pluralism is characterised in Beall et al. (2006) by acceptance of what the authors called the Generalised Tarski Thesis (p.29).

> **GTT**: An argument is valid$_x$ if and only if, in every case$_x$ in which the premises are true, so is the conclusion.

where valid$_x$ can be made precise in different ways, in which one of them may lead to intuitionistic logic and another to classical logic. This can certainly be expressed via the topological models that I have introduced, namely by introducing (or lifting) certain restrictions on how the interior operation is supposed to be understood (possibly accompanied by different interpretations of the notion of a warrant).

But the topological approach also suggests the idea that intuitionistic logic acts as a base line, where classical instances may be incorporated when required. This also fits nicely with a proof theoretic perspective and arguments from harmony. What harmony is and what the consequences are if it is lacking has been specified in a number of different ways (cf. Dummett 1991b, Steinberger 2011, Rumfitt 2017), but it seems that the most plausible of those (as well as the one closest to our concern) is to say that the rules for a logical connective are harmonious if their addition to a logical system forms a conservative extension, and that the lack of such harmony is taken to impede the innocence of logic.[8] Classical logic is certainly non-conservative over intuitionistic logic. Furthermore, proofs via natural deduction in intuitionistic logic have the subformula property which proofs in classical natural deduction lack. The subformula property may be taken as a measure of analyticity (cf. Restall 2022, p. 20), such that lack of analyticity and thus lack of innocence may be used to question to what extent the classical laws for

[8]This is actually only focused on one half of the harmony thesis, according to which the rules for a logical connective should not be too strong to license 'unwarranted' inferences. A corresponding dual, according to which they shouldn't be too weak is also being discussed, but does not relate to the issues of concern here.

negation may be called logical laws. From this perspective, Neil Tennant, for instance, has characterised LEM as a principle that is "synthetic a priori"(Tennant 2002, p. 239).

If indeed the attribution of the term 'logical' should be used this way and thus LEM to be denied this status, then the topological perspective provides an adequate assessment of the contribution that an assumption of LEM makes as well. Connecting the validity of LEM to a general feature of the topological space may be understood to showcase that it is of a different nature than the (other) logical laws. However, it's nature is not arbitrary. The conception of determinacy it engenders reflects a fundamental property of the space or object of discourse and may as such have meaning constitutive status. In other words, LEM may not hold generally as a logical law (in all topological spaces), but if it holds, then it does so a priori and with necessity. And this is all that is required for an assessment of the argument from indeterminacy.

## 4.3. Conclusion

This chapter has made a first distinction between traditional arguments against LEM and the argument from indeterminacy that is analysed in this thesis. In this respect some principles of traditional intuitionism were mentioned, such as the rejection of the following:

- LEM and DN and deMorgan with regards to propositional logic.
- The principles of BOM, and **Markov** with regards to first order logic.
- The use of classical logic on the meta-level.

The first of those was analysed in this chapter, pointing to a distinction between LEM and DN on the one hand, and deMorgan on the other. This analysis was done using a topological model which provided a neutral ground to assess the contribution that LEM makes while still adhering to a truth-conditional understanding of the logical constants.

The next chapter will focus on generality and quantification over indeterminate domains, and in doing so address the remaining principles as well as the question of the correct meta-logic.

# 5. Generic and Instance Based Generality

While the last chapter has been concerned with LEM and other propositional non-validities, this chapter will look closer into its application to quantified statements. Indeterminacy is given insofar as we cannot form a sufficiently precise conception of the elements that a domain of quantification contains. The standard (i.e. classical) model theoretic account of the quantifiers, however, always requires a determinately circumscribed domain as their range. If such a domain is unavailable, any assessment of LEM will require an alternative understanding of quantification and corresponding semantics first. Such an understanding has been suggested by Hermann Weyl and further developed by Øystein Linnebo. The idea is that generality expressions should be understood as pertaining to essential properties and conceptual entailments rather than with respect to a collection of instances. Such generalisations are called *generic generalisations* in opposition to *instance based generalisations*. Moreover, on both Weyl's and Linnebo's understanding, generic generalisations can only guarantee intuitionistic and not fully classical logic.

There are, however, some open questions regarding generic and instance based generality as they originate with Weyl's introduction of the concepts (and, insofar as he remains true to his source, as they transfer to Linnebo's formalisation, as well). These I will address in this chapter. In particular, I argue that there are different ways of understanding the negation of statements of generic generality to which, in turn, correspond different existential statements. I will give a formalisation of this using a joint object- and meta-logic. This will allow for a more refined account of the circumstances in which classical and intuitionistic logic is to be used in conjunction with generic generality.

In the first section, Weyl's original introduction of the notion of generic generality will be discussed and a number of problems will be noted. Section 2 presents Linnebo's truth-maker model and investigates how it fares with respect to the problems noted in section 1. This will be continued in section 3 where I introduce the meta-logical machinery needed to address these problems. In an appendix I extend the procedure to topological models and with that show how the notion of a warrant as introduced in chapter 4 can be cashed out solely in terms of the availability of objects.

## 5.1. Weyl on Generic and Instance Based Generality

In the previous chapter, LEM was discussed on the propositional level. Here I am interested in its relation to quantified statements. I will discuss LEM in the form

$$\forall x P(x) \vee \neg\forall x P(x), \tag{5.1}$$

where it is required that $P(a)$ is decidable for any $a$ at the outset.

Two points can be made about this on the outset. First, if $P(\cdot)$ is atomic, then it can be taken to be decidable. This is because unlike traditional intuitionism, the argument from indeterminacy only turns on the availability of objects. Thus, if an object $a$ is given, then $P(a)$ has a determinate truth value. As we go on, we will see that this can be extended to requiring that $P$ should not contain any quantifiers that equally range over the whole indeterminate domain it is interpreted over.

Second, the above expression of LEM should not be conflated with the duality of the quantifiers, i.e. with the validity of the formula

$$\forall x P(x) \vee \exists x \neg P(x). \tag{5.2}$$

Classically, the quantifiers are dual, i.e. $\neg\forall\neg \equiv \exists$, which relinquishes the distinction between (5.1) and (5.2) such that both become an equivalent expression of LEM. But to be precise, even though LEM needs to fail, else $\forall x P(x) \vee \exists x \neg P(x)$ would be derivable, if the quantifiers are *not* dual, then $\exists x \neg P(x)$ is not the negation of $\forall x P(x)$ but $\neg\forall x P(x)$

is.

This leads to two questions:

(i) can the quantifiers be understood as dual to each other?

If the answer to this question is No, we know that LEM cannot be ubiquitously assertible. But it does not predetermine an answer to

(ii) can LEM in the particular instances of $\forall x P(x) \vee \neg \forall x P(x)$ or $\exists x P(x) \vee \neg \exists x P(x)$ still be asserted?

This section discusses Hermann Weyl's answers to these questions.

### 5.1.1. The difference between generic and instance based generality

For Weyl (1921) and Linnebo (2022a), the duality of the quantifiers and the underlying notion of negation depends on the notion of generality that they express. We can distinguish from the outset between two such conceptions. These are called *generic* and *instance based* generality. The difference between generic and instance based generalisations lies *in virtue of what* makes a generality statement true. Linnebo illustrates the difference with the following examples:

a) Every whale is a mammal.

b) Everyone in the room is born on a Monday.

The first is an example of generic generality. It is true because being a whale *conceptually entails* being a mammal. The second one is an example of instance based generality. It is true when *every* '*instance*' in the room has the property in question.

Instance based generality is familiar from the standard model theoretic account of quantification and is the default way of understanding generality statements. Though never given a thorough formalisation, generic generality has also been considered by a variety of thinkers (cf. the list of references in Linnebo (2022a, p. 350)). Generic generality has the potential to handle generalisations over structures like the ordinals and even the sets more generally, where the assumption that the objects in question form a totality leads to a contradiction (due to the possibility of diagonalisation of forcing new elements). Further well known examples close to our topic are

- Russell's notion of typical ambiguity (Whitehead et al. 1927) and Hilbert's schematic generalisations (Hilbert 1925).

- Inductive definitions: Consider an inductively defined predicate $P$. In the clause

$$\frac{P(n)}{P(f(n))}$$

  the range of the variable $n$ need not be understood as given by a domain. Even though they were defined in this way in chapter 1, the more natural perspective would be to understand the natural numbers, for instance, to be given by the conceptual specification of the zero element and the successor function as antecedent to the availability of any objects.

Well beyond the scope of this thesis, but highlighting further potentially fruitful applications, are:

- The notion of a reducibility candidate that is employed in the proof of the normalisability of second order logic (cf. Pistone 2018).

- The epistemic circularity in impredicative higher order comprehension (cf. Wright 2021).

- The impredicativity of traditional intuitionistic implication (cf. van Atten 2018, Tabatabai 2017).

And even beyond the philosophy of mathematics, generic generality has applications in:

- generalisations involving an indeterminate future. Example a) in fact belongs to this category, for it says that any whale which might come into existence will in fact be a mammal.

- generalisations regarding unfinished works of fiction, where the question is if they follow from the hitherto created content.

The question, however, is: What kind of logic does generic generality in opposition to instance based generalisations license?

### 5.1.2. Connections to LEM

Relating the instance-based conception of generality to LEM is straightforward. Instance based generalisations are contained in the classical truth definition employed in first order logic. As Dummett (1991a) points out, in such a case the truth of a universal generalisation can be expressed as a complete conjunction, and the truth of an existential statement as a complete disjunction. Let $[P]$ be a domain of quantification that is given by a concept $P$. Then instance based generalisations can be expressed as:

$$\forall x \in [P] : \varphi(x) \equiv \bigwedge_{x \in [P]} \varphi(x) \qquad \exists x \in [P] : \varphi(x) \equiv \bigvee_{x \in [P]} \varphi(x)$$

Furthermore, an instance-based universal general statement fails to be true if there is an instance that does not have the property whose generality is in question. For this reason, the negation of such a statement yields a claim regarding the existence of a counterexample:

$$\neg\forall x \in [P] : \varphi(x) \;\equiv\; \neg(\bigwedge_{x \in [P]} \varphi(x)) \;\equiv\; (\bigvee_{x \in [P]} \neg\varphi(x)) \;\equiv\; \exists x \in [P] : \neg\varphi(x),$$

and thereby the duality of the quantifiers. This argument relies on three points: (i) the fact that $\varphi(x)$ is decidable (which was explicitly assumed in the beginning), (ii) the 'availability' of the conjunction, and (iii) the applicability of the deMorgan law. By the argument from indeterminacy, the duality of the quantifiers should fail on account of (ii), and conversely, as the analysis of the last chapter suggests, the deMorgan law should be applicable once the conjunction of the $\varphi(x)$ for $x \in [P]$ is given.

The notion of generic generality was introduced in Weyl's 1921 paper "On the New Foundational Crisis in Mathematics", which marks his brief conversion to intuitionism.[1] For Weyl, the meaning of a universal generalisation over an infinite domain cannot be

[1] Weyl adopts his own version of Brouwer's theses, and even thought their general direction is the same, there are some differences in detail between the two (cf. van Dalen 1995).

tied up with the availability of any instances, because the possibility of attaining them all is out of the question:

> [T]his point of view of a completed run through an infinite sequence is nonsensical. I cannot get general judgements about numbers by looking at the individual numbers but only by looking at the essence of number [*das Wesen der Zahl*].(Weyl 1998, p. 95)

The alternative that he suggests is that such a universal generalisation has to reflect something that lies in the "essence of the concept"(Weyl 1998, p. 95) of which a generality claim is expressed. In addition to that, Weyl understands the existential quantifier constructively:

> Existential states of affairs are empty inventions of logicians. “2 is an even number”: This is an actual judgement expressing a state of affairs; “there is an even number” is merely a judgement abstract gained from this judgement.(Weyl 1998, p. 95)

Thus, “[o]nly the successful construction can provide justification for [an existential statement]; the mere possibility is out of the question.”(Weyl 1998, p. 96) From this it follows that neither quantifier can be defined in terms of the other.

According to Weyl there is a *conceptual difference* between such existential and universal statements, which leads to the contention that “it would be absurd to think of a complete disjunction between the cases”(Weyl 1998, p. 95). Therefore $\forall x\varphi(x) \vee \exists x\neg\varphi(x)$ cannot be a logical truth. But far from leaving it at that, Weyl goes on to also call into question the intelligibility of the negations of both generalisations with their respective understanding. According to him, “Neither the negation of the one nor of the other makes any comprehensible sense.”(Weyl 1998, p. 97) That something is not in the essence of a concept does not necessarily yield a statement about one particular instance, and similarly, what is to count as the negation of a particular construction need not amount to an essential property itself. In other words, the negations of statements of generic generality and of explicit constructions are not statements, but the acknowledgement of a distinct absence of a statement. Thus, Weyl rejects both (5.1) and (5.2) and answers questions i) and ii) negatively.

### 5.1.3. Some problems with Weyl's account

A first point to make is that Weyl's rejection of the classical reading of both quantifiers is motivated by epistemic concerns of the sort that were discussed at the beginning of chapter 4.[2] But this should not prohibit an application of the notion of generic generality beyond traditional intuitionism. Instead of Weyl's constructive premise, we can reject instance based generalisations by reference to our notion of indeterminacy. This generalises Weyl's objection and, remarkably, leaves his positive contention of expressing (and justifying) universal generalisations via conceptual entailment still applicable. This extended notion, however, requires us to address some additional points.

One issue is with the appropriate understanding of the existential quantifier. The traditional intuitionistic explanation of the non-duality of the quantifiers is straightforward (cf. chapter 4.1.1): Just because the universal generalisation leads to absurdity does not mean that we have found a concrete counterexample. But this line of reasoning is no longer available for the argument from indeterminacy. For what would it mean to say that (without requirements of constructivity) that there *is* no counterexample? Of course, it cannot mean that it follows from the essence of the concept that there *cannot* be one. Since $\neg\exists x\neg\varphi(x)$ entails $\forall x\neg\neg\varphi(x)$ even intuitionistically, and since we require that $\varphi$ is decidable, this would imply that $\forall x\varphi(x)$. But to say that there 'just' is no counterexample, we can neither use any epistemic notion, nor quantification over all the objects in a domain—for these are precisely the resources the lack of which drives the use of generic generality. Of course, just because this kind of explanation fails, does not mean that there isn't another one that works. In chapter 6.4.1 and 6.4.2 we will see that only in certain selected cases an alternative explanation is available.

A second issue is Weyl's stance on negation. There are two possible ways in which negation with respect to generic generality can be understood. Consider the following two examples:

c) It does not lie in the rules of the test that everyone will pass it.

d) Whatever happens to Sansa Stark, she will not become a cyber-security engineer.

[2] For a more thorough analysis of this claim, see Engler (2023).

The first is of the form “It is not determinate that $p$” while the second is of the form “It is determinate that not-$p$”. Both of them—expressing two different notions of negation—seem to be perfectly intelligible.

c) is akin to Weyl’s point. It rejects a particular conceptual entailment (it would not be a test worth taking if everyone automatically passes it), but it leaves open the possibility that the fact does obtain (it might just happen that there is a particularly good group of students). Negation in this sense has the character of a meta-judgement

$$\nvDash \forall x \varphi(x) \qquad \nvDash \exists x \varphi(x)$$

saying that $\forall x \varphi(x)$, for instance, cannot be established, but need not be refuted. Since this does not contain any constructive content, it is not relevant to Weyl. But from our perspective it is. Especially when considering independence phenomena, one might want to allow for cases where a construction is possible with the help of an additional assumption that is itself independent of the concepts involved.

Regarding d), however, we can say something even stronger. We know that the story has not yet finished (properly!), but even regarding the most atrocious accounts of fan fiction, not affirming d) would be in conflict with the whole setting of the story. This is the way in which the intuitionist commonly understands negation, which states that one is able to derive absurdity from the assumption that an un-negated proposition is true. This involves more than the mere *absence* of conceptual entailment. Negation understood in this way is thus quite strong.

There is also a connection to generic generalisations. As noted before, Weyl calls into question that an existential statement that is detached from its verifying instance and that does not have an *actual* construction backing it up can be seen as a statement at all. He calls them instead “judgement abstracts” (Weyl 1998, p. 98). But it seems quite natural to object that if something does not lie in the essence of a concept (on pain of contradiction), it must at least in principle be *possible* to produce a counterexample. There is a difference between asserting the possibility of a counterexample and the actual counterexample, but this need not mean that the notion of a possible counterexample has to be devoid of all meaning. For instance, there could be different but incompatible

| | Negation as affirmation of the opposite | Negation as mere absurdity |
|---|---|---|
| Object-level | classical logician | intuitionist |
| Meta-level | classical logician | intuitionist |

ways of producing a witness, and it hasn't been decided yet which one to take. In this case, the mere possibility of a counterexample is given, but a particular (unique) one is not determined. Such scenarios, which would lead to a supervaluationist understanding of the existential quantifier, should not be excluded outright.

However, it should be noted that this type of negation is not necessarily restricted to be intuitionistic. As the previous chapter has shown, if the conditions are right, then negation as absurdity might be tantamount to an affirmation of the opposite. In this respect, we have a two-by-two distinction regarding negation, where the two types of negation that were exemplified in c) and d) may be fittingly called **meta-** and **object level** negation.[3] The situation can be depicted in the following square.
The classical logician would land squarely on the left while the traditional intuitionist lands squarely on the right. The pressing question now is: Where does the argument from indeterminacy lead?

The rest of the chapter will take some steps towards answering this question. My approach is based on the formal account of generic and instance based generality developed in Linnebo (2022a). I will enlarge the class of Linnebo's models to incorporate a supervaluationist reading of existential claims, and add a formalised meta-logic to integrate

[3]The distinction between object- and meta-level negation is reminiscent of the syllogistic distinction between *sentential negation* and *predicate term negation* (cf. Horn et al. 2020). This distinction goes back to Aristotle and we still find it, for instance, in Kant's table of judgements (cf. A70/B90). The sentential negation comprises cases like "Socrates is not immortal" which denies the connection between the subject 'Socrates' and the predicate term 'immortal'. Expression like this essentially reject a certain judgement, but as such they do not produce another judgement, i.e. they don't affirm the truth of any proposition. For this reason the sentential negation (unlike its modern pendant as we shall see) cannot be iterated. The predicate term negation on the other hand *is* a judgement in that it ascribes a property to an individual. For instance, in the judgement 'Socrates is un-happy', 'un-happy' characterises Socrates by saying that he is in a certain state of being, which is more than the mere observation that he is not in a certain state.

the two notions of negation. It will then argue that the argument from indeterminacy should use a classical meta-logic, and show how object and meta negation interact on both, instance based and generic generality. This will then shed some new light on the question which logic the statements of generic generality license, and also provide a richer explanation of when and why LEM is not a logical truth with respect to generic generalisations.

## 5.2. Linnebo's "Generality Explained"

When considering the difference between instance based and generic generality, the most important (and only) resource besides Weyl is Øystein Linnebo's paper "Generality Explained"(Linnebo 2022a), where he develops a truthmaker semantics to illustrate their difference. Start out with a sentential model $\mathcal{S} = \langle S, \leq, \Vdash \rangle$ in which $S$ is a set of truth-makers called **states** and $(S, \leq)$ is a join semi-lattice with an initial state $\mathbb{0}$ and a terminal state $\mathbb{1}$. For all states $s, t$ there is a state $s \sqcup t$, their fusion, and it holds that $s, t \leq s \sqcup t$. For any atomic formula $A$, and state $s$, it is the case that either $s \Vdash A$ or $s \nVdash A$ with a monotonicity requirement that

$$s \Vdash A \qquad \Rightarrow \qquad t \Vdash A \text{ for all } t \geq s$$

$\mathbb{1}$ is the unique inconsistent state which is said to verify every formula $\varphi$.

For formulas $\varphi$, $\psi$ the clauses governing the sentential connectives are:

$$\begin{array}{lll}
\mathbb{1} \Vdash \varphi & & \\
\text{If } s \Vdash \bot & \text{then} & s = \mathbb{1} \\
s \Vdash \varphi \wedge \psi & \text{iff} & s \Vdash \varphi \text{ and } s \Vdash \psi \\
s \Vdash \varphi \vee \psi & \text{iff} & s \Vdash \varphi \text{ or } s \Vdash \psi \\
s \Vdash \varphi \rightarrow \psi & \text{iff} & \text{for each } t, \text{ if } t \Vdash \varphi, \text{ then } s \sqcup t \Vdash \psi,
\end{array}$$

and we define $\neg\varphi$ as $\varphi \rightarrow \bot$. Thus a state $s$ verifies $\neg\varphi$ if $\varphi$ is excluded from all consistent states extending $s$. The monotonicity condition on atomic formulas leads to monotonicity for every $\varphi$ (which can be proven by induction):

**Lemma 5.1.** *Let $\varphi$ be an arbitrary formula with no free variables. Then*

$$\textit{if } s \Vdash \varphi \textit{ then } t \Vdash \varphi \textit{ for all } t \textit{ s.t. } s \leq t$$

*Proof.* By induction. □

We say that a state space model $\mathcal{S}$ validates a formula $\mathcal{S} \vDash \varphi$ iff $s \Vdash \varphi$ for all states $s \in S$. Given monotonicity, this is equivalent to saying that $\mathbb{O} \Vdash \varphi$. Since not every state needs to verify either $\varphi$ or $\neg\varphi$, the sentential logic of truthmaking can be shown to be intuitionistic. However, the space may contain *maximally consistent* states which are characterised as "being opinionated" on every formula $\varphi$, i.e. for which either $s \Vdash \varphi$ or $s \Vdash \neg\varphi$ holds, and which consequently makes all instances of LEM valid.

To extend this system to first order truthmaking introduce a domain of objects $D$ and for each state $s$ a function $[\![\cdot]\!]_s$ from predicate letters to subsets of $D$ such that

$$s \Vdash P(a) \qquad \text{iff} \qquad a \in [\![P]\!]_s.$$

Just like in the propositional case, $[\![\cdot]\!]_s$ respects monotonicity.

$$[\![P]\!]_s \subseteq [\![P]\!]_t \text{ for all } t \geq s$$

We can introduce an existence predicate by using $[\![=]\!]_s$ which we take to be the domain $D(s)$ of objects that exists at stage $s$ or that stage $s$ is about. Furthermore, Linnebo (only) requires that a state does not differentiate between objects that it is not about, i.e.

$$\text{if } a, a' \notin D(s) \text{ then } s \Vdash \varphi(a) \Leftrightarrow s \Vdash \varphi(a')$$

The additional semantic clauses for the quantifiers are then:

$$s \Vdash \exists x \varphi(x) \qquad \text{iff} \qquad s \Vdash \varphi(a) \text{ for some } a \in D(s)$$

$$s \Vdash \forall x\varphi(x) \qquad \text{iff} \qquad s \sqcup t \Vdash \varphi(a) \text{ for every } t \text{ and every } a \in D(t)$$

Especially the clause for the universal quantifier exhibits the idea that even though a state $s$ may not contain all objects, nonetheless "material intrinsic to $s$"(Linnebo 2022a, p. 357) suffices to establish a universal generalisation. Thus, in this system it is possible to verify generalisations without requiring the availability of any of its instances, which captures adequately the character of *generic generality*. All that is required is that when a state $s$ satisfies a statement $\forall x\varphi(x)$ of generic generality, the information in $s$ is sufficient to determine that for any state $t$ and for any $a \in D(t)$ we have that $s \sqcup t \vDash \varphi(a)$. For example, any intuitionistic tautology is satisfied by the initial state $\mathbb{O}$ without it needing to contain all (or any) of the instances in the domain $D$. For all we know, $D(\mathbb{O})$ might be empty, but it is still the case that $\mathbb{O} \Vdash \forall x\neg(Rx \wedge \neg Rx)$.

To capture the notion of an *instance based generalisation* we can define the notion of a *totality-state* which serves to validate it. To establish notation, write $[\![\varphi]\!]_s$ for the extension of $\varphi$ in $s$, i.e. $[\![\varphi]\!]_s = \{a \in D \mid s \Vdash \varphi(a)\}$.

**Definition 5.1.** A state $s$ is a **totality state** for $\varphi$ iff

a) $s$ is consistent

b) $[\![\varphi]\!]_s \subseteq D(s)$

c) $[\![\varphi]\!]_t = [\![\varphi]\!]_s$ for every consistent $t$ such that $s \leq t$.

The point of this definition is that totality states for formulas verify instance based generalisations when (i) quantification is restricted to their respective domains and (ii) the formula in question is decidable. More technically:

**Lemma 5.2.** *Let $s$ be a totality state for $\varphi$. Then*

$$s \Vdash \forall y(\psi(y) \vee \neg\psi(y)) \rightarrow [\forall x(\varphi(x) \rightarrow \psi(x)) \vee \exists x(\varphi(x) \wedge \neg\psi(x))] \tag{5.3}$$

*Proof.* Assume that the antecedent holds for a state $t \geq s$. We need to show that

$$t \Vdash \forall x(\varphi(x) \rightarrow \psi(x) \vee \exists x(\varphi(x) \wedge \neg\psi(x))$$

The crucial case is given by the assumption that $t \nVdash \forall x(\varphi(x) \to \psi(x))$. Then there is a state $u \geq t$ and $a \in D(u)$ such that $u \Vdash \varphi(a) \wedge \neg\psi(a)$. But since $s$ is a totality state for $\varphi$, we have that $[\![\varphi]\!]_s = [\![\varphi]\!]_t = [\![\varphi]\!]_u$, and hence $a \in D(t)$. Now, because $\varphi$ is decidable, we additionally have $t \Vdash \neg\psi(a)$ and hence $t \Vdash \exists x\varphi(x) \wedge \neg\psi(x)$. □

However, just by the formulation of (5.3), it is not obvious when a certain state actually is a totality state for $\varphi$. What is needed is a candidate for the formula $\varphi$ such that if $s \Vdash \varphi(a)$ for some $a$ it automatically follows that $s$ is a totality state for $\varphi$. This is plausibly the case when $\varphi$ is of the form 'being the member of a set' or 'being a member of a plurality'. Both sets and pluralities are commonly characterised as having rigid membership and existence conditions. Hence for a state $s$, a set $y$ or a plurality $xx$, we understand that if $s$ verifies $x \in y$ or $x \prec xx$ then $y$, $xx$ and their members exist at $s$ and it is also the case that these conditions are rigid: neither pluralities nor sets may gain members. Adding, for instance, pluralities to our system, we can show that $\mathsf{BOM}$:

$$\mathcal{S} \vDash \forall y(\psi(y) \vee \neg\psi(y)) \to [\forall x(x \prec xx \to \psi(x)) \vee \exists x(x \prec xx \wedge \neg\psi(x))] \tag{5.4}$$

holds globally for any assignment to the plurality $xx$. A logical system that contains all intuitionistic validities, $\mathsf{BOM}$ and the decidability of atomic formulas is called **semi-intuitionistic**. Systems of this sort, which obtain a middle ground between intuitionistic- and full classical logic, were introduced and investigated in Feferman (2010).

This completes our characterisation of Linnebo's system. There are two points that are worth emphasizing. First, as the interpretation of the negation operator shows, Linnebo's systems reflects the strong understanding of negation in an intuitionistic setting. But what about the weaker notion of negation? Can this be incorporated as well? Second, as mentioned before, characteristic of his approach is the requirement that existential generalisations have a unique witness, which means that if $\mathbb{0} \Vdash \exists x\varphi(x)$ then $\varphi(a)$ for some $a$ such that $a \in D(s)$ for all $s$. This is enforced by the fact that the model is based on a rooted frame. Yet, one might ask if this might not be too strong of a condition. In line with the aim of modelling indeterminacy one might want to hold that there could be the possibility of filling in conceptual content in mutually inconsistent ways. In this respect, we might still be entitled to claim an existential generalisation if every such way conceivable would yield one (but not necessarily the same) witness. This results in a supervaluationist understanding of indeterminacy. Both of these points will be addressed in the following section.

## 5.3. Meta-logic and Negation

Using meta-logic we can distinguish between the two versions of negation and the two types of existential statements that were explicated in section 1. The logical system introduced in this section primarily aims at capturing these differences formally and applying them to Linnebo's system $\mathcal{S}$.

### 5.3.1. Object- and Meta-logic

I will begin by formally defining a meta-logic and arguing that its use for the argument from indeterminacy should be classical.

**Definition 5.2.** Let a **meta-language** be constructed from the symbols $\|, \&, \Rightarrow, \curlywedge$ which correspond to the disjunction, conjunction, implication, and the symbol for absurdity. Meta-quantifiers are expressed by $(x)$ for universal quantification and $\langle x \rangle$ for existential quantification.

**Formulas of the meta-language** are built up from formulas of an object language using meta-logical connectives. Thus each object logic formula is an atomic meta-formula and if $\varphi$ and $\psi$ are meta-logic formulas, then $\varphi \parallel \psi$ and $\varphi \Rightarrow \curlywedge$, etc., are meta-logical formulas. The **rules of the meta-logic** can be given in a standard natural deduction style, and the logic can be classical as well as intuitionistic.

The following theorem illustrates how object- and meta-logic work together.

**Theorem 5.1.** *Let $\varphi$ and $\psi$ be object logic formulas. The following principles hold for intuitionistic as well as for classical object* and *metalogic.*

a) $\varphi \land \psi \;\Leftrightarrow\; \varphi \& \psi$

b) $\forall x\varphi(x) \;\Leftrightarrow\; (x)\varphi(x)$

c) $\curlywedge \;\Leftrightarrow \bot$

d) $\varphi \;\|\; \psi \Rightarrow \; \varphi \vee \psi$

e) $\langle x\rangle\varphi(x) \;\Rightarrow\; \exists x\varphi(x)$

f) $\varphi \to \psi \;\Rightarrow\; (\varphi \Rightarrow \psi)$

*Proof.* By application from the rules and principles (Melikhov 2015, p.86). For instance, to prove a), observe from $\varphi \wedge \psi$ follows $\varphi$ and $\psi$, and since these are also atomic meta formulas, we have $\varphi\&\psi$ by the rules of the meta-conjunction. The same holds for the other direction. To prove f), assume $\varphi \to \psi$ and $\varphi$ to obtain $\psi$. This gives us a derivation from $\varphi$ to $\psi$, and hence $\varphi \Rightarrow \psi$, and furthermore, as this is a derivation from $\varphi \to \psi$ to $\varphi \Rightarrow \psi$, we obtain $\varphi \to \psi \Rightarrow (\varphi \Rightarrow \psi)$. $\square$

The reverse direction of d), e), and f) already fail in the case of classical object- and meta-logic. It is precisely the asymmetry in those cases that will account for different ways of understanding negation (case f) as well as offer a refined notion of Weyl's "judgement abstracts" (cases d and e). In order to give some counterexamples, however, we need to introduce some semantics.

**Definition 5.3.** (Melikhov 2015) Let $\mathcal{M}$ be a model of an object-logic with a domain $D$. A **two-valued classical meta model** $\mathcal{M}^+$, defined on top of $\mathcal{M}$, is given by the interpretation $|\cdot|_{\mathcal{M}+}$, which is defined for atomic meta-formulas $A$ by setting $|A|_{\mathcal{M}+} = \mathbf{1}$ iff $\mathcal{M} \vDash A$ and else $|A|_{\mathcal{M}+} = \mathbf{0}$. $|\cdot|_{\mathcal{M}+}$ is then extended to compound meta-formulas in the usual way:

| | | |
|---|---|---|
| $\lvert\varphi \;\|\; \psi\rvert_{\mathcal{M}+} = \mathbf{1}$ | iff | $\lvert\varphi\rvert_{\mathcal{M}+} = \mathbf{1}$ or $\lvert\psi\rvert_{\mathcal{M}+} = \mathbf{1}$ |
| $\lvert\varphi \Rightarrow \psi\rvert_{\mathcal{M}+} = \mathbf{1}$ | iff | $\lvert\varphi\rvert_{\mathcal{M}+} = \mathbf{0}$ or $\lvert\psi\rvert_{\mathcal{M}+} = \mathbf{1}$ |
| $\lvert(x)\varphi(x)\rvert_{\mathcal{M}+} = \mathbf{1}$ | iff | $\lvert\varphi(a)\rvert_{\mathcal{M}+} = \mathbf{1}$ for all $a \in D$ |
| $\lvert\langle x\rangle\varphi(x)\rvert_{\mathcal{M}+} = \mathbf{1}$ | iff | $\lvert\varphi(a)\rvert_{\mathcal{M}+} = \mathbf{1}$ for some $a \in D$. |

Now, to construct counterexamples to the reverse direction of cases d), e), and f) of theorem 5.1, consider a model $\mathcal{B}$ given by the boolean algebra on $\mathscr{P}(\{1,0\})$ in which the logical operations are interpreted by their set-theoretic counterparts. Thus, e.g. $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$ and $\mathcal{B} \vDash \varphi$ iff $[\![\varphi]\!] = \{1,0\}$. Extend this model to interpret meta-formulas according to the above definition to obtain $\mathcal{B}^{+}$.

To falsify $(\varphi \vee \psi) \Rightarrow (\varphi \parallel \psi)$ let $[\![\varphi]\!] = \{1\}$ and $[\![\psi]\!] = \{0\}$, thus $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!] = \{1,0\}$ in the object logic, so $|\varphi \vee \psi|_{\mathcal{B}+} = \mathbf{1}$, but since neither $|\varphi|_{\mathcal{B}+} = \mathbf{1}$ nor $|\psi|_{\mathcal{B}+} = \mathbf{1}$ we don't have $|\varphi \parallel \psi|_{\mathcal{B}+} = \mathbf{1}$, so $|\varphi \vee \psi \Rightarrow \varphi \parallel \psi|_{\mathcal{B}+} = \mathbf{0}$.

To falsify $\exists x\varphi(x) \Rightarrow \langle x\rangle\varphi(x)$ follow the same method. Enrich the above model with a domain $D = \{a,b\}$ and set $[\![\varphi]\!](a) = \{0\}$ and $[\![\varphi]\!](b) = \{1\}$. Then $[\![\exists x\varphi(x)]\!] = [\![\varphi(a)]\!] \cup [\![\varphi(b)]\!] = \{1,0\}$, but $|\langle x\rangle\varphi(x)|_{\mathcal{B}+} = \mathbf{0}$.

To falsify $(\varphi \Rightarrow \psi) \Rightarrow (\varphi \rightarrow \psi)$ consider the case of $\psi = \perp$ and $[\![\varphi]\!] = \{0\}$. This gives us $|\varphi|_{\mathcal{B}+} = \mathbf{0}$ and hence $|\varphi \Rightarrow \perp|_{\mathcal{B}+} = \mathbf{1}$, but $[\![\varphi \rightarrow \perp]\!] = [\![\varphi]\!]^{c} \neq \{1,0\}$. This example also admits an intuitive explanation. While $\varphi \Rightarrow \curlywedge$ notes that we cannot establish $\varphi$ (for whatever reason), $\varphi \rightarrow \perp$ says something stronger, namely that we are indeed able to derive a contradiction from its assumption. This should suffice to give an initial impression how the evaluation of meta-logical statements work.

### The question of higher order indeterminacy

Extending $\mathcal{S}$ in a similar way to a meta-logic will allow us to accommodate the weaker notion of negation. But when doing so, we need to answer the question: Is a two valued and hence classical meta-logic adequate for this task? Contrary to the position of the traditional intuitionist, for the argument from indeterminacy, it seems that a classical meta-logic is unproblematic, and, in fact, even desired. It was mentioned before that the negation of a meta-judgement characterises the fact that a certain conceptual entailment does not obtain. But since we are dealing with facts in this respect, we should be entitled to claim that a certain conceptual entailment either obtains or doesn't obtain. In other words: there is no indeterminacy about what's indeterminate. Conversely, any rejection of a classical meta-logic would have to be based on an account of higher-order indeterminacy. But on this account it should not matter whether *we* can establish that something is indeterminate (although in most cases we can). For instance, a statement $\varphi$ is either independent of ZFC and hence indeterminate, or it isn't; either there is a proof of it or of its negation from ZFC or there isn't. To introduce indeterminacy about this, we would have to locate it in the notion of proof itself (and not in our lack of knowledge of it), and while I don't want to exclude the possibility of this outright, it does not seem to be entailed by the analysis of indeterminacy given in chapters 1 and 2. Thus, the possibility to use classical meta-reasoning in this respect is the second major point (besides the use of a truth-conditional semantics) that sets the argument from indeterminacy apart from traditional intuitionism.

### 5.3.2. Varieties of negation and generality statements

Now that the logical machinery is in place, we can use it to investigate the interaction between different kinds of negations and different kinds of generality. First, however, note a more troublesome feature of $\mathcal{S}$ that comes to the fore when extending it to a meta-logic:

**Lemma 5.3.** *Let $\mathcal{S}$ be a state space model. Then $\mathcal{S}^+ \vDash \langle x \rangle \varphi(x) \Leftrightarrow \exists x \varphi(x)$.*

*Proof.* a) The definition of satisfaction in $\mathcal{S}^+$ gives us:

$$\begin{aligned} \mathcal{S}^+ \vDash \langle x \rangle \varphi(x) \quad & \text{iff} \quad \mathcal{S} \vDash \varphi(a) \text{ for } \textit{some} \text{ parameter a} \in D \\ & \text{iff} \quad \mathcal{S} \vDash \exists x \varphi(x) \end{aligned}$$

where the last inference follows, because the frame is rooted, i.e. because it has an initial node $\mathbb{O}$. □

This feature takes away a layer of differentiation that will become important in understanding Weyl's notion of judgement abstract, which in turn plays a role in the evaluation of object level negation. In order to take this into account, we can give a slight extension to Linnebo's system. Leaving the interpretation of the logical connectives, constants, and variables identical, we obtain an enlarged class of models $\mathcal{C}$ by extending the class of join-semi lattices to those which don't have a root state $\mathbb{O}$. We can then extend this to a system $\mathcal{C}^+$ to incorporate the meta-logical machinery.

### Object and meta-level generalisations

For the interpretation of universal and existential generalisations, we now have:

**Lemma 5.4.** a) $\mathcal{C}^+ \Vdash (x)\varphi(x) \;\Leftrightarrow\; \mathcal{C}^+ \Vdash \forall x\varphi(x)$

b) $\mathcal{C}^+ \Vdash \langle x\rangle\varphi(x) \;\Rightarrow\; \mathcal{C} \Vdash \exists x\varphi(x)$

c) $\mathcal{C} \Vdash \exists x\varphi(x) \;\not\Rightarrow\; \mathcal{C}^+ \Vdash \langle x\rangle\varphi(x)$

*Proof.*

$$
\begin{aligned}
\text{For a) consider:}\quad & \mathcal{C}^+ \Vdash (x)\varphi(x) &&\Leftrightarrow\; s \Vdash \varphi(a) \text{ for all } s \in C \text{ and all } a \text{ in } D(s)\\
& &&\Leftrightarrow\; \mathcal{C} \Vdash \forall x\varphi(x)\\
\text{For b) consider:}\quad & \mathcal{C}^+ \Vdash \langle x\rangle\varphi(x) &&\Leftrightarrow\; \mathcal{C}^+ \Vdash \varphi(a) \text{ for some } a \in \bigcap_s D(s)\\
& &&\Leftrightarrow\; \mathcal{C} \Vdash \varphi(a) \text{ for some } a \in \bigcap_s D(s)\\
& &&\Rightarrow\; \mathcal{C} \Vdash \exists x\varphi(x) \;\Leftrightarrow\; s \Vdash \exists x\varphi(x) \text{ for all } s \in C.
\end{aligned}
$$

For a counterexample to c) consider a model $\mathcal{C}$ with a frame that has two roots $s$ and $t$ and $D = \{a, b\}$, s.t. $s \Vdash \varphi(a)$ and $t \Vdash \varphi(b)$ but not vice versa. Then $\mathcal{C} \Vdash \exists x\varphi(x)$, but there is no $c \in \bigcap_s D(s)$ s.t. $\mathcal{C} \Vdash \varphi(c)$. $\square$

As expected, the truth of an object-level universal generalisation is equivalent to that of a meta-level generalisation. While the meta-existential generalisation corresponds to an actual construction, the object-level one, I now claim, offers a refinement to Weyl's notion of a judgement abstract. For, in addition to Weyl's characterisation, the truth of $\exists x\varphi(x)$ does not merely entail the possibility of a construction, but *also guarantees that there is a witness on any possible state.* The difference between the two existential statements could best be seen as a difference in demand on truthmaking. While the first one requires a *unique* truthmaker for all refinements, the second one just requires there to be one. With that in place, we can now consider negation. How should we conceive of the different ways of negating statements of generic and instance based generality?

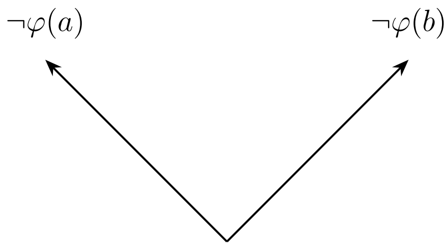
**Negations of generic generality**

From theorem 5.1 it follows that meta-negation does not generally imply object-level negation:

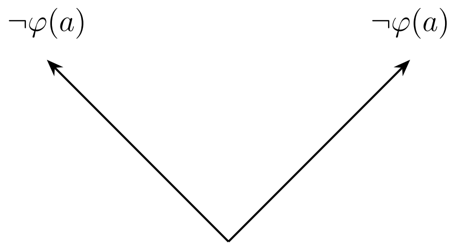
$$[\,(x)\varphi(x) \Rightarrow \curlywedge\,] \not\Rightarrow\ \neg\forall x\varphi(x)$$
$$[\,\langle x\rangle\varphi(x) \Rightarrow \curlywedge\,] \not\Rightarrow\ \neg\exists x\varphi(x)$$

Thus, when dealing with generality statements on the meta-level, it is in fact our understanding of the negation operation and thereby of meta-implication that limits our inferences. Applied to statements of generic generality, this captures the intuition that denying that something lies in the essence of a concept does not yet entail that any absurdities follow from additional assumptions to establish that it does. Considering again example c), the fact that it doesn't lie in the rules of the test that everyone passes does not imply that there cannot be a possible world or state of affairs in which everyone does.

Now what to make of the stronger negation of generic generality in the object logic such as $\neg\forall x\varphi(x)$ and $\neg\exists x\varphi(x)$? Assume that $\neg\forall x\varphi(x)$ holds. Previously, there have been two possible scenarios:

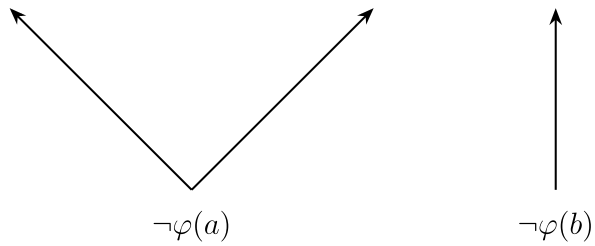



$$\models \neg\forall x\varphi(x) \quad \not\models \neg\varphi(a)$$
$$\not\models \exists x\neg\varphi(x) \quad \not\models \neg\varphi(b)$$




$$\models \neg\forall x\varphi(x) \quad \models \neg\varphi(a)$$
$$\models \exists x\neg\varphi(x)$$

Now there is a third one, namely




$$\begin{array}{ll} \vDash \neg\forall x\varphi(x) & \nvDash \neg\varphi(a) \\ \vDash \exists x\neg\varphi(x) & \nvDash \neg\varphi(b) \end{array}$$

This option reflects the fact that denying that there is a unique counterexample does not imply that there cannot be a *global covering of local counterexamples.*

**Negations of instance based generality**

To evaluate the negation of instance based generalisations, consider a state $s$ and assume that it is a totality state for $\varphi$, and that $\psi$ is decidable. Now, for object level negation of course it still holds that

$$s \Vdash \neg\forall x(\varphi(x) \to \psi(x)) \quad \Rightarrow \quad s \Vdash \exists x(\varphi(x) \wedge \neg\psi(x))$$

as shown in the previous section. What about meta-negation? Note that in light of theorem 5.1, $\forall x\psi(x) \Rightarrow \curlywedge$ generally does not even entail $\neg\forall x\psi(x)$, let alone $\exists x\neg\psi(x)$. But if $\forall x\psi(x)$ is an instance based generalisation, then its meta negation even allows us to infer $\langle x\rangle\neg\psi(x)$. To see this, let $s$ be a totality state for $\varphi(x)$ and consider the model $\mathcal{C} \restriction R^{\uparrow}(s)$ based on $\mathcal{C}$ by restricting its frame to $(S \restriction R^{\uparrow}(s), \leq)$. Then

$$(\mathcal{C} \restriction R^{\uparrow}(s))^{+} \Vdash \forall x(\varphi(x) \to \psi(x)) \Rightarrow \curlywedge \quad \text{entails} \quad (\mathcal{C} \restriction R^{\uparrow}(s))^{+} \Vdash \langle x\rangle(\varphi(x) \wedge \psi(x))$$

This should illustrate the remarkable power that instance based generalisations contain. Still, understanding them this way has a high degree of plausibility. For instance, if I say that it is not the case that all of *these* people will pass the test, then it means that one of *them* will not pass it.

The final task for this chapter is to make a connection to the general framework developed in chapter 4, and thus to connect the analysis of generality and negation, and in particular the inferences that it licenses to the notion of a warrant, and to show that is can be understood exclusively in terms of the availability of objects without predetermining (like the traditional intuitionist) a different meaning of the logical constants in the process. As this is an important but mostly a technical issue, it will be relegated to an appendix.

## 5.4. Conclusion

In this chapter the notion of generic generality was introduced and distinguished from instance based generalisations. It was argued that there are two ways to understand in particular the negation of statements of generic generality: an object level negation and a meta-negation. It was also argued that there are two ways to understand an existential statement: a strong one, requiring the existence of a unique witness; and a weaker one, based on a supervaluationist understanding which (only) requires a complete covering of localised witnesses. Regarding meta-level negation, a case was made for the argument from indeterminacy to license a classical meta-logic. Regarding object level negation, it is the case that

$$\mathcal{M} \Vdash \neg\forall x\varphi(x) \quad \Rightarrow \quad \mathcal{M} \Vdash \exists\neg\varphi(x)$$

either if there is a global counterexample or if on every possible refinement the existence of a counterexample is determined.

What this shows is that whether LEM holds cannot simply be decided by considering the type of generality employed. Generic generality can be understood to give us a lower bound on the logic that is applicable. But this does not mean that there might not be applications of generic generality for which classical logic (in a supervaluationist understanding) is appropriate. A decision between the two now comes down to how these individual notions of indeterminacy are being modelled. This will be the topic of part III.

## 5.A. Appendix: Generalised Notion of Instance Based and Generic Generality

This appendix shows how the difference between intuitionistic and classical object level negation can be explained in terms of warrants that are exclusively specified by the availability of objects. It thus gives us the assurance that we can use frame based models like Linnebo's and their extensions all the while maintaining a neutral view on the meaning of the logical constants as their are evaluated in them. In order to show this, we need to express the difference between generic and instance based generality in topological models. This requires some adjustments to the standard topological semantics.

### 5.A.1. A topological model for generic generality

Usually, first order topological models are defined in the following way:

**Definition 5.4.** A **first order topological model** is a triple $\mathcal{T} = \langle (X, \mathcal{O}_X), D, [\![\cdot]\!] \rangle$, where $D$ is a domain of objects, and $[\![\cdot]\!]$ interprets each atomic formula $P(\cdot)$ as a function $[\![P]\!](\cdot)$ from $D$ into $\mathcal{O}_X$. $[\![\cdot]\!]$ is then defined on sentential connectives as in definition (4.4) and on quantified formulas as

$$[\![\forall x \varphi(x)]\!] = (\bigcap_{a \in D} [\![\varphi]\!](a))^\circ$$
$$[\![\exists x \varphi(x)]\!] = \bigcup_{a \in D} [\![\varphi]\!](a)$$

But since it is exactly the correspondence between warrants (expressed by states in Linnebo's model) and the availability of objects that we are interested in, such a global domain won't do.

#### The availability of objects in topological models

Instead, consider the following. We need for each open subset $U$ a corresponding domain $D_U$ (in analogy to the states in $\mathcal{S}$). We then require for the order of inclusions in the open sets

$$U \supset V \supset W$$

a corresponding reverse order of domain inclusions

$$D_U \subseteq D_V \subseteq D_W,$$

i.e. the finer the refinement, the more objects it may contain.

This can be worked out the following way: given a topology $\mathcal{O}_X$ and a domain of objects $D$, define a contravariant functor $\mathcal{F} : \mathcal{O}_X \to \mathscr{P}(D)^{op}$ with $\mathcal{O}_X$ and $\mathscr{P}(D)$ both ordered by inclusion. I.e. $\mathcal{F}$ is given by the diagram:

$$\begin{tikzcd} U \arrow[r] & \mathcal{F}(U) \arrow[d, hook] \\ V \arrow[r] \arrow[u, hook] & \mathcal{F}(V) \end{tikzcd}$$

with $V \subset U$ for $V, U \in \mathcal{O}_X$. This construction guarantees that to each subset there corresponds a certain collection of elements (i.e. a subset of $D$) that satisfy monotonicity.

Then, to define satisfaction for atomic predicates, we use an auxiliary function

$$[\![P]\!]_U(x) : \mathcal{F}(U) \to \{U, \emptyset\}$$

for each $U$, such that $[\![P]\!]_U(a) = U$, when refinement $U$ establishes $P(a)$, $[\![P]\!]_U(a) = \emptyset$ when $a$ is available at $U$ but it does not hold that $P(a)$ and, importantly, $[\![P]\!]_U(a)$ is undefined for $a \notin \mathcal{F}(U)$. (Here the model differs from Linnebo's. While Linnebo just requires that states don't distinguish between objects that don't yet exist, the topological models leave these cases undefined.) Now it follows that if $\mathcal{F}(U) \subset \mathcal{F}(V)$, then $dom([\![P]\!]_U) \subset dom([\![P]\!]_V)$, corresponding again to $U \supset V$. To arrive at the 'usual' interpretation, set

$$[\![P]\!](a) = \bigcup_U [\![P]\!]_U(a).$$

We can use this info to give an account of *instance based generality.* For this, the notion of a totality state is required. I modify Linnebo's definition slightly in order to accommodate the above difference.

**Definition 5.5.** A **totality state** for a formula $\varphi(x)$ is an open set $U$ such that for any $a$

$$[\![\varphi]\!]_U(a) = \emptyset \text{ or undefined} \qquad \text{iff} \qquad [\![\varphi]\!]_V(a) = \emptyset \text{ for any } V \subset U$$

In other words, further refinements do not introduce satisfaction of $\varphi(x)$ for further elements. This means that if $U$ is a totality state, then there is no further refined $V \subset U$ such that $[\![\varphi]\!]_V(a) = V$ for any $a \in \mathcal{F}(V) \setminus \mathcal{F}(U)$.

#### Expressing generic generality in topological models

As the matter stands right now, there is a slight problem with expressing generic generality. If we consider $[\![\varphi]\!](a) = \bigcup_U [\![\varphi]\!]_s(a)$ with $[\![\cdot]\!]_U$ as a measure of existence at $U$, then $\forall x\varphi(x) = \bigcap_{a\in D} [\![\varphi]\!](a)$ is, if true, always instance based: For each $a$ we would need to have $[\![\varphi]\!](a) = X$. For a statement of generic generality to be true, however, it has to be sufficient that it follows from the concepts involved, which means that it should not make any overt existence claims. To model this, we need to adjust the interpretation of the universal quantifier with an existence predicate $x = x$, such that

$$[\![\forall x\varphi(x)]\!] = (\bigcap_{a\in D} [\![a = a]\!]^c \cup [\![\varphi]\!](a))^\circ$$

The interpretation of the existence predicate is similar to the other primitive ones $[\![x = x]\!] = \bigcup_U [\![x = x]\!]_U$, with

$$[\![x = x]\!]_U = \begin{cases} U & \text{if } x \in \mathcal{F}(U) \\ \emptyset & \text{else} \end{cases}$$

Furthermore, it follows from $U$ being a totality state for $\varphi$ that all objects falling under $\varphi$ exist at $U$. The interpretation of the existential quantifier remains unchanged in all this. Taking all together, we arrive at the following model.

**Definition 5.6.** A **topological model for generic generality** is a quadruple $\mathcal{T} = \langle (X, \mathcal{O}_X), D, [\![\cdot]\!]_\xi, \mathcal{F}\rangle$ where $\mathcal{F}$ and $[\![\cdot]\!]_\xi$ are defined as above. For atomic predicates, set

$$[\![P]\!](a) = \bigcup_U [\![P]\!]_U(a).$$

The sentential connectives are the usual ones from definition (4.4), and quantified statements are evaluated as:

$$[\![\forall x\varphi(x)]\!] = (\bigcap_{a\in D}[\![a = a]\!]^c \cup [\![\varphi]\!](a))^\circ$$
$$[\![\exists x\varphi(x)]\!] = \bigcup_{a\in D}[\![\varphi]\!](a)$$

With respect to the clauses for the quantifiers we can observe again that the truth of a universally quantified statement does not require the existence of any objects while the truth of an existential quantifier also allows for global covering of localised counterexamples.

### Linnebo's state space as a topological model

We can conceive of Linnebo's state space models as a special case of our topological models. We can construct a topology out of a state space frame by taking for each node $s$ its upset $R^\uparrow(s) = \{t|s \leq t\}$ to be a basic open set, where we only choose the consistent states, and where we then have in particular $R^\uparrow(\mathbb{O}) = S$. Call a model with the interpretation of the universal quantifier like above *existentially guarded*, then we should have the following:

**Theorem 5.2.** *To each rooted state space model $\mathcal{S}$ there corresponds a topological model $\mathcal{T}_\mathcal{S} = \langle (\mathcal{O}_S, S), D(\xi), [\![\cdot]\!]_\xi\rangle$, where $(\mathcal{O}_S, S)$ is given by the basis $\mathcal{B} = \{R^\uparrow(s) \mid s \in \mathcal{S}\}$ with the following specifications:*

- *$\mathcal{T}_\mathcal{S}$ is existentially guarded.*

- *To each predicate $P$ there exists a family of functions $[\![P]\!]_\xi : D(\xi) \to \mathcal{O}_S$ with*

$$[\![P]\!]_r(a) = \begin{cases} R^{\uparrow}(r) & \text{if } r \Vdash P(a) \\ \emptyset & \text{else} \end{cases}$$

*such that $[\![P]\!] = \bigcup [\![P]\!]_\xi$,*

*such that for any formula $\varphi$*

$$[\![\varphi(x)]\!] = S \quad \text{iff} \quad \mathcal{S} \Vdash \varphi$$

We can also note the following

**Corollary 5.1.** *$s$ is a totality state for $\varphi$ in $\mathcal{S}$ iff $[\![\varphi]\!]_{R^{\uparrow}(s)}[D] = [\![\varphi]\!]_{R^{\uparrow}(s')}$ for all $R(s') \subset R(s)$.*

Finally we obtain

**Lemma 5.5.** *The direction of theorem 4 is not reversible, i.e. there is a topological model to which there is no rooted state space model $\mathcal{S}$ such that $\mathcal{F}(U) = D(U)$ in $\mathcal{S}$ for all $U$ and for any formula $\varphi$*

$$[\![\varphi(x)]\!] = S \quad \text{iff} \quad \mathcal{S} \Vdash \varphi$$

*Proof.* Consider a model validating $\exists x \varphi(x)$, where sets $[\![\varphi]\!](a)$ for $a \in D$ is a non-trivial open cover of $\mathcal{O}_S$. Then there is no $a$ such that $[\![\varphi]\!](a) = S$ and hence there is no $a$ such that $\mathbb{O} \Vdash \varphi(a)$, and thus $\mathbb{O} \nVdash \exists x \varphi(x)$. □

Incidentally, this construction is also a case of the refined notion of judgement abstract discussed at the end of section 5.1.3.

## 5.A.2. Negation and generality in the generalised setting

With these definitions in place, let's first make sure that instance based generality indeed yields classical logic. In the following, let $\mathcal{T}$ be a topological model for generic generality as in definition 5.6. Let $\mathcal{T}^+$ be its extension to a two-valued meta-model in the way introduced in section 5.3.1.

### Negations of instance based generality

**Lemma 5.6.** *Let $U$ be a totality state for $\varphi$ and (for simplification) let $\psi$ be decidable. Consider the model $\mathcal{T} \upharpoonright U = \langle \mathcal{O}_X \upharpoonright U, D, [\![\cdot]\!]_V, \mathcal{F} \upharpoonright (\mathcal{O}_X \upharpoonright U)\rangle$ where $\mathcal{O}_X \upharpoonright U = \{V \cap U \mid V \in \mathcal{O}_X\}$ and $D$, $[\![\cdot]\!]_V$, and $\mathcal{F}$ are like in $\mathcal{T}$. Then*

$$(\mathcal{T} \upharpoonright U)^+ \Vdash \forall x(\varphi(x) \to \psi(x)) \Rightarrow \curlywedge \quad \textit{entails} \quad (\mathcal{T} \upharpoonright U)^+ \Vdash \langle x\rangle(\varphi(x) \wedge \psi(x))$$

*Proof.* Assume $[\![\forall x\varphi(x)(\psi(x))]\!] \neq U$. Then there is an $V \subset U$ and an $a \in D_V$, such that $[\![\varphi]\!]_V(a) = V$, but $[\![\psi]\!]_V(a) \neq V$. Since $U$ is a totality state for $\varphi$, we have $[\![\varphi]\!]_U(a) = U$ and hence $a$ exists at $D_U$. We thus need to show that $[\![\neg\psi]\!](a) = U$. But this follows from the fact that $\psi$ is decidable. □

Insofar as we are dealing with a totality state $U$, for all objects $a$ over which we quantify, it is *a fortiori* the case that $U \subset [\![a = a]\!]$. If we restrict the topology to $U$, then either $[\![\psi]\!](a) = U$ and hence $([\![\psi]\!](a))^c = (([\![\psi]\!](a))^c)^\circ = \emptyset$ or the other way around. This reflects the fact that the existence or availability of an object guarantees all required warrants. Again, this lets us infer the strongest notion of existence from the weakest notion of negation, thereby showcasing once again the power of instance based generalisations.

### Negations of generic generality

To investigate negation when applied to statements of generic generality consider $\forall x\varphi(x) \Rightarrow \curlywedge$ first. According to the two-valued semantics, the meta negation is true if the antecedent is not satisfied. Under this assumption we have

$$\bigcap_a ([\![a = a]\!]^c \cup [\![\varphi]\!](a))^\circ < X$$

which implies that $(\llbracket a = a \rrbracket^c \cup \llbracket \varphi \rrbracket(a))^\circ < X$ for at least one $a \in D$. But that does not imply that there is an open set $U$ such that $\llbracket a = a \rrbracket \cup (\llbracket \varphi \rrbracket(a)^c)^\circ = U$. Thus, the meta-negation is already valid if there is one possible refinement for which we can't establish that $\varphi$ for some object $a$.

Regarding object level negation, however, it matters if the topology is connected or not.

$$\mathcal{T} \Vdash \neg \forall x \varphi(x) \qquad \text{iff} \qquad (((\bigcap_a (\llbracket a = a \rrbracket^c \cup \llbracket \varphi \rrbracket(a))^\circ)^c)^\circ = X$$

For simplicity, assume that $D = \{a, b\}$. Then it has to be the case that

$$(\llbracket a = a \rrbracket^c \cup \llbracket \varphi \rrbracket(a))^\circ \cap (\llbracket b = b \rrbracket^c \cup \llbracket \varphi \rrbracket(b))^\circ = \emptyset \tag{5.5}$$

The question now is whether it is also the case that

$$((\llbracket \varphi \rrbracket(a)^c)^\circ \cap \llbracket a = a \rrbracket) \cup ((\llbracket \varphi \rrbracket(b)^c)^\circ \cap \llbracket b = b \rrbracket = X \tag{5.6}$$

and what is at issue if not.

#### Warrants as availability of objects

This is exactly the scenario that was discussed at the beginning of section 5.1.2, were it was noted that insofar as the conjunction is given, the deMorgan rule should be applicable. Consequently, there are two reasons for (5.6) to fail even though (5.5) is true. One reason is that either one of $(\llbracket \varphi \rrbracket(a)^c)^\circ$ and $(\llbracket \varphi \rrbracket(b)^c)^\circ$ is too small, the other is that one of $\llbracket a = a \rrbracket$ and $\llbracket b = b \rrbracket$ is too small. But since we are only interested in the availability of objects this means that insofar as $a$ or $b$ exists, they either have $\varphi$ or they have $\neg\varphi$. Therefore, only option two is a possible scenario for the argument from indeterminacy as it is the only option that reflects whether or not the conjunction may count as given. The conjunction may *not* be considered as given, precisely if the existence of $a$ and $b$ is accordingly limited.

In this respect, note that even if (5.5) holds it can be the case that

$$[\![a = a]\!]^c \cap [\![b = b]\!]^c \neq \emptyset,$$

which is due to the outer interior operations. In this case, we have that

$$[\![a = a]\!] \cup [\![b = b]\!] \neq X$$

And this is the point where the story fully connects to the last chapter. If we have that $([\![a = a]\!]^c)^\circ = [\![a = a]\!]^c$ and the same for $b$, then the previous two inequalities would become equalities and thus, all other things equal, we could infer a weak existential statement, i.e. we'd get a covering of local counterexamples, and thus LEM. In this case, there is no indeterminacy about the availability (or unavailability) of $a$ and $b$. This illustrates how indeterminacy regarding the (un)availability of objects *and that alone* may account for the failure of LEM.

Thus, the investigation of generic generality via topological models has shown how failure of the duality of the quantifiers can solely be due to the (un)availability of some objects, while retaining the neutral perspective on the meaning of the logical constants that was introduced in the last chapter.

# Part III.

# LEM and Indeterminacy

# 6. LEM and Indefinitely Extensible Domains

Now that our conception of indeterminacy as well as the requirements of LEM in connection to the availability of objects is in place, we can ask how this affects quantification. This chapter focuses on indeterminacy of height (i.e. indefinite extensibility) while the next one focuses on indeterminacy of width. According to Dummett, it is not possible to form a general conception of the domain of an indefinitely extensible concept, and for this reason, general statements over an indefinitely extensible domain need to "be able to cite an instance or an effective computation"(Dummett 1991a, p. 315), else they won't yield determinate truth conditions. These requirements, so he claims, can no longer sustain the assertability of LEM. This chapter assesses the ins and outs of this claim.

I will give an analysis of Dummett's claim that with respect to indefinitely extensible concepts it is not possible to form a general conception of their domain, and further show how this excluded an instance based way of quantifying over them. I will then suggest an alternative way of understanding generality over indefinitely extensible domains which will be expressed in a potentialist way. The modal resources, however, that were evoked to solve the problem that was initially noted are under suspicion to succumb to an analogue (revenge) problem. A closer investigation into the modality used to explicate the generalisations in question will help to dispel such worries. With the language enriched with modal operators, however, we need to reformulate the question regarding LEM. This will be done via well known translation procedures from modal formulas to non-modal formulas.

To this extent I will discuss three variants of potentialism: liberal-, branching-, and strict potentialism. Liberal potentialism will turn out to be the most natural representation of reasoning over indefinitely extensible concepts but it also validates LEM. Branching potentialism and strict potentialism do not validate LEM. Branching potentialism does not seem to be applicable to the cases of indefinite extensibility that we are interested in, while strict potentialism seems to flirt with intensionality.

Section 1 analyses the problem of quantifying over indefinitely extensible domains, section 2 gives a formalisation of an alternative proposal involving modal resources, section 3 discusses the nature of the modality involved, while section 4 addresses the varieties of potentialism, and section 5 establishes the connection to the previous chapters. In an appendix I will give proofs of the theorems mentioned in the chapter and in doing so provide a connection to the notion of a warrant that is exclusively given by the availability of objects.

## 6.1. Quantification over Indefinitely Extensible Totalities

In order to pinpoint the particular problem that indefinitely extensible domains might pose to quantification, it is instructive to see why instance based generality is not applicable to them. This section follows Dummett's analysis of the consequences that indefinite extensibility has for instance based generalisations in order to glean the required resources for an alternative proposal.

As noted in the previous chapter, on an instance based notion of generality the statements $\forall x\varphi(x)$ and $\exists x\varphi(x)$ can be understood as conjunction and disjunction of individual applications of $\varphi(x)$ to each object in the domain. This way of understanding quantification, Dummett notes, has two important requirements:

> [w]e understand the universally quantified statement because we have, as it were, a *general* grasp of the totality which constitutes the domain of quantification [...] and because we know what it is for the predicate to which the quantifier is attached to be true or false for any arbitrary element of this domain.(Dummett 1973, 517f.)

Since the extensibility of the domain of quantification is the only source of indeterminacy, we already know from the previous chapter that we can count on the second requirement to be fulfilled for a formula $\varphi$ insofar as $\varphi$ does not itself contain a quantifier whose meaning is in question. In this case, it is determined whether $\varphi(a)$ or $\neg\varphi(a)$ for any given object $a$. The central question is thus: What counts as a *general grasp* of a totality, and in particular, why does indefinite extensibility prohibit us from even forming such a general conception of the domain?

There are a few candidates to what one might understand by such a conception:

a) Being able to decide whether a given element is part of the domain or not.

b) Given an arbitrary element of the totality, being able to infer properties of such an element solely based on knowledge of its membership in the totality.

c) Being able to give a general description of an arbitrary element of the totality.

While a) seems to be an undisputed assumption in this context, there is an important difference between the last two points. In particular, c), which is a stronger requirement than b), seems to be what is needed for instance based generalisations. Only such a general description manages to collect all the elements that are to form part of the conjunction. It is crucial, however, to come to terms with what such a general description can look like. Dummett specifies this as saying that such a general description is given by the "kind of expression which can stand for an element of the totality"(Dummett 1973, p. 532). This is naturally inspired by the Fregean account of objects as possible references of singular terms (Frege et al. 1951, cf. Dummett 1973, ch.14, Linnebo 2018d, ch.2). As is well known, not every object needs to actually have a corresponding referencing expression on this account, it is merely characterised as the sort of entity which could be picked out by such an expression.[1] Presumably, Dummett means with "kind of expression" a canonical referring expression that manifests the fact that we are talking about an object of the respective domain. This allows us to express properties of those objects by reference to those expressions, and thereby incorporates our conception of language into our conception of objecthood.

Thus, our grasp of the domain of quantification, when taken to be a grasp of a general description of an arbitrary element of it is given in terms of our understanding of the

[1]In this respect, no restrictions will come from the mere countability of our language.

kind of expressions that serve to pick out such an element. There may, however, be a problem with that if such expressions themselves involve quantification.

> If such expressions include ones involving the use of variables ranging over that totality, then their reference is not independent of the specification of the totality, and we have indeed fallen into vicious circularity. (Dummett 1973, p. 532)

The problem is that we cannot simply determine the reference of such a variable by saying that it refers to an arbitrary element of the domain because such a characterisation is what we are looking for in the first place. In other words, we fail to "find a non-circular means to saying of what they consist"(Dummett 1973, p. 534). Using an impredicative definition leaves us therefore with a lack of specification. We haven't succeeded in specifying the objects we sought to determine, because for the specification to succeed it would require the range of objects of the impredicative quantifier to be determinate. This way of specifying the extension of a reflexively indefinitely extensible concept is thus circular.[2]

But why is this circularity vicious? This relates to the discussion of reflexively(!) indefinitely extensible concepts from chapter 1. Use as an example the ordinals and assume that we have succeeded in specifying a domain $\Omega$ of ordinals. Since we know that for any collection of ordinals we can specify an ordinal greater than each in the collection, we can apply the diagonal function $\delta$ to $\Omega$ such that $\delta(\Omega) = \Omega \cup \{\Omega\}$. Now since $\Omega \cup \{\Omega\}$ is still an ordinal, and since we have assumed that $\Omega$ comprises all of the ordinals, we are forced to wrongly conclude $\Omega \cup \{\Omega\} \in \Omega$ and thereby $\Omega \in \Omega$. Hence, by using the conception "greater than any ordinal in the collection" (which in the von Neumann way is just given by set formation operator) we can enrich any given collection of ordinals and thereby add to the concept of what an arbitrary element of the domain might consist in.

Following the discussion in chapter 1.3, the same can be expressed somewhat more perspicuous by using pluralities. There it was noted that the principles of *Plural Col-*

[2]The same worry is purported in the context of higher order impredicative quantification in Wright (2021). Note that even if we assume that the elements of the domain exist independently, as suggested by Ramsey (1990) and later on by Gödel (1944) consequently only goes so far in resolving the issue. The problem about *our way of characterising them* nonetheless persists. This was already noted in the discussion of realism at the end of chapter 3. Even if we hold that the ordinals exist in any realist fashion, this would be of no help in determining their length.

*lapse*,

$$\forall xx \exists y \forall u (u \in y \leftrightarrow u \prec xx),$$

according to which for any objects there is a set whose members are exactly those objects, and *Plural Comprehension*,

$$\exists xx \forall u (u \in xx \leftrightarrow \varphi(u)),$$

according to which any condition defines a plurality, are mutually inconsistent. If we let $\varphi$ be the condition of being an ordinal, then *Plural Comprehension* gives us a plurality of ordinals. Now, on the von Neumann conception, the collapse of these ordinals into a set (i.e. the set of all ordinals) is itself an ordinal, such that *Plural Collapse* can itself be seen as forming the new collection $\Omega \cup \{\Omega\}$ on basis of applying the diagonal function to a set $\Omega$. But now we have come to a contradiction. The set of ordinals cannot itself be part of the sequence, because if that were the case, there would be a bigger plurality of ordinals leading to a new set of ordinals $\Omega^+$ bigger than $\Omega$.

Thus, given a certain domain of quantification $D$, insofar as *Plural Comprehension* is permissible, *Plural Collapse* has to fail, i.e. there are instances of

$$\forall xx \neg \exists y \forall u (u \in y \leftrightarrow u \prec xx).$$

Insofar as this is accepted, the viciousness that was troubling Dummett ceases to be a problem. However, this need not mean that there isn't a bigger domain $D^+$ which contains the set in question. If we use $\exists^+$ for quantification over $D^+$, we can express this as saying that

$$\forall xx \exists^+ y \forall u (u \in y \leftrightarrow u \prec xx).$$

$D^+$ then naturally allows for an extension of the plural quantifier $\forall^+ xx$ as well. Hence, $D$ fails to encompass all the objects that we intended to quantify over and our grasp of the domain of all ordinals was thus incomplete. As Dummett points out, this process then "fails to determine the limits of acceptable specification of something to be acknowledged as [an object of the domain in question]"(Dummett 1991a, p. 315).

Thus the problem with instance based generalisations is that the characterisation of the domain of objects that they require is either incomplete or leads to a contradiction. But generic generalisations should not involve the extension of the concept in the first place. Just because we cannot specify the extension of the concept in a non-circular manner doesn't mean that we can't ground generalisations in the essence that is expressed or contained in the concept. Now, even though we are not able to characterise the type of expression which stands for the objects in a non-circular manner, what the above does show is that we do have a general conception of the extensibility of the domain via the diagonal function. In this sense, the concept of an ordinal may be grasped by the understanding of the iterative application of the diagonal function in combination with the determinate criterion of membership. In the following, I argue that this is an appropriate ground for expressing generality.

## 6.2. Formalizing Quantification over Indefinitely Extensible Domains

To reiterate: $P$ is indefinitely extensible with respect to a basis $\Pi$ if there is a diagonal function $\delta$ definable on each subconcept or subset $X$ of the extension of $P$ which yields a new object $\delta(X)$ that falls under $P$ but is not part of $X$ and $X \cup \delta(X)$ adhere to the condition set by $\Pi$. Given this definition, what does it mean to express a general claim for $P$? For one, generality over such a domain can only be expressed by use of what we do have, and that is, as is noted, the understanding of the iterative application of the diagonal function. This understanding can be modelled using modal resources, where the understanding of the modality will be informed by and serve to elucidate our understanding of indefinite extensibility. This section will introduce the required technical details while the next section discusses the nature of the modality itself.

The following draws heavily on the potentialist positions that were initially developed to give an account of the iterative conception of set and the set theoretic universe. In the literature we can distinguish broadly between two kinds of approaches: the modal structuralist approach, initiated by Putnam (1967) and further developed in Hellman (1989), Hellman (1996), and Berry (2022), in which the set theoretic universe is understood in terms of possible isomorphic extensions of set theoretic *structures*; and the more fine grained approach initiated by Parsons (1983b) and Parsons (1983a) and developed further in Linnebo (2013), Linnebo (2018d), Studd (2013), and Studd (2019), whose idea is to "locate some modally characterised features in the mathematical objects themselves"(Linnebo 2018b).[3] The approach given here is essentially the one given in Linnebo (2018d), but following Studd in its primary focus on semantics.

Given an indefinitely extensible concept $P$, a set $A$ containing only members of $P$, and a suitable diagonalisation procedure via a function $\delta$ from subsets of $P$ into elements of $P$, we can recursively define a series of sets

$$\begin{aligned} A_0 &:= A \\ A_{i+1} &:= A_i \cup \delta(A) \\ A_\lambda &:= \bigcup_{i<\lambda} A_i \qquad \text{for possible limits } \lambda \end{aligned}$$

The $A_i$ are ordered by the subset relation, which, of course, bears strong resemblance to the definition of the lattice of subsets in chapter 1.2.2. Now, it may be the case that there is more than one diagonal function definable on $A$ and more than one initial set to start with.[4] Pooling all of them together, we get:

**Definition 6.1.** Let $S = \{A, B, C, \dots\}$ be a collection of subsets of an indefinitely extensible concept $P$ and let $\Delta = \{\delta_A, \eta_A, \dots, \delta_B, \eta_B, \dots\}$ be a collection of diagonal functions defined on basis of elements in $S$. Then define $G_P$ as follows:

a) If $A \in S$ then $A \in G_P$

b) If $X \in G_P$ then $X \cup \delta(X) \in G_P$ for any $\delta \in \Delta$ defined for $X$.

c) $G_P$ is the smallest set closed under a) and b)

[3] Button (2021), although he is not a potentialist himself, contains a similar exposition.

[4] Even though the central examples that we consider here only have one starting set and one diagonal function, there are other possible instances like the collection of subsets where this is not the case.

Then for any $A, B \in G_P$ set

$$A \leq B \textit{ iff } A \subseteq B$$

Call the pair $\langle G_P, \leq \rangle$ a **frame for an indefinitely extensible concept** $P$.

For now, I will gloss over the distinction between $G$ forming a set or indeed any other entity of similar kind as the elements it contains. $G$ is needed to give the formal definition of modal satisfaction below, but as the following section argues, this common way of defining modal satisfaction is somewhat artificial and in case of indefinitely extensible concepts certainly gets it the wrong way round.

This construction is quite general and hence similarly versatile. We can use it to mirror, for example,

1. Our conception of the natural numbers as the initial section of the von Neumann ordinals. In this case, we start with the empty set $\emptyset$ and define $\delta(A_i) = A_i \cup \{A_i\}$. This allows to distinguish between each finite set which can be taken to be determinate, and their union, the whole collection $A_\omega$, that may not be considered as determinate, i.e. not as part of the frame.

2. This can be extended to a conception of the ordinals. In this case we allow the limit stages to be incorporated into the frame, e.g. $A_\omega = \bigcup_{i<\omega} A_i \in G_{Ord}$ and similarly for higher limits.

There are two points worth noting. One is that insofar the ordinals (or even the natural numbers qua being an infinite collection) are considered to be indeterminate, this also applies to our means of indexing these progressions and hence to the definition of the frame itself. This problem will be addressed in more detail in the next section. Another point concerns the accessibility relation. From the definition of the frame, we know that it is **reflexive** and **transitive**. In the examples, we have considered so far, the frame is also

**directed**, i.e. $\forall A, B\, \exists C$ s.t. $A \leq C \wedge B \leq C$,

and even

**linear**, i.e. $\forall A, B : A < B \vee A = B \vee B > A$.

Especially these last two features will turn out to play an important role regarding which logic indefinitely extensible concepts will turn out to license. In order to see how that comes about, we need to find a way to express generalisations over these frames.

To this end, let $\mathcal{L}^{\Diamond}_{\prec,\in}$ be the language of first order modal logic with the elementhood relation $\in$ and plural containment $\prec$.[5]

**Definition 6.2.** The triple $\mathcal{K}_P = \langle G_P, \leq, \vDash \rangle$ is a **Kripke model associated with the indefinitely extensible concept** $P$, where $\langle G_P, \leq \rangle$ is a frame based on $P$, and $\vDash$ is a mapping from $G$ and the set of sentences in $\mathcal{L}^{\Diamond}_{\prec,\in}$ to truth values. Each $A \in G_P$ will also be its own associated domain, such that a singular variable $x$ can be assigned an element an element $a \in A$ and a plural variable $xx$ a subset of $A$. Satisfaction for atomic formulas $x \in y$ and $x \prec xx$, worlds $A$ and assignments $\sigma(x), \sigma(y) \in A$ and $\sigma(xx) \subset A$ is then defined as:

$$
\begin{aligned}
A &\vDash_\sigma x \in y && \text{iff} \quad \sigma(x) \in \sigma(y) \\
A &\vDash_\sigma x \prec xx && \text{iff} \quad \sigma(x) \in \sigma(xx)
\end{aligned}
$$

Note that on the right hand side we use the actual membership and subset predicates to interpret the left one. We extend the definition of $\vDash$ to arbitrary formulas $\varphi$ and $\psi$ by

$$
\begin{aligned}
A &\vDash_\sigma \neg\varphi && \text{iff} \quad A \nvDash_\sigma \varphi \\
A &\vDash_\sigma \varphi \wedge \psi && \text{iff} \quad A \vDash_\sigma \varphi \text{ and } A \vDash_\sigma \psi \\
A &\vDash_\sigma \varphi \vee \psi && \text{iff} \quad A \vDash_\sigma \varphi \text{ or } A \vDash_\sigma \psi \\
A &\vDash_\sigma \varphi \rightarrow \psi && \text{iff} \quad A \nvDash_\sigma \varphi \text{ or } A \vDash_\sigma \psi \\
A &\vDash_\sigma \Box\varphi && \text{iff} \quad \text{for all } B \in G_P \text{ such that } A \leq B\text{: } B \vDash_\sigma \varphi \\
A &\vDash_\sigma \Diamond\varphi && \text{iff} \quad \text{there is some } B \in G_P \text{ such that } A \leq B \text{ and } B \vDash_\sigma \varphi
\end{aligned}
$$

[5] In the following (and as before) I will not use function symbols because their interpretation in the modal model would compel us to revert to a free logic in order to express the truth of statements like $\Box\forall x(0 + s0 = s0)$ at worlds where $s0$ does not have a denotation (like in the starting world). For the details of this, see Brauer (2021).

and for the quantifiers

$$A \vDash_\sigma \forall x\, \varphi(x) \quad \text{iff} \quad \text{for all } x\text{-variant assignments } \sigma^*, \text{ if } \sigma^*(x) \in A, \text{ then } A \vDash_{\sigma^*} \varphi(x)$$
$$A \vDash_\sigma \forall xx\, \varphi(xx) \quad \text{iff} \quad \text{for all } xx\text{-variant assignments } \sigma^*, \text{ if } \sigma^*(xx) \subset A, \text{ then } A \vDash_{\sigma^*} \varphi(xx)$$

This structure has some further properties worth noting.[6] The monotonicity of the frames assures that it is a model of the converse Barcan Formula,

$$\Box\forall x\varphi(x) \to \forall x\Box\varphi(x),$$

and since the interpretation of the set-membership and the plural membership predicates are the actual membership relations in the singular and plural domains, both are stable:

$$x \in y \to \Box(x \in y)$$
$$x \notin y \to \Box(x \notin y)$$
$$x \prec yy \to \Box(x \prec yy)$$
$$x \not\prec yy \to \Box(x \not\prec yy)$$

Now we are in a position to reassess the principles of plural comprehension and plural collapse.[7] To make distinction between formulas that express content with respect to an individual world, and formulas that express generality claims across all worlds, we highlight those $\varphi$ that don't contain any modal operators and those formulas $\varphi^\Diamond$ which are obtained by replacing every occurrence of $\forall$ and $\exists$ by the strings $\Box\forall$ and $\Diamond\exists$. This is called the **potentialist translation** $\cdot^\Diamond$. In the following, I will use $\varphi$ to refer to formulas without modal vocabulary and $\varphi^\Diamond$ to refer to formulas obtained by the potentialist translation.

Now, the structure satisfies ordinary plural comprehension with respect to any par-

[6]These features will also ensure that the structure so defined is an instance of a **potentialist system**, which is defined as a set of structures $\mathcal{W}$ for a language $\mathcal{L}$ ordered by the substructure relation (cf. Hamkins and Linnebo 2017).

[7]For the following discussion, see also Linnebo (2010), Linnebo (2018d, ch.3), and Florio and Linnebo (2021, ch.11)

ticular world

$$A \vDash \exists xx \forall u(u \in xx \leftrightarrow \varphi(u)).$$

If we are using the structure to express quantification over the ordinals, for instance, there is no problem with assuming that at each world $A$ there exists a plurality of ordinals, which in fact is just $A$ itself. However, $A$ does not validate *Plural Collapse*

$$A \nvDash \forall xx \exists y \forall u(u \in y \leftrightarrow u \prec xx)$$

else we would be able to derive $A \in A$ for the case where $\varphi$ is the property of being an ordinal. But it is possible to have a modalised version of *Collapse*

$$A \vDash \Box \forall xx \Diamond \exists y \Box \forall u(u \in y \leftrightarrow u \prec xx)$$

which effectively says that $A$ is available as an element in a subsequent world $A'$. Insofar as we are using the structure to express our understanding of the von Neumann ordinals, the combination of *Plural Comprehension* and *Modalised Plural Collapse* is in effect the diagonal function producing the next ordinal. Finally, the modalised version of *Plural Comprehension* is no longer satisfied:

$$A \nvDash \Diamond \exists xx \Box \forall u(u \in xx \leftrightarrow \varphi^{\Diamond}(u))$$

In our example, it would amount to the claim that there is a world in whose domain we have collected the plurality of all the ordinals. But of course, we can introduce a new ordinal as the set-collapse of this plurality in the next world, contradicting the rigidity of pluralities. That there is no such world reflects the incompleteness of any attempt to define a domain of ordinals as noted at the end of the previous section.

However, this does not prohibit us from expressing general claims about ordinals via the modalised quantifier $\Box \forall x$, as this expresses the fact that any object obtained by the iteration of the diagonal procedure has the property in question. In this way, we can understand generic generalisations over an indefinitely extensible concept. Generic generality was introduced as a type of generality pertaining to the essence of the concept, which, in the case of an indefinitely extensible concept is given by the iterated applicability of the diagonal function. And this was also the sole input for designing the whole model. Thus, we cannot specify the kind of expression that would stand for an arbitrary element of the domain, but we can characterise the *way in which* that is impossible and just take *that* to be the conceptual content relied upon in making our generalisation.

But some might not think so. In the next section I will address an important (and thus far unanswered) objection to the applicability of this model to in fact mirror statements of generic generality over indefinitely extensible concepts. This objection has been raised in the debate on absolute generality and rests on an interpretation of the modality (and our understanding of the collection $G_P$ involved). The next section addresses this objection by giving an interpretation of the modality as it related to generic generality.

## 6.3. Modality and Supermodality

One common objection against the claim that the above set-up actually manages to express generic or absolute generality is to construct a revenge problem. In its most detailed version, the argument is given in Studd (2019, ch.7.5).[8] The question is whether it is possible to apply the same reasoning that lead to the dynamic understanding of iteration to the whole structure itself. Why, for instance, can there be no ordinal number corresponding to the whole recursive series $\{A\}_\delta$. This number, of course, cannot be part of the series and hence must lay outside it. But why shouldn't we introduce a new super-modality $\blacksquare$ specified in a similar way as above, that subsumes and extends the ordinary modality $\Box$. We can then claim that comprehensibility failures with respect to the ordinary modality can form pluralities in the new super modality, and thus the principle:

$$\blacklozenge\exists xx\Box\forall u(u \in xx \leftrightarrow \varphi^{\Diamond}(u)).$$

[8]See also Fritz (2016), Warren (2017), and Studd (2023, ch.7.5).

The manner can also be expressed in a way involving the modalities that is analogous to the initial Burali-Forti paradox. The following two statements are jointly inconsistent (cf. Fritz 2016):

(1) *Necessarily*, for any actually existing elements, there is a *possible* element distinct from all of them.

(2) *Possibly*, all possibilities can be actualised.

Especially if (2) were the case, then, the modality with which we started, and indeed any modality we would subsequently introduce, would be insufficient to establish generic or universal generality, because it falls under the very same mechanism that prompted its application in the first place.[9]

But why should we accept such a super modality? Studd motivates it by the following analogy. If we take the perspective of the finitist, then the extension of the ordinals $\textsc{Ord}^{\Diamond}(x)$ is just the extension of the finite ordinals $\textsc{Fin-Ord}^{\Diamond}(x)$, but it is well known that the finite ordinals may simply be extended beyond $\omega$, and from this perspective, a restriction to $\textsc{Fin-Ord}^{\Diamond}(x)$ seems artificially limiting. But why can't something analogue apply to $\textsc{Ord}^{\Diamond}(x)$? In Studd's words:

> Why think that the ABSOLUTE non-comprehensibility of $\textsc{Fin-Ord}^{\Diamond}(x)$ or $\textsc{Ord}^{\Diamond}(x)$ is due to some intrinsic features of the condition rather than simply the point where the hierarchy happens to give out? Without a good reason to think that the hierarchy must give out here, the would-be explanation of comprehensibility failure seems unpleasantly arbitrary. (Studd 2019, p. 212)

There are two points about this analogy. The first thing to note is that from the perspective of the finitist, of course, there is no conceivable way of extending the finite ordinals, such that they won't accept a distinction between the two in the first place. If we evoke the difference between reflexively and non-reflexively indefinitely extensible concepts again, we may conceive of the finite ordinals to be completely collectable, precisely because they are not reflexively indefinitely extensible. The finitist, however,

[9] In addition to that, as pointed out by Luca Incurvati in his review of Studd's book, we would run into the very same expressibility problem that overshadows the claim of the generality relativist. This is a point that strikes me as a fundamental problem for the generality relativist that is hardly noticed by proponents of the revenge argument (cf. Incurvati 2022).

takes them to indeed be *reflexively* indefinitely extensible, and as such to just *be* the ordinals.

But Studd is right in that we need to give an intrinsic reason that the hierarchy itself cannot be collapsed to a world (as the above principle (2) suggests). In the analysis of generic generality given in the previous chapter, it was said that a statement of generic generality is true at a state $s$ iff "material intrinsic to $s$" suffices to validate it. This says that the extent of any extension of the concept is irrelevant to the truth *and* the meaning of such a generalisation.[10] But this only goes so far as an explanation in the current setting, for we have evoked the modal notions precisely because we intended to give an account of what such a generic generalisation consists in. And insofar as the modality is there to explicate *that*, we cannot explain its non-collectablilty by reference to material intrinsic to an information state. Thus an (informative) response to the revenge challenge in the current setting needs to come form the conception of the modality employed—the analysis of which I now turn to.

A first step in this direction is to note the following. Usually, we express the meaning of the $\Box$-operator by quantifying over all worlds (as in definition 6.2). However, our conception of all the worlds to which we refer to in this respect is itself indeterminate (if it weren't we would be able to collect them all into one). This traces back to the problem of specifying the indexing procedure and the alleged circularity in the definition of the set $G_P$. The meaning of the $\Box$-operator thus cannot be reduced to any collection of worlds—for this collection is indeterminate as well. However, when we use it to express a universal generic generalisation, we mean to express a truth that is invariant to any introduction of new elements. In this respect, it is to be understood as *primitive*:

$$w \vDash \Box p \quad \text{iff} \quad \text{from } w \text{ on } p \text{ is true invariant to the introduction of further objects}$$

Of course, this cannot be a definition, because the invariance clause contains reference to the modality on the right side as well. But, even though the $\Box$-operation is thus primitive, our understanding of it is as good as our understanding of the applicability of

[10] Linnebo (2023) suggests that the existence of such a truthmaker can be understood as a $\Sigma_1$ statement that is upwards absolute.

the diagonalisation procedure itself—and as such it should be invariant to the number of worlds that there are. Nonetheless, in order to explain why there cannot be a revenge modality, we need to go some further steps in elucidating what it consists in.

There are (almost) as many understandings of the modality as there are potentialists. The most substantial discussion of the notion can be found in Berry (2022). A very coarse distinction can be made between a so-called "circumstantial" and "interpretational" understanding of possibility (cf. Fine 2006). While circumstantial possibility is concerned with things in the world, interpretational possibility concerns only changes in our conceptual apparatus which does not affect what things there actually are. Consider circumstantial possibility first. It can be further divided into:

**Physical Possibility**, according to which it is possible, for instance, for there to be three pigs on the moon.

**Metaphysical Possibility**, according to which it is possible for there to be a pig with wings.

**Mathematical Possibility**, which is understood as the existence of a model which faithfully interprets mathematical vocabulary.[11] According to this one, for instance, it is possible for there to be $2^{\omega^{\omega}}$ many pigs.

**Conceptual Possibility**, according to which only analytic truths are preserved. This seems to hit the target directly. Unsurprisingly, however, it is exactly the notion that we are trying to explicate when we investigate the laws governing generic generality. Therefore, a reference to it only possesses limited explanatory force.

**Logical Possibility**, which only preserves validity, but need not even preserve analytic truths, e.g. according to which the sentence $\exists x Pig(x) \wedge \neg Mammal(x)$ has a model (in which either the expression mammal or pig gets reinterpreted). Thus logical possibility can be characterised as a possibility that is invariant to the size of the domain as well as to the interpretation of the vocabulary.

Proponents of the interpretational understanding of the modality object to each of these notions. Parsons (1983b) argues that we can't appeal to physical and metaphysical

[11] This would exclude, for instance, non-well founded set theoretic models as they are not understood to reflect the nature of the sets (even though they can, of course, be models of other mathematical concepts).

possibility, because the existence of sets isn't contingent this way. Similarly, mathematical possibility may be ruled out as a candidate, because mathematical reality might outrun any existing model, as it can be understood to be the case for the set theoretic universe $V$ in comparison to any model of the ZFC axioms (cf. Berry 2022). Furthermore, according to Parsons we cannot use logical possibility because sets are not extensional by necessity. In the tradition building on Parson's work, the notion of possibility has thus shifted to an interpretational understanding in which there is no claim involved in how the world may change. On the contrary, the actual world (including our concept of sets) stays the same when a possibility is exercised, it is merely reconceptualised to make additional objects available in the domain of quantification.

The interpretational possibility has been used in Fine (2006), Linnebo (2018d, ch 3.), and Studd (2019). According to Linnebo, when an interpretational possibility is realised there is a **shift in the interpretation** of the language which may result in an expansion of the domain of quantification. The existence of the objects newly introduced in this way then ontologically depends on those previously existing objects by reference to which the language shift was introduced. Objects whose existence ontologically depends on (or even involves) other objects in this way are called "thin"(ch.3 Linnebo 2018d). This is exactly the understanding of the objects of an indefinitely extensible domain that was given in section 1. Studd's understanding of the modality is very close to that as well. He takes it to be the possibility of an **admissible re-interpretation of the lexicon**, which is an idealised notion that is explicated via a Kripke model in much the same style as given above.

Studd already emphasises that the notion of an admissible re-interpretation of the lexicon is a constrained version of logical possibility. In recent work by Sharon Berry, this idea and with it the whole notion of logical possibility has been revitalised. Berry argues that the interpretational modality can be supplanted by the logical notion once it is properly clarified. To countenance the objection made by Parsons, she introduces the notion of conditional possibility according to which the interpretation of some non-logical vocabulary may count as fixed, such that we may conceive of something to be logically possible conditioned on something else—for instance the extensionality of sets. This is also supported by Linnebo's discussion of the Putnam-Hellman approach, where thinking about the possibility in logical terms remains in relatively good standing (Linnebo 2018b). Logical possibility can thus be taken to support the interpretational possibility of Linnebo and Studd.[12]

Another attempt to supplement the interpretational possibility is made in Warren (2017) who understands the iterative procedure of language expansion to be given by physical possibility. A first thing to note about this suggestion is that in such a case each application step must permit the addition of more than one element if the procedure is meant to incorporate any infinite cardinality. What is surprising about this is that Warren also holds that there always is a revenge modality such that generic or absolute generality cannot be achieved. But why can't we refer to the sum total of all physical possibilities when the modality is so construed? Presumably, this sum total of physical possibilities is not a physical possibility itself, but that shouldn't stop us form quantifying over it.

Similar problems beset the interpretational as well as the logical understanding of the modality. If we accept the existence of an actual world on the interpretational understanding, what non-epistemic reasons are there that we cannot simply refer to it? Moreover, if the possibility is understood to be instantiated by the availability of objects, why can't we refer to all of them in order to gain a larger one? In addition to that, we may ask on any of these understandings, which interpretational or physical facts are there that determine which sets actually exist, and which ones are merely potential? These questions seem uncanny, for the possibility of asking them suggest that there is something wrong with the understanding of the modality. In the following I will suggest a way to understand the modality that avoids both problems.

In this respect, note that our understanding of the modality is supplied by our understanding of indefinite extensibility and not the other way around. The iterability of the diagonalisation operation is not characterised in extensional terms by quantification over an index set. As such the modality is also not characterised in terms of any circumscription of a space of what instances count as possible. When we say that any application of

[12]It should be noted that the debate between these two camps is not solely about the correct understanding of the modality, but it extends to its application to structures or to objects. Many points in this respect also relate to the problem of justifying the modalised versions of the ZFC axioms on their respective accounts, which is a problem I am not presently concerned with.

the diagonal function is possible, we don't understand it as saying 'any of *those* options is exercisable', where 'those' is taken to refer to a determinate collection (like the actual world or the collection of physical objects). If that were the case, we are indeed able to come up with a revenge scenario. No, the modality is chiefly characterised by the *absence of any intrinsic restrictions* to the procedure. And the absence of a restriction cannot be cashed out by providing an external measure of length. In this sense, we are dealing with a modality that *does not have* any corresponding actuality. The absence of a limitation to the applicability of a certain procedure or rule does not entail the presence of a space or totality of its applications.[13]

Thus, the point is not that we always expand what we mean, for instance, by "all ordinals", the point is that the procedure of expansion itself *is* the meaning of "all" when we say "all ordinals". And in this respect there cannot be any revenge phenomenon. The modality is characterised by the absence of something in opposition to being characterised by a space of options that it could take. Positing a revenge modality rests on a misunderstanding of this distinction and hence the modality involved. In this way, I submit, absolute generality on generic generalisations can be coherently defended.

To summarise: the truth of a claim of generic generality is anchored in the understanding of the iterability of the diagonalisation procedure, which is a modal notion not given by any extension, but by the mere absence of any restrictions. For this reason, the truth of a statement of generic generality is indeed independent of the availability of any objects–and no revenge phenomenon arises.

[13]Historically, it seems that Wittgenstein was the first to isolate such a notion of modality to which no actuality corresponds. For instance, he writes:

> Generality in Mathematics is a direction, an arrow pointing along the series generated by an operation. And you can even say that the arrow points to infinity; but does that mean that there is something—infinity—at which it points, as at a thing? Construed in this way, it must of course lead to endless nonsense.(Wittgenstein 1975, p. 163)

Wittgenstein is actually trying to be funny here, because positing such an end point, i.e. a final modality is simple nonsense. The nonsense becomes endless, because such a position then prompts us to go further and thus to repeat the nonsensical construction a second time, and so on. If the joke gets funnier by being explained or not, I leave to the judgement of the reader. For a discussion of the historical position of Wittgenstein on this respect, see Methven (2016).

## 6.4. Varieties of Potentialism

Even though the modal vocabulary highlights the extensibility of the domain, we are still concerned with "standard reasoning" over the domain, i.e. reasoning that employs non-modal language. In this sense, certain statements that are meaningful in the model might not have their meaning transcend to such non-modal reasoning. In other words: the modal language equips us with considerable fine structure and not all of that fine structure is needed. For instance, $\forall x\varphi(x) \vee \exists x\neg\varphi(x)$ and $\Box(\forall x\varphi(x) \vee \exists x\neg\varphi(x))$ hold in the model, since each world is a classical model, but it doesn't seem to reflect what we mean with LEM for the 'whole domain'. The task is now to find a class of formulas that can be understood as representing our standard reasoning and to use a suitable translation and mirroring theorem to determine the logic that it obeys.

This section presents a variety of translations from modal to non-modal vocabulary that could be used for this and discusses their accompanying philosophical convictions. I will address liberal, strict, and branching potentialism and their ability to adequately represent reasoning with indefinitely extensible concepts. The proofs of the respective mirroring theorems that will be mentioned and their connection to the topological models (and thus to the notion of a warrant) will be given in the appendix.

One candidate translation that can be excluded outright is the **Gödel translation**. It is given by adding $\Box$ in front of atomic sentences, the negation sign, the implication clause and both quantifiers (cf. Gödel 1933). Let $\vDash_{\mathcal{K}}$ indicate satisfaction for any first order Kripke model $\mathcal{K}$ and $\vDash_{\mathcal{T}}$ first order topological satisfaction. This connection leads to the following theorem.

**Theorem 6.1.** (Gödel-Tarski) *Let $\varphi_1, \cdots, \varphi_n, \psi$ be a collection of formulas, and $\varphi_1^\Box, \cdots, \varphi_n^\Box, \psi^\Box$ be their Gödel translations. Let $\mathcal{K}$ be a Kripke model. Then there is a corresponding topological model $\mathcal{T}$ such that:*

$$\varphi_1, \cdots, \varphi_n \vDash_{\mathcal{T}} \psi \qquad \textit{iff} \qquad \varphi_1^\Box, \cdots, \varphi_n^\Box \vDash_{\mathcal{K}} \psi^\Box.$$

If successful, would show that upon this understanding which class of formulas in the modal language is representative of our standard reasoning we don't get LEM any longer. But the class of formulas in the Gödel-translation cannot adequately capture the potentialist idea. To see this, consider the principle of *Plural Collapse*, $\forall xx \exists y \forall z(z \in y \leftrightarrow z \prec xx)$. The translation of this yields for a world $w$:

$$w \vDash \Box\forall x\Box\exists y\Box\forall z(z \in y \leftrightarrow z \prec xx)$$

and due to reflexivity we would get $w \vDash \forall xx \exists y \forall z(z \in y \leftrightarrow z \prec xx)$ which contradicts the set up of the frame.

### 6.4.1. Liberal potentialism

The liberal potentialist simply relies on the potentialist translation (from where it got its name in the first place). However, in order to prove a suitable mirroring theorem, we need to require that the frames are directed.

Reasons to assume directedness may be the possibility of the mirroring theorem itself, as well as the idea that every set or plurality should be collectible. Furthermore, there might even be arguments to assume a stronger notion of linearity, which equally serves to prove the theorem.[14] Indeed, both of our cases, the natural number and the ordinals are taken to lead to linear frames.

However, it turns out that the logic that this semantics licenses is thoroughly classical. Let $\vDash_{\mathcal{K}^d}$ indicate satisfaction for Kripke models with directed frames, then

**Theorem 6.2** (Linnebo 2013)**.** *Let $\varphi_1, \cdots, \varphi_n, \psi$ be a collection of formulas, and $\varphi_1^\Diamond, \cdots, \varphi_n^\Diamond, \psi^\Diamond$ their potentialist translations. Let $\mathcal{K}^d$ be a Kripke model with a directed frame. Then there is a corresponding topological model $\mathcal{T}$ such that:*

$$\varphi_1, \cdots, \varphi_n \vDash_{\mathcal{T}} \psi \qquad \textit{iff} \qquad \varphi_1^\Diamond, \cdots, \varphi_n^\Diamond \vDash_{\mathcal{K}^d} \psi^\Diamond,$$

*and since $\vDash_{\mathcal{K}^d} \varphi^\Diamond \vee \neg\varphi^\Diamond$, it is the case that*

$$\vDash_{\mathcal{T}} \varphi \vee \neg\varphi.$$

[14] This is an assumption that has also been made at the very least for the modal interpretation of set theory in Studd (2019), Button (2021), Berry (2022). The Maximality Principle that is assumed in Linnebo (2013) and Linnebo (2018d) serves a similar function.

Insofar as we take the characterisation of natural number as well as that of ordinal by our understanding of initial element and successor function to be determinate, then the linear frame as well as the potentialist understanding of the existential quantifier seem to be appropriate and deliver LEM. This theorem also shows that on this account the distinction between generic and instance based generality loses its bite—and, if the frames are rooted, so does the distinction between the weak and the strong existential statements.

This also has to do with the fact (argued for in the last two sections) that we are not first and foremost thinking about an indefinitely extensible concept in terms of its extensions, but in terms of its components, the initial element(s) and the diagonal function(s). For this reason, a possible failure of the duality of the quantifiers is not primarily conceptualised by the lack of a witness for the existential quantifier—there is no such collection of objects from which it is missing in the first place. Thinking solely in terms of initial element(s) and diagonal function(s), indeed it is hard to see where bivalence ought to fail (at least when it comes to a directed frame).

In chapter 5.1.3 it was argued that the standard explanation why the quantifiers are not dual, according to which the truth of a universal generalisation may lead to absurdity all the while there is no counterexample, covertly relies on an illegitimate use of instance based generality. But it was left open whether another explanation of the failure of duality may be given. The above semantics, however, makes a strong case that maintaining the non-duality of the quantifiers without involving additional (e.g. epistemic) demands on the existential quantifier is actually incoherent. If we doubt the intelligibility of the claim that eventually a counterexample will come up, then we should also doubt the intelligibility of the negation operation in $\neg\forall x\varphi(x)$. In other words, insofar as we are only concerned with truth conditions solely dependent on the availability of objects, a distinction between $\neg$ and $\exists$ in this respect seems artificial. The matter looks different, though, for branching frames.

### 6.4.2. Branching potentialism

Insofar as the model is no longer directed (and the mirroring theorem fails), we may ask what to make of a statement like $\neg\Box\forall x\varphi(x)$ or $\Diamond\exists\neg\varphi(x)$. It would be a suitable example for Weyl's critique of the "judgement abstract". For saying that there *might be* a possible witness/counterexample to a property $\varphi$ may indeed be denied the status of a proper statement about the structure under investigation, especially if it might also not be the case that there is one. Insofar as the existence of these possible "future" objects are not sufficiently determined one might have reasonable doubt about understanding the expression $\Diamond\exists$ as a part of mathematical reasoning.

There is another well known translation which appends $\Box\Diamond$ to the existential quantifier and which thus seems to be usable in this situation. $\Box\Diamond\exists x\neg\varphi(x)$ reflects the fact that there is a possible counterexample on any possible development. The full **$\Box\Diamond$-translation** is given by the clauses:

$$
\begin{aligned}
P &\mapsto \Box\Diamond P \\
\neg\varphi &\mapsto \Box\neg\varphi^{\Box\Diamond} \\
\varphi\wedge\psi &\mapsto \varphi^{\Box\Diamond}\wedge\psi^{\Box\Diamond} \\
\varphi\vee\psi &\mapsto \Box\Diamond(\varphi^{\Box\Diamond}\vee\psi^{\Box\Diamond}) \\
\varphi\to\psi &\mapsto \Box(\varphi^{\Box\Diamond}\to\psi^{\Box\Diamond}) \\
\exists x\varphi(x) &\mapsto \Box\Diamond\exists x\varphi^{\Box\Diamond}(x) \\
\forall x\varphi(x) &\mapsto \Box\forall x\varphi^{\Box\Diamond}(x)
\end{aligned}
$$

It can be seen as a combination of the double negation translation combined with the Gödel translation. Using this translation we obtain a mirroring theorem for classical logic that does not require directedness.

**Theorem 6.3.** *Let $\varphi_1,\cdots,\varphi_n,\psi$ be a collection of formulas, and $\varphi_1^{\Box\Diamond},\cdots,\varphi_n^{\Box\Diamond},\psi^{\Box\Diamond}$ their $\Box\Diamond$-translations. Let $\mathcal{K}$ be a Kripke model. Then there is a corresponding topological model $\mathcal{T}$ such that:*

$$\varphi_1,\cdots,\varphi_n\vDash_{\mathcal{T}}\psi \qquad \textit{iff} \qquad \varphi_1^{\Box\Diamond},\cdots,\varphi_n^{\Box\Diamond}\vDash_{\mathcal{K}}\psi^{\Box\Diamond},$$

*and since* $\vDash_{\mathcal{K}} \Box\Diamond(\varphi^{\Box\Diamond} \vee \Box\neg\varphi^{\Box\Diamond})$*, it is the case that*

$$\vDash_{\mathcal{T}} \varphi \vee \neg\varphi.$$

But even though this technically works, we might argue that the interpretation of $\Box\Diamond\exists$ no longer serves its purpose. $\Box\Diamond\exists x\varphi(x)$ only says that it is necessary to remain possible that a witness of $\varphi$ comes up, but it does not claim it to be inevitable. Consider a model consisting of an infinite path which branches of at each node to a node that contains an $a$ such that $\varphi(a)$. Then this model verifies $\Box\Diamond\exists\varphi(x)$, but we could go on infinitely along the path without verifying the existence of an witness to the claim. $\Box\Diamond\exists x\varphi(x)$ therefore is too weak to express what we want. An analysis of this point in terms of warrants given exclusively be the existence of objects will be given in appendix 6.A.1. There we will also see that this point doubles down as a criticism of the double negation translation.

For this reason, Brauer (2020) introduced an inevitability operator into the modal system. The inevitability operator has its origin in temporal logic, and is used to express inevitability claims with respect to branching time (cf. Gabbay, Reynolds, et al. 2000, ch. 4, Gabbay, Hodkinson, et al. 1994, ch. 3). To define it, we need the notion of a path.

**Definition 6.3.** Given a partial order $\langle W, \leq\rangle$, a **chain** above a node $w \in W$ is a set of nodes $X \subseteq G$ such that for all $x, y \in X$, $w \leq x, y$ and $x \leq y$ or $y \leq x$.

A chain is a **path** if it is maximal in $G$.

We may now define

$$w \vDash \mathcal{I}\varphi \quad \text{iff} \quad \text{any path trough } w \text{ has a world } w' \text{ s.t. } w' \vDash \Box\varphi.$$

If we add the inevitability operator to S4, we obtain a system called S4+, which is no longer axiomatizable (for a proof, see Brauer (2020)). This has two interesting consequences. For one, this assures us that the inevitability operator cannot be expressed using $\Box$ and $\Diamond$. Another consequence is that that, as Brauer notes, there are no prospects of finding a full and faithful translation into first order classical logic. However, as shown in Brauer et al. (2022), there is a full and faithful translation of intuitionistic logic into a suitably axiomatised fragment of $S4+$. In my discussion of branching potentialism, I will focus on the semantical side of things. The translation used in the mirroring theorem is the following:

**Definition 6.4** (Brauer et al. 2022)**.** The **Beth Kripke (BK) translation** $\cdot^B$ is given by the clauses:

$$\begin{aligned}
P &\mapsto \mathcal{I}\Box P \\
\neg\varphi &\mapsto \Box\neg\varphi^B \\
\varphi \wedge \psi &\mapsto \varphi^B \wedge \psi^B \\
\varphi \vee \psi &\mapsto \mathcal{I}(\varphi^B \vee \psi^B) \\
(\varphi \to \psi) &\mapsto \Box(\varphi^B \to \psi^B) \\
\exists x\varphi(x) &\mapsto \mathcal{I}\exists x\varphi^B(x) \\
\forall x\varphi(x) &\mapsto \Box\forall x\varphi^B(x)
\end{aligned}$$

And with that we have the following theorem

**Theorem 6.4** (Brauer et al. 2022)**.** *Let $\varphi_1, \cdots, \varphi_n, \psi$ be a collection of formulas, and $\varphi_1^{BK}, \cdots, \varphi_n^{BK}, \psi^{BK}$ their BK-translations. Let $\mathcal{K}$ be a Kripke model. Then there is a corresponding topological model $\mathcal{T}$ such that:*

$$\varphi_1, \cdots, \varphi_n \vDash_{\mathcal{T}} \psi \qquad \textit{iff} \qquad \varphi_1^B, \cdots, \varphi_n^B \vDash_{\mathcal{K}_P} \psi^B$$

*where generally $\nvDash_{\mathcal{K}} \mathcal{I}(\varphi^B \vee \Box\neg\varphi^B)$, and thus*

$$\nvDash_{\mathcal{T}} \varphi \vee \neg\varphi.$$

Hence, with the BK-translation, we do have a suitable candidate for a possible route to a rejection of LEM. There is, however, another problem with this. Can we actually find examples of indefinite extensibility that lend themselves to be modelled via a branching frame? There are indeed potentialist structures that exhibit branching behaviour, the most prominent of examples would be choice sequences. But can the natural numbers and the ordinals be understood this way? We know that the potentialist structures of end-extensions of models arithmetic obey a branching frame (cf. Hamkins 2018), but as pointed out in Brauer (2020), the indeterminacy in question here is concerned with extensions of models which are all said to contain all the standard numbers. Consequently, the branching behaviour is exhibited above the standard numbers. As such, this situation doesn't reflect the indeterminacy that we might attribute to the natural numbers when they are considered as indefinitely extensible.[15] The matter looks even bleaker for the ordinals as a branching type scenario. It seems that there is no way that their indefinite extensibility can be understood in two mutually inconsistent ways (see for this also the discussion in chapter 2.3.3).

If there is no branching obtainable, perhaps one could try yet to obtain a stricter reading of the existential quantifier?

### 6.4.3. Strict potentialism

Linnebo and Shapiro introduce a position they call strict potentialism, which requires that any true statement needs to be "made true"(Linnebo and Shapiro 2019, p. 179) at a certain (finite) stage. The strict potentialist thus interprets an existentially quantified statement to require the actual availability of a witness, while they would need to provide some other means of verifying a universal generalisation if it is supposed to be true of a whole range if potential objects. In the paper, they make two (interrelated) suggestions to model the semantics of strict potentialism. I will discuss them in turn and argue that only one of them is promising.

Linnebo and Shapiro (2019) show via proof theoretic means an analogue to the potentialist mirroring theorem.[16] (For the definition of $\vDash_{\mathcal{H}}$ see below.)

**Theorem 6.5** (Linnebo and Shapiro 2019, Brauer 2021)**.** *Let $\varphi_1, \cdots, \varphi_n, \psi$ be a collection of formulas, and $\varphi_1^\Diamond, \cdots, \varphi_n^\Diamond, \psi^\Diamond$ their potentialist translation. Let $\mathcal{H}$ be a Kripke*

[15] It is another question whether the existence of non-standard models itself is a source of indeterminacy that may lead to a rejection of LEM. But this is a different question and will be addressed in the last chapter.

[16] An analogue of the theorem using topologies can also be proven. The prove is left out due to space constraints. For a proof using possible, worlds, see Brauer (2021)

*model for intuitionistic modal logic. Then there is an intuitionistic Kripke model $\mathcal{K}$ such that:*

$$\varphi_1, \cdots, \varphi_n \vDash_{\mathcal{K}} \psi \qquad \textit{iff} \qquad \varphi_1^{\Diamond}, \cdots, \varphi_n^{\Diamond} \vDash_{\mathcal{H}} \psi^{\Diamond},$$

*where generally $\nvDash_{\mathcal{H}} \varphi^{\Diamond} \vee \neg\varphi^{\Diamond}$, and thus*

$$\nvDash_{\mathcal{K}} \varphi \vee \neg\varphi.$$

*Proof.* See Brauer (2021). □

The theorem shows that insofar as we revert to an intuitionistic modal logic, we are able to extract intuitionistic logic proper as the adequate way for strict potentialist reasoning. But why should the strict potentialist adapt intuitionistic modal logic in the first place? A closer look at its semantics suggests that they shouldn't.

To begin, it should be noted that there are actually several ways to set-up intuitionist modal logic. The modal logic referred to in the theorem is given in Simpson (1994). It is determined on basis of some requirements intended to capture intuitionistic acceptability (for instance, that the disjunction property holds, that the addition of LEM causes the modal operators to be dual, and that they express an intuitionistically acceptable understanding of the modality). The semantics I present here is a combination of a Kripke frame for intuitionistic logic and a Kripke frame for modal logic, and builds on the definition given in Simpson (1994, p. 88).

**Definition 6.5.** A **Kripke model for intuitionistic first order modal logic with plurals** is a quintuple $\mathcal{H} = \langle H, \leq, \{W_h\}_h, \{R_h\}_h, \{D(w) \mid w \in \bigcup_h W_h\}, \Vdash\rangle$ which consist of a set of information states $H$, partially ordered by $\leq$, and for each $h \in H$, there is a set of worlds $W_h$ ordered by $R_h$ and for each $w \in W_h$ there is a corresponding domain of objects $D(w)$.

We require that $W_h \subseteq W_{h'}$ for $h' \geq h$ and $D(w) \subseteq D(w')$ for $wRw'$ to account for the fact that we are dealing with information growth. We also hold that $R_h \subset R_h'$ to guarantee that no accessibility relations gets lost when we advance form one information state to another. We will also impose the familiar requirements of reflexivity, transitivity, directedness, or linearity on the $R_h$s.[17]

Satisfaction $h, w \Vdash_\sigma \varphi$ is defined as a relation between information states $h$, worlds $w \in W_h$, assignments $\sigma$ and modal formulas as follows. For atomic formulas $x \in y$, $x \prec xx$ and assignments $\sigma$ with $\sigma(x), \sigma(y) \in D(w)$ and $\sigma(xx) \subset D(w)$

| | | |
|---|---|---|
| If $h, w \Vdash_\sigma x \in y$ | then | for all $h' \geq h, h', w \vDash_\sigma x \in y$ |
| If $h, w \Vdash_\sigma x \prec xx$ | then | for all $h' \geq h, h', w \vDash_\sigma x \in xx$ |
| $h, w \nVdash_\sigma \bot$ | | |

And for complex formulas[18]:

| | | |
|---|---|---|
| $h, w \Vdash_\sigma \varphi \wedge \psi$ | iff | $h, w \Vdash_\sigma \varphi$ and $h, w \Vdash_\sigma \psi$ |
| $h, w \Vdash_\sigma \varphi \vee \psi$ | iff | $h, w \Vdash_\sigma \varphi$ or $h, w \Vdash_\sigma \psi$ |
| $h, w \Vdash_\sigma \varphi \to \psi$ | iff | for all $h' \geq h$ if $h', w \Vdash_\sigma \varphi$ then $h', w \Vdash_\sigma \psi$ |
| $h, w \Vdash_\sigma \Box\varphi$ | iff | for all $h' \geq h$ and for all $w' \in W_h$ s.t. $wR_{h'}w' : h', w' \Vdash_\sigma \varphi$ |
| $h, w \Vdash_\sigma \Diamond\varphi$ | iff | there is some $w' \in W_h$ s.t. $wR_hw'$ and $h, w' \Vdash_\sigma \varphi$ |
| $h, w \Vdash_\sigma \forall x\varphi(x)$ | iff | for all $h' \geq h$, $w' \in W_{h'}$ with $wR_{h'}w'$ and $x$-variant assignments $\sigma'$ $h', w' \Vdash_{\sigma'} \varphi(x)$ |
| $h, w \Vdash_\sigma \exists x\varphi(x)$ | iff | there is some $w' \in W_h$ with $wR_hw'$ and $x$-variant assignment $\sigma'$ such that $h', w' \Vdash_{\sigma'} \varphi(x)$ |
| $h, w \Vdash_\sigma \forall xx\varphi(xx)$ | iff | for all $h' \geq h$, $w' \in W_{h'}$ with $wR_{h'}w'$ and $xx$-variant assignments $\sigma'$ $h', w' \Vdash_{\sigma'} \varphi(xx)$ |
| $h, w \Vdash_\sigma \exists xx\varphi(xx)$ | iff | there is some $w' \in W_h$ with $wR_hw'$ and $xx$-variant assignment $\sigma'$ |

[17] In fact, for the mirroring theorem we only require a feature called $I-$directedness: For any $h \in H$ and $w, u, v \in W_h$ with $wR_hu$ and $wR_hv$ there is an $h' \geq h$ and $w' \in W_{h'}$ such that $uR_{h'}w'$ and $vR_{h'}w'$ (cf. Brauer 2021).

[18] The clauses for the quantifiers follow the definition given in Brauer (2021). Defining the universal quantifier in this way has the effect that there is no real distinction between $\forall$ and $\Box\forall$. A weaker reading of the universal quantifier, for instance in analogy to the clause for the intuitionistic conditional, is also conceivable and has, due to the potentialist translation appending an $\Box$ to it anyway, no effect on the proof of the mirroring theorem.

$$\text{such that } h', w' \Vdash_{\sigma'} \varphi(xx)$$

Finally, we fix the stability of the atomic formulas in the modal sense.

Now, in what way does $\mathsf{LEM}^{\Diamond}$ via the duality of the modalised quantifiers fail on this structure? Assume that $h, w \Vdash \neg\Box\forall x\varphi(x)$, where $\varphi(x)$ is decidable. Then there is some $i \geq h$ such that $i, w \nVdash \Box\forall x\varphi(x)$. Hence, there is some $j \geq i$ and $u \in W_j$ such that $j, u \nVdash \forall x\varphi(x)$, and therefore some $k \geq j$ with an element $a$ in a world $v \in W_k$ such that $k, v \nVdash \varphi(a)$. By the decidability of $\varphi$ it follows that $k, v \Vdash \neg\varphi(a)$. But since it is not necessarily the case that $v \in W_h$, we may have that $h, w \nVdash \Diamond\exists x\neg\varphi(x)$. The reason the inference fails is thus a more elaborated form of the non-duality of the modal operators. New possible worlds are thus like atomic facts. Their existence and a subsequent collection of modal formulas can be validated at a finite information state, just like the strict potentialist requires.

There is, however, a problem with this semantics that is similar to the one that plagued the use of the Gödel-translation. Even the strict potentialist would like to have a statement like “for every natural number, there is a successor” come out true. This may be expressed in our language as an instance of modalised plural collection:

$$h, w \Vdash \Box\forall xx\Diamond\exists y\Box\forall u(u \in y \leftrightarrow u \prec xx)$$

If this is the case, however, $h, w \Vdash \neg\Box\forall x\varphi(x)$, when it is understood as a generalisation over the natural numbers, would not only imply that there is a world $w'$ and $a \in D(w')$ such that for some $h'$, with $h', w' \Vdash \neg\varphi(a)$, but also that $w' \in W_h$. Hence, insofar as we allow for the truth of nested quantification like above, we do get the corresponding instances of $\mathsf{LEM}^{\Diamond}$ on these frames.

This problem also arises in possible world semantics for other variants of intuitionistic modal logic, such as those studied in Wijesekera (1990) and Alechina et al. (2001). This leads to the question whether there are other possible semantics that might explain the difference we are after. In this respect, note that for the strict potentialist, the objects seems to be given in an intensional fashion, where, for instance the successor for any number is always given, but the witness to a certain other property is not. In this respect

Linnebo and Shapiro (2019) suggest that the strict potentialist might be better of with a realisability semantics.

A realisability semantics has the advantage that realisers are finite objects and as such an ideal tool for the strict potentialist. Let $e$ be the code of a Turing machine. Then we can define $e \mathsf{r} \varphi$ (read: $e$ realises $\varphi$) recursively in the following way. (The first clause says that any number $e$ realises a true atomic sentence.)

| | | |
|---|---|---|
| $e \mathsf{r} A$ | iff | $A$ |
| $e \mathsf{r} \varphi \wedge \psi$ | iff | $(e)_0 \mathsf{r} \varphi$ and $(e)_1 \mathsf{r} \psi$ |
| $e \mathsf{r} \varphi \vee \psi$ | iff | $[(e)_0 = 0 \to (e)_1 \mathsf{r} \varphi]$ and $[(e)_0 = 1 \to (e)_1 \mathsf{r} \psi]$ |
| $e \mathsf{r} \varphi \to \psi$ | iff | for any $n$, if $n \mathsf{r} \varphi$ then $\{e\}(n) \mathsf{r} \psi$ |
| $e \mathsf{r} \exists x \varphi(x)$ | iff | $(e)(1) \mathsf{r} \varphi(\bar{n})$ for some n |
| $e \mathsf{r} \forall x \varphi(x)$ | iff | $\{e\}(n) \mathsf{r} \varphi(\bar{n})$ for all n |

Of course, the most important clauses for us are the ones for the quantifiers. In the clause for the existential quantifier we may say that we require the realiser $e$ and *additionally* the number $n$ to be available; in the clause for the universal quantifier, the realiser $e$ itself is enough. This makes for their non-duality. Now, this semantics indeed seems to give us intuitionistic logic for reasoning with the natural numbers.

**Theorem 6.6.** *Every theorem of Heyting Arithmetic has a realiser, but not every theorem of* PA *has a realiser (cf. van Dalen 2002).*

There are a couple of points that I want to note about this. One possible objection against the use of realisability semantics was given by Dummett, who points out that the realisability interpretation is not epistemically transparent (Dummett 2000, p. 224). Given a natural number $n$, for instance, it cannot be effectively decided whether $n \mathsf{r} \forall x \varphi(x)$. As such, realisability is not suitable to explicate *traditional* intuitionistic truth. There might, however, be an argument that this is not a pressing problem for the strict potentialist, since those epistemic concerns were largely rejected in chapter 4.

Somewhat more concerning is the fact that since we are no longer using possible world semantics we lose an account of the availability of objects and their conditional existence on each other. At the very least, the strict potentialist would need to give an account of how realisers (which are natural numbers encoding turing machines) for universal statements are outright available while individual natural numbers in the clause of the interpretation of the existential quantifier are not.

Furthermore, there are some statements like Church's Thesis and the Extended Church Thesis that are not provable in Heyting Arithmetic, but that nonetheless have a realiser (cf. van Dalen 2002), and there are statements that have a realiser that are inconsistent with Peano Arithemtic (cf. Troelstra 1998, p. 409). If the strict potentialist were to rely on realisability to explicate their position, they would have to give an account of this within their own framework.

Finally, it should be noted that there are (apart from the basic numerical realisability used above) a wide variety of realisability semantics, and it is not obvious which particular one the strict potentialist should choose. One other possible candidate for instance would be Lifschitz realisability, which is characterised by interpreting the existential quantifier not by a single witness, but by a set of possible witnesses (cf. Troelstra 1998, p. 437).

Still, given the previous two approaches, strict potentialism seems to be the best way to develop the argument against LEM based on indefinite extensibility at the very least for directed or linear frames.

## 6.5. Conclusion

This chapter has (finally) provided some answers to the initial questions of the thesis. Using modal logic, I have given an account of what it means to generalise over concepts with indefinitely extensible domains. This involved an analysis of why instance based generalisations are not suitable for such domains as well as the introduction of a modality to provide an alternative means of generalisation. In this respect, it was argued that the modality is suitable to express generic (absolute) generality over indefinitely extensible domains and this was defended against a possible revenge scenario from the literature.

However, it turned out that generalisations regarding the particular concepts that we are interested in, namely the natural numbers (as a toy case) as well as the ordinals, still license LEM via its respective translations. Two alternative proposals in the form of strict and branching potentialism were also discussed. While branching potentialism indeed managed a rejection of LEM it did not prove to be suitable for the natural numbers or the ordinals, for it does not seem to be comprehensible how there are diverging routes in collecting them. Strict potentialism also leads to a rejection of LEM, but it was pointed out that the standard possible world semantics that is normally used to explicate it does not suitably express nested quantification. For this reason, an alternative realisability semantics would have to be chosen, which, however, comes with its own commitments.

In the appendix the respective theorems for the required translations will now be proven using the topological semantics developed in the previous two chapters. This again connects the validity of LEM with the notion of a warrant that is exclusively given by the availability of objects, and it also leads to an argument against the use of the Double Negation translation to establish classical logic upon a rejection of LEM.

## 6.A. Appendix. LEM and Modal Space

This appendix contains the proofs of theorems (6.2) to (6.4). By using topologies on the one non-modal side of the equivalence, I hope to provide a unified account of all the previous mirroring theorems, and with that I also hope to show how the notion of the availability of objects (on which the status of LEM is based according to the investigation in chapter 4 and 5) can be extended to modal space as well. In this respect, the discussion should establish that LEM holds (in its translated version) insofar as the modal space characterising our reasoning over indeterminate domains is disconnected. Furthermore, this will also highlight how the difference between instance based and generic generalisations can be incorporated into the above discussions. In order to avoid issues with impredicativity that are not directly related to the question of LEM, I will only focus on first order logic and leave out the use of pluralities.

## 6.A.1. The semantics of the $\Box\Diamond$- and of the potentialist translation

Throughout this section, let $\mathcal{K}$ denote a Kripke structure with frame $\langle W, \leq \rangle$ and let $\mathcal{O}_W$ be its corresponding up-set topology which is given by the basis $\mathcal{B} = \{R^{\uparrow}(w) \mid w \in W\}$, and let $\mathcal{O}_{w^{\uparrow}} = \{U \cap R^{\uparrow}(w) \mid U \in \mathcal{O}_W\}$ be the topology that we obtain by restricting $\mathcal{O}_W$ to the space above a certain world $w \in W$.

In order to prove semantic analogues to the mirroring theorems, a small lemma will be helpful.

**Lemma 6.1.** *Let $\mathcal{K}$ be a reflexive and transitive Kripke structure and $\mathcal{O}_W$ its corresponding topology. Then for arbitrary closed formulas $\varphi$ it is the case that*

$$\{w \mid w \vDash \varphi\}^{\circ} = \{w \mid w \vDash \Box\varphi\}$$
$$cl(\{w \mid w \vDash \varphi\}) = \{w \mid w \vDash \Diamond\varphi\}$$

*Proof.* If $w' \in \{w \mid w \vDash \varphi\}^{\circ}$, then $w'' \in \{w \mid w \vDash \varphi\}^{\circ}$ for all $w'' \geq w'$, hence $w' \vDash \Box\varphi$. If $w' \vDash \Box\varphi$ then for all $w'' \geq w'$, $w'' \vDash \Box\varphi$, hence $\{w \mid w \vDash \Box\varphi\}$ is upwards closed and thus open.

If $w' \in cl(\{w|w \vDash \varphi\})$ then $w' \leq w$ for some $w \vDash \varphi$, hence $w' \vDash \Diamond\varphi$, and vice versa. $\Box$

We can now state and prove the semantic analogues of the previous theorems. I will address the case of the $\Box\Diamond$-translation first, and present the others as modifications of it.

### Mirroring the $\Box\Diamond$-translation

**Theorem 6.3** (restated)**.** *Let $\mathcal{K}$ be a Kripke structure that is reflexive and transitive but* not necessarily directed *(thus satisfying at least S4). Then for each world $w \in W$ there is a topological model $\mathcal{T}_w = \langle D(\xi), \mathcal{O}_{w^{\uparrow}}, [\![\cdot]\!]_{\xi} \rangle$ such that*

$$\mathcal{T}_w \vDash \varphi \qquad \textit{iff} \qquad \mathcal{K}, w \vDash \varphi^{\Box\Diamond} \tag{6.1}$$

*Proof.* For each formula $\varphi$, we want its interpretation in $\mathcal{T}_w$ to be the set of worlds satisfying $\varphi^{\Box\Diamond}$. We then need to show that for each formula $\varphi$, the definition $[\![\varphi]\!] = \{w \mid w \vDash \varphi^{\Box\Diamond}\}$ commutes with the operations in the topology.

We show this by induction on the complexity of $\varphi$. For the atomic predicates, we have that

$$[\![P]\!] = \{w \mid w \vDash \Box\Diamond\Box P\}$$
$$[\![\neg P]\!] = \{w \mid w \vDash \Box\Diamond\Box\neg P\}$$

by stability. For the induction step, note that

$$\begin{aligned}
[\![\varphi \wedge \psi]\!] &= \{w \mid w \vDash \varphi^{\Box\Diamond} \wedge \psi^{\Box\Diamond}\} \\
&= \{w \mid w \vDash \varphi^{\Box\Diamond}\} \cap \{w \mid w \vDash \psi^{\Box\Diamond}\} \\
&= [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &= \{w \mid w \vDash \Box\Diamond(\varphi^{\Box\Diamond} \vee \psi^{\Box\Diamond})\} \\
&= \{w \mid w \vDash \Diamond(\varphi^{\Box\Diamond} \vee \psi^{\Box\Diamond})\}^\circ \\
&= (cl(\{w \mid w \vDash \varphi^{\Box\Diamond} \vee \psi^{\Box\Diamond})\}))^\circ \\
&= (cl(\{w \mid w \vDash \varphi^{\Box\Diamond}\} \cup \{w \mid w \vDash \psi^{\Box\Diamond}\}))^\circ \\
&= (cl([\![\varphi]\!] \cup [\![\psi]\!]))^\circ \\
[\![\varphi \to \psi]\!] &= \{w \mid w \vDash \Box(\varphi^{\Box\Diamond} \to \psi^{\Box\Diamond})\} \\
&= \{w \mid w \vDash \varphi^{\Box\Diamond} \to \psi^{\Box\Diamond}\}^\circ \\
&= (\{w \mid w \nvDash \varphi^{\Box\Diamond}\} \cup \{w \mid w \vDash \psi^{\Box\Diamond})\})^\circ \\
&= ([\![\varphi]\!]^c \cup [\![\psi]\!])^\circ \\
[\![\forall x\varphi(x)]\!] &= \{w \mid w \vDash \Box\forall x\varphi^{\Box\Diamond}(x)\} \\
&= \{w \mid w \vDash \forall x\varphi^{\Box\Diamond}(x)\}^\circ \\
&= \{w \mid w \nvDash a = a \text{ or } w \vDash \varphi^{\Box\Diamond}(a) \text{ for } a \in D\}^\circ \\
&= \bigcap_a([\![a = a]\!]^c \cup [\![\varphi(a)]\!])^\circ \\
[\![\exists x\varphi(x)]\!] &= \{w \mid w \vDash \Box\Diamond\exists x\varphi^{\Box\Diamond}(x)\}
\end{aligned}$$

$$= (cl(\{w \mid w \vDash \exists x \varphi^{\Box\Diamond}(x)\}))^{\circ}$$
$$= (cl(\bigcup_a [\![\varphi(a)]\!]))^{\circ}$$

$\square$

**LEM on the $\Box\Diamond$-translation**

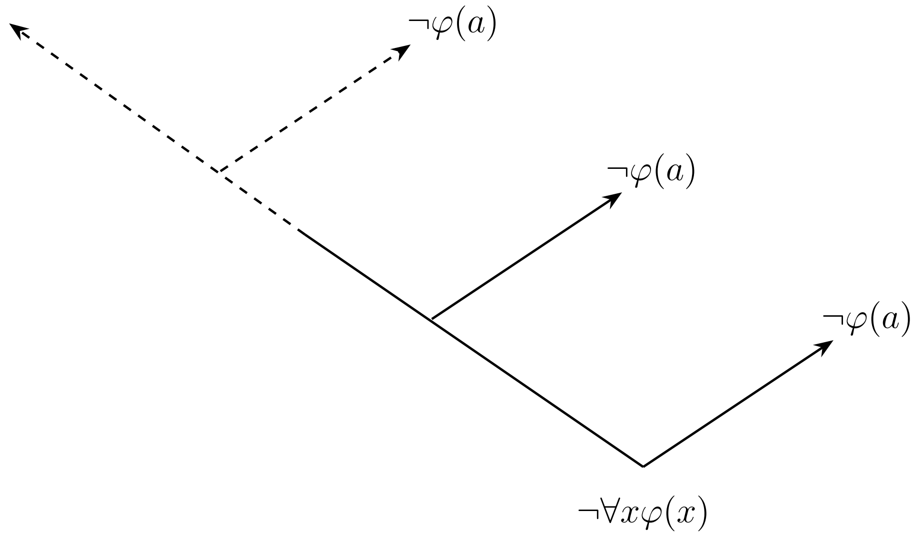



Figure 6.1.

Now, why does $\mathcal{T}_w$ license LEM? And, moreover, why does this not adequately capture our reasoning with indeterminacy? Consider the Kripke model in figure (6.1). In the corresponding topology $\mathcal{T}_{\mathcal{K}}$ we have that

$$[\![\forall x \varphi(x)]\!] = ((\bigcap_a ([\![a = a]\!]^c \cup [\![\varphi(a)]\!])^{\circ})^c)^{\circ} = W \tag{6.2}$$

To simplify things, the model only deals with the availability of one element $a$. To ask whether $\mathcal{T}_w$ satisfies LEM is to ask whether (6.2) entails that

$$[\![\exists x \neg\varphi(x)]\!] = (cl(\bigcup_x [\![\neg\varphi(x)]\!]))^{\circ} = (cl([\![\neg\varphi(a)]\!]))^{\circ} = W.$$

Furthermore, assume that $\varphi(x)$ is decidable, i.e. that $[\![a = a]\!] = [\![\varphi(a)]\!] \cup [\![\neg\varphi(a)]\!]$. In case of our example, this means that $a$ does not exist on any of the worlds on the leftmost branch. Hence $[\![a = a]\!]^c \neq \emptyset$. However, since the leftmost branch does not contain any non-trivial open sets, $([\![a = a]\!]^c)^\circ = \emptyset$. Hence the leftmost branch itself is the measure of $a$'s indeterminacy, which is expressed by the fact that $\partial[\![a = a]\!] = [\![a = a]\!]^c$; on any world on the branch it is possible to add $a$ but it is never forced.

From what has been said, it follows that $[\![\neg\varphi(a)]\!] \neq W$. However, when understanding the existential quantifier as suggested by the $\Box\Diamond$-translation, we simply add the boundary case to our evaluation:

$$\begin{aligned} [\![\exists x \neg\varphi(x)]\!] &= (cl([\![\neg\varphi(a)]\!]))^\circ \\ &= (\partial[\![a = a]\!] \cup [\![\neg\varphi(a)]\!])^\circ \\ &= W^\circ = W \end{aligned}$$

This example can of course be extended to cases involving multiple elements. The mechanism stays the same, we get the inference from $\neg\forall x\varphi(x)$ to $\exists x\neg\varphi(x)$ by simply incorporating the boundary cases of the existence of certain objects. What this shows is that by using the $\Box\Diamond$-translation, formula evaluation happens in ignorance of the relevant cases of indeterminacy that we are concerned with.

There is, however, an unproblematic case, namely when the frames are directed. In such a case, we can even simplify the used translation and obtain the proof of theorem (6.2) as a corollary.

### Mirroring the potentialist translation

For the proof of Linnebo's mirroring theorem we need one more lemma.

**Lemma 6.2.** *Let $\mathcal{K}^d$ be a* directed *Kripke model, and $w \in W$. Then*

$$\mathcal{K}, w \vDash \varphi^\Diamond \quad \textit{iff} \quad \mathcal{K}, w \vDash \Diamond\varphi^\Diamond \quad \textit{iff} \quad \mathcal{K}, w \vDash \Box\varphi^\Diamond$$

*Proof.* By induction on the length of the formula. The only interesting case is that of implication. We thus need to show that $w \vDash \varphi^\Diamond \rightarrow \psi^\Diamond$ iff $w \vDash \Diamond(\varphi^\Diamond \rightarrow \psi^\Diamond)$ iff $w \vDash \Box(\varphi^\Diamond \rightarrow \psi^\Diamond)$.

Consider the left equivalence first. If $w \vDash \varphi^\Diamond \to \psi^\Diamond$ then the right side follows by reflexivity. Now assume that $w \nvDash \varphi^\Diamond \to \psi^\Diamond$, but $w \vDash \Diamond(\varphi^\Diamond \to \psi^\Diamond)$. Then we have $w \vDash \varphi^\Diamond$ and $w \nvDash \psi^\Diamond$ by the first assumption. By the induction hypothesis $w \vDash \Box\varphi^\Diamond$, so $w' \vDash \psi^\Diamond$ for some $w' \geq w$. But then $w \vDash \Diamond\psi^\Diamond$, and by the induction hypothesis again, $w \vDash \psi^\Diamond$. Contradiction.

Now assume that $w \vDash \varphi^\Diamond \to \psi^\Diamond$, but $w \nvDash \Box(\varphi^\Diamond \to \psi^\Diamond)$ (the other direction follows immediately). If it is the case that $w \vDash \psi^\Diamond$, by induction hypothesis, $w \vDash \Box\psi^\Diamond$. Contradiction. If $w \vDash \neg\varphi^\Diamond$ then either $w \vDash \Box\neg\varphi^\Diamond$ and hence $w \vDash \Box(\varphi^\Diamond \to \psi^\Diamond)$, or there needs to be a world $w' \geq w$ such that $w' \vDash \varphi^\Diamond$, and $w' \nvDash \psi^\Diamond$. But then $w \vDash \Diamond\varphi^\Diamond$ and again by induction hypothesis $w \vDash \varphi^\Diamond$. Contradiction. $\Box$

**Theorem 6.2** (Linnebo 2013, restated)**.** *Let $\mathcal{K}$ be a Kripke model that is reflexive, transitive, and* directed. *Then for each world $w \in W$ there is a topological model $\mathcal{T}_w = \langle D(\xi), \mathcal{O}_{w\uparrow}, [\![\cdot]\!]_\xi \rangle$ such that*

$$\mathcal{T}_w \vDash \varphi \qquad \textit{iff} \qquad \mathcal{K}, w \vDash \varphi^\Diamond \tag{6.3}$$

*and $\mathcal{T}_w$* satisfies LEM*.*

*Proof.* The proof is exactly the same to the previous one, except some small modifications to account for the difference in translation. For the base case, set:

$$[\![P]\!] = \{w \mid w \vDash \Box P\}$$
$$[\![\neg P]\!] = \{w \mid w \vDash \Box\neg P\}$$

For the induction step, note that the cases for $\varphi \wedge \psi$, $\forall x\varphi(x)$ are exactly analogue, and that the case of $\varphi \vee \psi$ is even simpler than before. Some more attention has to be paid to the following two cases:

$$\begin{aligned} [\![\varphi \to \psi]\!] &= \{w \mid w \vDash \varphi^\Diamond \to \psi^\Diamond\} \\ &= \{w \mid w \vDash \Box(\varphi^\Diamond \to \psi^\Diamond)\} \\ &= \{w \mid w \vDash \varphi^\Diamond \to \psi^\Diamond\}^\circ \end{aligned}$$

$$= (\{w \mid w \nvDash \varphi^{\Diamond}\} \cup \{w \mid w \vDash \psi^{\Diamond})\})^{\circ}$$
$$= ([\![\varphi]\!]^{c} \cup [\![\psi]\!])^{\circ},$$

where the second step follows from lemma (6.2).

$$\begin{aligned}[\![\exists x\varphi(x)]\!] &= \{w \mid w \vDash \Diamond\exists x\varphi^{\Diamond}(x)\}\\ &= cl(\{w \mid w \vDash \exists x\varphi^{\Diamond}(x)\}\\ &= cl(\bigcup_{a}[\![\varphi(a)]\!])\end{aligned}$$

□

That LEM holds in $\mathcal{T}_{\omega}$ in the unproblematic fashion, follows from the above explanation as the directed frames manage to exclude all the problematic cases.

### 6.A.2. The semantics of the BK-translation

To address the case of branching potentialism, we need to adjust our definition of the corresponding topology. In the following let $U_w = \{\alpha \mid w \in \alpha\}$ be the set of paths $\alpha$ running though a node/world $w$. Given a Kripke frame $\langle W, \leq\rangle$, let $X = \{\alpha \mid \alpha \text{ is a path in } \langle W, \leq\rangle\}$ and define $\mathcal{O}_X$ to be given by the basis $\mathcal{B} = \{U_w \mid w \in W\}$. Then, we can note another version of our lemma.

**Lemma 6.3.** *Let $\mathcal{K}$ be a reflexive and transitive Kripke structure and $\mathcal{O}_X$ its corresponding path-based topology. Then for arbitrary closed formulas $\varphi$ it is the case that*

$$\{\alpha \mid w \in \alpha \textit{ s.t. there is a } w' \geq w \textit{ with } w' \vDash \Box\varphi\}^{\circ} = \{\alpha \mid w \in \alpha \textit{ and } w \vDash \Box\varphi\}$$

*Proof.* If $\alpha \in \{\alpha \mid w \in \alpha \text{ s.t. there is a } w' \geq w \text{ with } w' \vDash \Box\varphi\}^{\circ}$, then $\alpha \in U_w$ for some $w$ with $U_w \subseteq \{\alpha \mid w \in \alpha \text{ s.t. there is a } w' \geq w \text{ with } w' \vDash \Box\varphi\}^{\circ}$. But then it follows that all worlds $w' \geq w$ have $w' \vDash \Box\varphi$ and hence all paths thought $w$ are in $\{\alpha \mid w \in \alpha w \vDash \Box\varphi\}$. For the other direction just note that if $w \vDash \Box\varphi$, then $U_w \subseteq \{\alpha \mid w \in \alpha \text{ s.t. there is a } w' \geq w \text{ with } w' \vDash \Box\varphi\}^{\circ}$. □

### Mirroring the BK-translation

With this preamble, we can state the following

**Theorem 6.3** (Brauer et al. 2022, restated)**.** *Let $\mathcal{K} = \langle W, \leq, D(w), \vDash \rangle$ be a structure for S4+, where $\langle W, \leq \rangle$ is a partial order and $D(w)$ its corresponding domains. For each world $w \in W$, there is a topological model $\mathcal{T}_w = \langle D(\xi), \mathcal{O}_X, [\![\cdot]\!]_\xi \rangle$ s.t. for all formulas $\varphi$:*

$$\mathcal{T}_w \vDash \varphi \qquad \textit{iff} \qquad \mathcal{K}, w \vDash \varphi^B \tag{6.4}$$

*Proof.* Given $\mathcal{K}$ and a world $w$, let a path-based topology $\mathcal{T}_w$ be given by the basis $\mathcal{B}_w = \{U_{w'} | w \leq w'\}$ and collection of domains $D(w')$ for $w' \geq w$. For any formula $\varphi$, we want its interpretation in $\mathcal{T}_w$ to be the set of paths (eventually) satisfying $\varphi^B$ in $\mathcal{K}$, i.e., $[\![\varphi]\!] = \{\alpha \mid \exists w \in \alpha : w \vDash \varphi^B\}$. Analogous to the cases above, we then obtain

$$\begin{aligned}\mathcal{T}_w \models \varphi^B \qquad &\text{iff} \qquad [\![\varphi]\!] = X = \{\alpha \mid w \in \alpha : w \vDash \varphi^B\} \\ &\text{iff} \qquad \mathcal{K}, w \vDash \varphi^B\end{aligned}$$

The induction is essentially the same as well: Given a world $w$ and an atomic predicate $P$, if $w \vDash \mathcal{I}\Box P$, then $w' \vDash \mathcal{I}\Box P$ for all $w'$ on all paths through $w$, which thus delivers an open set. Now we can set

$$[\![P]\!] = \bigcup \{\alpha | w \in \alpha : w \vDash \mathcal{I}\Box P\}$$

For the induction step, note that

$$\begin{aligned}
[\![\varphi \wedge \psi]\!] &= \{\alpha \mid w \in \alpha \text{ and } w \vDash \varphi^B \wedge \psi^B\} \\
&= \{\alpha \mid w \in \alpha \text{ and } w \vDash \varphi^B\} \cap \{\beta \mid w \in \beta \text{ and } w \vDash \psi^B\} \\
&= [\![\varphi]\!] \cap [\![\psi]\!] \\
[\![\varphi \vee \psi]\!] &= \{\alpha \mid w' \in \alpha \text{ and } w' \vDash \mathcal{I}(\varphi^B \vee \psi^B)\} \\
&= \{\alpha \mid w' \in \alpha \text{ and } w' \vDash \varphi^B \vee \psi^B\} \\
&= \{\alpha \mid w' \in \alpha \text{ and } w' \vDash \varphi^B\} \cup \{\beta \mid w' \in \beta \text{ and } w' \vDash \psi^B\}
\end{aligned}$$

$$= [\![\varphi]\!] \cup [\![\psi]\!]$$

The first step regarding disjunction might require some explanation. For each path $\beta$ in $U_w$, there is a world $w'$ such that $w' \vDash \varphi^B \vee \psi^B$, but this means that each $\alpha$ with $w \in \alpha$ has a world $w' \geq w$ such that $w \vDash \varphi^B \vee \psi^B$.

For the conditional, note that

$$\begin{aligned}
[\![\varphi \to \psi]\!] &= \{\alpha \mid w \in \alpha \text{ and } w \vDash \Box(\varphi^B \to \psi^B)\} \\
&= (\{\alpha \mid w \in \alpha \text{ and } w \nvDash \varphi^B\} \cup \{\beta \mid w \in \beta \text{ and } w \vDash \psi^B\})^\circ \\
&= (\{\alpha \mid w \in \alpha \text{ and } w \vDash \varphi^B\}^c \cup \{\beta \mid w \in \beta \text{ and } w \vDash \psi^B\})^\circ \\
&= ([\![\varphi]\!]^c \cup [\![\psi]\!])^\circ
\end{aligned}$$

To give the clauses for the quantifiers, it is helpful to keep in mind that each path $\alpha$ can be associated with an (indeterminately) growing series of domains $\{D(w)\}_{w\in\alpha}$ linearly ordered by inclusion. We can then interpret the quantifiers by:

$$\begin{aligned}
[\![\exists x\varphi(x)]\!] &= \{\alpha \mid \text{there is a } w \in \alpha \text{ and } w \vDash \mathcal{I}\exists\varphi^B(x)\} \\
&= \{\alpha \mid \text{there is a } w \in \alpha \text{ with a } b \in D(w) \text{ and } w \vDash \varphi^B(b)\} \\
&= \bigcup_\alpha [\![\varphi(b)]\!] \text{ for some } w \in \alpha \text{ and } b \in D(w) \\
[\![\forall x\varphi(x)]\!] &= \{\alpha \mid \text{ for all } w \in \alpha \text{ and } w \vDash \Box\forall x\varphi^B(x)\} \\
&= (\{\alpha \mid \text{ for all } w \in \alpha \text{ and objects } a: \text{ If } w \vDash a = a \text{ then } w \vDash \varphi^B(a)\})^\circ \\
&= \bigcap_a ([\![a = a]\!]^c \cup [\![\varphi(a)]\!])^\circ
\end{aligned}$$

where $\Box\forall x\varphi^B(x)$ expresses that for $w' \geq w$ and any $a \in D(w')$ we get that $w \vDash \varphi(a)$. This is, of course, an instance of generic generality. $\Box$

### LEM on the BK-translation

In case the model $\mathcal{K}$ is converging, $\Diamond\exists x\varphi^\Diamond$, $\Box\Diamond\exists x\varphi^{\Box\Diamond}$, and $\mathcal{I}\exists x\varphi^B$ are equivalent , and more generally it is the case that $\varphi^B$ iff $\varphi^\Diamond$ iff $\varphi^{\Box\Diamond}$. But the interesting case is between the interpretation of the existential quantifier in the BK- and the $\Box\Diamond$-translation with respect to branching frames. Above, it was noted that the $\Box\Diamond$-translation effectively takes no account of the availability of objects in branching frames. The BK-translation on the other hand does precisely that. Consider, again, the model in figure (6.1). Unlike for the case of the $\Box\Diamond$-translation, in the resulting topology the interpretation of the existential quantifier of the formula $\exists x \neg\varphi(x)$ excludes exactly the leftmost path. Hence $\partial[\![a = a]\!] \nsubseteq [\![\exists x \neg\varphi(x)]\!]$ and therefore, the existential claim is not validated. This shows that only the $BK$ translation adequately captures the notion of availability of objects in the potentialist setting and assesses the duality of the quantifiers and thus the validity of LEM accordingly.

Hence, on this account, LEM is valid, just like it was argued in chapters 4 and 5, only if the topology is disconnected (i.e. if it allows for two open sets whose union is the entire space). Here, disconnectedness comes down to the observation that any path which does not verify $\forall x \varphi(x)$ necessarily leads to the production of a counterexample. Thus any path that we may choose yields one of those determinate outcomes. It is important to note, however, that this no longer gives us unity of truthmakers, at least not for the interpretation of $\exists x$. There might be different objects coming into existence on different paths that serve as a witness.

### On the double negation translation

This fact can also be used to reject the usage of the double negation translation (DN) to reintroduce classical logic. First off, note that we cannot just apply DN to the formulas in the intuitionistic calculus. Of course, it is the case that $\mathcal{T}_w \vDash \neg\neg\varphi \vee \neg\neg\neg\varphi$, but this is actually just a simpler way of saying that $\mathcal{K} \vDash \mathcal{I}(\Box\neg\Box\neg\Box\varphi^B \vee \Box\neg\Box\neg\Box\neg\varphi^B)$. Now, this can be captured by using the nested translation $(\varphi^{\mathsf{DN}})^B$.[19] Unpacking this, however, shows the following inadequacy. Take an atomic predicate $P(\cdot)$. Then we can perform the following simplifications

$$((\exists x P(x))^{\mathsf{DN}})^B \Leftrightarrow (\neg\neg\exists x \neg\neg P(x))^B$$

[19]Note that $(\varphi^B)^{\mathsf{DN}}$ is not an option, because it is not in the domain of the $BK$-translation, i.e. there is generally no $\psi$ such that $\psi^B = (\varphi^B)^{\mathsf{DN}}$.

$$\Leftrightarrow \Box\neg\Box\neg\mathcal{I}\exists x\Box\neg\Box\neg P(x)$$
$$\Leftrightarrow \Box\Diamond\mathcal{I}\exists x\Box\Diamond P(x)$$
$$\Leftrightarrow \Box\Diamond\exists x\Box\Diamond P(x)$$

But this is exactly the use of the $\Box\Diamond$-translation which was said to be ill equipped to actually reflect reasoning with indeterminacy. Of course, there is always the option to reject indeterminacy outright, but this use of the DN-translation comes down to accepting indeterminacy and ignoring it at the same time.

### Generic and instance based generality

What has been expressed with $\Box\forall$ and (to some extend) with $\mathcal{I}E$ where manifestations of generic generality. Now considering instance based generality, we should again be aware that even if $a_1, \ldots, a_n$ satisfy $\varphi$ it is not enough to express an instance based generalisation by the conjunction of $\psi(a_1) \wedge \cdots \wedge \psi(a_n)$. One also needs to express that the $a_i$'s are all the individuals there are that satisfy $\varphi$. As a reminder, $w \in W$ is a totality state for a formula $\varphi$ iff for any $t \geq w$ and any element $a \in D(t)$ for which $t \vDash \varphi(a)$ it is the case that $a \in D(w)$ and $w \vDash \varphi(a)$.

We can express this in the language of topologies as well. Let $\mathcal{T}_w = \langle X_w, \mathcal{O}_{X_w}\rangle$ be given by the basis $\mathcal{B}_w$. As in definition 5.5 let $X_w$ be a totality state for $\varphi$ iff for any object $a$:

$$\text{If } [\![\varphi]\!]_{X_w}(a) = \emptyset \text{ or undefined} \qquad \text{iff} \qquad [\![\varphi]\!]_U(a) = \emptyset \text{ for any open } U \subset X_w$$

Hence, if $X_w$ is a totality state, then $[\![\varphi]\!]_U(x) = [\![\varphi]\!]_{X_w}(x)$ for all $U \subset X_w$. This corresponds to the fact in the Kripke model that on any path extending from $w$ the collection of objects falling under $\varphi(x)$ remains constant. Now, under the assumption that $\psi(a) \vee \neg\psi(a)$ for any given $a$ we may assert that $\psi$ holds for all instances falling under $\varphi$ or provide a counterexample to it. This, like chapter 5, illustrates the claim that LEM is available on instance based generalisations and in particular, that the weak meta-negation entails the existence of a unique counterexample.

**Corollary 6.1.** *Let $X_w$ be a totality state for $\varphi$ and let $\psi$ be decidable. Then:*

$$(\mathcal{T}_w)^+ \vDash \forall x(\varphi(x) \to \psi(x)) \Rightarrow \curlywedge \qquad \textit{entails} \qquad (\mathcal{T}_w)^+ \vDash \langle x \rangle \varphi(x) \wedge \neg\psi(x)$$

*Proof.* This follows directly from lemma 5.6. □

Furthermore, insofar as we require stability for $\in$ and (in case the models are supplemented accordingly) for plural membership $\prec$ the following instances of BOM are valid in $\mathcal{T}_w$ for any $w$.

$$\mathcal{T}_w \vDash \forall y(\psi(y) \vee \neg\psi(y)) \to (\forall x \in a)\psi(x) \vee (\exists x \in a)\neg\psi(x)$$
$$\mathcal{T}_w \vDash \forall y(\psi(y) \vee \neg\psi(y)) \to (\forall x \prec yy)\psi(x) \vee (\exists x \prec yy)\neg\psi(x)$$

Thereby the topological models give us a spectrum ranging from full intuitionisitic logic, corresponding to fully connected topologies, over disconnected ones validating different degrees of semi-intuitionistic logics by satisfying more or fewer instances of BOM, up to fully discrete topologies, where classical logic holds.

# 7. LEM and Indeterminacy of Width

While the last chapter dealt with indeterminacy of height, this chapter focuses on arguments against LEM based on the indeterminacy of the notion of "all subsets". It attempts to extract a logic for reasoning with the domain of sets based on various ways of modelling this indeterminacy and discusses the philosophical principles and motivations involved in doing so.

The first one to make an argument to that effect was, to my knowledge, Ian Rumfitt (cf. Rumfitt 2015, ch.5). His approach aims to exploit the role that axiomatisation plays for set theoretic concept formation and to this extent it relies on the use of categoricity. I will give a complete formalisation of his approach and point out what I take to be its disadvantages. The discussion of Rumfitt's argument will leave us with a choice to either focus on axiomatic extensions of ZFC or to look at classes of its models that we take in some sense to be representative of the set concept. The discussion in this chapter will proceed with the latter, while axiomatic extensions are investigated in the next and final chapter.

We will thus look into conceptions of the set theoretic multiverse and ask which logic can be extracted from them. Ways of modelling the set theoretic universe have become quite diverse and there won't be room to discuss all the details. To assess which logic they license, I will discuss two paradigmatic approaches. I will first present the generalised approach to the multiverse that given in Väänänen (2014) and apply the analysis of generic and instance based generality from chapter 5 to it. It will turn out, however, that under the weak understanding of the existential quantifier this approach still yields classical logic. But the multiverse given by Väänänen does not take any accessibility relation between universes into account. I will then show how this particular feature of the multiverse can be used to obtain a way to reject LEM. It will turn out, though, that this approach does not license all the axioms of Feferman's SCS (or $\mathsf{SCS}^+$) in that it provides counterexamples to $\Delta_0$-Markov's principle.

Section 1 discusses Rumfitt's approach, section 2 Väänänen's multiverse, and section 3 the intuitionistic multiverse.

## 7.1. Rumfitt's Argument against Classical Semantics in Set Theory

Rumfitt's approach is based on a distinctive notion of determinacy, which is given by a second order axiomatisation that yields categorical models. The two prime examples of this are PA2 and ZFC2 as they were discussed in chapter 3. While PA2 can be used, according to Rumfitt, to tell a "coherent story" why arithmetic is determinate, the same cannot be said for ZFC2. The reason for this is that its models are merely quasi-categorical, i.e. for any two non-isomorphic models, one is an initial segment of the other, and if ZFC2 is augmented by the existence of a certain number of inaccessible cardinals, its corresponding models are at least the size of the next bigger inaccessible. This results in a process somewhat analogous to that characterised by indefinite extensibility. 'Reflecting' on the minimum size of the corresponding models leads us to state the existence of larger inaccessible cardinals explicitly as an axiom, which yields a model, at least the size of the next inaccessible.[1]

Of course, there is also a striking difference to the regular understanding of indefinite extensibility. While the subconcepts of an indefinitely extensible concept $P$ and their extensions are trivially not full extensions of $P$, each of the models understood here *is* a complete instantiation of the set theoretic axioms. In this sense, ZFC2 and each of its axiomatic extensions is taken to be incomplete upon reflection on its models and will get extended accordingly. In this respect, it is not just its extension, but also our (explicit) characterisation of the set concept that gets extended (modulo quasi-categorical models).

[1] Of course, quasi-categoricity is not indefinite extensibility *per se*, because a diagonal function is defined on a set and not with respect to axiom systems. Nonetheless, we can understand Rumfitt as providing another way of specifying the notion of determinacy that was left open in the definition of indefinite extensibility. For further details on this see Linnebo (2018c).

There is, however, a certain problem with this idea, which is connected to its reliance on categoricity, and which will be raised at the end of this section.

To represent our set concept Rumfitt uses an intuitionistic Kripke model where each world corresponds to an axiom system. A world $s$ has access to a world $t$ just in case the system corresponding to $t$ is an axiomatic extension of the system corresponding to $s$. Let $S$ be the space of consistent quasi-categorical axiom systems extending ZFC2. Each node $s \in S$ is understood as a particular axiom system, in case of $s \leq t$, $t$ is a consistent extension of $s$ which can be understood as saying that $t$ **semantically entails** $s$, which is understood as saying that any model of $t$ is also a model of $s$:

$$s \leq t \qquad \text{iff} \qquad \forall \mathcal{M} : \mathcal{M} \vDash t \Rightarrow \mathcal{M} \vDash s$$

This can be used to state:

**Definition 7.1.** A **Kripke model for (quasi-categorical) axiomatic extensions of** ZFC2 is a quadruple $\mathcal{R} = \langle R, \leq, D(\xi), \Vdash \rangle$, where $\langle R, \leq \rangle$ is a frame of quasi-categorical axiomatic extensions of ZFC2 ordered by semantic entailment, and for each $r \in R$, $D(r)$ is defined as the domain of the smallest model satisfying $r$. The forcing relation $\Vdash$ is defined for a world $r$, an atomic formula $x \in y$, and an assignment $\sigma$ with $\sigma(x), \sigma(y) \in D(r)$ as:

$$r \Vdash_\sigma x \in y \qquad \text{iff} \qquad \sigma(x) \in \sigma(y).$$

The clauses for complex formulas are then just as in definition 4.1.

Note that due to quasi-categoricity we have $D(r) \subseteq D(s)$ for $r \leq s$. From this it follows that $r \leq s$ and $r \Vdash P$ entail $s \Vdash P$ for atomic formulas. This can be extended to arbitrary formulas:

**Lemma 7.1.** *$r \leq s$ and $r \Vdash \varphi$ entail $s \Vdash \varphi$ for arbitrary formulas $\varphi$.*

*Proof.* Proof by induction on the length of formulas. □

Furthermore, the above definition entails the decidability of atomic formulas:

**Lemma 7.2.** *For any node $r \in R$, atomic formula $x \in y$ and assignment $\sigma$ it is the case that $r \Vdash_\sigma x \in y \vee x \notin y$*

*Proof.* Assume that $\sigma(x), \sigma(y) \in D(r)$. Then either $\sigma(x) \in \sigma(y)$, which means that $r \Vdash_\sigma x \in y$, or $\sigma(x) \notin \sigma(y)$. In the latter case, since sets are extensional, we have that $s \nVdash_\sigma x \in y$ for all $s \geq r$ and thus $r \Vdash x \notin y$. $\square$

Rumfitt takes $\mathcal{R}$ to show that LEM is not semantically entailed by the set concept as it is represented by the ZFC2 axioms and their extensions. He writes: "There may well be an axiom system which does not force $\exists x \neg \phi(x)$—because it does not entail that some set is not $\phi$—but which equally does not force $\forall x \phi(x)$—because it has an extension whose domain contains a set that is not $\phi$."(Rumfitt 2015, p. 280) However, in construction of such a conterexample, we have to be cautious about the fact that the logical vocabulary (and in particular negation) in $\phi$ is now interpreted within the Kripke model. One concrete example that largely circumvents this problem is following: Extend the bottom node $\mathbb{O}$ with two nodes: One of them is the axiom system ZFC2+ $\neg\exists\kappa$ $\texttt{Inacc}(\kappa)$, the other one is ZFC2+ $\exists!\kappa$ $\texttt{Inacc}(\kappa)$. Since any further extension of the first node doesn't change the model any more, and hence verifies all the same statements at subsequent nodes, we may treat it as terminal and thus as a classical model. From this it follows that $\mathbb{O} \nVdash \exists\kappa$ $\texttt{Inacc}(\kappa)$. Similar reasoning with respect to the second node establishes that $\mathbb{O} \nVdash \neg\exists\kappa$ $\texttt{Inacc}(\kappa)$. This is a counterexample to at least illustrate the workings of the semantics.

Rumfitt argues that this shows that we should not use classical semantics for the domain of all sets, insofar as we take it to be given by this very concept. Moreover, he also takes this to imply that the Powerset axiom should be rejected and that Separation and Replacement should be restricted to $\Delta_0$-Separation and $\Delta_0$-Collection. Based on intuitive notions of determinacy he argues that $\Delta_0$-LEM and $\Delta_0$-Markov are acceptable.[2] He retains the axiom of Infinity and replaces the axiom of Foundation with $\in$-induction, such that we end up with so-called intuitionistic Kripke-Platek set theory (iKP$\omega$) as the representative of a determinate semantics for the set concept.

Rumfitt's argument, however, does not finish there. Even though he takes the *semantics* of set theory to be intuitionistic as a consequence of the above, he argues that the *logic* of set theoretic reasoning should nonetheless be classical. For this, he applies

[2]For a precise statement of these axioms, see definition 7.7.

the double negation translation to the intuitionistic clauses. Since iKP$\omega$ + $\Delta_0$-LEM + $\Delta_0$-Markov proves the DN-translation of every theorem of KP$\omega$, its classical cousin, KP$\omega$ is taken to be the appropriate choice for reasoning with indeterminacy.

When it comes to indeterminacy of height, it was already noted in chapter 6.A.2 that the use of the double negation translation obscures the representation of indeterminacy. A related argument can be made for its application to indeterminacy of width (see section 7.3.4). But before this becomes relevant, there is a more substantial problem with the approach in the first place: The selection of the axioms of iKP$\omega$ is not connected to the above Kripke model, but given by Rumfitt on an intuitive understanding of constructive acceptability. Rumfitt doesn't show that these (and only these) hold in $\mathcal{R}$. In order to investigate this question, we need to give a detailed account of the structure of the model. In this respect, however, we can distinguish between two ways of understanding the indeterminacy that is involved in the design of the model. It can be understood to lead to a linear frame insofar as only its height is concerned (which would exclude the above given counterexample as being artificial), but it can also be understood as a branching frame, insofar as it is the case that "there are incompatible ways of making the iterative conception more precise"(Rumfitt 2015, p. 278). The latter is understood to be reflected by the existence of pairwise inconsistent axiomatic extensions of ZFC like $V = L$ and the existence of a measurable cardinal (rather than a conflict about the exact number of inaccessible cardinals).

In the first case, however, it seems that one could simply use the potentialist translation to even secure full ZFC2. In fact, the whole model then appears to be very similar to the modal structuralism proposed in Hellman (1989) and Berry (2022). Thus, only the second case seems to be a suitable account of indeterminacy of width. However, the conflict between $V = L$ and the existence of a measurable is impossible to represent in Rumfitt's model, because of its reliance on quasi-categoricity. As a recent study by Hamkins and Solberg has shown, there is only a limited number of so-called categorical cardinals, i.e. cardinals for which the axiomatic assertion of their existence yields a categorical model. Furthermore, these cardinals are bounded in the large cardinal hierarchy and their bound is below those cardinals whose existence has a decisive influence on the width of the hierarchy. In particular, theorem 12 of Hamkins and Solberg (2020) entails that all categorical cardinals are below the measurable ones. This means that when it comes to determining the width of the set theoretic hierarchy, $\mathcal{R}$ cannot represent conflicting proposals.

This leaves us with a choice when developing indeterminacy of width: we can either focus exclusively on axiomatic extensions of ZFC and eschew any demand on categoricity, or we can forego any further axiomatisation and just take an appropriately conceived multiverse corresponding to ZFC as the representation of our set concept. The remainder of this chapter focuses on the multiverse approach, while the next chapter is concerned with axiomatic extensions in more general terms.

## 7.2. The Classical Set Theoretic Multiverse

The basic idea underlying the multiverse is that its elements (i.e. universes) are all models of ZFC that stand in relation to one another if they can be obtained from each other via forcing or certain other suitable operations (the least demanding of which would be the simple embedding relation). There have been different proposals for what should be allowed in the multiverse. The most liberal proposal is given in Hamkins (2012), which consists in providing a list of principles the multiverse should adhere to by which inner models and other constructions are included, too. This allows for the highest degree of structures to be relativised to the multiverse, including possibly the natural numbers.[3] However, if we restrict the class of models in a suitable way we end up with a multiverse in which the accessibility relation between universes can be described in a uniform way. Some such candidates for the multiverse are:

The multiverse underlying **Forcing Potentialism** (Hamkins and Linnebo 2017):

$$\mathbb{H} = \{M[G] \mid G \text{ is } M\text{-generic}\}$$

For the notion of genericity, see p.52.

[3]However, as Barton (2016) argues, there is a limit to the relativism one can allow without self-undermining the means to describe the multiverse in the first place, including the concepts of a well formed formula and that of proof.

The multiverse underlying **Countable Transitive Model (ctm) Potentialism** (Hamkins and Linnebo 2017):

$$\mathbb{C} = \{M \mid M \vDash \mathsf{ZFC}, M \text{ is countable and transitive}\}.$$

**Woodin's generic multiverse** (Woodin 2011a): $M$ is a countable transitive model, $M \in \mathbb{M}$ and $\mathbb{M}$ contains all generic extensions and grounds of $M$. This multiverse is conceived such that it can be reduced to one universe, and which is essentially designed to show the inconsistency of the multiverse conception. For a discussion, see Antos, Friedman, et al. (2015), and Meadows (2021).

**Steel's multiverse** (Steel 2014): Where the multiverse is conceived in a way that it has a core. For a criticism of this point, see Antos, Friedman, et al. (2015). For extended discussion, see Maddy et al. (2022).

**Väänänen's multiverse** (Väänänen 2014), which provides a general framework under which the previous conceptions (and more) can be subsumed.

Steel's and Woodin's multiverse conceptions are too involved to be discussed here. Moreover, the case regarding LEM can already illustrated with respect to the 'simpler' conceptions of the multiverse. However, it is worth mentioning that Steel's and Woodin's multiverse have directed frames, while Forcing Potentialism as well as ctm Potentialism have branching frames. The latter is due to the following result:

**Theorem 7.1** (cf. Hamkins 2016)**.** *If $W$ is any countable transitive model of set theory, then there are '$W$-generic' Cohen reals[4] $c$ and $d$, for which the corresponding forcing extension $W[c]$ and $W[d]$ have no common extension to a model of set theory with the same ordinals.*

Thus there are generic sets $G$ and $H$ whose corresponding forcing extensions $V[G]$ and $V[H]$ have no common further extension. The approaches by Woodin and Steel avoid this result by restricting the multiverse in a way that we get directedness. The propositional modal logic of forcing potentialism and ctm potentialism is nonetheless S4.2

[4]A Cohen real is a real number that is identified with a subset of $\omega$.

because an extension of $V$ containing both $G$ and $H$ can be obtained by product forcing $V[G \times H]$ (cf. Hamkins and Löwe 2008, Scambler 2021).[5] For the two approaches to the multiverse discussed in this chapter, the distinction between branching and directed frames, however, does not matter. The approach discussed in this section abstracts from the accessibility relation altogether, while the next section advocates the use of the Gödel translation which is indifferent to the distinction. I will begin with an approach to the multiverse which leads to a vindication of LEM and show how it relates to the notions of generic and instance based generality.

### 7.2.1. Multiverse dependence logic

Väänänen (2014) presents an approach to the multiverse that can be understood as a generalisation of the above multiverse views. He imagines the multiverse as a product of universes such that instead of a cumulative $V$ we get a cumulative $\mathbb{V}$ in which many $V$s are defined in parallel. All the universes are said to contain the same ordinals which can then be used to index the following operation(s):

$$\mathbb{V}_0 = \emptyset \qquad \mathbb{V}_\alpha = \mathscr{P}(\mathbb{V}_{\alpha+1}) \qquad \mathbb{V}_\lambda = \bigcup_{\alpha<\lambda} \mathbb{V}_\alpha \text{ for limit ordinal } \lambda$$

$$\mathbb{V} = \bigcup_{\alpha} \mathbb{V}_\alpha$$

In each universe there is a powerset-operation, although it might differ from universe to universe. Similarly, there are choice sets which may differ from universe to universe, and so on. One important aspect of this multiverse conception is that the individual universes are hidden from us. In general, we don't have names for them and there are only limited ways to refer to them. On the face of it, we cannot distinguish the multiverse case from the simple universe case in a first order setting. This is the way in which the Väänänen's multiverse abstracts from the accessibility relation between

[5] It should also be mentioned that looking beyond the simplest cases of the multiverse a more radical version of branching is possible when considering intensional objects like classes. A potentialist systems for certain conceptions of classes has been developed in Barton and Williams (2021) and shown to have the modal logic of $S4$ as an upper bound. But this focus on classes and (especially) intensional objects constitutes a considerable change in topic such that I won't discuss it here.

worlds (i.e. universes). Väänänen additionally introduces clauses from dependence logic to give us a machinery to distinguish among worlds. We will use this machinery to give an account of generic and instance based generality with respect to the multiverse, and thus provide a connection to the theoretical discussion of chapter 5.

### Multiverse truth

In order to define truth and validity in the multiverse, we need the notion of a multiverse assignment.

**Definition 7.2.** A **multiverse assignment** $\mu$ is a mapping from $\mathbb{M}$ into assignments $\mu(M)$ for each universe $M \in \mathbb{M}$

**Definition 7.3. Validity of a formula** $\varphi$ in the multiverse $\mathbb{M}$ under an assignment $\mu$ is defined as:

$$\mathbb{M} \vDash_{\mu} \varphi \quad \Leftrightarrow \quad \forall M \in \mathbb{M}\, (M \vDash_{\mu(M)} \varphi)$$

Consequentially, a formula is valid if it is satisfied for all assignments $\mu$.

The assignment $\mu$ can be understood via the mechanism of team semantics where a formula does not get a single assignment, but a set of assignments which is called a **team**. A team is said to satisfy a formula if all its assignments do. In the multiverse logic the set of assignments $\mu(\mathbb{M}) = \{\mu(M) \mid M \in \mathbb{M}\}$ can be understood as a team. This gives the following clauses for satisfaction.

**Definition 7.4.** Let $\mu$ be a multiverse assignment. **Satisfaction of formulas** is given by:

| | | |
|---|---|---|
| $\mathbb{M} \vDash_{\mu} t = t'$ | iff | $\mu(M)(t) = \mu(M)(t')$ for all $M \in \mathbb{M}$ |
| $\mathbb{M} \vDash_{\mu} \neg t = t'$ | iff | $\mu(M)(t) \neq \mu(M)(t')$ for all $M \in \mathbb{M}$ |
| $\mathbb{M} \vDash_{\mu} R(t_1, \dots, t_n)$ | iff | $\langle \mu(M)(t_1), \dots, \mu(M)(t_n) \rangle \in R^M$ for all $M \in \mathbb{M}$ |
| $\mathbb{M} \vDash_{\mu} \neg R(t_1, \dots, t_n)$ | iff | $\langle \mu(M)(t_1), \dots, \mu(M)(t_n) \rangle \notin R^M$ for all $M \in \mathbb{M}$ |
| $\mathbb{M} \vDash_{\mu} \varphi \wedge \psi$ | iff | $\mathbb{M} \vDash \varphi$ and $\mathbb{M} \vDash \psi$ |

| | | |
|---|---|---|
| $\mathbb{M} \vDash_\mu \varphi \vee \psi$ | iff | there are $\mathbb{M}_0 \subset \mathbb{M}$ and $\mathbb{M}_1 \in \mathbb{M}$ such that $\mathbb{M}_0 \vDash \varphi$ and $\mathbb{M}_1 \vDash \psi$ and $\mathbb{M}_0 \cup \mathbb{M}_1 = \mathbb{M}$ |
| $\mathbb{M} \vDash_\mu \exists x\varphi(x)$ | iff | for each $M \in \mathbb{M}$ there is an $a \in M$ such that $M \vDash_{\mu(M)} \varphi(a)$ |
| $\mathbb{M} \vDash_\mu \forall x\varphi(x)$ | iff | for all $M \in \mathbb{M}$ and all $a \in M$, $M \vDash_{\mu(M)} \varphi(a)$ |

The semantic clauses from the definition tell us that $\mathbb{M} \nvDash \varphi$ does not entail $\mathbb{M} \vDash \neg\varphi$, but it is nonetheless the case that every sentence of first order classical logic holds in $\mathbb{M}$. For each $M \in \mathbb{M}$ either $M \vDash \varphi$ or $M \vDash \neg\varphi$. By the clause for disjunction this suffices to validate the conclusion that $\mathbb{M} \vDash \varphi \vee \neg\varphi$.

**Homogenisation in the multiverse**

Väänänen has two ways to introduce *homogenisation* into the multiverse. To begin with, he defines another operation called **boolean disjunction**:

$$\mathbb{M} \vDash \varphi \vee_B \psi \qquad \text{iff} \qquad \mathbb{M} \vDash \varphi \text{ or } \mathbb{M} \vDash \psi,$$

which can be used to define the property

$$Det(\varphi) := \varphi \vee_B \neg\varphi,$$

which says that $\varphi$ has one single truth value throughout the universe. Importantly, this is not to be confused with the tautology $\varphi \vee \neg\varphi$. It is also the case that boolean disjunction does not commute with other logical connectives. While $(\chi \vee \psi) \vee_B \varphi$ is fine, $\chi \vee (\psi \vee_B \varphi)$ cannot be evaluated in the semantics as it is given. As our analysis of generic and instance based generality will reveal, $\vee_B$ is actually a meta-expression that only commutes with its own kind.

Väänänen also introduces so-called dependence atoms from dependence logic (cf. Väänänen 2007) of the form $=(y, x)$, which say that the values of $x$ in the team/multiverse depend on the values of $y$, with the satisfaction clause:

$$\mathbb{M} \vDash_\mu {=}(y,x) \qquad \text{iff} \qquad \forall M, N \in \mathbb{M}\, [\mu(M)(y) = \mu(M)(y) \to \mu(M)(x) = \mu(M)(x)].$$

Their purpose is to define other important expressions like $=(x)$ which says that all universes agree on the interpretation of $x$, in symbols

$$\mathbb{M} \vDash_\mu {=}(x) \qquad \text{iff} \qquad \forall M, N \in \mathbb{M}\; \mu(M)(x) = \mu(M)(x).$$

This can be formulated by setting $y = \langle\rangle$ in the expression $=(y, x)$. The formula $=(x)$ can be used to express uniformity over models, for instance in statements like $\mathbb{M} \vDash \exists x\, {=}(x)$ which says that all models have a common element. One can go further and define agreement with respect to the extension of arbitrary formulas $=(x : \varphi(x))$, i.e.

$$\mathbb{M} \vDash {=}(x : \varphi(x)) \qquad \text{iff} \qquad \forall M, N \in \mathbb{M}\, \{a \in M \mid M \vDash \varphi(a)\} = \{a \in N \mid N \vDash \varphi(a)\}$$

As already noted, the multiverse logic validates LEM. The next task is to integrate this into our framework of generic and instance based generality. For this, we need to define a complete joint meta- and object logic on the multiverse dependence logic.

### 7.2.2. Generic and instance based generality

The object logic quantifiers featured in the above definition are understood to express statements of generic generality. In chapter 5 we required an additional existential guard for generic generality, whereas for instance based generality we used the notion of a totality state. Because of the way the object logic universal quantifier was defined, there is no need for the existential guard—it is already built in: At each universe $M$ the universal quantifier is evaluated on those $a$ for which $a \in M$. Furthermore, from the definition of satisfaction it is apparent that we are working with a weak (supervaluationist) understanding of the existential quantifier. Since we already know that the object logic is classical, their duality follows

### Meta-Logic for $\mathbb{M}$

Let $\mathbb{M}^+$ be the two-valued meta-model defined on the object logic multiverse model $\mathbb{M}$. Then it is immediate to see that boolean disjunction corresponds to meta-disjunction, i.e

$$\varphi \vee_B \neg\varphi \equiv \varphi \parallel \neg\varphi$$

The existential meta-quantifier can be defined by using an expression from dependence logic

$$\mathbb{M}^+ \vDash \langle x \rangle \varphi(x) \qquad \text{iff} \qquad \mathbb{M} \vDash \exists x =\!(x) \wedge \varphi(x),$$

which can similarly be expressed as a "long meta-disjunction" $\mathbb{M} \vDash \varphi(a) \parallel \varphi(b) \parallel \ldots$ for $a, b \in \bigcup \mathbb{M}$ such that $\mathbb{M} \vDash =\!(a)$ and $\mathbb{M} \vDash =\!(b)$ etc.

To define meta-implication $\varphi \Rightarrow \psi$ we cannot use boolean disjunction á la $(\neg\varphi \vee_B \psi)$ since we don't have meta-negation yet. Instead set

$$\mathbb{M}^+ \vDash \varphi \Rightarrow \psi \qquad \text{iff} \qquad \mathbb{M} \nvDash \varphi \text{ or } \mathbb{M} \vDash \psi$$

which, again, differs from $\mathbb{M} \vdash \varphi \to \psi$ which would be evaluated as $\mathbb{M} \vDash (\neg\varphi \vee \psi)$. Note that $\mathbb{M}^+ \nvDash (\varphi \Rightarrow \psi) \Rightarrow (\varphi \to \psi)$ because assuming $\mathbb{M} \nvDash \varphi$ which validates the antecedent does not tell us anything about whether $\mathbb{M} \vDash \varphi \to \psi$.

### Instance based generality

We have already seen that object level negation for generic generality is in order. The next question is to investigate the meta-negation of an instance based generalisation. A statement of instance based generality is characterised by the unicity of truthmakers. For this, we need a definition of a totality state that is suitable for the multiverse. This can be expressed with the help of dependence logic.

**Definition 7.5.** A sub-multiverse $\mathbb{M}_0$ is a **totality state** for a formula $\varphi$ iff $\mathbb{M}_0 \vDash =\!(x : \varphi(x))$.

But this alone is not sufficient to guarantee the existence of global counterexamples. Let $\mathbb{M}_0 \subset \mathbb{M}$ be a totality state for $\varphi$ such that $\mathbb{M}_0 \vDash {=}(x : \varphi(x))$. Now, if $\mathbb{M}_0 \nvDash \forall x(\varphi(x) \to \psi(x))$, this only means that there is an $a$ in all $M \in \mathbb{M}_0$ such that $\mathbb{M}_0 \vDash \varphi(a)$ but $\mathbb{M}_0 \nvDash \psi(a)$—but the latter is the case already if there is only one $M' \in \mathbb{M}_0$ such that $M' \vDash \neg\psi(a)$. But note that the formulation of BOM from chapter 5 required that $\psi$ was decidable. We thus need a multiverse analogue of this property. Since we are not dealing with epistemic concerns here, the corresponding formulation will be given in terms of absoluteness of $\psi$ across universes.

Recall from definition (2.3) that $\psi$ is **absolute** with respect to $\mathbb{M}_0$ iff for all $M, N \in \mathbb{M}_0$, if $a \in M$ and $a \in N$, then $M \vDash \psi(a)$ iff $N \vDash \psi(a)$. In short, the models/universes $M$ and $N$ agree with respect to $\psi$ on all the elements that they share. This requirement corresponds to the requirement of decidability for $\psi$ in the simple finitary set-up in Linnebo's approach. This now gets us:

**Lemma 7.3.** *Let $\mathbb{M}_0$ be a totality state for $\varphi$ and let $\psi$ be absolute with respect to $\mathbb{M}_0$. Then*

$$(\mathbb{M})^+ \vDash [\forall x(\varphi(x) \to \psi(x))] \Rightarrow \curlywedge \qquad \textit{implies} \qquad (\mathbb{M})^+ \vDash \langle x\rangle(\varphi(x) \wedge \neg\psi(x))$$

*Proof.* If $\mathbb{M}_0 \nvDash \forall x(\varphi(x) \to \psi(x))$ then there must be a world $M \in \mathbb{M}_0$ and $a \in M$ such that $M \vDash \varphi(a)$ and $M \vDash \neg\psi(a)$. Because ${=}(x : \varphi(x))$ holds in $\mathbb{M}_0$ $a$ must exist in all $N \in \mathbb{M}$, and by the absoluteness of $\psi$, it must hold that $N \vDash \neg\psi(a)$ for all $N \in \mathbb{M}$. Hence, $N \vDash {=}(\neg\psi(a))$ which is just another way of saying that $(\mathbb{M})^+ \vDash \langle x\rangle\neg\psi(x)$. $\square$

In summary, Väänänen's model and its supervaluationist reading of disjunction and existential quantification provides us with a counterexample to any claim that generic generality automatically entails intuitionistic logic. This, however, doesn't mean that there is no difference between negations of generic and instance based universal generalisations. While the negation of a generic universal generalisation gives us a global covering of local counterexamples, the negation of instance based generalisations yields a unique *global* counterexample.

## 7.3. Towards the Intuitionistic Multiverse

Until now, the multiverse gave us classical logic. Formulas were evaluated with respect to each universe individually. But what happens if we take the accessibility relation into account as well—after all, some elements of the multiverse depend on others and this dependency also reflects the notion that we are always able to disturb any determination of the size of the continuum. Any attempt to model the indeterminacy that forcing introduces should thus reflect this possibility. We will take (some version of) the multiverse as the correct account of indeterminacy within set theory and ask what may determinately hold in it. Methodically we will again use modal logic to describe the fine structure of our reasoning with this type of indeterminacy, and then define a suitable translation to extract universally valid statements. Which translation will be used corresponds to different conceptions of how exactly the multiverse expresses the indeterminacy of the sets.

### 7.3.1. The forcing multiverse as a potentialist system

As the study of the multiverse can grow quite complex, this section will focus only on introducing the general idea, and thereby exclusively on the behaviour of CH.

#### Satisfaction in the multiverse

Each forcing multiverse $\mathbb{F}$ gives rise to a Kripke frame, in which the nodes $M, N$ are the models $M[G]$. To define their accessibility relation, let $\mathbb{B}$ be a boolean algebra, and $G_{\mathbb{B}}$ be an ultrafilter on $\mathbb{B}$. Then set:

$$M \leq N \qquad \text{iff} \qquad \exists \mathbb{B} \in M \; \exists G_{\mathbb{B}} : N = M[G]$$

The frame is reflexive and transitive, but because of theorem 7.1 not necessarily directed.

**Definition 7.6.** The triple $\mathcal{K}_{\mathbb{F}} = \langle \mathbb{F}, \leq, \vDash \rangle$ is a **Kripke model of the forcing multiverse** $\mathbb{F}$, where $\leq$ is the substructure relation, and $\vDash$ is a mapping from sentences of $\mathcal{L}_{\in}^{\Diamond}$ to truth values. Let $\sigma$ be an assignment from variables to elements in the universes

$M \in \mathbb{F}$. Then, for any $M \in \mathbb{F}$, atomic formula $x \in y$ and assignment $\sigma$ such that $\sigma(x), \sigma(y) \in M$, we have

$$M \vDash_\sigma x \in y \quad \text{iff} \quad \sigma(x) \in \sigma(y)$$

Note, again, that on the right hand side we use the actual membership relation. The clauses for arbitrary formulas $\varphi$ and $\psi$ are then

| | | |
|---|---|---|
| $M \vDash_\sigma \neg\varphi$ | iff | $M \nvDash_\sigma \varphi$ |
| $M \vDash_\sigma \varphi \wedge \psi$ | iff | $M \vDash_\sigma \varphi$ and $M \vDash_\sigma \psi$ |
| $M \vDash_\sigma \varphi \vee \psi$ | iff | $M \vDash_\sigma \varphi$ or $M \vDash_\sigma \psi$ |
| $M \vDash_\sigma \varphi \rightarrow \psi$ | iff | $M \nvDash_\sigma \varphi$ or $M \vDash_\sigma \psi$ |
| $M \vDash_\sigma \forall x \varphi(x)$ | iff | for all $x$-variant assignments $\sigma^*$ with $\sigma^*(x) \in M, M \vDash_{\sigma^*} \varphi(x)$ |
| $M \vDash_\sigma \Box\varphi$ | iff | for all $N \in \mathbb{F}$ such that $M \leq N$: $N \vDash_\sigma \varphi$ |
| $M \vDash_\sigma \Diamond\varphi$ | iff | there is some $N \in \mathbb{F}$ such that $M \leq N$ and $N \vDash_\sigma \varphi$ |

Just like in the previous chapter, these definitions entail the converse Barcan formula plus the stability for atomic sentences and their negations.

For the purpose of this section, I take the forcing multiverse to be the collection $\mathbb{H} = \{\mathrm{M[G]} \mid G \text{ is } M\text{-generic}\}$ that is closed under the operation $M \mapsto M[G]$ for $G$ being $M$-generic. Let $\mathcal{K}_{\mathbb{H}}$ denote its **corresponding Kripke model**.

### The behaviour of CH in the multiverse

To investigate the behaviour of CH in the multiverse under this semantics, I will introduce the following convention. I will use the letters $x, y, z, \ldots$ as bound variables, the letters $u, v, w, p, q, r, \ldots$ as free variables and the letters $a, b, c, \ldots$ for actual elements in the universes. Also, note that since we are working with countable transitive models, $\omega$ is the same in all universes (cf. Jech 2002, Thrm. 6.15). Let $CS(x) = \forall y(y \in x \rightarrow y^+ \in x)$ express that $x$ is closed under successor. Then this means that for any assignment $\sigma$

and variable $u$, if

$$M \vDash_\sigma \emptyset \in u \wedge CS(u) \wedge \forall y(CS(y) \to u \subset y)$$

then for all $N \in \mathbb{H}$

$$N \vDash_\sigma \emptyset \in u \wedge CS(u) \wedge \forall y(CS(y) \to u \subset y).$$

In the following, we write $\omega$ in the style of a free variable and understand it as saying that the assignment that we are using interprets $\omega$ as the set of natural numbers, i.e. such that the above statement comes out true in all universes.

In general, $\mathsf{CH}$ can be represented as the claim that

$$\forall x \subset \mathscr{P}(\omega)(x \neq \emptyset \to (\exists f : \omega \twoheadrightarrow x \vee \exists g : x \twoheadrightarrow \mathscr{P}(\omega)))$$

where the two-headed arrows indicate that the respective functions are surjective. Now, assume that $\mathsf{CH}$ holds in $M \in \mathbb{H}$. Then we have an assignment $\sigma$ and variable $u$, where $u$ is interpreted to denote the object $a$ which acts as the powerset of $\omega$ in $M$, i.e. we have that

$$M \vDash_{\sigma[u/a]} \forall x(x \subseteq \omega \leftrightarrow x \in u),$$

which we can abbreviate as $M \vDash_{\sigma[u/a]} u = \mathscr{P}(\omega)$, and where it is the case that

$$M \vDash_{\sigma[u/a]} \forall x \subset u(x \neq \emptyset \to (\exists f : \omega \twoheadrightarrow x \vee \exists g : x \twoheadrightarrow u)). \tag{7.1}$$

Let the formula in (7.1) be abbreviated as $\mathsf{SU}(u)$.

As we move through $\mathbb{H}$, the object that is taken to be the powerset in the respective worlds changes from $M$ to an $M[G]$, such that

$$M[G] \nvDash_{\sigma[u/a]} u = \mathscr{P}(\omega)$$

and instead there is some other variable $v$ with $\sigma(v) \in M[G]$ denoting an object $b$ which is the (new) powerset of $\omega$ in $M[G]$, i.e.

$$M[G] \vDash_{\sigma[u/a,v/b]} v = \mathscr{P}(\omega)$$

Now, if it is the case that $M[G] \vDash \neg\mathsf{CH}$, then there is a new subset of $\omega$ in $M[G]$ for which there is no surjection into $b$. Hence for some $x$-variant assignment $\sigma^*[u/a, v/b]$

$$M[G] \vDash_{\sigma^*[u/a,v/b]} \neg\exists g : x \to v.$$

But if this is the case, then we can force an $M[G][F]$, in which CH is true again, such that there exist additional functions from $\omega$ into the element denoted by $x$, and hence

$$M[G][F] \vDash_{\sigma^*[u/a,v/b]} \exists g : \omega \to x.$$

This will have some repercussions when we consider the behaviour of modalised claims. The question that we are now concerned with is to choose a suitable class of modal formulas which represent determinate reasoning over the multiverse, and determine which principles hold for them. I will consider two possible choices: The class of formulas given by the potentialist translation and by the Gödel translation. This will also be connected to two different philosophical views regarding the status of the multiverse with respect to claims regarding the indeterminacy of the set concept. One of them has only emerged recently, and is referred to as *countabilism*, the other one will be sketched below and is to some extent connected to Feferman's ideas.

A most immediate consequence, however, is that no $M$ satisfies a modalised rendering of the powerset axiom on any of the two translations. The reason for this is that on any $M \in \mathbb{H}$

$$M \nvDash \exists y \Box \forall x (\Box x \subset \omega \to \Box x \in y),$$

which is the case, because there always is an $M[G]$ which contains a subset of $\omega$ that is not in $M$, and hence cannot be in $y$. Thus, we don't get the powerset axiom no matter what translation we choose.

**Lemma 7.4.** *Let $\cdot^{\Diamond}$ be the potentialist translation and $\cdot^{\Box}$ be the Gödel translation. Then*

$\mathcal{K}_{\mathbb{H}} \nvDash [\mathscr{P}(\omega) \text{ exists }]^{\Diamond}$

$\mathcal{K}_{\mathbb{H}} \nvDash [\mathscr{P}(\omega) \text{ exists }]^{\Box}$

### 7.3.2. Countabilism and the weak iterative conception of set

The multiverse has been investigated using the potentialist-translation and first order plural logic in Scambler (2021), where a system was given that interprets ZFC without the Powerset axiom. This idea can be understood as one way of addressing the indeterminacy of width. By showing that the underlying logic of the modality in $\mathcal{K}_{\mathbb{H}}$ is still S4.2, Scambler proves a mirroring theorem for the $\cdot^{\Diamond}$-translation into classical logic and thus vindicates $\mathsf{LEM}^{\Diamond}$ (Scambler 2021, Lemma 13.). However, in order to differentiate this approach from the one that is to come, it is instructive to see why the most natural cases for failure of LEM don't yield this result under the $\Diamond$ translation.

The prime candidate for the failure of $\mathsf{LEM}^{\Diamond}$ would have been the modalised version of CH. Note, however, that for all $M \in \mathbb{H}$,

$$M \vDash \Box\neg\exists y(y = \mathscr{P}^{\Diamond}(\omega) \wedge \mathsf{SU}^{\Diamond}(u)),$$

simply because the first conjunct doesn't hold in any world. For what it's worth, this statement does not seem to be a faithful rendering of CH in the first place, since it additionally incorporates the claim that the (modalised) powerset exists, which is something that CH rather presupposes. The next best candidate for a failure of LEM would have been $\mathsf{SU}^{\Diamond}(u)$ for an appropriate $u$. In this respect, however, note that for the crucial case in which $u$ is assigned the powerset of $\omega$ in $M$, $\mathsf{SU}^{\Diamond}(u)$ comes out true in $M$. More formally, given a variable $u$, fixing $\sigma(u) = a$, where $a$ denotes the powerset of $\omega$ in $M$, it is the case that

$$M \vDash_{\sigma[u/a]} \Box\forall x \subset u(x \neq \emptyset \rightarrow (\Diamond\exists f : \omega \twoheadrightarrow x \vee \Diamond\exists g : x \twoheadrightarrow u))$$

This is true because for any world $N \geq M$ and $x$-variant assignment $\sigma^*[u/a]$, there is a world $O \geq N$ such that either $O \vDash_{\sigma^*[u/a]} \exists f : x \twoheadrightarrow u$ or $O \vDash_{\sigma^*[u/a]} \exists g : \omega \twoheadrightarrow x$.

That $\mathsf{LEM}^{\Diamond}$ holds for formulas like $\mathsf{SU}^{\Diamond}(u)$ in any world $M$ is because there is always an $O \geq M$ in which there is the required surjection for any suitable assignment $\sigma$ of $x$. But this doesn't show us that there is a mapping from $\sigma(x)$ to the powerset *in* $O$, nor that $\sigma(x)$ was countable in $M$. And even if this were the case, this would not show us that there will be such a surjection to the respective powersets in subsequent worlds. Thus, one might ask, does $\Diamond\exists$ really express existence in the face of indeterminacy of width? Scambler (2021) affirms and interprets this as an argument for countability, i.e. the claim that every set is (potentially) countable (cf. Meadows 2014, Pruss 2020, Builes et al. 2022). In a sense, this form of potentialism is quite close to the one treated in the last chapter. It rests on the heuristic that we manage to introduce new surjections from $\omega$ onto any cardinal that might appear to be uncountable. Hence, forcing, in close analogy to indefinite extensibility, is understood first and foremost as a procedure to generate new elements (in this case the respective surjections). This idea has been developed further in recent literature as the 'weak iterative conception of set' (cf. Barton n.d.).

But there is another perspective that one might take on the multiverse, namely that each universe is taken to be (to some extent) *a separate instantiation of the set concept.* This is unparalleled for the case of height indeterminism, where none of the intermediate worlds have that status. This was already mentioned in chapter 2 and philosophically interpreted in chapter 3. It pushes in the opposite direction of the analogy of forcing with indefinite extensibility. Under this perspective, $\Diamond\exists$ no longer seems to be the right way to express existential claims. That it is possible to add a surjection from $\omega$ to any subset of $\omega$ does not express determinately that $\mathsf{CH}$ is true or that it eventually becomes (determinately) true, because, as we have seen above, the interpretation of the powerset may change accordingly from universe to universe. We are not just adding new elements, we are also *changing the roles* that each of the elements fulfils in their respective ways of instantiating the set concept. In this regard, then, the strategy is not to *construct* the domain of reasoning in the first place (as it was done in chapter 6), but to extract invariant principles from the multiverse as it is an antecedently given representation of set theoretic indeterminacy. For this kind of endeavour, the Gödel translation with its use of $\Box\exists$ to interpret existential claims seems to be the more suitable choice.

However, it seems that there is at least a tension between the idea that universes instantiate the set concept individually, and that they also depend on each other. But one might just point out that it just is a feature of the set concept that one can apply constructions to different instances of it all the while staying within the boundaries that the concept draws—and that might just be taken as the sort of indeterminacy that is contained in it. In any case, a full philosophical development and defence of the view is still required but cannot be developed here. The purpose of mentioning it is simply to indicate that there is a possible philosophical position which leads to the technical consequences that will be developed in the next section.

### 7.3.3. Feferman's SCS and Scambler's $\mathsf{SCS}^+$

The motivation for both, Feferman's SCS and Scamblers $\mathsf{SCS}^+$, comes from the assumption that indeterminacy only licenses intuitionistic logic. In explicit reference to Feferman, Scambler states one of the guiding principles for motivating $\mathsf{SCS}^+$:

> What's determinate is the domain of classical logic, what's not is the domain of intuitionistic logic.(Scambler 2020, p. 562)

We are now finally able to investigate this claim and provide an argument for it. We will, however, see that the way of arguing for this principle as it is presented here is incompatible with Markov's Principle..

#### LEM in the intuitionistic multiverse

In the last chapter, it was shown that the Gödel translation was not suitable to express reasoning over indefinitely extensible concepts, because when it was applied to nested quantifiers like $\forall x \exists y$ it was shown to collapse all possible elements into one world. In case of indeterminacy of width, things look differently. Insofar as each universe is understood as a separate instantiation of the set concept, the Gödel translation seems to be the more appropriate choice to capture what is determinate. Using the Gödel translation, we do get failures of $\mathsf{LEM}^\Box$ that are related to $\mathsf{CH}$.

**Lemma 7.5.** $\mathcal{K}_{\mathbb{H}} \not\models \mathsf{LEM}^\Box$.

*Proof.* We need to show that there is an (open) formula $\varphi^{\Box}(x)$ and an assignment $\sigma$ such that for some $M$, $M \nvDash_{\sigma} \varphi^{\Box}(x)$ and $M \nvDash_{\sigma} \Box\neg\varphi^{\Box}(x)$.

Let $\varphi(u) = \mathsf{SU}(u)$ and pick a world $M$ such that $M \vDash \neg\mathsf{CH}$. Then there is an assignment $\sigma$ with $\sigma(u) = a$, where $a$ is the powerset of $\omega$ in $M$ such that

$$M \nvDash_{\sigma[u/a]} \Box\forall x \subset u(x \neq \emptyset \rightarrow (\Box\exists f : \omega \twoheadrightarrow x \vee \Box\exists g : x \twoheadrightarrow u)),$$

because there is at least one $x$-variant assignment $\sigma^*[u/a]$ such that

$$M \nvDash_{\sigma^*[u/a]} x \subset u \wedge x \neq \emptyset \wedge \neg\exists f : \omega \twoheadrightarrow x \wedge \neg\exists g : x \twoheadrightarrow u$$

and the above follows by reflexivity.

Furthermore, it should also be the case that

$$M \nvDash_{\sigma[u/a]} \Box\neg\Box\forall x \subset u(x \neq \emptyset \rightarrow (\Box\exists f : \omega \twoheadrightarrow x \vee \Box\exists g : x \twoheadrightarrow u))$$

because this transforms to

$$M \nvDash_{\sigma[u/a]} \Box\Diamond\exists x \subset u(x \neq \emptyset \wedge (\Diamond\neg\exists f : \omega \twoheadrightarrow x \wedge \Diamond\neg\exists g : x \twoheadrightarrow u))$$

and, eventually, one of the surjections will be introduced, thus refuting the necessitated claim. To see this, pick a forcing extension $M[G] \geq M$ such that $M[G] \vDash \mathsf{CH}$. Then $M[G] \vDash_{\sigma[u/a]} u \subseteq v$ for some variable $v$ that denotes the powerset $b$ of $\omega$ in $M[G]$. Since $\mathsf{CH}$ holds in $M[G]$ we have for every $x$-variant assignment $\sigma^*[u/a, v/b]$ denoting a non-empty subset of $b$,

$$M[G] \vDash_{\sigma^*[u/a,v/b]} \exists f : \omega \twoheadrightarrow x \vee \exists g : x \twoheadrightarrow v.$$

But this implies

$$M[G] \vDash_{\sigma^*[u/a,v/b]} \exists f : \omega \twoheadrightarrow x \vee \exists g : x \twoheadrightarrow u.$$

The reason for this is that if there is no surjection from $\omega$ into $x$ then, because $\mathsf{CH}$ holds in $M[G]$, $M[G] \vDash \exists g : x \to v$, but since $u \subseteq v$ there has to be a surjection from $x$ onto $u$ as well. Now, this means that if no further subsets of $u$ are introduced, then the claim is refuted.

Hence, there is an $M$ and assignment $\sigma$, such that $M \nvDash \mathsf{SU}^{\Box}(u) \vee \Box\neg\mathsf{SU}^{\Box}(u)$. $\Box$

The matter can be further illustrated by the failure of the duality of the modalised quantifiers, i.e. by the claim that there is a formula $\varphi^{\Box}(x, y)$ such that $\Box\neg\Box\forall x\varphi^{\Box}(x, y)$ does not entail $\Box\exists x\Box\neg\varphi^{\Box}(x, y)$. For this, assume that $M \vDash_{\sigma} \neg\mathsf{CH}$, $M \vDash_{\sigma} u = \mathscr{P}(\omega)$, and that

$$M \vDash_{\sigma[u/a]} \Box\neg\Box\forall x \subset u(x \neq \emptyset \to (\Box\exists f : \omega \twoheadrightarrow x \vee \Box\exists g : x \twoheadrightarrow u)).$$

We can show that this is compatible with

$$M \nvDash_{\sigma[u/a]} \Box\exists x \subset u\Box\neg(x \neq \emptyset \to (\Box\exists f : \omega \twoheadrightarrow x \vee \Box\exists g : x \twoheadrightarrow u)),$$

hence refuting the duality of the modalised quantifiers. Because the latter transforms into

$$M \nvDash_{\sigma[u/a]} \Box\exists x \subset u\Box(x \neq \emptyset \wedge (\Diamond\neg\exists f : \omega \twoheadrightarrow x \wedge \Diamond\neg\exists g : x \twoheadrightarrow u)),$$

and if the above holds, then there is a world $M[G]$ with $M[G] \vDash \mathsf{CH}$ and

$$M[G] \vDash_{\sigma[u/a, v/b]} \forall x(x \subset \omega \leftrightarrow x \in v)$$

and $a \subseteq b$. Then either $M[G] \vDash_{\sigma[u/a,v/b]} \exists f : \omega \twoheadrightarrow x$ or, by the same reasoning as above, $M[G] \vDash_{\sigma[u/a,v/b]} \exists g : x \twoheadrightarrow u$, and that doesn't change for subsequent worlds. We thus have for all $x$-variant assignments that $\sigma^*[u/a, v/b]$

$$M[G] \vDash_{\sigma^*[u/a,v/b]} \Box\exists f : \omega \twoheadrightarrow x \vee \Box\exists g : x \twoheadrightarrow u$$

which entails that

$$M \nvDash_{\sigma^*[u/a,v/b]} \Box(x \neq \emptyset \wedge (\Diamond\neg\exists f : \omega \twoheadrightarrow x \wedge \Diamond\neg\exists g : x \twoheadrightarrow u))$$

Hence, the quantifiers are not dual.

This is a good point to stop and reflect on the overall argument that the thesis and this chapter in particular is concerned with. We have just seen an argument saying that intuitionistic logic is indeed a *consequence* of the indeterminacy in question insofar as

a) it is modelled via the forcing multiverse $\mathbb{H}$ and its corresponding semantics $\mathcal{K}_{\mathbb{H}}$, where

b) each universe is understood as a separate instantiation of the set concept.

This is a non-trivial result, and the first one where pure indeterminacy actually *entails* the rejection of LEM! Let us now develop some immediate consequences of this approach.

### SCS and the intuitionistic multiverse

The next question is, which (modalised) set theoretic principles hold determinately in $\mathcal{K}_{\mathbb{H}}$—and thus with respect to the selected notion of indeterminacy. Feferman has argued on multiple occasions that a theory called SCS is the right environment to express the difference between determinacy and indeterminacy.

**Definition 7.7.** SCS is the theory consisting of the following axioms.

The axioms of intuitioinistic first order logic

$\Delta_0$-LEM, i.e. $\neg\varphi \vee \varphi$ for all $\Delta_0$-formulas $\varphi$

$\Delta_0$-Markov : $\neg\neg\exists x\varphi(x) \rightarrow \exists x\varphi(x)$ for all $\Delta_0$-formulas $\varphi$

Extensionality: $\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b$

Pairing: $\forall x\forall y\exists a(x \in a \wedge y \in a)$

Union: $\forall x\exists y\forall z\forall a[a \in z \wedge z \in x \rightarrow a \in y]$

Infinity: $\exists x[\emptyset \in x \wedge \forall y(y \in x \rightarrow y^+ \in x)]$

$\Delta_0$-Separation: $\forall z \exists y \forall x[x \in y \leftrightarrow ((x \in z) \wedge \varphi(x))]$ for all $\Delta_0$-formulas $\varphi$

$\in$ -induction : $\forall x[(\forall y \in x\, \varphi(y)) \rightarrow \varphi(x)] \rightarrow \forall z \varphi(z)$

$\mathsf{AC}_{Set} : \forall x \in a \exists y \varphi(x, y) \rightarrow \exists f \forall x \in a[Fun(f) \wedge dom(f) = a \wedge \varphi(x, f(x))]$

As it turns out, $\mathcal{K}_{\mathbb{H}}$ licenses modalised versions of all axioms of $\mathsf{SCS}$ except $\Delta_0$-Markov. I will focus on the positive aspects first and afterwards discuss the failure of Markov's Principle.

**Theorem 7.2.** $\mathcal{K}_{\mathbb{H}} \vDash (\mathsf{SCS} - \Delta_0\text{-Markov})^{\Box}$

*Proof.* That intuitionistic first order logic holds, follows from the fact that the model validates all instances of the Gödel translation. This can be shown by induction. For instance, consider the intuitionistic axiom $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$. Its translation is $\Box(\Box\neg\varphi^{\Box} \rightarrow \Box(\varphi^{\Box} \rightarrow \psi^{\Box})$. Now assume that $M \vDash \Box\neg\varphi^{\Box}$. Then $N \nvDash \varphi^{\Box}$ for all $N \geq M$ and therefore also $N \vDash (\varphi^{\Box} \rightarrow \psi^{\Box})$. Similarly for quantified statements, consider $\varphi^{\Box}(t) \rightarrow \exists x \varphi^{\Box}(x)$ and its modal translation $\Box(\Box\varphi^{\Box}(t) \rightarrow \Box\exists x \Box\varphi^{\Box}(x))$. If we assume $M \vDash_{\sigma[t]} \Box\varphi^{\Box}(t)$ for some $M$ and assignment $\sigma$ of $t$, then $M \vDash_{\sigma[t]} \exists x \Box\varphi^{\Box}(t)$ and by stability $M \vDash_{\sigma[t]} \Box\exists x \Box\varphi^{\Box}(x)$.

We now check for each axiom if its Gödel-translation holds in $\mathcal{K}_{\mathbb{H}}$.

$\boldsymbol{\Delta_0}$$-\mathsf{LEM}$: We show that for any $M \in \mathbb{H}$, $M \vDash \varphi^{\Box} \vee \Box\neg\varphi^{\Box}$ for a $\Delta_0$-formula $\varphi$ by induction. Assume that $M \nvDash \varphi^{\Box}$. The interesting two cases are, where $\varphi^{\Box}$ is of the form $\Box\forall x \in a \psi^{\Box}(x)$ or of the form $\Box\exists x \in a \psi^{\Box}(x)$, for some fixed assignment $\sigma[a]$ of $a$.

If $M \nvDash_{\sigma[a]} \Box\forall x \in a \psi^{\Box}(x)$, then $M \vDash_{\sigma[a]} \Diamond\exists x \in a \wedge \neg\psi^{\Box}(x)$. But then by the Induction Hypothesis and the fact that the assignment of $a$ already exists in $M$, we have that $M \vDash_{\sigma[a]} \exists x \in a \wedge \neg\psi^{\Box}(x)$, and by stability $M \vDash_{\sigma[a]} \Box\exists x \in a \wedge \neg\psi^{\Box}(x)$, which transforms into $M \vDash_{\sigma[a]} \Box\neg\forall x \in a \wedge \psi^{\Box}(x)$.

Similarly, if $M \nvDash_{\sigma[a]} \Box\exists x \in a \psi^{\Box}(x)$, then $M \vDash_{\sigma[a]} \Diamond\forall x \in a \wedge \neg\psi^{\Box}(x)$, but then by the existence of $\sigma(a)$ in $M$ and by the induction hypothesis $M \vDash_{\sigma[a]} \forall x \in a \wedge \neg\psi^{\Box}(x)$, and by stability $M \vDash_{\sigma[a]} \Box\forall x \in a \wedge \neg\psi^{\Box}(x)$.

**Extensionality**: Fix an arbitrary assignment $\sigma[a,b]$ for $a$ and $b$. Then

$$M \vDash_{\sigma[a,b]} \Box\forall x\Box[(\Box x \in a \leftrightarrow \Box x \in b) \rightarrow \Box a = b]$$

is the case, if for any $N \geq M$, we have that

$$N \vDash_{\sigma[a,b]} \forall x\Box[(\Box x \in a \leftrightarrow \Box x \in b) \rightarrow \Box a = b].$$

For any $x$-variant assignment $\sigma^*[a,b]$ of $\sigma[a,b]$, it is the case that

$$M \vDash_{\sigma^*[a,b]} \Box[(\Box x \in a \leftrightarrow \Box x \in b) \rightarrow \Box a = b].$$

By the stability of $\in$ (i.e. of atomic sentences), the same holds in $N$.

The proofs for Pairing and Union follow the same idea:

**Pairing**: $\Box\forall x\Box\forall y\Box\exists a\Box(x \in a \wedge y \in a)$. Consider an arbitrary element $x$ in world $M$ then go to an arbitrary world $M[G]$ and consider an arbitrary element $y$. Since $x$ and $y$ both exists at $M[G]$ and the axiom of Pariring holds at it, the required $a$ exists as well.

**Union**: $\Box\forall x\Box\exists y\Box\forall z\Box\forall a\Box[a \in z \wedge z \in x \Rightarrow a \in y]$. Given an arbitrary world $M$, if $x$ exists at that world, then so do its elements, and so does their union.

**Infinity**: $\Box\exists x[\emptyset \in x \wedge \Box\forall y\Box(y \in x \rightarrow y^+ \in x)]$. This follows from the fact that there is such an $\omega$ in each universe, and furthermore, that this $\omega$ is identical in all of those.

**$\Delta_0$-Separation**: $\Box\forall z\Box\exists y\Box\forall x\Box[x \in y \leftrightarrow ((x \in z) \wedge \varphi^\Box(x))]$. Fix an arbitrary world $M$, an arbitrary $N \geq M$, and an arbitrary assignment $\sigma[z]$ for $z$ at $N$. It needs to hold that:

$$N \vDash_{\sigma[z]} \Box\exists y\Box\forall x\Box[x \in y \leftrightarrow ((x \in z) \wedge \varphi^\Box(x))]$$

Since the value of $z$ is fixed, we know that all its elements exist in $N$. Now, show that there exists at $N$ set $\sigma(y)$ which is separable out of $\sigma(z)$ by the condition $\varphi^\Box(x)$. This is done by induction on the complexity of $\varphi^\Box$. There are two cases of interest.

First, $\varphi^\Box(x) = \forall u \in v\psi^\Box(u,x)$. Since the interpretation of $v$ needs to exists at $N$, by the induction hypothesis we know that for each $u$ in $v$, the extension of $\psi^{\Box}(u,x)$ exists at $N$ as well. By Separation in $N$, it follows that the extension of $\forall u \in v\psi^{\Box}(u,x)$ does as well. Hence, again by Separation in $N$, we know that there is a set corresponding to $\varphi^{\Box}(x)$.

Second, $\varphi^{\Box}(x) = \exists u \in v\psi^{\Box}(u,x)$. Since the interpretation of $v$ needs to exists at $N$, by the induction hypothesis, there must be an assignment $\sigma(u)$ such that the extension of $\psi^{\Box}(u,x)$ exists at $N$. Then, by separation in $N$ we know that the extension of $\varphi^{\Box}(x)$ exists as well.

Note that $\Delta_0$-Separation essentially works, because $\Delta_0$-formulas are absolute for the class of models we are considering (see p.58).

**$\in$-induction**: $\Box[\Box\forall x\Box((\Box\forall y \in x\, \varphi^{\Box}(y)) \to \varphi^{\Box}(x)) \to \Box\forall z\varphi^{\Box}(z)]$ for any formula $\varphi$.

Show the contrapositive. Assume that $M \nvDash \Box\neg\forall z\varphi^{\Box}(z)$ and show that the antecedent cannot hold at $M$ either.

From our assumption it follows that there is an $N \geq M$ such that $N \vDash \exists z\neg\varphi^{\Box}(z)$. Fix a $z$-variant assignment $\sigma[z]$ such that $N \vDash_{\sigma[z]} \neg\varphi^{\Box}(z)$.

For all $y$-variant assignments $\sigma[z]^*$ of $\sigma[z]$ such that $N \vDash_{\sigma[z]^*} y \in z$, it is either the case that $N \vDash_{\sigma[z]^*} \varphi^{\Box}(y)$ or there is one such $\sigma[z]^* = \sigma[z,y]$ such that $N \vDash_{\sigma[z,y]} \neg\varphi^{\Box}(y)$. In the first case, we have a counterexample for the antecedent. In the second case, apply the same reasoning to all $x$-variant assignments $\sigma[z,y]^*$ with $N \vDash_{\sigma[z,y]^*} x \in y$. Due to $N$ being well founded, this process, if it does not terminate earlier, will eventually arrive at an assignment $\sigma^{\dagger}$ for some variable $v$ that denotes the empty set and that has $N \nvDash_{\sigma^{\dagger}} \varphi^{\Box}(v)$. But in this case we know that $N \vDash_{\sigma^{\dagger}} \forall u \in v\varphi^{\Box}(v)$ on account of $\sigma^{\dagger}(v) = \emptyset$.

From this it follows that $M \nvDash \Box\forall x\Box((\Box\forall y \in x\, \varphi^{\Box}(y)) \to \varphi^{\Box}(x))$.

**$\mathbf{AC_{Set}}$**: $\Box\forall x \in a\Box[\Box\exists y\varphi^{\Box}(x,y) \to \Box\exists f\Box\forall x \in a\Box[Fun(f) \wedge dom(f) = a \wedge \varphi^{\Box}(x, f(x))]]$. Fix an arbitrary assignment $\sigma[a]$ for $a$ on an arbitrary $M$ such that $M \vDash_{\sigma[a]} \Box\exists y\varphi^{\Box}(x,y)$ for any assignment $\sigma[a,x]$ such that $M \vDash_{\sigma[a,x]} x \in a$. By AC holding in $M$ it is the case that

$$M \vDash_{\sigma[a]} \exists f\forall x \in a[Fun(f) \wedge dom(f) = a \wedge \varphi^{\Box}(x, f(x))]].$$

And since all these conditions are in fact $\Delta_0$ and by stability, it is the case that

$$M \vDash_{\sigma[a]} \Box\exists f \Box\forall x \in a\Box[Fun(f) \wedge dom(f) = a \wedge \varphi^{\Box}(x, f(x))]].$$

$\Box$

Furthermore, we can also apply the notion of a totality state here. We already know that each set is determinate, and hence provides a totality state for quantification over it. This directly implies

**Corollary 7.1.** *The extension of each $\Delta_0$ formula determines a* **totality state** *in $\mathcal{K}_{\mathbb{H}}$.*

**Corollary 7.2.** *$\mathcal{K}_{\mathbb{H}}$ satisfies $\mathsf{BOM}^{\Box}$:*

$$\mathcal{K}_{\mathbb{H}} \vDash \Box\forall x(\psi^{\Box}(x) \vee \Box\neg\psi^{\Box}(x)) \rightarrow [\Box\forall y \Box(\varphi^{\Box}(y) \rightarrow \psi^{\Box}(y)) \vee \Box\exists y(\varphi^{\Box}(y) \wedge \Box\neg\psi^{\Box}(y))$$

*with with respect to $\Delta_0$-formulas $\varphi$.*

Do we get anything more? Scambler (2020) has argued that a system called $\mathsf{SCS}^{+}$ which is $\mathsf{SCS} + \Delta_1$-$\mathsf{LEM} + \Delta_1$-Markov is the axiom system describing an indeterminate universe of sets. In arguing for this, Scambler relies on an additional principle, which he calls the standard transitive model hypothesis (STMH).

> STMH: A set-theoretic statement is of determinate sense only if it is absolute between standard transitive models of ZFC (cf. Scambler 2020, p. 560).

For transitive models it is the case that $\Delta_1$-formulas become absolute as well (cf. Jech 2002, p. 185), which should allow us to claim the modalised versions of $\Delta_1$-LEM and $\Delta_1$-Separation$^{\Box}$ and the corresponding $\Delta_1$ instances of $\mathsf{BOM}^{\Box}$.

To this end, let $\mathcal{K}_{\mathbb{C}}$ be the **Kripke model corresponding to** $\mathbb{C} = \{M[G] \mid M \vDash \mathsf{ZFC}$, $M$ is countable and transitive and $G$ is $M$-generic.$\}$.

**Corollary 7.3.** $\mathcal{K}_{\mathbb{C}} \vDash \Delta_1$-$\mathsf{LEM}^{\Box} + \Delta_1$-Separation$^{\Box} + \Delta_1$-$\mathsf{BOM}^{\Box}$

But there is an upper bound to which form of comprehension/separation is acceptable.

**Lemma 7.6.** $\mathcal{K}_{\mathbb{C}} \nvDash \Pi_1\text{-Separation}^{\Box}$

$\mathcal{K}_{\mathbb{C}} \nvDash \Sigma_1\text{-Separation}^{\Box}$

*Proof.* An example for a $\Pi_1$ statement that doesn't hold in $\mathcal{K}_{\mathbb{C}}$ is $x = \mathscr{P}(\omega)$, i.e. $\forall y(y \subset a \to y \in x)$. Similarly, the $\Sigma_1$ formula $\exists f : \omega \twoheadrightarrow x$ is not absolute for the class of models that we are considering. $\Box$

### 7.3.4. Differences to traditional intuitionism

Finally, there are some interesting differences to the common path of the intuitionistic argument: There are failures of Markov's principle, the validity of the axiom of choice, and the status of the double negation translation.

#### Markov's principle

That the modalised version of $\Delta_0$-Markov holds, means that for any world $M \in \mathcal{K}_{\mathbb{C}}$ and $\Delta_0$-formula $\varphi$:

$$M \vDash \Box\neg\Box\neg\Box\exists x\varphi^{\Box}(x) \to \Box\exists x\varphi^{\Box}(x),$$

which transforms into

$$M \vDash \Box\Diamond\Box\exists x\varphi^{\Box}(x) \to \Box\exists x\varphi^{\Box}(x).$$

To this, however, we find a counterexample:

**Lemma 7.7.** $\mathcal{K}_{\mathbb{C}} \nvDash \Delta_0\text{-Markov}^{\Box}$

*Proof.* Let $\varphi(f, x, y)$ express the fact that $f$ is a surjection from $x$ into $y$. This is a $\Delta_0$-formula. Now, fix an $M \in \mathbb{C}$ with $M \vDash \neg\mathsf{CH}$ and an assignment $\sigma$ for the variable $u$ such that $M \vDash_\sigma u = \mathscr{P}(\omega)$. Furthermore, let $\sigma(v)$ be such that

$$M \vDash_\sigma v \subset u \wedge v \neq \emptyset \wedge \neg\exists f\varphi(f, \omega, v) \wedge \neg\exists g\varphi(g, v, u)$$

However, this is not to say that we can introduce some further function, such that there will be such a surjection from $\sigma(v)$ into $\sigma(u)$. In other words, there is always going to be an $M[G]$ such that

$$M[G] \vDash_\sigma \exists g \varphi(g, v, u).$$

Since $\varphi(f, v, u)$ is a $\Delta_0$-formula, this implies that $M[G] \vDash_\sigma \Box \exists g \varphi^\Box(g, v, u)$, and, since $M$ can always be extended in this way:

$$M \vDash_\sigma \Box \Diamond \Box \exists g \varphi^\Box(g, v, u).$$

But it is also the case that $M \nvDash_\sigma \Box \exists g \varphi^\Box(g, v, u)$, and thus

$$M \nvDash_\sigma \Box \Diamond \Box \exists g \varphi^\Box(g, v, u) \to \Box \exists g \varphi^\Box(g, v, u))$$

$\Box$

This is somewhat surprising, since $\Delta_0$-Markov enjoys some prima facie plausibility with respect to traditional intuitionism. This is entirely appropriate when it is viewed under a picture that is different from the one that we are considering. In his discussion of the axioms of SCS, Michael Rathjen, for instance, justifies it in the following way:

> So assume that $\neg\neg\exists x \varphi(x)$ is true where $\varphi(x)$ is $\Delta_0$. If $\neg\neg\exists x \varphi(x)$ is true its truth must be based on facts that only refer to a definite part $B$ of $V$ combined with generic knowledge about the collection of all sets. We know that the assumption of the non-existence of a set $x$ with $\varphi(x)$ leads to a contradiction. But the only fact about $V$ that could yield a contradiction from $\neg\exists x \varphi(x)$ would be a set $u$ such that $\varphi(u)$ holds. If we can systematically search through $V$ as more and more parts become actual, such a search, set by set, will eventually be successful since the predicate $\varphi(u)$ is checkable (on account of being determinately true or false for every set $x$) and we have a guarantee stemming from the truth of $\neg\exists x \varphi(x)$ that we will eventually hit upon a set $b$ such that $\varphi(b)$ holds. (Rathjen 2019, p. 509)

This justification and its reference to $V$ pertains to the idea that we are concerned with the height of the hierarchy, or at the very least that forcing is understood primarily as a way to add new sets, instead, as suggested in this section, as a way to produce a different instantiation of the set concept that is, however, dependent on the previous one. It seems that this idea conflicts with the set up that has motivated the switch to the Gödel-translation and that has only made the rejection of LEM possible. There is no real search for objects with certain properties upwards or 'horizontally' if each world changes the role that those objects play. In this respect, the argument from indeterminacy that was put forward in this section differs to a very interesting degree from traditional intutionism. It allows for certain intuitionistically non-valid principles, but it also rejects some commonly intuitionistically acceptable principles.

One final remark on this. Usually, $\Delta_0$-Markov's principle is needed to prove the well-foundedness of the $\in$-relation. This, however, is not needed here, because the well-foundedness follows directly from the well-foundedness of all the universes in the multiverse:

**Lemma 7.8.** $\mathcal{K}_{\mathbb{H}} \vDash \text{Foundation}^{\Box}$

*Proof.* We need to show that $\mathcal{K}_{\mathbb{H}} \vDash \Box\forall x\Box(x \neq \emptyset \to (\Box\exists y\Box(y \in x) \wedge \Box(y \cap x = \emptyset))$. Pick an arbitrary world $M$ and assignment $\sigma(x)$, such that $M \vDash_\sigma x \neq \emptyset$. Because Foundation holds in $M$, we have that

$$M \vDash_\sigma \exists y \in x \wedge y \cap x = \emptyset$$

and because all these are $\Delta_0$ formulas, it follows, that

$$M \vDash_\sigma \Box\exists y\Box(y \in x) \wedge \Box(y \cap x = \emptyset).$$

$\Box$

However, Foundation, similarly to the Axiom of Choice, is know to entail LEM over certain intuitionistic set theories (cf. Bell 2009). That this doesn't happen here will be discussed now.

### The Axiom of Choice

It is possible to show that the modalisation of 'regular' AC holds over $\mathcal{K}_{\mathbb{H}}$ as well.

**Lemma 7.9.** $\mathcal{K}_{\mathbb{H}} \vDash \mathsf{AC}^{\Box}$.

*Proof.* Fully written out, $\mathsf{AC}^{\Box}$ is the statement

$$\Box\forall x\Box[\Box\emptyset \notin x \to \Box\exists f(Fun^{\Box}(f) \wedge \Box dom(f) = \bigcup x \wedge \Box\forall y \in x : \Box f(y) \in y)].$$

Assume that for an arbitrary world $M$ and an arbitrary $x$-variant assignment $\sigma^*$ that $M \vDash_{\sigma^*} \Box\emptyset \in x$. Then in $M$ there is a choice function $f$ for $\sigma^*(x)$, hence

$$M \vDash_{\sigma^*} \exists f[Fun(f) \wedge dom(f) = \bigcup x \wedge \forall y \in x(f(y) \in y)].$$

By the fact that the universal quantifier is bounded, all other subformulas are $\Delta_0$ and by stability, the modalised version of this claim follows as well. □

Now, why do these two results not entail the validity of $\mathsf{LEM}^{\Box}$? As it turns out, the theory that we are concerned with is too weak to derive $\mathsf{LEM}^{\Box}$ from $\mathsf{AC}^{\Box}$ or from Foundation$^{\Box}$. In the following I only discuss the case for $\mathsf{AC}^{\Box}$, as the case for Foundation$^{\Box}$ is analogous.

The derivation of $\mathsf{LEM}^{\Box}$ from its modalised version $\mathsf{AC}^{\Box}$—just as the derivation of $\mathsf{LEM}$ from $\mathsf{AC}$—relies on a number of principles. As Bell (2009) (cf. Bell 2008) points out, it requires at least the *Binary Quotient Principle* or the *Comprehension Principle* in addition to the *Principle of Extensionality* for functions. Since we have already shown that extensionality has to hold, we will have a look at the other two principles.

Consider again the formula $\mathsf{SU}(u) := \forall x \subset u(x \neq \emptyset \to (\exists f : \omega \twoheadrightarrow x \vee \exists g : x \twoheadrightarrow u))$. The derivation of $\mathsf{SU}^{\Box}(u) \vee \Box\neg\mathsf{SU}^{\Box}(u)$ from $\mathsf{AC}^{\Box}$ using the **Comprehension Principle** would then require the definition of the sets

$$t = \{x \in 2 : x = 0 \vee \mathsf{SU}^{\Box}(u)\}$$
$$v = \{x \in 2 : x = 1 \vee \mathsf{SU}^{\Box}(u)\}$$

on whose doubleton one would assume the existence of a choice function. However, since $\mathsf{SU}(u)$ is a $\Pi_1$ formula, these sets simply don't exist, and hence the derivation of $\mathsf{LEM}^{\Box}$ cannot proceed. This generalises to all those formulas for which $\mathsf{LEM}^{\Box}$ does not hold in

$\mathcal{K}_{\mathbb{H}}$.

Similarly, the **Binary Quotient Principle** says that every formula $\varphi(x, y)$ that defines an equivalence relation on the set $2 = \{0, 1\}$ gives rise to the quotient set $2/\varphi$ such that there are at most $0_\varphi, 1_\varphi \in 2/\varphi$ and $0_\varphi = 1_\varphi$ iff $\varphi(0, 1)$. Then one would define the equivalence relation $\varphi(x, y) \equiv (x = y \vee \mathsf{SU}^\Box(u))$ and derive $\mathsf{LEM}$ from it. However, by the same reasoning, $\varphi$ does not actually define a set over the multiverse.

Should this be surprising? On second thought, I think not. For it fits very well with the general path from indeterminacy to intuitionistic logic. To reiterate, it is not about us being able to find/describe the choice functions in question. The only question is whether they are determinately 'there'.

### The double negation translation

Finally, what about the argument using the double negation translation again? In fact, the argument given against the use of the potentialist translation doubles down as an argument against the double negation translation as well. This can be seen by looking at the clause for the existential quantifier again. If we apply the double negation translation to a formula $\exists x \varphi(x)$ and afterwards the Gödel translation, we get the clauses

$$((\exists x \varphi(x))^{\mathsf{DN}})^\Box \equiv \Box\neg\Box\neg\Box\exists x(\varphi^{\mathsf{DN}})^\Box(x) \equiv \Box\Diamond\Box\exists x(\varphi^{\mathsf{DN}})^\Box(x)$$

Now, if $(\varphi^{\mathsf{DN}})^\Box$ expresses the condition that $x$ is a surjection from $\omega$ to a set $y$, then this is compatible with the existence of an $M$ and an $N$ such that

$$M \nvDash \exists x(\varphi^{\mathsf{DN}})^\Box(x)$$

and

$$N \vDash \exists x(\varphi^{\mathsf{DN}})^\Box(x)$$

and hence with $M \vDash \Box\Diamond\Box\exists x(\varphi^{\mathsf{DN}})^\Box(x)$. However, if $\varphi$ is taken to express countability again, this does not adequately reflect that $y$ is determinately countable (at least not from our perspective). If we take each $M$ and $N$ to be legitimate instantiations of the set concept, then the familiar potentialist argument from chapter 6 should not apply here. Thus the fact that we cannot use the double negation translation here differs from the fact that we cannot use it for the case of indeterminacy of height. In that case, the possibility of continuing without ever getting a witness (but always retaining the possibility for getting one) was left open. Here the reason is that its application is not sensitive to the relevant context that is taken to essentially represent the indeterminacy in question. In both cases, though, using of the double negation translation at this point of the argument bears some incoherence, for it comes down to accepting the indeterminacy and subsequently ignoring it.

## 7.4. Conclusion

This chapter was an investigation into possible consequences that follow from a certain understanding of the indeterminacy of width. I have discussed three possible ways of understanding this indeterminacy and subsequently extracting a semantics and logic from it. It was argued that the approach by Rumfitt, which was a combination of axiomatic extensions and model theory, does not achieve its aim. Then the multiverse by Väänänen was presented and it was shown that it fits with the analysis of generic and instance based generality that was given in chapter 5. However, it turned out that this approach licensed classical logic. This provided evidence that all the while intuitionistic logic is always available on generic generalisations, it is not the case that intuitionistic logic is also entailed by generic generality.

Finally, I have discussed a possible argument for intuitionistic logic that is based on applying the Gödel translation to the forcing multiverse. Different ways of restricting the forcing multiverse were shown to license either $(\mathsf{SCS} - \Delta_0\text{-Markov})^\Box$ or $(\mathsf{SCS}^+ - \Delta_0\&\Delta_1\text{-Markov})^\Box$. This provided support to the claims made by Scambler and Feferman according to which the indeterminacy caused by the independence of CH actually leads to a rejection of classical logic. In this respect, also the attempt to reintroduce classical reasoning by using the double negation translation was rejected. Thus, in the end, there *is* a route from indeterminacy of width to a rejection of LEM.

Further details of these views pending, do we have an answer to *all* the open questions raised in earlier chapters? Not quite yet. In chapter 5 we have distinguished between two types of negation in the face of generic generality: A claim can be said to be negated if its assumption leads to absurdity, and a claim can be said to be negated if it just doesn't follow from the concepts involved. Chapters 6 and 7 focused on the first version of negation, which was investigated via necessitated negation claims over indeterminate domains. In the final chapter of the thesis, I develop an approach to the other type of negation. This will lead to a notion of theoretical completeness of the respective concepts involved and may present another way to address the question of LEM.

# 8. LEM and Conceptual Completeness

The last two chapters focused on indeterminacy of domains. But as it was noted in the introduction, in chapter 1.4, as well as in the discussion of the two notions of negation in chapter 5.1.3, that one can also think of the natural numbers and the sets as having an *indefinite concept*. In this respect, chapter 1.4 already pointed out that the concept of natural number may be considered indefinite due to the incompleteness results and the ensuing existence of non-standard models for first order PA. This is taken to show that PA does not offer an adequate characterisation of the number concept, since it has models that don't validate everything we intuitively consider to be a truth of arithmetic. Similarly, the set concept may be considered indefinite due to the independence of CH from ZFC plus any known large cardinal axioms. In this chapter, I ask whether any of these situations provides ground for a rejection of LEM.

In order to do so, I discuss and formalise an argument initially sketched in Dummett (1978) which connects the incompleteness results to the notion of indefinite extensibility. This argument, although a closer defence of its premises is wanting, has somewhat more success than the argument from indefinite extensibility that was discussed in chapter 6. I will then extend the approach to the case of axiomatic extensions of ZFC in another attempt to argue for Feferman's claim that indeterminacy should lead to a rejection of LEM in set theory. This chapter provides a weaker claim in this regard, namely that under a certain conception of what a truth of set theory consists in, one may hold that set-theoretic truths are not bivalent and that LEM may or may not hold with respect to them. How this affects whether a set theorist should stop using LEM in practice, however, will turn out to be another question.

Section 1 extracts some of the ideas underlying Dummett's argument. Section 2 will give a formalisation of those and discuss some alterantives to Dummett's argument.

Section 3 then provides some steps towards an application to set theory.

## 8.1. Dummett's Initial Idea

Let $\theta_{\mathsf{PA}}$ be a Gödel sentence of PA, i.e.

$$\mathsf{PA} \vdash \theta_{\mathsf{PA}} \leftrightarrow \forall x \neg Prov(x, \ulcorner \theta_{\mathsf{PA}} \urcorner)$$

Commonly $\theta_{\mathsf{PA}}$ is considered to be true, since there is no numeral $\bar{n}$ such that $\mathsf{PA} \vdash Prov(\bar{n}, \ulcorner \theta_{\mathsf{PA}} \urcorner)$, which is exactly what $\theta_{\mathsf{PA}}$ seems to be expressing. By the orthodox conclusion PA thus does not seem to capture what we intuitively recognize as true statements of arithmetic (cf. Mostowski 1952, p. 107). Furthermore, adding this sentence to PA just leads to the construction of another Gödel sentence for the newly enriched theory $\mathsf{PA} + \theta_{\mathsf{PA}}$, and so on. In reference to this phenomenon, Dummett claims that insofar as a formal theory such as PA is taken to express our conception of natural number, this conception is bound to be "inherently vague"(Dummett 1978), and that, as a result of this, the validity of LEM with respect to quantified statements over the natural numbers should be called into question. The aim of this and the next section is to clarify these claims and assess their coherence. This section focuses on the first part of the argument and provides an answer to the question: When is a concept, according to Dummett, inherently vague, or, conversely, when is it definite? And how does the incompleteness phenomenon impact it?

In his preferred notion of definiteness, Dummett makes a distinction similar to the one in chapter 6. We distinguish the decidability of membership, which is granted (as it was in the case of the ordinals), from the determinacy of generalisations involving the concept. Whenever we have a canonical representation of a number, we know whether its a standard number or not, and we are also able to recognize each non-standard model as such. This is already more than we are able to do with respect to models of set theory. But Dummett nonetheless lays down a stronger criterion. We not only need to recognize each non-standard number/model as such, but we need to provide a general "criterion by which something is recognized as a ground for asserting that something is true of the natural numbers"(Dummett 1978, p. 194).[1]

The most natural candidate for such a ground is, of course, 'being provable by PA'. This, however, is now put in jeopardy by the incompleteness phenomenon:

> The only way to explain the meanings of the quantifiers whose variables range over the natural numbers is to state principles for recognising as true a statement which involves it; Gödel's discovery amounted to the demonstration that the class of these principles cannot be specified exactly once and for all, but must be acknowledged to be an indefinitely extensible class. (Dummett 1978, p. 199)

As mentioned above, due to incompleteness, PA does not seem to capture all that we take to be intuitively true about the natural numbers. In this respect, the quote suggests that a "principle for recognising"/a "ground for asserting" something true of the natural numbers can be identified more generally with a truth of arithmetic. Hence, the natural number concept may then be understood, according to Dummett, as characterised by the collection of all arithmetical truths.

But why does Dummett claim that they cannot be specified "once and for all"? Why can't the natural number concept simply be specified as the collection of all arithmetic truths, i.e. as the theory of the standard model of arithmetic? Dummett's answer, it seems, is that specifiability comes with certain demands that require recursive enumerability. PA and all its theorems give us a recursively enumerable collection of arithmetical truths. In chapter 1.4, recursively enumerable sets of arithmetic truths were shown to be indefinitely extensible with respect to being recursively enumerable, such that no recursive enumerable collection can comprise all arithmetic truths. This allows us to explain why Dummett says that they cannot be specified "once and for all"—and hence why our concept of natural number retains its indefiniteness throughout those possible sharpenings.

As already mentioned in chapter 1.4, the same can be expressed in terms of its extensions. This will later on allow us to make the connection to quantification more straightforward. Dummett puts this matter in the following way:

> If we tried to interpret 'natural number' as inherently vague with respect to its extension, we should have to regard it as the reverse of indefinitely extensible: any

[1] This is in analogy to chapter 6.1, where we required a general characterisation of an arbitrary object of the domain in addition to decidability of membership in order to secure its determinacy.

> particular definite characterisation allows the possibility of elements in the totality not attainable from 0 by reiteration of the successor operation; by successive further formal characterisations we may exclude some and more of these non-standard elements, without ever being able to exclude them all.(Dummett 1978, p. 197)

And similarly, Hartry Field suggests that:

> It would be natural to argue that if the concept of property of natural numbers is the concept of an indefinitely extensible totality, then the concept of natural numbers is the concept of an indefinitely contractible totality (since an increase in the supply of properties can lead to a decrease in the things closed under these properties).(Field 1994, p. 411)

What I take Dummett and Field here to say is roughly the following: Given the Gödel sentence $\theta_{\mathsf{PA}}$ of PA, there is a non-standard model $\mathcal{N}$ of PA in which $\neg\theta_{\mathsf{PA}}$ is satisfied. This means that in $\mathcal{N}$ there is a non-standard number $e$ which encodes a proof of $\theta_{\mathsf{PA}}$. Now, of course, when $\theta_{\mathsf{PA}}$ is added as an axiom to PA, $\mathcal{N}$ can no longer be a model of it, since no model of $\mathsf{PA} + \theta_{\mathsf{PA}}$ can contain a non-standard number encoding a proof of $\theta_{\mathsf{PA}}$.

This idea of specifying our concept of natural number by enriching our collection of arithmetical truths (and its dual of cutting down its models) via indefinitely extending PA provokes two questions: What are plausible candidates for such an extension? And how are they justified? Dummett himself only considers Gödel sentences, about which he claims: "There is indeed something that leads us to recognize the statement $\theta_{\mathsf{PA}}$ as true, and therefore goes beyond the characterisation of natural number which is embodied in the formal system."(Dummett 1978, p. 192) Thus, similar to our informal gloss at the beginning of the section, he takes it to be informally shown or proven that the Gödel sentence is true and hence it (and not its negation) can be legitimately added to PA. The reason that we are, according to Dummett, justified to assert the truth of the Gödel sentence is due to the fact that we recognize PA to be consistent by an informal inductive argument on the length of its proofs, which establishes that any proof leads from intuitively correct premises to an intuitively correct conclusion (Dummett 1978, pp. 194–5). Such a proof is not epistemically informative, because it essentially relies on the conviction that the chosen axioms and proof rules are adequate to begin with, and it just reflects our intuitive knowledge of the number concept.[2]

[2]A similar sentiment is expressed in Girard (1987, p. 64) and Shapiro (1998, p. 505).

It should be noted, though, that this is not the only way one can justify the addition of $\theta_{\mathsf{PA}}$ to the system. A less demanding approach has been discussed in recent literature surrounding the so called *implicit commitment thesis* (cf. Dean 2014). Implicit commitment asks whether one can on reflection of one's commitment to a theory (like PA) transfer justified belief or mere acceptance to its consistency statements on basic grounds of rationality (cf. Horsten 2021). This transfer of commitment or justification can also be applied to other types of sentences like local or even global reflection principles (cf. Łełyk et al. 2022). Dummett has something stronger in mind than transfer of epistemic status based on acceptance of the initial theory. He seems to think that we are convinced of the truth of the Gödel sentence on intuitive grounds, which is stronger than merely being rationally justified to adopt it. From Dummett's perspective, the consistency of PA and the truth of the Gödel sentence are not mere artefacts of our thinking that PA expresses our concept of natural number; and grounds for accepting PA not merely lead to grounds for believing the truth of the Gödel sentence. The two are in fact identical: It is our intuitive concept that provides ground for both, thinking that PA adequately reflects our concept of natural number and that it is consistent.

Dummett's concerns are thus not primarily about justifying truths of arithmetic, but, as the following quote shows, rather with *explicating* them:

> If we think about it in terms of extension, then it is not the concept that is inherently vague, but the means available to us for precise expressions are intrinsically defective; the concept itself is perfectly definite, but our language prohibits us from giving complete expression to it; the concept guides us, however, in approaching ever more closely to this unattainable ideal. (Dummett 1978, p. 197)

What Dummett means here with 'extensional definiteness' is the decidability of membership and our intuitive ability to recognize non-standard numbers and non-standard models as such.[3] Our implicit grasp of PA as reflecting or expressing the natural number concept is thereby also an implicit grasp of its consistency and of the truth of its Gödel sentences. The issue is rather with the fact that we cannot explicate the concept that

[3]The use of 'extensional definiteness' for this is confusing when compared with its use in Dummett's later works, where extensional definiteness or determiancy not only required decidability of membership but a general conception of the domain in question (cf. the discussion in chapter 6.1).

we do have in a sufficiently complete manner[4]; any explication—insofar as it is bound to recursive enumerability— will never attain completeness. This suggests that Dummett's point is best represented by a distinction between *implicit* and *explicit knowledge* of the concept of natural number. Dummett seems to say that we are simply not able to state explicitly what is contained implicitly in the number concept.

Of course, any of these points can be contested. Should we think about the extensions of PA in terms of explicating implicit knowledge in the first place or rather in terms of implicit commitment? If the former, should our lack of implicit specification really provide ground to doubt the indeterminacy of quantification? And if so, should recursive enumerability be required for specifiability? In the following, I will leave these points uncontested. The only requirement that I have is that they are coherent (which at certain points will be tested). The main point of interest in this chapter is the conclusion that Dummett aims to draw from all this, namely that LEM should be rejected for quantification over the natural numbers.

Dummett draws this conclusion in the following way:

> Intuitionists hold that the classical explanations of the logical constants, and of the quantifiers in particular, in terms of the truth-conditions for statements in which they occur, are faulty because circular; and that the right method for explaining them, which avoids vicious circularity, is not to lay down, for each constant, conditions under which a statement for which that constant is the main operator is true, but rather to lay down criteria for recognising something to be a proof of such a statement. As far as quantifiers whose variables range over an infinite totality are concerned, this coincides exactly with what I have here asserted. (Dummett 1978, p. 199)

Taking the perspective of indefinite contractability, what Dummett seems to say here is that quantification over the natural numbers should be understood intuitionistically, because we cannot adequately state truth-conditional satisfaction clauses for quantifiers, and we can't do that because we cannot sufficiently characterise their domain by PA and its recursively enumerable extensions. Again, however, the transition from this failure of sufficient characterisation to a rejection of LEM is, as Field (2003) rightly notes, not obvious. In the next section, we will see how it might come about.

[4]In this respect, there is a difference to the ordinals for which it was argued in chapters 1.3 and 6.3 that our concept just *is* their incompleteability.

## 8.2. Potentialism for Theories of Arithmetic

This section gives a formalisation of Dummett's argument, focusing on the extensional perspective. I will use modalities, like before, to think about these questions in a similar manner as in chapter 6. One reason for using modalities is to show that one can apply potentialist ideas to the area of theory extensions as well. Another reason is that the modalities will give us a way to think of the extensional perspective as it is developed in the quotes by Dummett and Field, and it will give us a way to formalise conceptual containment (and lack thereof) by 'pulling down' the meta-logical perspective developed in chapter 5 into the object theory.

Contrary to previous cases of indefinite extensibility, we are not dealing with collections of elements that are stepwise enriched, but with theories which gain new axioms, by adding Gödel sentences to them. To assess the way in which *that* affects the extension of the natural number concept, we focus not on individual models but rather on the whole corresponding class of models of a theory. Let $\mathbb{M} = \text{Mod}(\mathsf{PA}) = \{\mathcal{M} \mid \mathcal{M} \vDash \mathsf{PA}\}$ be the class of models of $\mathsf{PA}$. Then for each consistent extension $\mathsf{PA}^* = \mathsf{PA} + \theta_{\mathsf{PA}}$ or $\mathsf{PA}^* = \mathsf{PA} + Con(\mathsf{PA})$ of $\mathsf{PA}$ it is the case that $\mathbb{M}^* = \text{Mod}(\mathsf{PA}^*) \subseteq \mathbb{M}$. Correspondingly, there are more formulas true in all models in $\mathbb{M}^*$ than in all models in $\mathbb{M}$ (most notably $\theta_{\mathsf{PA}}$ or $Con(\mathsf{PA})$). In this respect the characterisation of the natural number concept is said to get sharper when progressing from $\mathsf{PA}$ to $\mathsf{PA}^*$. This process can of course be iteratied by extending further to $\mathsf{PA}^{*^*} = \mathsf{PA}^* + \theta_{\mathsf{PA}*}$ or $\mathsf{PA}^{*^*} = \mathsf{PA}^* + Con(\mathsf{PA}^*)$

This leads to two immediate questions:

(i) What is the precise frame of these extensions?

(ii) How are general claims over the natural numbers expressed in this enriched setting?

To better appreciate the consequences of a complete assessment of (i), it is best to start with (ii).

### 8.2.1. A model for indefinite extensibility of theories for the extensional perspective

Start with ordinary PA and the class $\mathbb{M}$ of all its models. Assume, for the time being, that PA *completely* expresses our concept of natural number. Then something that is true of all natural numbers holds for every model in the class $\mathbb{M}$, and something that is consistent with our notion of all numbers should hold in at least one model. Use $\Box$ to express that a sentence holds universally and $\Diamond$ to say that it holds in at least one model. Consequently, $\mathbb{M}$ can be treated as a system for the modal logic $S5$ where $\Box$ and $\Diamond$ express universal necessity and possibility, respectively.

But when we try to give this idea a complete formulation in first order modal logic (as it has been done in other potentialist settings), we quickly run into trouble. The problem lies with the identity conditions of objects across possible worlds when it comes to quantifying into modal contexts. For instance, how do we express the truth of $\forall x \Box x = x$ at a world $\mathcal{M}$? Formally, this statement is true at $\mathcal{M}$ if for all variable assignments $\sigma$, it is the case that $\mathcal{M} \vDash_\sigma \Box x = x$. This is unproblematic for the case in which $\sigma(x)$ denotes a standard number, since we can be sure that it is available in all models accessible from $\mathcal{M}$. But if $\sigma(x)$ denotes a non-standard number, then we wouldn't be sure that it exists in a world $\mathcal{N} \geq \mathcal{M}$ and hence that $\sigma(x)$ has a value in it. For this reason, variable assignments and quantification into modal contexts need to be treated with care. One possible response would be the use of a free-logic. However, for our purposes, there is a simpler approach available: The application of modal operators will be restricted to closed formulas only. Quantification into modal contexts is thus no longer permissible, from which it follows that the Converse Barcan formula, $\Box \forall x \varphi(x) \rightarrow \forall x \Box \varphi(x)$, which usually holds in potentialist systems, isn't just not satisfied, it cannot even be properly expressed.

#### The initial fragment

**Definition 8.1.** The triple $\mathfrak{M} = \langle \mathbb{M}, R, \vDash \rangle$ is a modal model in which $\mathbb{M}$ is a collection of models of PA, considered as worlds, $R$ is a relation on $\mathbb{M}$ such that for all $\mathcal{M}, \mathcal{N} \in \mathbb{M}$, $\mathcal{M} R \mathcal{N}$, and $\vDash$ is a mapping from elements of $\mathbb{M}$, sentences of $\mathcal{L}^{\Diamond}_{\mathsf{PA}}$ into truth values. In each $\mathcal{M} \in \mathbb{M}$, formulas of $\mathcal{L}_{\mathsf{PA}}$ are interpreted as usual:

| | |
|---|---|
| $\sigma(c) = 0$ | for the constant $c$ denoting 0. |
| $(f(x))^{\mathcal{M}} = f^{\mathcal{M}}(\sigma(x))$ | if $\sigma(x)$ is defined in $\mathcal{M}$. |

For formulas $t = t'$ and assignments $\sigma(t), \sigma(t') \in \mathcal{M}$:

| | | |
|---|---|---|
| $\mathcal{M} \vDash_\sigma t = t'$ | iff | $\sigma(t) = \sigma(t')$ |
| $\mathcal{M} \vDash_\sigma \varphi \wedge \psi$ | iff | $\mathcal{M} \vDash_\sigma \varphi$ and $\mathcal{M} \vDash_\sigma \psi$ |
| $\mathcal{M} \vDash_\sigma \varphi \vee \psi$ | iff | $\mathcal{M} \vDash_\sigma \varphi$ or $\mathcal{M} \vDash_\sigma \psi$ |
| $\mathcal{M} \vDash_\sigma \varphi \to \psi$ | iff | $\mathcal{M} \nvDash_\sigma \varphi$ or $\mathcal{M} \vDash_\sigma \psi$ |
| $\mathcal{M} \vDash_\sigma \forall x \varphi(x)$ | iff | $\mathcal{M} \vDash_{\sigma'} \varphi(x)$ for every $x$-variant assignment $\sigma'$ of $\sigma$ with values in $\mathcal{M}$ |
| $\mathcal{M} \vDash_\sigma \exists x \varphi(x)$ | iff | $\mathcal{M} \vDash_{\sigma'} \varphi(x)$ for some $x$-variant assignment $\sigma'$ of $\sigma$ with values in $\mathcal{M}$ |

For the modal clauses, let $\sigma^\star$ denote the trivial assignment.[5]

| | | |
|---|---|---|
| $\mathcal{M} \vDash_\sigma \Box\varphi$ | iff | $\mathcal{N} \vDash_{\sigma^\star} \varphi$ for every $\mathcal{N}$ such that $\mathcal{M}R\mathcal{N}$ |
| $\mathcal{M} \vDash_\sigma \Diamond\varphi$ | iff | $\mathcal{N} \vDash_{\sigma^\star} \varphi$ for some $\mathcal{N}$ such that $\mathcal{M}R\mathcal{N}$ |

Note that the clauses for the non-modal connectives are exactly those for models of PA. Insofar as a formula does not contain any modal operators, it can just as well be understood as a formula of PA. In the following, we will adopt the convention that all modal content of a formula will be made explicit and I will use Roman uppercase letters $A$ and $B$ for formulas in the language of PA, i.e. those with no modal content.

Now, under the assumption that PA in fact completely characterises the concept of natural number, we can express that a formula $A$ is a truth of arithmetic by saying that $\mathfrak{M} \vDash \Box A$, which is the case if $\mathcal{N} \vDash A$ for all $\mathcal{N} \in \mathbb{M}$, i.e. if $PA \vdash A$. But the modal vocabulary allows further reasoning from there. To express that either $A$ is a truth of arithmetic or that $B$ is a truth of arithmetic, we may say that $\mathcal{M} \vDash \Box A \vee \Box B$. But note that if $A \vee B$ is a truth of arithmetic, which can be expressed by $\mathcal{M} \vDash \Box(A \vee B)$, then

[5] The trivial assignment $\sigma^\star$ is defined to assign 0 to every variable, and hence denotes in every world.

this doesn't entail that either $A$ or $B$ is a truth of arithmetic. A similar difference holds regarding the conditional and negation. Note that

$$\mathcal{M} \vDash \Box(A \to B) \quad \Rightarrow \quad \mathcal{M} \vDash (\Box A \to \Box B) \tag{8.1}$$

by the Kripke Schema, but

$$\mathcal{M} \vDash \Box A \to \Box B \quad \not\Rightarrow \quad \mathcal{M} \vDash \Box(A \to B) \tag{8.2}$$

If negation is understood as implication to absurdity $\bot$, then $\mathcal{M} \vDash \Box\neg A$ expresses that $A$ is refuted, whereas $\mathcal{M} \vDash \neg\Box A$ says that there is a model $\mathcal{N}$ in which $\neg A$ holds, and thus that $A$ is not a consequence of PA.

In this way, the formalism distinguishes between the two types of negation that were introduced in chapter 5.1.3. $\mathcal{M} \vDash \neg\Box A$ expresses the fact that $A$ does not lie in the essence of the concept of natural number (insofar as it is taken to be entirely characterised by PA), which makes it an instance of meta-negation. This is equivalent to saying that $PA \nvdash A$, which need not necessarily entail that $PA \vdash \neg A$, i.e. $\mathcal{M} \vDash \Box\neg A$. I will call a situation in which $\mathsf{PA} \nvdash A$ but not $\mathsf{PA} \vdash \neg A$ an instance of *conceptual incompleteness*, where the fact that something doesn't lie in the concept does not entail that its negation does. The argument against LEM that I ascribe to Dummett (and to some extent also to Feferman for the case of set theory) will turn out to revolve around this notion: Instances of conceptual incompleteness do not validate LEM.[6] There is indeed a clear connection of this and what is to follow to the meta-logical perspective developed in chapter 5. In order to completely state Dummett's argument first, I defer discussion of this to appendix 8.A.

[6]Reference to completeness as a non-technical term has been made before, most notably by Daniel Isaacson. What is known as *Isaacson's thesis* is the claim that PA is in fact complete with respect to arithmetic truths and that any statements not provable in it are not purely arithmetical but contain higher order concepts (cf. Isaacson 1987, 157ff. see also Horsten 2001 and Incurvati 2008b for a related discussion with respect to set theory). What I mean (following Dummett's idea) with conceptual completeness, however, incorporates the possibility of adding additional axioms to PA based on our antecendent understanding of their truth (whatever concepts there may be involved in it). Barton (2022, p. 164) uses the notion of theoretical completeness in a way that is similar to mine. However, he distinguishes between object level and meta-theoretic sentences which I don't. (But, see the discussion at the end of section 8.2.2.)

### The extended model

Of course, the above considerations only serve to illustrate the formalism. Until now, we have only looked at one class of models corresponding to PA itself. But Dummett doesn't hold that PA exhaustively characterises the concept of natural number. Going beyond this simple case, the indefinite extensibility phenomenon that we are interested in was concerned with *a whole progression of theories* and their corresponding classes of models, and *this* is what may have some claim to be expressing the content of the natural number concept. So, in order to evaluate this structure for conceptual completeness, we need an account of how to model and evaluate general statements for it.

Given a progression of classes of models $\mathbb{M} \subset \mathbb{M}^* \subset \mathbb{M}^{*^*}$ corresponding to a sequence of theories $\mathsf{PA}$, $\mathsf{PA}^*$, $\mathsf{PA}^{*^*}$ which are extensions of each other, construct a modal model encompassing all of them in the following way. Since a model of $\mathsf{PA}^*$ is also a model of PA, we need to duplicate it in order to distinguish its use as a model of the one from its use as a model of the other. For this reason, we consider possible worlds to be pairs $(\mathcal{M}, \mathsf{PA}^*)$, where $\mathcal{M} \in \mathbb{M}^* = \mathrm{Mod}(\mathsf{PA}^*)$. In this sense the set of worlds can also be viewed as the disjoint union $\bigsqcup_i \mathbb{M}_i$ of worlds/models indexed by the theories under consideration. We can then define a new relation $\leq$ between all the elements of $\bigsqcup_i \mathbb{M}_i$. Call the collection of all $(\mathcal{M}, \mathsf{PA}^*)$ for some $\mathsf{PA}^*$ a **cluster**. Within a cluster we have the S5 accessibility relation as before, i.e.

$$(\mathcal{M}, \mathsf{PA}^*) \leq (\mathcal{N}, \mathsf{PA}^*) \qquad \text{for all } \mathcal{M}, \mathcal{N} \in \mathbb{M}^*,$$

and if $\mathsf{PA}^{*^*}$ is an axiomatic extension of $\mathsf{PA}^*$, set

$$(\mathcal{M}, \mathsf{PA}^*) \leq (\mathcal{N}, \mathsf{PA}^{*^*}) \qquad \text{for all } \mathcal{M} \in \mathbb{M}^* \text{ and all } \mathcal{N} \in \mathbb{M}^{*^*}$$

but *not* vice versa. This second clause (which actually subsumes the previous one) can be understood to reflect the accessibility of two clusters to each other. Hence, to each theory there corresponds a cluster of models, and the clusters themselves are ordered reflexively and transitively corresponding to the order of theory extensions (cf. figure 8.1). Call $\leq$ a **preorder of clusters**.

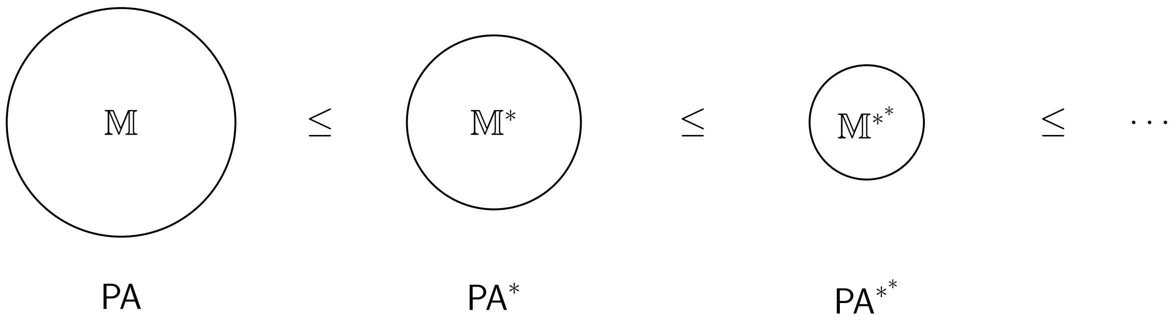



Figure 8.1.: A quasi-linear frame of clusters

**Definition 8.2.** Let $\mathsf{T}$ be a theory and $\{\mathsf{T}_i : i \in I\}$ a set of its extensions. A **potentialist system for theories extending** $\mathsf{T}$ is a triple $\mathfrak{M}_{\mathsf{T}} = \langle \bigsqcup_i \mathbb{M}_i, \leq, \vDash \rangle$, where $\bigsqcup_i \mathbb{M}_i$ is a collection of pairs of models and theories, $\leq$ is a pre-order of clusters, and $\vDash$ is as in definition 8.1.

Now, if we take our theory to be $\mathsf{PA}$, we can understand $\mathfrak{M}_{\mathsf{PA}}$ to express our conception of natural number as it is reflected by its axiomatic extensions and the corresponding classes of its models (pending an exact account of the iteration according to which we obtain the respective extensions of $\mathsf{PA}$ which will be discussed in section 8.2.2). The next question is then imminent: What in this setting is to be understood by a *truth of arithmetic*? If our concept of natural number is something that is indefinitely specifiable in the way that it is illustrated by $\mathfrak{M}_{\mathsf{PA}}$, a truth of arithmetic is best considered to be something that *will eventually be proven* along the progression of richer and richer theories, i.e. as something that is or eventually will become necessary. Then being a truth of arithmetic can be expressed for a formula $A$ by saying that,

$$A \text{ is a truth of arithmetic} \qquad \text{iff} \qquad \mathfrak{M}_{\mathsf{PA}} \vDash \Box\Diamond\Box A \tag{8.3}$$

It should be stressed again that this characterisation relies on the informal notion of provability that also Dummett alluded to. Hence we take it that insofar as a formula is independent of $\mathsf{PA}$, there are informal reasons according to which it expresses a property of the natural numbers.

With respect to this notion of arithmetical truth, the meaning of $\mathsf{LEM}$ needs to be reassessed. For this, we need to express what is meant by negation and implication with respect to arithmetic truths. Again, we can distinguish between object and meta-level negation:

$$\Box\Diamond\Box\neg A \qquad \text{meaning} \qquad \text{“A will eventually be refuted”}, \tag{8.4}$$

and $\Box\Diamond(\Box\Diamond\Box A \rightarrow \Box\Diamond\Box\perp)$ which transforms to $\Box\Diamond\neg\Box\Diamond\Box A$ and simplifies to

$$\Box\Diamond\neg A \qquad \text{meaning} \qquad \text{“A will never be proven”}. \tag{8.5}$$

The first one (which also entails the second one) obviously captures what the classical mathematician as well as the intuitionist mean with negation, namely that the assumption of $A$ leads to an absurdity. Thus, the adequate formulation of LEM with respect to our notion of arithmetic truth should be the schematic expression:

$$\Box\Diamond\Box A \vee \Box\Diamond\Box\neg A, \tag{8.6}$$

where $A$ stands for any formula in the language of PA. (8.6) is the claim that any formula will either eventually be proven or eventually be refuted, thus that any question will be settled.

Yet, the second understanding of negation, namely "$A$ will never be proven", is the one that is more readily accessible and more appropriately expresses the fact that $A$ doesn't lie in the concept of natural number (given the position developed in the section 8.1). It is natural to say that something which will never be proven cannot be a truth of arithmetic, but the crucial question is whether this also entails that its negation is a truth of arithmetic, i.e. whether (8.5) implies (8.4). The extent to which the former entails the latter can be understood as a measure of the *completeness* of the concept involved, and hence as the litmus-test for the assertibility of LEM with respect to arithmetic truths.

This is an important point for clarifying the type of argument from conceptual incompleteness that is analysed here. It characterises how LEM hinges on the question whether one can infer from the fact that something does not lie in the concept (represented by (8.5)) that its negation does (represented by (8.4)). For this reason, this holds all the same for universal as well as existential quantification. Usually, when it comes to criticism of LEM with respect to infinite domains, the issue is with the quantifiers. The traditional intuitionist argument targets the duality of the quantifiers, i.e. whether we can infer $\exists x \neg\varphi(x)$ from $\neg\forall x\varphi(x)$, and from there on the validity of LEM (cf. chapter 4.1.1). The argument developed in this chapter, by contrast, is directly concerned with the validity of $\forall x\varphi(x) \vee \neg\forall x\varphi(x)$. From this angle, quantified formulas don't play any different role than regular formulas. To obtain the duality of the quantifiers from (8.6), just note that from $\Box\Diamond\Box\neg\forall x\varphi(x)$ we can infer $\Box\Diamond\Box\exists x\neg\varphi(x)$.[7]

Now, to assess whether LEM holds in this setting, we need to get a better grip on the nature of its extensions and thereby address question (i).

### 8.2.2. Frame conditions and the validity of LEM with respect to arithmetic truths.

First note that since each world is classical, the model validates

$$\mathfrak{M}_{\mathsf{PA}} \vDash \Box\Diamond\Box A \vee \neg\Box\Diamond\Box A \tag{8.7}$$

which expresses that either $A$ is a truth of arithmetic or it is not. We take this to adequately represent the phenomenon we are interested in, as it corresponds to the position that there is no indeterminacy about indeterminacy which was developed in chapter 5.3.1.

Now, consider the case in which we extend PA finitely many times with consistency statements (the reason to use consistency statements instead of Gödel sentences will become clearer further below). Thus we define for all natural numbers $i$:

$$\mathsf{T}_0 = \mathsf{PA}$$
$$\mathsf{T}_{i+1} = \mathsf{T}_i + Con(\mathsf{T}_i)$$

This gives us a linear frame of clusters. Note that on such frames $\neg\Box\Diamond\Box A$, which

[7]However, one might be inclined to argue that the cases where (8.6) doesn't hold, effectively always involve quantification, such that the difference between this argument and the traditional intuitionistic ones might not be so large.

transforms to $\Diamond\Box\Diamond A$, implies $\Box\Diamond\Box\Diamond\neg A$. If $\mathfrak{M}_{\mathsf{PA}} \vDash \Diamond\Box\Diamond\neg A$ then this means that eventually it becomes necessary that we retain the possibility of $\neg A$ throughout the frame. But since the frame is directed, this means that 'eventually' entails 'always', hence $\mathfrak{M}_{\mathsf{PA}} \vDash \Box\Diamond\Box\Diamond\neg A$.

Now $\Box\Diamond\Box\Diamond\Box\neg A$ transforms into $\Box\Diamond\neg A$, which, however, generally does not imply $\Box\Diamond\Box\neg A$. But the only way the inference can fail is if there is still the possibility of $\neg A$ in all clusters. And that is the case if the statement $A$ hasn't been considered at all in the progression of extending PA. Yet, when we only add finitely many consistency statements, this is bound to be the case for some $A$. This is an interesting departure from potentialism with respect to domains as discussed in chapter 6. For this type of potentialism, we have the mirroring theorem into classical logic for linear frames. With respect to theory extensions, however, we can get failures of classicality even for linear frames, precisely when they are too short and thereby signal conceptual incompleteness. The reason for this is the fact that we are here using the weaker meta-negation rather than the object level negation as in chapter 6.

However, what happens if we consider iterating extensions beyond the finite, i.e, if we want to consider theories like

$$\mathsf{T}_\omega = \bigcup_{i<\omega} \mathsf{T}_i$$

$$\mathsf{T}_{\omega+1} = \mathsf{T}_\omega + Con(\mathsf{T}_\omega)$$

The question that we then need to address is: can *we express* a consistency statement for the whole union $\mathsf{T}_\omega$ of theories expressed in the language of arithmetic? In order to be able to do so by only using means available in the language of arithmetic as well, we need to find a way to index theory extensions beyond the finite by using natural numbers. Furthermore, since we aim to be staying within the restrictions provided by Dummett, all these expressions need to be at least recursively enumerable. Luckily, this problem has already been addressed in early proof theory, leading to some results that are quite interesting for our endeavour (cf. Franzén 2004, Rathjen and Sieg 2018). To express how far along such progressions of theory extensions we can go while still retaining our grip on recursive enumerability is dealt with in Kleene's ordinal notation $\mathcal{O}$. For this, note that a *computable* function on the ordinals is one that can be given by a Turing machine, which can be coded by a natural number. Use $\{e\}(n)$ to express that $n$ is input into the function given by the Turing machine coded by $e$.

**Definition 8.3.** Let $a$ and $e$ be natural numbers, and let $suc(a) = 2^a$ and $lim(e) = 2 \cdot 3^e$. Then the **class of ordinal notations** $\mathcal{O}$, the order $<_\mathcal{O}$ and $|a|$, the ordinal denoted by $a$, is defined by:

(i) $0 \in \mathcal{O}$ and $|0| = 0$

(ii) If $a \in \mathcal{O}$, then $suc(a) \in \mathcal{O}$, $a <_\mathcal{O} suc(a)$, and $|suc(a)| = a + 1$.

(iii) If $\{e\}(n) <_\mathcal{O} \{e\}(n+1)$ for all natural numbers $n$, then $lim(e) \in \mathcal{O}$ and $|lim(e)| = sup\{|\{e\}(n)|\}$.

(iv) If $a <_\mathcal{O} b$ and $b <_\mathcal{O} c$, then $a <_\mathcal{O} c$

The **recursive ordinals** are exactly those that have a notation in $\mathcal{O}$.

Using this notation, we can define a **consistency progression of theories**, which is a function that takes a natural number into the code of a $\Sigma_1$-formula $\psi_n$ that defines the axioms of a theory $\mathsf{T}_n$, and for which $\mathsf{PA}$ proves:

$$\begin{aligned}
\mathsf{T}_0 &= \mathsf{PA} \\
\mathsf{T}_{suc(n)} &= \mathsf{T}_n + Con(\mathsf{T}_n) \\
\mathsf{T}_{lim(n)} &= \bigcup_x \mathsf{T}_{\{n\}(x)}
\end{aligned}$$

This shows that we are indeed allowed to use this machinery to investigate Dummett's position. Each ordinal notation, qua being a natural number, is finite, and hence readily available for us; and the fact that something is an ordinal notation and the way we use it to recursively enumerate theory extensions is available by means provided by $\mathsf{PA}$.

Now, with respect to these progressions, Turing has shown that there is a way of iterating consistency statements such as to prove any $\Pi_1$ truth of arithmetic:

**Theorem 8.1** (Turing 1939, cf. Franzén 2004)**.** *For any true $\Pi_1$ sentence $A$ of arithmetic, a number $a_A \in \mathcal{O}$ can be constructed such that $|a_A| = \omega + 1$ and $\mathsf{T}_{a_A} \vdash A$ and $A \mapsto a_A$ is given by a primitive recursive function.*

What are the consequences of this with respect to Dummett's argument? Does this mean that we can determinately express all true $\Pi_1$ statements of arithmetic? And does it mean that a $\Pi_1$ statement is false when it does not feature in any of the progressions? Regarding the first question, note that $\mathcal{O}$ might itself not be directed such that we do not get one recursively enumerable sequence of theories that eventually captures all $\Pi_1$ truths of arithmetic. This is also the reason that, while the second question has a positive answer, what it is saying is not accessible to us by Dummett's standard.

Going back to Dummett, what has been said till now goes some way to fulfilling his following demand:

> When, as with 'natural number', the expression has an inherently vague meaning, it will be essential to the characterisation of its use to formulate the general principle according to which any precise formal characterisation can always be extended.(Dummett 1978, p. 198)

Such a general principle of extension has just been formulated. For this principle to be determinate, we do not require a limit of the procession, which is, as the above considerations show, also not available by the same standard.

Thus, Dummett seems to hit the mark (intentionally or not) by stating:

> [T]here is no ground for recourse to the conception of a mythical limit to the process of extension, a perfectly definite concept incapable of a complete description but apprehended by an ineffable faculty of intuition, which guides us in replacing our necessarily incomplete descriptions by successively less incomplete ones.

In light of this quote, Dummett's aim in pointing our that we can think about the definiteness of our number concept in this way may also be understood as refuting a possible transcendental argument for the determinacy of our conception of the standard model based on our intuitive notion of arithmetic truth. We can link our intuitive notion of arithmetic truth to recursive enumerability and consider recursively enumerable extensions of PA as representing our concept of natural number without making the leap to true arithmetic.

Now that the argument has been presented in full, we can ask if any changes in its initial set-up may cause it to significantly alter its outcome. One option, as already mentioned before would be to consider different ways of extending PA. As suggested in the discussion of the implicit commitment thesis, one might consider adding uniform reflection principles instead. To this, we find a corresponding theorem in Feferman (1962), which says that we can get a result similar to Turing's for *any* truth of arithmetic by iterating global reflection along relatively short well-orderings. However, here, too, there is no recursively enumerable theory that corresponds to all true statements, such that we would still obtain failures of conceptual completeness in much the same way as in the case of true $\Pi_1$ sentences.

In this respect, one might try to weaken the definition of what we mean by an arithmetic truth and use $\Diamond\Box A$ instead of $\Box\Diamond\Box A$. The idea here is to say that something is an arithmetic truth if it gets proven along at least one of the ways of encoding a consistency progression. One argument for this is that—directed frames or not—from the model theoretic perspective, there should be an intersection between all the models obtained by all the sequences that are in $\mathcal{O}$ (which is precisely the class of those models that satisfy all $\Pi_1$ truths of arithmetic). This would correspond to a somewhat artificial way of enforcing the usual potentialist result. However, to this intersection would correspond the models obtained by $\bigcup\limits_{a \in \mathcal{O}} \mathsf{T}_a$ and that itself is not a recursively enumerable theory. And in this respect it also fails by Dummett's standard.

These responses all stay within the boundaries set by Dummett. A more radical position would be to take issue with his set-up in general. One might discard recursive enumerability as a criterion for specifiability, and accept a supervaluationist understanding of the modality after all. Even more radically, one could deny that lack of specifiability should actually influence our reasoning with the concept and claim that intuitive knowledge is enough. I don't want to make any stances on these issues. Finally, however, the defendant of LEM might point out that the statements independent of PA that we know of so far are only meta-theoretical statements, and there is, to my knowledge, no statement that number theorists are actually concerned about which has been shown to be independent of PA. The instances of conceptual incompleteness that we would thus get may seem highly artificial and we might wonder whether they should actually impact our way of thinking with natural numbers. This matter, however, changes when we look into similar cases for set theory.

## 8.3. Potentialism for Theories of Sets

Looking at ZFC has the advantage that we are not restricted to axiomatic extensions that were only motivated by meta-theoretical considerations, or for which there is a standard model functioning as a measuring stick. This reflects a more genuine sort of indeterminacy. In this section, the consequences of these observations and their possible bearing on the claims in Feferman (2011) will be addressed.

### 8.3.1. Unique truthmakers and set theoretic ontology

Of course, this argument relies heavily on the fact that the axioms of ZFC themselves play an essential role in characterising (the determinateness of) the set concept and that furthermore, any development of it should be reflected by the addition of further axioms to ZFC. This position was illustrated in chapters 2.3.2 and 3.4. Of course, to a certain extent, this perspective requires that further development of the set concept is not predetermined, and neither that there simply is a plurality of co-existing set concepts. The position is thus incompatible with universism as well as multiversism. It is best be understood as a middle ground between the two.

I light of these ideas, we may thus consider the set concept to be most adequately reflected by a model of the sort $\mathfrak{M}_{\mathsf{ZFC}}$ which incorporates all practically plausible extensions of ZFC. This notion of a practically plausible extension is, of course, somewhat vague as it depends on what axioms mathematicians consider there to be practically plausible candidates for such an extension—and there might not be consensus about that. However, to illustrate the workings of conceptual indeterminacy, the toy models provided below will suffice. We may then say about a formula $A$ that:

$$A \text{ is a truth of set theory} \qquad \text{iff} \qquad \mathfrak{M}_{\mathsf{ZFC}} \vDash \Box\Diamond\Box A \tag{8.8}$$

and LEM for set theoretical truths is again expressed by:

$$\mathfrak{M}_{\mathsf{ZFC}} \vDash \Box\Diamond\Box A \lor \Box\Diamond\Box\neg A. \tag{8.9}$$

To illustrate some possible consequences of this view, consider the statement CH and look at the following three scenarios depicted in figure (8.2). I take them to be paradigmatic candidates to reflect conceptual development in mathematical practice. Their ordering may be taken to reflect a temporal development but it may also be understood in a non-temporal way as corresponding to certain dependencies illustrated by certain research programs.

In $\mathfrak{M}_1$, we start with a cluster of models of ZFC from which there are two clusters of models branching off: One in which CH holds and another one in which ¬CH holds. This may be understood as a scenario in which two competing solutions to CH are being developed and face off. In $\mathfrak{M}_2$, we start with a cluster corresponding to ZFC and then branch off one cluster in which CH holds and another cluster in which CH remains unsettled. No further developments are being made. In $\mathfrak{M}_3$, we keep on developing the right branch, occasionally branching off when an extension is reached that negates CH. The length of the right branch is left open, and it remains indeterminate if it will have a final cluster. This depicts a scenario in which the set concept undergoes a continuous strand of specifications, none of which settle CH.[8]

For each of those models $\mathfrak{M}_i$ for $i = 1, 2, 3$, it is the case that

$$\mathfrak{M}_i \nvDash \Box\Diamond\Box\mathsf{CH} \lor \Box\Diamond\Box\neg\mathsf{CH} \tag{8.10}$$

This constitutes a straightforward and very simple objection to bivalence as well as LEM with respect to set theoretic truths. This is a position that aligns very well with Feferman's remarks regarding CH. In fact, Feferman's position seems to be even stricter than this. Feferman argues that incorporating any attempts to settle CH on basis of adding additional axioms would constitute a change in conceptual background and thus

[8]Such a scenario might become reality if the current research programs to settle CH don't turn out to yield the required technical results or are otherwise deemed philosophically inadequate. Cf. Lingamneni (2020), ch. 2.3.2 and 3.4.

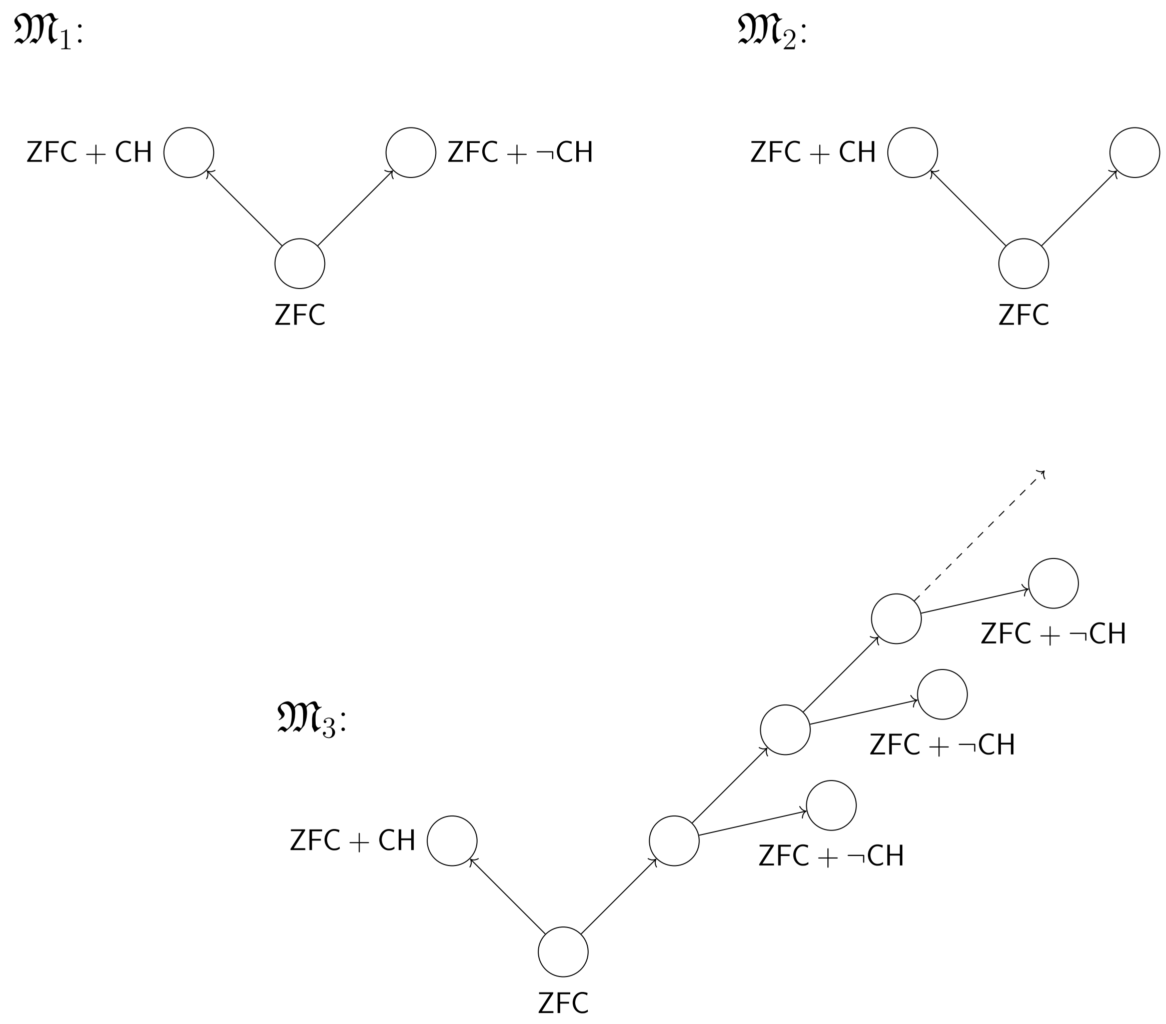



Figure 8.2.: Each point represents a cluster of models/worlds. CH or its negation besides a point indicates that the corresponding cluster belongs to an extension of ZFC in which it is settled.

no longer serve to settle the original question of CH, which he takes to be indeterminate simpliciter. In Feferman (2011) he particularly mentions the approach using $\Omega$-logic and the ultimate-V approach as candidates that might superficially fix the truth value of CH but in fact, so he argues, change its meaning. On this conception, there is no further development of the set concept possible, for it would be the instantiation of a different concept.

There is more that can be said about this, of course. New axioms may be motivated not only by them reflecting a certain informal notion, they may be viewed (and justified) as refinements, improvements or even replacements of certain aspects of the concept under consideration. I will leave the precise nature of selecting axiomatic extensions aside an instead focus on another possibility of interpreting the frames that they might give rise to.

### 8.3.2. Supervaluationist set theoretic truth

An alternative to the strict approach illustrated above that might establish LEM even on failures of bivalence, is to weaken the understanding of what LEM means with respect to set theoretic truths. Similarly to what has been discussed with respect to Väänänen's multiverse in chapter 7.2.1, we might supervaluate on the truth-values that statements determinately receive in the clusters. According to this perspective it should be a set theoretic truth that CH or $\neg$CH if any possibility of extending the ZFC axioms eventually leads to settling the matter. From this standpoint, $\mathfrak{M}_1$ is considered to actually reflect a case of determinacy—there are only two possible ways of development and each of them settles the question. Hence, even though $\mathfrak{M}_1 \not\models \mathsf{CH}$ and $\mathfrak{M}_1 \not\models \neg\mathsf{CH}$, it should be the case that $\mathsf{CH} \vee \neg\mathsf{CH}$ is considered an arithmetic truth in $\mathfrak{M}_1$. To accommodate this, the expression of LEM should change accordingly.

One possible option is to say that in analogy to the $\Box\Diamond$-translation LEM holds supervaluationistically with respect to set theoretical truths iff for any formula $A$, we have

$$\mathfrak{M}_{\mathsf{ZFC}} \models \Box\Diamond(\Box\Diamond\Box A \vee \Box\Diamond\Box\neg A). \tag{8.11}$$

This seems to be adequate to reflect determinacy in $\mathfrak{M}_1$.[9] Note, however, that

$$\mathfrak{M}_2 \not\models \Box\Diamond(\Box\Diamond\Box\mathsf{CH} \vee \Box\Diamond\Box\neg\mathsf{CH}). \tag{8.12}$$

But the fact that $\mathfrak{M}_2$ fails in this respect is not too bothersome. As far as any of these models is supposed to reflect the actual development of the set concept, $\mathfrak{M}_2$ seems to be the least plausible candidate. It represents the case where set theorists just stop thinking about axiomatic extensions and them settling CH, and even if that were the case, it would be more adequately represented by branching off the model at this point by appending $\mathfrak{M}_1$ to the 'undecided' cluster. In this respect only $\mathfrak{M}_1$ and $\mathfrak{M}_3$ seem to be plausible candidates.

However, looking at $\mathfrak{M}_3$ now, we might doubt that (8.11) adequately captures our intention, and, furthermore, that $\Box\Diamond\Box A$ is still an adequate expression of what we mean by set theoretic truth. The reason for this is that the rightmost branch also encodes the possibility of adding axioms to ZFC for an indeterminate amount of time without actually settling CH either to the positive or to the negative. This is, unsurprisingly, the same situation that arose with respect to branching potentialism in chapter 6.4.2. What we would want to say thus is that a statement is a set theoretic truth if and only if it *inevitably* gets settled in the process of axiomatically extending ZFC. In order to express this stronger claim, we can make use of the inevitability operator $\mathcal{I}$ again. To state its semantics in the present context, we need the notion of a path of clusters.

**Definition 8.4.** Given a partial order of clusters $\langle \bigsqcup_i \mathbb{M}_i, \leq\rangle$, a **chain** above a cluster $\mathbb{M}_i$ is a set of clusters $X = \bigsqcup_{j\leq i} \mathbb{M}_j \subseteq \bigsqcup_i \mathbb{M}_i$ such that for all $\mathbb{M}_j, \mathbb{M}_k \subseteq X$ and all $(\mathcal{M}, \mathsf{T}_j) \in \mathbb{M}_j$ and $(\mathcal{M}, \mathsf{T}_k) \in \mathbb{M}_k$, we have that $(\mathcal{M}, \mathsf{T}_j) \leq (\mathcal{M}, \mathsf{T}_k)$ or $(\mathcal{M}, \mathsf{T}_k) \leq (\mathcal{M}, \mathsf{T}_j)$.

A chain is a **path** if it is maximal in $\bigsqcup_i \mathbb{M}_i$.

[9] Note that in directed frames $\Box\Diamond\Box A \vee \Box\Diamond\Box B$ is equivalent to $\Box\Diamond(\Box\Diamond\Box A \vee \Box\Diamond\Box B)$, but not in branching frames. The reason for this is that if the second is true in directed frames, any way we go it is always possible to either prove $A$ or $B$, and because of directedness this means that any way we go it is possible to prove $A$ or it is possible to prove $B$, hence $\Box\Diamond\Box A \vee \Box\Diamond\Box B$. If the frame is branching, this no longer works. If the model consists of two branches, we could have one branch only satisfying $\Box\Diamond\Box A$ and another one only satisfying $\Box\Diamond\Box B$. This leads to satisfaction of $\Box\Diamond(\Box\Diamond\Box A \vee \Box\Diamond\Box B)$ but not of $\Box\Diamond\Box A \vee \Box\Diamond\Box B$.

We may now define

$$(\mathcal{M}, \mathsf{T}_i) \vDash \mathcal{I}A \qquad \text{iff} \qquad \text{any path of clusters } X \text{ with } (\mathcal{M}, \mathsf{T}_i) \in X \text{ has a cluster } \mathbb{M}_k \text{ with a world } (\mathcal{M}, \mathsf{T}_k) \text{ s.t. } (\mathcal{M}, \mathsf{T}_k) \vDash A.$$

With that, we are able to amend our notion of set theoretic truth accordingly and state that

$$A \text{ is a truth of set theory} \qquad \text{iff} \qquad \mathfrak{M}_{\mathsf{ZFC}} \vDash \mathcal{I}\Box A \tag{8.13}$$

we then express the disjunction of set theoretic truths (understood supervaluationistically) as[10]

$$\mathcal{I}(\mathcal{I}\Box A \vee \mathcal{I}\Box B). \tag{8.14}$$

In this respect, however, we also need to adjust how we understand our notion of meta-implication and negation. For this consider the corresponding expression of the conditional as

$$\mathcal{I}(\mathcal{I}\Box A \to \mathcal{I}\Box B),$$

which says that any path that verifies $A$ also verifies $B$. (For a more thorough discussion, see appendix 8.A).

The corresponding notion of negation is:

$$\mathcal{I}\neg\mathcal{I}\Box\mathsf{CH}. \tag{8.15}$$

It entails that $\mathcal{I}\Box\Diamond\neg\mathsf{CH}$ which says that there is a bar after which all clusters (theories) retain the possibility that $\neg\mathsf{CH}$. This certainly entails that $\mathsf{CH}$ cannot be a set theoretic truth, because it either means that inevitably $\neg\mathsf{CH}$ will be established or that we might indefinitely remain indecisive between the two. Consequently, (8.15) doesn't hold in

[10] Note that on directed frames, this expression is equivalent to $\Box\Diamond(\Box\Diamond\Box A \vee \Box\Diamond\Box B)$ and to $\Box\Diamond\Box A \vee \Box\Diamond\Box B$.

any of the models. Corresponding to this, we also have an expression of meta-logical definiteness, which is now given as

$$\mathfrak{M}_i \models \mathcal{I}(\mathcal{I}\Box\mathsf{CH} \vee \mathcal{I}\neg\mathcal{I}\Box\mathsf{CH}), \tag{8.16}$$

which says that CH either is or is not a set theoretic truth, and which holds in all three models.

Finally, however, considering the model $\mathfrak{M}_3$, the right branch has a path at which neither CH nor $\neg$CH will eventually be the case, which implies that

$$\mathfrak{M}_3 \not\models \mathcal{I}(\mathcal{I}\Box\mathsf{CH} \vee \mathcal{I}\Box\neg\mathsf{CH}) \tag{8.17}$$

Thus even on a supervaluationist understanding of set theoretic truth explicated in the form above, there is a conception of set theoretic development in which LEM does not hold with respect to set theoretic truths.

These results lead to two important questions:

(i) What is the correct reflection of actual set theoretical practice? Is a model of type $\mathfrak{M}_1$ or of type $\mathfrak{M}_3$ a more adequate description (under broad acceptance of the picture sketched in chapters 2 and 3)? Intriguing in this respect is the possibility that there might not be a decisive answer to this question (yet). But if we don't know, we should admit that we don't know whether the set theoretical truths obey LEM or not.

(ii) Let it be granted that we either hold a non-supervaluationist understanding of set theoretic truth (like Feferman), or that we allow for a supervaluationists understanding, but the reality of mathematical practice is best captured by $\mathfrak{M}_3$. What consequences do either of these options have for the actual use of LEM in mathematical practice? An opponent of LEM might be inclined to argue that insofar as LEM is uncritically used in actual mathematical practice, it might not be certain whether it is possible to make one of its disjuncts true. But it is not obvious to what extend this should lead to a rejection of its use. A defendant of LEM might respond that the ultimate endeavour of set-theory is to characterise a model (or a class of models), and in those LEM *prima facie* holds. The models discussed here provide a first step towards the connection between indeterminacy and a rejection of LEM, but in doing so they raise another question: Perhaps the logic of set theoretic truths or set theoretic discovery should differ from the logic that set theorists actually use in proving theorems. Feferman himself was aware that there might be a gap between these two types of considerations:

> [T]he use of truth in ordinary mathematical parlance is deflationary and the reasons for accepting such and such principles as true has either been made without question or for mathematical reasons in the course of the development of the subject. The philosopher, by contrast, is concerned to explain in what sense the notion of truth is applicable to mathematical statements, in a way that cannot be considered in ordinary mathematical parlance. Whether the mathematician should pay attention to either of these aims of the philosopher is another matter.(Feferman 2014, p. 90)

Indeed, why should or shouldn't that be the case? This remains an open question, but I submit that due to the above analysis this question now has a precise formulation.

## 8.4. Conclusion

This chapter discussed another type of argument against LEM. Contrary to chapters 6 and 7, it was not based on indeterminacy of domains, but on conceptual indefiniteness. The basic question was whether it is possible to infer from the fact that something does not lie in the concept of set or natural number that its negation must. With respect to the natural numbers it was found that unlike his other appeal to indefinite extensibility, this argument by Dummett—pending the acceptability of its premises—actually yields ground to reject LEM with respect to arithmetic truths. A similar point was observed with respect to set theory, with the crucial advantage that the statements that were discussed were not exclusively meta-theoretical. This allowed for restrictions of LEM with respect to set theoretic truths, both on a singular as well as on a supervaluationistic understanding of them. As the discussion of set theory, however, has shown, there might still be a difference between the laws that govern the notion of set theoretic truth as developed in this chapter and the rules of inference that mathematicians use in practice.

# 8.A. Appendix

This appendix serves to clarify the link between the two types of negation discussed in chapter 5 and their formalisation in this chapter. I will show how using the modalities allows us to pull down meta reasoning and thus negation into the object level system, and thereby treat both on the same level. In order to formally define meta-negation, however, we need an object-model and a meta-model.

## 8.A.1. Meta and object reasoning in the initial fragment $\mathfrak{M}$

Strictly speaking, the meta-model we are trying to define is not a meta-model with respect to $\mathfrak{M}$, but only with respect to its evaluation of formulas that only contain non-modal vocabulary. For this purpose, again, let $A$ be a closed formula in the language of PA.

**Definition 8.5.** Let $\mathfrak{M}$ be given as in definition 8.1 with respect to a theory $\mathsf{T}$. $\mathfrak{M}^+$ is a **two valued meta model** for satisfaction of formulas of $\mathsf{T}$ in $\mathfrak{M}$, that is given by the clauses:

$$
\begin{array}{lll}
\mathfrak{M}^+ \vDash A & \text{iff} & \mathfrak{M} \vDash A \text{ for formulas of } \mathsf{T} \\
\mathfrak{M}^+ \vDash \varphi \mathbin{\&} \psi & \text{iff} & \mathfrak{M}^+ \vDash \varphi \text{ and } \mathfrak{M}^+ \vDash \psi \\
\mathfrak{M}^+ \vDash \varphi \parallel \psi & \text{iff} & \mathfrak{M}^+ \vDash \varphi \text{ or } \mathfrak{M}^+ \vDash \psi \\
\mathfrak{M}^+ \vDash \varphi \Rightarrow \psi & \text{iff} & \mathfrak{M}^+ \nvDash \varphi \text{ or } \mathfrak{M}^+ \vDash \psi
\end{array}
$$

For this we can define a translation from meta-formulas into modal formulas holding in $\mathfrak{M}$. Let $\varphi$ be a formula of the meta language of $\mathsf{T}$, and let $\varphi^M$ be defined recursively for such formulas by

$$
\begin{array}{rcl}
A^M & \mapsto & \Box A, \text{ for closed formulas of } \mathsf{T} \\
(\varphi \mathbin{\&} \psi)^M & \mapsto & \varphi^M \wedge \psi^M \\
(\varphi \parallel \psi)^M & \mapsto & \varphi^M \vee \psi^M \\
(\varphi \Rightarrow \psi)^M & \mapsto & \varphi^M \to \psi^M
\end{array}
$$

**Proposition 8.1.** *Let $\varphi$ be a meta-formula that does not contain any modal vocabulary, then*

$$(\mathfrak{M})^+ \vDash \varphi \qquad \textit{iff} \qquad \mathfrak{M} \vDash \varphi^M.$$

*Proof.* By Induction on length of formulas. For the base case:

$$\mathfrak{M}^+ \vDash A \quad \text{iff} \quad \mathfrak{M} \vDash \Box A \quad \text{iff} \quad \mathcal{M} \vDash A \text{ for all } \mathcal{M} \in \mathfrak{M}.$$

And for complex formulas:

$$\begin{aligned}
\mathfrak{M}^+ \vDash \varphi \mathbin{\&} \psi \quad &\text{iff} \quad \mathfrak{M}^+ \vDash \varphi \text{ and } \mathfrak{M}^+ \vDash \psi \\
&\text{iff} \quad \mathfrak{M} \vDash \varphi^M \text{ and } \mathfrak{M} \vDash \psi^M \\
&\text{iff} \quad \mathfrak{M} \vDash \varphi^M \wedge \psi^M \\
\mathfrak{M}^+ \vDash \varphi \parallel \psi \quad &\text{iff} \quad \mathfrak{M}^+ \vDash \varphi \text{ or } \mathfrak{M}^+ \vDash \psi \\
&\text{iff} \quad \mathfrak{M} \vDash \varphi^M \text{ or } \mathfrak{M} \vDash \psi^M \\
&\text{iff} \quad \mathfrak{M} \vDash \varphi^M \vee \psi^M \\
\mathfrak{M}^+ \vDash \varphi \Rightarrow \psi \quad &\text{iff} \quad \mathfrak{M}^+ \nvDash \varphi \text{ or } \mathfrak{M}^+ \vDash \psi \\
&\text{iff} \quad \mathfrak{M} \nvDash \varphi^M \text{ or } \mathfrak{M} \vDash \psi^M \\
&\text{iff} \quad \mathfrak{M} \vDash \varphi^M \to \psi^M
\end{aligned}$$

□

This idea can be extended to the whole model $\mathfrak{M}_\mathsf{T}$. In order to do so, we need to, again, define a meta-model. Only this time, it is not straightforward what the corresponding object model is supposed to be. But we can use $\mathfrak{M}_\mathsf{T}$ just like in the chapter to define satisfaction for a group of formulas that we take to be governing our reasoning with the concept connected to $\mathsf{T}$.

## 8.A.2. Meta and object reasoning in $\mathfrak{M}_{\mathsf{T}}$

First, note that each cluster can be treated as a meta-model just before. Hence for each $\mathsf{T}^*$ and corresponding cluster $\mathbb{M}^*$, there is an $(\mathfrak{M}^*)^+$ as defined above. Now, to express that $A$ is a truth of the concept that is modelled via $\mathfrak{M}_{\mathsf{T}}$, means to say that on any path extending $\mathsf{T}$, $A$ will eventually become proven, i.e. there is a cluster $\mathfrak{M}^*$ such that $\mathfrak{M}^* \vDash \Box A$. Similar clauses we can define for the other meta-connectives. In this respect, however, it becomes relevant whether the frames are directed or not. To give the definition in full generality, we will assume that the frames are branching.

However, branching frames introduce a novelty with respect to meta-reasoning that was not discussed in chapter 5, and that is the question whether the meta-disjunction should be understood supervaluationistic or not. Thinking about it non-supervaluationistically, we'd say that

> $\varphi \parallel \psi$ is true iff *either* on any path there is an extension $\mathsf{T}^*$ of $\mathsf{T}$ such that $(\mathfrak{M}^*)^+ \vDash \varphi$ *or* on any path there is an extension $\mathsf{T}^*$ of $\mathsf{T}$ such that $(\mathfrak{M}^*)^+ \vDash \psi$.

Thinking about it supervaluationistically, we'd say that

> $\varphi \parallel \psi$ is true iff on any path there is an extension $\mathsf{T}^*$ of $\mathsf{T}$ such that $(\mathfrak{M}^*)^+ \vDash \varphi$ or $(\mathfrak{M}^*)^+ \vDash \psi$.

In what is to come, I will follow the supervaluationist understanding.

Finally, the meta-conditional warrants some attention as well. Informally, $\varphi \Rightarrow \psi$ seems to say that if $\varphi$ lies in the concept, then so does $\psi$. This can be expressed in the model as saying that any path that verifies the antecedent should also verify the consequent, i.e.

> $\varphi \Rightarrow \psi$ is true iff any path that has an extension $\mathsf{T}^*$ of $\mathsf{T}$ such that $(\mathfrak{M}^*)^+ \vDash \varphi$ then it should also have an extension $\mathsf{T}^{*^*}$ of $\mathsf{T}^*$ such that $(\mathfrak{M}^{*^*})^+ \vDash \psi$.

As we will see below, this lends itself to the classical material understanding of the conditional. This might seem counterintuitive because it essentially comes down to collapsing upwards. But that is exactly what we want to reflect when understanding conceptual containment in this dynamic way. For instance, given this understanding of meta-implication, we would get true instances of $Con(\mathsf{T}) \Rightarrow Con(\mathsf{T}^*)$, which can be read as saying that our knowledge of $\mathsf{T}$ entails knowledge of its consistency and of the consistency of the consistency claim, and so on. But note that this is importantly distinct from $Con(\mathsf{T}) \to Con(\mathsf{T}^*)$, which says that if we have proved $Con(\mathsf{T})$, we can prove $Con(\mathsf{T}^*)$ by the same formal means—and for which there is at least one cluster in which it is false.

This gives us the following satisfaction clauses for meta-formulas in a suitably defied model.

**Definition 8.6.** Let $\mathfrak{M}_{\mathsf{T}}$ be as in definition 8.2. Let $\mathcal{B}$ be the set of all paths in $\mathfrak{M}_{\mathsf{T}}$, then $(\mathfrak{M}_{\mathsf{T}})^+ = \langle \mathcal{B}, [\![\cdot]\!] \rangle$ is **a model for conceptual containment connected to** $\mathsf{T}$, where $[\![\cdot]\!]$ is defined as

$$[\![A]\!] = \{\beta \in \mathcal{B} \mid \text{ there is an extension } \mathsf{T}^* \text{ of } \mathsf{T} \text{ on } \beta \text{ such that } (\mathfrak{M}^*)^+ \vDash A\}$$

for formulas of $\mathsf{T}$, and

$$\begin{aligned} [\![\varphi \mathbin{\&} \psi]\!] &= [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\varphi \parallel \psi]\!] &= [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\varphi \Rightarrow \psi]\!] &= [\![\varphi]\!]^c \cup [\![\psi]\!] \\ [\![\curlywedge]\!] &= \emptyset \end{aligned}$$

for complex formulas.

Now, we can define the corresponding class of modal formulas. This was already implicitly used in the chapter. Let $\varphi$ be a meta-formula of arithmetic, define $\varphi^+$ by

$$\begin{aligned} A^+ &\mapsto \mathcal{I}\Box A \\ (\varphi \mathbin{\&} \psi)^+ &\mapsto \varphi^+ \wedge \psi^+ \\ (\varphi \parallel \psi)^+ &\mapsto \mathcal{I}(\varphi^+ \vee \psi^+) \\ (\varphi \Rightarrow \psi)^+ &\mapsto \mathcal{I}(\varphi^+ \to \psi^+) \end{aligned}$$

**Theorem 8.2.** *Let $\varphi$ be a meta-formula of $\mathsf{T}$. Let $\mathfrak{M}^+_{\mathsf{T}}$ be the corresponding model for conceptual containment connected to $\mathsf{T}$, then*

$$\mathfrak{M}^+_{\mathsf{T}} \vDash \varphi \qquad \textit{iff} \qquad \mathfrak{M}_{\mathsf{T}} \vDash \varphi^+.$$

*Proof.* By Induction on length of formulas. For the base case:

$$\mathfrak{M}^+_{\mathsf{T}} \vDash A \quad \text{iff} \quad [\![A]\!] = \mathcal{B} \quad \text{iff} \quad \mathfrak{M}_{\mathsf{T}} \vDash \mathcal{I}\Box A$$

And for complex formulas, start with conjunction:

$$\begin{aligned} \mathfrak{M}^+_{\mathsf{T}} \vDash \varphi \,\&\, \psi \quad & \text{iff} \quad [\![\varphi]\!] = [\![\psi]\!] = \mathcal{B} \\ & \text{iff} \quad \mathfrak{M}_{\mathsf{T}} \vDash \varphi^+ \text{ and } \mathfrak{M}_{\mathsf{T}} \vDash \psi^+ \\ & \text{iff} \quad \mathfrak{M}_{\mathsf{T}} \vDash \varphi^+ \wedge \psi^+ \end{aligned}$$

Now, for disjunction consider that

$$\mathfrak{M}^+_{\mathsf{T}} \vDash \varphi \;\|\; \psi \quad \text{iff} \quad [\![\varphi]\!] \cup [\![\psi]\!] = \mathcal{B},$$

which means that any path either verifies $\varphi$ or it verifies $\psi$, which is the same as $\mathfrak{M}_{\mathsf{T}} \vDash \mathcal{I}(\varphi^+ \vee \psi^+)$.

Finally, for meta-conditional

$$\mathfrak{M}^+_{\mathsf{T}} \vDash \varphi \Rightarrow \psi \quad \text{iff} \quad [\![\varphi]\!]^c \cup [\![\psi]\!] = \mathcal{B},$$

which means that any path which verifies $\varphi$ needs to verify $\psi$. This means that if there is a cluster $\mathbb{M}^*$ and a path $\beta$ such that $\mathbb{M}^* \in \beta$ and $\mathfrak{M}^* \vDash \varphi^+$ then any path going through it should have a cluster $\mathbb{M}^{*^*}$ for which $\mathfrak{M}^{*^*} \vDash \psi^+$. But this entails that already $\mathfrak{M}^* \vDash \psi^+$ and hence $\mathfrak{M}^{*^*} \vDash \varphi^+ \to \psi^+$. Since $\beta$ was arbitrary, it follows that $\mathfrak{M}^{*^*} \vDash \mathcal{I}(\varphi^+ \to \psi^+)$. □

Finally, note that we always get the classicality of the meta-logic, i.e. $\mathfrak{M}^+_{\mathsf{T}} \vDash A \parallel A \Rightarrow \curlywedge$, which says that $A$ is a truth corresponding to $\mathsf{T}$ or it isn't. But we do not necessarily get $\mathfrak{M}^+_{\mathsf{T}} \vDash A \parallel \neg A$, saying that $A$ or $\neg A$ is a truth corresponding to $\mathsf{T}$. For instance

$$(\mathsf{CH} \parallel \mathsf{CH} \Rightarrow \curlywedge)^+ = \mathcal{I}(\mathcal{I}\Box\mathsf{CH} \vee \mathcal{I}\neg\mathcal{I}\Box\mathsf{CH})$$

which holds in $\mathfrak{M}_{\mathsf{ZFC}}$ but

$$(\mathsf{CH} \parallel \neg\mathsf{CH})^+ = \mathcal{I}(\mathcal{I}\Box\mathsf{CH} \vee \mathcal{I}\Box\neg\mathsf{CH})$$

which does not (cf. the discussion on p.231).

# 9. Conclusion

Judging by the course of the thesis, it seems that we have set ourselves up for disappointment. The type of argument against LEM under development promised to be a genuine alternative to traditional intuitionism and to avoid some of its major pitfalls. In the end, however, it turned out that the standard of success was lowered to the mere existence and coherence of the respective arguments that could be made based on the relevant notions of indeterminacy and indefiniteness.

### The general set-up

An argument against LEM based on indeterminacy and indefiniteness promised to

- avoid the identification of provability as an essential part of truth, and with it its dubious consequences for the nature of mathematical statements;
- not rely on the standard intuitionist interpretation of the logical constants and the inadequacy of known formal and informal semantics for them;
- avoid the use of an intuitionistic meta-logic which put the proof of completeness for any kind of formal semantics in jeopardy.

In this respect, it presented itself as a significant improvement on many central problems of the intuitionist tradition (ch. 4.1).

Instead of a connection between truth and provability, the rejection of LEM was based on an account of indeterminacy (chs. 1.1, 1.3-1.4, 2) with a corresponding anti-realist ontology (ch. 3.4). The notion of indeterminacy was semantically modelled in terms of the availability of objects for the case of indeterminacy of domains. This was done by introducing the notion of a warrant that was characterised as exclusively given by the availability of objects (chs. 4.2, 5.A, 6.A). For the case of indefiniteness of concepts, indeterminacy was said to be a form of conceptual incompleteness that consists in the lack of certain inferences from meta-level to object-level statements (chs. 5.3, 8.3).

### The contributions made in this thesis

Based on this idea, the main contributions of this thesis are:

- The investigation of the two notions of negation as an extension to Weyl's and Linnebo's treatment of generic generality and its application to reasoning with indeterminacy (chs. 5, 6.A, 8.A). As it turned out, object level negation was the suitable candidate for indeterminacy of domains while meta-level negation was the right tool to model conceptual indefiniteness.

- The account of indefinite extensibility, in particular the clarification of the modality involved in order to exclude revenge and attain absolute generality (ch. 6.3). This is the most substantial philosophical point made in this thesis.

- The use of the Gödel translation as a way to *actually derive* (and not just postulate) a weakening of $\mathsf{ZFC}$ to $\mathsf{SCS} - \Delta_0$-Markov based on the multiverse model of indeterminacy (ch. 7.3).

- The development of an argument by Dummett based on indefinite extensibility related to Gödel's incompleteness theorems, the formalisation of potentialism with respect to theory extensions in the course of that development (chs. 8.1-8.2), and its subsequent application to set theory (ch. 8.3).

In general, the thesis shows that it is possible to reject the use of $\mathsf{LEM}$ on basis of indeterminacy in a coherent argument.

- Indeterminacy of height may lead to a rejection of $\mathsf{LEM}$ in cases where it is modelled via branching frames.

- It may also lead to a rejection of $\mathsf{LEM}$ when paired with a strict potentialist understanding.

- Indeterminacy of width may lead to a rejection of LEM, and indeed a restriction of ZFC to SCS $-$ $\Delta_0$-Markov.
- Indefiniteness of concepts, when understood as indefinite extensibility of the theories characterising them, can lead to certain cases of conceptual incompleteness that may also involve failures of bivalence and LEM.

These results thus lend some credibility to Dummett's and Feferman's claims. But the above arguments only work in a select range of cases and involves subscribing to some positions that are philosophically not uncontroversial. We are here, but should we have come?

**Critical assessment of the arguments against LEM**

Throughout the thesis I have already noted some possible reservations against the arguments that are being developed. Some of them simply don't work outright and others rest on premises that require a more thorough defence and development than it was possible to give them:

- First off, it should be noted that the rejection of LEM based on indefinite extensibility of *domains* is actually incoherent when it involves linear frames (ch. 6.4.1).
- Indefinite extensibility on branching frames does lead to a rejection of LEM, however, the phenomena actually modelled by branching frames do not reflect the sort of indeterminacy via indefinite extensibility that Dummett was interested in (ch. 6.4.2).
- The standard possible world semantics for strict potentialism does not seem to work (if it is meant to be including nested quantification). An alternative based on realisability seems to be more promising, but requires a more thorough development. Additionally, it seems that this form of potentialism does flirt with intensionality and thus constitutes at least some departure from the 'orthodox' argument from indeterminacy (ch. 6.4.3).

- The rejection of LEM with respect to set theory via the use of the Gödel translation requires a tension-ridden stance on how the multiverse reflects the set concept (ch. 7.3.2).

- Dummett's argument against LEM based on axiomatic extensions of PA gives us a technical result in which LEM is not applicable to certain statements, but the sort of statements that this concerns might elude any statements of actual interest to number theorists (ch. 8.2.2). A related problem was also noted with respect to set theory. Thus even though theory extensions might give us a rejection of LEM, assessing their consequences for actual mathematical practice are a different matter (ch. 8.3.2).

A complete argument against the use of LEM for any of these domains would thus have to address these issues. In this respect the work done in this thesis leads to a number of questions:

**Further questions**

Some of these questions call for a more thorough investigation into some of the positions initially developed here, but some also suggest more independent future research. The most interesting questions in this respect, are:

- An extension of the use of generic generality to other domains, as listed in ch. 5.1.1. Given the fact that the notion is so fundamental, and versatile, and still has only been even investigated by two authors (or three, if I may), this seems to be the most substantial future contribution to be made.

- A more detailed development of strict potentialism and discussion of its similarities and dissimilarities with traditional intuitionism.

- As it was noted, there is a certain tension in the philosophical underpinning of the use of the Gödel translation as a measure of determinacy in the set theoretic multiverse. Some philosophical background on this position is required for a more thorough rendering of the argument.

Furthermore, it should also be possible to develop the technical side of the argument in a more differentiated way. In particular, the resulting logic and set forming principles extractable from the multiverse should be fine grained enough to reflect determinacy/indeterminacy on the levels of $V_{\omega+1}$ and $V_{\omega+2}$.

- The investigation launched in chapter 8 could be extended, on the technical side, with respect to the relation between potentialism for theories and provability logics (Beklemishev 1991), and on the philosophical side with respect to its relation to the implicit commitment thesis.

**Final verdict**

So, what should the ultimate verdict be? Should we reject LEM when we reason with indeterminate domains or concepts of the sort discussed in this thesis? The answer is, if you really want to, you can. But I have no illusion that the material presented in the thesis will cause any mathematician to change their mind about using the law in their daily practice. (But was this ever an option?) From a philosophical perspective, however, it does strike me as an insight that a rejection of LEM in the vicinity of a truth theoretical semantics can be coherently maintained. This at the very least serves to show that LEM does have a peculiar status among the other logical laws. And since no other law seemed to be in jeopardy at least for the structures under consideration here, it might be safe to say that intuitionistic reasoning is indeed more fundamental than classical reasoning. But that does not mean that for this reason alone, we should reject LEM. But the same holds also for its adoption. LEM might be a priori valid, and it might be taken to be so for any of the structures where it could be coherently rejected. However, any decision to this effect, whether it is a rejection or adoption of it, must be underpinned (explicitly or not) by a philosophical conviction of what constitutes the domain or concept in question. The matter might be obvious, but it is not trivial.

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