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What can graphs and algebraic structures say to each other?

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ABSTRACT

In the last couple of decades, there has been a big upsurge of research on graphs defined on algebraic structures (groups, rings, vector spaces, semigroups, and others). Much of this has concerned detailed graph-theoretic properties and parameters of these graphs. However, my concern here is to consider how this research can benefit both graph theory and algebra. I am mainly concerned with graphs on groups, and will give three types of interaction between graphs and groups, with examples of each taken from recent research. The paper also contains a number of open questions. This talk was presented at the conference ICRAGAA 2023 held in Thrissur in Kerala, India. I am grateful to the organizers of the conference, and also to Ambat Vijayakumar and Aparna Lakshmanan S, who organized a very productive on-line research discussion on graphs and groups in 2021. Much of what I report has its roots in that discussion. I am grateful to them for organizing this discussion, as well as to the conference organizers, and all my many coauthors.

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1. Introduction

The subject of graphs defined on groups goes back to the work of Cayley [17] in 1878. But Cayley graphs are not my topic here. I will be discussing graphs on algebraic structures which directly reflect some aspect of the structure. Examples for groups G include

- the commuting graph: vertices x, y are joined if $xy = yx$;
- the generating graph: x and y are joined if $\langle x, y\rangle = G$.

For rings, an example is the zero-divisor graph, where x and y are joined if they are nonzero but $xy = 0$.

There are many other examples of such graphs, on algebraic structures of many kinds. I will concentrate mainly on groups, since that is my main interest. (In the ICRAGAA 2023 conference, T. Tamizh Chelvam spoke about rings, and Vinayak Joshi about partially ordered sets.)

The history of graphs defined on groups in this sense goes back to 1955, where Brauer and Fowler, in the seminal paper [10], used the commuting graph to show that there are only finitely many simple groups of even order which have a given structure for the centralizer of some involution. (The rider about even order was necessary for them since it was several more years before Feit and Thompson showed that a nonabelian finite simple group must have even order.) The centralizer of z is the set of elements of the group which commute with z, that is, the closed neighborhood of z in the commuting graph. Thus, a bound on the diameter of the graph can give, by techniques familiar to graph theorists, a bound on the number of vertices. (Brauer and Fowler did not use the term "graph" in their paper, though the graph distance is fundamental to their arguments. However, the commuting graph of a group has become the subject of a lot of research since then.)

Another type of graph which has been considered is the *intersection graph* of subalgebras of an algebra, or perhaps of subalgebras of some given type. For example, given a group G, we could make a graph whose vertices are the nontrivial proper subgroups of G, where H and K are joined if $H \cap K \neq \{1\}$; or we could restrict our attention to maximal subgroups, or to cyclic subgroups.

The large body of research about graphs on groups has mainly concentrated on calculating various graph-theoretic parameters. This is valuable for filling in details of the picture, but I am going to step back and ask a different question: what do the theories of graphs and groups gain from this conversation? I will distinguish three main areas where this happens:

- Using graphs, we may find new results about groups.
- We may be able to define or characterize interesting classes of groups by putting conditions on various graphs defined on them.
- We may find beautiful and interesting graphs in the process.

Similar remarks apply to other types of algebraic structure, though I will say less about this.

After a section in which I ask whether there are any general principles at work here, I will treat each of the areas listed, and give recent results illustrating them.

As mentioned in the Abstract, I will be citing work by many coauthors, and a secondary purpose of the paper is to showcase some of these results.

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2. General principles?

What I have in mind is the possibility of recognizing that graphs defined on different algebraic structures, or in different ways on the same algebraic structure, may have connections, or be special cases of more general classes of graphs.

The best example of this was discussed by Vinayak Joshi in his talk at the conference [28, 37]. He defined the *zero-divisor graph* of a partially ordered set with unique least element zero as follows: we join two nonzero elements if their greatest lower bound is 0; then we discard the isolated vertices.

This class of graphs, allowing small modifications such as taking the join with a complete graph, includes various types. Among them are

- zero-divisor graphs (in the usual sense) of commutative rings with identity [4];
- component intersection and union graphs of vector spaces [20, 21] (see also [29]);
- the complement of the *enhanced power graph* of a finite group (to be defined below).

For example, it is the case that for certain types of poset, the clique number and chromatic number of the zero-divisor graph are equal; this enables the same conclusion to be drawn in some of the above cases.

The other two principles I will describe are both taken from my survey paper [12]. First, let me define the classes of graphs on groups I will be talking about. Let G be a finite group. For each of the graphs below, the vertex set is G (though for many purposes we might want to discard isolated or dominating vertices of G), and I give the joining rule for vertices x and y.

- The *power graph*: one of x and y is a power of the other. This was first defined as a directed graph [30], and studied as an undirected graph in [18]. See [31] for a recent survey.
- The *enhanced power graph*: x and y are both powers of an element z; equivalently, the group $\langle x, y\rangle$ generated by x and y is cyclic. The complement of this graph was defined in [2], the graph in the form given was defined in [1].
- The *commuting graph*: $xy = yx$; equivalently, $\langle x, y\rangle$ is abelian. As we saw, this goes back to [10] and is the oldest of the graphs considered here.
- The *solvability graph*: $\langle x, y\rangle$ is solvable.
- The *non-generating graph*: $\langle x, y\rangle \neq G$.

The first four classes form a hierarchy; each is a spanning subgraph of the next. Moreover, the commuting graph is a spanning subgraph of the non-generating graph if G is non-abelian; and the solvability graph is a subgraph of the non-generating graph is G is non-solvable.

When we have a hierarchy of this form, then a couple of possibilities open up:

- for any graph parameter which is monotone on edge sets (such as clique number, chromatic number, matching number, etc.), its values on the graphs in the hierarchy are nondecreasing.
- We can form various difference graphs, where the edges are those of a specified graph which are not edges of another graph lower in the hierarchy.

As just one example of the second point, I give a theorem shown in [12, Theorem 5.9].

Theorem 2.1. Given any coloring of the edges of a finite complete graph with red, green and blue, there is an embedding into a finite group so that the red edges belong to the enhanced power graph, the green edges to the commuting graph but not the enhanced power graph, and the blue edges are not in the commuting graph.

This shows that the enhanced power graph and the commuting graph, as well as their difference, are universal. A similar result was shown for the zero-divisor graph of R^2 and the two-dimensional dot product graph of R, for any finite commutative ring R with identity by G. Arunkumar, T. Kavaskar, T. Tamizh Chelvam and me [5].

Question 1. Under what conditions can a 3-edge-coloured finite complete graph be embedded in a finite group so that the red edges belong to the power graph, the green edges to the enhanced power graph but not the power graph, and the blue edges are not in the enhanced power graph?

Not every 3-edge coloring can satisfy this. The power graph is the comparability graph of a partial order (see Theorem 5.1), and so there are certain forbidden induced subgraphs, which must also be forbidden as red edges in the input graph if it is to be embeddable.

The final principle is the following. I call two graphs Γ_1 and Γ_2 *dual* if there is a bipartite graph Δ with no isolated vertices such that Γ_1 and Γ_2 are the induced subgraphs of the distance-2 graph on the two bipartite blocks. It is easy to see that dual graphs have the same number of connected components, and the diameters of the components differ by at most one.

For example, the non-generating graph of a group G (with the identity removed) is dual to the intersection graph of nontrivial proper subgroups of G. To see this, form the bipartite graph whose vertices are the nonidentity elements of G and the nontrivial proper subgroups, an element joined to a subgroup which contains it. Now two nonidentity elements which are joined in the non-generating graph lie in a proper subgroup, while two proper subgroups with nontrivial intersection share a nonidentity element.

Freedman [24], building on work of a number of authors including Csákány and Pollák [19], showed that if the intersection graph of nontrivial subgroups is connected then its diameter is at most 5, equality being realized by (among others) the Baby Monster simple group. This then gives a bound for the diameter of the non-generating graph, at least for simple groups.

Some open problems from this section:

Question 2. Describe the class of posets with 0 for which the complement of the zero-divisor graph is perfect or weakly perfect, and describe the graphs on algebraic structures covered by such a result.

Question 3. Investigate difference graphs defined by the hierarchy of graphs on groups. (Apart from the universality result given above, there is a paper on the difference of the power graph and enhanced power graph [8], and a paper on the difference of the non-generating graph and the commuting graph, concentrating on questions of connectedness [13]. There is also a graph, the *deep commuting graph*, interpolated between the enhanced power graph and the commuting graph [14], but it has not been much studied.)

Question 4. Are there other properties, apart from connectedness and diameter, which are shared by a dual pair of graphs?

Question 5. One can build a hierarchy of graphs from other algebraic structures such as rings. (One universality result was mentioned above.) Investigate this in a similar manner to the hierarchy of graphs on groups.

3. Digression: some related constructions

Based on an idea by Lavanya Selvaganesh, the authors of [6] gave a construction of what they called super graphs on groups. This extends the hierarchy into a second dimension, as follows.

Given a graph Γ defined on a group G, and an equivalence relation R on G, we define the R superΓ graph as follows. The vertices are, as usual, the elements of G; we join x to y if there are elements x' and y', R-equivalent to x and y respectively, which are joined in Γ. (By convention, each R-equivalence class induces a complete graph.) The cited paper studies this concept for the power graph, enhanced power graph, and commuting graph, with the relations of equality, conjugacy, and same order. This gives us nine graphs forming a two-dimensional hierarchy, of which two (the order superenhanced power graph and the order supercommuting graph) are equal for any group G, but all other pairs are in general distinct.

An alternative is to produce the *reduced* versions of these graphs, where the vertices are the R-equivalence classes, two vertices being joined if those classes contain elements which are joined in Γ. These graphs are constructed from the corresponding super graphs by shrinking each R-class to a single vertex, or alternatively by taking just one vertex from each R-class. These reduced graphs have been studied, in some cases, under other names. For example, the reduced conjugacy supercommuting and supersolvability graphs are known respectively as the *commuting* and *solvable conjugacy class graphs* of G. Thus, the vertices are the conjugacy classes, two vertices joined if those classes contain elements which generate an abelian (resp. solvable) group.

The solvable conjugacy class graph will appear in the next section.

Question 6. Lavanya Selvagenesh, who introduced the order superpower graph, has begun investigating its properties. But surely there is more to be said about all these graphs, in particular about relationships between them.

4. Theorems on groups proved using graphs

The best example of a theorem on groups proved using graphs is certainly the Brauer–Fowler theorem, discussed earlier. I will give a different example here. It does not strictly fit my pattern, since a graph appears in the statement of the theorem. But I include it because it strengthens an old theorem of Landau [32] from 1904. The theorem is in the paper [7].

Landau's theorem states:

Theorem 4.1. Given a positive integer k, there are only finitely many finite groups which have exactly k conjugacy classes.

This theorem asserts that the order of a finite group is bounded by a function of the number of conjugacy classes. No such bound holds for infinite groups; there are infinite groups with all nonidentity elements conjugate, so that there are just two conjugacy classes; and the upward Löwenheim–Skolem theorem of model theory then guarantees that there exist such groups of arbitrarily large cardinality, and hence infinitely many different groups, with just two conjugacy classes. But the proof of Landau's theorem is quite simple, and I will give it.

Consider the action of a finite group G on itself by conjugation. Each conjugacy class is an orbit, and the stabilizer of a point x is its centralizer $C_G(x)$. Thus, if $x_1, \dots, x_k$ are conjugacy class representatives, then

$$\sum_{i=1}^{k} \frac{|G|}{|C_G(x_i)|} = |G|.$$

Dividing by $|G|$ and letting $n_i = |C_G(x_i)|$, we have

$$\sum_{i=1}^{k} \frac{1}{n_i} = 1.$$

It is easily checked that, for given k, this equation has only finitely many solutions. If x_1 is the identity, then $n_1 = |G|$ is the largest of the n_i; so it can only take finitely many values.

Our strengthening uses the *solvable conjugacy class graph*, or for short SCC-graph, of G. As in the last section, its vertices are the conjugacy classes, and two vertices are joined if there are elements of those classes generating a solvable group. Landau's theorem asserts that $|G|$ is bounded by a function of the number of vertices of the SCC-graph. Our improvement states that the clique number of this graph will suffice:

Theorem 4.2. Given any positive integer k, there are only finitely many finite groups G whose solvable conjugacy class graph has clique number k.

In the case of Landau's theorem, there has been a lot of work on finding upper bounds for the order of a group with a given number of conjugacy classes. We do not have any such results for our theorem. The proof of the theorem is not elementary, involving the Classification of Finite Simple Groups (CFSG), but only in a rather low-key way.

I leave this section with a couple of questions.

Question 7. Find an explicit upper bound for $|G|$ in terms of the clique number of the SCC-graph of G.

Question 8. Does a similar theorem hold for the similarly defined *nilpotent conjugacy class graph* or *commutative conjugacy class graph*?

Question 9. Can our theorem be proved without recourse to CFSG?

5. Classes of groups defined by graphs

There are a number of ways we can use graphs to define classes of groups. It happens quite often that interesting classes arise in this way. Here are two which have been considered: there are others.

- We can take a significant class of finite graphs, such as perfect graphs, and a particular graph type Γ on groups, and ask for which groups G it is true that $\Gamma(G)$ belongs to the chosen class. If the graph type has the property that $\Gamma(H)$ is the induced subgraph of $\Gamma(G)$ on H for any subgroup H of G, then the class of groups is subgroup-closed.
- We can take two graph types Γ_1 and Γ_2, and ask for which groups G does $\Gamma_1(G) = \Gamma_2(G)$.

On the first method, I will mention the work of Britnell and Gill [11] classifying the quasi-simple groups whose commuting graph is perfect, and my work with Pallabi Manna and Ranjit Mehatari [15, 35] on groups whose power graph is a cograph. I will say a bit about the latter since this class of graphs will reappear in the next section.

A *cograph* is a graph Γ satisfying the following equivalent conditions:

- Γ does not contain an induced subgraph isomorphic to P_4, the 4-vertex path;
- Γ can be constructed from the one-vertex graph by the operations of complementation and disjoint union.

We will meet another characterization in the next section. Cographs form a well-studied class of graphs which has been rediscovered a number of times and given several different names, such as N-free graphs and hereditary Dacey graphs. In the cited papers we determine all nilpotent groups and all simple groups whose power graph is a cograph, and make some progress toward the complete classification.

I mention here the following theorem [3, 23]. The *comparability graph* of a partial order has as vertices the elements of the order, two vertices x and y joined if $x \leq y$ or $y \leq x$.

Theorem 5.1. For any finite group G, the power graph of G is the comparability graph of a partial order, and hence is perfect.

In fact, the directed power graph (with $x \preceq y$ if y is a power of x) is a *partial preorder* (a reflexive and transitive relation), and it is easy to see that the classes of comparability graphs of partial orders and partial preorders are the same. The perfectness of comparability graphs of partial orders is (the easy part of) Dilworth's theorem.

For a similar example in ring theory, it is shown in [5] that the zero-divisor graph of a finite commutative local ring whose maximal ideal is principal is a threshold graph. The class of rings whose zero-divisor graphs are threshold is wider than this, and is investigated in the paper.

Now I turn to the other method of defining classes of groups. I will give a few examples.

A finite group G is an *EPPO group* if all its elements have prime power order. This class was introduced by Higman in the 1950s, who classified the solvable EPPO groups. In the 1960s, Suzuki, in the course of discovering his family of simple groups, determined all the simple EPPO groups. The complete characterization was given by Brandl [9] in 1981, in a paper which is difficult to find now, and whose results were rediscovered several times.

The next theorem characterizes groups in which pairs of graphs in our hierarchy are equal [12, Propositions 3.1, 3.2].

Theorem 5.2. (a) A finite group is an EPPO group if and only if its power graph and enhanced power graph are equal.
(b) A finite group has all its Sylow subgroups cyclic or generalised quaternion if and only if its enhanced power graph and commuting graph are equal.
(c) A finite non-abelian group is minimal non-abelian (that is, all proper subgroups are abelian) if and only if its commuting graph and non-generating graph are equal.

I note that the second class of groups in the theorem can be determined using results of Glauberman [25] and Gorenstein and Walter [26], while the third class (minimal non-abelian groups) were determined by Miller and Moreno [36] in 1903.

If we include supergraphs, then several more classes of groups occur.

A finite group G is a *Dedekind group* if all its subgroups are normal. Dedekind [22] showed that a group is Dedekind if and only if either it is abelian, or it has the form $Q \times A \times B$, where Q is the quaternion group of order 8, A an abelian group of exponent 2, and B an abelian group of odd order.

A group G is a 2-*Engel group* if $[x, y, y] = 1$ for all $x, y \in G$, where $[x, y]$ is the commutator $x^{-1}y^{-1}xy$, and $[x, y, z] = [[x, y], z]$. These groups are equivalently defined by either of the conditions that conjugates commute, or that centralizers are normal subgroups. Any nilpotent group of class 2 is 2-Engel (these groups satisfy $[x, y, z] = 1$ for all x, y, z), and Hopkins [27] and Levi [33] showed that 2-Engel groups are nilpotent of class 3.

The next result is [6, Theorems 2 and 3].

Theorem 5.3. (a) For a finite group G, the following are equivalent:

- the power graph of G is equal to the conjugacy superpower graph;
- the enhanced power graph of G is equal to the conjugacy superenhanced power graph;
- G is a Dedekind group.

(b) For a finite group G, the commuting graph is equal to the conjugacy supercommuting graph if and only if G is a 2-Engel group.

Some open problems on this material:

Question 10. Determine the groups for which the power graph is a cograph.

Question 11. Determine the groups for which the enhanced power graph, or the commuting graph, is perfect. Similarly for other graph types and for other classes of graphs (such as chordal, split or threshold graphs). Note that, in a paper in preparation, Xuanlong Ma, Natalia Maslova and I [34] have determined the finite simple groups whose commuting graph is a cograph: they are the groups $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$ over finite fields of even order.

Question 12. Examine the differences of pairs of graphs in the hierarchy and determine for which groups these have various properties.

Several other questions along the same lines could be asked.

6. Some beautiful graphs from groups

This section grew out of my surprise when, with my colleague Colva Roney-Dougal, I was investigating generating graphs of finite groups. At one point we wondered about the automorphism group of the generating graph of the alternating group A_5, the smallest non-abelian finite simple group. The computer told us that the order of this automorphism group was 23482733690880. (Had we chosen the power graph instead, the answer would have been even more extreme, a number with 33 digits.)

Indeed, all the graphs associated with groups have extremely large automorphism groups. Usually, a large automorphism group indicates a beautiful symmetric graph; but here the reason is different. These graphs tend to have a lot of inessential rubbish which can be stripped away. Sometimes we are left with nothing, or something dull; but occasionally we do find a genuinely interesting and beautiful graph.

The reason for this behavior is that the groups we are considering contain many pairs of twins. Two vertices v and w in a graph are *twins* if they have the same neighbors, possibly excepting one another. We could distinguish two types of twins, closed twins (whose closed neighborhoods are equal, so that they are joined but their remaining neighbors coincide) or open twins (which are not joined but have the same neighbors); but this distinction is unimportant in what follows.

If v and w are twins in a graph, then the transposition interchanging them and fixing all other vertices is an automorphism of the graph. The twin relation is an equivalence relation, and so the automorphism group of Γ has a normal subgroup which is the direct product of symmetric groups on the equivalence classes. This is part of the explanation of the very large groups we see: for all of the graphs considered have many pairs of twins. If an element x of a group G has order $k > 2$, then for any integer d with $1 < d < k$ and $\gcd(k, d) = 1$, the elements x and x^d generate the same cyclic subgroup, and so are twins in the power graph, enhanced power graph, commuting graph, generating graph, and several other such graphs.

Twin reduction is the process of finding a pair of twins, identifying them, and repeating as long as twins exist. It can be shown that the final outcome of twin reduction (up to isomorphism) is independent of the order in which the reduction is done. I will call the resulting graph the *cokernel* of the original graph, for reasons which the next theorem makes clear. Recall the definition of a *cograph* from the preceding section. Now we have [12, Proposition 7.2]:

Theorem 6.1. The cokernel of a graph Γ consists of a single vertex if and only if Γ is a cograph.

This gives some added point to the problem of deciding for which groups G a particular graph defined on G is a cograph.

The power graph and the enhanced power graph are, in several senses, fairly close together: in their definitions, in the results about their equality, and in several other results. For example, with Swathi V V and M S Sunitha, I showed that they have the same matching number [16]. (The proof requires arguments in the spirit of classical matching theory.) This suggests that the difference $D(G)$ of these two graphs (whose edges are those of the enhanced power graph not in the power graph) will be fairly sparse, and may give rise to graphs with good network properties. In the paper [8], Sucharita Biswas, Angsuman Das, Hiranya Kishore Dey and I looked at the difference $D(G)$ from this point of view, primarily for simple groups.

Empirically, we found that four possibilities could occur (though we have no proof that these are all the possible behaviors):

- The simplest is the class of EPPO groups where, as we saw, the power graph and enhanced power graph are equal, so that $D(G)$ has no edges. As we saw, these groups were determined by Brandl [9]. Among simple groups we have a small finite number of groups $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$ and the group $\mathrm{PSL}(3, 4)$.
- Next come the groups whose difference graph is a cograph, so that the cokernel has just a single vertex. We determined the simple groups for which this condition holds. We get a few more groups $\mathrm{PSL}(2, q)$ and $\mathrm{Sz}(q)$.
- Next come groups where the cokernel of the difference graph consists of a number of small components, pairwise isomorphic. For the groups $\mathrm{PSL}(2, 23)$ and $\mathrm{PSL}(2, 25)$, we obtain respectively 253 or 325 copies of the graph $K_5 - P_4$. We do not know why the components are the same in the two cases.
- In the final case, we find an interesting connected graph with low valency and high girth.

We do not know whether we have seen the full range of behavior.

Here are the three interesting graphs that we found.

- $G = \mathrm{PSL}(3, 3)$. This group acts on the projective plane over the finite field with three elements, with 13 points and 13 lines. The cokernel of the difference graph has 169 vertices, which can be identified with the ordered pairs (P, L) where P is a point and L a line of the plane. These pairs are of two types, *flags* (with $P \in L$) and *antiflags* (with $P \notin L$); the adjacency is defined by the rule that the antiflag (P, L) is incident with the flag (Q, M) if $P \neq Q$, $L \neq M$, and $P \in M$, $Q \in L$. This graph is bipartite semiregular, with blocks of size 52 and 117; the valencies in the two blocks are 9 and 4. The graph has diameter 5 and girth 6.
- $G = M_{11}$. The cokernel has 385 vertices, and is semiregular bipartite, with bipartite blocks of sizes 165 and 220 and valencies 4 and 3. It has diameter and girth 10.
- $G = \mathrm{P\Gamma L}(2, 8)$, the smallest Ree group (which is not simple, but has a simple subgroup of index 3). The cokernel of $D(G)$ has 147 vertices, and is semiregular bipartite, with bipartite

blocks of sizes 63 and 84 with valencies 4 and 3; it has diameter 5 and girth 6.

Some questions on the material in this section:

Question 13. The main question is, of course, find more examples of beautiful and interesting graphs produced by this process.

Question 14. Can one describe in general the cokernels of the difference graphs for groups $\mathrm{PSL}(2, q)$ for prime powers q?

Question 15. Investigate other types of graphs on groups, and non-simple groups, in a similar way. (The power graph of M_{11} is considered in [31].)

Question 16. Can the behavior be more complicated than that we have observed? In particular, can interesting connected non-bipartite graphs be obtained from larger simple groups?

Question 17. In the interesting cases, we obtain bipartite graphs; by reversing the duality construction, each gives rise to a pair of graphs (the distance-2 graphs on the bipartite blocks). What properties do these graphs have?

Question 18. Is it true that, if G is a finte simple group and the cokernel of $D(G)$ is connected, then its automorphism group is the automorphism group of G? This is the case in the first two examples above. More generally, what can be said about the connectedness of the cokernel of $\Gamma(G)$, and its automorphism group, for simple (or arbitrary) groups G and any graph type?

Question 19. (A question for finite geometers or algebraic graph theorists.) The graph we found above for $\mathrm{PSL}(3, 3)$ can be defined for the projective plane over any finite field. What properties does it have? What is its automorphism group?

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