ALGEBRAIC COMBINATORICS

Marina Anagnostopoulou-Merkouri & Peter J. Cameron

Association schemes with given stratum dimensions: on a paper of Peter M. Neumann

Volume 6, issue 5 (2023), p. 1189-1210.
https://doi.org/10.5802/alco.307

Algebraic Combinatorics is published by The Combinatorics Consortium and is a member of the Centre Mersenne for Open Scientific Publishing
www.tccpublishing.org www.centre-mersenne.org
e-ISSN: 2589-5486

Algebraic Combinatorics
Volume 6, issue 5 (2023), p. 1189–1210
https://doi.org/10.5802/alco.307

Association schemes with given stratum dimensions: on a paper of Peter M. Neumann

Marina Anagnostopoulou-Merkouri & Peter J. Cameron

In memory of Peter Neumann: teacher, colleague, friend

Abstract In January 1969, Peter M. Neumann wrote a paper entitled "Primitive permutation groups of degree $3p$". The main theorem placed restrictions on the parameters of a primitive but not 2-transitive permutation group of degree three times a prime. The paper was never published, and the results have been superseded by stronger theorems depending on the classification of the finite simple groups, for example a classification of primitive groups of odd degree.

However, there are further reasons for being interested in this paper. First, it was written at a time when combinatorial techniques were being introduced into the theory of finite permutation groups, and the paper gives a very good summary and application of these techniques. Second, like its predecessor by Helmut Wielandt on primitive groups of degree $2p$, it can be re-interpreted as a combinatorial result concerning association schemes whose common eigenspaces have dimensions of a rather limited form. This result uses neither the primality of p nor the existence of a permutation group related to the combinatorial structure. We extract these results and give details of the related combinatorics.

1. Introduction

In 1956, Helmut Wielandt [18] proved the following result:

Theorem 1.1. *Let G be a primitive permutation group of degree $2p$, where p is prime. If G is not 2-transitive, then $n = 2a^2 + 2a + 1$ for some positive integer a, and G has rank 3 and subdegrees $a(2a+1)$ and $(a+1)(2a+1)$.*

The proof of this theorem is also given in Chapter 5 of his book [19]. It illustrates an extension of the methods of Schur rings using representation theory. He mentioned that, for $a = 1$, we have two examples: the groups S_5 and A_5, acting on the set of 2-element subsets of $\{1, \ldots, 5\}$.

Now it is possible to show that there are no others. For example, using the Classification of Finite Simple Groups, all the finite primitive rank 3 permutation groups have been determined [8, 10, 12], and the observation can be verified by checking the list.

Manuscript received 7th September 2022, revised 17th January 2023 and 18th March 2023, accepted 26th March 2023.

Keywords. association scheme, permutation group, strongly regular graph.

Acknowledgements. The research of the first author was supported by an Undergraduate Research Bursary, number XCLM18, from the London Mathematical Society.
The second author acknowledges the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme *Groups, representations and applications: new perspectives* (supported by EPSRC grant no. EP/R014604/1), where he held a Simons Fellowship.

However, there is more to be said. Wielandt's proof falls into two parts. The first involves showing that the permutation character of G decomposes as $1_G + \chi_1 + \chi_2$, where 1_G is the principal character of G and χ_1, χ_2 are irreducibles with degrees $p-1$ and p. It follows from this that G has rank 3 and is contained in the automorphism group of a strongly regular graph, having the property that the eigenvalues of its adjacency matrix have multiplicities 1, $p-1$, and p. Now the argument shows something much more general. Neither the existence of a rank 3 group of automorpisms nor the primality of p are needed.

First, a definition: a graph Γ is *strongly regular* with parameters (n, k, λ, μ) if it has n vertices, every vertex has k neighbours, and two vertices have λ or μ common neighbours according as they are joined by an edge or not. Every rank 3 group of even order is an automorphism group of a strongly regular graph, but not conversely; many strongly regular graphs have no non-trivial automorphisms. Any regular graph has the all-1 vector as an eigenvector; a regular graph is strongly regular if and only if its adjacency matrix, acting on the space orthogonal to the all-1 vector, has just two eigenvalues.

Theorem 1.2. *Let Γ be a strongly regular graph on $2n$ vertices, with the property that the eigenvalues of the adjacency matrix, on the space of vectors orthogonal to the all-1 vector, have multiplicities $n-1$ and n. Then either*

(a) *Γ is a disjoint union of n complete graphs of size 2, or the complement of this; or*
(b) *for some positive integer a, we have $n = 2a^2 + 2a + 1$, and up to complementation the parameters of the graph Γ are given by*

$$2n = (2a+1)^2 + 1, \quad k = a(2a+1), \quad \lambda = a^2 - 1, \quad \mu = a^2.$$

We are not aware of who first pointed this out. The result is given, for example, as [1, Theorem 2.20].

In the case $a = 1$, the complementary strongly regular graphs are the line graph of the complete graph K_5 and the Petersen graph. But, unlike in Wielandt's case, there are many others. For example, suppose that there exists a Steiner system $S(2, a+1, 2a^2 + 2a + 1)$. Then the strongly regular graph whose vertices are the blocks, two vertices adjacent if the corresponding blocks intersect, has the parameters given in the theorem. For example, when $a = 2$, the two Steiner triple systems on 13 points give non-isomorphic strongly regular graphs on 26 vertices. (We discuss examples further in the last section.)

Now to the subject of this paper. In 1969, Peter Neumann wrote a long paper [13] extending Wielandt's result from $2p$ to $3p$, where p is prime. His conclusion is that, if such a group is not 2-transitive, then p is given by one of three quadratic expressions in a positive integer a, or one of three sporadic values; the rank is at most 4, and the subdegrees are given in each case.

Like Wielandt's, Neumann's proof falls into two parts: first find the decomposition of the permutation character, and then in each case find the combinatorial implications for the structure acted on by the group. In contrast to Wielandt, the first part is much easier, since in the intervening time, Feit [3] had given a characterisation of groups with order divisible by p having a faithful irreducible representation of degree less than $p-1$. On the other hand, the second part is much harder; rather than just one possible decomposition of the permutation character, he finds eight potential decompositions, some of which require many pages of argument.

Again like Wielandt's, Neumann's conclusions have been superseded by results obtained using the classification of finite simple groups. For example, all the primitive permutation groups of odd degree have been classified [7, 11].

The paper was never published. It happened that both Leonard Scott and Olaf Tamaschke had produced similar results. There was a plan for Neumann and Scott to collaborate on a joint paper, but for unknown reasons this never happened. The authors are grateful to Leonard Scott [17] for providing a scan of Peter Neumann's original typescript together with some historical material about the proposed collaboration. The second author has re-typed the paper and posted it on the arXiv [14].

Our task is to produce a combinatorial version of this, as we have seen for Wielandt's theorem. We give some historical background to the theorem with some comments on the place of Neumann's paper in the introduction of combinatorial methods into the study of permutation groups, and to check in detail that his arguments give combinatorial results which do not depend on either the existence of a primitive group or the primality of p. Indeed we find some families of parameters which do not occur in Neumann's case since the number of vertices is even.

2. History

The 1960s saw a unification of combinatorial ideas which had been developed independently in three different areas of mathematics. In statistics, R C. Bose and his colleagues and students developed the concept of an *association scheme*. Extracting information from experimental results requires inversion of a large matrix, and Bose realised that the task would be much simpler if the matrix belonged to a low-dimensional subalgebra of the matrix algebra; requiring entries to be constant on the classes of an association scheme achieves this. In the former Soviet Union, Boris Weisfeiler and his colleagues were studying the graph isomorphism problem, and developed the concept of a *cellular algebra*, an isomorphism invariant of graphs, to simplify the problem, and an algorithm, the *Weisfeiler–Leman algorithm*, to construct it. In Germany, Helmut Wielandt was extending the method of *Schur rings* to study permutation groups with a regular subgroup; by using methods from representation theory he was able to dispense with the need for the regular subgroup. These techniques were further developed by Donald Higman in the USA, under the name *coherent configuration.*

The three concepts are very closely related. We begin with Higman's definition. A *coherent configuration* consists of a set Ω together with a set $\{R_1, R_2, \ldots, R_r\}$ of binary relations on Ω with the properties

(a) $\{R_1, \ldots, R_r\}$ form a partition of $\Omega \times \Omega$;
(b) some subset of $R_1, \ldots, R_r$ is a partition of the *diagonal* $\{(\omega, \omega) : \omega \in \Omega\}$ of Ω^2;
(c) the converse of each relation R_i is another relation in the set;
(d) for any triple (i, j, k) of indices, and any $(\alpha, \beta) \in R_k$, the number p_{ij}^k of $\gamma \in \Omega$ such that $(\alpha, \gamma) \in R_i$ and $(\gamma, \beta) \in R_j$ depends only on (i, j, k) and not on the choice of $(\alpha, \beta) \in R_k$.

The number r is the *rank* of the configuration. Combinatorially, a coherent configuration is a partition of the edge set of the complete directed graph with loops.

If G is a permutation group on Ω, and we take the relations R_i to be the orbits of G on Ω^2, we obtain a coherent configuration. This was Higman's motivating example, which he called the *group case.* Not every coherent configuration falls into the group case; indeed, our task is to extend Neumann's results from the group case to the general case.

A coherent configuration is *homogeneous* if the diagonal is a single relation. In the group case, this means that the group is transitive. All the configurations in this paper will be homogeneous.

The notion of a cellular algebra is the same apart from an inessential small difference (the diagonal is replaced by some equivalence relation). Association schemes form a special case, where all the relations R_i are symmetric. It follows that, in an association scheme, the diagonal is a single relation. (Statisticians deal with symmetric matrices, for example covariance matrices.)

A coherent configuration with rank 2 is *trivial*: one relation is the diagonal, the other is everything else. For rank 3, we can suppose without loss that R_1 is the diagonal. There are then two possibilities:

- R_3 is the converse of R_2. Then R_2 is a *tournament* (an orientation of the edges of the complete graph on Ω); condition (d) shows that it is a *doubly regular* tournament [16].
- R_2 and R_3 are symmetric. Then each is the edge set of a graph, and these graphs are *strongly regular* [1, Chapter 2].

The definition of coherent configuration has an algebraic interpretation. Let A_i be the *adjacency matrix* of the relation R_i, the $\Omega \times \Omega$ matrix with (α, β) entry 1 if $(\alpha, \beta) \in R_i$. Then $A_1, \ldots, A_r$ are zero-one matrices satisfying the following conditions:

(a) $A_1 + \cdots + A_r = J$, the all-1 matrix;
(b) there is a subset of these matrices whose sum is the identity I;
(c) for any i there is a j such that $A_i^\top = A_j$;
(d) $A_i A_j = \displaystyle\sum_{k=1}^{r} p_{ij}^k A_k$.

Condition (d) says that the linear span over $\mathbb{C}$ of $A_1, \ldots, A_r$ is an algebra (closed under multiplication), and condition (c) implies that this algebra is semi-simple. In the group case, it is the *centraliser algebra* of the permutation group, consisting of matrices which commute with every permutation matrix in the group. In the case of association schemes, it is known as the *Bose–Mesner algebra* of the scheme. In this case, all the matrices are symmetric, the algebra is commutative, and we can work over $\mathbb{R}$. In the group case, the centraliser algebra is commutative if and only if the permutation character is multiplicity-free.

If the algebra is commutative, then the matrices are simultaneously diagonalisable; the common eigenspaces are called the *strata* of the configuration. In the rank 3 case where we have a strongly regular graph and its complement, the stratum dimensions are simply the multiplicities of the eigenvalues. We occasionally extend the use of the word “stratum” to the non-commutative case, where it means a submodule for the algebra spanned by the matrices which is maximal with respect to being a sum of isomorphic submodules.

In all cases which arise in Peter Neumann’s paper, the algebra turns out to be commutative, although there are two potential cases where the permutation character is not multiplicity-free; both of these are eliminated.

It seems clear to the authors that, had the paper been published in 1969, it would have been very influential: it provides both a clear account of the theory and how it can be used to study permutation groups, and also a non-trivial example of such an application. The second author of the present paper read it at the start of his DPhil studies in Oxford under Peter Neumann’s supervision, and considers himself fortunate to have been given such a good grounding in this area; he has worked on the interface of group theory and combinatorics ever since.

3. The results

The main theorems in this paper are the following. They are numbered to correspond to the eight cases in Neumann’s paper.

Theorem 3.1. *Let $\mathcal{A} = \{I_n, A_1, A_2\}$ be a coherent configuration of $n \times n$ matrices. If the eigenvalues of A_1 have multiplicities $1, \frac{n-1}{2}, \frac{n-1}{2}$ then one of the two following cases must hold:*

- $n \equiv 1 \pmod 4$ *and A_1 and A_2 are the adjacency matrices of conference graphs;*
- $n \equiv 3 \pmod 4$ *and A_1 and A_2 are the adjacency matrices of doubly regular tournaments.*

Theorem 3.2. *Let G be a strongly regular graph on $3n$ vertices. If the multiplicities of the eigenvalues of G are $1, n, 2n-1$ then G or its complement have the following parameters in terms of a non-negative integer a:*

- $3n = 144a^2 + 54a + 6$, $k_1 = 48a^2 + 14a + 1$, $\lambda = 16a^2 + 6a, \mu = 16a^2 + 2a$;
- $3n = 144a^2+90a+15$, $k_1 = 48a^2+34a+6, \lambda = 16a^2+10a+1, \mu = 16a^2+14a+3$;
- $3n = 144a^2 + 198a + 69$, $k_1 = 48a^2 + 62a + 20, \lambda = 16a^2 + 22a + 7, \mu = 16a^2 + 18a + 5$;
- $3n = 144a^2 + 234a + 96$, $k_1 = 48a^2 + 82a + 35, \lambda = 16a^2 + 26a + 10, \mu = 16a^2 + 30a + 14$.

Theorem 3.3. *Let G be a strongly regular graph on $3n$ vertices. If the multiplicities of the eigenvalues of G are $1, 2n, n-1$ then either G or its complement is a disjoint union of n copies of K_3 or G or its complement have the following parameters for some non-negative integer a:*

- $3n = 9a^2 + 9a + 3$, $k_1 = 3a^2 + 5a + 2, \lambda = a^2 + 3a + 1, \mu = (a+1)^2$;
- $3n = 9a^2 + 9a + 3$, $k_1 = 3a^2 + a, \lambda = a^2 - a - 1, \mu = a^2$.

Theorem 3.4. *Let $\mathcal{A} = \{I_{3n}, A_1, A_2, A_3\}$ be a coherent configuration of $3n \times 3n$ matrices. If the multiplicities of the eigenvalues of A_1, A_2, A_3 are $1, n, n, n-1$ then one of the following hold:*

- $A_2 = A_3^T$ *and the row sums of $A_1, A_2,$, and A_3 are $n-2a-1, n+a$, and $n+a$ respectively for some even integer a;*
- $A_2 = A_3^T$ *and the row sums of A_1, A_2, and A_3 are $n+2a+1, n-a-1$, and $n-a-1$ respectively for some odd integer a;*
- *All matrices are symmetric and the row sums of A_1, A_2, A_3 are $n+2a+1, n-a-1$, and $n-a-1$ respectively for some non-negative integer a;*
- *all matrices are symmetric and the row sums of A_2, A_3 and A_4 are 2, $n-1$ and $2(n-1)$;*
- $A_3 = A_2^\top$ *and the row sums of A_2, A_3 and A_4 are 1, 1 and $3(n-1)$.*

Theorem 3.5. *There exists no coherent configuration $\mathcal{A} = \{I_{3n}, A_1, A_2, A_3, A_4, A_5\}$ of $3n \times 3n$ matrices such that the multiplicities of the eigenvalues of $A_1, \dots, A_5$ are $1, n, n, n-1$.*

Theorem 3.6. *There is no strongly regular graph on $3n$ vertices with eigenvalue multiplicities $1, n+1, 2(n-1)$.*

Theorem 3.7. *Let $\mathcal{A} = \{I_{3n}, A_1, A_2, A_3\}$ be a coherent configuration of $3n \times 3n$ matrices. If the eigenvalues of $A_1, \dots, A_3$ have multiplicities $1, n+1, n-1, n-1$, then $\mathcal{A}$ is an association scheme and one of the following hold:*

- $n = 7$ *and the row sums of A_1, A_2, A_3 are $4, 8$, and 8;*
- $n = 19$ *and the row sums of A_1, A_2, A_3 are $6, 20$, and 30;*
- $n = 31$ *and the row sums of A_1, A_2, A_3 are $32, 40$, and 20.*

Theorem 3.8. *There exists no coherent configuration $\mathcal{A} = \{I_{3n}, A_1, A_2, A_3, A_4, A_5\}$ of $3n \times 3n$ matrices, where the multiplicities of the eigenvalues of $A_1, \dots, A_5$ are $1, n+1, n-1, n-1$.*

4. The proofs

4.1. A lemma.

We start with a lemma that will be used throughout the paper.

Lemma 4.1. *Let $\mathcal{A}$ be a homogeneous coherent configuration on n points. Suppose that the dimension of a non-trivial stratum for $\mathcal{A}$ is at least $n/3-1$. Then one of the following happens:*

(a) *One of the relations in $\mathcal{A}$ has at least $n/3$ connected components.*
(b) *Any matrix in $\mathcal{A}$ has the property that any eigenvalue λ apart from the row sum r satisfies $|\lambda| < r$.*

Proof. We use the *Perron–Frobenius Theorem*, see [4]. For any non-negative matrix A, one of the following holds:

- Under simultaneous row and column permutations, A is equivalent to a matrix of the form $\begin{pmatrix} B & O \\ O & C \end{pmatrix}$. In our case the constancy of the row sum r means that r has multiplicity equal to the number of connected components; so there are at least $n/3$ connected components, and (a) holds.
- A is decomposable, that is, under simultaneous row and column permutations it is equivalent to a matrix of the form $\begin{pmatrix} B & X \\ O & C \end{pmatrix}$, where $X \neq O$. But this contradicts the fact that the row sum is constant.
- A is imprimitive, that is, equivalent under simultaneous row and column permutations to a matrix of the form

$$\begin{pmatrix} O & B_1 & \dots & \dots & 0 \\ O & O & B_2 & \dots & O \\ \dots & \dots & \dots & \dots & \dots \\ B_t & O & O & \dots & O \end{pmatrix}.$$

  But then $re^{2\pi \mathrm{i}k/t}$ is a simple eigenvalue for $k = 0, 1, \dots, t-1$, contrary to assumption.
- A is primitive. Then the Perron–Frobenius Theorem asserts that there is a single eigenvalue with largest absolute value, as required.

□

4.2. Proof of Theorem 3.1.

We first prove a lemma about strongly regular graphs that will be used in the proof of Theorem 3.1.

Firstly we define a special type of strongly regular graphs. A *conference graph* is a strongly regular graph on v vertices with parameters $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$.

Lemma 4.2. *Let G be a strongly regular graph with parameters (n, k, λ, μ) and let k, r, s be the eigenvalues of the adjacency matrix of G. If r and s have equal multiplicities then G is a conference graph.*

Proof. It is known for a strongly regular graphs that the multiplicities of r and s are

$$f, g = \frac{1}{2}(n - 1 \pm \frac{(n-1)(\mu-\lambda) - 2k}{\sqrt{(\mu-\lambda)^2 - 4(k-\mu)}})$$

respectively. Hence, if $f = g$ then it follows that

$$(n-1)(\mu-\lambda) - 2k = -(n-1)(\mu-\lambda) + 2k \Rightarrow 2k = (n-1)(\mu-\lambda)$$

and thus G is a conference graph, as required. Moreover, $f = g = \frac{n-1}{2}$. □

Proof of Theorem 3.1. Since $\mathcal{A}$ is a coherent configuration, $A_0 + A_1 + A_2 = J_n$ and moreover $A_i^T = A_j$ for $i, j \in \{1, 2\}$. Hence, there are two possibilities. Either $A_i = A_i^T$ for $i \in \{1, 2\}$ or $A_i = A_j^T$ for $i, j \in \{1, 2\}$ and $i \neq j$.

In the first case, the graphs with adjacency matrices A_1 and A_2 are undirected. Moreover, since A_1 and A_2 are symmetric, $\mathcal{A}$ is an association scheme and hence those graphs are strongly regular and one is the complement of the other. It follows by Lemma 4.2 that A_1 and A_2 are the adjacency matrices of conference graphs and in fact two copies of the same conference graph. Moreover, for a conference graph to exist, it is known that $n \equiv 1 \pmod 4$.

In the second case, since $\mathcal{A}$ is a coherent configuration, it follows that A_1 and A_2 must have constant row and column sums and hence their digraphs are regular. Let G_1, G_2 be the digraphs with adjacency matrices A_1 and A_2 respectively and V be the vertex set of those digraphs. For $u, v \in V$, we write $u \to_{G_1} v$ if v is an out-neighbour of u in G_1 and similarly $u \to_{G_2} v$ if v is an out-neighbour of u in G_2. Since $A_1 + A_2 = J - I$, it follows that $u \to_{G_1} v \iff u \not\to_{G_2} v$ and vice versa and also that either $(A_k)_{ij} = 1$ or $(A_k)_{ji} = 1$ for $k \in \{1, 2\}$. Hence, G_1 and G_2 are regular tournaments. Also, notice that since $\mathcal{A}$ is a coherent configuration, it follows that for $m, n \in \{1, 2\}, m \neq n$, there exists a constant p_{mn}^m such that for any $i, j \in V$, such that $(A_m)_{ij} = 1$, $|\{k \mid (A_m)_{ik} = 1, (A_n)_{kj} = 1\}| = |\{k \mid (A_m)_{ik} = 1, (A_n)_{jk} = 1\}| = p_{mn}^m$. Hence, both G_1 and G_2 are doubly regular, and it is known that $n \equiv 3 \pmod 4$ for doubly regular tournaments. □

4.3. Proof of Theorem 3.2.

Proof. Let A_1 be the adjacency matrix of G and A_2 be the adjacency matrix of its complement. Since G is strongly regular, the eigenvalues of A_1 and A_2 have the same multiplicities. Moreover, if A_1 has eigenvalues k_1, r_1, s_1 then A_2 has eigenvalues $k_2 = 3n - k_1 - 1, r_2 = -1 - r_1, s_2 = -1 - s_1$. We know that for $i \in \{1, 2\}$

$$\mathrm{Tr}(A_i) = k_i + nr_i + (2n-1)s_i = 0$$

Reducing modulo n gives that $k_i \equiv s_i \pmod n$. Therefore, since $k_i > s_i$ by Lemma 4.1, it follows that $k_i - s_i = \epsilon_i n$ for $\epsilon_i \in \{1, 2\}$. Therefore,

$$k_1 + k_2 - s_1 - s_2 = (\epsilon_1 + \epsilon_2)n \Rightarrow 3n - 1 - s_1 + 1 + s_1 = (\epsilon_1 + \epsilon_2)n \Rightarrow \epsilon_1 + \epsilon_2 = 3.$$

Assume without loss of generality that $\epsilon_1 = 1$ and $\epsilon_2 = 2$. Then, $k_1 = n + s_1$ and also

$$n + s_1 + nr_1 + (2n-1)s_1 = 0 \Rightarrow r_1 = -1 - 2s_1.$$

Also, we have that

$$\mathrm{Tr}(A_1^2) = k_1^2 + nr_1^2 + (2n-1)s_1^2 = 3nk_1.$$

Appropriate substitution gives

$$(n + s_1)^2 + n(1 + 2s_1)^2 + (2n-1)s_1^2 = 3n(n + s_1)$$

which simplifies to

$$6s_1^2 + 3s_1 + 1 - 2n = 0.$$

Therefore,

$$s_1 = \frac{1}{4}\left(-1 \pm \sqrt{\frac{16n-5}{3}}\right).$$

Since G is strongly regular and its eigenvalues have different multiplicities, it is not a conference graph, and hence its eigenvalues are integers. Hence, $16n - 5 = 3b^2$ for some non-negative integer b. This gives us that $3b^2 + 5 \equiv 0 \pmod{16}$. It follows that $b = 3, 5, 11$ or $13 \pmod{16}$. We therefore need to examine the following four cases:

Case 1: $b = 16a + 3$.

In this case we get:

$$16n = 3(16a+3)^2 + 5 \Rightarrow n = 48a^2 + 18a + 2.$$

and $s_1 = -4a - 1$. Notice that only the negative solution works, since $16a + 2$ is not divisible by 4. Consequently $k_1 = 48a^2 + 14a + 1$. We also get $r_1 = 8a + 1$

Now, using the formulae for the eigenvalues of strongly regular graphs, namely

$$r_1, s_1 = \frac{1}{2}\left((\lambda - \mu) \pm \sqrt{(\lambda-\mu)^2 + 4(k_1 - \mu)}\right)$$

we get

$$\begin{aligned} \lambda - \mu &= r_1 + s_1 \\ 4\mu &= (\lambda-\mu)^2 - (r_1 - s_1)^2 + 4k_1. \end{aligned}$$

Solving this system we obtain $\lambda = 16a^2 + 6a$ and $\mu = 16a^2 + 2a$.

Case 2: $b = 16a + 5$.

In this case we get:

$$16n = 3(16a+5)^2 + 5 \Rightarrow n = 48a^2 + 30a + 5.$$

and $s_1 = 4a + 1$. Hence, $k_1 = 48a^2 + 34a + 6$. We also get $r_1 = -8a - 3$
As above, knowing r_1, s_1 we can obtain λ and μ which in this case are equal to $16a^2 + 10a + 1$ and $16a^2 + 14a + 3$ respectively.

Case 3: $b = 16a + 11$.

In this case we get:

$$16n = 3(16a+11)^2 + 5 \Rightarrow n = 48a^2 + 66a + 23.$$

and $s_1 = -4a - 3$. Hence, $k_1 = 48a^2 + 62a + 20$. Also, $r_1 = 8a + 5$ and routine calculation as above gives $\lambda = 16a^2 + 22a + 7, \mu = 16a^2 + 18a + 5$.

Case 4: $b = 16a + 13$.

In this case we get:

$$16n = 3(16a+13)^2 + 5 \Rightarrow n = 48a^2 + 78a + 32.$$

and $s_1 = 4a+3$. Hence, $k_1 = 48a^2 + 82a + 35$, $r_1 = -8a - 7$, $\lambda = 16a^2 + 26a + 10, \mu = 16a^2 + 30a + 14$. □

4.4. Proof of Theorem 3.3.

Proof. Let A_1 be the adjacency matrix of G and A_2 be the adjacency matrix of its complement. Since G is strongly regular we know that the eigenvalues of A_1 and A_2 have the same multiplicities. Also, if A_1 has eigenvalues k_1, r_1, s_1, then A_2 has eigenvalues $k_2 = 3n - k_1 - 1, r_2 = -1 - r_1, s_2 = -1 - s_1$. We know that for $i \in \{1, 2\}$

$$\mathrm{Tr}(A_i) = k_i + 2nr_i + (n-1)s_i = 0.$$

Reducing modulo n gives that $k_i \equiv s_i \pmod{n}$, and since by Lemma 4.1 either one of A_1, A_2 is the disjoint union of n copies of K_3 or $k_i > |s_i|$. In the second case, it follows that $k_i - s_i = \epsilon_i n$ for $\epsilon_i \in \{1, 2\}$. Also, as before, $\epsilon_1 + \epsilon_2 = 3$ and hence we may suppose without loss of generality that $\epsilon_1 = 1$ and $\epsilon_2 = 2$. Then, $k_1 = n + s_1$ and $r_1 = \frac{-s_1 - 1}{2}$. We therefore get

$$\mathrm{Tr}(A_1^2) = (n+s_1)^2 + 2n\left(\frac{s_1+1}{2}\right)^2 + (n-1)s_1^2 = 3n(n+s_1).$$

and simplifying gives $3s_1^2 = 4n - 1$. Therefore,

$$s_1^2 = \frac{4n-1}{3}.$$

We can thus write s_1^2 as $(2a+1)^2$ for some $a \geqslant 0$ and we get

$$(2a+1)^2 = \frac{4n-1}{3} \Rightarrow n = 3a^2 + 3a + 1$$

and $s_1 = \pm(2a+1)$. We therefore get the following cases:

Case 1: $s_1 = 2a + 1$.

In this case we get $k_1 = 3a^2 + 5a + 2$ and $r_1 = -a - 1$, and computing λ and μ as in the proof of Theorem 3.2 we obtain $\lambda = a^2 + 3a + 1$ and $\mu = (a+1)^2$.

Case 2: $s_1 = -2a - 1$.

Here, routine calculation gives $k_1 = 3a^2 + a$, $r_1 = a$, $\lambda = a^2 - a - 1, \mu = a^2$. □

4.5. Proof of Theorem 3.6.

Proof. Suppose for a contradiction that there exists such a strongly regular graph, and let A_1 be its adjacency matrix and A_2 be the adjacency matrix of its complement and suppose that k_1, r_1, s_1 and k_2, r_2, s_2 are the eigenvalues of A_1 and A_2 respectively. Then, for $i \in \{1,2\}$ we get

$$\mathrm{Tr}(A_i) = k_i + (n+1)r_i + 2(n-1)s_i = 0$$

and

$$\mathrm{Tr}(A_i^2) = k_i^2 + (n+1)r_i^2 + 2(n-1)s_i^2 = 3nk_i.$$

Reducing modulo n gives

$$k_i \equiv 2s_i - r_i \pmod{n}$$
$$k_i^2 \equiv 2s_i^2 - r_i^2 \pmod{n}.$$

Hence, $(2s_i - r_i)^2 \equiv 2s_i^2 - r_i^2 \pmod{n}$. By routine calculation, it follows that $s_i \equiv r_i \pmod{n}$ and consequently $k_i \equiv r_i \pmod{n}$. Therefore, $k_i = \epsilon_i n + r_i$ and $s_i = \eta_i n + r_i$ for some $\epsilon_i, \eta_i \in \{1,2\}$.

Substituting into the trace equations and reducing modulo n^2 gives

$$\epsilon_i n + r_i + (n+1)r_i + 2(n-1)r_i - 2\eta_i n \equiv 0 \pmod{n^2}$$
$$2\epsilon_i n r_i + r_i^2 + (n+1)r_i^2 + 2(n-1)r_i^2 - 4r_i\eta_i n \equiv 3nr_i \pmod{n^2}.$$

We now collect terms and divide by n and we get

$$\epsilon_i + 3r_i - 2\eta_i \equiv 0 \pmod{n}$$
$$3r_i^2 + r_i(2\epsilon_i - 4\eta_i - 3) \equiv 0 \pmod{n}.$$

Since $1 + r_1 + r_2 = 0$ it cannot be the case that both r_1 and r_2 are divisible by n. Hence, interchanging A_1 and A_2 if necessary we may assume that $r_1 \not\equiv 0 \pmod{n}$. Then,

$$3r_1 \equiv 2\eta_1 - \epsilon_1 \pmod{n}$$
$$3r_1 \equiv 4\eta_1 - 2\epsilon_1 + 3 \pmod{n}.$$

Eliminating $2\eta_1 - \epsilon_1$ gives $r_1 \equiv -1 \pmod{n}$. Therefore, since $k_1 \equiv r_1 \pmod{n}$, either $k_1 = n-1$ or $k_1 = 2n-1$. If $k_1 = n-1$, then since $r_1 \equiv s_1 \equiv -1 \pmod{n}$ and by Lemma 4.1 $|r_1| < k_1$ and $|s_1| < k_1$, it follows that $r_1 = s_1 = -1$. However, by looking at the formulae for r_1 and s_1 for a strongly regular graph, we deduce that $r_1 \neq s_1$, a contradiction. Similarly, if $k_1 = 2n - 1$, then $k_2 = n$ which forces $r_2 = s_2 = 0$, again a contradiction. Hence, there is no strongly regular graph with those eigenvalue multiplicities. □

4.6. Proof of Theorem 3.4.

Proof. Suppose first that at least some A_i corresponds to a disconnected graph. There are two possibilities: either A_2 (say) is an undirected graph which consists of n disjoint triangles; or $A_3 = A_2^\top$, corresponding to a converse pair of directed triangles. In the second case, $I, A_2 + A_3, A_4$ satisfy the hypotheses of Theorem 3.3, and the last statement holds. So suppose that the first possibility occurs.

Let $T_1, \ldots, T_n$ be the connected components of A_2 (each a triangle), and let $T_i = \{v_{i1}, v_{i2}, v_{i3}\}$. Now the common eigenspaces of I and A_2 have dimensions n and $2n$. The first of these has the form $V_0 \oplus V_1$, where V_0 is the space of constant vectors and

$$V_1 = \sum_{i=1}^{n} c_i(v_{i1} + v_{i2} + v_{i3})$$

where $\sum_1^n c_i = 0$. Then V_1 is an eigenspace for A_2 as well; suppose that the eigenvalue is λ. Take a vector of the above form with $c_i = 1$, $c_j = -1$, and $c_k = 0$ for $k \neq i, j$. The fact that it is an eigenvector of A_2 shows that v_{i1} is joined to $-\lambda$ of the vertices of T_j. So this number is independent of the choice of i and j and the particular vertex $v_{i1} \in T_i$ chosen; that is, in the graph A_2, each vertex of T_i is joined to $-\lambda$ vertices of T_j for $j \neq i$. We can suppose that $\lambda = -1$, so each vertex of T_i is joined to one vertex of T_j by an A_2 edge, and to two by A_3 edges. Hence A_2 has valency $n-1$ and eigenvalue -1 on W; for A_3 these numbers are $2(n-1)$ and -2. Thus we have the penultimate case in the Theorem. So we may suppose that all the orbital graphs are connected.

Let k_i, r_i, s_i, t_i be the eigenvalues of A_i, $i \in \{1,2,3\}$, with multiplicities $1, n, n, n-1$ respectively. Firstly notice that t_i must be a rational integer and r_i and s_i must either both be rational integers or algebraically conjugate algebraic integers. Then, we get

$$\mathrm{Tr}(A_i) = k_i + nr_i + ns_i + (n-1)t_i = 0.$$

Hence, n must divide $k_i - t_i$, and since by Lemma 4.1 and the connectivity assumption $k_i > t_i$, it follows that $k_i = \epsilon_i n + t_i$ for some $\epsilon_i > 0$. Moreover, by [13, Equation (6.9)], $\epsilon_1 + \epsilon_1 + \epsilon_3 = 3$ and hence $\epsilon_i = 1$ for all $i \in \{1,2,3\}$. Thus, $k_i = n + t_i$.

There are now two cases to consider. Either all matrices are symmetric or two of them, say A_2 and A_3 without loss of generality are such that $A_2^T = A_3$. We first consider the second case. In this case the eigenvalues of A_2 and A_3 are the same. Hence, $t_2 = t_3$ and either $r_2 = r_3$ and $s_2 = s_3$ or $r_2 = s_3$ and $r_3 = s_2$. Notice that the algebra spanned by the matrices of this coherent configuration is commutative and therefore A_2 and A_3 can be simultaneously diagonalised. Let U be the matrix that simultaneously reduces A_2 and A_3. If $r_2 = r_3$ and $s_2 = s_3$ then $U^{-1}A_2U = U^{-1}A_3U$, which implies that $A_2 = A_3$, a contradiction. Hence, $r_2 = s_3$ and $r_3 = s_2$.

Now adding A_2 and A_3 together produces an association scheme of the type arising in Theorem 3.3. Hence, $n = 3a^2+3a+1$ and either $k_1 = n-2a-1$ and $k_2 = k_3 = n+a$ or $k_1 = n + 2a + 1$ and $k_2 = k_3 = n - a - 1$.

We now show that if $k_1 = n - 2a - 1$ then a is even and if $k_1 = n + 2a + 1$ then a is odd. In the first case, the remaining eigenvalues of A_1, A_2, and A_3 are as shown below:

$$\begin{aligned} r_1 = a, s_1 = a, t_1 = -2a - 1 \\ r_2 = r, s_2 = s, t_2 = a \\ r_3 = s, s_3 = r, t_3 = a \end{aligned}$$

where $r + s = -a - 1$. Now [13, Equation (6.7)] gives

$$rs = \frac{1}{2}(2n - a - a^2) = \frac{1}{2}(5a + 2)(a + 1)$$

and [13, Equation (6.8)] gives

$$3n(n + a)p_{22}^3 = (n + a)^3 + nrs(r + s) + (n - 1)a^3.$$

where p_{22}^3 is as defined on page 3. Eliminating rs and simplifying gives $p_{22}^3 = a^3 + \frac{3a}{2}$ and since $p_{22}^3 \in \mathbb{Z}$, a must be even.

In the second case, the eigenvalues of A_1, A_2, and A_3 are the ones given below:

$$\begin{aligned} r_1 = -a - 1, s_1 = -a - 1, t_1 = 2a + 1 \\ r_2 = r, s_2 = s, t_2 = -a - 1 \\ r_3 = s, s_3 = r, t_3 = -a - 1 \end{aligned}$$

where $r + s = 1$ by [13, Equation (6.6)]. And [13, Equation (6.7)] gives $rs = \frac{1}{2}a(5a+3)$ and from [13, Equation (6.8)] we get

$$3n(n - a - 1)p_{22}^3 = (n - a - 1)^3 + nrs(r + s) - (n - 1)(a + 1)^3.$$

Simplifying gives $p_{22}^3 = a^2 + \frac{a-1}{2}$, and since $p_{22}^3 \in \mathbb{Z}$, it follows that a is odd, as claimed.

We now consider the symmetric case. We get the following equations

$$\begin{aligned} s_i + r_i &= -1 - t_i \\ \mathrm{Tr}(A_i^2) = k_i^2 + nr_i^2 + ns_i^2 + (n - 1)t_i^2 &= 3nk_i \\ (t_i + n)^2 + nr_i^2 + ns_i^2 + nt_i^2 - t_i^2 &= 3n(n + t_i) \\ r_i^2 + s_i^2 &= -t_i^2 + t_i + 2n. \end{aligned}$$

From this we get $2r_is_i = 1 + t_i + 2t_i^2 - 2n$ and hence we deduce that t_i is odd. Also, we can calculate r_i and s_i and we find that $r_i, s_i = \frac{1}{2}(-1 - t_i \pm \sqrt{4n - 1 - 3t_i^2})$. Without loss of generality we set

$$\begin{aligned} r_i &= \frac{1}{2}\left(-1 - t_i + \sqrt{4n - 1 - 3t_i^2}\right) \\ s_i &= \frac{1}{2}\left(-1 - t_i - \sqrt{4n - 1 - 3t_i^2}\right). \end{aligned}$$

Since A_i is symmetric for all $i \in \{1, 2, 3\}$, it has real eigenvalues, and therefore

$$3t_i^2 \leqslant 4n - 1. \tag{1}$$

Now, from [13, Equation (6.9)] we get

$$\begin{cases} t_1 + t_2 + t_3 = -1 \\ \sqrt{4n - 1 - 3t_1^2} + \sqrt{4n - 1 - 3t_2^2} + \sqrt{4n - 1 - 3t_3^2} = 0. \end{cases} \tag{2}$$

Now eliminating t_3 and rationalising gives us

$$\begin{aligned} t_1^2(3t_2 + 2n + 1) + t_1(3t_2^2 + 2nt_2 + 4t_2 + 2n + 1) \\ +(2n + 1)(t_2^2 + t_2) - 2n(n - 1) = 0. \end{aligned}$$

Notice that

$$3t_2^2 + 2nt_2 + 4t_2 + 2n + 1 = (3t_2 + 2n + 1)(t_2 + 1).$$

Therefore, $3t_2+2n+1$ divides $(2n+1)(t_2^2+t_2)-2n(n-1)$. Now consider the equation

$$2(2n+1)(t_2^2 + t_2) - 4n(n-1) \equiv 0 \pmod{3t_2 + 2n + 1}.$$

If we eliminate n from the equation, we deduce that $3t_2 + 2n + 1$ must divide $3(t_2 + 1)^2(2t_2 + 1)$.

Notice that there is complete symmetry between t_1, t_2, and t_3. Hence, we deduce that

$$3(t_i + 1)^2(2t_i + 1) \equiv 0 \pmod{3t_i + 2n + 1} \tag{3}$$

for all $i \in \{1, 2, 3\}$.

Using the equation for $\mathrm{Tr}(A_i^3)$ we deduce that $3nk_i$ must divide $k_i^3 + n(r_i^3 + s_i^3) + (n-1)t_i^3$. Substitution for k_i, r_i, s_i in terms of t_i and algebraic manipulation gives

$$2n^2 - 6n + 6t_i^2 + 2t_i^3 + (1 + t_i)(4t_i^2 - t_i + 1) \equiv 0 \pmod{6(n + t_i)}.$$

Reducing modulo $2(n + t_i)$, we have $2n^2 - 6n \equiv 2t_i(t_i + 3)$, and after simplifying we deduce that

$$(t_i + 1)(2t_i + 1)(3t_i + 1) \equiv 0 \pmod{2(n + t_i)}. \tag{4}$$

Since $t_1 + t_2 + t_3 = -1$ and $t_i \in \mathbb{Z}$ for all $i \in \{1, 2, 3\}$, not all of them can be negative. Let b be one of them such that $b \geqslant 0$. Then, it follows by (3) and (4) that

$$\begin{aligned}(b + 1)(2b + 1)(3b + 1) &= u.(2n + b)\\ (b + 1)(2b + 1)(3b + 3) &= v.(2n + 3b + 1)\end{aligned}$$

for some $u, v \in \mathbb{Z}$. Now subtracting gives

$$2(b + 1)(2b + 1) = 2(v - u)(n + b) + v(b + 1) + bu.$$

Now set $w = v - u$. We want to show that $w = 0$. Firstly notice that

$$\begin{aligned}w &= (b + 1)(2b + 1)\left(\frac{3b + 3}{2n + 3b + 1} - \frac{3b + 1}{2(n + b)}\right)\\ &= \frac{(b + 1)(2b + 1)(4n - 1 - 3b^2)}{2(2n + 3b + 1)(n + b)}\end{aligned} \tag{5}$$

and hence, by Equation (1), $w \geqslant 0$. Rearranging gives

$$3(b + 1)^3(2b + 1) = (2n + 3b + 1)\left(2(b + 1)(2b + 1) - 2w(n + b)\right).$$

Setting $n + b = x$ and refactorising we get the following quadratic in terms of x:

$$4wx^2 - 2(n + 1)(4b + 2 - w)x + (b + 1)^2(2b + 1)(3b + 1) = 0.$$

By definition x is real and hence, the discriminant of this quadratic must be non-negative. Therefore,

$$4(b + 1)^2(4b + 2 - w)^2 - 16w(b + 1)^2(2b + 1)(3b + 1) \geqslant 0$$

and hence

$$\begin{aligned}(4b + 2 - w)^2 &\geqslant 4w(2b + 1)(3b + 1)\\ &= w(4b + 2)(6b + 2).\end{aligned} \tag{6}$$

By (5) we have that $w < 2b+1$. Now since $w \geqslant 0$ it follows that $2b+1 < 4b+2-w \leqslant 4b+2$. Now, by (6), we get that $w \leqslant 0$ and hence $w = 0$, as claimed. Therefore, by (5), $4n - 1 = 3b^2$ and hence b must be odd. We therefore set $b = 2a + 1$ for $a \geqslant 0$ and it follows that $n = 3a^2 + 3a + 1$. Now suppose without loss of generality that t_1 was b. Then from (2)

$$t_2^2 = t_3^2$$

and therefore

$$t_2 = \pm t_3.$$

But we know that $t_2 + t_3 = -1 - t_1 \neq 0$ and hence

$$t_2 = t_3 = \frac{-1 - t_1}{2}.$$

Hence, $t_1 = 2a + 1, t_2 = t_3 = -a - 1$.

Moreover, since we have shown that t_i is odd, a must be even and

$$k_1 = n + 2a + 1$$
$$k_2 = k_3 = n - a - 1$$

as required. □

4.7. Proof of Theorem 3.5.

Proof. As in 3.4, assume first that at least one of the graphs A_i is disconnected, say A_2. if $A_2^\top = A_3$, then symmetrising as before gives an example with multiplicities $1, n-1, 2n$, as in Theorem 3.4; but before this symmetrisation there are six linearly independent matrices, which cannot have just four eigenspaces. So suppose that A_2 is symmetric, and corresponds to n disjoint triangles as in the case of Theorem 3.4. Then again the space V_1 of that proof is an eigenspace for all the matrices, and so for $i \neq j$ a vertex in T_i is joined to a constant number of vertices in T_j. Then there can be at most three such matrices, or five altogether in the configuration (including I and A_2), contrary to assumption.

Let k_i, r_i, s_i, t_i be the eigenvalues of A_i for $i \in \{1, \ldots, 5\}$ with multiplicities $1, n, n, n-1$ respectively. If the matrices $\Theta_{i,1}$ are as in [13], then they must be 2×2 matrices with eigenvalues r_i, s_i, where r_i and s_i are the eigenvalues of A_i with multiplicity n. We know that r_i, s_i must necessarily be rational integers. Now from the linear trace equation

$$\mathrm{Tr}(A_i) = k_i + n(r_i + s_i) + (n-1)t_i$$

we deduce that n must divide $k_i - t_i$ and since by Lemma 4.1 $|t_i| < k_i$, it follows that $k_i = \epsilon_i n + t_i$ for $\epsilon_i \geqslant 1$ for all i. Therefore, $\sum_{i=1}^{5} \epsilon_i \geqslant 5$. On the other hand,

$$3n - 1 = \sum_{i=1}^{5} k_i = (\sum_{i=1}^{5} \epsilon_i)n + \sum_{i=1}^{5} t_i = (\sum_{i=1}^{5} \epsilon_i)n - 1$$

and hence $\sum_{i=1}^{5} \epsilon_i = 3$, a contradiction. Therefore, this type of coherent configuration cannot exist. □

4.8. Proof of Theorems 3.7 and 3.8. In this section we deal with the cases arising in Theorems 3.7 and 3.8 together. We prove both statements through a series of lemmas that eliminate the case arising in Theorem 3.8 and force the parameters stated in Theorem 3.7.

Lemma 4.3. *If $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ is a homogeneous coherent configuration of rank 4, where its matrices have eigenvalue multiplicities $1, n+1, n-1$, and $n-1$, then all matrices are symmetric.*

Proof. Suppose for a contradiction that this is not the case. Then, since $\mathcal{A}$ is a homogeneous coherent configuration, one of the matrices say A_1 must be symmetric and A_2, A_3 are such that $A_2^T = A_3$. Then, A_2 and A_3 would have the same eigenvalues. Let k_i, r_i, s_i, t_i for $i \in \{1,2,3\}$ be the eigenvalues of A_1, A_2, A_3 respectively with multiplicities $1, n+1, n-1, n-1$ respectively. Then, since $A_2 = A_3^T$, A_2 and A_3 have the same eigenvalues with the same multiplicities. Hence, $s_2 + t_2 = s_3 + t_3$. But then, since by [13, Equation (6.9)]

$$\begin{aligned} s_1 + s_2 + s_3 &= -1 \\ t_1 + t_2 + t_3 &= -1 \end{aligned}$$

it follows that $s_1 = t_1$ and thus $\mathcal{A}_1$ has three eigenvalues with multiplicities $1, n+1$, and $2(n-1)$. However, Theorem 3.6 such a matrix cannot exist, a contradiction. Therefore, all matrices of $\mathcal{A}$ must be symmetric. ☐

For the remainder of the section, given a coherent configuration $\mathcal{B}$ we consider the association scheme $\mathcal{A}$ arising by adding every non-symmetric matrix and its transpose together to make a symmetric matrix. In this case notice that if B_i has eigenvalues $n_i, \lambda_i, \mu_i, \nu_i$ then $A_i = B_i + B_i^T$ has eigenvalues $k_i = 2n_i, r_i = 2\lambda_i, s_i = 2\mu_i, t_i = 2\nu_i$ again with eigenvalue multiplicities $1, n+1, n-1, n-1$ respectively.

Lemma 4.4. *If $\mathcal{A}$ is as defined above, then $k_i = \epsilon_i(n-1) - 2r_i$ for some $\epsilon_i \geqslant 0$ for all i. Moreover, $\sum \epsilon_i = 3$.*

Proof. By the linear trace relation for A_i we get

$$\mathrm{Tr}(A_i) = k_i + (n+1)r_i + (n-1)(s_i + t_i).$$

Hence, $k_i \equiv -2r_i \pmod{n-1}$ and we can write

$$k_i = \epsilon_i(n-1) - 2r_i$$

as claimed.

Also, notice that

$$3n - 1 = \sum_i k_i = (n-1)\sum_i \epsilon_i - 2\sum_i r_i.$$

Since by [13, Equation (6.9)] $\sum_i r_i = -1$, it follows that $\sum_i \epsilon_i = 3$.

Now suppose for a contradiction that $\epsilon_i < 0$. Since $k_i \geqslant 0$ it follows that $r_i < 0$. In particular, since by Lemma 4.1 $|r_i| < k_i$ we have that $-r_i < (n-1)\epsilon_i - 2r_i$ and hence $|r_i| > n-1$ and thus $|r_i| \geqslant n$. By the quadratic trace relation we get

$$k_i^2 + (n+1)r_i^2 \leqslant \mathrm{Tr}(A_i^2) = 3nk_i.$$

Hence, $(n+1)r_i^2 \leqslant k_i(3n - k_i)$, and basic calculus shows that $k_i(3n-k_i)$ is maximised at $k_i = \frac{3n}{2}$. Hence,

$$nr_i^2 < (n+1)r_i^2 \leqslant \left(\frac{3n}{2}\right)^2.$$

Dividing through by n and applying sqare roots gives us $|r_i| < \frac{3\sqrt{n}}{2} < n$, a contradiction. Hence $\epsilon_i \geqslant 0$ for all i. At this point, notice also that this bound on r_i holds independently of the assumption on ϵ_i and this claim will also be used later. ☐

Now considering the quadratic trace equation again and reducing modulo $n-1$ we get

$$\begin{aligned} \mathrm{Tr}(A_i^2) = k_i^2 + (n+1)r_i^2 + (n-1)(\lambda_i^2 + \mu_i^2) &= 3nk_i \\ (-2r_i)^2 + 2r_i^2 &\equiv -6r_i \\ 6r_i(r_i+1) &\equiv 0 \pmod{n-1}. \end{aligned}$$

We now show that in fact $n-1$ divides $3r_i(r_i+1)$.

Lemma 4.5. *If r_i is as defined above, then $n-1$ divides $3r_i(r_i+1)$.*

Proof. The trace equations give

$$s_i + t_i = -\epsilon_i - r_i$$
$$(n-1)(s_i^2+t_i^2) = 3n(\epsilon_i(n-1)-2r_i) - (\epsilon_i(n-1)-2r_i)^2 - (n+1)r_i^2$$

Now s_it_i is a rational integer by assumption and also $2s_it_i = (s_i+t_i)^2 - (s_i^2+t_i^2)$. Calculating modulo $2(n-1)$ we get

$$\begin{aligned} 0 &\equiv (n-1)(\epsilon_i+r_i)^2 - 3n(\epsilon_i(n-1)-2r_i) + (\epsilon_i(n-1)-2r_i)^2 + (n+1)r_i^2 \\ &\equiv (n-1)(\epsilon_i^2+r_i^2) - \epsilon_i n(n-1) + 6nr_i + (n-1)^2\epsilon_i^2 + 4r_i^2 + (n-1)r_i^2 + 2r_i^2 \\ &\equiv n(n-1)(\epsilon_i^2-\epsilon_i) + (6n-6)r_i + 6r_i + 6r_i^2 \\ &\equiv n(n-1)(\epsilon_i^2-\epsilon_i) + 6r_i + 6r_i^2. \end{aligned}$$

Since $\epsilon_i^2-\epsilon_i$ is a product of consecutive integers it is even and hence $2(n-1)$ must divide $6r_i(r_i+1)$ and hence $n-1$ divides $3r_i(r_i+1)$, as claimed. □

We now prove another inequality that we will use later.

Lemma 4.6. $\epsilon_i(n-1)(6n-2\epsilon_i n+\epsilon_i) - 6r_i(2n-\epsilon_i n+\epsilon_i) - (3n+9)r_i^2 \geqslant 0$

Proof. Consider the quadratic equation whose roots are s_i and t_i. Since s_i and t_i are real, it follows that the discriminant of this equation, namely $(s_i+t_i)^2 - 4s_it_i = (s_i-t_i)^2$ is non-negative. Notice that $(s_i-t_i)^2 = 2(s_i^2+t_i^2) - (s_i+t_i)^2$ and hence using the trace equations we get

$$6n(\epsilon_i(n-1)-2r_i) - 2(\epsilon_i(n-1)-2r_i)^2 - 2(n+1)r_i^2 - (n-1)(\epsilon_i+r_i)^2 \geqslant 0.$$

This can be rearranged to give the required statement. □

From Lemma 4.4 we know that either one of the ϵ_i's is zero say ϵ_1 without loss of generality, or there are just three non-identity matrices and $\epsilon_1=\epsilon_2=\epsilon_3=1$. We first consider the former case.

Proposition 4.7. *If $\epsilon_1 = 0$, then $n = 7$ or 19 and the coherent configurations are symmetric.*

Proof. If $\epsilon_1 = 0$, then $k_1 = -2r_1$ and since $k_1 > 0$, it follows that $r_1 < 0$. Using Lemma 4.6 we get

$$-12nr_1 - (3n+9)r_1^2 \geqslant 0$$

and hence $r_1 \geqslant \frac{-4n}{n+3} > -3$.

Therefore, $r_1 = -3$ or $r_1 = -2$, or $r_1 = -1$ and $k_1 = 6, 4$, or 2. Consider the case where $k_1 = 2$ and $r_1 = -1$. The trace equations give us

$$s_1 + t_1 = 1$$
$$(n-1)(s_1^2+t_1^2) = 5n-5$$
$$s_1^2+t_1^2 = 5.$$

Therefore, s_1 and t_1 are equal to 2 and -1 respectively. However, $r_1 = -1$ and $k_1 = 2$ but by Lemma 4.1 $|s_1| < k_1$, a contradiction. Hence, $k_1 = 2$ cannot hold.

It now follows by Lemma 4.5 that $n-1$ divides 18 or $n-1$ divides 6. Using the inequality from Lemma 4.6 we deduce that either $r_1 = -3$ and $n = 10$ or $n = 19$, or $r_1 = -2$ and $n = 3, 4$, or 7.

Now define $A = \sum\{A_i \mid \epsilon_i = 0\}$. Then, A must be a symmetric matrix of row sum $k = \sum k_i$ and eigenvalue $r = \sum r_i$. What we have said above for matrices A_i

with $\epsilon_i = 0$ applies to A as well and therefore A must consist of only one summand, A_1 without loss of generality. Now since by Lemma 4.4 $\sum \epsilon_i = 3$ there are two possibilities. There are either 5 matrices and $\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$ or there are 4 matrices and $\epsilon_2 = 2$ and $\epsilon_3 = 1$.

Now we check this case individually to see which of those can hold.

Case 1: $r_1 = -2, n = 3$.

In the case that $\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$ the inequality from Lemma 4.6 gives us

$$13 - 12r_i - 9r_i^2 \geqslant 0$$

for $i \in \{2, 3, 4\}$ and since r_i is integer, $-2 \leqslant r_i \leqslant 0$. Since by [13, Equation (6.9)] r_1, r_2, r_3, r_4 must sum up to -1, it follows that r_2, r_3, r_4 must sum up to 1, but this cannot hold since none of them can be positive.

Now we examine the case where we have four matrices and $\epsilon_2 = 1$ and $\epsilon_3 = 2$. In this case Lemma 4.6 gives us

$$-2 \leqslant r_2 \leqslant 0$$
$$-1 \leqslant r_3 \leqslant 1.$$

The only way r_2 and r_3 could sum up to 1 is $r_2 = 0$ and $r_3 = 1$. In this case we get $k_1 = 4, k_2 = 2, k_3 = 2$ and checking for such coherent configurations in [6] we find that there is a unique coherent configuration with such row and column sums, but checking the rational eigenvalues using GAP [5] shows that the r_i's are not equal to $-2, 0, 1$ as we wish and hence there is no such association scheme.

Case 2: $r_1 = -2, n = 4$.

First we look at the case where $\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$. By Lemma 4.6 we get

$$-7r_i^2 - 10r_i + 17 \geqslant 0$$

Since r_i is integer for $i \in \{2, 3, 4\}$ this gives

$$-2 \leqslant r_i \leqslant 1$$

Again in this case we want the r_i's for $i \in \{2, 3, 4\}$ to sum up to 1 but none of them is positive, so this case cannot hold.

Now let $\epsilon_2 = 1$ and $\epsilon_3 = 2$. In this case Lemma 4.6 gives

$$-2 \leqslant r_2 \leqslant 0$$
$$-2 \leqslant r_3 \leqslant 1.$$

The only combination that could work is $r_2 = 0$ and $r_3 = 1$. In this case we would get $k_1 = 4, k_2 = 3, k_3 = 4$. Checking in [6] we do not find any coherent configurations with such row and column sums and appropriate eigenvalues and hence $n = 4$ cannot hold either.

Case 3: $r_1 = -2, n = 7$.

In the case that $\epsilon_2 = \epsilon_3 = \epsilon_4 = 1$ Lemma 4.6 gives us

$$-15r_i^2 - 24r_i + 87 \geqslant 0$$

and hence, since $r_i \in \mathbb{Z}$ for $i \in \{2, 3, 4\}$,

$$-3 \leqslant r_i \leqslant 1.$$

The only combinations (up to permutation) that would give us $r_1 + r_2 + r_3 + r_4 = -1$ are $r_2 = 1, r_3 = 0, r_4 = 0$ and $r_2 = -1, r_3 = 1, r_4 = 1$. We then get $k_1 = 4, k_2 = 4, k_3 = 6, k_4 = 6$ or $k_1 = 4, k_2 = 4, k_3 = 4, k_4 = 8$ respectively. Looking at [6], we deduce that there are not any coherent configurations with such matrix row and column sums.

For $\epsilon_2 = 1, \epsilon_3 = 2$, as shown in [13] we need $k_1 = 4, k_2 = 8, k_3 = 8$ and looking at [6] we deduce that there is a unique coherent configuration with such matrix row and column sums and hence, it is the one arising in [13]. The corresponding s_i's and t_i's can be calculated to be

$$\begin{aligned} s_1 &= 1+\sqrt{2}, \quad t_1 = 1-\sqrt{2} \\ s_2 &= -2\sqrt{2}, \quad t_2 = 2\sqrt{2} \\ s_3 &= -2+\sqrt{2}, \quad t_3 = -2-\sqrt{2}. \end{aligned}$$

Case 4: $r_1 = -3, n = 10$.

In this case it suffices to check the subcase $\epsilon_2 = 1, \epsilon_3 = 2$, since r_1 is odd and hence it cannot be the case that A_1 is the sum of a matrix and its transpose. Therefore, all the matrices in the initial coherent configuration must be symmetric and we must have four of them. In this case by Lemma 4.6 we get

$$\begin{aligned} -13r_2^2 - 22r_2 + 123 &\geqslant 0 \\ -39r_3^2 - 12r_3 + 396 &\geqslant 0 \end{aligned}$$

which gives

$$\begin{aligned} -4 \leqslant r_2 \leqslant 2, \\ -3 \leqslant r_3 \leqslant 3. \end{aligned}$$

The (r_2, r_3) pairs consistent with [13, Equation (6.9)] are $(2,0), (1,1), (0,2), (-1,3)$ and all of those give row and column sums for which an association scheme does not exist.

Case 5: $r_1 = -3, n = 19$.

In this case, as shown in [13] $k_1 = 6, k_2 = 20, k_3 = 30$ and the corresponding s_i's and t_i's are

$$\begin{aligned} s_1 &= \frac{3+\sqrt{5}}{2}, \quad t_1 = \frac{3-\sqrt{5}}{2} \\ s_2 &= -2\sqrt{5}, \quad t_2 = 2\sqrt{5} \\ s_3 &= \frac{-5+3\sqrt{5}}{2}, \quad t_3 = \frac{-5-3\sqrt{5}}{2}. \end{aligned}$$

□

We now deal with the case where $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. Notice that in this case, since the ϵ_i's are all odd, $\mathcal{B} = \mathcal{A}$ and by Lemma 4.3 all matrices are symmetric.

Lemma 4.8. *If $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ then r_1, r_2, r_3 are all different.*

Proof. Suppose for a contradiction that this is not the case and without loss of generality, let $r_1 = r_2$. Then, since $\epsilon_1 = \epsilon_2$, it follows that $k_1 = k_2$. Thus, either $s_1 = t_1$ and $s_2 = t_2$ or $s_1 = t_2$ and $s_2 = t_1$, and since our coherent configuration has rank 4, the matrices are simultaneously diagonalisable and it follows that

$$\begin{aligned} s_1 + s_2 + s_3 &= -1, \\ t_1 + t_2 + t_3 &= -1. \end{aligned}$$

But this means that $s_3 = t_3$ and thus A_3 is a matrix of the kind that Theorem 3.6 forbids, a contradiction. □

Lemma 4.9. *Let $a_i = \frac{3r_i(r_i+1)}{n-1}$. Then, $a_i \leqslant 4$ and if $r_i \geqslant 0$, then $a_i \leqslant 3$.*

Proof. Firstly notice that by Lemma 4.5, $a_i \in \mathbb{Z}$. By Lemma 4.6 we get

$$(n-1)(4n+1) - 6(n+1)r_i - (3n+9)r_i^2 \geqslant 0.$$

Therefore,

$$(3n+9)(r_i^2 + r_i) \leqslant (n-1)(4n+1) - (3n-3)r_i$$

and hence

$$\begin{aligned} a_i &= \frac{3r_i(r_i+1)}{n-1} \\ &\leqslant \frac{4n+1}{n+3} - \frac{3r_i}{n+3} \\ &= 4 - \frac{11}{n+3} - \frac{3r_i}{n+3}. \end{aligned}$$

Now, if $r_i \geqslant 0$, we get $a_i < 4$, and hence $a_i \leqslant 3$. If $r_i < 0$ and $n \geqslant 19$, using the inequality from Lemma 4.4 stating that $r_i < \frac{3\sqrt{n}}{2}$, we deduce that $\frac{-3r_i}{n+3} \leqslant 1$ and hence $a_i < 5$ and so $a_i \leqslant 4$. Now, if $n < 19$ and $r_i \leqslant 0$ checking gives that $a_i \leqslant 3$. ☐

Lemma 4.10. *If none of a_1, a_2, a_3 are zero, then a_1, a_2, a_3 are all different.*

Proof. Suppose for a contradiction that without loss of generality, $a_1 = a_2$. Then, both r_1 and r_2 are roots of the equation

$$3r(r+1) - a_1(n-1) = 0.$$

Since by Lemma 4.8 $r_1 \neq r_2$, we must have $r_1 + r_2 = -1$. But from [13, Equation (6.9)], $r_1 + r_2 + r_3 = -1$ and hence $r_3 = 0$. But then, $a_3 = 0$, a contradiction. ☐

Lemma 4.11. *If $a > 0$ and r is a root of the equation*

$$x^2 + x - a = 0$$

then $r = -\frac{1}{2} \pm \sqrt{a} + \eta$, where $|\eta| < \frac{1}{8\sqrt{a}}$.

Proof. Notice that $\left(r + \frac{1}{2}\right)^2 = r^2 + r + \frac{1}{4} = a + \frac{1}{4}$.

Now, by squaring both $\sqrt{a + \frac{1}{4}}$ and $\sqrt{a} + \frac{1}{8\sqrt{a}}$ we see that $|\eta| < \frac{1}{8\sqrt{a}}$, as claimed. ☐

Lemma 4.12. *One of a_1, a_2, a_3 must be zero.*

Proof. Suppose that this is not the case. Then, by Lemma 4.10, a_1, a_2, a_3 are all different. Since $a_i = \frac{3r_i(r_i+1)}{n-1}$, it follows that r_i is a root of the equation

$$x^2 + x - \frac{a_i(n-1)}{3} = 0.$$

By Lemma 4.11 we get that

$$r_i = -\frac{1}{2} \pm \sqrt{\frac{a_i(n-1)}{3}} + \eta_i$$

where $|\eta_i| < \frac{1}{8}\sqrt{\frac{3}{a_i(n-1)}} < \frac{1}{8}$.

Now, it follows by [13, Equation (6.9)] that

$$r_1 + r_2 + r_3 = -1$$

and hence

$$-\frac{3}{2} + \sqrt{\frac{n-1}{3}}(\pm\sqrt{a_1} \pm \sqrt{a_2} \pm \sqrt{a_3}) + \eta_1 + \eta_2 + \eta_3 = -1.$$

Rearranging and taking absolute values gives

$$\left|\sqrt{\frac{n-1}{3}}(\pm\sqrt{a_1}\pm\sqrt{a_2}\pm\sqrt{a_3})\right| < \frac{7}{8}.$$

Since $a_i \neq 0$, by Lemmas 4.9 and 4.10 we get that a_1, a_2, a_3 must be among the numbers $1, 2, 3, 4$ and all different. Hence, crude approximations to $\sqrt{2}$ and $\sqrt{3}$ give the estimate

$$|\pm\sqrt{a_1}\pm\sqrt{a_2}\pm\sqrt{a_3}| > \frac{4}{10}$$

and hence

$$\frac{4}{10}\sqrt{\frac{n-1}{3}} < \frac{7}{8}.$$

This gives $n < 15$, but checking all cases shows that no integer less than 15 has three different representations in the form $1 + \frac{3r_i(r_i+1)}{a_i}$ with r_i, a_i integral, all different for every i, and $1 \leqslant a_i \leqslant 4$, a contradiction. Hence, one of a_1, a_2, a_3 must be zero, as claimed. □

We now choose notation such that $a_1 = 0$.

Lemma 4.13. *If r_1 is as defined above, then $r_1 = -1$.*

Proof. Since $a_1 = 0$, $r_1 = 0$ or $r_1 = -1$. Assume now that $r_1 = 0$. One of a_2, a_3 must be non-zero, for otherwise, all r_i's would be solutions of the equation $x^2 + x = 0$ and hence they would not all be different, as Lemma 4.8 states. Suppose without loss of generality that $a_2 \neq 0$. Then,

$$n = \frac{3r_2(r_2+1)}{a_2} + 1.$$

If 3 does not divide a_2 then $n \equiv 1 \pmod 3$. If 3 divides a_2 then by Lemma 4.9, it follows that $a_2 = 3$ and $n = r_2^2 + r_2 + 1$. Hence $n \equiv 1 \pmod 3$ or $n \equiv 0 \pmod 3$. From the linear and quadratic trace equations for A_1 we get

$$\begin{aligned} s_1 + t_1 &= -1 \\ s_1^2 + t_1^2 &= 3n - (n-1) = 2n + 1. \end{aligned}$$

Now $p_{11}^1 = |\{j \in \{1, \dots, 3n\} \mid (A_1)_{ij} = 1, (A_1)_{jk} = 1\}|$ is an integer constant for any $i, k \in \{1, \dots, 3n\}$ such that $(A_1)_{ik} = 1$. Moreover, the cubic trace equation for A_1 gives

$$\begin{aligned} 3np_{11}^1 &= (n-1)^2 + \frac{3}{2}(s_1+t_1)(s_1^2+t_1^2) - \frac{1}{2}(s_1+t_1)^3 \\ &= (n-1)^2 - \frac{3}{2}(2n+1) + \frac{1}{2} \\ &= n^2 - 5n. \end{aligned}$$

Thus, $3p_{11}^1 = n - 5$ and since $p_{11}^1 \in \mathbb{Z}$, it follows that $n \equiv 5 \pmod 3$, a contradiction. Hence $r_1 \neq 0$ and therefore $r_1 = -1$. □

Lemma 4.14. $n = 31$.

Proof. Since $r_1 = -1$ and $r_1 + r_2 + r_3 = -1$ by [13, Equation (6.9)], it follows that $r_2 = -r_3 = r \in \mathbb{Z}$. Then, since a_2, a_3 are integers,

$$\begin{aligned} &n - 1 \text{ divides } 3r(r+1) \\ &n - 1 \text{ divides } 3(-r)(-r+1). \end{aligned}$$

Hence, $n - 1$ divides $3r(r+1) - 3(-r)(-r+1) = 6r$.

By interchanging A_2 and A_3 if necessary we may assume that $r \geqslant 0$. Then since by Lemma 4.8, r_1, r_2, r_3 are all different, it follows that $r \neq 0$ and $r \neq 1$. Hence, $r \geqslant 2$. Moreover, from Lemma 4.9 we know that

$$\frac{6r}{n-1} \cdot \frac{r+1}{2} \leqslant 3.$$

It follows that $r+1 \leqslant 6$ and if $6r \neq n-1$ then since $n-1$ divides $6r$, $r+1 \leqslant 3$. Now considering that $\frac{6r}{n-1} \cdot \frac{r+1}{2}$ must be integer and that the above inequality must hold for our choices of n and r we can check all cases and find that the only possibilities are:

$$\begin{aligned} 6r &= n-1, \quad r = 5, \quad n = 31 \\ 6r &= n-1, \quad r = 3, \quad n = 19 \\ 3r &= n-1, \quad r = 2, \quad n = 7. \end{aligned}$$

For $n = 7$ we see that k_1, k_2, k_3 are equal to $8, 2, 10$ respectively and checking in [6], we see that there is no association scheme with such row and column sums.

If $n = 19$, then the trace equations give

$$\begin{aligned} s_1, t_1 &\text{ are } \pm 2\sqrt{5} \\ s_2, t_2 &\text{ are } \pm -2 \pm \sqrt{6} \\ s_3, t_3 &\text{ are } \pm 5, -3 \end{aligned}$$

Now, no possible tuple (s_1, s_2, s_3) satisfies $s_1 + s_2 + s_3 = -1$ and hence this case cannot arise.

Finally, for $n = 31$ for suitable choices of roots we get

$$\begin{aligned} s_1 &= 4\sqrt{2}, \quad t_1 = -4\sqrt{2} \\ s_2 &= -3 - \sqrt{2}, \quad t_2 = -3 + \sqrt{2} \\ s_3 &= 2 - 3\sqrt{2}, \quad t_3 = 2 + 3\sqrt{2}. \end{aligned}$$

□

Proof of Theorem 3.7. Follows directly by Proposition 4.7 and Lemma 4.14. □

Proof of Theorem 3.8. Follows directly by proposition 4.7. □

5. Examples

In this section we provide examples with the parameters found in Theorems 3.1 to 3.8, in cases where they are known to exist.

5.1. Theorem 3.1.

The classic examples of symmetric conference graphs are the Paley graphs. The vertex set of such a graph is the set of elements of a finite field whose order is congruent to 1 (mod 4), and two vertices are connected by an edge if and only if their difference is a square in the field.

Similarly, the classic examples of doubly regular tournaments are the Paley tournaments; the vertex set is the set of elements of a finite field of order congruent to 3 (mod 4), with an arc from a to b if $b - a$ is a square.

5.2. Theorem 3.2.

For the second set of parameters arising in Theorem 3.2, a known example (with $a = 0$) is the triangular graph $T(6)$ and its complement; no further examples are known. For the other sets of parameters, no known example with fewer than 512 vertices is known. Moreover, due to the large number of vertices that the given parameters force, it would be very hard to construct one.

5.3. Theorem 3.3. For the first set of parameters arising in Theorem 3.3 and for $a \geqslant 2$, the graphs arising from Steiner systems of the type $S(2, a+1, n)$ with $a \in \{1, 2, 3\}$ are known examples. The number of non-isomorphic Steiner systems $(2, 3, 19)$ is $11, 084, 874, 829$ (see [9]); these give pairwise non-isomorphic graphs. There is no known example of graphs with the second set of parameters, and the nonexistence in the case $a = 2$ has been shown by Wilbrink and Brouwer [20].

5.4. Theorem 3.4. We do not have any examples for the first three cases of this theorem. For the last two (imprimitive) cases, there are examples. Neumann gives examples for the fourth case based on the group $\mathrm{PSL}(2, p-1)$ where p is a Fermat prime. For the fifth case, the wreath product of the cyclic group of order 3 with any 2-transitive group of degree n gives an example.

5.5. Theorem 3.7. The cases $n = 21$ and $n = 57$ are realised by the groups $\mathrm{PGL}(2, 7)$ and $\mathrm{PSL}(2, 19)$ respectively. These can be found in the GAP [5] database of primitive permutation groups as `PrimitiveGroup(21,1)` and `PrimitiveGroup(57,1)` respectively.

The database [6] gives the basis matrices for the first of these, and certifies its uniqueness. In the second case, the association scheme is also known to be unique [2]; the graph of valency 6 is the distance-transitive *Perkel graph* [15]. Existence in the final case with 93 points is undecided, as far as we know.

Acknowledgement. We are grateful to two reviewers for their careful reading of the paper and helpful comments, which in particular have rectified a gap in our original argument for Theorem 3.4.

References

[1] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, London Mathematical Society Student Texts, vol. 22, Cambridge University Press, Cambridge, 1991.

[2] K. Coolsaet and J. Degraer, *A computer assisted proof of the uniqueness of the Perkel graph*, Design Code Cryptogr **34** (2005), 155–171.

[3] W. Feit, *On finite linear groups*, J ALgebra **5** (1967), 378–400.

[4] Felix Gantmacher, *The Theory of Matrices*, vol. 2, AMS Chelsea Publishing, New York, 1998, reprint of 1959 edition.

[5] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.12.2*, 2022, `https://www.gap-system.org`.

[6] Akihide Hanaki and Izumi Miyamoto, *Classification of association schemes of small vertices*, visited on 3 July 2022, `http://math.shinshu-u.ac.jp/~hanaki/as/`.

[7] William M. Kantor, *Primitive permutation groups of odd degree, and an application to finite projective planes*, J. Algebra **106** (1987), no. 1, 15–45.

[8] William M. Kantor and Robert A. Liebler, *The rank 3 permutation representations of the finite classical groups*, Trans. Amer. Math. Soc. **271** (1982), no. 1, 1–71.

[9] Petteri Kaski and Patric R. J. Östergård, *The Steiner triple systems of order 19*, Math. Comp. **73** (2004), no. 248, 2075–2092.

[10] Martin W. Liebeck, *The affine permutation groups of rank three*, Proc. London Math. Soc. (3) **54** (1987), no. 3, 477–516.

[11] Martin W. Liebeck and Jan Saxl, *The primitive permutation groups of odd degree*, J. London Math. Soc. (2) **31** (1985), no. 2, 250–264.

[12] ———, *The finite primitive permutation groups of rank three*, Bull. London Math. Soc. **18** (1986), no. 2, 165–172.

[13] Peter M. Neumann, *Primitive permutation groups of degree 3p*, unpublished manuscript, 1969.

[14] ———, *Primitive permutation groups of degree 3p*, 2022, `https://arxiv.org/abs/2204.06926`.

[15] Manley Perkel, *Bounding the valency of polygonal graphs with odd girth*, Canadian J. Math. **31** (1979), no. 6, 1307–1321.

[16] K. B. Reid and Ezra Brown, *Doubly regular tournaments are equivalent to skew Hadamard matrices*, J. Combinatorial Theory Ser. A **12** (1972), 332–338.

[17] L. L. Scott, *Personal communication*, 2022.

[18] Helmut Wielandt, *Primitive Permutationsgruppen vom Grad 2p*, Math. Z. **63** (1956), 478–485.

[19] ———, *Finite permutation groups*, Academic Press, New York-London, 1964, translated from the German by R. Bercov.

[20] H. A. Wilbrink and A. E. Brouwer, *A* (57, 14, 1) *strongly regular graph does not exist*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 1, 117–121.

Marina Anagnostopoulou-Merkouri, School of Mathematics, University of Bristol, Bristol BS8 1QU, UK
E-mail : `marina.anagnostopoulou-merkouri@bristol.ac.uk`

Peter J. Cameron, School of Mathematics and Statistics, University of St Andrews, St Andrews, Fife KY16 9SS, UK
E-mail : `pjc20@st-andrews.ac.uk`