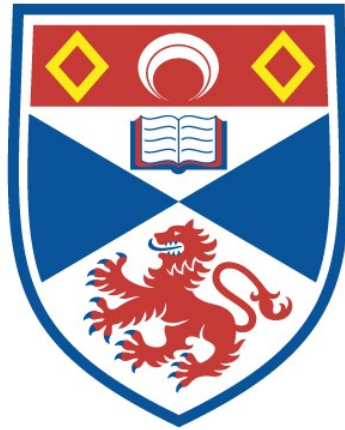


On constructing topology from algebra

Luke Elliott

A thesis submitted for the degree of PhD
at the
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ON CONSTRUCTING TOPOLOGY FROM ALGEBRA

Abstract

In this thesis we explore natural procedures through which topological structure can be constructed from specific semigroups. We will do this in two ways: 1) we equip the semigroup object itself with a topological structure, and 2) we find a topological space for the semigroup to act on continuously.

We discuss various minimum/maximum topologies which one can define on an arbitrary semigroup (given some topological restrictions). We give explicit descriptions of each these topologies for the monoids of binary relations, partial transformations, transformations, and partial bijections on the set \mathbb{N} . Using similar methods we determine whether or not each of these semigroups admits a unique Polish semigroup topology. We also do this for the following semigroups: the monoid of all injective functions on \mathbb{N} , the monoid of continuous transformations of the Hilbert cube $[0, 1]^{\mathbb{N}}$, the monoid of continuous transformations of the Cantor space $2^{\mathbb{N}}$, and the monoid of endomorphisms of the countably infinite atomless boolean algebra. With the exception of the continuous transformation monoid of the Hilbert cube, we also show that all of the above semigroups admit a second countable semigroup topology such that every semigroup homomorphism from the semigroup to a second countable topological semigroup is continuous.

In a recent paper, Bleak, Cameron, Maissel, Navas, and Olukoya use a theorem of Rubin to describe the automorphism groups of the Higman-Thompson groups $G_{n,r}$ via their canonical Rubin action on the Cantor space. In particular they embed these automorphism groups into the rational group \mathcal{R} of transducers introduced by Grigorchuk, Nekrashevich, and Sushchanskii. We generalise these transducers to be more suitable to higher dimensional Cantor spaces and give a similar description of the automorphism groups of the Brin-Thompson groups dV_n (although we do not give an embedding into \mathcal{R}). Using our description, we show that the outer automorphism group $\text{Out}(dV_2)$ of dV_2 is isomorphic to the wreath product of $\text{Out}(1V_2)$ with the symmetric group on d points.

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Part 1. Introduction

This thesis is concerned with methods for choosing an appropriate or “best” topological structure in a given context. We are particularly interested in doing this when we are provided with a nice algebraic object to work with. This usually involves finding a topological structure which is in some way “compatible” with our algebraic object and investigating its uniqueness given some “reasonable” topological assumptions.

We have three main parts. In Part 2, we establish the definitions and background knowledge that we require in the remaining parts. In particular we include proofs of some of the key theorems in the literature which Part 3 and Part 4 are dependant on. The most important of these being:

- Comparable Polish (3.25) topologies on a group are equal (Theorem 4.27).
- All homomorphisms from the symmetric group on \mathbb{N} to second countable groups are continuous, and similarly all homomorphisms from the homeomorphism group of the Cantor space to second countable groups are continuous (Theorem 5.39)
- Rubin’s Theorem (Theorem 5.49).

The term “automatic continuity” is often used in the literature to describe properties like the one in Theorem 5.39. However to avoid confusion we have avoided this terminology in all the formal statements made in this document.

In Part 3, we are concerned with identifying natural ways of defining topologies on a given semigroup. We are also interested in finding topologies compatible with important semigroups which are unique up to some “reasonable” topological restrictions.

To this end, we will give five ways of defining a topology on an arbitrary semigroup, each of which is in some sense a bound on the set of “nice” topologies compatible with the semigroup. These topologies are the minimum Fréchet topology (6.6), the Fréchet/Hausdorff-Markov topologies (6.7), the Zariski topology (6.9), and the second countable continuity topology (6.11). Notably, some of these topologies are always comparable (see Proposition 6.15).

We explore and describe these topologies for various naturally occurring semigroups and as a result show that each of them is compatible with either 0, 1 or many Polish topologies (see Theorems 9.4, 9.10, 9.18, 9.21, 9.23, 9.29, 9.35 and 9.42).

Clones (8.1) are a natural generalisation of monoids, and many of the monoids we discuss have natural clone analogues. We also extend some of the results in Part 3 to these analogues (see Corollaries 9.11, 9.30, 9.36 and 9.43).

In Part 4, we use Rubin’s Theorem (see Corollary 5.50) to study the automorphisms of the Brin-Thompson groups dV_n (see Definition 10.2 and [7]). In [3] it was shown that, via Rubin’s Theorem, it is possible to categorise which homeomorphisms of the Cantor set correspond to automorphisms of the Higman-Thompson groups $G_{n,r}$. This description is based on the combinatorial properties of the minimum transducers representing these homeomorphisms as described in [15].

We extend the notion of a transducer given in [15] so as to allow transducers to describe continuous maps on “higher dimensional” Cantor spaces (see Theorem 11.16). We then extend the methods of [3] to give an analogous description of the groups $\text{Aut}(dV_n)$ (see Theorem 12.18). We then use this description to draw an algebraic connection between the outer automorphism groups of the groups dV_n for different values of d (see Theorems 14.19 and 14.18).

Part 2. Assumed Knowledge and Definitions

The purpose of this part is to establish the previously known results from the literature which we need for later parts, as well as introducing some relevant notation and terminology. It is also assumed that the reader is familiar with the following concepts:

- Zorn's Lemma.
- Cardinals and taking the cardinalities of arbitrary sets (the cardinality of a set X is denoted by $|X|$).
- The claims about compact and Hausdorff spaces in Remark 3.18.
- Brouwer's Theorem (Theorem 3.20).
- The Baire Category Theorem (Theorem 3.28).

In the Hilbert cube subsection (9.5) of Part 3 we also make use of Theorem 9.27 (Theorem 5.2.4 of [34]), and in the boolean algebra subsection (9.7) we make use of the Stone Duality Theorem (9.41) but no other subsections of this document are dependent on these results.

1. SETS

Most of the concepts in this section are well-known. However it will be important (particularly in Part 3) that the reader is comfortable with how these objects are viewed in this thesis. At times we will write “:=” instead of “=” to indicate that the given equality is defining the object on the left.

Definition 1.1 (Numbers).

We denote the set of natural numbers (including 0) by \mathbb{N} . We also denote the set of integers by \mathbb{Z} , the set of rational numbers by \mathbb{Q} , and the set of real numbers by \mathbb{R} .

If $a, b \in \mathbb{R}$, then we write $[a, b]$, $(a, b]$, $[a, b)$ or (a, b) to denote the usual closed, half-open and open intervals in \mathbb{R} . If $a, b \in \mathbb{Z}$, then $\{a, a + 1, \dots, b\} := \left\{ i \in \mathbb{Z} \mid a \leq i \leq b \right\}$.

Definition 1.2 (Power set).

If X is a set, then we denote the *power set* of X (the set of subsets of X) by $\mathcal{P}(X)$.

Definition 1.3 (Functions, Assumed Knowledge: 1.2).

If X, Y are sets, then we say that f is a *function* from X to Y (denoted $f : X \rightarrow Y$) if f is a subset of the set $X \times Y$ which satisfies the following condition:

- (1) For all $x \in X$, there is a unique $y \in Y$ with $(x, y) \in f$.

We will compose our functions from left to right. Note that we later (Definition 1.5) define the set $X \times Y$ using functions, so this is technically a circular definition. Formally this can be avoided by defining $X \times Y$ without the use of functions first and then redefining the notation later, but we avoid doing this for simplicity.

Definition 1.4 (Function sets, Assumed Knowledge: 1.3).

If X and Y are sets, then we define X^Y to be the set of all functions from Y to X . If Y is a natural number n , then we treat n as the set $\{0, 1, \dots, n - 1\}$ for these purposes.

Definition 1.5 (Product sets and projection maps, Assumed Knowledge: 1.4).

If $(X_i)_{i \in I}$ are sets, then we define

$$\prod_{i \in I} X_i := \left\{ t \in \left(\bigcup_{i \in I} X_i \right)^I \mid \text{for all } i \in I \text{ we have } (i)t \in X_i \right\}.$$

Note that if X, Y are sets and for all $y \in Y$ we define $X_y := X$, then $X^Y = \prod_{y \in Y} X_y$. If X_0, X_1, \dots, X_{n-1} are sets, then we will also use the notation:

$$X_0 \times X_1 \times \dots \times X_{n-1} := \prod_{i \in \{0, 1, \dots, n-1\}} X_i.$$

We will often use the notation $((0)t, (1)t, \dots, (n-1)t)$, to define an element t of $X_0 \times X_1 \times \dots \times X_n$. Moreover, we will generally denote the *projection maps* $\pi_i : \prod_{j \in I} X_j \rightarrow X_i$ (defined by $(t)\pi_i = (i)t$) by π_i . We rely on context for the domain of the map π_i . In particular if X, Y are sets and $(x, y) \in X \times Y$, then $(x, y)\pi_0 = x$ and $(x, y)\pi_1 = y$.

Definition 1.6 (Binary relations, Assumed Knowledge: 1.2, 1.5).

If X and Y are sets and $b \subseteq X \times Y$, then we say that b is a *binary relation from X to Y* (or a *binary relation on X* if $X = Y$). Let b be a binary relation from a set X to a set Y . If $S \subseteq X$, then we define the *image of S under b* by

$$(S)b := \left\{ y \in Y \mid \text{there is } x \in S \text{ with } (x, y) \in b \right\}.$$

If $S \subseteq \mathcal{P}(X)$, then we similarly define the image of S by $(S)b := \left\{ (A)b \mid A \in S \right\}$, we also extend this to $\mathcal{P}(\mathcal{P}(X)), \mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ etc. We define the *inverse* of b by

$$b^{-1} := \left\{ (y, x) \in Y \times X \mid (x, y) \in b \right\}.$$

If b' is a binary relation from Y to another set Z , then we define the *composition* of b and b' by

$$bb' := \left\{ (x, z) \in X \times Z \mid \text{there is } y \in Y \text{ with } (x, y) \in b \text{ and } (y, z) \in b' \right\}.$$

We define the *domain*, *image* and *kernel* of b by $\text{dom}(b) := (Y)b^{-1}$, $\text{img}(b) := (X)b$ and $\text{ker}(b) := \left\{ (x, y) \in \text{dom}(b)^2 \mid (\{x\})b = (\{y\})b \right\}$ respectively. Note that functions are examples of binary relations and composition of functions is a special case of composition of binary relations, so these definitions can be applied to functions.

If b is a binary relation from X to Y and S is a set, then we define the *restriction* of b to S by

$$b \upharpoonright_S := b \cap (S \times Y).$$

We denote the relation $\left\{ (x, x) \mid x \in X \right\}$ by id_X (this relation is often called the identity function, the diagonal relation or equality relation on X). We say that a binary relation b from X to Y is *surjective* if $\text{img}(b) = Y$ and *injective* if $\text{ker}(b) = \text{id}_X$. If f is a function, then we say that f is *bijective* if it is both injective and surjective.

Definition 1.7 (Countable, Assumed Knowledge: 1.1, 1.6).

We say that a set X is *countable* if there is an injective function from X to \mathbb{N} . In particular, finite sets are countable.

Definition 1.8 (Union and intersection conventions).

If X is a set, then we define

$$\bigcup X := \bigcup_{S \in X} S, \quad \text{and} \quad \bigcap X := \bigcap_{S \in X} S.$$

Where $\bigcup_{S \in \emptyset} S = \emptyset$, and $\bigcap_{S \in \emptyset} S$ will be the “universe” or “space” we are working in (when this is ambiguous the notation is avoided).

Definition 1.9 (Complement).

If X is a set and $S \subseteq X$, then we will denote the *complement* of S in X by S^c . That is

$$S^c := X \setminus S = \left\{ x \in X \mid x \notin S \right\}.$$

In the rare cases when the set X is ambiguous the notation S^c is avoided.

Definition 1.10 (Types of binary relation, Assumed Knowledge: 1.6).

Suppose that X is a set and b is a binary relation on X . We will use the following terms to describe these potential properties of b :

- (1) Reflexive: if $x \in X$ then we have $(x, x) \in b$.
- (2) Transitive: if $(x, y) \in b$ and $(y, z) \in b$ we also have $(x, z) \in b$.
- (3) Anti-Symmetric: if $(x, y) \in b$ and $(y, x) \in b$ then $x = y$.
- (4) Symmetric: if $(x, y) \in b$ then $(y, x) \in b$.
- (5) Total: if $x, y \in X$ then either $(x, y) \in b$ or $(y, x) \in b$.

We say that b is a *preorder* if it is reflexive and transitive. We say that a preorder b is a *partial order* if it is anti-symmetric. We say that a preorder b is an *equivalence relation* if it is symmetric. We say that a partial order b is a *total order* if it is total. If \leq is one of these types of binary relations we will often write $x \leq y$ to mean $(x, y) \in \leq$, and $x < y$ to mean that x, y are distinct and satisfy $x \leq y$.

If X is a set and \leq is a partial order on X , then we say that $x \in X$ is

- (1) *Minimum* if $x \leq y$ for all $y \in X$.
- (2) *Minimal* if for all $y \in X$ we have $y \leq x \Rightarrow y = x$.
- (3) *Maximum* if $y \leq x$ for all $y \in X$.
- (4) *Maximal* if for all $y \in X$ we have $x \leq y \Rightarrow y = x$.

If \sim is an equivalence relation on a set X and $x \in X$, then we define the *equivalence class* of x by $[x]_\sim := \left\{ y \in X \mid x \sim y \right\}$. Furthermore we define $X / \sim := \left\{ [x]_\sim \mid x \in X \right\}$, and we define the *index* of \sim to be $|X / \sim|$.

Definition 1.11 (Filters, Assumed Knowledge: 1.10).

Suppose that (X, \leq) is a partially ordered set. We say that a subset F of X is a *filter* if:

- (1) For all $x, y \in F$, there is $z \in F$ with $z \leq x$ and $z \leq y$.
- (2) If $z \in X$ and there is $x \in F$ with $x \leq z$, then $z \in F$.

We say that a filter F on X is *proper* if $F \neq X$, and we say that F is a *ultrafilter* if it is maximal (with respect to containment) among the proper filters on X .

Filters are often considered on the power set of some set ordered by inclusion, in this case it follows from Zorn's Lemma that every proper filter is contained in an ultrafilter.

2. CATEGORIES

We will use many categories throughout this document. In particular we introduce a category of transducers which is used heavily throughout Part 4, and we have a categorical perspective when discussing clones in Part 3. Most of the categories of this document have products (Definition 2.3). We make heavy use of this fact throughout the document and in particular the notation

$$\langle (f_i)_{i \in I} \rangle_P$$

(where I is an index set, $(f_i)_{i \in I}$ are morphisms with a common source object, and P is a product object) will be used heavily.

The following lengthy “Definition” is for the purposes of those who (like the author) are scared when proper classes are used at length. If the reader is comfortable with proper classes then they are encouraged to skip it.

Definition 2.1 (Proper classes, Assumed Knowledge: 1.3, 1.5).

If P is a property which assigns the value true or false to every set (being countable for example), then it will often be useful to talk simultaneously about all the sets which satisfy P . We will informally use the term *class* to simultaneously refer to all the sets with some property. Recall that every element of a set is itself a set, we make no distinction between a set S and the class associated with the property of being an element of S . Many classes are not sets, a famous example being the class of sets which do not contain themselves. We refer to a class that is not a set as a *proper class*. If \mathcal{C} is a class defined by a property P , then we will often write $c \in \mathcal{C}$ or $P(c)$ to mean that c satisfies P .

As proper classes are not formal ZFC objects, much care is required when working with them. Nevertheless as we will see later, many natural and useful examples of categories make use of proper classes. Thus it will be useful to be able to make subclasses, tuples of classes, products of classes, and make functions between classes.

Note that we cannot define a function whose image consists of classes. In particular when we give subscripts to classes here we are not defining a function, we are only saying that we allow this notation at different times with a different number of classes being discussed.

- (1) If \mathcal{A}, \mathcal{B} are classes defined by properties P_A, P_B respectively, then we say that \mathcal{A} is a subclass of \mathcal{B} if P_A implies P_B .
- (2) If $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{n-1}$ are classes (including at least one proper class) defined by the properties P_0, P_1, \dots, P_{n-1} respectively, then we write

$$(\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{n-1})$$

to refer to the class of pairs (c_i, i) where c_i satisfies property P_i . This is very different to how tuples were defined for sets (Definition 1.5), it is in fact much more like how disjoint unions are typically defined, but all that will matter in this document is that our original classes are “recoverable” from this class. Note that \mathcal{C}_i is the class of c such that $(c, i) \in (\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{n-1})$.

- (3) If $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_{n-1}$ are classes defined by the properties P_0, P_1, \dots, P_{n-1} respectively, then we write

$$\mathcal{C}_0 \times \mathcal{C}_1 \times \dots \times \mathcal{C}_{n-1}$$

to refer to the class of n -tuples $(c_0, c_1, \dots, c_{n-1})$ such that each c_i is a set which satisfies property P_i . Note that in the case that our classes are sets, this definition agrees with Definition 1.5.

- (4) If \mathcal{A}, \mathcal{B} are classes defined by properties P_A, P_B respectively, then we say that $f : \mathcal{A} \rightarrow \mathcal{B}$ is a *class function* to mean that f is a subclass of $\mathcal{A} \times \mathcal{B}$ and for all $a \in \mathcal{A}$, there is a unique $b \in \mathcal{B}$ with $(a, b) \in f$. If $a \in \mathcal{A}$, then we write $(a)f$ to denote the unique $b \in \mathcal{B}$ with $(a, b) \in f$. Note that this agrees with Definition 1.3.

Definition 2.2 (Categories and functors, Assumed Knowledge: 2.1).

We say that \mathcal{C} is a *category* to mean that \mathcal{C} is a tuple $(\mathcal{O}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}}, s_{\mathcal{C}}, t_{\mathcal{C}}, \circ_{\mathcal{C}})$, where

- (1) \mathcal{O}_C is a class (called the *objects* of the category).
- (2) \mathcal{M}_C is a class (called the *morphisms* of the category).
- (3) $s_C : \mathcal{M}_C \rightarrow \mathcal{O}_C$ and $t_C : \mathcal{M}_C \rightarrow \mathcal{O}_C$ are class functions (which assign to morphisms a *source* and *target* respectively).
- (4) Comp_C is the subclass of $\mathcal{M}_C \times \mathcal{M}_C$ consisting of those (m_0, m_1) with $(m_0)t_C = (m_1)s_C$ (called the *composable pairs*).
- (5) $\circ_C : \text{Comp}_C \rightarrow \mathcal{M}_C$ is a class function (called *composition*) such that if $(a, b), (b, c) \in \text{Comp}_C$ then

$$((a, b)\circ_C)s_C = (a)s_C, \quad ((a, b)\circ_C)t_C = (b)t_C, \quad ((a, b)\circ_C, c)\circ_C = (a, (b, c)\circ_C)\circ_C.$$

- (6) For all $O \in \mathcal{O}_C$, there is a unique $1_O \in \mathcal{M}_C$ such that $(1_O)s_C = (1_O)t_C = O$ and if $(1_O, m_0), (m_1, 1_O) \in \text{Comp}_C$ then

$$(1_O, m_0)\circ_C = m_0 \quad \text{and} \quad (m_1, 1_O)\circ_C = m_1.$$

If \mathcal{C} is a category and A, B are objects of \mathcal{C} , then we say A, B are *isomorphic* in \mathcal{C} if there are morphisms f_A, f_B of \mathcal{C} with

$$(f_A, f_B)\circ_C = 1_A \quad \text{and} \quad (f_B, f_A)\circ_C = 1_B.$$

If \mathcal{A}, \mathcal{B} are categories, then we write $f : \mathcal{A} \rightarrow \mathcal{B}$ is a *functor* to mean that f is a tuple $(f_{\mathcal{O}}, f_{\mathcal{M}})$ where

- (1) $f_{\mathcal{O}} : \mathcal{O}_{\mathcal{A}} \rightarrow \mathcal{O}_{\mathcal{B}}$ is a class function.
- (2) $f_{\mathcal{M}} : \mathcal{M}_{\mathcal{A}} \rightarrow \mathcal{M}_{\mathcal{B}}$ is a class function.
- (3) For all $a, b \in \text{Comp}_C$ and $O \in \mathcal{O}_C$ we have

$$(1_O)f_{\mathcal{M}} = 1_{(O)f_{\mathcal{O}}}, \quad ((a)t_{\mathcal{A}})f_{\mathcal{O}} = ((a)f_{\mathcal{M}})t_{\mathcal{B}}, \quad ((b)s_{\mathcal{A}})f_{\mathcal{O}} = ((b)f_{\mathcal{M}})s_{\mathcal{B}}$$

$$((a, b)\circ_{\mathcal{A}})f_{\mathcal{M}} = ((a)f_{\mathcal{M}}, (b)f_{\mathcal{M}})\circ_{\mathcal{B}}.$$

We say that a category is *small* if it is a tuple of sets and *big* if it is not.

The notation π_i in the following definition is used because in the category of sets and functions (see Example 2.4) the natural projection maps are precisely these maps from Definition 1.5. While these are not the only products or projection maps, all concrete examples discussed in this document will be (essentially) of this type. For example in Definition 11.14, the projections will not even be functions, but they are triples of the usual projection maps.

Definition 2.3 (Categorical products, Assumed Knowledge: 1.1, 2.2).

If \mathcal{C} is a category, I is a set and $(O_i)_{i \in I}$ are objects of \mathcal{C} , then a *product* of $(O_i)_{i \in I}$ in \mathcal{C} is a pair $(P, (\pi_i)_{i \in I})$ such that

- (1) P is an object of \mathcal{C} (called the *product object*).
- (2) For each $i \in I$, π_i is a morphism of \mathcal{C} with source P and target O_i (called the *projection morphisms*).
- (3) If B is an object of \mathcal{C} and $(f_i)_{i \in I}$ are morphisms of \mathcal{C} with $f_i : B \rightarrow O_i$, then there is a unique morphism $\langle (f_i)_{i \in I} \rangle_P : B \rightarrow P$ of \mathcal{C} such that

$$\langle (f_i)_{i \in I} \rangle_P, \pi_i \circ_C = f_i$$

for all $i \in I$.

The morphism described in the third condition above is dependant not only on the product object P , but also on the choice of projection morphisms. However in this document we will never consider a single object as a product in multiple ways so the notation will always be unambiguous.

We are often interested in the cases when I is $\{0, 1, \dots, n-1\}$ for some $n \in \mathbb{N}$. In this case we will often write $\langle f_0, f_1, \dots, f_{n-1} \rangle_P$ instead of $\langle (f_i)_{i \in \{0, 1, \dots, n-1\}} \rangle_P$.

Example 2.4 (Important categories, Assumed Knowledge: 1.5, 2.2, 2.3).

The most notable example of a category is the category of sets and functions. All sets are objects of this category and the morphisms are tuples (A, f, B) where $f : A \rightarrow B$ is a function. The source and target of (A, f, B) are A, B respectively, and the composition of $(A, f, B), (B, g, C)$ is (A, fg, C) . We cannot simply use functions as the morphisms of this category because the target of a function is not deducible from the function itself, however we will sometimes abuse language and treat functions as morphisms to avoid clunky notation. As one would expect this category has products for arbitrary collections of sets and they are precisely as described in Definition 1.5.

The “category of categories and functors” is also a natural candidate, but this fails to be a category as big categories cannot be elements of classes. However small categories and functors do form a category in the natural fashion.

3. TOPOLOGY

This section consists mostly of introducing terminology and facts from topology which will be using throughout the document. Polish spaces (3.25) will be of particular interest, and Baire Category (3.28) plays a prominent role in the study of these spaces.

In our view of topology we are occasionally interested the topology of a space as an object in its own right. For example in the Rubin construction (5.47) the partial order on a given topology is crucial, and in Part 3 we frequently compare multiple topologies on the same set.

Definition 3.1 (Topological space, Assumed Knowledge: 1.2, 1.8, 1.9).

A *topological space* is a pair (X, \mathcal{T}) where X is a set and $\mathcal{T} \subseteq \mathcal{P}(X)$ satisfies:

- (1) If $F \subseteq \mathcal{T}$ is finite, then $\bigcap F \in \mathcal{T}$.
- (2) If $S \subseteq \mathcal{T}$, then $\bigcup S \in \mathcal{T}$.
- (3) Both \emptyset and X are elements of \mathcal{T} .

In this case we say that \mathcal{T} is a topology on X . Recall that we allow empty unions/intersections (Definition 1.8) so the third condition above is technically redundant. When convenient we will often identify the pair (X, \mathcal{T}) with the set X , particularly in the use of the symbols \in, \subseteq etc. We will refer to elements of \mathcal{T} as *open sets* and sets whose complement is in \mathcal{T} as *closed sets*. If a set is both open and closed then we will call it *clopen*.

Definition 3.2 (Subspaces, Assumed Knowledge: 3.1).

If (X, \mathcal{T}) is a topological space and $A \subseteq X$, then we will view A as a topological space with the topology

$$\mathcal{T} \upharpoonright_A := \left\{ U \cap A \mid U \in \mathcal{T} \right\}.$$

This is called the *subspace topology* on A .

Definition 3.3 (Continuous maps and homeomorphisms, Assumed Knowledge: 1.6, 2.2, 2.4, 3.1).

If X and Y are topological spaces, then we say that a function $f : X \rightarrow Y$ is *continuous* if whenever

$U \subseteq Y$ is open, $(U)f^{-1}$ is also open. We say that f is a *homeomorphism* if it is a bijection and f^{-1} is also continuous (in this case we say that X and Y are *homeomorphic*).

It is routine to verify that topological spaces and continuous maps form a category (2.2), and homeomorphisms are the isomorphisms of this category.

Definition 3.4 (Closure and interior, Assumed Knowledge: 3.1).

If X is a topological space and $S \subseteq X$, then we define

$$S^- := \bigcap \left\{ F \subseteq X \mid F \text{ is closed and } F \supseteq S \right\}, \quad S^\circ := \bigcup \left\{ U \subseteq X \mid U \text{ is open and } U \subseteq S \right\}.$$

These are called the *closure* and *interior* of S respectively. Note that the closure of a set is always closed and the interior of a set is always open. We often use these notations together with the notation for complement (1.9) horizontally together in one superscript to denote doing them in succession, for example $S^{-c}(= S^{c\circ})$ denotes the complement of the closure of the set S .

Definition 3.5 (Neighbourhoods, Assumed Knowledge: 3.4).

If X is a topological space $S \subseteq X$ and $x \in X$, then we say that S is a *neighbourhood* (often written *nbhd*) of x if $x \in S^\circ$. We denote the set of all neighbourhoods of x by $\text{Nbhd}_X(x)$. Note that a set S is open if and only if S is a neighbourhood of each of its elements.

The set of neighbourhoods (3.5) of a point in a topological space form a filter (1.11). We will later see that it is precisely these filters which allow the Rubin construction (5.47) to construct the elements of Rubin spaces.

Definition 3.6 (Discrete and trivial topologies, Assumed Knowledge: 3.1).

If X is a set, then the *discrete topology on X* is $\mathcal{P}(X)$ and the *trivial topology on X* is $\{\emptyset, X\}$. A topological space equipped with the discrete topology is called a *discrete space*.

Definition 3.7 (Subbasis, Assumed Knowledge: 3.1, 3.6).

Suppose that X is a set, B is a collection of subsets of X , and \mathcal{T} is the smallest topology on X containing B (this topology always exists because B is contained in the discrete topology on X). In this case we say that B is a *subbasis for \mathcal{T}* or that \mathcal{T} is the *topology generated by B* .

Definition 3.8 (Basis, Assumed Knowledge: 3.7).

Suppose that X is a set, B is a collection of subsets of X , and $\mathcal{T} = \left\{ \bigcup S \mid S \subseteq B \right\}$ is a topology. In this case we say that B is a *basis for \mathcal{T}* . In particular any basis for a topology is a subbasis for that topology. Also if S is a subbasis for a topology \mathcal{T} on X , then $\left\{ \bigcap F \mid F \subseteq S \text{ is finite} \right\}$ is a basis for \mathcal{T} .

Definition 3.9 (Second countable, Assumed Knowledge: 1.7, 3.8).

We say that a topology/topological space is *second countable* if the topology admits a countable basis.

Definition 3.10 (Isolated points, Assumed Knowledge: 3.1).

If X is a topological space and $x \in X$, then we say that x is an *isolated point* of X if $\{x\}$ is open.

Definition 3.11 (Product topology, Assumed Knowledge: 1.5, 2.3, 3.3, 3.7).

If $(X_i)_{i \in I}$ are topological spaces, then we view $\prod_{i \in I} X_i$ as a topological space with the topology generated by the sets of the form $(U)\pi_i^{-1}$ where U is open in X_i . We call this topology the *pointwise topology* or the *product topology*.

Note that (with respect to the product topology) the projections maps are all continuous and map open sets to open sets. It is routine to verify that this construction gives a categorical product in the category of topological spaces and continuous functions (see Definition 2.3).

As we will see later, the products above are very useful for constructing continuous maps. They are also very useful for proving that nice looking maps are indeed continuous, as we can often show that they are continuous by constructing them as compositions of maps of the form $\langle f, g \rangle_P$ (see Definition 2.3) for appropriate choices of f, g and P .

Definition 3.12 (Union topology, Assumed Knowledge: 3.2).

If $(X_i)_{i \in I}$ are disjoint topological spaces, then we view $\bigcup_{i \in I} X_i$ as a topological space with the topology

$$\left\{ U \subseteq \bigcup_{i \in I} X_i \mid U \cap X_i \text{ is open in } X_i \text{ for all } i \in I \right\}.$$

We call this topology the *disjoint union topology*.

It is routine to verify that this is a topology and for all $i \in I$, the original topology on X_i agrees with the subspace topology on $X_i \subseteq \bigcup_{i \in I} X_i$.

Definition 3.13 (Zero-dimensional, Assumed Knowledge: 3.7).

We say that a topological space X is *zero-dimensional* if it has a subbasis consisting of clopen sets.

Definition 3.14 (Hausdorff, Assumed Knowledge: 1.6, 3.5, 3.11, 3.18).

We say that a topological space X is Hausdorff if one of the following equivalent (see Remark 3.18) conditions holds :

- (1) If $x, y \in X$ and $x \neq y$, then there exist disjoint neighbourhoods U_x, U_y of x and y respectively.
- (2) The diagonal id_X (recall Definition 1.6) of X is a closed subset of $X \times X$.

Definition 3.15 (Fréchet, Assumed Knowledge: 3.1).

We say that a topological space X is *Fréchet* if for all $x \in X$, the set $\{x\}$ is a closed subset of X .

The notion of a Fréchet space is a weakening of the notion of a Hausdorff space which is particularly useful as it gives us concrete examples of sets which must be open/closed with respect to our topology. In particular in Part 3, we will often make use of the observation that every finite subspace of a Fréchet space is discrete.

Definition 3.16 (Compact, Assumed Knowledge: 1.2, 1.11, 3.1, 3.5, 3.18).

We say that a topological space X is *compact* if any of the following equivalent (see Remark 3.18) conditions holds:

- (1) If S is a collection of open subsets of X with $\bigcup S = X$, then there is a finite $F \subseteq S$ with $\bigcup F = X$.
- (2) If S is a collection of closed subsets of X and $\bigcap F \neq \emptyset$ for all finite $F \subseteq S$, then $\bigcap S \neq \emptyset$.
- (3) If F is a proper filter on $(\mathcal{P}(X), \subseteq)$, then there is a proper filter F' on $(\mathcal{P}(X), \subseteq)$ and $x \in X$ such that $\text{Nbhd}_X(x) \cup F \subseteq F'$.

Definition 3.17 (Locally compact, Assumed Knowledge: 3.2, 3.16).

We say that a topological space X is *locally compact* if every element of X has a compact neighbourhood.

Proposition 3.18 (Compactness and Hausdorffness facts, Assumed Knowledge: 1.11, 3.3, 3.11, 3.14, 3.16).

The following facts about compactness and Hausdorffness are well known and we use them frequently throughout the document.

- (1) *The three definitions of compactness given in Definition 3.16 are equivalent.*

- (2) The two definitions of Hausdorffness given in Definition 3.14 are equivalent.
- (3) If $\phi : X \rightarrow Y$ is a continuous bijection between compact Hausdorff spaces then ϕ^{-1} is also continuous (Theorem 26.6 of [25]).
- (4) A closed subspace of a compact space is compact (Theorem 26.2 of [25]).
- (5) A continuous image of a compact space is compact (Theorem 26.5 of [25]).
- (6) A compact subspace of a Hausdorff space is closed (Theorem 26.3 of [25]).
- (7) An arbitrary product of Hausdorff spaces is Hausdorff.
- (8) An arbitrary product of compact spaces is compact (Tychonoff's Theorem) (Theorem 37.3 of [25]).
- (9) If X is a compact Hausdorff space and $x \in X$ then every neighbourhood of x contains a compact neighbourhood of x (Theorem 26.4 of [25]).
- (10) If X is a compact topological space and \mathcal{F} is an ultrafilter (Definition 1.11) on X , then there is $x \in X$ such that $\text{Nbhd}_X(x) \subseteq \mathcal{F}$.

Proof. (1) : For (1) \iff (2) see Theorem 26.9 of [25]. The equivalence of (2) and (3) follows from the observation that a filter $F \subseteq \mathcal{P}(X)$ is proper if and only if $\emptyset \notin F$, and thus $\text{Nbhd}_X(x) \cup F$ is contained in a proper filter if and only if $x \in \bigcap_{S \in F} S^-$.

(2) : Note that $x, y \in X$ and $x \neq y$, then U_x, U_y are neighbourhoods of x, y respectively if and only if $U_x \times U_y$ is a neighbourhood of (x, y) . Moreover U_x, U_y are disjoint if and only if $U_x \times U_y$ is disjoint from id_X .

(7) : Let $(X_i)_{i \in I}$ be Hausdorff topological spaces. If $x, y \in \prod_{i \in I} X_i$, and $i \in I$ is such that $(x)\pi_i \neq (y)\pi_i$, then there are disjoint neighbourhoods U_x, U_y of $(x)\pi_i, (y)\pi_i$ respectively (in X_i). The sets $(U_x)\pi_i^{-1}$ and $(U_y)\pi_i^{-1}$ are then the required neighbourhoods of x and y .

(10) : This is immediate from the third definition of compactness in Definition 3.16. \square

Definition 3.19 (Cantor space, Assumed Knowledge: 3.11, 3.6).

We define the *Cantor space* $2^{\mathbb{N}}$ to be the set $\{0, 1\}^{\mathbb{N}}$ with the product topology (where $\{0, 1\}$ has the discrete topology).

Theorem 3.20 (Brouwer's Theorem, Assumed Knowledge: 3.3, 3.9, 3.10, 3.13, 3.14, 3.16, 3.19).

Any second countable, zero-dimensional, Hausdorff, compact topological space with no isolated points is homeomorphic to either \emptyset or $2^{\mathbb{N}}$.

There are many known proofs of Brouwer's Theorem, see for example Theorem 7.4 of [20].

Definition 3.21 (Metric spaces, Assumed Knowledge: 1.1, 1.5).

If (X, d) is a pair where $d : X \times X \rightarrow [0, \infty)$ is a function satisfying the following for all $x, y, z \in X$:

- (1) $(x, y)d = 0 \iff x = y$.
- (2) $(x, y)d = (y, x)d$.
- (3) $(x, z)d \leq (x, y)d + (y, z)d$.

then we call (X, d) a *metric space* and we call d a *metric on X* .

Definition 3.22 (Topology of a metric space, Assumed Knowledge: 3.7, 3.21).

If (X, d) is a metric space, then we define the topology *induced by d* to be the topology \mathcal{T} generated by the sets of the form

$$B_d(x, r) := \left\{ y \in X \mid (x, y)d < r \right\}$$

for all $(x, r) \in X \times \mathbb{R}$. In this case we say that \mathcal{T} is *compatible with d* . We will always assume a metric space is equipped with this topology unless otherwise stated. We say that a topological space is *metrizable* if it is compatible with a metric.

Definition 3.23 (Convergence of sequences, Assumed Knowledge: 3.5, 3.22).

If X is a topological space and $(x_i)_{i \in \mathbb{N}}$ is a sequence of points in X , then we say that $(x_i)_{i \in \mathbb{N}}$ *converges to a limit $x \in X$* (written $(x_i)_{i \in \mathbb{N}} \rightarrow x$) if:

- (1) For all $N \in \text{Nbhds}_X(x)$, the set $\left\{ i \in \mathbb{N} \mid x_i \notin N \right\}$ is finite.

This concept is particularly useful in metrizable spaces as in these cases a function is continuous if and only if it preserves limits of sequences.

Definition 3.24 (Complete metric, Assumed Knowledge: 3.23, 3.22).

Let (X, d) be a metric space. A sequence $(x_i)_{i \in \mathbb{N}}$ in X is called *Cauchy* if it satisfies the following condition:

- (1) For all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $(x_n, x_m)d \leq \varepsilon$.

We say that d is *complete* if all Cauchy sequences in (X, d) converge.

Definition 3.25 (Polish spaces, Assumed Knowledge: 3.9, 3.24).

We say that a topological space X is *Polish*, if it is second countable and there is a complete metric on X which induces its topology.

Note that it is not true in general that if X is a Polish space, then all metrics compatible with X are complete. For example the open unit interval with the standard metric is not complete but it is homeomorphic to \mathbb{R} which is complete with respect to the standard metric.

Definition 3.26 (Denseness, Assumed Knowledge: 3.4).

Suppose that X is a topological space and $A, B \subseteq X$. We say that A is *dense in B* if $B \subseteq A^-$.

We say that A is *somewhere dense* (in X) if it is dense in a non-empty open subset of X (equivalently if $A^{-\circ} \neq \emptyset$). We say that A is *nowhere dense* (in X) if A is not somewhere dense (in X) (equivalently if $A^{-c} = A^{c\circ}$ is dense in X).

Definition 3.27 (Meagreness, Assumed Knowledge: 1.7, 3.26).

If X is a topological space and $A \subseteq X$, then we say that A is *meagre* (in X) if A is a countable union of nowhere dense sets (or equivalently if A is disjoint from the intersection of a countable collection of dense open sets).

We say that A is *comeagre* if A^c is meagre in X (or equivalently if A contains a countable intersection of dense open sets).

Theorem 3.28 (Baire Category Theorem, Assumed Knowledge: 3.22, 3.24, 3.27).

If X is a topological space which is compatible with a complete metric, then all comeagre subsets of X are dense in X .

There are many known proofs of the Baire Category Theorem, see for example Theorem 8.4 of [20].

Definition 3.29 (Almost containment, Assumed Knowledge: 1.10, 3.27).

If X is a topological space, then we define a preorder \lesssim_X on the subsets of X by

$$A \lesssim_X B \iff \text{there exists } C \text{ comeagre in } X \text{ such that } A \cap C \subseteq B \cap C.$$

We also say $A \approx_X B$ if $A \lesssim_X B \lesssim_X A$. This notation is of particularly use due to the following observations:

$$\begin{aligned} A \approx_X B &\iff \text{there is } C \text{ comeagre in } X \text{ with } C \cap A = C \cap B, \\ A \approx_X \emptyset &\iff A \text{ is meagre in } X, \\ A \approx_X X &\iff A \text{ is comeagre in } X. \end{aligned}$$

Remark 3.30 (Meagreness in subspaces, Assumed Knowledge: 3.2, 3.29).

If X is a topological space and $A \subseteq B$ are subspaces of X , then $A \approx_B \emptyset$ implies that $A \approx_X \emptyset$ (the converse is not always true).

We conclude this section with a proof of the Kuratowski-Ulam Theorem which we will use when giving a proof that all homomorphisms from groups with ample generics to second countable groups are continuous (Theorem 5.38).

The converse of the Kuratowski-Ulam Theorem also holds for sets which are almost open (Definition 4.21). However we only provide the version of the result which we will be needing. For more information see for example Theorem 8.41 of [20].

Theorem 3.31 (Kuratowski-Ulam, Assumed Knowledge: 3.9, 3.11, 3.29).

Suppose that X, Y are second countable topological spaces and $C \approx_{X \times Y} X \times Y$. Then

$$X \approx_X \left\{ x \in X \mid (\{x\} \times Y) \approx_{(\{x\} \times Y)} (\{x\} \times Y) \cap C \right\}.$$

Proof. Let $(B_i)_{i \in \mathbb{N}}$ be a countable basis for Y . Let $(U_i)_{i \in \mathbb{N}}$ be a sequence of dense open subsets of $X \times Y$ such that

$$C \supseteq \bigcap_{i \in \mathbb{N}} U_i.$$

For all $i, j \in \mathbb{N}$ let $D_{i,j} := (U_i \cap (X \times B_j))\pi_0$. As π_0 maps open sets to open sets, the sets $D_{i,j}$ are all open. Moreover as π_0 is a continuous surjection and each of the sets $U_i \cap (X \times B_j)$ is dense in $X \times B_j$, the sets $D_{i,j}$ are dense in X . So for all $i, j \in \mathbb{N}$ we have $D_{i,j} \approx_X X$ and thus

$$\bigcap_{i,j \in \mathbb{N}} D_{i,j} \approx_X X.$$

Whenever $x \in D_{i,j}$ it follows that $U_i \cap (X \times B_j) \cap (\{x\} \times Y) \neq \emptyset$. So if $x \in \bigcap_{i,j \in \mathbb{N}} D_{i,j}$ then $U_i \cap (\{x\} \times Y)$ is open and dense in $\{x\} \times Y$ for all $i \in \mathbb{N}$. The result follows. \square

4. STRUCTURES

In this section, we introduce the model theoretic view of the structures we will be working with. This allows us to talk about topological structures and define notation in a more general context. This will be useful as while we are primarily concerned with groups and semigroups we will occasionally need to use other types of structures.

Definition 4.1 (Signature, Assumed Knowledge: 1.1, 1.5).

A *signature* is a 3-tuple $\sigma = (F_\sigma, R_\sigma, \text{ar}_\sigma)$, where

- (1) F_σ is a set (whose elements are called function symbols or operation symbols).
- (2) R_σ is a set (whose elements are called relation symbols) disjoint from F_σ .
- (3) $\text{ar}_\sigma : F_\sigma \cup R_\sigma \rightarrow \mathbb{N}$ is a function (assigns each symbol an arity).

Definition 4.2 (Structure, Assumed Knowledge: 4.1).

If σ is a signature, then we say that $\mathbb{M} = (M, \sigma, I)$ is a σ -structure if

- (1) M is a set (called the *universe* of \mathbb{M}).
- (2) $I : F_\sigma \cup R_\sigma \rightarrow \bigcup_{n \in \mathbb{N}} (M^{M^n} \cup \mathcal{P}(M^n))$ is such that

$$(F)I \in M^{M^n} \iff (F \in F_\sigma \text{ and } (F) \text{ ar}_\sigma = n) \text{ and}$$

$$(R)I \in \mathcal{P}(M^n) \iff (R \in R_\sigma \text{ and } (R) \text{ ar}_\sigma = n).$$

The idea being that I interprets each element of F_σ as an operation and each element of R_σ as a relation. If $S \in F_\sigma \cup R_\sigma$ then we will use the notation $S^{\mathbb{M}}$ to denote $(S)I$. We often treat a structure \mathbb{M} as if it were equal to its universe M for ease of notation. We say that a σ -structure is *algebraic* if $R_\sigma = \emptyset$.

Definition 4.3 (Homomorphisms, Assumed Knowledge: 1.6, 2.2, 2.3, 4.2).

If \mathbb{X} and \mathbb{Y} are σ -structures and $f : \mathbb{X} \rightarrow \mathbb{Y}$ is a function, then we say that f is a *homomorphism* if for all $F \in F_\sigma$ and $R \in R_\sigma$ we have

$$F^{\mathbb{X}} \circ f = \mathbf{f}_{(F) \text{ ar}_\sigma} \circ F^{\mathbb{Y}} \quad \text{and} \quad (R^{\mathbb{X}}) \mathbf{f}_{(R) \text{ ar}_\sigma} \subseteq R^{\mathbb{Y}}$$

where $\mathbf{f}_n := \langle (\pi_i f)_{i \in \{0, 1, \dots, n-1\}} \rangle_{X^n}$ (recall Definition 2.3). We say that a homomorphism is an *embedding* if it is injective and the containment above can be strengthened to $(R^{\mathbb{X}}) \mathbf{f}_{(R) \text{ ar}_\sigma} = R^{\mathbb{Y}} \cap (\text{img}(f)^n)$.

We say that a homomorphism is an *isomorphism* if it is a surjective embedding. If there is an isomorphism between structures \mathbb{X} and \mathbb{Y} , then we say that they are *isomorphic* (denoted $\mathbb{X} \cong \mathbb{Y}$).

It is routine to verify that for a signature σ , the class of σ -structures and homomorphisms forms a category (2.2), and the notions of isomorphism from Definitions 2.2 and 4.3 coincide.

Definition 4.4 (Semigroup signature, Assumed Knowledge: 4.1).

We define σ_S to be the signature $(\{*\}, \emptyset, \{(*, 2)\})$, where $*$ is some fixed symbol. The choice of $*$ is not very important, but for example one could define $*$:= (19, 20, 01, 18).

Definition 4.5 (Semigroups and monoids, Assumed Knowledge: 4.4).

We say a σ_S -structure S is a *semigroup* if for all $a, b, c \in S$ we have $((a, b)*^S, c)*^S = (a, (b, c)*^S)*^S$. If S is a semigroup, and $a, b \in S$ then we will often write ab as an abbreviation of $(a, b)*^S$.

If S is a semigroup and $s \in S$, then we define the maps $\rho_s : S \rightarrow S$ and $\lambda_s : S \rightarrow S$ by $(t)\rho_s = ts$, and $(t)\lambda_s = st$ for all $t \in S$. These maps are dependent on the semigroup S but when used the intended semigroup is to be inferred from context.

If S is a semigroup and there is an element $e \in S$ such that $ea = ae = a$ for all $a \in S$, then we say that S is a *monoid* and e is an *identity* for S . It is routine to verify that if M is a monoid then it has a unique identity element, we will denote this element by 1_M .

Definition 4.6 (Inverse semigroup signature, Assumed Knowledge: 4.4).

We define the σ_I to be the signature $(\{*, \iota\}, \emptyset, \{(*, 2), (\iota, 1)\})$ (extending the signature σ_S) where ι is some fixed symbol. The actual value of the symbol ι is not very important, but for example one could define ι := (09, 14, 22, 05, 18, 19, 05).

Definition 4.7 (Inverse semigroup, Assumed Knowledge: 4.6).

We say a σ_I -structure $\mathbb{S} = (S, \sigma_I, I)$ is an *inverse semigroup* if $(S, \sigma_S, I \upharpoonright_{\{*\}})$ is a semigroup and for

all $a, b \in \mathbb{S}$ we have

$$((a)\iota^{\mathbb{S}})\iota^{\mathbb{S}} = a, \quad a (a)\iota^{\mathbb{S}} = a, \quad a (a)\iota^{\mathbb{S}} b (b)\iota^{\mathbb{S}} = b (b)\iota^{\mathbb{S}} a (a)\iota^{\mathbb{S}}.$$

If \mathbb{S} is an inverse semigroup and $a \in \mathbb{S}$, then we will usually denote $(a)\iota^{\mathbb{S}}$ by a^{-1} for brevity.

Definition 4.8 (Units, Assumed Knowledge: 4.2, 4.5, 4.9).

If S is a semigroup and $g \in S$, then we say that g is a *unit* if there is an element $g^{-1} \in S$ (called the *inverse* of g) such that for all $s \in S$ we have $g^{-1}gs = s = sgg^{-1}$. It is routine to verify that the inverse of any unit is unique.

Definition 4.9 (Groups, Assumed Knowledge: 4.7, 4.8).

It is routine to verify that if S is a semigroup, then the set of units $\mathcal{U}(S)$ of S is a substructure of S . We consider $\mathcal{U}(S)$ to be an inverse semigroup with the added unary operation being the map sending an element to its inverse.

We refer to the inverse semigroup $\mathcal{U}(S)$ as the *group of units* of S , and we say that an inverse semigroup is a *group* if it is the group of units of a semigroup. Note that if G is a group and $g \in G$, then gg^{-1} is an identity for G .

It is important to note that (as defined in this section), we do not consider groups or inverse semigroups to be the same objects as the semigroups obtained from them by removing their inverse operation. It will be important to keep this distinction in mind as it will have a real impact when we start talking about topological groups and topological inverse semigroups.

4.1. Topological structures.

Definition 4.10 (Topological structures, Assumed Knowledge: 2.2, 3.11, 4.2, 4.3, 4.9).

A topological σ -structure \mathbb{S} is a pair $(S, \mathcal{T}_{\mathbb{S}})$, where

- (1) S is a σ -structure.
- (2) $\mathcal{T}_{\mathbb{S}}$ is a topology on S .
- (3) For all $F \in F_{\sigma}$, the function $F^{\mathbb{S}}$ is continuous with respect to $\mathcal{T}_{\mathbb{S}}$ and the corresponding product topology.
- (4) For all $R \in R_{\sigma}$ the set $R^{\mathbb{S}}$ is closed with respect to $\mathcal{T}_{\mathbb{S}}$ and the corresponding product topology.

In this case we say that $\mathcal{T}_{\mathbb{S}}$ is *compatible* with S . We will use the established structure terminology and topology terminology for topological structures by ignoring the topological or structural part of the object as required. For example a topological group is a topological structure whose underlying structure is a group, and we consider a topological structure to be discrete if its associated topology is discrete.

We will sometimes think of a topological space as a topological structure in the signature with no symbols, and think of a structure without topology as a discrete structure.

If $f : \mathbb{S} \rightarrow \mathbb{T}$ is a function between topological structures which is both a homeomorphism of spaces and an isomorphism of structures, then we call f a *topological isomorphism*.

Topological structures and continuous homomorphisms form a category (2.2), this is essentially the intersection of the usual categories of structures and topological spaces.

Definition 4.11 (Substructures, Assumed Knowledge: 4.10).

If \mathbb{S} is a topological σ -structure and $\mathbb{M} \subseteq S$ is closed under the operations $\{F^{\mathbb{S}} : F \in F_{\sigma}\}$, then we say

that \mathbb{M} is a topological *substructure* of \mathbb{S} , and we view \mathbb{M} as a topological structure in the signature σ by restricting the operations, relations and topology in the natural fashion.

One should verify that such an object always satisfies the conditions required to be a topological structure. This follows from the observation that if X is a topological space, $A \subseteq X$ and V is a set, then the topology on A^V is the same regardless of whether we view it as a subspace of a power of X or a power of a subspace of X .

The following example of a topological structure is particularly important as many of the topological semigroups discussed in Part 3 are topological substructures of it.

Example 4.12 (Binary relation monoids are topological, Assumed Knowledge: 1.6, 3.5, 3.7, 4.5, 4.10).

If X is a set, then we define \mathcal{B}_X to be the monoid of binary relations from X to X , with composition as the binary operation (recall Definition 1.6). We define \mathcal{TB}_X to be the topology X generated by the sets of the form

$$U_{x,y} := \left\{ b \in \mathcal{B}_X \mid (x,y) \in b \right\}$$

for all $x, y \in X$. With these definitions, $(\mathcal{B}_X, \mathcal{TB}_X)$ is a topological semigroup and moreover the map $b \mapsto b^{-1}$ is continuous.

Proof. Let $x, y \in X$ be arbitrary. We first need to show that the set

$$V := \left\{ (a,b) \in \mathcal{B}_X \mid ab \in U_{x,y} \right\}$$

is open. Let $(a,b) \in V$ be arbitrary, to conclude that V is open we need only show that V is a neighbourhood of (a,b) .

As $ab \in U_{x,y}$, we have $(x,y) \in ab$. Thus by the definition of ab , there is some $z \in X$ such that $(x,z) \in a$ and $(z,y) \in b$. Thus $(a,b) \in U_{x,z} \times U_{z,y}$. By the definition of composition of binary relations we have $U_{x,z}U_{z,y} \subseteq U_{x,y}$, thus $U_{x,z} \times U_{z,y} \subseteq V^\circ$. In particular, V is a neighbourhood of (a,b) .

It remains to show that the map $b \mapsto b^{-1}$ is continuous. For all $x, y \in X$ we have

$$U_{x,y}^{-1} = \left\{ b^{-1} \in \mathcal{B}_X \mid (x,y) \in b \right\} = \left\{ b \in \mathcal{B}_X \mid (y,x) \in b \right\} = U_{y,x}.$$

Thus the map $b \mapsto b^{-1}$ fixes the subbasis for \mathcal{TB}_X (setwise) and hence is continuous as required. \square

Example 4.13 (The full transformation monoid, Assumed Knowledge: 3.11, 4.12).

If X is a set, then we define the *full transformation monoid* of X to be the set X^X with composition of functions as the operation. We also define the *pointwise topology* \mathcal{PT}_X on X^X to be the product topology obtained by viewing each copy of X with the discrete topology. The pair (X^X, \mathcal{PT}_X) is notably a topological subsemigroup of $(\mathcal{B}_X, \mathcal{TB}_X)$ (recall Example 4.12),

Definition 4.14 (Product structures, Assumed Knowledge: 2.3, 3.11, 4.10).

If $(\mathbb{S}_i)_{i \in I}$ are topological σ -structures, then we view $\mathbb{P} := \prod_{i \in I} (\mathbb{S}_i)_{i \in I}$ as a topological σ -structure as follows:

If $f \in F_\sigma$ has $(f) \text{ ar}_\sigma = n$ and $(s_{i,0})_{i \in I}, (s_{i,1})_{i \in I}, \dots, (s_{i,n-1})_{i \in I} \in \mathbb{P}$ then

$$((s_{i,0})_{i \in I}, (s_{i,1})_{i \in I}, \dots, (s_{i,n-1})_{i \in I}) f^\mathbb{P} = ((s_{i,0}, s_{i,1}, \dots, s_{i,n-1}) f^{\mathbb{S}_i})_{i \in I}.$$

If $R \in R_\sigma$ has $(R) \text{ ar}_\sigma = n$ and $(s_{i,0})_{i \in I}, (s_{i,1})_{i \in I}, \dots, (s_{i,n-1})_{i \in I} \in \mathbb{P}$ then

$$((s_{i,0})_{i \in I}, (s_{i,1})_{i \in I}, \dots, (s_{i,n-1})_{i \in I}) \in R^\mathbb{P} \iff ((s_{i,0}, s_{i,1}, \dots, s_{i,n-1}) \in R^{\mathbb{S}_i} \text{ for all } i \in I).$$

It is routine to verify that the above construction is a product in the category of topological σ -structures and continuous homomorphisms (recall Definition 2.3).

Example 4.15 (Product of posets, Assumed Knowledge: 1.10, 4.14).

If (X, \leq) is a partially ordered set (which we view as a structure in the signature with one binary relation symbol) and $n \in \mathbb{N}$, then the relation on X^n as given in Definition 4.14 is also a partial order and is defined by

$$(x_0, x_1, \dots, x_{n-1}) \leq (y_0, y_1, \dots, y_{n-1}) \iff (x_i \leq y_i \text{ for all } i < n)$$

where $x_0, x_1, \dots, x_{n-1}, y_0, y_1, \dots, y_{n-1} \in X$ are arbitrary.

Definition 4.16 (Congruences and quotients, Assumed Knowledge: 1.6, 1.10, 4.11, 4.14).

If \mathbb{S} is a structure, and \sim is an equivalence relation on \mathbb{S} , then we say that \sim is a *congruence* on \mathbb{S} if \sim is a substructure of \mathbb{S}^2 . In this case we view the set $Q := \mathbb{S}/\sim$ as a σ -structure (called the *quotient* of \mathbb{S} by \sim) as follows:

If $f \in F_\sigma$, $R \in R_\sigma$ and $s_0, s_1, \dots, s_{n-1} \in \mathbb{S}$ then

$$\begin{aligned} ([s_0]_\sim, [s_1]_\sim, \dots, [s_{n-1}]_\sim) f^Q &= [(s_0, s_1, \dots, s_{n-1}) f^{\mathbb{S}}]_\sim, \\ ([s_0]_\sim, [s_1]_\sim, \dots, [s_{n-1}]_\sim) \in R^Q &\iff (([s_0]_\sim \times [s_1]_\sim \times \dots \times [s_{n-1}]_\sim) \cap R^{\mathbb{S}}) \neq \emptyset. \end{aligned}$$

This is well-defined precisely because \sim is a congruence.

For an important example of this, note that if \mathbb{S}, \mathbb{T} are σ -structures and $f : \mathbb{S} \rightarrow \mathbb{T}$ is a homomorphism, then $\ker(f)$ (recall Definition 1.6) is a congruence on \mathbb{S} . Moreover the induced map $f^* : \mathbb{S}/\ker(f) \rightarrow \mathbb{T}$ is an injective homomorphism. When this induced map f^* is an isomorphism, we call f a *quotient map*.

4.2. Polish semitopological groups are nice. In this subsection we give a proof of a key result from the literature: That comparable Polish topologies compatible with a group are always equal (Theorem 4.27). This fact is the primary reason we are interested in Polish spaces as opposed to any other class. While the topological assumptions are quite strong, this result is very useful when investigating the potential topologies on a group. While this result does not hold for semigroups in general or even inverse semigroups, as we will see in Part 3, it will still be useful to us in these more general algebraic contexts.

Definition 4.17 (Semitopological semigroups, Assumed Knowledge: 4.5, 4.7, 4.10, 4.12).

We define a *semitopological semigroup* to be a pair (S, \mathcal{T}) such that

- (1) S is a semigroup.
- (2) \mathcal{T} is a topology on S .
- (3) For all $s \in S$, the maps λ_s and ρ_s are continuous (recall Definition 4.5).

Similarly we define a *semitopological inverse semigroup* to be a pair (I, \mathcal{T}) such that (S, \mathcal{T}) is a semitopological semigroup (where S the semigroup obtained by removing the unary operation of I). In these cases we say that \mathcal{T} is *semicompatible* with S or I .

For each $s \in S$ let $c_{S,s} : S \rightarrow S$ be the constant map with value s . As

$$\lambda_s = \langle c_{S,s}, \text{id}_S \rangle_{S^2} \circ *^S \quad \text{and} \quad \rho_s = \langle \text{id}_S, c_{S,s} \rangle_{S^2} \circ *^S \quad (\text{recall Definition 2.3})$$

it follows that all topologies compatible with S are also semicompatible with S .

The contents of Lemma 4.18, Lemma 4.19 and Corollary 4.20 are well-known. However how these facts are stated and grouped tends to vary depending on the author's needs. For example in [20], these facts can be found as (parts of) Proposition 3.3, Theorem 3.11, Lemma 13.2 and Lemma 13.3.

Lemma 4.18 (Nice Polish subspaces, Assumed Knowledge: 3.2, 3.25).

If (X, \mathcal{T}) is a Polish topological space and $U \subseteq X$ is open or closed, then the topological space $(U, \mathcal{T} \upharpoonright_U)$ is Polish.

Proof. First observe that a subspace of a second countable space is always second countable so we need only show that the subspaces are compatible with complete metrics. Let d be a complete metric compatible with (X, \mathcal{T}) . First suppose that U is closed. In this case any Cauchy sequence of $(U, d \upharpoonright_{U \times U})$ has a limit in X , which must be an element of U due to U being closed. We next suppose that U is open. We define a map $d' : U \rightarrow \mathbb{R}$ by

$$(x)d' = \inf \left(\left\{ (x, v)d \mid v \in X \setminus U \right\} \right).$$

Note that if $\varepsilon > 0$, $x, y \in U$ and $(x, y)d \leq \varepsilon$ then $|(x)d' - (y)d'| \leq \varepsilon$, so the map d' is continuous.

As $X \setminus U$ is closed, it follows that $\text{img}(d') \subseteq (0, \infty)$. We define a new map $\delta : U \times U \rightarrow \mathbb{R}$ by

$$(x, y)\delta = (x, y)d + \left| \frac{1}{(x)d'} - \frac{1}{(y)d'} \right|.$$

It is routine to verify that δ satisfies the first two conditions of being a metric on U . For the third condition, note that if $x, y, z \in U$, then

$$\begin{aligned} (x, z)\delta &= (x, z)d + \left| \frac{1}{(x)d'} - \frac{1}{(z)d'} \right| \\ &\leq (x, y)d + (y, z)d + \left| \frac{1}{(x)d'} - \frac{1}{(y)d'} \right| + \left| \frac{1}{(y)d'} - \frac{1}{(z)d'} \right| = (x, y)\delta + (y, z)\delta. \end{aligned}$$

If $(x_i)_{i \in \mathbb{N}}$ is a sequence in U , then if $(x_i)_{i \in \mathbb{N}}$ is Cauchy with respect to δ , then $(x_i)_{i \in \mathbb{N}}$ is Cauchy with respect to d and hence converges with respect to $\mathcal{T} \upharpoonright_U$. Similarly if $(x_i)_{i \in \mathbb{N}}$ converges to $x \in U$ with respect to $\mathcal{T} \upharpoonright_U$, then

$$\lim_{i \in \mathbb{N}} (x, x_i)\delta = \lim_{i \in \mathbb{N}} \left((x, x_i)d + \left| \frac{1}{(x)d'} - \frac{1}{(x_i)d'} \right| \right) = 0.$$

Thus the metric δ is complete and compatible with $\mathcal{T} \upharpoonright_U$ as required. \square

Lemma 4.19 (More nice Polish spaces, Assumed Knowledge: 2.3, 3.11, 4.18).

Suppose that $((X_i, \mathcal{T}_i)_{i \in \mathbb{N}}$ are Polish topological spaces and consider the space (I, \mathcal{T}_I) where I is the intersection of the sets X_i and \mathcal{T}_I is the topology on I generated by all of the subspace topologies $\mathcal{T}_i \upharpoonright_I$.

The space $\prod_{i \in \mathbb{N}} X_i$ is also Polish (with the usual product topology). Moreover, if there is a Hausdorff space (X, \mathcal{T}) such that for all $i \in \mathbb{N}$, we have $X_i \subseteq X$ and $\mathcal{T} \upharpoonright_{X_i} \subseteq \mathcal{T}_i$, then \mathcal{T}_I is Polish.

Proof. For each $i \in \mathbb{N}$, let d_i be a metric compatible with X_i . Then for each $i \in \mathbb{N}$, let $d'_i : X_i \times X_i \rightarrow \mathbb{R}$ be defined by $(x, y)d'_i = \min(\{(x, y)d_i, \frac{1}{2^i}\})$. We now define a metric d on $\prod_{i \in \mathbb{N}} X_i$ by

$$(x, y)d = \max_{i \in \mathbb{N}} ((x)\pi_i, (y)\pi_i)d'_i.$$

It is routine to verify that this is a well defined complete metric compatible with the product topology on $\prod_{i \in \mathbb{N}} X_i$. Thus $\prod_{i \in \mathbb{N}} X_i$ is Polish. Consider the subspace

$$D := \left\{ x \in \prod_{i \in \mathbb{N}} X_i \left| \begin{array}{l} (x)\pi_i = (x)\pi_j \\ \text{for all } i, j \in \mathbb{N} \end{array} \right. \right\} = \left\{ x \in \prod_{i \in \mathbb{N}} X_i \left| \begin{array}{l} (x)\langle \pi_i, \pi_j \rangle_{X^2} \in \text{id}_X \\ \text{for all } i, j \in \mathbb{N} \end{array} \right. \right\} = \bigcap_{i, j \in \mathbb{N}} (\text{id}_X)\langle \pi_i, \pi_j \rangle_{X^2}^{-1}$$

of $\prod_{i \in \mathbb{N}} X_i$. As (X, \mathcal{T}) is Hausdorff, the space D is a closed subspace of the Polish space $\prod_{i \in \mathbb{N}} X_i$. By Lemma 4.18, it follows that D is a Polish space. It follows from the definition of D that $\langle (\text{id}_I)_{i \in \mathbb{N}} \rangle_{\prod_{i \in \mathbb{N}} X_i} : I \rightarrow D$ is a continuous bijection. As π_i is the inverse of this map for every choice of $i \in \mathbb{N}$, it follows that this is a homeomorphism from I to a Polish space. The result follows. \square

Corollary 4.20 (Even more nice Polish spaces, Assumed Knowledge: 2.3, 3.11, 4.18).

Suppose that (X, \mathcal{T}) is a Polish topological space and $(V_i)_{i \in \mathbb{N}}$ are closed subsets of X . In this case the subspace $\bigcap_{i \in \mathbb{N}} X \setminus V_i$ of (X, \mathcal{T}) is Polish, and the topology generated by \mathcal{T} and all of the sets $(V_i)_{i \in \mathbb{N}}$ is also Polish.

Proof. From Lemma 4.18 each of the spaces $X \setminus V_i$ is Polish, so their intersection is also Polish from Lemma 4.19.

Let $i \in \mathbb{N}$ be arbitrary. By Lemma 4.18, each of V_i and $X \setminus V_i$ is a Polish subspace of X . As the topology \mathcal{T}_i generated by \mathcal{T} and V_i is equal to the disjoint union topology of V_i and $X \setminus V_i$, it follows that it is Polish. Thus from Lemma 4.19, it follows that the topology generated by all of the topologies \mathcal{T}_i is also Polish as required. \square

Definition 4.21 (Almost open sets, Assumed Knowledge: 3.1, 3.29).

If X is a topological space and $S \subseteq X$, then we say that S is *almost open* if there is an open set U such that $U \approx_X S$ (recall Definition 3.29).

Note that there is no relation between algebraic σ structures as defined earlier and σ -algebras from the following definition.

Definition 4.22 (σ -algebras, Assumed Knowledge: 1.2, 1.7).

Suppose that X is a set and $\Sigma \subseteq \mathcal{P}(X)$ is such that

- (1) If $A \in \Sigma$, then $A^c = X \setminus A \in \Sigma$.
- (2) If $A \subseteq \Sigma$ is countable, then $\bigcup A \in \Sigma$.
- (3) The empty set is an element of Σ .

In this case we say that Σ is a σ -algebra on X . Recall that we allow empty unions (Definition 1.8) so the third condition is technically redundant.

Lemma 4.23 (cf Proposition 8.22 of [20], Almost open σ -algebra, Assumed Knowledge: 4.21, 4.22).
If X is a topological space, then the almost open subsets of X form a σ -algebra.

Proof. Let $A \subseteq X$ be almost open. Let U be an open subset of X such that $U \approx_X A$. Note that

$$U^c = U^{c\circ} \cup (U^c \setminus U^{c\circ}) = U^{c\circ} \cup (U^c \cap U^{c\circ c}) = U^{c\circ} \cup (U^c \cap U^{c\circ -}) = U^{c\circ} \cup (U^c \cap U^-).$$

As $U^c \cap U^-$ is closed and $(U^c \cap U^-)^\circ = U^{c\circ} \cap U^{-\circ} = U^{-c} \cap U^{-\circ} = \emptyset$, it follows that $U^c \cap U^- \approx_X \emptyset$. Thus $A^c \approx_X U^c = U^{c\circ} \cup (U^c \cap U^-) \approx_X U^{c\circ}$ and so A^c is almost open.

Suppose that A is a countable collection of almost open sets. If A is empty then $\bigcup A = \emptyset$ is almost open, otherwise let $A = \{A_i \mid i \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let U_i be open and let C_i be comeagre such

that $U_i \cap C_i = A_i \cap C_i$. It follows that

$$\bigcup A \approx_X \left(\bigcap_{i \in \mathbb{N}} C_i \right) \cap \left(\bigcup_{i \in \mathbb{N}} A_i \right) = \left(\bigcap_{i \in \mathbb{N}} C_i \right) \cap \left(\bigcup_{i \in \mathbb{N}} U_i \right) \approx_X \bigcup_{i \in \mathbb{N}} U_i.$$

As $\bigcup_{i \in \mathbb{N}} U_i$ is open, the result follows. \square

Definition 4.24 (Analytic spaces, Assumed Knowledge: 3.3, 3.25).

We say that a topological space X is *analytic* if it is the image of a Polish space under a continuous map.

Lemma 4.25 (cf. Theorem 7.9 of [20], $\mathbb{N}^{\mathbb{N}}$ is a nice space, Assumed Knowledge: 1.2, 3.3, 3.25, 4.13, 4.19).

The topological space $(\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$ from Example 4.13 is Polish. Moreover if X is a non-empty Polish topological space, then X is the image of a continuous map from $(\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$.

Proof. The fact that $\mathbb{N}^{\mathbb{N}}$ is Polish follows from the fact that \mathbb{N} is Polish together with Lemma 4.19.

Let d be a complete metric compatible with X and let $B := \{U_i \mid i \in \mathbb{N}\}$ be a countable basis for X consisting of non-empty sets. For each $i \in \mathbb{N}$, we choose a fixed $d_i \in U_i$. As B is a basis for X , it follows that $D := \{d_i \mid i \in \mathbb{N}\}$ is dense in X and so for all $x \in X$, there are sequences of points in D converging to x .

We say that a sequence $(x_i)_{i \in \mathbb{N}}$ in X is strongly Cauchy if for all $i \in \mathbb{N}$ we have $(x_i, x_{i+1})d \leq \frac{1}{2^i}$. It is routine to verify that strongly Cauchy sequences are Cauchy.

We now define

$$C := \left\{ f \in \mathbb{N}^{\mathbb{N}} \mid (d_{(i)f})_{i \in \mathbb{N}} \text{ is a strongly Cauchy sequence} \right\}.$$

From the definition of a strongly Cauchy sequence, it follows that the subspace C of $\mathbb{N}^{\mathbb{N}}$ is closed. As strongly Cauchy sequences are Cauchy and d is complete, it follows that the map $\phi : C \rightarrow X$ given by

$$(f)\phi = \lim_{i \rightarrow \infty} d_{(i)f}$$

is well defined. It is routine to verify that the map ϕ is also continuous. Moreover as D is dense in X , the map ϕ is surjective as well.

Let $c : \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \setminus \{\emptyset\} \rightarrow \mathbb{N}^{\mathbb{N}}$ be such that $(S)c \in S$ for all $S \in \mathcal{P}(\mathbb{N}^{\mathbb{N}}) \setminus \{\emptyset\}$. For each $f \in \mathbb{N}^{\mathbb{N}} \setminus C$, we define $m_f := \max \left(\left\{ m \in \mathbb{N} \mid \text{there exists } h \in C \text{ with } h \upharpoonright_m = f \upharpoonright_m \right\} \right)$ (note that this is only well defined because C is closed). We then define a map $\psi : \mathbb{N}^{\mathbb{N}} \rightarrow C$ as follows

$$(f)\psi = \left\{ \begin{array}{ll} f & \text{if } f \in C \\ \left(\left\{ g \in C \mid g \upharpoonright_{m_f} = f \upharpoonright_{m_f} \right\} \right) c & \text{if } f \notin C \end{array} \right\}.$$

This map $\psi : \mathbb{N}^{\mathbb{N}} \rightarrow C$ is surjective by definition, it is also routine to verify that ψ is continuous. It follows that $\psi\phi$ is a continuous surjection from $\mathbb{N}^{\mathbb{N}}$ to X as required. \square

Lemma 4.26 (cf. Theorem 14.7 of [20], Lusin Separation Theorem, Assumed Knowledge: 3.2, 3.14, 4.22, 4.24, 4.25).

Suppose that (X, \mathcal{T}) is a Hausdorff topological space and Σ is a σ -algebra on X with $\mathcal{T} \subseteq \Sigma$. If $\{U, V\}$ is a partition of X into analytic subspaces of (X, \mathcal{T}) , then $\{U, V\} \subseteq \Sigma$.

Proof. If $U = \emptyset$ or $V = \emptyset$, then the result is clear. Otherwise both U and V are images of non-empty Polish spaces under continuous maps. Thus from Lemma 4.25, there are continuous maps $\phi_U : \mathbb{N}^{\mathbb{N}} \rightarrow X, \phi_V : \mathbb{N}^{\mathbb{N}} \rightarrow X$ with $\text{img}(\phi_U) = U$ and $\text{img}(\phi_V) = V$.

We say $A, B \subseteq X$ are *separated by* Σ if there are $A', B' \in \Sigma$ such that $A \subseteq A', B \subseteq B'$ and $A' \cap B' = \emptyset$. It suffices to show that U and V are separated by Σ .

Claim: Suppose that $(U_i)_{i \in \mathbb{N}}, (V_i)_{i \in \mathbb{N}}$ are sequences of subsets of X such that $\bigcup_{i \in \mathbb{N}} U_i$ and $\bigcup_{i \in \mathbb{N}} V_i$ are not separated by Σ . In this case there are $i, j \in \mathbb{N}$ such that U_i and V_j are not separated by Σ .

Proof of Claim: Suppose for a contradiction that U_i and V_j are separated by Σ for all $i, j \in \mathbb{N}$. For each $i, j \in \mathbb{N}$ let $U'_{i,j}, V'_{i,j} \in \Sigma$ be disjoint such that $U_i \subseteq U'_{i,j}$ and $V_j \subseteq V'_{i,j}$. We define

$$U' := \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}} U'_{i,j} \quad \text{and} \quad V' := \bigcup_{j \in \mathbb{N}} \bigcap_{i \in \mathbb{N}} V'_{i,j}.$$

We have that $U', V' \in \Sigma$ are disjoint, $\bigcup_{i \in \mathbb{N}} U_i \subseteq U'$ and $\bigcup_{i \in \mathbb{N}} V_i \subseteq V'$. As $U' \cap V' = \emptyset$, it follows that $\bigcup_{i \in \mathbb{N}} U_i$ and $\bigcup_{i \in \mathbb{N}} V_i$ are separated by Σ . This is a contradiction. \diamond

Suppose for a contradiction are U and V are not separated by Σ . For each $n \in \mathbb{N}$, we define $f_n : \{0, 1, \dots, n-1\} \rightarrow \mathbb{N}$, and $g_n : \{0, 1, \dots, n-1\} \rightarrow \mathbb{N}$ as follows:

- (1) $f_0 = g_0 = \emptyset$.
- (2) If f_i is already defined and the sets

$$\left(\left\{ f \in \mathbb{N}^{\mathbb{N}} \mid f_i \subseteq f \right\} \right) \phi_U \quad \text{and} \quad \left(\left\{ g \in \mathbb{N}^{\mathbb{N}} \mid g_i \subseteq g \right\} \right) \phi_V$$

are not separated by Σ , then by the claim choose f_{i+1}, g_{i+1} such that $f_i \subseteq f_{i+1}, g_i \subseteq g_{i+1}$ and $\left(\left\{ f \in \mathbb{N}^{\mathbb{N}} \mid f_{i+1} \subseteq f \right\} \right) \phi_U$ and $\left(\left\{ g \in \mathbb{N}^{\mathbb{N}} \mid g_{i+1} \subseteq g \right\} \right) \phi_V$ are not separated by Σ .

We can then define $f := \bigcup_{i \in \mathbb{N}} f_i$ and $g := \bigcup_{i \in \mathbb{N}} g_i$. As X is Hausdorff and $(f)\phi_U \neq (g)\phi_V$ (ϕ_U and ϕ_V has disjoint images), we can choose open $U_f \in \text{Nbhd}_X((f)\phi_U)$ and $V_g \in \text{Nbhd}_X((g)\phi_V)$ such that $U_f \cap V_g = \emptyset$. As ϕ_U, ϕ_V are both continuous, there is some $N \in \mathbb{N}$ such that

$$(U_f)\phi_U^{-1} \supseteq \left\{ f \in \mathbb{N}^{\mathbb{N}} \mid f_N \subseteq f \right\} \quad \text{and} \quad (V_g)\phi_V^{-1} \supseteq \left\{ g \in \mathbb{N}^{\mathbb{N}} \mid g_N \subseteq g \right\}.$$

As $U_f, V_g \in \Sigma$, it follows that $\left(\left\{ f \in \mathbb{N}^{\mathbb{N}} \mid f_N \subseteq f \right\} \right) \phi_U$ and $\left(\left\{ g \in \mathbb{N}^{\mathbb{N}} \mid g_N \subseteq g \right\} \right) \phi_V$ are separated by Σ , this is a contradiction. \square

Theorem 4.27 (Comparable Polish group topologies, Assumed Knowledge: 3.25, 4.10, 4.17).

Suppose that G, H are groups, and $\mathcal{T}_G, \mathcal{T}_H$ are Polish topologies semicompatible with G, H respectively. If $\phi : G \rightarrow H$ is an isomorphism and ϕ is continuous with respect to \mathcal{T}_G and \mathcal{T}_H , then ϕ^{-1} is also continuous.

Proof. We need only show that if $(x_i)_{i \in \mathbb{N}}$ is a sequence of elements of H which converge to some $x \in H$ with respect to \mathcal{T}_H , and $S := \{x\} \cup \{x_i \mid i \in \mathbb{N}\}$, then $\phi^{-1} \upharpoonright_S$ is continuous.

Let $B = \{U_i \mid i \in \mathbb{N}\}$ be a countable basis for the topology \mathcal{T}_G . By Lemma 4.18, the sets U_i and $X \setminus U_i$ are analytic (with respect to \mathcal{T}_G) for each $i \in \mathbb{N}$. Therefore as ϕ is continuous, it follows that $(U_i)\phi$ and $(X \setminus U_i)\phi$ are analytic with respect to \mathcal{T}_H . From Lemmas 4.23 and 4.26, it follows that U_i and $X \setminus U_i$ are almost open with respect to \mathcal{T}_H . Thus there is a comeagre subset C of H such that for all $i \in \mathbb{N}$, the set $U_i \cap C$ is open in $\mathcal{T}_H \upharpoonright_C$. In particular $\phi^{-1} \upharpoonright_C$ is continuous.

As C is comeagre in the semitopological group (H, \mathcal{T}_H) , it follows that $C \cap Cx^{-1} \cap (\bigcap_{i \in \mathbb{N}} Cx_i^{-1})$ is also comeagre in (H, \mathcal{T}_H) . So from Theorem 3.28, it follows that there is some $h \in C \cap Cx^{-1} \cap (\bigcap_{i \in \mathbb{N}} Cx_i^{-1})$. Thus $(S)\lambda_h \subseteq C$. It follows that $\phi^{-1} \upharpoonright_S = \lambda_h \upharpoonright_S \circ \phi^{-1} \upharpoonright_C \circ \lambda_{(h^{-1})\phi^{-1}}$, and is thus continuous as required. \square

4.3. Words and terms. In this subsection we introduce words and terms. Words are used constantly throughout Part 4 and terms give us access to a large collection of continuous maps to work with given a topological structure (they are also used to define the Zariski topology in Part 3).

Definition 4.28 (Words, Assumed Knowledge: 1.4, 1.6, 1.10, 4.5).

If X is a set, then we denote the *free monoid* on X by X^* . The universe of this monoid is the set

$$X^* := \bigcup_{n \in \mathbb{N}} X^n.$$

If $w \in X^* \cup X^{\mathbb{N}}$, then we call w a *word* or *string* in the *alphabet* X . In this context we will refer to the elements of X as letters. We denote by $|w|$ the *length* of the word w , that is the value $n \in \mathbb{N} \cup \{\aleph_0\}$ such that $w \in X^n$ (conveniently this is the same as the cardinality of w so this notation is unambiguous). If $n \in \mathbb{N}$ then we write $w \upharpoonright_n$ to mean the restriction of the function w to the set $\{0, 1, \dots, n-1\}$, we call such restrictions the *prefixes* of w . The set $X^* \cup X^{\mathbb{N}}$ is partially ordered by the relation $v \leq w$ if v is a prefix of w (this is the same as the \subseteq relation).

If $v = (v_0, v_1, \dots, v_{|v|-1}) \in X^*$ and $w = (w_0, w_1, \dots) \in X^* \cup X^{\mathbb{N}}$, then the *concatenation* of v and w is defined by

$$vw = (v_0, v_1, \dots, v_{|v|-1}, w_0, w_1, \dots).$$

Restricting concatenation to X^* gives the binary operation of X^* as a monoid. The identity of X^* is the element of length 0 which we call the *empty word* and denote by ε . Although technically $X \neq X^1$ (Definition 1.4), we will often for convenience treat letters and words of length 1 as the same objects.

It is routine to verify that concatenation is always *cancellative*, that is if v, u, w are finite words then

$$vu = vw \Rightarrow u = w, \quad \text{and} \quad uv = wv \Rightarrow u = w.$$

It follows from this that maps ρ_w and λ_w (using the monoid X^*) are injective for all $w \in X^*$. When infinite words are being used, we extend the domain of λ_w to allow for infinite words (note that λ_w is still injective).

Definition 4.29 (Terms of a signature, Assumed Knowledge: 4.1, 4.28).

For this definition we will need sets to represent the symbols comma, open bracket and closed bracket. To avoid confusion with the symbols being used for their English function, we use \textasciitilde to denote the set representing a comma, we use \lceil to denote the set representing an open bracket and we use \rfloor to denote the set representing a closed bracket. The actual choice of these sets is unimportant as long as they are distinct from each other and anything else they could be confused with. For example one could define

$$\begin{aligned} \text{\textasciitilde} &:= (03, 15, 13, 13, 01) \\ \lceil &:= (15, 16, 05, 14, 00, 02, 18, 01, 03, 11, 05, 20) \\ \rfloor &:= (03, 12, 15, 19, 05, 04, 00, 02, 18, 01, 03, 11, 05, 20). \end{aligned}$$

If V is a set and σ is a signature, then we inductively define a *term* of σ over the variable set V to be a string in the alphabet $\{ _ , \lceil , \rceil \} \cup F_\sigma \cup V$ which is one of:

- (1) The string x for some $x \in V$.
- (2) The string $\lceil t_0, t_1, \dots, t_n \rceil F$ where $F \in F_\sigma$ has arity n and t_0, t_1, \dots, t_n are terms.

Definition 4.30 (Term operations, Assumed Knowledge: 2.3, 3.11, 4.10, 4.29).

If \mathbb{X} is a σ -structure, and V is a set (called a *variable set*), then for each term t of σ over V we will define a corresponding *term operation* $t_V^\mathbb{X} : \mathbb{X}^V \rightarrow \mathbb{X}$. We do this inductively on the length of t as follows:

- (1) If $t \in V$, then we define $t_V^\mathbb{X}$ to be the projection map π_t .
- (2) If $t = \lceil t_0, t_1, \dots, t_n \rceil F$ where $F \in F_\sigma$ has arity n and t_0, t_1, \dots, t_n are terms, then we define $t_V^\mathbb{X} : \mathbb{X}^V \rightarrow \mathbb{X}$ by

$$t_V^\mathbb{X} = \langle t_{0V}^\mathbb{X}, t_{1V}^\mathbb{X}, \dots, t_{n-1V}^\mathbb{X} \rangle_{\mathbb{X}^n} \circ F^\mathbb{X} \quad (\text{recall Definition 2.3}).$$

It is routine to check that any term of σ over V is considered exactly once in exactly one of the above cases. Note that by construction, if \mathbb{X} is a topological structure, then all term operations of \mathbb{X} over any variable set are continuous (using the appropriate product topologies).

Example 4.31 (Term operations in semigroups and inverse semigroups, Assumed Knowledge: 4.30).

It is routine to verify that if S is a topological semigroup and V is a variable set, then the term operations $t : S^V \rightarrow S$ are precisely the maps of the form

$$((s_v)_{v \in V})t = s_{v_0} s_{v_1} s_{v_2} \dots s_{v_{n-1}}$$

where $n \in \mathbb{N}$ and $v_0, v_1, \dots, v_{n-1} \in V$. In particular these maps are continuous. Similarly if S is a topological inverse semigroup and V is a variable set, then the term operations $t : S^V \rightarrow S$ are precisely the maps of the form

$$((s_v)_{v \in V})t = s_{v_0}^{i_0} s_{v_1}^{i_1} \dots s_{v_{n-1}}^{i_{n-1}}$$

where $n \in \mathbb{N}$, $i_0, i_1, \dots, i_{n-1} \in \{1, -1\}$ and $v_0, v_1, \dots, v_{n-1} \in V$.

5. NICE ACTIONS

In this section we introduce actions of semigroups on objects from categories. We also provide proofs of the remaining key results from the literature motioned earlier. We first show that all homomorphisms from $\text{Aut}(2^{\mathbb{N}})$ or $\text{Sym}(\mathbb{N})$ to second countable groups are continuous, and we conclude by proving Rubin's Theorem which given a nice group allows one to construct a canonical action of that group on a nice topological space.

Definition 5.1 (Endomorphisms and automorphisms, Assumed Knowledge: 1.6, 2.2, 4.5, 4.9, 4.10).

If \mathcal{C} is a category, O is an object of \mathcal{C} , and the class of morphisms in \mathcal{C} with O as both source and target form a set, then we denote this set by $\text{End}(O)$ (the category being used will be given by the context).

We give $\text{End}(O)$ a monoid structure by defining the product of $f, g \in \text{End}(O)$ to be $(f, g) \circ_{\mathcal{C}}$. We then define $\text{Aut}(O)$ to be the group of units of $\text{End}(O)$ (recall Definition 4.9).

Definition 5.2 (Actions, Assumed Knowledge: 4.14, 5.1).

If S is a topological semigroup, \mathbb{X} is a topological structure and $a : \mathbb{X} \times S \rightarrow \mathbb{X}$ is a function, then we say that a is an *action* (of S on \mathbb{X}) if the map $\phi_a : S \rightarrow \text{End}(\mathbb{X})$ given by

$$(x)((f)\phi_a) = (x, f)a$$

is a well-defined homomorphism of semigroups. We define actions analogously for groups using $\text{Aut}(\mathbb{X})$ in place of $\text{End}(\mathbb{X})$. Recall that as per Definition 4.10, we consider both structures with no topology and topological spaces with no structure as topological structures. In the case that an action being used is clear from context we will sometimes write xf instead of $(x, f)a$. We say that the action a is:

- (1) *Continuous* if the map $a : \mathbb{X} \times S \rightarrow \mathbb{X}$ is continuous.
- (2) *Transitive* if for all $x, y \in \mathbb{X}$, there is $f \in S$ with $(x, f)a = y$.
- (3) *Faithful* if the map ϕ_a is injective.

Definition 5.3 (Powers of actions, Assumed Knowledge: 5.2).

If a is an action of a group G on a set X , then we define a^n to be the action of G on X^n given by:

$$((x_0, x_1 \dots x_{n-1}), g)a^n = ((x_0, g)a, (x_1, g)a, \dots, (x_{n-1}, g)a).$$

This notation always takes precedent over the notation in Definition 1.4.

Definition 5.4 (Action orbits, Assumed Knowledge: 5.2).

If a is an action of a group G on a topological structure \mathbb{X} , and $x \in \mathbb{X}^n$ then we call the set $(\{x\} \times G)a$ the *orbit* of x under a .

The following type of action (ample generics) frequently occurs in the literature in the case of a group acting on itself by conjugation. Indeed we will only be using it in this case. There are many nice examples of actions with ample generics (see Theorem 5.18 and Theorem 5.33) but proving this tends to be quite involved.

Definition 5.5 (Ample generics, Assumed Knowledge: 3.27, 5.3, 5.4).

If a is an action of a topological group G on a topological structure \mathbb{X} , then we say that a has *ample generics* if for all $n \in \mathbb{N}$ the action a^n has an orbit comeagre in \mathbb{X}^n .

Definition 5.6 (Centralizer and normalizer, Assumed Knowledge: 4.9, 4.11).

Recall that two semigroup elements f, g *commute* if $fg = gf$. If G is a group and $S \subseteq G$, then we define the *centralizer* and *normalizer* of S in G by

$$C_G(S) := \left\{ f \in G \mid f \text{ commutes with all elements of } S \right\},$$

$$N_G(S) := \left\{ f \in G \mid fS = Sf \right\}$$

respectively. It is routine to verify that $C_G(S)$ and $N_G(S)$ are always subgroups of G . In the case that S is a singleton $\{f\}$, we will often write $C_G(f)$ in place of $C_G(\{f\})$ for brevity.

If G is a group, S is a subgroup of G and $N_G(S) = G$, then we say that S is a *normal subgroup* of G (denoted $S \trianglelefteq G$). In this case we define the *quotient* of G by S by:

$$G/S := \left\{ gS \mid g \in G \right\}.$$

The set G/S is viewed as a group, and sets of elements of G are multiplied in the usual fashion.

Definition 5.7 (Conjugation action, Assumed Knowledge: 5.2, 5.6).

If G is a group, then we define $\text{conj}_G : G^2 \rightarrow G$ to be the action

$$(g, h) \text{conj}_G = h^{-1}gh.$$

We call the group $(G)\phi_{\text{conj}_G}$ (recall Definition 5.2), the *inner automorphism* group of G (denoted $\text{Inn}(G)$). If $\psi \in \text{Aut}(G)$, and $h \in G$, then $\psi^{-1} \circ ((h)\phi_{\text{conj}_G}) \circ \psi = ((h)\psi)\phi_{\text{conj}_G}$. In particular $\text{Inn}(G) \trianglelefteq \text{Aut}(G)$. We call the group $\text{Aut}(G)/\text{Inn}(G)$ the *outer automorphism group* of G and denote this group by $\text{Out}(G)$.

Before proceeding to $\text{Sym}(\mathbb{N})$ and $\text{Aut}(2^{\mathbb{N}})$, we give a nice and easy consequence of a group having ample generics (Theorem 5.9). This fact seems to be well known but we do not know of a specific reference for it.

Definition 5.8 (Commutators, Assumed Knowledge: 4.10).

If G is a topological group, then we define the *commutator map* $[\cdot, \cdot] : G^2 \rightarrow G$ by $[f, g] = f^{-1}g^{-1}fg$.

Theorem 5.9 (Commutator width 1, Assumed Knowledge: 3.28, 5.5, 5.8).

If G is a completely metrizable topological group and conj_G has a comeagre orbit (in particular if it has ample generics), then the commutator map $[\cdot, \cdot] : G^2 \rightarrow G$ is surjective.

Proof. Let C be a comeagre orbit of conj_G and let $f \in G$ be arbitrary. Then it follows that Cf is also comeagre, and thus by the Baire Category Theorem (Theorem 3.28) the set $C \cap Cf$ is dense in G . It follows that there are $g, h \in C$ such that $g = hf$. Thus $h^{-1}g = f$. As $h, g \in C$, there is $k \in G$ such that $h = k^{-1}gk$. Thus

$$f = h^{-1}g = (k^{-1}gk)^{-1}g = k^{-1}g^{-1}kg = [k, g]$$

as required. □

Definition 5.10 (Symmetric groups, Assumed Knowledge: 3.2, 4.10, 4.13).

If X is a set, then we define $\text{Sym}(X)$ to be the group of all bijections from X to itself (if $X \in \mathbb{N}$, we treat X as $\{0, 1, \dots, X-1\}$). We define the *pointwise topology* on $\text{Sym}(X)$ to be the topology $\mathcal{TB}_X \upharpoonright_{\text{Sym}(X)} = \mathcal{PT}_X \upharpoonright_{\text{Sym}(X)}$ (recall Examples 4.12 and 4.13). The collection of sets of the form

$$U_{x,y} := \left\{ f \in \text{Sym}(X) \mid (x)f = y \right\}$$

for $x, y \in X$ form a subbasis for this topology, and the collection of sets of the form

$$U_h := \left\{ f \in \text{Sym}(\mathbb{N}) \mid h \subseteq f \right\}$$

for finite $h \subseteq X^2$ form a basis. From Example 4.12, we know that this topology makes $\text{Sym}(X)$ into a topological group.

Definition 5.11 (Spaces of continuous functions, Assumed Knowledge: 3.3, 3.16, 3.7).

If X, Y are topological spaces. Then we define $C(X, Y)$ to be the set of continuous maps from X to Y . If $X = Y$ then we write $C(X)$ for brevity. The *compact-open topology* on $C(X, Y)$ is then the topology generated by the sets of the form

$$\left\{ f \in C(X, Y) \mid (F)f \subseteq U \right\}$$

where $F \subseteq X$ is compact and $U \subseteq Y$ is open.

Theorem 5.12 (cf Proposition 1.3.3 of [34], Polish compact-open topologies, Assumed Knowledge: 3.4, 3.25, 5.11).

If X and Y are compact metrizable spaces, then $C(X, Y)$ with the compact-open topology is a Polish space. Moreover if B_X and B_Y are countable bases for X and Y respectively, then the collection of sets of the form

$$B(U_X, U_Y) := \left\{ f \in C(X, Y) \mid (U_X^-)f \subseteq U_Y \right\}$$

for $U_X \in B_X$ and $U_Y \in B_Y$ generate the topology on $C(X, Y)$. Moreover if d is any metric compatible with Y , then the metric d_∞ on $C(X, Y)$ defined by

$$(f, g)d_\infty = \sup_{x \in X} ((x)f, (x)g)d$$

is complete and compatible with the compact-open topology.

It is well known (see for example Proposition 4.5 of [20]) that if $f : (X, d_x) \rightarrow (Y, d_y)$ is a continuous map between compact metric spaces then f is uniformly continuous. Meaning that for all $\varepsilon > 0$, then there is $\delta > 0$ such that

$$(a, b)d_x < \delta \Rightarrow ((a)f, (b)f)d_y < \varepsilon.$$

As a result we obtain the following corollary to Theorem 5.12.

Corollary 5.13 (Composition of continuous maps is continuous, Assumed Knowledge: 3.11, 5.11, 5.12).

Let $(X, d_x), (Y, d_y), (Z, d_z)$ be compact metric spaces. The composition map from $C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)$ is continuous (using the compact-open topologies (Definition 5.11) and the product topology (Definition 3.11)).

Proof. Let $(f, g) \in C(X, Y) \times C(Y, Z)$ and $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ be such that

$$(a, b)d_y < \delta \Rightarrow ((a)g, (b)g)d_x < \frac{\varepsilon}{2}.$$

It follows that if

$$\sup_{x \in X} ((x)f, (x)f')d_y < \delta \quad \text{and} \quad \sup_{y \in Y} ((x)g, (x)g')d_z < \frac{\varepsilon}{2},$$

then

$$\sup_{x \in X} ((x)f'g', (x)fg)d_z \leq \sup_{x \in X} (((x)f'g, (x)fg)d_z + ((x)f'g, (x)f'g')d_z) < \varepsilon.$$

The result then follows from Theorem 5.12. □

Proposition 5.14 (Polish homeomorphism groups, Assumed Knowledge: 3.18, 4.20, 5.13).

If X is a compact metrizable topological space, then the group $\text{Aut}(X)$ of homeomorphisms of X is a Polish space with the compact-open topology.

Proof. Let B be a countable basis for X with $\emptyset \notin B$. From Theorem 5.12, the sets $B(U, V) := \left\{ f \in C(X) \mid (U^-)f \subseteq V \right\}$ for $U, V \in B$ form a basis for the compact-open topology.

For all $x, y \in X$ with $x \neq y$, let

$$P_{x,y} := \left\{ (U, V) \in B^2 \mid \begin{array}{l} \text{there exist } U', V' \in B \text{ such that } x \in U \subseteq U^- \subseteq U', \\ y \in V \subseteq V^- \subseteq V', \text{ and } U' \cap V' = \emptyset \end{array} \right\}$$

Note that (by Remark 3.18 part 9) the set $P_{x,y}$ is never empty and that the set $P := \bigcup_{x,y \in X} P_{x,y} \subseteq B^2$ is countable. From Remark 3.18, $\text{Aut}(X)$ consists of the elements of $C(X)$ which are bijections. Thus

$$\text{Aut}(X) = \left(\bigcap_{(U,V) \in P} \bigcup_{\substack{U',V' \text{ open} \\ \text{and disjoint}}} B(U,U') \cap B(V,V') \right) \cap \left(\bigcap_{V \in B} \bigcup_{U \in B} B(U,V) \right).$$

is a countable intersection of sets which are open with respect to the compact-open topology. It follows from Corollary 4.20 that this subspace of $C(X)$ is also Polish. \square

5.1. The symmetric group. In [21], it is stated that “one can easily show that S_∞ has ample generics”. We do not know of a reference for this fact so we give our own proof (Theorem 5.18). As we see below, the key in showing that $\text{Sym}(\mathbb{N})$ has ample generics will be viewing elements of $\text{Sym}(\mathbb{N})$ as algebraic objects.

Definition 5.15 (Unary algebras, Assumed Knowledge: 4.1).

If $n \in \mathbb{N}$, then we define the signature $\sigma_{u,n} := (\{0, 1, \dots, n-1\}, \emptyset, \{(0,1), (1,1), \dots, (n-1,1)\})$. That is $\sigma_{u,n}$ has the n unary function symbols $\{0, 1, \dots, n-1\}$ and no other symbols. We call a $\sigma_{u,n}$ -structure an n -unary algebra. Moreover, we say that an n -unary algebra \mathbb{U} is *bijective* if for all $i \in \{0, 1, \dots, n-1\}$ the map $i^\mathbb{U}$ is a permutation of \mathbb{U} .

For example when $n = 2$ a bijective n -unary algebra consists of a set X and pair of permutations of X .

Definition 5.16 (Connected components, Assumed Knowledge: 1.10, 4.11, 5.15).

The *connected components* of an n -unary algebra \mathbb{U} are the equivalence classes of the least equivalence relation on \mathbb{U} containing the set

$$\bigcup_{i < n} i^\mathbb{U}.$$

We say that an n -unary algebra is *connected* if this relation is all of $\mathbb{U} \times \mathbb{U}$. In particular, the connected components of a unary algebra partition it into connected substructures.

Proposition 5.17 (Polish symmetric groups, Assumed Knowledge: 4.20, 4.26, 5.10).

The group $\text{Sym}(\mathbb{N})$ is a Polish space with the pointwise topology.

Proof. By Lemma 4.25, the space $\mathbb{N}^\mathbb{N}$ is Polish. As

$$\text{Sym}(\mathbb{N}) = \left(\bigcap_{a \in \mathbb{N}} \bigcup_{b \in \mathbb{N}} \{f \in \mathbb{N}^\mathbb{N} \mid (b, a) \in f\} \right) \cap \left(\bigcap_{a \neq b} \bigcup_{c \neq d} \{f \in \mathbb{N}^\mathbb{N} \mid (a, c) \in f \text{ and } (b, d) \in f\} \right),$$

the result follows from Corollary 4.20. \square

Theorem 5.18 ($\text{Sym}(\mathbb{N})$ has ample generics, Assumed Knowledge: 1.4, 3.29, 5.5, 5.7, 5.16, 5.10).

The action $\text{conj}_{\text{Sym}(\mathbb{N})}$ has ample generics (using the pointwise topology).

Proof. Let $n \in \mathbb{N}$ be fixed, and let

$$C := \left\{ y \in \text{Sym}(\mathbb{N})^n \left| \begin{array}{l} \text{the connected components of the unary algebra } (\mathbb{N}, \sigma_{u,n}, y) \\ \text{consist of countably infinitely many copies of every finite} \\ \text{bijective connected } n\text{-unary algebra.} \end{array} \right. \right\}$$

We will first show that C is an orbit of $\text{conj}_{\text{Sym}(\mathbb{N})}^n$.

Note that $x, y \in \text{Sym}(\mathbb{N})^n$ are in the same orbit precisely when the corresponding unary algebras $(\mathbb{N}, \sigma_{u,n}, x)$ and $(\mathbb{N}, \sigma_{u,n}, y)$ are isomorphic (a conjugating element of $\text{Sym}(\mathbb{N})$ is the same as an isomorphism). As the property defining C is preserved by isomorphisms, it follows that C is closed under the action of $\text{Sym}(\mathbb{N})$.

We now show that all elements of C are in the same orbit. Suppose that $x, y \in C$. Let $\{C_{i,j} : i, j \in \mathbb{N}\}$ be the connected components of $(\mathbb{N}, \sigma_{u,n}, y)$, indexed in such a way that $C_{i_0, j_0} \cong C_{i_1, j_1}$ if and only if $i_0 = i_1$. Similarly let $\{D_{i,j} : i, j \in \mathbb{N}\}$ be the set of connected components of $(\mathbb{N}, \sigma_{u,n}, x)$, indexed in such that way that $D_{i_0, j_0} \cong D_{i_1, j_1}$ if and only if $i_0 = i_1$, and $C_{i_0, j} \cong D_{i_1, j}$ if and only if $i_1 = i_2$.

For all $i, j \in \mathbb{N}$, let $\phi_{i,j} : C_{i,j} \rightarrow D_{i,j}$ be an isomorphism of unary algebras. We now define $\phi := \bigcup_{i,j \in \mathbb{N}} \phi_{i,j}$. It follows that $\phi : (\mathbb{N}, \sigma_{u,n}, y) \rightarrow (\mathbb{N}, \sigma_{u,n}, x)$ is an isomorphism of unary algebras and hence $(y, \phi) \text{conj}_{\text{Sym}(\mathbb{N})}^n = x$.

Now that we have identified our orbit, we show that it is comeagre. Let R be a countable set whose elements are bijective connected n -unary algebras and such that each isomorphism class of finite bijective n -unary algebras has precisely one representative in R . Our description of C gives us that if we define the sets

$$B_m := \left\{ y \in \text{Sym}(\mathbb{N})^n \mid \begin{array}{l} \text{the connected component of } m \text{ in } (\mathbb{N}, \sigma_{u,n}, y) \\ \text{is finite} \end{array} \right\}$$

$$B_{\mathbb{A}, m} := \left\{ y \in \text{Sym}(\mathbb{N})^n \mid \begin{array}{l} (\mathbb{N}, \sigma_{u,n}, y) \text{ has at least } m \text{ connected components} \\ \text{isomorphic to } \mathbb{A} \end{array} \right\}$$

for $m \in \mathbb{N}$ and $\mathbb{A} \in R$ then

$$C = \left(\bigcap_{m \in \mathbb{N}} B_m \right) \cap \left(\bigcap_{\substack{\mathbb{A} \in R \\ m \in \mathbb{N}}} B_{\mathbb{A}, m} \right).$$

So it suffices to show that all of these sets we are open and dense. They are open because the properties defining them are determined by only finitely many points.

We must show that they are dense. Let $m \in \mathbb{N}$ and $\mathbb{A} \in R$ be arbitrary. Let $U := \prod_{i < n} U_i$ be a non-empty basic open set in $\text{Sym}(\mathbb{N})^n$, and for all $k < n$ let f_k be finite bijections between subsets of \mathbb{N} such that $U_k = \left\{ g \in \text{Sym}(\mathbb{N}) \mid f_k \subseteq g \right\}$. Let

$$P := \bigcup_{k \in \{0, 1, \dots, n-1\}} \text{dom}(f_k) \cup \text{img}(f_k).$$

We first show that $B_m \cap U \neq \emptyset$. For all $k < n$, let f'_k be a permutation of $P \cup \{m\}$ which contains f_k , and let $g_k \in \text{Sym}(\mathbb{N})$ contain f'_k . Then $(g_0, g_1, \dots, g_{n-1}) \in B_m \cap U$.

It remains to show that $B_{\mathbb{A}, m} \cap U \neq \emptyset$. Let $\phi_0, \phi_1, \dots, \phi_{m-1} : \mathbb{A} \rightarrow \mathbb{N} \setminus P$ be injections with disjoint images. For each $k \in \{0, 1, \dots, n-1\}$ let $g_k \in \text{Sym}(\mathbb{N})$ contain the partial bijection

$$f_k \cup \bigcup_{j \in m} \phi_j^{-1}(k^{\mathbb{A}}) \phi_j.$$

Then $(g_0, g_1, \dots, g_{n-1}) \in B_{\mathbb{A}, m} \cap U$ as required. \square

Corollary 5.19 (Permutations are commutators, Assumed Knowledge: 5.9, 5.18).

Every element of the group $\text{Sym}(\mathbb{N})$ is a commutator.

5.2. Homeomorphisms of Cantor space. Our main aim in this subsection is to provide an argument that the action $\text{conj}_{\text{Aut}(2^{\mathbb{N}})}$ has ample generics (where $\text{Aut}(2^{\mathbb{N}})$ is the group of homeomorphisms of the Cantor set equipped with the compact-open topology).

The argument we present is that of Aleksandra Kwiatkowska [22], which shows that a generic homeomorphism of $2^{\mathbb{N}}$ can be constructed as a sort of limit of a nice class of finite topological structures. This limit is in the sense of ‘‘Projective Fraïssé Theory’’ as introduced by Irwin and Solecki in [19].

The conditions defining a projective Fraïssé class are analogous to those for the usual definition of a Fraïssé class, except with quotient maps used in place of embeddings (however a projective Fraïssé class need not be closed under quotients).

Definition 5.20 (Projective Fraïssé class, Assumed Knowledge: 4.3, 4.16).

We say a class \mathcal{C} of discrete σ -structures is a *projective Fraïssé class* if it satisfies the following properties:

- (1) \mathcal{C} is closed under topological isomorphism.
- (2) \mathcal{C} contains countably infinitely many structures up to isomorphism.
- (3) \mathcal{C} consists of finite structures.
- (4) If $\mathbb{D}, \mathbb{E} \in \mathcal{C}$, then there is $\mathbb{F} \in \mathcal{C}$ and quotient maps $\phi_{\mathbb{D}}, \phi_{\mathbb{E}}$ from \mathbb{F} onto \mathbb{D} and \mathbb{E} respectively.
- (5) If $\mathbb{C}, \mathbb{D}, \mathbb{E} \in \mathcal{C}$ and $\phi_{\mathbb{D}, \mathbb{C}} : \mathbb{D} \rightarrow \mathbb{C}, \phi_{\mathbb{E}, \mathbb{C}} : \mathbb{E} \rightarrow \mathbb{C}$ are quotient maps, then there is $\mathbb{F} \in \mathcal{C}$ and quotient maps $\phi_{\mathbb{F}, \mathbb{D}}, \phi_{\mathbb{F}, \mathbb{E}}$ from \mathbb{F} onto \mathbb{D} and \mathbb{E} respectively, such that $\phi_{\mathbb{F}, \mathbb{D}} \phi_{\mathbb{D}, \mathbb{C}} = \phi_{\mathbb{F}, \mathbb{E}} \phi_{\mathbb{E}, \mathbb{C}}$.

Example 5.21 (Non-empty finite semigroups, Assumed Knowledge: 4.3, 4.16).

The class of non-empty finite discrete semigroups form a projective Fraïssé class.

Proof. The first three conditions of Definition 5.20 are immediate. We first verify Condition 4. If D and E are finite discrete semigroups, then so is $F := D \times E$. Moreover if D, E are non-empty then the projection maps $\pi_0 : F \rightarrow D$ and $\pi_1 : F \rightarrow E$ are surjective. As semigroups are algebraic structures, it follows that these are quotient maps, thus Condition 4 follows.

It remains to verify Condition 5. Suppose that C, D and E are finite discrete semigroups and $\phi_{D, C} : D \rightarrow C, \phi_{E, C} : E \rightarrow C$ are quotient maps. Let

$$F := \left\{ (d, e) \in D \times E \mid (d)\phi_{D, C} = (e)\phi_{E, C} \right\}.$$

It is routine to verify that F is a subsemigroup of $D \times E$. If we define $\phi_{F, D} : F \rightarrow D$ and $\phi_{F, E} : F \rightarrow E$ to be the restrictions of the projection maps π_0 and π_1 respectively to the set F , then we have $\phi_{F, D} \phi_{D, C} = \phi_{F, E} \phi_{E, C}$.

It thus suffices to show that $\phi_{F, D}$ and $\phi_{F, E}$ are quotient maps (which for algebraic structures is equivalent to being a surjective homomorphism). If $d \in D$ is arbitrary, then we have

$$\{d\} \times (\{(d)\phi_{D, C}\})\phi_{E, C}^{-1} \subseteq F$$

(by the definition of F). As $\phi_{E, C}$ is a quotient map, it follows that $(\{(d)\phi_{D, C}\})\phi_{E, C}^{-1} \neq \emptyset$, hence $\phi_{F, D}$ is surjective. By the same argument, $\phi_{F, E}$ is also surjective as required. \square

Theorem 5.22 (Projective Fraïssé limit, Assumed Knowledge: 3.20, 3.22, 5.20).

If \mathcal{C} is a projective Fraïssé class, then there is a unique (up to topological isomorphism) second countable, compact, metrizable, zero-dimensional topological structure $\text{Flim}_{\leftarrow}(\mathcal{C})$ satisfying

- (1) *For all $\mathbb{D} \in \mathcal{C}$, there is a continuous quotient map $\phi_{\mathbb{D}} : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \mathbb{D}$.*

- (2) If X is a discrete, finite set and $f : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow X$ is a continuous surjection, then there is $\mathbb{S}_f \in \mathcal{C}$, a continuous quotient map $\phi_f : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \mathbb{S}_f$ and a surjective map $s_f : \mathbb{S}_f \rightarrow X$ such that $f = \phi_f s_f$.
- (3) If $\mathbb{A}, \mathbb{B} \in \mathcal{C}$, and $\psi_1 : \mathbb{A} \rightarrow \mathbb{B}$, $\psi_2 : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \mathbb{B}$ are continuous quotient maps, then there is a continuous quotient map $\psi_3 : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \mathbb{A}$ such that $\psi_3 \psi_1 = \psi_2$.

We call such an object $\text{Flim}_{\leftarrow}(\mathcal{C})$ a **projective Fréissé limit** of \mathcal{C} .

Proof. We first show existence. Let $S = \{\mathbb{S}_i \mid i \in \mathbb{N}\}$ be a countable set of structures in \mathcal{C} , such that every element of \mathcal{C} is isomorphic to a unique element of S .

Build by the following algorithm a sequence of structures $(\mathbb{P}_i)_{i \in \mathbb{N}}$ in \mathcal{C} , and a sequence of quotient maps $(\phi_{i+1} : \mathbb{P}_{i+1} \rightarrow \mathbb{P}_i)_{i \in \mathbb{N}}$:

- (1) Choose $N \subseteq \mathbb{N}$ containing 0 such that $|N| = |N^c|$, define $i := 0$, choose $t : N \rightarrow S$ to be a bijection, and define $\mathbb{P}_0 := \mathbb{S}_0$.
- (2) Choose $M \subseteq \mathbb{N} \setminus (N \cup \{0, 1, \dots, i-1\})$ such that $|N| = |\mathbb{N} \setminus (M \cup N \cup \{0, 1, \dots, i-1\})|$ and choose a bijection q from M to the set

$$\bigcup_{\substack{j < i \\ \mathbb{A}, \mathbb{B} \in S}} \left\{ (\psi_0, \psi_1) \in \text{Hom}(\mathbb{A}, \mathbb{B}) \times \text{Hom}(\mathbb{P}_j, \mathbb{B}) \mid \psi_0, \psi_1 \text{ are quotient maps} \right\}.$$

- (3) Redefine $t := t \cup q$, $N := N \cup M$ and $i := i + 1$.
- (4) If $(i)t$ is undefined, then define $\mathbb{P}_i := \mathbb{P}_{i-1}$ and define ϕ_i to be the identity map. If $(i)t$ is a structure, then by Definition 5.20 Condition 4, let $\mathbb{P}_i \in \mathcal{C}$ be a structure which $(i)t$ and \mathbb{P}_{i-1} are both quotients of and choose $\phi_i : \mathbb{P}_i \rightarrow \mathbb{P}_{i-1}$ to be a quotient map. If $(i)t = (\psi_0, \psi_1)$ is a pair of quotient maps $\psi_0 : \mathbb{A} \rightarrow \mathbb{B}$ and $\psi_1 : \mathbb{P}_j \rightarrow \mathbb{B}$, then by Definition 5.20 Condition 5, choose \mathbb{P}_i to be a structure and $\phi_i : \mathbb{P}_i \rightarrow \mathbb{P}_{i-1}$ be a quotient map such that there is a quotient map $\psi : \mathbb{P}_i \rightarrow \mathbb{A}$ satisfying

$$\psi \circ \psi_0 = \phi_i \circ (\phi_{i-1} \circ \dots \circ \phi_{j+1} \circ \psi_1).$$

- (5) Go to step 2.

In particular, this sequence has the property that every structure in \mathcal{C} is a quotient of a structure in the sequence, and if $\mathbb{A}, \mathbb{B} \in \mathcal{C}$ and $\psi_1 : \mathbb{A} \rightarrow \mathbb{B}$, $\psi_2 : \mathbb{P}_j \rightarrow \mathbb{B}$ are quotient maps, then there is $i \in \mathbb{N}$ and a quotient map from $\psi_3 : \mathbb{P}_i \rightarrow \mathbb{A}$ such that $\psi_3 \psi_1 = \phi_i \phi_{i-1} \dots \phi_{j+1} \psi_2$.

We now define \mathbb{L} to be the topological structure

$$\left\{ (p_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} \mathbb{P}_i \mid \text{for all } i \in \mathbb{N} \text{ we have } (p_{i+1}) \phi_{i+1} = p_i \right\}.$$

We next need to verify that \mathbb{L} has the required properties. First note that \mathbb{L} is a closed subspace of $\prod_{i \in \mathbb{N}} \mathbb{P}_i$ which by Theorem 3.20 is a Cantor space, so \mathbb{L} satisfies the given topological conditions.

Note also that all the projection maps $\pi_i : \mathbb{L} \rightarrow \mathbb{P}_i$ are quotient maps. Thus \mathbb{L} has the first required property as each element of \mathcal{C} is a quotient of one of the \mathbb{P}_i .

To check the second condition, suppose that X is a finite discrete space and $f : \mathbb{L} \rightarrow X$ is a continuous surjection. The preimage of each point in X is a clopen and is hence a finite union of sets of the form $(\{a\})\pi_i^{-1}$ where $a \in \mathbb{P}_i$. It follows, by taking the max M of all such i , that f can be factored into maps $\pi_M s_f$ as required.

We now check the third condition. Let $\psi_1 : \mathbb{A} \rightarrow \mathbb{B}$ and $\psi_2 : \mathbb{L} \rightarrow \mathbb{B}$ be continuous quotient maps. As in the above paragraph, we can find $j \in \mathbb{N}$ and $\psi'_2 : \mathbb{P}_j \rightarrow \mathbb{B}$ such that $\psi_2 = \pi_j \psi'_2$. Thus by the construction of $(\mathbb{P}_i)_{i \in \mathbb{N}}$, there is $i \in \mathbb{N}$ and $\psi'_3 : \mathbb{P}_i \rightarrow \mathbb{A}$ with $\psi'_3 \psi_1 = \phi_i \phi_{i-1} \dots \phi_{j+1} \psi'_2$. Thus choosing $\phi_3 := \pi_i \psi'_3$ gives the required map.

It remains to show uniqueness. Let \mathbb{L}_0 and \mathbb{L}_1 be structures with the listed properties. As \mathbb{L}_0 is zero-dimensional and second countable, we can find a sequence of equivalence relations $(\sim_{0,i})_{i \in \mathbb{N}}$, such that the equivalence classes of these relations are all clopen and they together form a basis for the topology on \mathbb{L}_0 . Be redefining $\sim_{0,i} := \sim_{0,i-1} \cap \sim_{0,i}$, we can assume without loss of generality that each relation in this sequence is contained in the previous one (so the corresponding sequence of partitions is getting finer). Also by the second condition of the theorem (and refining more if necessary), we can assume that for all $i \in \mathbb{N}$, the structure $\mathbb{L}_0 / \sim_{0,i}$ is isomorphic to a structure in \mathcal{C} . We also define an analogous sequence of equivalence relations $(\sim_{1,i})_{i \in \mathbb{N}}$ on \mathbb{L}_1 .

As

- (1) the classes of $(\sim_{0,i})_{i \in \mathbb{N}}$ form a basis for \mathbb{L}_0 ,
- (2) $\mathbb{L}_1 / \sim_{1,0}$ is isomorphic to an element of \mathcal{C} , and
- (3) \mathbb{L}_0 satisfies the first condition of the theorem,

we can find $i_0 \in \mathbb{N}$ and a quotient map $\phi_{0,0} : \mathbb{L}_0 / \sim_{0,i_0} \rightarrow \mathbb{L}_1 / \sim_{1,0}$. Similarly using the fact that \mathbb{L}_1 satisfies the third condition of the theorem, we can find $j_0 \in \mathbb{N}$ and $\phi_{1,0} : \mathbb{L}_1 / \sim_{1,j_0} \rightarrow \mathbb{L}_0 / \sim_{0,i_0}$ such that $([x]_{\sim_{1,j_0}}) \phi_{1,0} \phi_{0,0} = [x]_{\sim_{0,i_0}}$ for all $x \in \mathbb{L}_1$.

Continuing in this fashion we construct strictly increasing sequences $(i_k)_{k \in \mathbb{N}}$, $(j_k)_{k \in \mathbb{N}}$, and sequences $(\phi_{0,k})_{k \in \mathbb{N}}$, $(\phi_{1,k})_{k \in \mathbb{N}}$ such that for all $x_0 \in \mathbb{L}_0$, $x_1 \in \mathbb{L}_1$ and $k \in \mathbb{N} \setminus \{0\}$ we have:

$$([x_0]_{\sim_{0,i_{k+1}}}) \phi_{0,k+1} \phi_{1,k} = [x_0]_{\sim_{1,i_k}}, \quad \text{and} \quad ([x_1]_{\sim_{1,j_k}}) \phi_{1,k} \phi_{0,k} = [x_1]_{\sim_{1,j_{k-1}}}.$$

We define a homomorphism $\phi_0 : \mathbb{L}_0 \rightarrow \mathbb{L}_1$ by defining $(x_0) \phi_0$ to be the unique element of the set $\bigcap_{k \in \mathbb{N}} ([x_0]_{\sim_{0,i_{k+1}}}) \phi_{0,k+1}$ (this exists because \mathbb{L}_1 is compact and is unique because \mathbb{L}_1 is Hausdorff). We define $\phi_2 : \mathbb{L}_1 \rightarrow \mathbb{L}_0$ analogously. By construction $\phi_0, \phi_1 \rightarrow$ continuous homomorphisms and they are inverses of each other, so the result follows. \square

We also give an alternative formulation of the above theorem which will be more useful at times.

Theorem 5.23 (Projective Fraïssé limit 2, Assumed Knowledge: 5.22).

If \mathcal{C} is a projective Fraïssé class, then $\text{Flim}_{\leftarrow}(\mathcal{C})$ is the unique (up to topological isomorphism) compact, second countable, Hausdorff, zero-dimensional topological structure satisfying:

- (1) *For all $\mathbb{D} \in \mathcal{C}$, there is a partition of $\text{Flim}_{\leftarrow}(\mathcal{C})$ into clopen sets such that the corresponding quotient structure is isomorphic to \mathbb{D} .*
- (2) *If P is a finite partition of $\text{Flim}_{\leftarrow}(\mathcal{C})$ into clopen sets, then there is a partition R of $\text{Flim}_{\leftarrow}(\mathcal{C})$ into clopen sets refining P and such that the quotient structure corresponding to R is an element of \mathcal{C} .*
- (3) *If P_0, P_1 are finite partitions of $\text{Flim}_{\leftarrow}(\mathcal{C})$ into clopen sets and $\phi : P_0 \rightarrow P_1$ is an isomorphism of the corresponding quotient structures, then there is a topological isomorphism $h : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \text{Flim}_{\leftarrow}(\mathcal{C})$ such that $(S)h = (S)\phi$ for all $S \in P_0$.*

Proof. The first two conditions are reformulations of the corresponding conditions in Theorem 5.22 using the fact that for any quotient map ϕ we have

$$\text{dom}(\phi)/\ker(\phi) \cong \text{img}(\phi).$$

We show that the third condition of Theorem 5.22 is equivalent to the third condition of Theorem 5.23.

((3) of Theorem 5.22 \Rightarrow (3) of Theorem 5.23): Let P_0, P_1 be finite partitions of $\text{Flim}_{\leftarrow}(\mathcal{C})$ into clopen sets. By the second condition, we can assume without loss of generality that the corresponding quotient structures are in \mathcal{C} . The result follows by revisiting the uniqueness proof from Theorem 5.22 using $\mathbb{L}_0 = \mathbb{L}_1 = \text{Flim}_{\leftarrow}(\mathcal{C})$ and choosing $\phi_{0,0}$ to be the map $x \rightarrow ([x]_{P_0})\phi$ (where $[x]_{P_0}$ is the element of P_0 containing x). The desired map h is then we map ϕ_0 constructed in that proof.

((3) of Theorem 5.23 \Rightarrow (3) of Theorem 5.22): Let $\mathbb{A}, \mathbb{B} \in \mathcal{C}$ and $\phi_1 : \mathbb{A} \rightarrow \mathbb{B}, \phi_2 : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \mathbb{B}$ are quotient maps. By renaming elements of \mathbb{A} using Condition (1) we can assume without loss of generality that \mathbb{A} is a quotient of $\text{Flim}_{\leftarrow}(\mathcal{C})$. By renaming elements of \mathbb{B} , we can assume without loss of generality that ϕ_2 is the map $x \mapsto [x]_{\ker(\phi_2)}$. Let $\psi : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \mathbb{A}$ be the quotient map mapping a point to the set containing it. We have a natural isomorphism $\phi : \text{Flim}_{\leftarrow}(\mathcal{C})/\ker(\phi_2) \rightarrow \text{Flim}_{\leftarrow}(\mathcal{C})/\ker(\psi\phi_1)$ defined by

$$([x]_{\ker(\phi_2)})\phi = (\{x\phi_2\})(\psi\phi_1)^{-1}.$$

So there is a topological isomorphism $h : \text{Flim}_{\leftarrow}(\mathcal{C}) \rightarrow \text{Flim}_{\leftarrow}(\mathcal{C})$ corresponding to this isomorphism. Then defining $\phi_3 = h\psi$ we have

$$([x]_{\ker(\phi_2)})\phi_3\phi_1 = (\{x\phi_2\})(\psi\phi_1)^{-1}\psi\phi_1 = \{x\phi_2\}$$

for all $x \in \text{Flim}_{\leftarrow}(\mathcal{C})$, and so $\phi_3\phi_1 = \phi_2$ as required. \square

We now establish the classes of structures whose limits will give our generic homeomorphisms.

Definition 5.24 (Cantor signatures, Assumed Knowledge: 4.1).

If $n \in \mathbb{N}$, then we define the signature $\sigma_{b,n} := (\emptyset, \{0, 1, \dots, n-1\}, \{(0, 2), (1, 2), \dots, (n-1, 2)\})$. That is $\sigma_{b,n}$ has the n binary relation symbols $\{0, 1, \dots, n-1\}$ and no other symbols.

Definition 5.25 (Amalgamation structures: 1.6, 4.2, 5.24).

For $n \geq 1$, we define an n -Cantor amalgamation structure (n -CA structure) to be a non-empty finite $\sigma_{b,n}$ -structure \mathbb{A} satisfying the following properties:

- (1) For each $i < n$, the binary relations $i^{\mathbb{A}}$ and $(i^{\mathbb{A}})^{-1}$ are surjective.
- (2) The binary relation

$$\Gamma_{\mathbb{A}} := \left\{ (a, (i, j)) \in A \times (\{0, 1, \dots, n-1\} \times \{-1, 1\}) \mid |(\{a\})(i^{\mathbb{A}})^j| \geq 2 \right\}$$

is a function with domain A .

- (3) If $(a, b) \in i^{\mathbb{A}}$, then either $(a)\Gamma_{\mathbb{A}} = (i, 1)$ or $(b)\Gamma_{\mathbb{A}} = (i, -1)$.

It is somewhat difficult to construct natural examples of n -CA structures, but the following theorem will allow us to construct a large collection of examples.

Theorem 5.26 (cf. Theorem 4.6 of [22], Cofinal amalgams, Assumed Knowledge: 1.5, 4.16, 5.25).

Suppose that \mathbb{S} is a finite non-empty $\sigma_{b,n}$ -structure, and each of $i^{\mathbb{S}}$ and $(i^{\mathbb{S}})^{-1}$ a surjective binary relation for all $i < n$. Then there is an n -CA structure $Q_{\mathbb{S}}$ which has \mathbb{S} as a quotient.

Proof. We define $\{0, 1, \dots, n-1\} \times \{-1, 1\} \times \{0, 1\} \times \mathbb{S}$ to be the universe of $Q_{\mathbb{S}}$. For all $s \in \mathbb{S}$ and $i < n$, we choose some $s_{i,l}, s_{i,r} \in \mathbb{S}$ with $(s_{i,l}, s), (s, s_{i,r}) \in i^{\mathbb{S}}$. We then define

$$\begin{aligned} i^{Q_{\mathbb{S}}} := & \left\{ ((i, 1, 0, s_{i,l}), (j, p, k, s)) \mid (j, p, k, s) \in Q_{\mathbb{S}} \right\} \\ & \cup \left\{ ((j, p, k, s), (i, -1, 0, s_{i,r})) \mid (j, p, k, s) \in Q_{\mathbb{S}} \right\} \\ & \cup \left\{ ((i, 1, j, a), (i, -1, k, b)) \mid (a, b) \in i^{\mathbb{S}}, j, k \in \{0, 1\} \right\}. \end{aligned}$$

First note that due to the third of the sets in the union above, the map $(a, b, c, d) \rightarrow d$ from $Q_{\mathbb{S}} \rightarrow \mathbb{S}$ is a quotient map. We now need to check that $Q_{\mathbb{S}}$ is an n -CA structure. First each of $i^{Q_{\mathbb{S}}}$ and $(i^{Q_{\mathbb{S}}})^{-1}$ is surjective because of the first two sets in the union.

We need to show that $\Gamma_{Q_{\mathbb{S}}}$ is a function. Let $(a, b, c, d) \in Q_{\mathbb{S}}$ be arbitrary. We must show that there is exactly one pair $(i, j) \in \{0, 1, \dots, n-1\} \times \{-1, 1\}$ with $|(a, b, c, d)(i^{Q_{\mathbb{S}}})^j| \geq 2$. By definition of $i^{Q_{\mathbb{S}}}$ any such pair (i, j) must be equal to the pair (a, b) . Moreover the last set in the union defining $(i^{Q_{\mathbb{S}}})^b$ must pair (a, b, c, d) with at least 2 elements. So $\Gamma_{Q_{\mathbb{S}}}$ is precisely the function $(a, b, c, d) \rightarrow (a, b)$. From this observation the third condition for being an n -CA structure is clear. \square

Our proof of Lemma 5.27 is quite different from the proof of the analogous statement given in [22], but it is more inline with the theme of this document.

Lemma 5.27 (cf. Theorem 4.1 of [22], Amalgams amalgamate, Assumed Knowledge: 1.5, 2.3, 4.28, 5.2, 5.26).

Suppose that \mathbb{A}, \mathbb{B} , and \mathbb{C} are n -CA structures, and $\phi_{\mathbb{B}}, \mathbb{B} \rightarrow \mathbb{A}, \phi_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbb{A}$ are quotient maps. Then we can find an n -CA structure \mathbb{D} and a pair of quotient maps $\phi_{\mathbb{D}, \mathbb{B}} : \mathbb{D} \rightarrow \mathbb{B}$ and $\phi_{\mathbb{D}, \mathbb{C}} : \mathbb{D} \rightarrow \mathbb{C}$ such that

$$\phi_{\mathbb{D}, \mathbb{C}} \phi_{\mathbb{C}} = \phi_{\mathbb{D}, \mathbb{B}} \phi_{\mathbb{B}}.$$

Proof. First we consider the following substructure of $\mathbb{B} \times \mathbb{C}$:

$$\mathbb{P} := \left\{ (b, c) \in \mathbb{B} \times \mathbb{C} \mid (b)\phi_{\mathbb{B}} = (c)\phi_{\mathbb{C}} \right\}.$$

Note that the homomorphisms $\pi_0 \phi_{\mathbb{B}}$ and $\pi_1 \phi_{\mathbb{C}}$ from $\mathbb{B} \times \mathbb{C}$ to \mathbb{A} are the same when restricted to \mathbb{P} , however \mathbb{P} might not be an n -CA structure so we need to do a bit more work.

We define the free group of rank n by F_n . That is F_n is the set of all finite words in the alphabet $\{0, 1, \dots, n-1\} \times \{-1, 1\}$ (where we consider $(i, 1)$ and $(i, -1)$ to be mutual inverses for each $i < n$) with the property that no letter can follow its inverse. The composition is given by concatenation and then removing all instances of a letter followed by its inverse.

We view F_n as an $\sigma_{b,n}$ -structure (its Cayley graph) by defining

$$i^{F_n} = \left\{ (x, y) \in F_n \mid xi = y \right\}.$$

Note that this structure is symmetric in that its automorphism group is transitive (the group F_n acts transitively on it by left multiplication). Note also that if we have a $\sigma_{b,n}$ -structure \mathbb{S} , then showing that all the relations $i^{\mathbb{S}}$ and $(i^{\mathbb{S}})^{-1}$ are surjective is equivalent to showing that every $s \in \mathbb{S}$ is the image of the empty word ε under a homomorphism from F_n to \mathbb{S} .

Consider the following substructure of \mathbb{P}

$$\mathbb{P}^{\text{core}} := \left\{ p \in \mathbb{P} \mid \text{there is a homomorphism } q : F_n \rightarrow \mathbb{P} \text{ with } (\varepsilon)q = p \right\}.$$

By the comment above, the binary relations $i^{\mathbb{P}^{\text{core}}}$ and $(i^{\mathbb{P}^{\text{core}}})^{-1}$ are surjective for all $i < n$.

Claim: The maps $\pi_0 \upharpoonright_{\mathbb{P}^{\text{core}}}$ and $\pi_1 \upharpoonright_{\mathbb{P}^{\text{core}}}$ are both quotient maps.

Proof of Claim: We show that $\pi_0 \upharpoonright_{\mathbb{P}^{\text{core}}}$ is a quotient map, the other case is symmetric. Let $i < n$ be arbitrary, and $(a, b) \in i^{\mathbb{B}}$ be arbitrary. It suffices to show that there is $(p, q) \in i^{\mathbb{P}^{\text{core}}}$ with $((p)\pi_0, (q)\pi_0) = (a, b)$. As \mathbb{A} is an n -CA structure, we either have $((a)\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i, 1)$ or $((b)\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i, -1)$. We assume without loss of generality that $((b)\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i, -1)$. There are 2 cases to consider:

- (1) There are finite sequences $b = b_0, a = b_{-1}, b_{-2}, \dots, b_{-l}$ in \mathbb{B} and $i_0, i_{-1}, \dots, i_{-l}$ in $\{0, 1, \dots, n-1\}$ such that $(b_{k-1}, b_k) \in i_k^{\mathbb{B}}$ and $((b_k)\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i_k, -1)$ for each $-l < k \leq 0$ and $((b_{-l})\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i_{-l}, 1)$.
- (2) There are infinite sequences $b = b_0, a = b_{-1}, b_{-2}, \dots$ in \mathbb{B} and $i_0, i_{-1}, i_{-2}, \dots$ in n such that $(b_{k-1}, b_k) \in i_k^{\mathbb{B}}$ and $((b_k)\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i_k, -1)$ for each $k \in -\mathbb{N}$.

For now we consider the first case. Let $(c_0, c_1) \in i_{-l+1}^{\mathbb{C}}$ be such that

$$((c_0)\phi_{\mathbb{C}}, (c_1)\phi_{\mathbb{C}}) = ((b_{-l})\phi_{\mathbb{B}}, (b_{-l+1})\phi_{\mathbb{B}}),$$

(these must exist as $\phi_{\mathbb{C}}$ is a quotient map). Let $q_{\mathbb{C}} : F_n \rightarrow \mathbb{C}$ be any homomorphism which maps (ε, i_{-l+1}) to (c_0, c_1) . Similarly, let $q_{\mathbb{B}} : F_n \rightarrow \mathbb{B}$ be any homomorphism which maps (ε, i_{-l+1}) to (b_{-l}, b_{-l+1}) , maps $(i_{-l+1}, i_{-l+1}i_{-l+2})$ to (b_{-l+1}, b_{-l+2}) and so on.

Note that $((b_{-l})\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i_{-l}, 1)$ and $((b_{-l+1})\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i_{-l+1}, -1)$. A routine inductive argument using this fact and the fact that \mathbb{A} is an n -CA structure shows that there is a unique homomorphism $q : F_n \rightarrow \mathbb{A}$ such that $(\varepsilon)q = (b_{-l})\phi_{\mathbb{B}}$ and $(i_{-l+1})q = (b_{-l+1})\phi_{\mathbb{B}}$. In particular it follows that $q_{\mathbb{C}}\phi_{\mathbb{C}} = q_{\mathbb{B}}\phi_{\mathbb{B}}$ as both maps have this property.

The map $\langle q_{\mathbb{B}}, q_{\mathbb{C}} \rangle_{\mathbb{B} \times \mathbb{C}} : F_n \rightarrow \mathbb{B} \times \mathbb{C}$ is a homomorphism (recall Definition 2.3). As $q_{\mathbb{C}}\phi_{\mathbb{C}} = q_{\mathbb{B}}\phi_{\mathbb{B}}$, it follows that the image of this homomorphism is contained in \mathbb{P} and hence \mathbb{P}^{core} , finally the image of the pair $(i_{-l+1}i_{-l+2} \dots i_{-1}, i_{-l+1}i_{-l+2} \dots i_0)$ under $\langle q_{\mathbb{B}}, q_{\mathbb{C}} \rangle_{\mathbb{B} \times \mathbb{C}}$ is mapped by π_0 to (a, b) as required.

We now consider the case that there are infinite sequences $b = b_0, a = b_{-1}, b_{-2}, \dots$ in \mathbb{B} and $i_0, i_{-1}, i_{-2}, \dots$ in n such that $(b_{k-1}, b_k) \in i_k^{\mathbb{B}}$ and $((b_k)\phi_{\mathbb{B}})\Gamma_{\mathbb{A}} = (i_k, -1)$ for each $k \in -\mathbb{N}$. As \mathbb{B} is finite, we can choose $s, t \in \mathbb{N}$ such that $-s < -t$ and $b_{-s} = b_{-t}$. Let $(c_0, c_1) \in i_{-s+1}^{\mathbb{C}}$ be such that $((c_0)\phi_{\mathbb{C}}, (c_1)\phi_{\mathbb{C}}) = ((b_{-s})\phi_{\mathbb{B}}, (b_{-s+1})\phi_{\mathbb{B}})$. We then inductively define a sequence c_0, c_1, c_2, \dots as follows:

- (1) Let c_0, c_1 be as before.
- (2) If c_k has already been defined and $(c_k)\phi_{\mathbb{C}} = (b_{-j})\phi_{\mathbb{B}}$ where $-s \leq -j < -t$, then we choose c_{k+1} to be such that $(c_k, c_{k+1}) \in i_{-j+1}^{\mathbb{C}}$.

As the sequence c_0, c_1, c_2, \dots is infinite it must repeat, thus we can assume without loss of generality that we chose it to be eventually be periodic, and thus by changing the choice of c_0, c_1 , we can assume without loss of generality that there is $k \in \mathbb{N}$ such that $c_l = c_{l \bmod k}$ for all $l \in \mathbb{N}$.

We are now in the position to define maps from F_n to \mathbb{B} and \mathbb{C} . We have established some ‘‘cycles’’ of note, that is $c_0, c_1, \dots, c_k = c_0$ and $b_{-s}, b_{-s-1}, \dots, b_{-t} = b_{-s}$. Moreover these cycles are mapped respectively by $\phi_{\mathbb{C}}$ and $\phi_{\mathbb{B}}$ to some common cycle $a_0, a_1, \dots, a_p = a_0$ in \mathbb{A} (hence both k and $s - t$ are multiples of p but are not necessarily equal). We define a map $q_{\mathbb{C}} : F_n \rightarrow \mathbb{C}$ by defining $(\varepsilon)q_{\mathbb{C}} = c_0$ and inductively extending this homomorphism to all of F_n as follows:

- (1) If $(w_0w_1w_2 \dots w_{l-1})q_{\mathbb{C}}$ is in the cycle $c_0, c_1, \dots, c_k = c_0$ and mapping $w_0w_1w_2 \dots w_l$ to the next/previous element in the cycle would preserve the fact that q is a homomorphism, then map $w_0w_1w_2 \dots w_l$ to the next/previous element in the cycle.
- (2) Otherwise map $w_0w_1w_2 \dots w_l$ to any element of \mathbb{C} which preserves the fact that $q_{\mathbb{C}}$ is a homomorphism.

We define a map $q_{\mathbb{B}}$ similarly using the cycle $b_{-s}, b_{1-s}, \dots, b_{-t} = b_{-s}$, and a map $q_{\mathbb{A}}$ using the cycle $a_0, a_1, \dots, a_p = a_0$. Note that we never make any arbitrary choices in the construction of $q_{\mathbb{A}}$ as if we ever had a choice it would follow that $\Gamma_{\mathbb{A}}$ was not a function. It follows that

$$q_{\mathbb{B}}\phi_{\mathbb{B}} = q_{\mathbb{A}} = q_{\mathbb{C}}\phi_{\mathbb{C}}.$$

Thus $\langle q_{\mathbb{B}}, q_{\mathbb{C}} \rangle_{\mathbb{B} \times \mathbb{C}}$ is a homomorphism into \mathbb{P}^{core} . It is routine to verify that the edge (a, b) is mapped onto by $q_{\mathbb{B}}$, so again we have the required outcome. \diamond

By Theorem 5.26, let \mathbb{D} be an n -CA structure such that there is a quotient map $\psi : \mathbb{D} \rightarrow \mathbb{P}^{\text{core}}$. The result follows by defining $\phi_{\mathbb{D}, \mathbb{C}} := \psi\pi_1$ and $\phi_{\mathbb{D}, \mathbb{B}} := \psi\pi_0$. \square

Theorem 5.28 (We have a Projective Fräissé Class, Assumed Knowledge: 5.20, 4.14, 5.26, 5.27).
For each $n \geq 1$, the n -CA structures form a projective Fräissé class.

Proof. We verify conditions (1)-(5) of Definition 5.20.

- (1) The defining property of the class is an isomorphism invariant, so the class is closed under isomorphisms.
- (2) The class consists of finite structures in a finite signature and so has countably many isomorphism types. It also has infinitely many isomorphism types as by Theorem 5.26, it contains structures of arbitrarily large finite cardinality.
- (3) The class consists of finite structures by definition.
- (4) Suppose that \mathbb{B} and \mathbb{C} are n -CA structures. Note that the structure $\mathbb{B} \times \mathbb{C}$ is a non-empty $\sigma_{b,n}$ -structure such that each relation and its inverse are surjective, moreover the projection maps π_0 and π_1 are quotient maps. By Theorem 5.26 we can find an n -CA structure \mathbb{D} such that there is a quotient map $\psi : \mathbb{D} \rightarrow \mathbb{B} \times \mathbb{C}$. The maps $\psi\pi_0$ and $\psi\pi_1$ are the required quotient maps.
- (5) This is immediate from Lemma 5.27.

\square

Definition 5.29 (The CA limit, Assumed Knowledge: 5.22, 5.28).

For each $n \geq 1$, define $n\mathbb{L}$ to be a projective Fräissé limit of the class of n -CA structures (see Theorem 5.22).

Now that for all $n \in \mathbb{N} \setminus \{0\}$ we have the objects $n\mathbb{L}$ that we wanted, we need to show that $n\mathbb{L}$ can be viewed as a generic element of $\text{Aut}(2^{\mathbb{N}})^n$.

Lemma 5.30 (Splitting points in n -CA structures, Assumed Knowledge: 5.26).

Suppose that \mathbb{S} is an n -CA structure, $i < n$, $p \in \mathbb{S}$, $k \geq 2$ and $(p)i^{\mathbb{S}} = \{q_0, q_1, \dots, q_{k-1}\}$. Then there is an n -CA structure \mathbb{S}_2 and a quotient map $\phi : \mathbb{S}_2 \rightarrow \mathbb{S}$ with $(p)\phi^{-1} \geq k$ and such that for each $p' \in (p)\phi^{-1}$ the set $((\{p'\})i^{\mathbb{S}_2})\phi$ is a singleton.

Proof. Let \mathbb{T} denote the structure with universe $(\mathbb{S} \setminus \{p\}) \cup (\{p\} \times \{0, 1, \dots, k-1\})$, and relations to be defined later (we assume without loss of generality that $\{p\} \times \{0, 1, \dots, k-1\}$ is disjoint from \mathbb{S}). Define $\phi_{\mathbb{T}} : \mathbb{T} \rightarrow \mathbb{S}$ by

$$(x)\phi_{\mathbb{T}} = \left\{ \begin{array}{ll} x & \text{if } x \in \mathbb{S} \setminus \{p\} \\ p & \text{if } x \in \{p\} \times \{0, 1, \dots, k-1\} \end{array} \right\}.$$

For $j \in \{0, 1, \dots, n-1\} \setminus \{i\}$, we define

$$j^{\mathbb{T}} := \left\{ (a, b) \mid ((a)\phi_{\mathbb{T}}, (b)\phi_{\mathbb{T}}) \in j^{\mathbb{S}} \right\}$$

and

$$i^{\mathbb{T}} := \left\{ (a, b) \mid ((a)\phi_{\mathbb{T}}, (b)\phi_{\mathbb{T}}) \in i^{\mathbb{S}} \text{ and if } a = (p, l) \text{ then } b = q_l \right\}.$$

Note that $\phi_{\mathbb{T}}$ is a quotient map from \mathbb{T} to \mathbb{S} and that each $j^{\mathbb{T}}$ is surjective with surjective inverse. Thus by Theorem 5.26 there is an n -CA structure \mathbb{S}_2 which quotients onto \mathbb{T} . The composite of these two quotient maps gives the required quotient map ϕ . \square

Lemma 5.31 (Cantor limit is a Cantor space, Assumed Knowledge: 3.20, 5.23, 5.30).

For each $n \geq 1$, the topological structure $n\mathbb{L}$ is homeomorphic to the Cantor set $2^{\mathbb{N}}$.

Proof. By Theorem 3.20 it suffices to show that $n\mathbb{L}$ is non-empty, second countable, zero-dimensional, Hausdorff, compact and has no isolated points. All of these except no isolated points are immediate from Theorem 5.22. So we show that $n\mathbb{L}$ has no isolated points.

Suppose for a contradiction that p is an isolated point of $n\mathbb{L}$. Thus $\{\{p\}, n\mathbb{L} \setminus \{p\}\}$ is a partition of $n\mathbb{L}$ into clopen sets. By Theorem 5.23 part (2), there is $k \in \mathbb{N}$ and a partition $\mathbb{S} := \{C_0, C_1, \dots, C_{k-1}, \{p\}\}$ of $n\mathbb{L}$ into clopen sets such that the corresponding quotient structure is an n -CA structure. By Lemma 5.30 there is an n -CA structure \mathbb{S}_2 and a quotient map $\phi : \mathbb{S}_2 \rightarrow \mathbb{S}$ with $(\{\{p\}\})\phi^{-1} \geq 2$ (we can use $\Gamma_{\mathbb{S}}$ to choose a relation allowing us the guarantee this).

By Theorem 5.23 part (1), we can assume without loss of generality that \mathbb{S}_2 is a quotient structure of $n\mathbb{L}$ (whose elements are clopen subsets of $n\mathbb{L}$). Thus the map sending $s \in \mathbb{S}$ to $\bigcup(\{s\})\phi^{-1}$ is an isomorphism from \mathbb{S} to a quotient structure of $n\mathbb{L}$. By Theorem 5.23 part (3), we can find a topological isomorphism $h : n\mathbb{L} \rightarrow n\mathbb{L}$ such that $(s)h = \bigcup(\{s\})\phi^{-1}$ for all $s \in \mathbb{S}$. Let $A, B \in (\{\{p\}\})\phi^{-1}$ be distinct. It follows that $(A)h^{-1}$ and $(B)h^{-1}$ are disjoint non-empty subsets of $\{p\}$, this is a contradiction. \square

Lemma 5.32 (Cantor limit relations are homeomorphisms, Assumed Knowledge: 1.5, 3.18, 5.31).

If $n \geq 1$ and $i \in \{0, 1, \dots, n-1\}$, then $i^{n\mathbb{L}}$ is a homeomorphism of $n\mathbb{L}$.

Proof. We first show that $i^{n\mathbb{L}}$ is a function. Let $x \in n\mathbb{L}$ be arbitrary, and suppose that $(x, y), (x, z) \in i^{n\mathbb{L}}$. It suffices to show that $y = z$. Suppose for a contradiction that $y \neq z$ and let $\{U_y, U_z\}$ be a partition of $n\mathbb{L}$ into disjoint clopen subsets of $n\mathbb{L}$ with $y \in U_y, z \in U_z$. By Theorem 5.23 part (2) there is a finite refinement \mathbb{S} of this partition into clopen sets such that if we view \mathbb{S} as quotient structure of $n\mathbb{L}$, then \mathbb{S} is an n -CA structure.

Using the element of \mathbb{S} containing x as p , let ϕ and \mathbb{S}_2 be as in Lemma 5.30. By Theorem 5.22 part (3), there is a continuous quotient map $\psi : n\mathbb{L} \rightarrow \mathbb{S}_2$ such that $\psi\phi$ is the map sending an element of $n\mathbb{L}$ to the element of \mathbb{S} containing it.

Thus by the choice of \mathbb{S}_2 , each element of $(\{[x]_{\ker(\psi)}\})i^{n\mathbb{L}/\ker(\psi)}$ is contained in a common element of \mathbb{S} (and hence only one element of $\{U_x, U_y\}$). This is a contradiction as x is related to both y and z by $i^{n\mathbb{L}}$.

We have shown that $i^{n\mathbb{L}}$ is a function, by symmetry $(i^{n\mathbb{L}})^{-1}$ is also a function and thus $i^{n\mathbb{L}}$ is a bijection. It remains to show that $i^{n\mathbb{L}}$ is continuous. As $n\mathbb{L}$ is a topological structure, we have that $i^{n\mathbb{L}}$ is closed and hence compact subspace of $n\mathbb{L}^2$ (Remark 3.18). As $i^{n\mathbb{L}}$ is a bijection, both the maps π_0 and π_1 from $i^{n\mathbb{L}}$ to $n\mathbb{L}$ are bijections. Moreover (as projections) both the maps $\pi_0 \upharpoonright_{i^{n\mathbb{L}}}$, $\pi_1 \upharpoonright_{i^{n\mathbb{L}}}$ are continuous and thus are homeomorphisms as they map between compact Hausdorff spaces. As the function $i^{n\mathbb{L}}$ is equal to the composition of the inverse of π_0 (using $i^{n\mathbb{L}}$ as the domain of π_0) with π_1 , the result follows. \square

Theorem 5.33 (cf. Theorem 4.7 of [22], $\text{Aut}(2^{\mathbb{N}})$ has ample generics, Assumed Knowledge: 4.10, 5.23, 5.1, 5.5, 5.26, 5.32, 5.13).

The action $\text{conj}_{\text{Aut}(2^{\mathbb{N}})}$ has ample generics (using the compact-open topology on $\text{Aut}(2^{\mathbb{N}})$).

Proof. For all $f = (f_0, f_1, \dots, f_{n-1}) \in \text{Aut}(2^{\mathbb{N}})^n$, let \mathbb{S}_f denote the $\sigma_{b,n}$ structure with universe $2^{\mathbb{N}}$ and $i^{\mathbb{S}_f} = f_i$ for all $i < n$. We show that the set

$$C := \left\{ f \in \text{Aut}(2^{\mathbb{N}})^n \mid \mathbb{S}_f \cong n\mathbb{L} \right\}$$

is comeagre.

Let $R = \left\{ \mathbb{R}_i \mid i \in \mathbb{N} \right\}$ be a collection of n -CA structures such that every n -CA structure is isomorphic to precisely one element of R . Let $\text{Clo}(2^{\mathbb{N}})$ denote the collection of clopen subsets of $2^{\mathbb{N}}$, let $\text{CloP}(2^{\mathbb{N}})$ denote the set of finite partitions of $2^{\mathbb{N}}$ into clopen sets. Let $\text{PS}(2^{\mathbb{N}})$ denote the set of surjections from elements of $\text{CloP}(2^{\mathbb{N}})$ to elements of R . Let QR denote the set of quotient maps between elements of R . Note that the sets $R, \text{Clo}(2^{\mathbb{N}}), \text{CloP}(2^{\mathbb{N}}), \text{QR}$ and $\text{PS}(2^{\mathbb{N}})$ are all countable.

By Theorem 5.12, it follows that the sets of the form

$$\left\{ f \in \text{Aut}(2^{\mathbb{N}}) \mid (U)f = V \right\} = B(U, V) \cap B(U^c, V^c)$$

where $U, V \in \text{Clo}(2^{\mathbb{N}})$, form a subbasis for the topology on $\text{Aut}(2^{\mathbb{N}})$.

Claim: The sets of the form

$$U_\phi := \left\{ f \in \text{Aut}(2^{\mathbb{N}})^n \mid \phi \text{ is a quotient map from } \mathbb{S}_f \text{ to } \text{img}(\phi) \right\}$$

where ϕ can be any surjection from an element of $\text{CloP}(2^{\mathbb{N}})$ to a $\sigma_{b,n}$ -structure, are a basis of clopen sets for the topology on $\text{Aut}(2^{\mathbb{N}})^n$.

Proof of Claim:

Let ϕ be an arbitrary surjection from a partition of $2^{\mathbb{N}}$ into clopen sets to a $\sigma_{b,n}$ -structure. To see that U_ϕ is clopen in the compact-open topology, note that it is the intersection of the following two sets

$$\begin{aligned} & \bigcap_{\substack{i < n \\ (a,b) \in i^{\text{img}(\phi)}}} \bigcup_{\substack{U, V \in \text{Clo}(2^{\mathbb{N}}) \\ U \subseteq \bigcup ((a)\phi^{-1}) \\ V \subseteq \bigcup ((b)\phi^{-1})}} \left\{ (f_0, f_1, \dots, f_{n-1}) \in \text{Aut}(2^{\mathbb{N}})^n \mid (U)f_i = V \right\}, \\ & \bigcap_{\substack{i < n \\ (a,b) \notin i^{\text{img}(\phi)}}} \bigcup_{V \in \text{Clo}(2^{\mathbb{N}})} \left\{ (f_0, f_1, \dots, f_{n-1}) \in \text{Aut}(2^{\mathbb{N}})^n \mid (U((a)\phi^{-1}))f_i = V \right\}. \end{aligned}$$

We now need to show that every set of the form

$$\bigcap_{j < k} \left\{ (f_0, f_1, \dots, f_{n-1}) \in \text{Aut}(2^{\mathbb{N}})^n \mid (U_j)f_i = V_j \right\}$$

for $i < n$, $k \in \mathbb{N}$, $U_0, U_1, \dots, U_{k-1} V_0, V_1, \dots, V_{k-1} \in \text{Clo}(2^{\mathbb{N}})$ is a union of sets of the form U_ϕ . Let $f = (f_0, f_1, \dots, f_{n-1})$ in the above set be arbitrary. Let P be a partition of $2^{\mathbb{N}}$ into clopen sets such that each of the sets U_j and V_j can be expressed as unions of elements of P and let ϕ be the identity map from P to P (where we view the second P as a quotient structure of \mathbb{S}_f). Note that $f \in U_\phi$ and let $g = (g_0, g_1, \dots, g_{n-1}) \in U_\phi$ be arbitrary. Then $(F)g_i \subseteq V_j$ for all $j < k$ and $F \in P$ with $F \subseteq U_j$. Similarly $(F)g_i \cap V_j = \emptyset$ for all $j < k$ and $F \in P$ with $F \cap U_j = \emptyset$. As g was arbitrary, it follows that

$$U_\phi \subseteq \bigcap_{j < k} \left\{ (f_0, f_1, \dots, f_{n-1}) \in \text{Aut}(2^{\mathbb{N}})^n \mid (U_j)f_i = V_j \right\}$$

as required. \diamond

We then define:

$$\begin{aligned} U_1 &:= \bigcap_{i \in \mathbb{N}} \bigcup_{\substack{\phi \in \text{PS}(2^{\mathbb{N}}) \\ \text{img}(\phi) = \mathbb{R}_i}} U_\phi, \\ U_2 &:= \bigcap_{P \in \text{CloP}(2^{\mathbb{N}})} \bigcup_{\substack{\phi \in \text{PS}(2^{\mathbb{N}}) \\ \ker(\phi) \text{ refines } P}} U_\phi, \\ U_3 &:= \bigcap_{\substack{\phi \in \text{PS}(2^{\mathbb{N}}) \\ g \in \text{QR} \\ \text{img}(g) = \text{img}(\phi)}} \left((\text{Aut}(2^{\mathbb{N}})^n \setminus U_\phi) \cup \left(\bigcup_{\substack{\psi \in \text{PS}(2^{\mathbb{N}}) \\ \text{dom}(g) = \text{img}(\psi) \\ \psi g = \phi}} U_\psi \right) \right). \end{aligned}$$

From Theorem 5.22, it follows that $C = U_1 \cap U_2 \cap U_3$.

We have now shown that C is a countable intersection of open sets. We need now only show that C is dense. Let $\phi \in \text{PS}(2^{\mathbb{N}})$ be an arbitrary surjection from a clopen partition of $2^{\mathbb{N}}$ to a $\sigma_{b,n}$ -structure. If $U_\phi \neq \emptyset$, then $\text{img}(\phi)$ has the property that all its relations are surjective with surjective inverse. Thus by Theorem 5.26, we can assume without loss of generality that $\text{img}(\phi)$ is an n -CA structure.

It suffices to show that $U_\phi \cap C$ is not empty. By Lemmas 5.31 and 5.32, let $f \in C$ be fixed. As $f \in U_1$, there is $\phi' \in \text{PS}(2^{\mathbb{N}})$ such that $\text{img}(\phi') = \text{img}(\phi)$ and $f \in U_{\phi'}$. Let $q : 2^{\mathbb{N}} \rightarrow \text{img}(\phi)$ be defined by $(x)q = (U)\phi'$ where U is the element of $\text{dom}(\phi')$ containing x . We have that $q : \mathbb{S}_f \rightarrow \text{img}(\phi)$ is a continuous quotient map. Let h be any homeomorphism of $2^{\mathbb{N}}$ such that for all $p \in \text{img}(\phi)$ we have $(\{p\}q^{-1})h = \bigcup(\{p\}\phi^{-1})$ (Theorem 5.23 (3)). It follows that $(f, h) \text{conj}_{\text{Aut}(2^{\mathbb{N}})}^n$ is an element of $C \cap U_\phi$ as required. \square

5.3. Automatic continuity for groups. In this subsection we use the ample generics results from the previous two subsection to show that all homomorphisms from either of $\text{Sym}(\mathbb{N})$ or $\text{Aut}(2^{\mathbb{N}})$ to second countable groups are continuous.

This is done via a result (Theorem 5.38) of the paper [21] of Kechris and Rosendal, which states that any Polish group whose conjugacy action has ample generics will satisfy this condition. The results in

this subsection are all individually present in some form in the literature but we include them so we can present a self contained proof of Theorem 5.39.

Lemma 5.34 (Countable covers, Assumed Knowledge: 3.5, 3.9, 3.26, 4.10).

Suppose that G is a second countable topological group with identity 1_G . If N is a neighbourhood of 1_G , then there is a sequence $(g_i)_{i \in \mathbb{N}}$ of elements of G such that $G = \bigcup_{i \in \mathbb{N}} Ng_i$.

Proof. As G is a topological group and $1_G = 1_G^{-1}$, it follows that $A := N \cap N^{-1}$ is a neighbourhood of 1_G . Let B be a countable basis for G with $\emptyset \notin B$. By the axiom of choice, let $(g_i)_{i \in \mathbb{N}}$ be a sequence of elements of G such that every element of B contains g_i for some $i \in \mathbb{N}$. It follows that $\{g_i \mid i \in \mathbb{N}\}$ is a dense subset of G . It suffices to show that $G = \bigcup_{i \in \mathbb{N}} Ng_i$. Let $g \in G$ be arbitrary. The set Ag is a neighbourhood of g , so there is $i \in \mathbb{N}$ such that $g_i \in Ag$. It follows that $g \in A^{-1}g_i = Ag_i \subseteq Ng_i$ as required. \square

The following theorem is often referred to as (part of) Effros' Theorem. For more information see [35] for example.

Theorem 5.35 (Effros' Theorem, Assumed Knowledge: 3.4, 3.22, 3.25, 3.27, 4.10, 5.2, 5.34).

Suppose that G is a Polish group and suppose that $a : X \times G \rightarrow X$ is a continuous and transitive action of G on a metrizable space X which is not meagre in itself. If N is a neighbourhood of 1_G in G and $x \in X$, then $(x, N)a$ is a neighbourhood of x in X .

Proof. Let d_G be a complete metric compatible with G , and d_X be a metric compatible with X .

Claim: If A is a neighbourhood of 1_G in G and $z \in X$, then $z \in ((z, A)a)^{-\circ}$.

Proof of Claim: Let A_2 be a neighbourhood of 1_G in G such that $A_2^{-1} = A_2$ and $A_2^2 \subseteq A$. By Lemma 5.34, let $(t_i)_{i \in \mathbb{N}}$ be a sequence of elements of G such that $\bigcup_{i \in \mathbb{N}} A_2 t_i = G$. It follows that $\bigcup_{i \in \mathbb{N}} (z, A_2 t_i)a = X$. As X is not meagre in itself, it follows that at least one of the sets $(z, A_2 t_i)a$ is somewhere dense and so $(z, A_2)a$ is somewhere dense. If $b \in A_2$ is such that $(z, b)a \in ((z, A_2)a)^{-\circ}$, then $z \in (((z, A_2)a)^{-\circ}, b^{-1})a = ((z, A_2 b^{-1})a)^{-\circ} \subseteq ((z, A)a)^{-\circ}$ as required. \diamond

Let V be an open neighbourhood of 1_G such that $V^{-1} = V$ and $(V^-)^2 \subseteq N$. By the claim, we have $x \in ((x, V)a)^{-\circ}$. It thus suffices to show that $((x, V)a)^{-\circ} \subseteq (x, N)a$.

Let $y \in ((x, V)a)^{-\circ}$ be arbitrary. As V is inverse closed, it follows that similarly $x \in ((y, V)a)^{-\circ}$. We define sequences $(g_i)_{i \in \mathbb{N}}$ and $(h_i)_{i \in \mathbb{N}}$ in G inductively as follows:

- (1) Choose $g_0 \in V$ to be such that

$$(x, g_0)a \in B_{d_X} \left(y, \frac{1}{2^0} \right) \cap \left(\left(y, B_{d_G} \left(1_G, \frac{1}{2^0} \right) \cap V \right) a \right)^{-\circ}$$

(we can do this as $(x, V)a$ is dense in a neighbourhood of y).

- (2) Choose $h_0 \in B_{d_G} \left(1_G, \frac{1}{2^0} \right) \cap V$ to be such that

$$(y, h_0)a \in B_{d_X} \left((x, g_0)a, \frac{1}{2^1} \right) \cap \left(\left(x, B_{d_G} \left(g_0, \frac{1}{2^1} \right) \cap V \right) a \right)^{-\circ}$$

(we can do this as $(y, B_{d_G} \left(1_G, \frac{1}{2^0} \right) \cap V)a$ is dense in a neighbourhood of $(x, g_0)a$).

- (3) Suppose that g_{i-1} and h_{i-1} are defined. Choose $g_i \in B_{d_G} \left(g_{i-1}, \frac{1}{2^i} \right) \cap V$ to be such that

$$(x, g_i)a \in B_{d_X} \left((y, h_{i-1})a, \frac{1}{2^i} \right) \cap \left(\left(y, B_{d_G} \left(h_{i-1}, \frac{1}{2^i} \right) \cap V \right) a \right)^{-\circ}$$

(we can do this as $(x, B_{d_G}(g_{i-1}, \frac{1}{2^i}) \cap V) a$ is dense in a neighbourhood of $(y, h_{i-1})a$).

(4) Suppose that g_i and h_{i-1} are defined. Choose $h_i \in B_{d_G}(h_{i-1}, \frac{1}{2^i}) \cap V$ to be such that

$$(x, h_i)a \in B_{d_X}\left((x, g_i)a, \frac{1}{2^{i+1}}\right) \cap \left(\left(x, B_{d_G}\left(g_i, \frac{1}{2^{i+1}}\right) \cap V\right)a\right)^{-\circ}$$

(we can do this as $(y, B_{d_G}(h_{i-1}, \frac{1}{2^i}) \cap V) a$ is dense in a neighbourhood of $(x, g_i)a$).

By construction, the sequences $(g_i)_{i \in \mathbb{N}}$ and $(h_i)_{i \in \mathbb{N}}$ are Cauchy and so they converge to some $g, h \in V^-$ respectively. As our action is continuous, it follows that $((x, g_i)a)_{i \in \mathbb{N}}$ and $((y, h_i)a)_{i \in \mathbb{N}}$ converge to $(x, g)a$ and $(y, h)a$ respectively.

For all $i \in \mathbb{N}$ we have $((x, g_i)a, (y, h_i)a)d_X \leq \frac{1}{2^i}$, so we must have that $(x, g)a = (y, h)a$. Thus $y = (x, gh^{-1})a$, and so $y \in (x, V^-(V^{-1})^-)a = (x, (V^-)^2)a \subseteq (x, N)a$ as required. \square

Lemma 5.36 (cf. Lemma 6.6 of [21], Extending generic tuples, Assumed Knowledge: 3.5, 3.25, 3.30, 3.31, 3.28, 5.5, 5.35).

Suppose that G is a Polish Group, the action conj_G has ample generics, $A \subseteq G$ is not meagre in G , $(x_0, x_1, \dots, x_{n-1}) \in G^n$ has a comeagre orbit under conj_G^n , $B \subseteq G$ is such that for all comeagre subsets C of G the set $B \cap C$ is dense in G , and N is a neighbourhood of the identity 1_G of G .

In this case there is $(a, g, b) \in A \times N \times B$ such that

$$((x_0, x_1, \dots, x_{n-1}, a), g) \text{conj}_G^n = (x_0, x_1, \dots, x_{n-1}, b)$$

has comeagre orbit under conj_G^{n+1} .

Proof. For the purposes of this proof, we will say that a element of G^n is *generic* if its orbit under conj_G^n is comeagre. By Theorems 3.28 and 3.31, there exists a generic $(z_0, z_1, \dots, z_{n-1}) \in G^n$ such that

$$G \approx_G \left\{ y \in G \mid (z_0, z_1, \dots, z_{n-1}, y) \text{ is generic} \right\}.$$

Let $h \in G$ be such that $(z_0, z_1, \dots, z_{n-1}) = ((x_0, x_1, \dots, x_{n-1}), h) \text{conj}_G^n$, we now have:

$$\begin{aligned} G &= hGh^{-1} \\ &\approx_G h \left\{ y \in G \mid (z_0, z_1, \dots, z_{n-1}, y) \text{ is generic} \right\} h^{-1} \\ &= \left\{ hyh^{-1} \mid (z_0, z_1, \dots, z_{n-1}, y) \text{ is generic} \right\} \\ &= \left\{ y \in G \mid (z_0, z_1, \dots, z_{n-1}, h^{-1}yh) \text{ is generic} \right\} \\ &= \left\{ y \in G \mid ((x_0, x_1, \dots, x_{n-1}), y, h) \text{conj}_G^n \text{ is generic} \right\} \\ &= \left\{ y \in G \mid (x_0, x_1, \dots, x_{n-1}, y) \text{ is generic} \right\}. \end{aligned}$$

Let $a \in A$ be such that $(x_0, x_1, \dots, x_{n-1}, a)$ is generic. Let $G_{x_0, x_1, \dots, x_{n-1}}$ be the subgroup of elements of G which commute with each of x_0, x_1, \dots, x_{n-1} . Note that

$$\left\{ y \in G \mid (x_0, x_1, \dots, x_{n-1}, y) \text{ is generic} \right\} = (a, G_{x_0, x_1, \dots, x_{n-1}}) \text{conj}_G.$$

Moreover this set is comeagre and so it is not meagre in G by Theorem 3.28. In particular the space $(a, G_{x_0, x_1, \dots, x_{n-1}}) \text{conj}_G$ is not meagre in itself by Remark 3.30. As $G_{x_0, x_1, \dots, x_{n-1}} \leq G$ is closed, it is Polish and so by Theorem 5.35, we have that $(a, G_{x_0, x_1, \dots, x_{n-1}} \cap N) \text{conj}_G$ is a neighbourhood of a in

the space $(a, G_{x_0, x_1, \dots, x_{n-1}}) \text{conj}_G$. By assumption $B \cap (a, G_{x_0, x_1, \dots, x_{n-1}}) \text{conj}_G$ is dense in G and so in particular, it follows that there is some $b \in B \cap (a, G_{x_0, x_1, \dots, x_{n-1}} \cap N) \text{conj}_G$ as required. \square

Lemma 5.37 (cf. Lemma 6.7 of [21], Generic Cantor map, Assumed Knowledge: 3.19, 4.28, 5.36).
Suppose that G is a Polish Group, the action conj_G has ample generics, $A \subseteq G$ is not meagre in G , $B \subseteq G$ is such that for all comeagre subsets C of G the set $B \cap C$ is dense in G .

In this case, there is a continuous map $a \mapsto h_a$ from $2^{\mathbb{N}} \rightarrow G$ such that if $p \in \{0, 1\}^$, $a, b \in 2^{\mathbb{N}}$, $p0 < a$ and $p1 < b$, then $h_a^{-1}Ah_a \cap h_b^{-1}Bh_b \neq \emptyset$ (recall Definition 4.28).*

Proof. Let d be a complete metric for G . We define a maps $w \rightarrow h_w$, $w \rightarrow x_w$ from $\{0, 1\}^* \rightarrow G$ inductively to satisfy for following properties:

- (1) If $w \in \{0, 1\}^n$ then the tuple $(x_{w \upharpoonright_0}, x_{w \upharpoonright_1}, x_{w \upharpoonright_2}, \dots, x_w)$ has a generic orbit under conj_G^{n+1} .
- (2) $h_\varepsilon = 1_G$.
- (3) If $w \in \{0, 1\}^n$ and $(n-1)w = 0$ then $x_w \in A$.
- (4) If $w \in \{0, 1\}^n$ and $(n-1)w = 1$ then $x_w \in B$.
- (5) If $w \in \{0, 1\}^n$ and $(n-1)w = 1$ then $h_w = h_{w \upharpoonright_{n-1}}$.
- (6) If $w_0, w_1 \in \{0, 1\}^n$, $(n-1)w_0 = 0$, $(n-1)w_1 = 1$ and $w_0 \upharpoonright_{n-1} = w_1 \upharpoonright_{n-1}$ then $(h_{w_0}, h_{w_1})d \leq \frac{1}{2^n}$, and moreover there is some $g \in G$ which commutes with each of $x_{w_0 \upharpoonright_0}, x_{w_0 \upharpoonright_1}, \dots, x_{w_0 \upharpoonright_{n-1}}$ and such that $(x_{w_0}, g) \text{conj}_G = x_{w_1}$ and $h_{w_0} = gh_{w_1}$.

We do this by defining $h_\varepsilon = 1_G$ and x_ε to have comeagre orbit under conj_G , we then define the rest by induction on n (the inductive step follows from Lemma 5.36).

For $w \in 2^{\mathbb{N}}$, we define h_w to be the limit of the sequence $(h_{w \upharpoonright_n})_{n \in \mathbb{N}}$ (by construction this sequence is Cauchy and so converges). By construction, if $a \neq b \in 2^{\mathbb{N}}$, $n = \min \{i \in \mathbb{N} \mid (i)a \neq (i)b\}$ and $(n)a = 0$, then $x_{a \upharpoonright_{n+1}} \in A$, $x_{b \upharpoonright_{n+1}} \in B$. Note also that (by conditions (5) and (6)) the sequences $((x_{a \upharpoonright_{n+1}}, h_{a \upharpoonright_k}) \text{conj}_G)_{k \geq n+1}$ and $((x_{b \upharpoonright_{n+1}}, h_{b \upharpoonright_k}) \text{conj}_G)_{k \geq n+1}$ are constant. Thus:

$$\begin{aligned} (x_{a \upharpoonright_{n+1}}, h_a) \text{conj}_G &= (x_{a \upharpoonright_{n+1}}, h_{a \upharpoonright_{n+1}}) \text{conj}_G \\ &= (x_{a \upharpoonright_{n+1}}, gh_{b \upharpoonright_{n+1}}) \text{conj}_G \quad \text{where } g \text{ is as in Condition (6)} \\ &= (x_{b \upharpoonright_{n+1}}, h_{b \upharpoonright_{n+1}}) \text{conj}_G \\ &= (x_{b \upharpoonright_{n+1}}, h_b) \text{conj}_G. \end{aligned}$$

In particular this value is an element of $h_a^{-1}Ah_a \cap h_b^{-1}Bh_b$ as required. \square

Theorem 5.38 (cf. Theorem 6.24 of [21], Ample continuity, Assumed Knowledge: 4.3, 5.37).

If G is a Polish Group and the action conj_G has ample generics, then every homomorphism from G to a second countable topological group is continuous.

Proof. Let H be a second countable topological group. Let $\phi : G \rightarrow H$ be a homomorphism. We need to show that ϕ is continuous at 1_G , (that the preimage of every neighbourhood of 1_H is a neighbourhood of 1_G). Let N be a neighbourhood of 1_H , let V be an inverse closed neighbourhood of 1_H with $V^{20} \subseteq N$. Define $A := (V)\phi^{-1}$, we have that

$$G = \bigcup_{h \in (G)\phi} (Vh)\phi^{-1}.$$

By Lemma 5.34 (as $(G)\phi$ is second countable) there is a sequence $(g_i)_{i \in \mathbb{N}}$ with

$$G = \bigcup_{i \in \mathbb{N}} (V(g_i)\phi)\phi^{-1} = \bigcup_{i \in \mathbb{N}} Ag_i.$$

Thus A is not meagre in G .

Claim: There is a non-empty open $U \subseteq G$ such that $U \cap A^5$ is comeagre in U .

Proof of Claim: Suppose that A^5 is not comeagre in any non-empty open set U . It follows that $B := (A^5)^c$ is not meagre in any non-empty open set. If C is comeagre in G , then C is comeagre in all non-empty open subsets of G so $B \cap C$ is dense in G . By Lemma 5.37, let $w \rightarrow h_w$ from $2^{\mathbb{N}}$ be a continuous map such that if $p \in \{0, 1\}^*$, $a, b \in 2^{\mathbb{N}}$, $p0 \leq a$ and $p1 \leq b$ then $(h_a^{-1}Ah_a) \cap (h_b^{-1}Bh_b) \neq \emptyset$.

If $h \in G$, then

$$(h^{-1}Ah) \cap (h^{-1}Bh) = (h^{-1}Ah) \cap (h^{-1}((A^5)^c)h) \subseteq (h^{-1}Ah) \cap (h^{-1}(A^c)h) = \emptyset.$$

It follows that the map $w \rightarrow h_w$ is injective. As $2^{\mathbb{N}}$ is uncountable, there must be $i \in \mathbb{N}$ and $a \neq b \in 2^{\mathbb{N}}$ such that $h_a, h_b \in Ag_i$. Assume without loss of generality, that there is $p \in \{0, 1\}^*$ such that $p0 \leq a$ and $p1 \leq b$.

Note that

$$\begin{aligned} (A, h_a h_b^{-1}) \text{conj}_G &= (A, h_a g_i^{-1} g_i h_b^{-1}) \text{conj}_G \\ &= (h_b g_i^{-1})(h_a g_i^{-1})^{-1} A (h_a g_i^{-1})(h_b g_i^{-1})^{-1} \\ &\subseteq A^5. \end{aligned}$$

But by assumption $(B, h_b) \text{conj}_G \cap (A, h_a) \text{conj}_G \neq \emptyset$, so $B \cap (A, h_a h_b^{-1}) \text{conj}_G \neq \emptyset$ which is a contradiction. \diamond

So there is a non-empty open set subset of G in which A^5 is comeagre. As A (and thus A^5) are inverse closed, it follows that A^{10} is comeagre in an open neighbourhood U of 1_G . Let $h \in U$ be arbitrary. We have that $U \cap Uh$ is an open neighbourhood of h . Thus $A^{10} \cap A^{10}h$ is comeagre in $U \cap Uh$. So there are $h_0, h_1 \in A^{10}$ such that $h_0 = h_1 h$. It follows that $h \in A^{20}$.

As h was arbitrary we obtain $U \subseteq A^{20} \subseteq (N)\phi^{-1}$, and ϕ is continuous as required. \square

Theorem 5.39 (Ample automatic examples, Assumed Knowledge: 4.3, 5.37).

If G is one of the following topological groups, then all homomorphisms from G to second countable topological groups are continuous.

- (1) $\text{Sym}(\mathbb{N})$ with the pointwise topology (recall Proposition 5.17).
- (2) $\text{Aut}(2^{\mathbb{N}})$ with the compact-open topology (recall Proposition 5.14).

Proof. The groups $\text{Sym}(\mathbb{N})$ and $\text{Aut}(2^{\mathbb{N}})$ are Polish groups by Propositions 5.17 and 5.14 respectively. Moreover, by Theorems 5.18 and 5.33 respectively we know that $\text{conj}_{\text{Sym}(\mathbb{N})}$ and $\text{conj}_{\text{Aut}(2^{\mathbb{N}})}$ have ample generics. The result then follows from Theorem 5.38. \square

5.4. Rubin's Theorem. Rubin's Theorem was first proved by Matatyahu Rubin in [31, Corollary 3.5]. The theorem shows that for a given group, there is at most one way for it to act "nicely" on a "nice" topological space (see Theorem 5.49). In particular, the topological space is constructable entirely from the group. In this subsection we give a proof of Rubin's Theorem, and conclude with

the corollary (Corollary 5.50) to Rubin's Theorem which we will require in Part 4. The author of this thesis learned of Rubin's proof via a set of notes taken by Francesco Matucci and James Belk on Rubin's original proof. What is presented here is a minimalistic version of the proof which avoids all references to boolean algebras and gives an explicit description of the Rubin action of a given group (see Definition 5.47).

Definition 5.40 (Support and fix, Assumed Knowledge: 3.14, 4.11, 5.2).

Suppose that G is a group, $a : X \times G \rightarrow X$ is a continuous action of G on a topological space X , and $g \in G$. We define

$$\text{supt}_a(g) := \{x \in X : (x, g)a \neq x\}, \quad \text{fix}_a(g) := \{x \in X : (x, g)a = x\}.$$

Moreover if $U \subseteq X$ then we define

$$G_{U,a} := \{g \in G : \text{supt}_a(g) \subseteq U\}.$$

It is routine to verify that $G_{U,a}$ is always a subgroup of G , and if X is Hausdorff, then the set $\text{supt}_a(g)$ is always open, and the set $\text{fix}_a(g)$ is always closed.

Definition 5.41 (Locally moving, Assumed Knowledge: 3.5, 3.26, 5.40).

If $a : X \times G \rightarrow X$ is an action of a group G on a topological structure X , then we say that a is *locally moving* if for all $x \in X$ and neighbourhoods U of x , we have that $x \in ((x, G_{U,a})a)^{-\circ}$.

The support sets (Definition 5.40), are precisely how the topological space will be constructed. Due to the restrictions we will put on the action, the interplay between the sets of this type will completely determine the topological space we need.

The following lemma (Lemma 5.42), is our means for understanding the action of the group on these sets using the group alone.

Lemma 5.42 (Support tricks, Assumed Knowledge: 3.5, 5.7, 5.8, 5.40).

If $a : X \times G \rightarrow X$ is a continuous action of a group G on a topological space X , and $f, g \in G$, then

$$\text{supt}_a((f, g) \text{conj}_G) = (\text{supt}_a(f), g)a,$$

$$\text{supt}_a([f, g]) \subseteq \text{supt}_a(f) \cup (\text{supt}_a(f), g)a, \quad \text{and} \quad \text{supt}_a([f, g]) \subseteq \text{supt}_a(g) \cup (\text{supt}_a(g), f)a.$$

Moreover if X is Hausdorff and $x \in \text{supt}_a(f)$, then there is an open neighbourhood U of x such that $U \subseteq \text{supt}_a(f)$ and $(U, f)a \cap U = \emptyset$.

Proof. The first equality can be seen as follows:

$$\begin{aligned} x \in \text{supt}_a((f, g) \text{conj}_G) &\iff (x, (f, g) \text{conj}_G)a \neq x \\ &\iff (x, g^{-1}fg)a \neq x \\ &\iff (x, g^{-1}f)a \neq (x, g^{-1})a \\ &\iff ((x, g^{-1})a, f)a \neq (x, g^{-1})a \\ &\iff (x, g^{-1})a \in \text{supt}_a(f) \\ &\iff x \in (\text{supt}_a(f), g)a. \end{aligned}$$

We now show the first of the inclusions, the second is proved similarly. From the first equality it follows that if $x \notin \text{supt}_a(f) \cup (\text{supt}_a(f), g)a$ then $x \in \text{fix}_a(f) \cap \text{fix}_a((f, g) \text{conj}_G) = \text{fix}_a(f^{-1}) \cap$

$\text{fix}_a((f, g) \text{conj}_G)$. So in particular, if $x \notin \text{supt}_a(f) \cup (\text{supt}_a(f), g)a$ then

$$(x, [f, g])a = (x, f^{-1}(f, g) \text{conj}_G)a = (x, (f, g) \text{conj}_G)a = x,$$

the inclusion follows.

Finally suppose that X is Hausdorff and $x \in \text{supt}_a(f)$. Let U_0, U_1 be disjoint open sets containing $x, (x, f)a$ respectively. As X is Hausdorff, $\text{supt}_a(f)$ is an open neighbourhood of both x and $(x, f)a$. Thus by replacing U_0, U_1 with their intersection with $\text{supt}_a(f)$, we can assume without loss of generality that U_0, U_1 are both contained in $\text{supt}_a(f)$. Let $U := U_0 \cap (U_1, f^{-1})a$. The set U is the required neighbourhood of x . \square

The main tool we still need to establish, before we proceed with the main proof, is a purely algebraic means of describing how the support sets (Definition 5.40) compare with each other. This role is served by the notion of ‘‘Algebraic Disjointness’’ (as we will see in Theorem 5.44).

Definition 5.43 (Algebraic disjointness, Assumed Knowledge: 5.8, 5.6).

If G is a group and $f, g \in G$, then we say that f is *algebraically disjoint* from g if for all $h \in G \setminus C_G(f)$ there are $h_{0,g}, h_{1,g} \in C_G(g)$ such that $[[h, h_{0,g}], h_{1,g}]$ is a non-identity element of $C_G(g)$.

Theorem 5.44 (Disjoint vs algebraically disjoint, Assumed Knowledge: 3.10, 3.14, 5.41, 5.42, 5.43). *Suppose that X is a Hausdorff topological space without isolated points, G is a group of homeomorphisms of X , and the natural action $a : X \times G \rightarrow X$ defined by $(x, g)a = (x)g$ is locally moving. If $f, g \in G$ then*

$$\begin{aligned} \text{supt}_a(f) \cap \text{supt}_a(g) = \emptyset &\Rightarrow f \text{ is algebraically disjoint from } g \\ &\Rightarrow \text{supt}_a(f) \cap \text{supt}_a(g^{12}) = \emptyset. \end{aligned}$$

Proof. First suppose that $\text{supt}_a(f) \cap \text{supt}_a(g) = \emptyset$. Let $h \in G \setminus C_G(f)$ and let $p \in \text{supt}_a(f) \cap \text{supt}_a(h)$. By Lemma 5.42, let U_0 be a neighbourhood of p contained in $\text{supt}_a(f) \cap \text{supt}_a(h)$ such that $(U_0, h)a \cap U_0 = \emptyset$. As a is locally moving and X is Hausdorff and has no isolated points, we may choose $h_{0,g} \in G_{U_0, a}$ such that $(p)h_{0,g} \neq p$. Similarly, let $U_1 \subseteq U_0$ be a neighbourhood of p such that $(U_1, h_{0,g})a \cap U_1 = \emptyset$, and let $h_{1,g} \in G_{U_1, a}$ be such that $(p)h_{1,g} \neq p$.

By Lemma 5.42, $\text{supt}_a(h^{-1}h_{0,g}^{-1}h) \cap U_1 = \emptyset$, so the actions of $[h, h_{0,g}]$ and $h_{0,g}$ agree on U_1 . Similarly the actions of $[h_{0,g}, h_{1,g}]$ and $h_{1,g}$ agree on U_1 , it follows that

$$(p)[[h, h_{0,g}], h_{1,g}] = (p)[h_{0,g}, h_{1,g}] = (p)h_{1,g} \neq p.$$

So $[[h, h_{0,g}], h_{1,g}] \neq \text{id}_X$. Also by Lemma 5.42 we have

$$\begin{aligned} \text{supt}_a([[h, h_{0,g}], h_{1,g}]) &\subseteq \text{supt}_a(h_{1,g}) \cup (\text{supt}_a(h_{1,g}), [h, h_{0,g}])a \\ &= \text{supt}_a(h_{1,g}) \cup (\text{supt}_a(h_{1,g}), h_{0,g})a \subseteq U_0 \cup U_1 \subseteq \text{supt}_a(f). \end{aligned}$$

As each of $h_{0,g}, h_{1,g}$ and $[[h, h_{0,g}], h_{1,g}]$ have supports disjoint from g , they all commute with g so the first implication follows.

Now suppose that f is algebraically disjoint from g . Suppose for a contradiction that $x \in \text{supt}_a(f) \cap \text{supt}_a(g^{12})$. Note that

$$\text{supt}_a(g^{12}) \subseteq \bigcap_{1 \leq i \leq 4} \text{supt}_a(g^i).$$

It follows from Lemma 5.42, that we can find a neighbourhood U of x such that

$$(U)f \cap U = (U)g \cap U = (U)g^2 \cap U = (U)g^3 \cap U = (U)g^4 \cap U = \emptyset.$$

Note that the sets $\{(U)g^i : 0 \leq i \leq 4\}$ are pairwise disjoint. As G is locally moving and X is Hausdorff without isolated points, we may choose an element h of $G_{U,a} \setminus \{\text{id}_X\}$. By Lemma 5.42, $\text{supt}_a(h) \cap \text{supt}_a(f^{-1}hf) = \emptyset$ and in particular $h \in G \setminus C_G(f)$. So by assumption we may find $h_{0,g}, h_{1,g} \in C_G(g)$ such that $h_2 := [[h, h_{0,g}], h_{1,g}] \in C_G(g) \setminus \{\text{id}_X\}$. Let $P := \{\text{id}_X, h_{0,g}, h_{1,g}, h_{0,g}h_{1,g}\} \subseteq C_G(g)$. By Lemma 5.42 (applied twice) we have

$$\text{supt}(h_2) \subseteq \bigcup_{p \in P} (U)p.$$

As $h_2 \neq \text{id}_X$, there is thus some $y \in U$ and $k \in P$ such that $((y)k)h_2 \neq (y)k$. As $g^{-1}h_2g = h_2$, we have $\text{supt}_a(h_2)g^i = \text{supt}_a(h_2)$ for all $i \in \mathbb{Z}$. Moreover, as the sets $\{(U)g^i : 0 \leq i \leq 4\}$ are pairwise disjoint, when $|i - j| \in \{1, 2, 3, 4\}$ we have $(y)kg^i = (y)g^i k \neq (y)g^j k = (y)kg^j$. So

$$|\{(y)kg^{-i} : 0 \leq i \leq 4\}| = 5.$$

As $|P| = 4$ and $\{(y)kg^{-i} : 0 \leq i \leq 4\} \subseteq \text{supt}_a(h_2) \subseteq \bigcup_{p \in P} (U)p$, there is some $p \in P$ and $0 \leq i < j \leq 4$ such that $\{(y)kg^{-i}, (y)kg^{-j}\} \subseteq (U)p$. Thus $(y)k \in (U)pg^i \cap (U)pg^j = (U)g^i p \cap (U)g^j p = ((U)g^i \cap (U)g^j)p = \emptyset$. This is a contradiction. \square

Lemma 5.45 (Non-trivial powers, Assumed Knowledge: 5.41, 5.42).

Suppose that X is a Hausdorff topological space without isolated points, G is a group of homeomorphisms of X , and the natural action $a : X \times G \rightarrow X$ defined by $(x, g)a = (x)g$ is locally moving. If $U \subseteq X$ is open and non-empty, then for all $n \in \mathbb{N} \setminus \{0\}$ there is $g \in G_{U,a}$ such that $g^n \neq \text{id}_X$.

Proof. We prove by induction on n , that for all $n \in \mathbb{N}$ there is a non-empty open set U_n and $g_n \in G_{U,a}$ such that $U_n \subseteq U$ and $\{(U_n)g_n^i : 0 \leq i < n\}$ are pairwise disjoint (note that this implies the lemma).

As a is locally moving, and X is Hausdorff with no isolated points, it follows that there is $g_1 \in G_{U,a} \setminus \{1_G\}$. The case for $n = 1$ now follows immediately from Lemma 5.42.

Suppose inductively that the claim is true for all $n < k$. If $U_{k-1} \not\subseteq \text{fix}_a(g_{k-1}^k)$ then let $x \in \text{supt}_a(g_{k-1}^k) \cap U_{k-1}$. For each $i \in \{0, 1, 2, \dots, k\}$ choose a neighbourhood V_i of $(x)g_{k-1}^i$ such that the sets V_i are pairwise disjoint (note that the points $(x)g_{k-1}^i$ are distinct by assumption). We can then define $U_k := \bigcap_{i \in \{0, 1, 2, \dots, k\}} V_i g_{k-1}^{-i}$ (which is notably a neighbourhood of x) and $g_k := g_{k-1}$.

If $U_{k-1} \subseteq \text{fix}_a(g_{k-1}^k)$, choose some $h \in G_{U_{k-1}, a} \setminus \{1_G\}$. If $x \in \text{supt}_a(h)$, then we have $(x)(hg_{k-1})^k = (x)hg_{k-1}^k = (x)h$ (the first equality follows from the fact that $(\text{supt}_a(h))g_{k-1}^i \cap \text{supt}_a(h) = \emptyset$ for $0 < i < k$). Moreover for $0 \leq i < k$, we have $(U_{k-1})(hg)^i = (U_{k-1})g^i$. Hence hg_{k-1} was an appropriate candidate for g_{k-1} and we can return to the case that $U_{k-1} \not\subseteq \text{fix}_a(g_{k-1}^k)$. \square

Definition 5.46 (Algebraic basis, Assumed Knowledge: 3.8, 5.1, 5.6, 5.43).

If G is a group and $f \in G$, then we define

$$G_f := C_G(\{g^{12} : f \text{ is algebraically disjoint from } g\}).$$

Note that if $\phi \in \text{Aut}(G)$, then $(G_f)\phi = G_{(f)\phi}$. Moreover, if $F \subseteq G$ is finite, then we define $G_F := \bigcap_{f \in F} G_f$.

Definition 5.47 (Canonical Rubin action, Assumed Knowledge: 1.11, 5.7, 5.46).

If G is a group, then we define the *canonical Rubin space* (X_G, \mathcal{T}_G) and *canonical Rubin action* $r_G : X_G \times G \rightarrow X_G$ of G as follows:

- (1) Let $B_G := \left\{ G_F \mid F \text{ is a finite subset of } G \right\}$. The set B_G is partially ordered by containment, and G acts on B_G by conjugation.
- (2) Let $\mathcal{U}_G := \{ \mathcal{F} \subseteq B_G : \mathcal{F} \text{ is an ultrafilter of } B_G \text{ as a poset} \}$ (recall Definition 1.11).
- (3) If $\mathcal{F} \in \mathcal{U}_G$ and F is a finite subset of G , then we say that $\mathcal{F} \in_G G_F$ if there is some non-identity element $g \in G$ with $G_g \leq G_F$ and such that

$$(\mathcal{F}, G_F) \text{ conj}_G = \left\{ (b, h) \text{ conj}_G \mid b \in \mathcal{F}, h \in G_F \right\} \supseteq \{ b \in B_G : b \leq G_g \}.$$

Note that if $g \in G$ and $\mathcal{F} \in_G G_F$, then $(\mathcal{F}, g) \text{ conj}_G \in_G G_{(F, g) \text{ conj}_G}$.

- (4) We define an equivalence relation \sim_G on \mathcal{U}_G by

$$\mathcal{F}_1 \sim_G \mathcal{F}_2 \iff \{ G_F \in B_G : \mathcal{F}_1 \in_G G_F \} = \{ G_F \in B_G : \mathcal{F}_2 \in_G G_F \}.$$

We also extend the relation \in_G to \mathcal{U}_G / \sim_G by $[\mathcal{F}]_{\sim_G} \in_G G_F \iff \mathcal{F} \in_G G_F$.

- (5) We define $X_G := \{ [\mathcal{F}]_{\sim_G} : \mathcal{F} \in \mathcal{U}_G \text{ such that there is } f \in G \text{ with } \mathcal{F} \in_G G_{\{f\}} = G_f \}$.
- (6) For each finite $F \subseteq G$, we define $G'_F := \left\{ x \in X_G \mid x \in_G G_F \right\}$.
- (7) We define \mathcal{T}_G , to be the topology on X_G generated by the sets G'_F for finite $F \subseteq G$.
- (8) We define $r_G : X_G \times G \rightarrow X_G$ by $([\mathcal{F}]_{\sim_G}, g)r_G = [(\mathcal{F}, g) \text{ conj}_G]_{\sim_G}$. It is routine to verify that this is indeed a well-defined action.

Definition 5.48 (Conjugate actions, Assumed Knowledge: 2.3, 5.2).

Suppose that G is a group, X_0, X_1 are topological structures and $a_0 : X_0 \times G \rightarrow X_0, a_1 : X_1 \times G \rightarrow X_1$ are actions of G on these structures. We say that a_0, a_1 are *topologically conjugate* if there is a topological isomorphism $\psi : X_0 \rightarrow X_1$ such that $a_0 \circ \psi = \langle \pi_0 \psi, \pi_1 \rangle_{X_1 \times G} \circ a_1$ (recall Definition 2.3).

Theorem 5.49 (Rubin's Theorem, Assumed Knowledge: 3.18, 3.17, 4.3, 5.44, 5.47, 5.48).

If $a : X \times G \rightarrow X$ is a continuous, faithful, locally moving action of a group G on a locally compact, Hausdorff topological space X without isolated points, then a is topologically conjugate to the canonical Rubin action r_G .

Proof. Let \mathcal{T} denote the topology on X partially ordered by \subseteq .

Claim 0: If $U, V \subseteq X$ are arbitrary open sets satisfying $G_{U^{-\circ}, a} \subseteq G_{V^{-\circ}, a}$, then $U^{-\circ} \subseteq V^{-\circ}$.

Proof of Claim: Let U' be an arbitrary non-empty open subset of U . As X is Hausdorff and has no isolated points, and a is locally moving, it follows that we can find some non-identity $h \in G_{U', a}$. As

$$h \in G_{U', a} \subseteq G_{U, a} \subseteq G_{U^{-\circ}, a} \subseteq G_{V^{-\circ}, a}$$

it follows that $\text{supt}_a(h) \subseteq V^{-\circ}$, so $U' \cap V^{-\circ} \neq \emptyset$, and thus $U' \cap V \neq \emptyset$. As U' was arbitrary, it follows that V is dense in U , so $U^- \subseteq V^-$, and $U^{-\circ} \subseteq V^{-\circ}$. \diamond

Claim 1: For all finite $F \subseteq G$, we have that $G_F = G_{\bigcap_{f \in F} \text{supt}_a(f)^{-\circ}, a}$. If we then define $\gamma : B_G \rightarrow \mathcal{T}$ by $(G_F)\gamma = \bigcap_{f \in F} \text{supt}_a(f)^{-\circ}$, then γ is a well-defined embedding of partially ordered sets. Moreover, for all $f, g \in G$ we have $((G_f, g) \text{ conj}_G)\gamma = ((G_f)\gamma, g)a$.

Proof of Claim:

As $G_{\bigcap_{f \in F} \text{supt}_a(f)^{-\circ}, a} = \bigcap_{f \in F} G_{\text{supt}_a(f)^{-\circ}, a}$, we need only show the first part of the claim in the case that F is a singleton $\{f\}$.

We first show that $G_f \subseteq G_{\text{supt}_a(f)^{-\circ}, a}$. Suppose that $h \notin G_{\text{supt}(f)^{-\circ}, a}$ is arbitrary, we will show that $h \notin G_f$. As $\text{supt}_a(h) \not\subseteq \text{supt}_a(f)^{-\circ}$, there is a non-empty open set $U_0 \subseteq \text{supt}_a(h) \setminus \text{supt}_a(f)$. By Lemma 5.42, let U_1 be a non-empty open subset of U_0 such that $(U_1, h)a \cap U_1 = \emptyset$. By Lemma 5.45, let $g \in G_{U_1, a}$ be such that $g^{12} \neq 1_G$. Then $\text{supt}_a(g) \subseteq U_1 \subseteq \text{supt}_a(h) \setminus \text{supt}_a(f)$. So by Theorem 5.44, f is algebraically disjoint from g . But g^{12} and $h^{-1}g^{12}h$ have disjoint supports, so g^{12} does not commute with h . Thus $h \notin G_f$.

We next show that $G_{\text{supt}_a(f)^{-\circ}, a} \subseteq G_f$. Let $h \in G_{\text{supt}_a(f)^{-\circ}, a}$ be arbitrary, and $g \in G$ be arbitrary such that f is algebraically disjoint from g . We show that h commutes with g^{12} . By Theorem 5.44, we have that $\text{supt}_a(f) \cap \text{supt}_a(g^{12}) = \emptyset$. As $\text{supt}_a(f)$ is dense in $\text{supt}_a(h)$ and $\text{supt}_a(g^{12})$ is open, it follows that $\text{supt}_a(h) \cap \text{supt}_a(g^{12}) = \emptyset$. Thus h commutes with g^{12} as required.

We now show that γ is well-defined. If $F \subseteq G$ is finite, then

$$(G_F)\gamma \subseteq (G_F)\gamma^{-\circ} = \left(\bigcap_{f \in F} \text{supt}_a(f)^{-\circ} \right)^{-\circ} \subseteq \left(\bigcap_{f \in F} \text{supt}_a(f)^{-\circ-\circ} \right)^{\circ} = \left(\bigcap_{f \in F} \text{supt}_a(f)^{-} \right)^{\circ} \subseteq (G_F)\gamma.$$

Thus $(G_F)\gamma$ is of the form in Claim 0. If $G_{U^{-\circ}, a} = (G_F)\gamma = G_{V^{-\circ}, a}$, then we have both $U^{-\circ} \subseteq V^{-\circ}$ and $V^{-\circ} \subseteq U^{-\circ}$, so $V^{-\circ} = U^{-\circ}$. Thus γ is a well-defined function (and is order preserving).

We next show that γ is an embedding. By the first part of the claim, the map $\gamma' : \mathcal{T} \rightarrow \mathcal{P}(G)$ defined by $(U)\gamma' = G_{U, a}$ satisfies $\gamma\gamma' = \text{id}_{B_G}$ (where id_{B_G} is the identity function on B_G). It follows that γ is an injective function. Moreover the map γ' preserves containment and hence so does $\gamma^{-1} \subseteq \gamma'$.

Finally by Lemma 5.42, for all $f, g \in G$ we have that

$$\begin{aligned} ((G_f, g) \text{conj}_G)\gamma &= (G_{g^{-1}fg})\gamma = (\text{supt}_a(g^{-1}fg))^{-\circ} \\ &= ((\text{supt}_a(f), g)a)^{-\circ} = (\text{supt}(f)^{-\circ}, g)a = ((G_f)\gamma, g)a. \diamond \end{aligned}$$

Claim 2: The set $\left\{ \left\{ G_f \mid f \in G \right\} \right\} \gamma \subseteq \text{img}(\gamma)$ is a basis for the topology \mathcal{T} .

Proof of Claim:

Let $x \in X$ and U an arbitrary open neighbourhood of x . Let V be a compact neighbourhood of x . It follows that $U \cap V$ is an open neighbourhood of x in the compact space V . By Remark 3.18, there is a compact neighbourhood W of x contained in $U \cap V$. In particular W is closed in X .

Let $f \in G_{W^{\circ}, a}$ be such that $(x)f \neq x$. We have

$$x \in \text{supt}_a(f) \subseteq \text{supt}_a(f)^{-\circ} \subseteq W^{\circ-\circ} \subseteq W \subseteq U.$$

As $\text{supt}_a(f)^{-\circ} = (G_f)\gamma$, the result follows. \diamond

Claim 3: If $\mathcal{F} \in \mathcal{U}_G$ and U is an open set such that $G_{U, a} \in B_G$, then

$$(\text{there is } x \in U \text{ such that } \text{Nbhd}_X(x) \subseteq ((\mathcal{F})\gamma)^*) \iff \mathcal{F} \in_G G_{U, a}.$$

Where $((\mathcal{F})\gamma)^* := \left\{ S \in \mathcal{P}(X) \mid S \text{ contains an element of } (\mathcal{F})\gamma \right\}$.

Proof of Claim:

(\Rightarrow): Suppose that $x \in U$ and $\text{Nbhd}_X(x) \subseteq ((\mathcal{F})\gamma)^*$. As a is locally moving, it follows from Claim 2 that there is a non-identity element $g \in G$ such that $(x, G_{U, a})a$ is dense in $(G_g)\gamma$. Let $b \in B_G$ be arbitrary with $b \leq G_g$. It suffices to show that $b \in (\mathcal{F}, G_{U, a}) \text{conj}_G$.

As $(x, G_{U, a})a$ is dense in $(G_g)\gamma$, there is some $f_b \in G_{U, a}$ such that $(x, f_b)a \in (b)\gamma$. It follows that $((b)\gamma, f_b^{-1})a \in \text{Nbhd}_X(x) \subseteq ((\mathcal{F})\gamma)^*$. As \mathcal{F} is a filter, we have $(\mathcal{F})\gamma = ((\mathcal{F})\gamma)^* \cap (B_G)\gamma$. Moreover,

by Claim 1 we have $((b, f_b^{-1}) \text{conj}_G)\gamma = ((b)\gamma, f_b^{-1})a$, so $((b)\gamma, f_b^{-1})a \in ((\mathcal{F})\gamma)^* \cap (B_G)\gamma = (\mathcal{F})\gamma$. Thus we have

$$(b)\gamma \in ((\mathcal{F})\gamma, f_b)a \Rightarrow (b)\gamma \in ((\mathcal{F}, f_b) \text{conj}_G)\gamma \Rightarrow b \in (\mathcal{F}, f_b) \text{conj}_G \Rightarrow b \in (\mathcal{F}, G_{U,a}) \text{conj}_G$$

as required.

(\Leftarrow): Suppose that $\mathcal{F} \in_G G_{U,a}$. Let $g \in G$ be such that $G_g \leq G_{U,a}$, and for all $h \in G$ with $G_h \leq G_g$, we have that $G_h \in (\mathcal{F}, G_{U,a}) \text{conj}_G$. As shown in the proof of Claim 2, $(G_g)\gamma$ contains a compact set with non-empty interior, and by Claim 2 we can always choose this set to be the closure of an element of $(B_G)\gamma$, thus by replacing G_g with a smaller element of B_G we can assume without loss of generality that $((G_g)\gamma)^-$ is a compact subset of U .

Let $h \in G$ be arbitrary such that $G_h \leq G_g$, and note that $((G_h)\gamma)^-$ is compact. There is some $f_h \in G_{U,a}$ such that $G_{f_h^{-1}hf_h} = (G_h, f_h) \text{conj}_G \in \mathcal{F}$. Let $k := f_h^{-1}hf_h$ and note that $G_k \in \mathcal{F}$ and

$$((G_k)\gamma)^- = (((G_h)\gamma)^-, f_h)a \subseteq (((G_g)\gamma)^-, f_h)a \subseteq (U, f_h)a = U.$$

By Remark 3.18, there is some $x \in (G_k)\gamma^- \cap \bigcap_{b \in \mathcal{F}} (b)\gamma^-$. Let $N \in \text{Nbhd}_X(x)$ be arbitrary. We need only show that $N \in ((\mathcal{F})\gamma)^*$. Let $N' := (G_k)\gamma \cap N$. As B_G and \mathcal{F} are closed under finite intersections, the set $\left\{ A \in B_G \mid A \supseteq b \cap (N')\gamma^{-1} \text{ for some } b \in \mathcal{F} \right\}$ is a proper filter on B_G containing \mathcal{F} . As \mathcal{F} is an ultrafilter, it follows that $(N')\gamma^{-1} \in \mathcal{F}$. Hence $N \in ((\mathcal{F})\gamma)^*$ as required. \diamond .

From Claim 3, for all $x \in X_G$ there is $(x)\psi \in X$ such that $\text{Nbhd}_X((x)\psi) \subseteq ((\mathcal{F})\gamma)^*$ for all $\mathcal{F} \in x$. As X is Hausdorff it follows from Claim 2 that this point is unique (thus we can define a map $\psi : X_G \rightarrow X$ in this fashion). It also immediate from Claim 3 and the definition of \sim_G that the function ψ is injective. Furthermore, by Claim 2 and the definition of \mathcal{T}_G , a subset $U \subseteq X_G$ is open if and only if it is the preimage of an open set under ψ .

If $x \in X$ is arbitrary, then $\text{Nbhd}_X(x)$ is a proper filter on $\mathcal{P}(X)$, thus it can be extended to some ultrafilter \mathcal{F} on $\mathcal{P}(X)$, thus $([(\mathcal{F} \cap (B_G)\gamma)\gamma^{-1}] \sim_G)\psi = x$. As x was arbitrary, it follows that ψ is surjective. We now have that $\psi : X_G \rightarrow X$ is a homeomorphism.

It now suffices to show that for all $g \in G$, and $x \in X_G$, we have $(x, g)r_G\psi = ((x)\psi, g)a$. Let $g \in G$ and $x \in X_G$ be arbitrary. By Claim 2 and the assumption that X is Hausdorff, it suffices to show that

$$\{f \in G : (x, g)r_G\psi \in (G_f)\gamma\} = \{f \in G : ((x)\psi, g)a \in (G_f)\gamma\}.$$

This can be seen as follows:

$$\begin{aligned} \{f \in G : (x, g)r_G\psi \in (G_f)\gamma\} &= \{f \in G : (x, g)r_G \in_G G_f\} \\ &= \{f \in G : x \in_G (G_f, g^{-1}) \text{conj}_G\} \\ &= \{f \in G : x \in_G G_{gf g^{-1}}\} \\ &= \{f \in G : (x)\psi \in (G_{gf g^{-1}})\gamma\} \\ &= \{f \in G : (x)\psi \in ((G_f, g^{-1}) \text{conj}_G)\gamma\} \\ &= \{f \in G : (x)\psi \in ((G_f)\gamma, g^{-1})a\} \\ &= \{f \in G : ((x)\psi, g)a \in (G_f)\gamma\}. \quad \square \end{aligned}$$

Now that we have proved Rubin's Theorem (Theorem 5.49), we know that any sufficiently nice action of a group on a sufficiently nice topological space is in fact an algebraic invariant of the group. This

is particularly useful when taking about isomorphisms between groups. In the following corollary, we show how Rubin's Theorem can help with understanding the automorphisms of a group (this corollary is crucial to Part 4).

Corollary 5.50 (Rubin automorphisms, Assumed Knowledge: 5.6, 5.49).

If G is a group and $a : X \times G \rightarrow X$ is a continuous, faithful, locally moving action of G on a locally compact, Hausdorff topological space X without isolated points, then

$$\text{Aut}(G) \cong N_{\text{Aut}(X)}((G)\phi_a)$$

(where ϕ_a is as in Definition 5.2) via an isomorphism which restricts to an isomorphism from $\text{Inn}(G)$ to $(G)\phi_a$.

Proof. We define a map $n : G \times N_{\text{Aut}(X)}((G)\phi_a) \rightarrow G$ of $N_{\text{Aut}(X)}((G)\phi_a)$ on G by:

$$(g, f)n = (f^{-1}(g)\phi_a f)\phi_a^{-1}.$$

Note that this is only well-defined because of the choice of the group $N_{\text{Aut}(X)}((G)\phi_a)$.

It suffices to show that n is a faithful action of $N_{\text{Aut}(X)}((G)\phi_a)$ on G and that the corresponding embedding $\phi_n : N_{\text{Aut}(X)}((G)\phi_a) \rightarrow \text{Aut}(G)$ is surjective.

We first show that n is an action. Let $g \in G$ and $f, h \in N_{\text{Aut}(X)}((G)\phi_a)$ be arbitrary, then

$$\begin{aligned} (((g, f)n, h)n &= (h^{-1}((f^{-1}(g)\phi_a f)\phi_a^{-1})\phi_a h)\phi_a^{-1} \\ &= (h^{-1}f^{-1}(g)\phi_a fh)\phi_a^{-1} \\ &= ((fh)^{-1}(g)\phi_a fh)\phi_a^{-1} \\ &= (g, fh)n. \end{aligned}$$

We next show that ϕ_n is injective. Suppose that $f, h \in N_{\text{Aut}(X)}((G)\phi_a)$ are such that $(f)\phi_n = (h)\phi_n$, we show that $f = h$. It suffices to show that fh^{-1} is the identity function on X . As ϕ_n is a homomorphism, we know that $(fh^{-1})\phi_n$ is the identity function on G . Thus by the definition of n , it follows that fh^{-1} commutes with all elements of $(G)\phi_a$. Suppose for a contradiction that fh^{-1} is non-trivial. It follows from Lemma 5.42, that there is a non-empty open subset U of X such that $(U)fh^{-1} \cap U = \emptyset$. As X is Hausdorff and has no isolated points, and a is locally moving, it follows that there is some non-identity element $g \in G_{U,a}$. By Lemma 5.42 it follows that no elements of $\text{supt}_a(g)$ are fixed by $((g)\phi_a, fh^{-1}) \text{conj}_{\text{Aut}(X)}$. In particular fh^{-1} does not commute with $(g)\phi_a$, this is a contradiction.

It remains to show that ϕ_n is surjective. By Rubin's Theorem (5.49) let $\psi : X_G \rightarrow X$ be a homeomorphism such that for all $x \in X_G$ and $g \in G$ we have $((x)\psi, g)a = (x, g)r_G\psi$. Recall that $X_G \subseteq \mathcal{P}(\mathcal{P}(\mathcal{P}(G)))$, and as per Definition 1.6, any automorphism φ of G can act on X_G . Thus for each $\varphi \in \text{Aut}(G)$, we define a map $\varphi' : X_G \rightarrow X_G$ by $(x)\varphi' = (x)\varphi$.

For all $f \in G$, we have $(G'_f)\varphi' = G'_{(f)\varphi}$, so in particular φ' is a homeomorphism of X_G . We now choose $\varphi \in \text{Aut}(G)$ to be arbitrary, it suffices to show that $\varphi \in \text{img}(\phi_n)$. For all $g \in G$, we define an automorphism c_g of G by $(f)c_g = (f, g) \text{conj}_G$. Note that for all $g \in G$ we have $\varphi^{-1}c_g\varphi = c_{(g)\varphi}$, and the condition on ψ together with the definition of r_G gives that $\psi((g)\phi_a)\psi^{-1} = c'_g$ for all $g \in G$.

Let $g \in G$ be arbitrary, it suffices to show that $(g)\varphi = (g)((\psi^{-1}\varphi'\psi)\phi_n)$. This can be seen as follows:

$$\begin{aligned}
(g)((\psi^{-1}\varphi'\psi)\phi_n) &= (g, \psi^{-1}\varphi'\psi)n \\
&= ((\psi^{-1}\varphi'\psi)^{-1}((g)\phi_a)(\psi^{-1}\varphi'\psi))\phi_a^{-1} \\
&= (\psi^{-1}(\varphi')^{-1}(\psi((g)\phi_a)\psi^{-1})\varphi'\psi)\phi_a^{-1} \\
&= (\psi^{-1}(\varphi')^{-1}(c'_g)\varphi'\psi)\phi_a^{-1} \\
&= (\psi^{-1}(\varphi^{-1}c_g\varphi)'\psi)\phi_a^{-1} \\
&= (\psi^{-1}c'_{(g)\varphi}\psi)\phi_a^{-1} \\
&= (g)\varphi.
\end{aligned}$$

□

Part 3. Semigroups And Clones

The results of this part are (for the most part) the results from [11] by J. Jonušas, Z. Mesyan, J. D. Mitchell, M. Morayne, Y. Péresse and the author of this document. The other authors have kindly given permission for the inclusion of these results here.

In this part we discuss various natural ways of defining topologies on arbitrary semigroups, and investigate how these topologies interact (see Proposition 6.15). Many of the definitions are minimal/maximal up to some topological condition so by understanding them we gain a better understanding of all the topologies compatible with the semigroups satisfying the required topological conditions.

The topological conditions we are particularly interested are those of being Hausdorff, second countable or Polish. Hausdorffness (Definition 3.14) is a useful and popular separation condition which forces the topologies in question to be “reasonably big” (in particular it allows us to exclude indiscrete/trivial topologies). Given that we will be primarily concerned with topological spaces of continuum cardinality, second countableness (Definition 3.9) similarly forces the topologies in question to be “reasonably small” (in particular it allows us to exclude discrete topologies). Polishness (Definition 3.25) is a much stronger restriction, but it turns out to be somewhat common for nice semigroups to admit Polish topologies. Polishness is also very useful as it implies the previous two conditions and gives us access to Theorem 4.27.

In this part we also consider the notion of a (topological) clone (for examples of work on topological clones in the literature see [1, 5, 29]). In the same sense that groups are a generalisation of permutation groups and semigroups are a generalisation of transformation semigroups, clones are a generalisation of operation clones (where here an “operation” is a finitary operation on a set in the sense of Definition 4.2).

For example the set of real polynomials in finitely many variables forms a clone, or more generally the set of finite arity term operations for any algebraic structure (Definition 4.30). Also if X is any object in a category with finite products (for example a graph, a semigroup, or a topological space) then the set of polymorphisms of X (morphisms between finite powers of X) form a clone.

The elements of a clone are “combined” in the same manner as term operations (see Definition 4.30). Clones give us another reasonable way of comparing algebraic structures, as if we have two algebraic structures such that every operation of one is a term operation of the other, it is reasonable to think

of them as equivalent although this does not imply that they are isomorphic. Although term operations are defined in such a way that an operation can only output a single element, it is sometimes useful to think of an n -tuple of m -ary operations $(t_0, t_1, \dots, t_{n-1})$ on a set X as the single object $\langle t_0, t_1, \dots, t_{n-1} \rangle_{X^n}$ (recall Definition 2.3). In particular, this reduces the way operations are combined to the usual composition of maps.

We describe the results of the topological constructions when applied to specific examples of semigroups and use this to show various uniqueness/non-uniqueness results about the possible topologies on the semigroups and clones. Specifically, we consider the following semigroups and clones:

- (1) The binary relation monoid $\mathcal{B}_{\mathbb{N}}$ (see Example 4.12 and Theorem 9.4).
- (2) The full transformation monoid $\mathbb{N}^{\mathbb{N}}$ (see Example 4.13 and Theorem 9.10).
- (3) The partial function monoid $\mathcal{P}_{\mathbb{N}}$ (see Example 9.5 and Theorem 9.10).
- (4) The symmetric inverse monoid $\mathcal{I}_{\mathbb{N}}$ (see Example 9.12 and Theorem 9.18).
- (5) The injective function monoid $\text{Inj}(\mathbb{N})$ (see Example 9.20, Theorem 9.21 and Theorem 9.23).
- (6) The continuous function monoid of the Hilbert cube $C([0, 1]^{\mathbb{N}})$ (See Proposition 5.12 and Theorem 9.29).
- (7) The continuous function monoid of the Cantor space $C(2^{\mathbb{N}})$ (See Proposition 5.12 and Theorem 9.35).
- (8) The endomorphism monoid of the countably infinite atomless boolean algebra B_{∞} (see Theorem 9.42).
- (9) The clone of all maps between finite powers of the set \mathbb{N} (see Corollary 9.11).
- (10) The polymorphism clone of the Hilbert cube $[0, 1]^{\mathbb{N}}$ (see Corollary 9.30).
- (11) The polymorphism clone of the Cantor set $2^{\mathbb{N}}$ (see Corollary 9.36).
- (12) The polymorphism clone of the countably infinite atomless boolean algebra B_{∞} (see Corollary 9.43).

Another related topic of interest which has received much attention in the literature (particularly for groups) is that of “automatic continuity” (see [24, 28, 33, 36] for example).

As seen earlier in Theorem 5.39, it is sometimes possible to conclude that a homomorphism between semigroups is continuous with little information about the spaces in question. In particular when we know that a large class of maps from a topological space are continuous, this allows us to use the topology on the spaces as an “upper bound” of sorts. In the case of Polish groups, such a bound can often allow us to immediately identify a unique Polish topology compatible with the group by using Theorem 4.27.

As we see throughout this part, while the same approach (see Theorem 9.18 for example) does not apply to arbitrary semigroups, such continuity results still give us a great deal of information about the potential topologies compatible with the semigroups.

6. NATURAL TOPOLOGIES FOR ARBITRARY SEMIGROUPS

In this section we introduce various natural ways of defining a topology on an arbitrary semigroup.

Every semigroup has a corresponding “dual semigroup” (Definition 6.2). This semigroup may not be isomorphic to the original semigroup (notably it is in the case of an inverse semigroup) although for our purposes, these objects are essentially the same.

Definition 6.1 (Anti-homomorphisms, Assumed Knowledge: 2.3, 4.5).

Suppose that S, T are semigroups and $\phi : S \rightarrow T$ is a function, then we say that ϕ is an *anti-homomorphism* if $*^S \circ \phi = \langle \pi_1 \phi, \pi_0 \phi \rangle_{S^2} \circ *^T$ (recall Definition 4.5). That is for all $a, b \in S$ we have $(ab)\phi = (b)\phi(a)\phi$. If ϕ is also a bijection, then we say that ϕ is a *anti-isomorphism* and that S and T are *anti-isomorphic*.

Definition 6.2 (Dual semigroups, Assumed Knowledge: 4.5, 6.1).

If S is a semigroup then we define the *dual semigroup* S^\dagger of S to be the semigroup with the same universe and such that for all $s, t \in S$ we have $(s, t)*^{S^\dagger} = (t, s)*^S$. By definition S^\dagger is anti-isomorphic to S via the identity map.

It will be useful going forward to be able to make semigroup topologies from other semigroup topologies without needing to reprove continuity, we give a few such ways here.

Proposition 6.3 (Pulling back topologies, Assumed Knowledge: 1.6, 2.3, 4.10, 4.17, 6.1, 6.2).

Suppose that S, T are algebraic structures and $\phi : S \rightarrow T$ is a homomorphism or anti-homomorphism and \mathcal{T} is a topology on the set T .

- (1) If (T, \mathcal{T}) is a semitopological semigroup, then $(S, (\mathcal{T})\phi^{-1})$ is a semitopological semigroup.
- (2) If (T, \mathcal{T}) is a topological algebraic structure, then $(S, (\mathcal{T})\phi^{-1})$ is a topological structure.

Proof. We first consider the case that ϕ is an anti-homomorphism of semigroups. Note that the map $\phi : (S, (\mathcal{T})\phi^{-1}) \rightarrow (T, \mathcal{T})$ is continuous by definition. If $s \in S$ is arbitrary, then we have

$$\phi^{-1}\lambda_s^{-1} = (\lambda_s\phi)^{-1} = (\phi\rho_{(s)\phi})^{-1} \quad \text{and} \quad \phi^{-1}\lambda_s^{-1} = (\rho_s\phi)^{-1} = (\phi\lambda_{(s)\phi})^{-1}$$

(here we are using composition of binary relations). In particular if T is a semitopological semigroup and U is an open subset of T , then

$$((U)\phi^{-1})\lambda_s^{-1} = (U)(\phi\rho_{(s)\phi})^{-1} \quad \text{and} \quad ((U)\phi^{-1})\rho_s^{-1} = (U)(\phi\lambda_{(s)\phi})^{-1}$$

are open sets, so S is also a semitopological semigroup as well. Similarly if T is a topological semigroup then

$$((U)\phi^{-1})(*^S)^{-1} = (U)(\langle \pi_1 \phi, \pi_0 \phi \rangle_{S^2} \circ *^T)^{-1}$$

is open and thus S is also a topological semigroup.

We now consider the case that ϕ is a homomorphism of semigroups. As the identity function is an anti-isomorphism from T to T^\dagger , it follows that (T^\dagger, \mathcal{T}) is a (semi)topological semigroup if and only if (T, \mathcal{T}) is as well. If $\phi : S \rightarrow T$ is a homomorphism, then ϕ is an anti-homomorphism from S to (T^\dagger, \mathcal{T}) and so the result follows from the first case.

If ϕ is a homomorphism of arbitrary algebraic structures, then for any open subset U of T and any n -ary function symbol F from the signature of S we have

$$((U)\phi^{-1})(F^S)^{-1} = (U)(\langle \pi_0 \phi, \pi_1 \phi, \dots, \pi_{n-1} \phi \rangle_{S^m} \circ F^T)^{-1}.$$

As before, it follows that $F^S : S^m \rightarrow S$ is continuous as required. \square

Lemma 6.4 (Combining topologies, Assumed Knowledge: 4.10).

If \mathbb{S} is an algebraic structure, I is a set, and $\left\{ \mathcal{T}_i \mid i \in I \right\}$ are topologies compatible with \mathbb{S} , then the topology generated by $\bigcup_{i \in I} \mathcal{T}_i$ is compatible with \mathbb{S} .

Proof. Let \mathcal{T} be the topology generated by $\bigcup_{i \in I} \mathcal{T}_i$. Let $U \in \bigcup_{i \in I} \mathcal{T}_i$ be arbitrary and let $f : \mathbb{S}^n \rightarrow \mathbb{S}$ be an arbitrary operation of \mathbb{S} . It suffices to show that $(U)f^{-1}$ is open with respect to \mathcal{T} .

Let $i \in I$ be such that $U \in \mathcal{T}_i$. As \mathcal{T}_i is compatible with \mathbb{S} , it follows that $(U)f^{-1}$ is an open subset of \mathbb{S}^n using the product topology obtained from using the topology \mathcal{T}_i on \mathbb{S} . As this topology on \mathbb{S}^n is contained in the topology that we are using, the result follows. \square

Lemma 6.5 (Intersecting somewhat nice topologies, Assumed Knowledge: 3.15, 4.17).

Suppose that S is a semigroup, I is a set and $\{\mathcal{T}_i \mid i \in I\}$ is a collection of Fréchet topologies semicompatible with S . In this case $\mathcal{T} := \bigcap_{i \in I} \mathcal{T}_i$ is also Fréchet and semicompatible with S .

Proof. Let $s \in S$ and $U \in \mathcal{T}$ be arbitrary. To show that (S, \mathcal{T}) is a semitopological semigroup, it suffices to show that $(U)\lambda_s^{-1}$ and $(U)\rho_s^{-1}$ are open with respect to \mathcal{T} . As U is open in all of the topologies \mathcal{T}_i and these topologies are semicompatible with S , it follows these two sets are also open in all of the topologies \mathcal{T}_i . Thus they are open in \mathcal{T} by definition.

If $s \in S$ is arbitrary, then $S \setminus \{s\}$ is an element of all Fréchet topologies on S . As \mathcal{T} is an intersection of Fréchet topologies, it follows that $S \setminus \{s\} \in \mathcal{T}$ as well and thus \mathcal{T} is Fréchet. \square

The first of the topologies we introduce (the minimum Fréchet topology) is very useful due to its simple definition, however when considered on a group it is less interesting as it always results in the cofinite topology.

Definition 6.6 (Minimum topologies, Assumed Knowledge: 1.10, 3.15, 4.17).

If S is an (inverse) semigroup, then we define the *minimum Fréchet topology* of S by

$$\mathcal{MF}(S) := \bigcap \left\{ \mathcal{T} \mid (S, \mathcal{T}) \text{ is a Fréchet semitopological (inverse) semigroup} \right\}$$

(by Lemma 6.5 this is in the minimum, with respect to containment of topologies, Fréchet topology semicompatible with S).

Definition 6.7 (Markov topologies, Assumed Knowledge: 3.15, 4.10).

If \mathbb{S} is an algebraic structure, then we define the *Fréchet-Markov topology* on \mathbb{S} by

$$\mathcal{FM}(\mathbb{S}) := \bigcap \left\{ \mathcal{T} \mid (\mathbb{S}, \mathcal{T}) \text{ is a Fréchet topological structure} \right\},$$

and the *Hausdorff-Markov topology* on \mathbb{S} by

$$\mathcal{HM}(\mathbb{S}) := \bigcap \left\{ \mathcal{T} \mid (\mathbb{S}, \mathcal{T}) \text{ is a Hausdorff topological structure} \right\}.$$

In contrast to the minimum Fréchet topology being semi-compatible, the Markov topologies on a structure need not be compatible with the structure.

Proposition 6.8 (Small topologies are nice, Assumed Knowledge: 6.3, 6.5, 6.6, 6.7).

If S is a semigroup, then $\mathcal{MF}(S)$, $\mathcal{FM}(S)$ and $\mathcal{HM}(S)$ are Fréchet and semicompatible with S . Moreover if ϕ is an automorphism or anti-automorphism of S as a discrete semigroup, then ϕ is continuous with respect to each of $\mathcal{MF}(S)$, $\mathcal{FM}(S)$ and $\mathcal{HM}(S)$.

Proof. The first statement follows from Lemma 6.5. Let ϕ be an arbitrary automorphism or anti-automorphism of S . Let $\mathcal{T}_0, \mathcal{T}_1$ and \mathcal{T}_2 be $\mathcal{MF}(S)$, $\mathcal{FM}(S)$ and $\mathcal{HM}(S)$ respectively. Similarly let P_0, P_1 and P_2 be the properties of being a Fréchet semitopological semigroup, Fréchet topological semigroup and Hausdorff topological semigroup respectively.

As ϕ is a bijection it follows that if \mathcal{T} is a Fréchet/Hausdorff topology on the set S , then so is $(\mathcal{T})\phi^{-1}$. So for each $i \in \{0, 1, 2\}$, it follows from Proposition 6.3 that

$$\begin{aligned} (\mathcal{T}_i)\phi &= \left(\bigcap \left\{ \mathcal{T} \mid (S, \mathcal{T}) \text{ satisfies } P_i \right\} \right) \phi = \bigcap \left\{ (\mathcal{T})\phi \mid (S, \mathcal{T}) \text{ satisfies } P_i \right\} \\ &= \bigcap \left\{ \mathcal{T} \mid (S, (\mathcal{T})\phi^{-1}) \text{ satisfies } P_i \right\} = \bigcap \left\{ \mathcal{T} \mid (S, \mathcal{T}) \text{ satisfies } P_i \right\} = \mathcal{T}_i. \end{aligned}$$

So ϕ is continuous as required. \square

Definition 6.9 (Zariski topology, Assumed Knowledge: 3.7, 4.10, 4.30).

Let \mathbb{S} be an algebraic σ -structure. We say that a subset $E \subseteq \mathbb{S}$ is *elementary algebraic* if there are $n, m \in \mathbb{N} \setminus \{0\}$, $s_1, s_2, \dots, s_{n-1}, t_1, t_2, \dots, t_{m-1} \in \mathbb{S}$, terms a, b of σ over $\{0, 1, \dots, n-1\}$ and $\{0, 1, \dots, m-1\}$ respectively such that

$$E = \left\{ s \in \mathbb{S} \mid (s, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^{\mathbb{S}} = (s, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^{\mathbb{S}} \right\}$$

(recall Definition 4.30). We then define the *Zariski topology* $\mathcal{Z}(\mathbb{S})$ of \mathbb{S} to be the topology on \mathbb{S} generated by the sets of the form $\mathbb{S} \setminus E$ where E is elementary algebraic.

For semigroups in particular, there is a rather limited scope for term operations, in fact every term operation is of the form

$$(s_0, s_1, s_2, \dots, s_{n-1}) \rightarrow s_{i_0} s_{i_1} \dots s_{i_{k-1}}$$

for some $n, k \in \mathbb{N}$ and $i_0, i_1, \dots, i_{k-1} < n$. Thus the elementary algebraic sets for the Zariski topology can be thought of as the “solution sets” to pairs of “polynomials”.

Proposition 6.10 (Zariski topologies are nice, Assumed Knowledge: 4.4, 6.1, 6.2, 6.9).

If S is a semigroup, then $\mathcal{Z}(S)$ is a Fréchet topology semicompatible with S . Moreover if $\phi : S \rightarrow T$ is an isomorphism or anti-isomorphism between S and T as discrete semigroups, then ϕ is continuous with respect to $\mathcal{Z}(S)$ and $\mathcal{Z}(T)$.

Proof. Let a, b be arbitrary terms of σ_S over the variable sets $\{0, 1, \dots, n-1\}$ and $\{0, 1, \dots, m-1\}$ respectively (recall Definition 4.4). Moreover let $u, s_1, s_2, \dots, s_{n-1}, t_1, t_2, \dots, t_{m-1} \in S$ be arbitrary. To conclude that $\mathcal{Z}(S)$ is semicompatible with S it suffices to show that the sets

$$\begin{aligned} &\left(\left\{ s \in S \mid (s, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^S = (s, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^S \right\} \right) \lambda_u^{-1}, \\ &\left(\left\{ s \in S \mid (s, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^S = (s, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^S \right\} \right) \rho_u^{-1} \end{aligned}$$

are closed in $\mathcal{Z}(S)$. We show that the first of these sets is closed, the second follows by a symmetric argument. We define term operations $a' : S^{n+1} \rightarrow S$ and $b' : S^{m+1} \rightarrow S$ by

$$\begin{aligned} (x_0, x_1, \dots, x_n)a' &= ((x_0, x_n)*^S, x_1, \dots, x_{n-1})a_{\{0,1,\dots,n-1\}}^S, \\ (x_0, x_1, \dots, x_m)b' &= ((x_0, x_m)*^S, x_1, \dots, x_{m-1})b_{\{0,1,\dots,m-1\}}^S. \end{aligned}$$

We now have

$$\begin{aligned}
& \left(\left\{ s \in S \mid (s, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^S = (s, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^S \right\} \right) \lambda_u^{-1} \\
&= \left\{ s \in S \mid ((s)\lambda_u, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^S = ((s)\lambda_u, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^S \right\} \\
&= \left\{ s \in S \mid ((u, s)*^S, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^S = ((u, s)*^S, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^S \right\} \\
&= \left\{ s \in S \mid (s, s_1, \dots, s_{n-1}, u)a' = (s, t_1, \dots, t_{m-1}, u)b' \right\}.
\end{aligned}$$

So the required set is elementary algebraic and hence closed. To see that \mathcal{Z} is Fréchet, note that the projection maps $\pi_0, \pi_1 : S^2 \rightarrow S$ are term operations, thus for all $x \in S$ the set $\{x\} = \left\{ s \in S \mid (s, x)\pi_0 = (s, x)\pi_1 \right\}$ is elementary algebraic and hence closed in $\mathcal{Z}(S)$.

If $\phi : S \rightarrow T$ is an isomorphism, then

$$\begin{aligned}
& \left(\left\{ s \in S \mid (s, s_1, \dots, s_{n-1})a_{\{0,1,\dots,n-1\}}^S = (s, t_1, \dots, t_{m-1})b_{\{0,1,\dots,m-1\}}^S \right\} \right) \phi \\
&= \left\{ t \in T \mid (t, (s_1)\phi, \dots, (s_{n-1})\phi)a_{\{0,1,\dots,n-1\}}^T = (t, (t_1)\phi, \dots, (t_{m-1})\phi)b_{\{0,1,\dots,m-1\}}^T \right\}.
\end{aligned}$$

In particular ϕ and ϕ^{-1} map elementary algebraic sets to elementary algebraic sets, so ϕ is a homeomorphism of the Zariski topologies.

If $\phi : S \rightarrow T$ is an anti-isomorphism, then ϕ is an isomorphism from S^\dagger to T . So $\phi : (S, \mathcal{Z}(S^\dagger)) \rightarrow (T, \mathcal{Z}(T))$ is a homeomorphism. As the operations of S and S^\dagger are term operations of each other, it follows that they have the same term operations and thus $\mathcal{Z}(S) = \mathcal{Z}(S^\dagger)$. Thus ϕ is continuous with respect to $\mathcal{Z}(S)$ and $\mathcal{Z}(T)$ as required. \square

The “second countable continuity topology” defined in the following definition is primarily of interest due to Proposition 6.12. In this part we will see many examples of semigroups for which this topology is second countable and is thus maximum among the second countable topologies compatible with the semigroup.

Definition 6.11 (Second countable continuity topology, Assumed Knowledge: 3.9, 4.10, 6.4).

If \mathbb{S} is an algebraic structure, then we define the *second countable continuity topology* $SCC(\mathbb{S})$ of \mathbb{S} to be the topology generated by the set

$$\bigcup \left\{ \mathcal{T} \mid (\mathbb{S}, \mathcal{T}) \text{ is a second countable topological structure} \right\}$$

(by Lemma 6.4, this topology is compatible with \mathbb{S}).

Proposition 6.12 (Second countable continuity topologies are nice, Assumed Knowledge: 4.9, 6.2, 6.11).

Suppose that (S, \mathcal{T}) is a topological semigroup. The following are equivalent:

- (1) $SCC(S) \subseteq \mathcal{T}$.
- (2) *If $\phi : (S, \mathcal{T}) \rightarrow T$ is a homomorphism, where T is a second countable topological semigroup, then ϕ is continuous.*
- (3) $SCC(S^\dagger) \subseteq \mathcal{T}$.
- (4) *If $\phi : (S, \mathcal{T}) \rightarrow T$ is an anti-homomorphism, where T is a second countable topological semigroup, then ϕ is continuous.*

Suppose further that there is an inverse semigroup I such that S is the semigroup obtained by removing the unary operation of I , and that (I, \mathcal{T}) is a topological inverse semigroup. In this case $\mathcal{SCC}(S) = \mathcal{SCC}(I)$ and the following condition is equivalent to the above conditions:

- (5) If $\phi : (I, \mathcal{T}) \rightarrow T$ is a homomorphism, where T is a second countable topological inverse semigroup, then ϕ is continuous.

Proof. (1 \Rightarrow 2) : Let $\phi : (S, \mathcal{T}) \rightarrow T$ be a homomorphism, where T is a second countable topological semigroup. Let \mathcal{U} be the topology on T . By Proposition 6.3, the topology $(\mathcal{U})\phi^{-1}$ is compatible with S . If B is a countable basis for \mathcal{U} , then $(B)\phi^{-1}$ is a countable basis for $(\mathcal{U})\phi^{-1}$, so $(\mathcal{U})\phi^{-1}$ is second countable. By the definition of $\mathcal{SCC}(S)$, it follows that $(\mathcal{U})\phi^{-1} \subseteq \mathcal{SCC}(S)$, and hence ϕ is continuous.

(2 \Rightarrow 1) : Let \mathcal{U} be an arbitrary second countable topology compatible with S . The identity map $\text{id}_S : (S, \mathcal{T}) \rightarrow (S, \mathcal{U})$ is a homomorphism, so by assumption it is continuous. Thus $\mathcal{U} \subseteq \mathcal{T}$. As \mathcal{U} was arbitrary and such topologies generate $\mathcal{SCC}(S)$, it follows that $\mathcal{SCC}(S) \subseteq \mathcal{T}$.

(1 \iff 3) : The topology $\mathcal{SCC}(S)$ is generated by the set

$$\bigcup \left\{ \mathcal{U} \mid (S, \mathcal{U}) \text{ is a second countable topological semigroup} \right\}$$

and the topology $\mathcal{SCC}(S^\dagger)$ is generated by the set

$$\bigcup \left\{ \mathcal{U} \mid (S^\dagger, \mathcal{U}) \text{ is a second countable topological semigroup} \right\}.$$

By Proposition 6.3 (using the anti-isomorphism id_S) it follows that these sets are the same and hence $\mathcal{SCC}(S^\dagger) = \mathcal{SCC}(S)$.

(3 \iff 4) : Note that $\phi : S \rightarrow T$ is an anti-homomorphism if and only if $\phi : S^\dagger \rightarrow T$ is a homomorphism. Thus by replacing S with S^\dagger , it follows that (1 \iff 2) \iff (3 \iff 4). As we have already shown (1 \iff 2), the result follows.

($\mathcal{SCC}(S) = \mathcal{SCC}(I)$) : Note that the map $x \rightarrow x^{-1}$ is an anti-isomorphism from I to itself. In particular if \mathcal{T} is a topology compatible with S , then \mathcal{T}^{-1} is also a topology compatible with S (by Proposition 6.3). Thus by Lemma 6.4, the topology generated by $\mathcal{T} \cup \mathcal{T}^{-1}$ is also compatible with S . As the set $\mathcal{T} \cup \mathcal{T}^{-1}$ is inverse closed, it follows that the topology generated by this set is compatible with I as well.

We have shown that if \mathcal{T} is a topology compatible with S , then the topology generated by $\mathcal{T} \cup \mathcal{T}^{-1}$ is compatible with I . If B is a countable basis for \mathcal{B} , then $\left\{ \bigcap F \mid \text{finite } F \subseteq B \cup B^{-1} \right\}$ is a countable basis for the topology generated by $\mathcal{T} \cup \mathcal{T}^{-1}$. Thus every second countable topology compatible with S is contained in a second countable topology compatible with I . The result then follows from the definitions of $\mathcal{SCC}(I)$ and $\mathcal{SCC}(S)$.

(1 \iff 5) : This proof is identical to the proof of (1 \Rightarrow 2) together with the proof of (2 \Rightarrow 1) if we replace S with I . □

Corollary 6.13 ($\text{Sym}(\mathbb{N})$ continuity, Assumed Knowledge: 5.39, 6.12).

The topology $\mathcal{SCC}(\text{Sym}(\mathbb{N}))$ is the pointwise topology on $\text{Sym}(\mathbb{N})$.

Proof. From Proposition 6.12 together with Theorem 5.39, we have that $\mathcal{SCC}(\text{Sym}(\mathbb{N}))$ is contained in the pointwise topology. As every second countable topology compatible with $\text{Sym}(\mathbb{N})$ is contained in $\mathcal{SCC}(\text{Sym}(\mathbb{N}))$, we also have the reverse inclusion. □

Lemma 6.14 (Hausdorff Zariski Comparison, Assumed Knowledge: 2.3, 3.14, 6.9).

If \mathbb{S} is an algebraic σ -structure and \mathcal{T} is a Hausdorff topology compatible with \mathbb{S} , then $\mathcal{Z}(\mathbb{S}) \subseteq \mathcal{T}$.

Proof. Let a, b be arbitrary terms of σ over $\{0, 1, \dots, n-1\}$ and $\{0, 1, \dots, m-1\}$ respectively. Suppose that $s_1, s_2, \dots, s_{n-1}, t_1, t_2, \dots, t_{m-1} \in \mathbb{S}$ are arbitrary and $c_{s,1}, c_{s,2}, \dots, c_{s,n-1}, c_{t,1}, c_{t,2}, \dots, c_{t,m-1} : \mathbb{S} \rightarrow \mathbb{S}$ are the constant maps with these values. We define continuous maps $a' : \mathbb{S} \rightarrow \mathbb{S}$ and $b' : \mathbb{S} \rightarrow \mathbb{S}$ as follows (recall Definition 2.3)

$$a' := \langle \text{id}_{\mathbb{S}}, c_{s,1}, \dots, c_{s,n-1} \rangle_{\mathbb{S}^n} \circ a_{\{0,1,\dots,n-1\}}^{\mathbb{S}},$$

$$b' := \langle \text{id}_{\mathbb{S}}, c_{t,1}, \dots, c_{t,m-1} \rangle_{\mathbb{S}^m} \circ b_{\{0,1,\dots,m-1\}}^{\mathbb{S}}.$$

Note that by construction the maps a', b' are continuous with respect to \mathcal{T} and

$$\begin{aligned} E &:= \left\{ s \in \mathbb{S} \mid (s, s_1, \dots, s_{n-1}) a_{\{0,1,\dots,n-1\}}^{\mathbb{S}} = (s, t_1, \dots, t_{m-1}) b_{\{0,1,\dots,m-1\}}^{\mathbb{S}} \right\} \\ &= \left\{ s \in \mathbb{S} \mid (s) a' = (s) b' \right\} = \left\{ s \in \mathbb{S} \mid (s) \langle a', b' \rangle_{\mathbb{S}^2} \in \text{id}_{\mathbb{S}} \right\} = (\text{id}_{\mathbb{S}}) \langle a', b' \rangle_{\mathbb{S}^2}^{-1} \end{aligned}$$

is an arbitrary elementary algebraic subset of \mathbb{S} . As \mathcal{T} is Hausdorff, it follows that $\text{id}_{\mathbb{S}}$ is a closed subset of \mathbb{S}^2 and hence E is closed as required. \square

Proposition 6.15 (Comparing topologies, Assumed Knowledge: 6.6, 6.7, 6.8, 6.10, 6.11, 6.14).

If S is a semigroup, then we have the following containments:

$$\begin{array}{ccc} & \mathcal{Z}(S) & \\ \mathcal{MF}(S) & \subsetneq & \mathcal{HM}(S) \\ & \mathcal{FM}(S) & \end{array}$$

Moreover if S is compatible with some second countable Hausdorff topology \mathcal{T} , then $\mathcal{HM}(S) \subseteq \mathcal{T} \subseteq \mathcal{SCC}(S)$.

Proof. ($\mathcal{MF}(S) \subseteq \mathcal{Z}(S)$ and $\mathcal{MF}(S) \subseteq \mathcal{FM}(S)$): By Propositions 6.8 and 6.10, both of the topologies $\mathcal{Z}(S)$ and $\mathcal{FM}(S)$ are Fréchet and semicompatible with S . The containments then follow from the definition of $\mathcal{MF}(S)$.

($\mathcal{Z}(S) \subseteq \mathcal{HM}(S)$): By the definition of $\mathcal{HM}(S)$, it suffices to show that $\mathcal{Z}(S)$ is contained in every Hausdorff topology compatible with S . This is immediate from Lemma 6.14.

($\mathcal{FM}(S) \subseteq \mathcal{HM}(S)$): As all Hausdorff spaces are Fréchet, the topologies in the intersection defining $\mathcal{HM}(S)$ are included in the intersection defining the topology $\mathcal{FM}(S)$.

($\mathcal{HM}(S) \subseteq \mathcal{T} \subseteq \mathcal{SCC}(S)$): This is immediate from the definitions of $\mathcal{HM}(S)$ and $\mathcal{SCC}(S)$. \square

7. THE SMALL INDEX PROPERTY AND PROPERTY **W**

We next introduce the “right/left small index properties” for topological semigroups and explain how they interact with the second countable continuity topology mentioned earlier (see Corollary 7.4). These notions generalise the notation of the “small index property” for topological groups which says that all countable index subgroups of a group are open. This property has been studied a lot in the literature (see for example [10, 13, 17, 18, 32]). It is routine to verify that each of the right and left small index properties coincide with the usual small index property when applied to a group.

Definition 7.1 (One-sided congruences, Assumed Knowledge: 1.10, 4.5).

Let S be a semigroup. We say that an equivalence relation \sim on S is a *right congruence* if for all $(s, t) \in \sim$ and $u \in S$ we have $(su, tu) \in \sim$. Similarly we say that \sim is a *left congruence* if for all $(s, t) \in \sim$ and $u \in S$ we have $(us, ut) \in \sim$.

Definition 7.2 (Small index properties, Assumed Knowledge: 4.10, 7.1).

If S is a semitopological semigroup, then we say that S has the *right small index property* if all countable index right congruences on S are open subsets of S^2 (recall Definition 1.10). We define the *left small index property* analogously.

Proposition 7.3 (Small index condition equivalence, Assumed Knowledge: 5.2, 6.2, 6.3, 4.13, 7.2).

If (S, \mathcal{T}) is a semitopological semigroup, then the following are equivalent:

- (1) All homomorphisms $\psi : (S, \mathcal{T}) \rightarrow (\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$ are continuous,
- (2) (S, \mathcal{T}) has the right small index property.

Similarly the following are equivalent (to each other not to the first two points):

- (3) All anti-homomorphisms $\psi : (S, \mathcal{T}) \rightarrow (\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$ are continuous,
- (4) (S, \mathcal{T}) has the left small index property.

Proof. (1 \Rightarrow 2) : Let \sim be an arbitrary countable index right congruence on S . We assume without loss of generality that $(S/\sim) \cap \mathbb{N} = \emptyset$. Let $\Phi : (S/\sim) \cup \mathbb{N} \rightarrow \mathbb{N}$ be a bijection. We define an action a of S on the set \mathbb{N} by

$$(p, s)a = \begin{cases} ([ts]_{\sim})\Phi & \text{if } p = ([t]_{\sim})\Phi \\ ([s]_{\sim})\Phi & \text{if } p \in (\mathbb{N})\Phi \end{cases}.$$

Note that the map $\phi_a : (S, \mathcal{T}) \rightarrow (\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$ defined by $(i)((s)\phi_a) = (i, s)a$ is a semigroup homomorphism. By assumption, it follows that ϕ_a is continuous. Let $s \in S$ be arbitrary and consider the open subset

$$U_{(1)\Phi, ([s]_{\sim})\Phi} := \left\{ f \in \mathbb{N}^{\mathbb{N}} \mid ((1)\Phi)f = ([s]_{\sim})\Phi \right\}$$

of $\mathbb{N}^{\mathbb{N}}$. As ϕ_a is continuous, it follows that

$$\begin{aligned} (U_{(1)\Phi, ([s]_{\sim})\Phi})\phi_a^{-1} &= \left\{ t \in S \mid ((1)\Phi)((f)\phi_a) = ([s]_{\sim})\Phi \right\} \\ &= \left\{ t \in S \mid ((1)\Phi, s)a = ([s]_{\sim})\Phi \right\} = [s]_{\sim} \end{aligned}$$

is an open subset of S . As $\sim = \bigcup_{s \in S} ([s]_{\sim} \times [s]_{\sim})$, it follows that \sim is open as required.

(2 \Rightarrow 1) : Let $\psi : (S, \mathcal{T}) \rightarrow (\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$ be a homomorphism, let $a, b \in \mathbb{N}$ be arbitrary and let

$$U_{a,b} := \left\{ f \in \mathbb{N}^{\mathbb{N}} \mid (a, b) \in f \right\}.$$

It suffices to show that $(U_{a,b})\psi^{-1}$ is an open subset of S . We define an equivalence relation \sim_a on S by

$$\sim_a := \left\{ (s, t) \in S \mid (a)((s)\psi) = (a)((t)\psi) \right\}.$$

As ψ is a homomorphism, it follows that \sim_a is a right congruence. Note also that

$$S/\sim_a = \left\{ \left\{ s \in S \mid (a)((s)\psi) = b \right\} \mid b \in \mathbb{N} \right\}$$

and so \sim_a has countable index. Thus \sim_a is open and $(U_{a,b})\psi^{-1}$ is an equivalence class of \sim_a . If $s \in S$, then $[s]_{\sim_a} = ((S \times \{s\}) \cap \sim_a)\pi_0$. As $(S \times \{s\}) \cap \sim_a$ is an open subset of $S \times \{s\}$ and the projection

map $\pi_0 : S \times \{s\} \rightarrow S$ is a homeomorphism, it follows that all equivalence classes of \sim_a are open. The result follows.

(3 \iff 4) : The identity map $\text{id}_S : (S, \mathcal{T}) \rightarrow (S^\dagger, \mathcal{T})$ is a homeomorphism of spaces and an anti-isomorphism of semigroups. Moreover by Proposition 6.3, the pair (S^\dagger, \mathcal{T}) is a topological semigroup. By the definition of S^\dagger , it follows that \sim is a left congruence on S if and only if \sim is a right congruence on S^\dagger . Moreover $\psi : S \rightarrow \mathbb{N}^\mathbb{N}$ is an anti-homomorphism if and only if $\phi : S^\dagger \rightarrow \mathbb{N}^\mathbb{N}$ is a homomorphism.

Thus we need only show that (S^\dagger, \mathcal{T}) having the right small index property is equivalent to all homomorphisms $\psi : (S^\dagger, \mathcal{T}) \rightarrow (\mathbb{N}^\mathbb{N}, \mathcal{PT}_\mathbb{N})$ being continuous. This follows from the equivalence (1 \iff 2). \square

Corollary 7.4 (Automatic continuity vs the small index property, Assumed Knowledge: 6.12, 7.3). *If S is a semigroup, then $(S, \text{SCC}(S))$ has both the right and left small index properties.*

Proof. As the topological semigroup $(\mathbb{N}^\mathbb{N}, \mathcal{PT}_\mathbb{N})$ is second countable, this follows from Propositions 6.12 and 7.3. \square

We now introduce “property **W**” (our version of what is called “property **X**” in the joint paper [11]), which is a key tool that we use to lift automatic continuity type results from groups to semigroups in which they are “prominently” contained (see Lemma 7.6).

Definition 7.5 (Property **W**, Assumed Knowledge: 3.5, 4.10, 4.30).

Suppose that \mathbb{S} is a topological algebraic σ -structure and $A \subseteq \mathbb{S}$. We say that \mathbb{S} has *property **W** with respect to A* if:

- (1) For all $s \in \mathbb{S}$ there is $t_s \in A$, $t_{1,s}, t_{2,s}, \dots, t_{m-1,s} \in \mathbb{S}$ and a term a of σ over $\{0, 1, \dots, m-1\}$ (for some $m \in \mathbb{N} \setminus \{0\}$) such that for all $N \in \text{Nbhd}_A(t_s)$ we have

$$(t_s, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,m-1\}}^{\mathbb{S}} = s \quad \text{and}$$

$$(N, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,m-1\}}^{\mathbb{S}} \in \text{Nbhd}_S(s).$$

Every topological structure has property **W** with respect to itself but more interesting examples tend to be quite technical to verify. Later on we will be showing that many semigroups have property **W** with respect to their groups of units (see Theorems 9.1, 9.8, 9.9, 9.17, 9.22, 9.28, and 9.32).

Lemma 7.6 (Property **W** is nice, Assumed Knowledge: 3.2, 4.3, 7.5).

*Suppose that \mathbb{S}, \mathbb{T} are topological σ -structures and \mathbb{S} has property **W** with respect to $A \subseteq \mathbb{S}$. If $\phi : \mathbb{S} \rightarrow \mathbb{T}$ is a homomorphism and $\phi \upharpoonright_A : A \rightarrow \mathbb{T}$ is continuous then ϕ is continuous.*

Proof. Let $A, \mathbb{S}, \mathbb{T}, \phi : \mathbb{S} \rightarrow \mathbb{T}$ be as hypothesised. Let $U \subseteq \mathbb{T}$ be open. We need to show that $(U)\phi^{-1}$ is open. Let $s \in (U)\phi^{-1}$ be arbitrary, it suffices to show that $(U)\phi^{-1} \in \text{Nbhd}_S(s)$.

Let $m \in \mathbb{N} \setminus \{0\}$, $t_s \in A$, $t_{1,s}, t_{2,s}, \dots, t_{m-1,s} \in \mathbb{S}$, and a term $a \in (\{ _ , _ , _ \} \cup F_\sigma \cup \{0, 1, \dots, m-1\})^*$ be as in the definition of property **W**. As U is open, it follows that

$$V := \left\{ t \in \mathbb{T} \mid (t, (t_{1,s})\phi, \dots, (t_{m-1,s})\phi)a_{\{0,1,\dots,m-1\}}^{\mathbb{T}} \in U \right\}$$

is an open subset of \mathbb{T} . Moreover as $(t_s, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,m-1\}}^{\mathbb{S}} = s$ and ϕ is a homomorphism, it follows that $(t_s)\phi \in V$.

As $V \in \text{Nbhds}_{\mathbb{T}}((t_s)\phi)$ and $\phi \upharpoonright_A$ is continuous, it follows that $(V)\phi^{-1} \cap A \in \text{Nbhds}_A(t_s)$. Thus by property **W**, it follows that

$$((V)\phi^{-1} \cap A, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,m-1\}}^{\mathbb{S}} \in \text{Nbhds}_{\mathbb{S}}(s).$$

By the definition of V we have

$$((V)\phi^{-1} \cap A, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,m-1\}}^{\mathbb{S}} \subseteq (U)\phi^{-1}$$

so $(U)\phi^{-1} \in \text{Nbhds}_{\mathbb{S}}(s)$ as required. \square

Lemma 7.7 (Property **W** is transitive lemma, Assumed Knowledge: 7.5).

*Suppose that \mathbb{S} is a topological algebraic σ -structure, \mathbb{T} is a substructure of \mathbb{S} and $A \subseteq \mathbb{T}$. If \mathbb{S} has property **W** with respect to \mathbb{T} , and \mathbb{T} has property **W** with respect to A , then \mathbb{S} has property **W** with respect to A .*

Proof. Let $s \in \mathbb{S}$ be arbitrary. Let $t_s \in \mathbb{T}$, $t_{1,s}, \dots, t_{m-1,s} \in \mathbb{S}$, $m \in \mathbb{N} \setminus \{0\}$ and a term a of σ over $\{0, 1, \dots, m-1\}$ be as in the definition of property **W** with respect to \mathbb{T} . Using the fact that \mathbb{T} has property **W** with respect to A , let $t_{t_s} \in A$, $t_{1,t_s}, \dots, t_{m-1,t_s} \in \mathbb{S}$, $n \in \mathbb{N} \setminus \{0\}$ and a term b of σ over $\{0, 1, \dots, n-1\}$ be such that for all $N \in \text{Nbhds}_A(t_{t_s})$ we have

$$(t_{t_s}, t_{1,t_s}, t_{2,t_s}, \dots, t_{m-1,t_s})b_{\{0,1,\dots,n-1\}}^{\mathbb{T}} = s \quad \text{and}$$

$$(N, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})b_{\{0,1,\dots,n-1\}}^{\mathbb{T}} \in \text{Nbhds}_{\mathbb{T}}(s).$$

Let $c : \mathbb{S}^{n+m-1} \rightarrow \mathbb{S}$ be the term operation such that $(x_0, x_1, \dots, x_{n+m-2})c$ is defined to be

$$((x_0, x_1, x_2, \dots, x_{n-1})b_{\{0,1,\dots,n-1\}}^{\mathbb{S}}, x_n, x_{n+1}, \dots, x_{n+m-2})a_{\{0,1,\dots,n-1\}}^{\mathbb{S}}.$$

We now have

$$\begin{aligned} & (t_{t_s}, t_{1,t_s}, t_{2,t_s}, \dots, t_{n-1,t_s}, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})c \\ &= ((t_{t_s}, t_{1,t_s}, t_{2,t_s}, \dots, t_{n-1,t_s})b_{\{0,1,\dots,n-1\}}^{\mathbb{S}}, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,n-1\}}^{\mathbb{S}} \\ &= (t_s, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})a_{\{0,1,\dots,n-1\}}^{\mathbb{S}} = s. \end{aligned}$$

Similarly if $N \in \text{Nbhds}_A(t_{t_s})$, then

$$(N, t_{1,t_s}, t_{2,t_s}, \dots, t_{n-1,t_s}, t_{1,s}, t_{2,s}, \dots, t_{m-1,s})c \in \text{Nbhds}_{\mathbb{S}}(s)$$

and so \mathbb{S} has property **W** with respect to A as required. \square

8. TOPOLOGICAL CLONES

In this section we introduce two ways of viewing topological clones and explain why they are actually equivalent. We also prove a proposition (Proposition 8.6) which will be using later to transfer topological semigroup results to the corresponding clones.

Definition 8.1 (Abstract clones, Assumed Knowledge: 2.2, 2.3).

An *abstract clone* is a pair $(\mathcal{C}, \text{Pow}_{\mathcal{C}})$ where

- (1) \mathcal{C} is a small category (recall Definition 2.2).
- (2) $\text{Pow}_{\mathcal{C}}$ is a bijection from the object set of \mathcal{C} to \mathbb{N} (the *power map*).

(3) For each $n \in \mathbb{N}$, there exist morphisms $\pi_0, \pi_1, \dots, \pi_{n-1}$ of \mathcal{C} such that

$$((n) \text{Pow}_{\mathcal{C}}^{-1}, (\pi_0, \pi_1, \dots, \pi_{n-1}))$$

is a product of n copies of $(1) \text{Pow}_{\mathcal{C}}^{-1}$ (recall Definition 2.3).

For ease of notation, we often treat a clone as if it were equal to its underlying small category.

Note that in the previous definition, the choice of morphisms $\pi_0, \pi_1, \dots, \pi_{n-1}$ is usually not unique. However we will always be using the “obvious” choice when such a choice exists.

Definition 8.2 (Topological abstract clones, Assumed Knowledge: 3.3, 4.10, 8.1).

A *topological abstract clone* is a triple $(\mathcal{C}, \mathcal{T}_{\mathcal{C}, \mathcal{O}}, \mathcal{T}_{\mathcal{C}, \mathcal{M}})$, where

- (1) \mathcal{C} is an abstract clone.
- (2) $\mathcal{T}_{\mathcal{C}, \mathcal{O}}$ is a topology on the object set $\mathcal{O}_{\mathcal{C}}$ of \mathcal{C} .
- (3) $\mathcal{T}_{\mathcal{C}, \mathcal{M}}$ is a topology on the morphism set $\mathcal{O}_{\mathcal{M}}$ of \mathcal{C} .
- (4) The power map $\text{Pow}_{\mathcal{C}} : (\mathcal{O}_{\mathcal{C}}, \mathcal{T}_{\mathcal{C}, \mathcal{O}}) \rightarrow \mathbb{N}$ is continuous (or equivalently $\mathcal{T}_{\mathcal{C}, \mathcal{O}}$ is discrete).
- (5) The source and target maps $s_{\mathcal{C}}, t_{\mathcal{C}} : (\mathcal{M}_{\mathcal{C}}, \mathcal{T}_{\mathcal{C}, \mathcal{M}}) \rightarrow (\mathcal{O}_{\mathcal{C}}, \mathcal{T}_{\mathcal{C}, \mathcal{O}})$ are continuous,
- (6) The composition map $\circ_{\mathcal{C}} : \text{Comp}_{\mathcal{C}} \rightarrow \mathcal{M}_{\mathcal{C}}$ is continuous (where $\mathcal{M}_{\mathcal{C}}$ has the topology $\mathcal{T}_{\mathcal{C}, \mathcal{M}}$ and $\text{Comp}_{\mathcal{C}}$ is viewed as a subspace of the product space $\mathcal{M}_{\mathcal{C}} \times \mathcal{M}_{\mathcal{C}}$).
- (7) For each $i, j \in \mathbb{N}$, the bijection

$$\phi_{i,j} : \left(\left\{ f \in \mathcal{M}_{\mathcal{C}} \mid (f)_{s_{\mathcal{C}}} \text{Pow}_{\mathcal{C}} = i, (f)_{t_{\mathcal{C}}} \text{Pow}_{\mathcal{C}} = 1 \right\} \right)^j \rightarrow \left\{ f \in \mathcal{M}_{\mathcal{C}} \mid (f)_{s_{\mathcal{C}}} \text{Pow}_{\mathcal{C}} = i, (f)_{t_{\mathcal{C}}} \text{Pow}_{\mathcal{C}} = j \right\}$$

defined by $(f_0, f_1, \dots, f_{j-1})\phi_{i,j} = \langle f_0, f_1, \dots, f_{j-1} \rangle_{X^j}$ is continuous.

In this case we say that $\mathcal{C}, \mathcal{T}_{\mathcal{C}, \mathcal{O}}$ and $\mathcal{T}_{\mathcal{C}, \mathcal{M}}$ are compatible. If P is a topological property, then we will describe $(\mathcal{C}, \mathcal{T}_{\mathcal{C}, \mathcal{O}}, \mathcal{T}_{\mathcal{C}, \mathcal{M}})$ as having this property if both of $\mathcal{T}_{\mathcal{C}, \mathcal{O}}$ and $\mathcal{T}_{\mathcal{C}, \mathcal{M}}$ do.

Definition 8.3 (Operation clones, Assumed Knowledge: 1.4, 2.3).

An *operation clone* (over a set X) is a collection of functions from finite powers of X to X such that

- (1) For all $n \in \mathbb{N}$, the projection maps $\pi_0, \pi_1, \dots, \pi_{n-1} : X^n \rightarrow X$ are elements of \mathcal{C} .
- (2) If $f_0, f_1, \dots, f_{n-1} \in \mathcal{C}$ have the same domain and $g \in \mathcal{C}$ has domain X^n , then

$$\langle f_0, f_1, \dots, f_{n-1} \rangle_{X^n} \circ g \in \mathcal{C}.$$

If $f \in \mathcal{C}$, then we define the *arity* (f) of f to be the unique $n \in \mathbb{N}$ such that $\text{dom}(f) = X^n$. We also denote the set $\{f \in \mathcal{C} \mid f \text{ has arity } n\}$ by $\mathcal{C}^{(n)}$.

Definition 8.4 (Topological operation clones, Assumed Knowledge: 3.3, 4.10, 8.1).

A *topological operation clone* is a pair $(\mathcal{C}, \mathcal{T}_{\mathcal{C}})$, where:

- (1) \mathcal{C} is an operation clone.
- (2) $\mathcal{T}_{\mathcal{C}}$ is a topology on the set \mathcal{C} .
- (3) For all $m, n \in \mathbb{N}$, the map $\circ_{m,n} : (\mathcal{C}^{(m)})^n \times \mathcal{C}^{(n)} \rightarrow \mathcal{C}^{(m)}$ defined by

$$((f_0, f_1, \dots, f_{n-1}), g)_{\circ_{m,n}} = \langle f_0, f_1, \dots, f_{n-1} \rangle_{X^n} \circ g \in \mathcal{C}$$

is continuous.

Proposition 8.5 (Topological clones are topological clones, Assumed Knowledge: 2.4, 8.2, 8.4).

If $(\mathcal{C}, \mathcal{T}_{\mathcal{C}})$ is a topological operation clone over a non-empty set X , then the object $(\mathcal{C}^*, \mathcal{T}_{\mathcal{C}^*, \mathcal{O}}, \mathcal{T}_{\mathcal{C}^*, \mathcal{M}})$ defined as follows is a topological abstract clone.

- (1) Let \mathcal{C}^* be the subcategory of the category of sets and functions with object set $\{X^n \mid n \in \mathbb{N}\}$ and morphism set consisting of the maps

$$\langle f_0, f_1, \dots, f_{n-1} \rangle_{X^n}$$

where $n, m \in \mathbb{N}$ and $f_0, f_1, \dots, f_{n-1} \in \mathcal{C}^{(m)}$ are arbitrary.

- (2) Let $\mathcal{T}_{\mathcal{C}^*, \mathcal{O}}$ be the discrete topology on $\mathcal{O}_{\mathcal{C}^*}$.
(3) Let $\mathcal{T}_{\mathcal{C}^*, \mathcal{M}}$ be the topology on $\mathcal{M}_{\mathcal{C}^*}$ generated by the sets of the forms

$$A_{n,m} := \left\{ f \in \mathcal{M}_{\mathcal{C}^*} \mid (f)_{s_{\mathcal{C}^*}} \text{Pow}_{\mathcal{C}^*} = n, (f)_{t_{\mathcal{C}^*}} \text{Pow}_{\mathcal{C}^*} = m \right\},$$

$$V_{n,U} := \left\{ f \in \mathcal{M}_{\mathcal{C}^*} \mid f \circ \pi_n \in U \right\}$$

where $n, m \in \mathbb{N}$ and $U \in \mathcal{T}_{\mathcal{C}}$ are arbitrary.

Moreover, the original topological operation clone is then precisely the subspace of $\mathcal{M}_{\mathcal{C}^*}$ consisting of morphisms with target X , and (up to topological isomorphism) $(\mathcal{C}^*, \mathcal{T}_{\mathcal{C}^*, \mathcal{O}}, \mathcal{T}_{\mathcal{C}^*, \mathcal{M}})$ is the only topological abstract clone with this property.

Proof. We need to verify the conditions of Definition 8.2. The first five conditions are immediate from the definition. We next show Condition 7. For each $i, j \in \mathbb{N}$, let $\phi_{i,j} : (A_{i,1})^j \rightarrow A_{i,j}$ be the bijection defined by $(f_0, f_1, \dots, f_{j-1})\phi_{i,j} = \langle f_0, f_1, \dots, f_{j-1} \rangle_{X^j}$. We need to show that $\phi_{i,j}$ is continuous with respect to $\mathcal{T}_{\mathcal{C}^*, \mathcal{M}}$. Let $n \in \mathbb{N}$ and $U \in \mathcal{T}_{\mathcal{C}}$ be arbitrary. We have

$$\begin{aligned} (V_{n,U})\phi_{i,j}^{-1} &= \left(\left\{ f \in A_{i,j} \mid f \circ \pi_n \in U \right\} \right) \phi_{i,j}^{-1} \\ &= \left\{ f \in (A_{i,1})^j \mid ((f)\phi_{i,j}) \circ \pi_n \in U \right\} \\ &= \left\{ f \in (A_{i,1})^j \mid (f)\pi_n \in U \right\} \\ &= \left\{ f \in (A_{i,1})^j \mid (f)\pi_n \in V_{0,U} \cap A_{i,1} \right\} \\ &= (V_{0,U} \cap A_{i,1})\pi_n^{-1}. \end{aligned}$$

So indeed $\phi_{i,j}$ is continuous. Moreover, as $\phi_{i,j}$ is a bijection and the topology on $A_{i,j}^j$ is generated by the sets of the form $(V_{0,U} \cap A_{i,1})\pi_n^{-1} = (U)\pi_n^{-1}$ for $n \in \mathbb{N}$ and $U \in \mathcal{T}_{\mathcal{C}}$, it follows from the above equality that $\phi_{i,j}$ is a homeomorphism.

It remains to show Condition 6. As the sets $(A_{n,i} \times A_{i,m})_{i,n,m \in \mathbb{N}}$ are a partition of $\text{Comp}_{\mathcal{C}^*}$ into open sets, it suffices to show that for all $n, m, i \in \mathbb{N}$, the map $\circ_{\mathcal{C}^*} \upharpoonright_{A_{n,i} \times A_{i,m}}$ is continuous. Let $i, n, m \in \mathbb{N}$ be arbitrary and let $(f, g) \in A_{n,i} \times A_{i,m}$. We have

$$\begin{aligned} (f, g)\circ_{\mathcal{C}^*} &= f \circ g \\ &= \langle f \circ \pi_0, f \circ \pi_1, \dots, f \circ \pi_{n-1} \rangle_{X^n} \circ g \\ &= ((f \circ \pi_0, f \circ \pi_1, \dots, f \circ \pi_{n-1}), g)\circ_{m,n} \\ &= ((f)\phi_{m,n}^{-1}, g)\circ_{m,n}. \end{aligned}$$

As $\phi_{m,n}^{-1}$ and $\circ_{m,n}$ are continuous, it follows that $\circ_{\mathcal{C}^*}$ is also continuous as required. \square

Proposition 8.6 (Corresponding clone and semigroup topologies, Assumed Knowledge: 3.12, 4.5, 4.10, 4.18, 6.14, 8.2).

Suppose that \mathcal{C} is an abstract clone, and M is the endomorphism monoid of the object $(1)\text{Pow}_{\mathcal{C}}^{-1}$ in the category \mathcal{C} . Suppose further that the objects $(1)\text{Pow}_{\mathcal{C}}^{-1}$ and $(2)\text{Pow}_{\mathcal{C}}^{-1}$ are isomorphic in the category \mathcal{C} .

If \mathcal{S} is a topology compatible with the semigroup M , then there is at most one pair of topologies $(\mathcal{T}_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}})$ compatible with \mathcal{C} such that $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_M = \mathcal{S}$ (recall Definition 3.2). Moreover the correspondence $\mathcal{S} \leftrightarrow (\mathcal{T}_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}})$ preserves containments of topologies as well as the properties of being Hausdorff, second countable or Polish.

Proof. For all possible $n \in \mathbb{N} \setminus \{0\}$, choose $\psi_{1,n} : (1)\text{Pow}_{\mathcal{C}}^{-1} \rightarrow (n)\text{Pow}_{\mathcal{C}}^{-1}$ to be an isomorphism (by assumption the map $\psi_{1,2}$ is defined). For all $n \in \mathbb{N} \setminus \{0\}$ the morphism

$$\langle \pi_0 \circ_{\mathcal{C}} \psi_{1,2} \circ_{\mathcal{C}} \pi_0, \pi_0 \circ_{\mathcal{C}} \psi_{1,2} \circ_{\mathcal{C}} \pi_1, \pi_1, \pi_2, \dots, \pi_{n-1} \rangle_{(n+1)\text{Pow}_{\mathcal{C}}^{-1}}$$

is an isomorphism from $(n)\text{Pow}_{\mathcal{C}}^{-1}$ to $(n+1)\text{Pow}_{\mathcal{C}}^{-1}$. So the morphisms $\psi_{1,n}$ are defined for all $n \in \mathbb{N} \setminus \{0\}$. For all $n \in \mathbb{N}$, let ι_n be the unique morphism $\langle \rangle_{(0)\text{Pow}_{\mathcal{C}}^{-1}}$ from $(n)\text{Pow}_{\mathcal{C}}^{-1}$ to $(0)\text{Pow}_{\mathcal{C}}^{-1}$.

Suppose that $(\mathcal{T}_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}), (\mathcal{T}'_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}})$ are pairs of topologies compatible with \mathcal{C} and such that $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_M = \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_M = \mathcal{S}$. We need to show that $(\mathcal{T}_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}) = (\mathcal{T}'_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}})$. As $\mathcal{T}_{\mathcal{C},\mathcal{O},\mathcal{S}}$ and $\mathcal{T}'_{\mathcal{C},\mathcal{O},\mathcal{S}}$ are both discrete, they must be equal. It remains to show that $\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} = \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}$. For all $i, j \in \mathbb{N}$, let

$$A_{i,j} := \left\{ f \in \mathcal{M}_{\mathcal{C}} \mid (f)_{s_{\mathcal{C}}} \text{Pow}_{\mathcal{C}} = i, (f)_{t_{\mathcal{C}}} \text{Pow}_{\mathcal{C}} = j \right\}.$$

As $s_{\mathcal{C}}, t_{\mathcal{C}}$ and $\text{Pow}_{\mathcal{C}}$ are all assumed to be continuous, the sets $A_{i,j}$ are open with respect to both $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}$ and $\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}}$. Note that the sets $A_{i,j}$ partition $\mathcal{M}_{\mathcal{C}}$ into sets which are open with respect to both of the topologies $\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}}$ and $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}$. Let $i, j \in \mathbb{N}$ be arbitrary. To conclude that $(\mathcal{T}_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}) = (\mathcal{T}'_{\mathcal{C},\mathcal{O},\mathcal{S}}, \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}})$, it suffices to show that the topologies $\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}}$ and $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}}$ are equal.

In the case that $i = j = 1$, both of these topologies are \mathcal{S} and in particular they are equal. If neither of i, j are 0, then since composition is continuous with respect to both of $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}, \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}}$ and $\psi_{1,i}^{-1}, \psi_{1,j}$ are both invertible, it follows that

$$\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}} = \psi_{1,i}^{-1} \circ_{\mathcal{C}} \mathcal{S} \circ_{\mathcal{C}} \psi_{1,j} = \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}}.$$

If $j = 0$, then $A_{i,j}$ is the singleton $\{t_i\} = \{\langle \rangle_{(0)\text{Pow}_{\mathcal{C}}^{-1}}\}$, so again $\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}} = \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}}$. It remains to consider the case that $i = 0$ and $j \neq 0$. If $A_{0,j} = \emptyset$, then the result is clear, otherwise let $z \in A_{0,j}$ be fixed.

Let $\phi_j : A_{0,j} \rightarrow A_{1,j}$ and $\phi'_j : \text{img}(\phi_j) \rightarrow A_{0,j}$ be defined by

$$(f)\phi_j = \iota_1 \circ_{\mathcal{C}} f \quad \text{and} \quad (f)\phi'_j = z \circ_{\mathcal{C}} \psi_{1,j}^{-1} \circ_{\mathcal{C}} f.$$

By definition, both of these maps are continuous with respect to both of $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}$ and $\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}}$. If $f \in A_{i,j}$ is arbitrary, then $(f)\phi_j\phi'_j = z \circ_{\mathcal{C}} \psi_{1,j}^{-1} \circ_{\mathcal{C}} \iota_1 \circ_{\mathcal{C}} f$. As $z \circ_{\mathcal{C}} \psi_{1,j}^{-1} \circ_{\mathcal{C}} \iota_1$ has both source and target $(0)\text{Pow}_{\mathcal{C}}^{-1}$, it follows that this is ι_0 (which is also the identity element of the object). Thus $(f)\phi_j\phi'_j = f$. Moreover if $(f)\phi_j \in \text{img}(\phi_j)$ is arbitrary, then $((f)\phi_j)\phi'_j\phi_j = ((f)\phi_j\phi'_j)\phi_j = (f)\phi_j$. Thus ϕ_j is a homeomorphism from $A_{0,j}$ to $\text{img}(\phi_j)$. As $\text{img}(\phi_j)$ is contained in $A_{1,j}$ and we have already shown that $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{1,j}} = \mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{1,j}}$, it follows that

$$\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}} = (\mathcal{T}'_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{\text{img}(\phi_j)})\phi'_j = (\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{\text{img}(\phi_j)})\phi'_j = \mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}} \upharpoonright_{A_{i,j}}.$$

We have now shown that the topology $\mathcal{T}_{\mathcal{C},\mathcal{M},\mathcal{S}}$ is a disjoint union of countably many topological spaces which are homeomorphic to specific subspaces of (M, \mathcal{S}) (either M or $\text{img}(\phi_j\psi_{1,j}^{-1})$ for some j). Thus the correspondence preserves containment as well as the properties of being second countable and Hausdorff. To show that Polishness is preserved, it suffices to show that if \mathcal{S} is Polish and $j \in \mathbb{N} \setminus \{0\}$, then so is $\mathcal{S} \upharpoonright_{\text{img}(\phi_j\psi_{1,j}^{-1})}$.

We show that $\text{img}(\phi_j) \circ_{\mathcal{C}} \psi_{1,j}^{-1}$ is closed (and is thus Polish from Lemma 4.18). As \mathcal{S} is Hausdorff and thus $\mathcal{Z}(M) \subseteq \mathcal{S}$ (Lemma 6.14), the set

$$RO := \bigcap_{g \in M} \left\{ f \in M \mid gf = f \right\}$$

is closed. We need only show that $\text{img}(\phi_j) \circ_{\mathcal{C}} \psi_{1,j}^{-1} = RO$. If $(f)\phi_j \in \text{img}(\phi_j)$, $g \in M$ are arbitrary, then

$$g \circ_{\mathcal{C}} (f)\phi_j \circ_{\mathcal{C}} \psi_{1,j}^{-1} = (g \circ_{\mathcal{C}} \iota_1) \circ_{\mathcal{C}} f \circ_{\mathcal{C}} \psi_{1,j}^{-1} = \iota_1 \circ_{\mathcal{C}} f \circ_{\mathcal{C}} \psi_{1,j}^{-1} = (f)\phi_j \circ_{\mathcal{C}} \psi_{1,j}^{-1}.$$

So indeed $(f)\phi_j \circ_{\mathcal{C}} \psi_{1,j}^{-1} \in RO$. If $f \in RO$ is arbitrary, then

$$f = (\iota_1 \circ_{\mathcal{C}} z) \circ_{\mathcal{C}} f = (\iota_1 \circ_{\mathcal{C}} z) \circ_{\mathcal{C}} f \circ_{\mathcal{C}} (\psi_{1,j} \circ_{\mathcal{C}} \psi_{1,j}^{-1}) = (z \circ_{\mathcal{C}} f \circ_{\mathcal{C}} \psi_{1,j})\phi_j \circ_{\mathcal{C}} \psi_{1,j}^{-1}.$$

So $f \in \text{img}(\phi_j) \circ_{\mathcal{C}} \psi_{1,j}^{-1}$ as required. \square

9. INVESTIGATING IMPORTANT SEMIGROUPS AND CLONES

In this section we investigate the topologies compatible with various important examples of semigroups using the tools introduced in the previous sections.

9.1. Binary relations. In this subsection we explore the binary relation monoids (see Example 4.12). These monoids contain many of the monoids we will be studying later.

Lemma 9.1 (Property **W** for \mathcal{B}_X , Assumed Knowledge: 3.5, 4.12, 7.5).

*If X is an infinite set, then the topological semigroup $(\mathcal{B}_X, \mathcal{T}\mathcal{B}_X)$ (recall Example 4.12) has property **W** with respect to $\text{Sym}(X)$.*

Proof. Let $s \in \mathcal{B}_X$ be arbitrary. It suffices to find $f_s, g_s \in \mathcal{B}_X$ and $h_s \in \text{Sym}(X)$, such that $f_s h_s g_s = s$ and for all $N \in \text{Nbhd}_{\text{Sym}(X)}(h_s)$ we have $f_s N g_s \in \text{Nbhd}_{\mathcal{B}_X}(s)$.

We first fix a bijection $\Phi : X \rightarrow X \times X \times \{0, 1\}$ (this must exist because X is infinite and $\{0, 1\}$ is finite). We then define a binary relation f from X to $X \times X \times \{0, 1\}$ by $f := \left\{ (x, (x, y, 0)) \mid x, y \in X \right\}$. We then define

$$f_s := f\Phi^{-1} \quad \text{and} \quad g_s := \Phi f^{-1}.$$

Claim: Suppose that $k \in \mathcal{B}_X$ and a is a bijection between finite subsets of $X \times X \times \{0, 1\}$ such that $f a f^{-1} \subseteq k$. In this case there is some $t_{k,a} \in \text{Sym}(X \times X \times \{0, 1\})$ such that $a \subseteq t_{k,a}$ and $k = f t_{k,a} f^{-1}$.

Proof of Claim: Let $B := (\text{dom}(a))\pi_0 \cup (\text{dom}(a))\pi_1 \cup (\text{img}(a))\pi_0 \cup (\text{img}(a))\pi_1$, $G := X \setminus B$ and $\phi : X \rightarrow G$ be a bijection. For each $x \in X$, let

$$t_x : \{x\} \times (\{x\})k\phi \times \{0\} \rightarrow (\{x\})k \times (\{x\})\phi \times \{0\}$$

be the bijection defined by

$$(x, (y)\phi, 0)t_x = (y, (x)\phi, 0).$$

Note that for each $x \in X$ we have $\text{dom}(t_x) \cap \text{dom}(a) = \emptyset = \text{img}(t_x) \cap \text{img}(a)$ and $ft_x f^{-1} = k \cap (\{x\} \times X)$. Moreover if $x \neq y$, then $\text{dom}(t_x) \cap \text{dom}(t_y) = \emptyset = \text{img}(t_x) \cap \text{img}(t_y)$. Thus $t := \bigcup_{x \in X} t_x$ is a bijection between a subsets of $X \times X \times \{0\}$ and $ftf^{-1} = k$.

We now define $R_d := (X \times X \times \{0\}) \setminus (\text{dom}(t) \cup \text{dom}(a))$ and $R_i := (X \times X \times \{0\}) \setminus (\text{img}(t) \cup \text{img}(a))$. Let $b \in G$ be fixed. As $|R_d|, |R_i| \leq |X|$, we can define injections $u_d : R_d \rightarrow G \times \{b\} \times \{1\}$ and $u_i : R_i \rightarrow G \times \{b\} \times \{1\}$. We have that $|G \times (G \setminus \{b\}) \times \{1\}| = |X|$, moreover $G \times (G \setminus \{b\}) \times \{1\}$ is disjoint from

$$\text{dom}(a) \cup \text{img}(a) \cup \text{dom}(t) \cup \text{img}(t) \cup \text{dom}(u_i) \cup \text{img}(u_i) \cup \text{dom}(u_d) \cup \text{img}(u_d).$$

So there is a bijection u from the set

$$(X \times X \times \{0, 1\}) \setminus (\text{dom}(a) \cup \text{dom}(u_d) \cup \text{img}(u_i) \cup \text{dom}(t))$$

to the set

$$(X \times X \times \{0, 1\}) \setminus (\text{img}(a) \cup \text{img}(u_d) \cup \text{dom}(u_i) \cup \text{img}(t)).$$

In particular the sets

$$\{\text{dom}(u), \text{dom}(u_d), \text{img}(u_i), \text{dom}(t), \text{dom}(a)\} \text{ and } \{\text{img}(u), \text{img}(u_d), \text{dom}(u_i), \text{img}(t), \text{img}(a)\}$$

are both partitions of $X \times X \times \{0, 1\}$. It follows that $t_{k,a} := u \cup u_d \cup u_i^{-1} \cup t \cup a$ is a bijection from $X \times X \times \{0, 1\}$ to itself. By definition $a \subseteq t_{k,a}$, so we need only show that $ft_{k,a}f^{-1} = k$. This can be seen as follows:

$$\begin{aligned} ft_{k,a}f^{-1} &= (fuf^{-1}) \cup (fudf^{-1}) \cup (fu_i^{-1}f^{-1}) \cup (ftf^{-1}) \cup (faf^{-1}) \\ &= (\emptyset) \cup (\emptyset) \cup (\emptyset) \cup (k) \cup (faf^{-1}) \\ &= k. \diamond \end{aligned}$$

By the claim (using $a = \emptyset$), we can choose $t_s \in \text{Sym}(X \times X \times \{0, 1\})$ such that $ft_s f^{-1} = k$. So if we define $h_s := \Phi t_s \Phi^{-1}$, then $f_s h_s g_s = s$.

Let $N \in \text{Nbhd}_{\text{Sym}(X)}(h_s)$ be arbitrary, it suffices to show that $f_s N g_s \in \text{Nbhd}_{\mathcal{B}_X}(s)$. As N is a neighbourhood of h_s , there is some finite $a \subseteq t_s$ such that

$$\Phi \left\{ t \in \text{Sym}(X \times X \times \{0, 1\}) \mid a \subseteq t \right\} \Phi^{-1} \subseteq N.$$

We define $U := \left\{ k \in \mathcal{B}_X \mid faf^{-1} \subseteq k \right\}$. As faf^{-1} is finite, the set U is an open subset of \mathcal{B}_X . Moreover $faf^{-1} \subseteq ft_s f^{-1} = s$, so $U \in \text{Nbhd}_{\mathcal{B}_X}(s)$. It therefore suffices to show that $U \subseteq f_s N g_s$. Let $k \in U$ be arbitrary. By the claim let $t_{k,a} \in \text{Sym}(X \times X \times \{0, 1\})$ be such that $ft_{k,a}f^{-1} = k$ and $a \subseteq t_{k,a}$. By the definition of a , it follows that $\Phi t_{k,a} \Phi^{-1} \in N$. Thus

$$k = ft_{k,a}f^{-1} = f_s \Phi t_{k,a} \Phi^{-1} g_s \in f_s N g_s$$

as required. \square

Definition 9.2 (Hausdorff binary relation topology, Assumed Knowledge: 3.14, 4.12, 9.3).

If X is a set, then we define the topology \mathcal{HTB}_X on the set \mathcal{B}_X to be the topology generated by the sets of the form

$$U_{Y,Z} := \left\{ f \in \mathcal{B}_X \mid (Y \times Z) \cap f \neq \emptyset \right\} \quad \text{and} \quad V_{Y,Z} := \left\{ f \in \mathcal{B}_X \mid (Y \times Z) \cap f = \emptyset \right\}$$

for all $Y, Z \subseteq X$ (We show in Lemma 9.3 that this is a Hausdorff semigroup topology).

Lemma 9.3 (Hausdorff binary relation topology is nice, Assumed Knowledge: 3.2, 3.5, 6.6, 9.2).

If X is a set, then the topology \mathcal{HTB}_X from Definition 9.2 is a Hausdorff topology compatible with \mathcal{B}_X . Moreover $\mathcal{HTB}_X = \mathcal{MF}(\mathcal{B}_X)$.

Proof. We first show that \mathcal{HTB}_X is Hausdorff. Let $f, g \in \mathcal{B}_X$ be arbitrary with $f \neq g$. We can assume without loss of generality that there is some pair $(x, y) \in f \setminus g$. It follows that $f \in U_{\{x\}, \{y\}}$ and $g \in \mathcal{B}_X \setminus U_{\{x\}, \{y\}} = V_{\{x\}, \{y\}}$. As f and g were arbitrary, it follows that \mathcal{HTB}_X is Hausdorff.

We next show that \mathcal{HTB}_X is compatible with \mathcal{B}_X . Let $Y, Z \subseteq X$ be arbitrary. We first show that $(U_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}$ is open. Let $(f, g) \in (U_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}$ be arbitrary. Note that $(Y \times Z) \cap fg \neq \emptyset \iff (Y)f \cap (Z)g^{-1} \neq \emptyset$. So there is some $p \in (Y)f \cap (Z)g^{-1}$. In particular we have

$$(f, g) \in U_{Y, \{p\}} \times U_{\{p\}, Z} \subseteq (U_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}.$$

So $(U_{Y,Z})(\ast^{\mathcal{B}_X})^{-1} \in \text{Nbhd}_{\mathcal{B}_X^2}((f, g))$ and hence $(U_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}$ is open. We next show that the set $(V_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}$ is open. Let $(f, g) \in (V_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}$ be arbitrary. As before $(Y \times Z) \cap fg = \emptyset \iff (Y)f \cap (Z)g^{-1} = \emptyset$. So we have

$$(f, g) \in V_{Y, X \setminus (Y)f} \times V_{X \setminus (Z)g^{-1}, Z} \subseteq (V_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}.$$

So $(V_{Y,Z})(\ast^{\mathcal{B}_X})^{-1} \in \text{Nbhd}_{\mathcal{B}_X^2}((f, g))$ and hence $(V_{Y,Z})(\ast^{\mathcal{B}_X})^{-1}$ is open.

It remains to show that $\mathcal{HTB}_X = \mathcal{MF}(\mathcal{B}_X)$. As \mathcal{HTB}_X is Fréchet and compatible with \mathcal{B}_X , it follows from the definition of $\mathcal{MF}(\mathcal{B}_X)$ that $\mathcal{MF}(\mathcal{B}_X) \subseteq \mathcal{HTB}_X$. To show the reverse inclusion, let \mathcal{T} be an arbitrary Fréchet topology semicompatible with \mathcal{B}_X and let $Y, Z \subseteq X$ be arbitrary.

It suffices to show that $U_{Y,Z}$ and $V_{Y,Z}$ are open in \mathcal{T} . As \mathcal{T} is Fréchet, the subspace

$$\text{img}(\lambda_{Y \times Y} \circ \rho_{Z \times Z}) = \{\emptyset, Y \times Z\}$$

of $(\mathcal{B}_X, \mathcal{T})$ is discrete. Moreover as \mathcal{T} is semicompatible with \mathcal{B}_X , it follows that $(\{\{\emptyset\}, \{Y \times Z\}\})(\lambda_{Y \times Y} \circ \rho_{Z \times Z})^{-1}$ is a partition of \mathcal{B}_X into open sets. As

$$(\{\{\emptyset\}, \{Y \times Z\}\})(\lambda_{Y \times Y} \circ \rho_{Z \times Z})^{-1} = \{V_{Y \times Z}, U_{Y \times Z}\},$$

the result follows. \square

Theorem 9.4 (The topologies of $\mathcal{B}_{\mathbb{N}}$, Assumed Knowledge: 3.7, 6.12, 6.13, 6.15, 7.6, 9.1, 9.3).

Every homomorphism from $(\mathcal{B}_{\mathbb{N}}, \mathcal{TB}_{\mathbb{N}})$ (recall Example 4.12) to a second countable topological semigroup is continuous and there are no second countable Fréchet topologies compatible with $\mathcal{B}_{\mathbb{N}}$. Moreover we have

$$\mathcal{TB}_{\mathbb{N}} = \text{SCC}(\mathcal{B}_{\mathbb{N}}) \subsetneq \mathcal{MF}(\mathcal{B}_{\mathbb{N}}) = \mathcal{Z}(\mathcal{B}_{\mathbb{N}}) = \mathcal{FM}(\mathcal{B}_{\mathbb{N}}) = \mathcal{HM}(\mathcal{B}_{\mathbb{N}}) = \mathcal{HTB}_{\mathbb{N}}.$$

Proof. From Lemma 9.3, we have that $\mathcal{HTB}_{\mathbb{N}} = \mathcal{MF}(\mathcal{B}_{\mathbb{N}})$ is Hausdorff and compatible with $\mathcal{B}_{\mathbb{N}}$. Thus from the definition of $\mathcal{HM}(\mathcal{B}_{\mathbb{N}})$ we have $\mathcal{HM}(\mathcal{B}_{\mathbb{N}}) \subseteq \mathcal{HTB}_{\mathbb{N}} = \mathcal{MF}(\mathcal{B}_{\mathbb{N}})$. Thus from Proposition 6.15, we have

$$\mathcal{MF}(\mathcal{B}_{\mathbb{N}}) = \mathcal{Z}(\mathcal{B}_{\mathbb{N}}) = \mathcal{FM}(\mathcal{B}_{\mathbb{N}}) = \mathcal{HM}(\mathcal{B}_{\mathbb{N}}) = \mathcal{HTB}_{\mathbb{N}}.$$

If $x, y \in \mathbb{N}$ are arbitrary, then the set $U_{x,y}$ from Example 4.12 coincides with the set $U_{\{x\}, \{y\}}$ from Definition 9.2. Thus $\mathcal{TB}_{\mathbb{N}} \subseteq \mathcal{HTB}_{\mathbb{N}}$. Moreover, the only neighbourhood of \emptyset in the topology $\mathcal{TB}_{\mathbb{N}}$ is the entire space, thus $\mathcal{TB}_{\mathbb{N}}$ is not Hausdorff and so $\mathcal{TB}_{\mathbb{N}} \subsetneq \mathcal{HTB}_{\mathbb{N}}$.

It remains to show that $\mathcal{TB}_{\mathbb{N}} = \mathcal{SCC}(\mathcal{B}_{\mathbb{N}})$ (recall Proposition 6.12). By definition, the topology $\mathcal{TB}_{\mathbb{N}}$ is second countable. As $\mathcal{TB}_{\mathbb{N}}$ is compatible with $\mathcal{B}_{\mathbb{N}}$, it follows from the definition of $\mathcal{SCC}(\mathcal{B}_{\mathbb{N}})$ that $\mathcal{TB}_{\mathbb{N}} \subseteq \mathcal{SCC}(\mathcal{B}_{\mathbb{N}})$. Let \mathcal{T} be an arbitrary second countable topology compatible with $\mathcal{B}_{\mathbb{N}}$. It suffices to show that $\mathcal{T} \subseteq \mathcal{TB}_{\mathbb{N}}$.

By Corollary 6.13, we have $\mathcal{SCC}(\text{Sym}(\mathbb{N})) = \mathcal{TB}_{\mathbb{N}} \upharpoonright_{\text{Sym}(\mathbb{N})}$. Thus $\mathcal{T} \upharpoonright_{\text{Sym}(\mathbb{N})} \subseteq \mathcal{TB}_{\mathbb{N}} \upharpoonright_{\text{Sym}(\mathbb{N})}$. By applying Lemma 7.6 to the identity homomorphism from $(\mathcal{B}_{\mathbb{N}}, \mathcal{TB}_{\mathbb{N}})$ to $(\mathcal{B}_{\mathbb{N}}, \mathcal{T}_{\mathbb{N}})$, it suffices to show that \mathcal{B}_X has property **W** with respect to $\text{Sym}(\mathbb{N})$. This is shown in Lemma 9.1. \square

Sets and binary relations form a category in a natural fashion. This category even has products, although the products of this category are very different from the usual products in the categories of sets and functions. There is thus a natural clone of binary relations to consider using Proposition 8.6, however from Theorem 9.4 there is no second countable Fréchet topology compatible with $\mathcal{B}_{\mathbb{N}}$, so the same is true of such a clone.

9.2. (Partial) transformations. In this subsection we explore the topologies of the transformation and partial transformation monoids (see Examples 4.13 and 9.5).

Example 9.5 (Partial function monoid, Assumed Knowledge: 4.12).

If X is a set then we define the *partial function monoid* \mathcal{P}_X to be the submonoid of \mathcal{B}_X (recall Example 4.12) consisting of those relations which are functions from subsets of X to X .

We define the *partial pointwise topology* \mathcal{PPT}_X on \mathcal{P}_X to be the topology generated by sets of the form

$$U_{x,y} := \left\{ f \in \mathcal{P}_X \mid (x,y) \in f \right\} \quad \text{or} \quad V_x := \left\{ f \in \mathcal{P}_X \mid x \notin \text{dom}(f) \right\}$$

for $x, y \in X$. We show in Lemma 9.6 that $(\mathcal{P}_X, \mathcal{PPT}_X)$ is a topological semigroup.

Lemma 9.6 (Topological partial functions, Assumed Knowledge: 3.2, 3.25, 9.5).

The pair $(\mathcal{P}_X, \mathcal{PPT}_X)$ from Example 9.5 is a topological semigroup.

Proof. Let $x, y \in X$ be arbitrary. It suffices to show that the sets

$$(U_{x,y})(*\mathcal{P}_X)^{-1} \quad \text{and} \quad (V_x)(*\mathcal{P}_X)^{-1}$$

are open subsets of \mathcal{P}_X^2 (using the topology \mathcal{PPT}_X). As $(\mathcal{P}_X, \mathcal{TB}_X \upharpoonright_{\mathcal{P}_X})$ is a topological subsemigroup of $(\mathcal{B}_X, \mathcal{TB}_X)$ and $\mathcal{TB}_X \upharpoonright_{\mathcal{P}_X} \subseteq \mathcal{PPT}_X$ (recall Example 4.12 and Definition 3.2), it follows that $(U_{x,y})(*\mathcal{P}_X)^{-1}$ is open.

It remains to show that $(V_x)(*\mathcal{P}_X)^{-1}$ is open. Note that if $a, b \in \mathcal{P}_X$ are arbitrary, then $x \notin \text{dom}(ab) \iff (x \notin \text{dom}(a) \text{ or } (x)a \notin \text{dom}(b))$. In other words

$$(V_x)(*\mathcal{P}_X)^{-1} = (V_x \times \mathcal{P}_X) \cup \left(\bigcup_{y \in X} (U_{x,y} \times V_y) \right).$$

So $(V_x)(*\mathcal{P}_X)^{-1}$ is open as required. \square

In this document we will only be using the following lemma for the semigroups \mathcal{P}_X and X^X (to establish the “lower bounds” in Theorem 9.10), but it also applies to many other potentially interesting semigroups ($\text{End}(\mathbb{Q}, \leq)$ for example).

Lemma 9.7 (Minimum partial function topologies, Assumed Knowledge: 3.2, 6.6, 9.5).

Suppose that X is a set and S is a subsemigroup of \mathcal{P}_X such that

- (1) For all $x \in X$ the constant map $c_{X,x} := X \times \{x\}$ is an element of S .
- (2) For all $x \in X$ there is some $f_x \in S$ such that $(\{x\})f_x^{-1} = \{x\}$ and $|\text{img}(f_x)|$ is finite.

In this case $\mathcal{MF}(S) = \mathcal{PPT}_X \upharpoonright_S$.

Proof. First note that by Lemma 9.6, the topology \mathcal{PPT}_X is compatible with S . Thus by the definition of $\mathcal{MF}(S)$ we have $\mathcal{MF}(S) \subseteq \mathcal{PPT}_X \upharpoonright_S$. It remains to show the reverse inclusion.

Let \mathcal{T} be an arbitrary Fréchet topology semicompatible with S and let $x, y \in X$ be arbitrary. It suffices to show that the sets $U_{x,y}, V_x$ from Example 9.5 are open in \mathcal{T} . Note that if $g \in S$ then

$$\begin{aligned} (x, y) \in g &\iff c_{X,x}gf_y = c_{X,y} \iff c_{X,x}gf_y \notin \{\emptyset\} \cup \left\{ c_{X,z} \mid z \in \text{img}(f_y) \setminus \{y\} \right\}, \\ x \notin \text{dom}(g) &\iff c_{X,x}gc_{X,x} \neq c_{X,x} \iff c_{X,x}gc_{X,x} \in \{\emptyset\}. \end{aligned}$$

In other words

$$\begin{aligned} U_{x,y} &= \mathcal{P}_X \setminus \left(\left(\{\emptyset\} \cup \left\{ c_{X,z} \mid z \in \text{img}(f_y) \setminus \{y\} \right\} \right) \lambda_{c_{X,x}}^{-1} \rho_{f_y}^{-1} \right), \\ V_x &= (\{\emptyset\}) \lambda_{c_{X,x}}^{-1} \rho_{c_{X,x}}^{-1}. \end{aligned}$$

As the topology \mathcal{T} is Fréchet, the finite sets $\{\emptyset\}$ and $\{\emptyset\} \cup \left\{ c_{X,z} \mid z \in \text{img}(f_y) \setminus \{y\} \right\}$ are closed with respect to \mathcal{T} . As \mathcal{T} is semicompatible with S , it follows from the above equalities that $U_{x,y}$ is open with respect to \mathcal{T} . \square

We next prove the lemmas for establishing the ‘‘upper bounds’’ in Theorem 9.10.

Lemma 9.8 (Property **W** for X^X , Assumed Knowledge: 3.5, 4.13, 7.5).

If X is a set, then the topological semigroup (X^X, \mathcal{PT}_X) has property **W** with respect to $\text{Sym}(X)$.

Proof. Let $s \in X^X$ be arbitrary. It suffices to find $f_s, g_s \in X^X$ and $h_s \in \text{Sym}(X)$, such that $f_s h_s g_s = s$ and for all $N \in \text{Nbhd}_{\text{Sym}(X)}(h_s)$ we have $f_s N g_s \in \text{Nbhd}_{X^X}(s)$.

If X is finite, then X^X is discrete and so we can choose $f_s = s, h_s = 1_{X^X}, g_s = 1_{X^X}$. If X is infinite then we fix a bijection $\Phi : X \rightarrow X \times X$ and $b \in X$.

We then define $f_s, g_s \in X^X$ by

$$(x)f_s = (x, b)\Phi^{-1} \quad \text{and} \quad (x)g_s = (x)\Phi\pi_0.$$

Claim: Suppose that $k \in X^X$ and a is a bijection between finite subsets of $X \times X$ such that whenever $(x, b) \in \text{dom}(a)$ we have $(x, b)a\pi_0 = (x)k$. In this case there is some $t_{k,a} \in \text{Sym}(X \times X)$ such that $a \subseteq t_{k,a}$ and $f_s \Phi t_{k,a} \Phi^{-1} g_s = k$.

Proof of Claim: Let $M \subseteq X$ be such that $|M| = |X \setminus M|$ and let $\phi : X \rightarrow M \setminus ((\text{img}(a))\pi_0 \cup (\text{img}(a))\pi_1)$ be an injection.

Define $t' : (X \times \{b\}) \setminus \text{dom}(a) \rightarrow X \times M$ by $(x, b)t' = ((x)k, (x)\phi)$. As $t'\pi_1$ is injective, it follows that t' is injective. Moreover

$$|X| = |X \times X| \geq |(X \times X) \setminus \text{dom}(t')| \geq |X \times (X \setminus \{b\})| \geq |X|,$$

$$|X| = |X \times X| \geq |(X \times X) \setminus \text{img}(t')| \geq |X \times (X \setminus M)| \geq |X|.$$

So $|(X \times X) \setminus \text{dom}(t')| = |(X \times X) \setminus \text{img}(t')| = |X|$. As a is finite, there is a bijection

$$u : (X \times X) \setminus ((\text{dom}(t') \cup \text{dom}(a))) \rightarrow (X \times X) \setminus ((\text{img}(t') \cup \text{img}(a))).$$

By construction each of t', a, u is injection and the sets

$$\{\text{dom}(t'), \text{dom}(a), \text{dom}(u)\} \quad \text{and} \quad \{\text{img}(t'), \text{img}(a), \text{img}(u)\}$$

are partitions of $X \times X$. Thus $t_{k,a} := t' \cup a \cup u$ is an element of $\text{Sym}(X \times X)$. Moreover by definition $a \subseteq t_{k,a}$ and if $x \in X$ then either $(x, b) \in \text{dom}(a)$ and

$$(x)f_s\Phi t_{k,a}\Phi^{-1}g_s = (x,b)t_{k,a}\pi_0 = (x,b)a\pi_0 = (x)k,$$

or $(x, b) \notin \text{dom}(a)$ and

$$(x)f_s\Phi t_{k,a}\Phi^{-1}g_s = (x,b)t_{k,a}\pi_0 = ((x)k, (x)\phi)\pi_0 = (x)k.\diamond$$

By the claim (using $a = \emptyset$), we can choose $t_s \in \text{Sym}(X \times X)$ such that $f_s\Phi t_s\Phi^{-1}g_s = k$. We then define $h_s := \Phi t_s\Phi^{-1}$.

Let $N \in \text{Nbhd}_{\text{Sym}(X)}(h_s)$ be arbitrary, it suffices to show that $f_sNg_s \in \text{Nbhd}_{X^X}(s)$. As N is a neighbourhood of h_s , there is some finite $a \subseteq t_s$ such that

$$\Phi \left\{ t \in \text{Sym}(X \times X) \mid a \subseteq t \right\} \Phi^{-1} \subseteq N.$$

We define

$$U := \left\{ k \in X^X \mid \text{whenever } (x, b) \in \text{dom}(a) \text{ we have } (x)k = (x, b)a\pi_0 \right\}.$$

As a is finite, the set U is an open subset of X^X . Moreover if $(x, b) \in \text{dom}(a)$, then $(x)s = (x)f_s h_s g_s = (x, b)a\pi_0$ and so $U \in \text{Nbhd}_{X^X}(s)$. It therefore suffices to show that $U \subseteq f_sNg_s$.

Let $k \in U$ be arbitrary. By the claim let $t_{k,a} \in \text{Sym}(X \times X)$ be such that $f_s\Phi t_{k,a}\Phi^{-1}g_s = k$ and $a \subseteq t_{k,a}$. By the definition of a , it follows that $\Phi t_{k,a}\Phi^{-1} \in N$. Thus

$$k = f_s\Phi t_{k,a}\Phi^{-1}g_s \in f_sNg_s$$

as required. \square

Lemma 9.9 (Property **W** for \mathcal{P}_X , Assumed Knowledge: 3.5, 7.5, 9.5).

If X is a set, then the topological semigroup $(\mathcal{P}_X, \mathcal{PPT}_X)$ has property **W** with respect to X^X .

Proof. Let $s \in X^X$ be arbitrary. It suffices to find $g_s \in \mathcal{P}_X$ and $h_s \in X^X$, such that $h_s g_s = s$ and for all $N \in \text{Nbhd}_{X^X}(h_s)$ we have $Ng_s \in \text{Nbhd}_{\mathcal{P}_X}(s)$.

If X is finite, then \mathcal{P}_X is discrete and so we can choose $h_s = 1_{\mathcal{P}_X}, g_s = s$. If X is infinite then fix some $p \in X$ and let $g_s : X \setminus \{p\} \rightarrow X$ be a bijection.

Claim: Suppose that $k \in \mathcal{P}_X$ and a is a function between finite subsets of X such that $ag_s \subseteq k$ and $(\{p\})a^{-1} \cap \text{dom}(k) = \emptyset$. In this case there is some $h_{k,a} \in X^X$ such that $a \subseteq h_{k,a}$ and $h_{k,a}g_s = k$.

Proof of Claim: Let $b : \text{dom}(k) \setminus \text{dom}(a) \rightarrow X$ and $c : X \setminus (\text{dom}(k) \cup \text{dom}(a)) \rightarrow X$ be defined by

$$(x)b = (x)kg_s^{-1} \quad \text{and} \quad (x)c = p.$$

As $\{\text{dom}(a), \text{dom}(b), \text{dom}(c)\}$ is a partition of X , it follows that $h_{k,a} := a \cup b \cup c \in X^X$.

Note that $\text{dom}(ag_s) = \text{dom}(a) \setminus (\{p\})a^{-1}$ and so by the hypothesis of the claim we have $\text{dom}(b) = \text{dom}(k) \setminus \text{dom}(ag_s)$. It remains to show that $h_{k,a}g_s = k$. There are 4 cases to consider:

- (1) If $x \in \text{dom}(k) \cap \text{dom}(ag_s)$ then $(x)h_{k,a}g_s = (x)ag_s = (x)k$.
- (2) If $x \in \text{dom}(k) \setminus \text{dom}(ag_s)$ then $(x)h_{k,a}g_s = (x)kg_s^{-1}g_s = (x)k$.
- (3) If $x \in \text{dom}(ag_s) \setminus \text{dom}(k)$ then (as $ag_s \subseteq k$) we have that $x \in \emptyset$ and hence $x \notin \text{dom}(h_{k,a}g_s)$.

- (4) If $x \in X \setminus (\text{dom}(ag_s) \cup \text{dom}(k))$ then $x \notin \text{dom}(ag_s) \cup \text{dom}(bg_s) = \text{dom}(ag_s) \cup \text{dom}(bg_s) \cup \text{dom}(cg_s) = \text{dom}(h_{k,a}g_s)$.

So $h_{k,a}g_s = k$ as required. \diamond

By the claim (using $s = k$ and $\emptyset = a$) let $h_s \in X^X$ be such that $h_s g_s = s$. Let $N \in \text{Nbhd}_{X^X}(h_s)$ be arbitrary. It suffices to show that $Ng_s \in \text{Nbhd}_{\mathcal{P}_X}(s)$. By the choice of N , there is a finite $a \subseteq h_s$ such that $N \supseteq \left\{ h \in X^X \mid a \subseteq h \right\}$. We define

$$U := \left\{ k \in \mathcal{P}_{\mathbb{N}} \mid ag_s \subseteq k \text{ and } (\{p\})a^{-1} \cap \text{dom}(k) = \emptyset \right\}.$$

Note that the set U is open in \mathcal{PPT}_X . Moreover we have $ag_s \subseteq h_s g_s = s$, and if $x \in (\{p\})a^{-1}$ then $(a)h_s = p$ and hence $x \notin \text{dom}(h_s g_s)$. Thus $U \in \text{Nbhd}_{\mathcal{P}_X}(s)$. It thus suffices to show that $U \subseteq \left\{ h \in X^X \mid a \subseteq h \right\} g_s \subseteq Ng_s$. This is immediate from the claim together with the definition of a , so the result follows. \square

Theorem 9.10 (The topologies of $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{P}_{\mathbb{N}}$, Assumed Knowledge: 5.39, 6.15, 7.6, 7.7, 9.7, 9.8, 9.9).

The topology $\mathcal{PT}_{\mathbb{N}}$ is the only second countable Fréchet topology compatible with the semigroup $\mathbb{N}^{\mathbb{N}}$ (recall Example 4.13). Similarly the topology $\mathcal{PPT}_{\mathbb{N}}$ is the only second countable Fréchet topology compatible with the semigroup $\mathcal{P}_{\mathbb{N}}$ (Recall Example 9.5). Moreover all semigroup homomorphisms from either of $(\mathbb{N}^{\mathbb{N}}, \mathcal{PT}_{\mathbb{N}})$ or $(\mathcal{P}_{\mathbb{N}}, \mathcal{PPT}_{\mathbb{N}})$ to second countable semigroups are continuous,

$$\mathcal{PT}_{\mathbb{N}} = \mathcal{Z}(\mathbb{N}^{\mathbb{N}}) = \mathcal{MF}(\mathbb{N}^{\mathbb{N}}) = \mathcal{FM}(\mathbb{N}^{\mathbb{N}}) = \mathcal{HM}(\mathbb{N}^{\mathbb{N}}) = \mathcal{SCC}(\mathbb{N}^{\mathbb{N}}),$$

$$\mathcal{PPT}_{\mathbb{N}} = \mathcal{Z}(\mathcal{P}_{\mathbb{N}}) = \mathcal{MF}(\mathcal{P}_{\mathbb{N}}) = \mathcal{FM}(\mathcal{P}_{\mathbb{N}}) = \mathcal{HM}(\mathcal{P}_{\mathbb{N}}) = \mathcal{SCC}(\mathcal{P}_{\mathbb{N}}),$$

and both of these topologies are Polish.

Proof. Let $d : \mathcal{P}_X^2 \rightarrow \mathbb{R}$ be defined by

$$(f, g)d = \inf \left(\left\{ \frac{1}{2^n} \mid f \upharpoonright_{\{0,1,\dots,n-1\}} = g \upharpoonright_{\{0,1,\dots,n-1\}}, n \in \mathbb{N} \right\} \right).$$

It is routine to verify that this is a complete metric compatible with \mathcal{PPT}_X , and it restricts to a complete metric compatible with \mathcal{PT}_X so these topologies are Polish. Let $S \in \{\mathbb{N}^{\mathbb{N}}, \mathcal{P}_{\mathbb{N}}\}$ be arbitrary. Let \mathcal{T} be an arbitrary second countable Fréchet topology compatible with S . By Lemma 9.7 the topology \mathcal{T} contains $\mathcal{PPT}_{\mathbb{N}} \upharpoonright_S$, so \mathcal{T} is Hausdorff. Thus from Proposition 6.15, we have

$$\begin{array}{ccc} & \mathcal{Z}(S) & \\ \mathcal{MF}(S) & \subsetneq & \mathcal{HM}(S) \subseteq \mathcal{T} \subseteq \mathcal{SCC}(S). \\ & \mathcal{FM}(S) & \end{array}$$

It suffices to show that all of these containments are equalities. Thus we need only show that $\mathcal{SCC}(S) \subseteq \mathcal{MF}(S)$.

By Lemma 9.7, we have that $\mathcal{MF}(S) = \mathcal{PPT}_{\mathbb{N}} \upharpoonright_S$ (so if $S = \mathbb{N}^{\mathbb{N}}$ then $\mathcal{MF}(S) = \mathcal{PT}_{\mathbb{N}}$). Thus it suffices to show that $\mathcal{SCC}(S) \subseteq \mathcal{PPT}_{\mathbb{N}} \upharpoonright_S$. Let ϕ be an arbitrary semigroup homomorphism from $(S, \mathcal{PPT}_{\mathbb{N}} \upharpoonright_S)$ to a second countable topological semigroup. From Proposition 6.12, it suffices to show that ϕ is continuous.

From Proposition 6.12 and Theorem 5.39, it follows that $\phi \upharpoonright_{\text{Sym}(\mathbb{N})}$ is continuous. From Lemma 7.6, it suffices to show that S has property **W** with respect to $\text{Sym}(\mathbb{N})$. If $S = \mathbb{N}^{\mathbb{N}}$, then this follows from Lemma 9.8 and if $S = \mathcal{P}_{\mathbb{N}}$, then this follows from Lemmas 9.8, 7.7 and 9.9. \square

We have now proved the main result of this subsection, and we can use Proposition 8.6 to extend it to the corresponding clones. The category of sets and functions, and the category of sets and partial functions are both nice enough for the proposition to apply. However in the category of sets and partial functions the products and projections are a bit unusual (the projection maps are not defined everywhere anymore and a product of no sets no longer contains the empty tuple). As a result we only state the clone corollary for the full function clone but for those interested, the argument for the partial function clone is the same.

Corollary 9.11 (The topologies of the full function clone, Assumed Knowledge: 8.2, 8.6, 9.10). *Let \mathcal{C} be the subcategory of sets and functions with the object set $\mathcal{O}_{\mathcal{C}} := \{\mathbb{N}^i \mid i \in \mathbb{N}\}$, and the morphism set $\mathcal{M}_{\mathcal{C}}$ consisting of the all functions between these objects. Let $\mathcal{P}_{\mathcal{O}}$ be the discrete topology on $\mathcal{O}_{\mathcal{C}}$ and let $\mathcal{P}_{\mathcal{M}}$ be the topology on $\mathcal{M}_{\mathcal{C}}$ generated by the sets of the form*

$$A_{i,j} := (\mathbb{N}^j)^{\mathbb{N}^i} \quad \text{and} \quad U_{x,y} := \left\{ f \in \mathcal{M}_{\mathcal{C}} \mid (x,y) \in f \right\}$$

for all $i, j \in \mathbb{N}$, $x \in \mathbb{N}^i$, and $y \in \mathbb{N}^j$.

The triple $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is a Polish topological abstract clone. Moreover if $(\mathcal{T}_{\mathcal{O}}, \mathcal{T}_{\mathcal{M}})$ are compatible with the clone \mathcal{C} then

- (1) If $\mathcal{T}_{\mathcal{M}}$ is Hausdorff then $\mathcal{T}_{\mathcal{M}} \supseteq \mathcal{P}_{\mathcal{M}}$.
- (2) If $\mathcal{T}_{\mathcal{M}}$ is second countable then $\mathcal{T}_{\mathcal{M}} \subseteq \mathcal{P}_{\mathcal{M}}$.

Proof. We first show that $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is a topological abstract clone. We need to verify the seven conditions in Definition 8.2. The first five conditions are immediate from the definition. We next show that $\circ_{\mathcal{C}}$ is continuous. As the sets $(A_{n,i} \times A_{i,m})_{i,n,m \in \mathbb{N}}$ are a partition of $\text{Comp}_{\mathcal{C}}$ into open sets, it suffices to show that for all $n, m, i \in \mathbb{N}$, the map $\circ_{\mathcal{C}} \upharpoonright_{A_{n,i} \times A_{i,m}}$ is continuous. For all $j, k \in \mathbb{N}$, $x \in \mathbb{N}^i$, $y \in \mathbb{N}^j$ the set $(A_{j,k})_{\circ_{\mathcal{C}}} \upharpoonright_{A_{n,i} \times A_{i,m}}^{-1}$ is either empty or $A_{n,i} \times A_{i,m}$ and is thus open, and

$$(U_{x,y})_{\circ_{\mathcal{C}}} \upharpoonright_{A_{n,i} \times A_{i,m}}^{-1} = \bigcup_{l \in \mathbb{N}} \bigcup_{z \in \mathbb{N}^l} U_{x,z} \times U_{z,y}$$

which is also open.

It remains to show that for all $i, j \in \mathbb{N}$, the bijection $\phi_{i,j} : A_{i,1}^j \rightarrow A_{i,j}$ from Definition 8.2 is continuous. This follows from the observation that if $x \in \mathbb{N}^i$ and $y \in \mathbb{N}^j$, then

$$(U_{x,y})_{\phi_{i,j}}^{-1} = \prod_{k \in \{0,1,\dots,j-1\}} U_{x,(y)\pi_k}.$$

Thus $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is a topological abstract clone. From Theorem 9.10, it follows that $\mathcal{P}_{\mathcal{M}} \upharpoonright_{\mathbb{N}^{\mathbb{N}}}$ is Polish and the required containments hold when all topologies are restricted to $\mathbb{N}^{\mathbb{N}}$. Thus the result follows from Proposition 8.6. \square

9.3. Partial bijections. In this subsection we explore the topologies on the symmetric inverse monoids (see Theorem 9.18), these monoids are the inverse semigroup analogues of the symmetric groups.

Example 9.12 (Symmetric inverse monoids, Assumed Knowledge: 4.7, 9.5).

If X is a set, then we define the *symmetric inverse monoid* \mathcal{I}_X of X to be the inverse semigroup with universe consisting of the bijections between subsets of X , and composition and inversion of binary relations as the operations. Note that the universe of \mathcal{I}_X is precisely $\mathcal{P}_X \cap \mathcal{P}_X^{-1}$.

Definition 9.13 (Topologies on symmetric inverse monoids, Assumed Knowledge: 3.2, 9.5, 9.12).

If X is a set, then we define the following topologies on the set \mathcal{I}_X

- (1) The *complemented pointwise topology* \mathcal{CPT}_X generated by the sets of the form

$$\left\{ f \in \mathcal{I}_X \mid (x, y) \in f \right\} \quad \text{and} \quad \left\{ f \in \mathcal{I}_X \mid (x, y) \in (X \times X) \setminus f \right\}$$

for all $x, y \in X$.

- (2) The *left pointwise topology* $\mathcal{LPT}_X := \mathcal{PPT}_X \upharpoonright_{\mathcal{I}_X}$.

- (3) The *right pointwise topology* $\mathcal{RPT}_X := \mathcal{LPT}_X^{-1}$,

- (4) The *inverse pointwise topology* \mathcal{IPT}_X , generated by $\mathcal{LPT}_X \cup \mathcal{RPT}_X$.

Lemma 9.14 (The complemented pointwise topology is nice, Assumed Knowledge: 3.14, 3.18, 6.6, 9.13).

If X is a set, then $\mathcal{CPT}_X = \mathcal{MF}(\mathcal{I}_X)$. Moreover, this topology is compact and Hausdorff.

Proof. We first show that this space is compact and Hausdorff. By Remark 3.18, if we view $\{0, 1\}$ as a discrete space, then $\{0, 1\}^{X \times X}$ is a compact space (using the product topology). For each $f \in \mathcal{B}_X$, we define $\chi_f \in \{0, 1\}^{X \times X}$ by:

$$(a, b)\chi_f := \begin{cases} 1 & \text{if } (a, b) \in f \\ 0 & \text{if } (a, b) \in (X \times X) \setminus f \end{cases}.$$

Let $\phi : \mathcal{B}_X \rightarrow \{0, 1\}^{X \times X}$ be the bijection $f \rightarrow \chi_f$. From the definition of \mathcal{CPT}_X , it follows that $\phi \upharpoonright_{\mathcal{I}_X}$ is a homeomorphism from $(\mathcal{I}_X, \mathcal{CPT}_X)$ to $(\mathcal{I}_X)\phi$.

Thus by Remark 3.18, to conclude that \mathcal{CPT}_X is compact and Hausdorff, it suffices to show that $(\mathcal{I}_X)\phi$ is a closed subset of $\{0, 1\}^{X \times X}$. Note that

$$\begin{aligned} (\mathcal{I}_X)\phi &= \left(\left\{ f \in \mathcal{B}_X \mid \text{for all } x \in X \text{ we have } |(\{x\})f| \leq 1 \text{ and } |(\{x\})f^{-1}| \leq 1 \right\} \right) \phi \\ &= \left\{ \alpha \in \{0, 1\}^{X \times X} \mid \begin{array}{l} \text{for all } x \in X \text{ we have } \left| \left\{ y \in X \mid (x, y)\alpha = 1 \right\} \right| \leq 1 \\ \text{and } \left| \left\{ y \in X \mid (y, x)\alpha = 1 \right\} \right| \leq 1 \end{array} \right\}. \end{aligned}$$

As the complement of the set above is

$$\left\{ \alpha \in \{0, 1\}^{X \times X} \mid \begin{array}{l} \text{there are } x, y, z \in X \text{ such that } y \neq z \text{ and either} \\ (x, y)\alpha = (x, z)\alpha = 1 \text{ or } (y, x)\alpha = (z, x)\alpha = 1 \end{array} \right\},$$

it follows that $(\mathcal{I}_X)\phi$ is closed.

We next show that $\mathcal{CPT}_X = \mathcal{MF}(\mathcal{I}_X)$. To show $\mathcal{CPT}_X \supseteq \mathcal{MF}(\mathcal{I}_X)$, we need to show that \mathcal{CPT}_X is semicompatible with \mathcal{I}_X (we have already shown that it is Fréchet). Let $f \in \mathcal{I}_X$ and $x, y \in X$ be arbitrary. If $y \notin \text{img}(f)$, then

$$\left(\left\{ g \in \mathcal{I}_X \mid (x, y) \in g \right\} \right) \rho_f^{-1} = \emptyset \quad \text{and} \quad \left(\left\{ g \in \mathcal{I}_X \mid (x, y) \in (X \times X) \setminus g \right\} \right) \rho_f^{-1} = \mathcal{I}_X$$

are both open in \mathcal{CPT}_X . If $y \in \text{img}(f)$, then we have

$$\begin{aligned} \left(\left\{ g \in \mathcal{I}_X \mid (x, y) \in g \right\} \right) \rho_f^{-1} &= \left\{ g \in \mathcal{I}_X \mid (x, (y)f^{-1}) \in g \right\}, \\ \left(\left\{ g \in \mathcal{I}_X \mid (x, y) \in (X \times X) \setminus g \right\} \right) \rho_f^{-1} &= \left\{ g \in \mathcal{I}_X \mid (x, (y)f^{-1}) \in (X \times X) \setminus g \right\}. \end{aligned}$$

Again these sets are both open in \mathcal{CPT}_X , so ρ_f is continuous. By symmetry, it follows that λ_f is continuous as well and thus \mathcal{CPT}_X is semicompatible with \mathcal{I}_X . It remains to show that $\mathcal{CPT}_X \subseteq \mathcal{MF}(\mathcal{I}_X)$.

Let \mathcal{T} be an arbitrary Fréchet topology semicompatible with \mathcal{I}_X . It suffices to show that $\mathcal{CPT}_X \subseteq \mathcal{T}$. Let $x, y \in X$ be arbitrary. As \mathcal{T} is Fréchet, it follows that $\text{img}(\lambda_{\{(x,x)\}} \circ \rho_{\{(y,y)\}}) = \{\emptyset, \{(x, y)\}\}$ is a discrete subspace of $(\mathcal{I}_X, \mathcal{T})$. Thus

$$(\{\{\emptyset\}, \{(x, y)\}\})(\lambda_{\{(x,x)\}} \circ \rho_{\{(y,y)\}})^{-1}$$

is a partition of \mathcal{I}_X into two sets which are open with respect to \mathcal{T} . This partition is precisely

$$\left\{ \left\{ f \in \mathcal{I}_X \mid (x, y) \in (X \times X) \setminus f \right\}, \left\{ f \in \mathcal{I}_X \mid (x, y) \in f \right\} \right\}$$

so indeed $\mathcal{T} \supseteq \mathcal{CPT}_X$ as required. \square

Lemma 9.15 (The left/right pointwise topologies are nice, Assumed Knowledge: 3.14, 6.3, 6.7, 9.6, 9.13, 9.14).

If X is a set and S is the semigroup obtained by removing the unary operation of \mathcal{I}_X , then every Fréchet topology compatible with S contains either \mathcal{LPT}_X or \mathcal{RPT}_X . Moreover these topologies are compatible with S and $\mathcal{FM}(S) = \mathcal{HM}(S) = \mathcal{LPT}_X \cap \mathcal{RPT}_X$.

Proof. If X is finite then the result is clear as all of these topologies are discrete, so we assume without loss of generality that X is infinite. As \mathcal{PPT}_X is a Hausdorff semigroup topology on \mathcal{P}_X (Lemma 9.6), it follows that \mathcal{LPT}_X is Hausdorff and compatible with S . Moreover as the inversion map is an anti-isomorphism of S , it follows from Proposition 6.3 that \mathcal{RPT}_X is also Hausdorff and compatible with S . Thus by the definitions of $\mathcal{FM}(S)$ and $\mathcal{HM}(S)$, it follows that $\mathcal{FM}(S) \subseteq \mathcal{HM}(S) \subseteq \mathcal{LPT}_X \cap \mathcal{RPT}_X$.

To conclude the second part of the lemma, it therefore suffices to show that $\mathcal{LPT}_X \cap \mathcal{RPT}_X \subseteq \mathcal{FM}(S)$. This follows from the first part of the lemma which we will now show.

Let \mathcal{T} be an arbitrary Fréchet topology compatible with S and let $x \in X$ be fixed. Recall that by Proposition 6.3, the topology \mathcal{T}^{-1} is also a Fréchet and compatible with S . By Lemma 9.14, we know that $\mathcal{CPT}_X \subseteq \mathcal{T}$. In particular the set $V := \left\{ f \in S \mid (x, x) \notin f \right\}$ is open with respect to \mathcal{T} . As $\emptyset \circ \emptyset = \emptyset \in V$ and \mathcal{T} is compatible with S , it follows that there is an open neighbourhood U of \emptyset (in \mathcal{T}) such that $UU \subseteq V$.

For each $W \subseteq S$, we define

$$A_W := X \setminus \bigcup_{a \in W} (\{x\})a, \quad \text{and} \quad B_W := X \setminus \bigcup_{b \in W} (\{x\})b^{-1}.$$

As $UU \subseteq V$, the sets $\bigcup_{a \in U} (\{x\})a$ and $\bigcup_{b \in U} (\{x\})b^{-1}$ are disjoint, and so $X = A_U \cup B_U$. As $A_U = B_{U^{-1}}$ and $B_U = A_{U^{-1}}$, it follows that either $|B_U| = |X|$ or $|B_{U^{-1}}| = |X|$. Let $U' \in \{U, U^{-1}\}$ be such that $|B_{U'}| = |X|$.

Let $B \subseteq B_{U'}$ be such that $|B| = |X| = |X \setminus B|$, and let $\phi' : B \rightarrow (X \setminus B)$ be a bijection. It follows that $\phi := \phi' \cup \phi'^{-1} \in \text{Sym}(X) \subseteq \mathcal{I}_X$, and $\phi = \phi^{-1}$.

If $f \in U'$, then by the choice of B we have $(\{x\})f^{-1} \cap B = \emptyset$. So if $f = \phi g \in U' \cap (U')\lambda_\phi$, then $(\{x\})f^{-1} = (\{x\})g^{-1}\phi = \emptyset$.

As $\lambda_\phi = \lambda_\phi^{-1}$, the set $U'' := U' \cap (U')\lambda_\phi$ is open with respect to either \mathcal{T} or \mathcal{T}^{-1} . As shown above, we also have $\emptyset \in U'' \subseteq \{f \in S \mid x \notin \text{img}(f)\}$. Thus if $f \in S$ is arbitrary, then

$$x \notin \text{img}(f) \Rightarrow f \circ \{(x, x)\} = \emptyset \Rightarrow (f)\rho_{\{(x, x)\}} \in U'' \Rightarrow x \notin \text{img}(f).$$

It follows that $(U'')\rho_{\{(x, x)\}}^{-1} = \{f \in S \mid x \notin \text{img}(f)\}$, thus this set is either open in \mathcal{T} or \mathcal{T}^{-1} . If $y \in X$ is arbitrary and $f \in \text{Sym}(X)$ is such that $f^{-1} = f$ and $(x)f = y$, then

$$\left(\{f \in S \mid x \notin \text{img}(f)\}\right)\rho_f^{-1} = \left(\{f \in S \mid x \notin \text{img}(f)\}\right)\rho_f = \{f \in S \mid y \notin \text{img}(f)\}.$$

So we either have $\mathcal{RPT}_X \subseteq \mathcal{T}$ or $\mathcal{RPT}_X \subseteq \mathcal{T}^{-1}$. As $\mathcal{RPT}_X \subseteq \mathcal{T}^{-1}$ if and only if $\mathcal{LPT}_X \subseteq \mathcal{T}$, the result follows. \square

Lemma 9.16 (The inverse pointwise topologies are nice, Assumed Knowledge: 6.4, 6.7, 9.13, 9.15). *If X is a set then $\mathcal{FM}(\mathcal{I}_X) = \mathcal{HM}(\mathcal{I}_X) = \mathcal{IPT}_X$ and this topology is compatible with \mathcal{I}_X .*

Proof. Let S be the semigroup obtained by removing the unary operation of \mathcal{I}_X . By Lemma 9.15, both \mathcal{LPT}_X and \mathcal{RPT}_X are compatible with S . So by Lemma 6.4, it follows that \mathcal{IPT}_X is compatible with S . By the definition of \mathcal{IPT}_X , the inversion map is also continuous with respect to this topology, so \mathcal{IPT}_X is compatible with \mathcal{I}_X as well.

From Lemma 9.15, it follows that every Fréchet topology compatible with \mathcal{I}_X contains either \mathcal{LPT}_X or \mathcal{RPT}_X . As inversion is required to be continuous and these topologies are mapped to each other by the inverse map, it follows that every Fréchet topology compatible with \mathcal{I}_X contains \mathcal{IPT}_X .

Thus $\mathcal{IPT}_X \subseteq \mathcal{FM}(\mathcal{I}_X) \subseteq \mathcal{HM}(\mathcal{I}_X) \subseteq \mathcal{IPT}_X$, so we have equality throughout as required. \square

Lemma 9.17 (Property **W** for \mathcal{I}_X , Assumed Knowledge: 7.5, 7.6, 9.13).

*If X is a set and S is the semigroup obtained by removing the unary operation from \mathcal{I}_X , then (S, \mathcal{IPT}_X) has property **W** with respect to $\text{Sym}(X)$.*

Proof. It suffices to show for all $s \in S$, that there are $f_s, g_s \in S$ and $h_s \in \text{Sym}(X)$ such that for all $N \in \text{Nbhd}_{\text{Sym}(X)}(h_s)$ we have $f_s N g_s \in \text{Nbhd}_S(s)$. If X is finite then S is discrete, so we can choose $f_s = h_s = 1_S, g_s = s$. Thus we can assume that X is infinite.

Let $s \in S$ be arbitrary. Let $\{M_0, M_1\}$ be a partition of X such that $|M_0| = |M_1| = X$, and choose $f_s : X \rightarrow M_0$ to be a bijection. We then define $g_s := f_s^{-1}$.

Claim: Suppose that $k \in S$ and a is a bijection between finite subsets of X satisfying:

- (1) $f_s a g_s \subseteq k$.
- (2) $\text{dom}(f_s a) \cap \text{dom}(k) \subseteq \text{dom}(f_s a g_s)$.
- (3) $\text{img}(a g_s) \cap \text{img}(k) \subseteq \text{img}(f_s a g_s)$.

In this case there is some $h_{k,a} \in \text{Sym}(X)$ such that $a \subseteq h_{k,a}$ and $f_s h_{k,a} g_s = k$.

Proof of Claim: If $x \in (\text{dom}(k))f_s \cap \text{dom}(a)$, then $(x)f_s^{-1} \in \text{dom}(f_s a) \cap \text{dom}(k) \subseteq \text{dom}(f_s a g_s)$. Thus $(x)f_s^{-1} k g_s^{-1} = (x)f_s^{-1} f_s a g_s g_s^{-1} = (x)a$. By a symmetric argument, if $x \in (\text{img}(k))g_s^{-1} \cap \text{img}(a)$, then $(x)(f_s^{-1} k g_s^{-1})^{-1} = (x)a^{-1}$. Thus if we define $b := f_s^{-1} k g_s^{-1} \setminus (\text{dom}(a) \times \text{img}(a))$, then $b = f_s^{-1} k g_s^{-1} \setminus a$.

In particular

$$\text{dom}(b) = (\text{dom}(k))f_s \setminus \text{dom}(a) \quad \text{and} \quad \text{img}(b) = (\text{img}(k))g_s^{-1} \setminus \text{img}(a).$$

As $\text{img}(b) \subseteq M_0$, and a is finite, it follows that $|M_1 \setminus (\text{img}(a) \cup \text{img}(b))| = |X|$. Thus we can choose an injection $c : M_0 \setminus (\text{dom}(a) \cup \text{dom}(b)) \rightarrow (M_1 \setminus (\text{img}(a) \cup \text{img}(b)))$ such that $|X \setminus (\text{img}(a) \cup \text{img}(b) \cup \text{img}(c))| = |X|$.

Similarly, as $\text{dom}(a)$ is finite and $\text{dom}(b) \cup \text{dom}(c) \subseteq M_0$, we can choose a bijection $d : X \setminus (\text{dom}(a) \cup \text{dom}(b) \cup \text{dom}(c)) \rightarrow X \setminus (\text{img}(a) \cup \text{img}(b) \cup \text{img}(c))$.

As each of a, b, c, d is a bijection between subsets of X and

$$\{\text{dom}(a), \text{dom}(b), \text{dom}(c), \text{dom}(d)\} \quad \text{and} \quad \{\text{img}(a), \text{img}(b), \text{img}(c), \text{img}(d)\}$$

are partitions of X , it follows that $h_{k,a} := a \cup b \cup c \cup d \in \text{Sym}(X)$.

By definition $a \subseteq h_{k,a}$, it remains to show that $f_s h_{k,a} g_s = k$. Let $x \in X$ be arbitrary, there are 5 cases to consider:

- (1) If $(x)f_s \in \text{dom}(a)$ and $(x)f_s a \in \text{dom}(g_s)$, then $x \in \text{dom}(f_s a g_s)$ and hence

$$(x)f_s h_{k,a} g_s = (x)f_s a g_s = (x)k.$$

- (2) If $(x)f_s \in \text{dom}(a)$, $(x)f_s a \notin \text{dom}(g_s)$ and $x \in \text{dom}(k)$, then

$$x \in (\text{dom}(f_s a) \cap \text{dom}(k)) \setminus \text{dom}(f_s a g_s) = \emptyset$$

a contradiction.

- (3) If $(x)f_s \in \text{dom}(a)$, $(x)f_s a \notin \text{dom}(g_s)$ and $x \notin \text{dom}(k)$, then

$$(\{x\})k = \emptyset = (\{(x)f_s a\})g_s = (\{x\})f_s h_{k,a} g_s.$$

- (4) If $(x)f_s \in \text{dom}(b)$, then $x \in \text{dom}(f_s b g_s)$ and hence

$$(x)f_s h_{k,a} g_s = (x)f_s b g_s = (x)f_s f_s^{-1} k g_s^{-1} g_s = (x)k.$$

- (5) If $(x)f_s \in \text{dom}(c)$, then $(x)f_s \notin \text{dom}(a) \cup ((\text{dom}(k))f_s \setminus \text{dom}(a))$. Thus $(\{x\})k = \emptyset$. Also by the choice of c , we have

$$(\{x\})f_s h_{k,a} g_s = (\{x\})f_s c g_s = (\{(x)f_s c\})g_s = \emptyset. \diamond$$

From the claim, using $a = \emptyset$ and $s = k$, let $h_s \in \text{Sym}(X)$ be such that $f_s h_s g_s = s$. Let $N \in \text{Nbhd}_{\text{Sym}(X)}(s)$ be arbitrary, we need only show that $f_s N g_s \in \text{Nbhd}_S(s)$.

Let a be a finite bijection between subsets of X such that $a \subseteq h_s$ and $\left\{ h \in \text{Sym}(X) \mid a \subseteq h \right\} \subseteq N$. We now define

$$U := \left\{ k \in S \mid \begin{array}{l} f_s a g_s \subseteq k, \text{dom}(f_s a) \cap \text{dom}(k) \subseteq \text{dom}(f_s a g_s), \\ \text{and } \text{img}(a g_s) \cap \text{img}(k) \subseteq \text{img}(f_s a g_s). \end{array} \right\}$$

Note that U is open with respect to \mathcal{IPT}_X . We first show that $s \in U$:

- (1) By the definitions of a and h_s we have $f_s a g_s \subseteq f_s h_s g_s = s$.
- (2) If $x \in \text{dom}(f_s a) \cap \text{dom}(k)$, then $(x)k = (x)f_s h_s g_s = (x)f_s a g_s$. So $x \in \text{dom}(f_s a g_s)$.
- (3) If $x \in \text{img}(a g_s) \cap \text{img}(k)$, then $(x)k^{-1} = (x)(f_s h_s g_s)^{-1} = (x)(f_s a g_s)^{-1}$. So $x \in \text{img}(f_s a g_s)$.

Thus $U \in \text{Nbhd}_S(s)$. We need only show that $U \subseteq f_s N g_s$. Let $k \in U$ be arbitrary. By the claim, there is $h_{k,a} \in \text{Sym}(X)$ such that $a \subseteq h_{k,a}$ and $f_s h_{k,a} g_s = k$. By the definition of a , it follows that $h_{k,a} \in N$. Thus $k = f_s h_{k,a} g_s \in f_s N g_s$ as required. \square

Theorem 9.18 (The topologies of \mathcal{I}_X , Assumed Knowledge: 4.20, 5.39, 6.15, 9.10, 9.14, 9.16, 9.17). *The topology $\mathcal{IPT}_{\mathbb{N}}$ is Polish and is the only second countable Fréchet topology compatible with the inverse semigroup $\mathcal{I}_{\mathbb{N}}$. Moreover*

$$\mathcal{CPT}_{\mathbb{N}} = \mathcal{MF}(\mathcal{I}_{\mathbb{N}}) \subsetneq \mathcal{FM}(\mathcal{I}_{\mathbb{N}}) = \mathcal{HM}(\mathcal{I}_{\mathbb{N}}) = \mathcal{SCC}(\mathcal{I}_{\mathbb{N}}) = \mathcal{IPT}_{\mathbb{N}}.$$

If S is the semigroup obtained by removing the unary operation of $\mathcal{I}_{\mathbb{N}}$, then

$$\mathcal{CPT}_{\mathbb{N}} = \mathcal{MF}(S) \subsetneq \mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}} = \mathcal{FM}(S) = \mathcal{HM}(S) \subsetneq \mathcal{SCC}(S) = \mathcal{IPT}_{\mathbb{N}}$$

and all homomorphisms from $(S, \mathcal{IPT}_{\mathbb{N}})$ to second countable topological semigroups are continuous.

Proof. From Theorem 9.10, the space $(\mathcal{P}_{\mathbb{N}}, \mathcal{PPT}_{\mathbb{N}})$ is Polish. As

$$\mathcal{P}_{\mathbb{N}} \setminus \mathcal{I}_{\mathbb{N}} = \left\{ f \in \mathcal{P}_{\mathbb{N}} \mid \text{there are } x, y \in \mathbb{N} \text{ with } x \neq y \text{ and } (x)f = (y)f \right\},$$

it follows that $\mathcal{I}_{\mathbb{N}}$ is closed with respect to $\mathcal{PPT}_{\mathbb{N}}$ and is hence $\mathcal{LPT}_{\mathbb{N}}$ is Polish by Lemma 4.18. As $\mathcal{IPT}_{\mathbb{N}}$ is generated by $\mathcal{LPT}_{\mathbb{N}}$ and the sets of the form

$$\left\{ f \in \mathcal{I}_{\mathbb{N}} \mid x \notin \text{img}(f) \right\}$$

for $x \in \mathbb{N}$ (which are closed with respect to $\mathcal{LPT}_{\mathbb{N}}$) it follows from Corollary 4.20 that $\mathcal{IPT}_{\mathbb{N}}$ is Polish.

By Proposition 6.12, we have $\mathcal{SCC}(\mathcal{I}_{\mathbb{N}}) = \mathcal{SCC}(S)$. Moreover as $\mathcal{IPT}_{\mathbb{N}}$ is second countable and compatible with S (Lemma 9.16), it follows from the definition of $\mathcal{SCC}(S)$, that $\mathcal{IPT}_{\mathbb{N}} \subseteq \mathcal{SCC}(S)$.

Let ϕ be an arbitrary homomorphism from $(S, \mathcal{IPT}_{\mathbb{N}})$ to a second countable topological inverse semigroup. To conclude that $\mathcal{IPT}_{\mathbb{N}} \supseteq \mathcal{SCC}(S)$, it suffices (by Proposition 6.12) to show that ϕ is continuous. By Theorem 5.39, we have that $\phi \upharpoonright_{\text{Sym}(\mathbb{N})}$ is continuous. Thus from Lemma 7.6 it suffices to show that S has property **W** with respect to $\text{Sym}(\mathbb{N})$. This is shown in Lemma 9.17.

We now have $\mathcal{IPT}_{\mathbb{N}} = \mathcal{SCC}(S) = \mathcal{SCC}(\mathcal{I}_{\mathbb{N}})$. By Lemma 9.16, it follows that this topology coincides with both of $\mathcal{FM}(\mathcal{I}_{\mathbb{N}})$ and $\mathcal{HM}(\mathcal{I}_{\mathbb{N}})$ as well. As every Fréchet second countable topology compatible with $\mathcal{I}_{\mathbb{N}}$, contains $\mathcal{FM}(\mathcal{I}_{\mathbb{N}})$ and is contained in $\mathcal{SCC}(\mathcal{I}_{\mathbb{N}})$, they all coincide. From Lemma 9.15 we have $\mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}} = \mathcal{FM}(S) = \mathcal{HM}(S)$, and from Lemma 9.14 we have $\mathcal{CPT}_{\mathbb{N}} = \mathcal{MF}(\mathcal{I}_X) = \mathcal{MF}(S)$.

It remains to show the containments. The non-strict versions of the containments are immediate from Proposition 6.15. So we need only show strictness of the containments. We start by showing that $\mathcal{IPT}_{\mathbb{N}} \neq \mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}}$. The set $U := \left\{ f \in \mathcal{I}_{\mathbb{N}} \mid 0 \notin \text{dom}(f) \right\}$ is open with respect to $\mathcal{LPT}_{\mathbb{N}}$. Moreover $\emptyset \in U$. If $V \in \mathcal{RPT}_{\mathbb{N}}$ is arbitrary such that $\emptyset \in V$, then by the definition of $\mathcal{RPT}_{\mathbb{N}}$, there is some finite $F \subseteq \mathbb{N}$ such that $\left\{ f \in \mathcal{I}_X \mid F \cap \text{img}(f) = \emptyset \right\} \subseteq V$. In particular if $n \in \mathbb{N} \setminus F$, then $\{(0, n)\} \in V \setminus U$. Thus $V \neq U$ and $U \in \mathcal{LPT}_{\mathbb{N}} \setminus \mathcal{RPT}_{\mathbb{N}} \subseteq \mathcal{IPT}_{\mathbb{N}} \setminus (\mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}})$.

We finally show $\mathcal{CPT}_{\mathbb{N}} \neq \mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}}$. Let $V := \left\{ \{(0, n), (n, 0)\} \mid n \in \mathbb{N} \right\}$ and note that

$$\mathcal{I}_{\mathbb{N}} \setminus V = \left\{ f \in \mathcal{I}_X \mid \begin{array}{l} \text{either } 0 \notin \text{dom}(f), \text{ there is } (a, b) \in ((\mathbb{N} \setminus \{0\}) \times (\mathbb{N} \setminus \{0\})) \\ \text{with } (a, b) \in f \text{ or there is } n \notin \text{dom}(f) \text{ with } (0, n) \in f \end{array} \right\}.$$

So $\mathcal{I}_{\mathbb{N}} \setminus V \in \mathcal{LPT}_{\mathbb{N}}$. Thus $\mathcal{I}_{\mathbb{N}} \setminus V = (\mathcal{I}_{\mathbb{N}} \setminus V)^{-1} \in (\mathcal{LPT}_{\mathbb{N}})^{-1} = \mathcal{RPT}_{\mathbb{N}}$. So $\mathcal{I}_{\mathbb{N}} \setminus V$ is a neighbourhood of \emptyset with respect to $\mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}}$. However every neighbourhood of \emptyset with respect to $\mathcal{CPT}_{\mathbb{N}}$ contains a set of the form

$$U_F := \left\{ f \in \mathcal{I}_{\mathbb{N}} \mid F \cap f = \emptyset \right\}$$

where $F \subseteq \mathbb{N} \times \mathbb{N}$ is finite. As all of the sets U_F intersect V , it follows that $\mathcal{I}_{\mathbb{N}} \setminus V$ is not a neighbourhood of \emptyset with respect to $\mathcal{CPT}_{\mathbb{N}}$ so $\mathcal{CPT}_{\mathbb{N}} \neq \mathcal{LPT}_{\mathbb{N}} \cap \mathcal{RPT}_{\mathbb{N}}$ as required. \square

We have now established the main result of this subsection (Theorem 9.18). The category of partial bijections does not have product objects, so we have no clones in this subsection. However, we can use this result to distinguish the left/right small index properties discussed earlier.

Example 9.19 (The small index properties are distinct, Assumed Knowledge: 7.4, 9.18).

If S is the semigroup obtained by removing the unary operation of $\mathcal{I}_{\mathbb{N}}$, then the topological semigroup $(S, \mathcal{LPT}_{\mathbb{N}})$ has the right small index property but not the left small index property.

Proof. We first show that $(S, \mathcal{LPT}_{\mathbb{N}})$ has the right small index property. By Theorem 9.18, we have that $\mathcal{IPT}_{\mathbb{N}} = \text{SCC}(S)$. Thus by Corollary 7.4, the topological semigroup $(S, \mathcal{IPT}_{\mathbb{N}})$ has the right small index property. Let \sim be an arbitrary right congruence on S , and let $s \in S$ be arbitrary. To conclude that $(S, \mathcal{LPT}_{\mathbb{N}})$ has the right small index property, it suffices to show that $[s]_{\sim}$ is a neighbourhood of s with respect to the topology $\mathcal{LPT}_{\mathbb{N}}$.

As $(S, \mathcal{IPT}_{\mathbb{N}})$ has the right small index property, the set $[s]_{\sim}$ is open in $\mathcal{IPT}_{\mathbb{N}}$. Thus there are finite $A, B \subseteq X$ and a finite bijection a between subsets of \mathbb{N} such that if

$$V := \left\{ f \in S \mid A \cap \text{dom}(f) = \emptyset, B \cap \text{img}(f) = \emptyset \text{ and } a \subseteq f \right\},$$

then $s \in V \subseteq [s]_{\sim}$. Let $U := \left\{ f \in S \mid A \cap \text{dom}(f) = \emptyset \text{ and } a \subseteq f \right\}$. By construction, U is a neighbourhood of s with respect to $\mathcal{LPT}_{\mathbb{N}}$. Let $u \in U$ be arbitrary. To conclude that $(S, \mathcal{LPT}_{\mathbb{N}})$ has the right small index property, it suffices to show that $u \in [s]_{\sim}$. Let $g : \mathbb{N} \rightarrow \mathbb{N} \setminus B$ be a bijection such that $g \upharpoonright_{\text{img}(a)}$ is the identity map. Note that $ug, sg \in V \subseteq [s]_{\sim}$, so $ug \sim sg$. As \sim is a right congruence, it follows that $s = sgg^{-1} \sim ugg^{-1} = u$. So $u \sim s$ as required.

It remains to show that $(S, \mathcal{LPT}_{\mathbb{N}})$ does not have the left small index property. We define a left congruence \sim on S by

$$\sim := \left\{ (f, g) \in S \times S \mid (\{0\})f^{-1} = (\{0\})g^{-1} \right\}.$$

Note that $[\emptyset]_{\sim} = \left\{ f \in S \mid 0 \notin \text{dom}(f) \right\}$. We show in the proof of Theorem 9.18 that this set is not open with respect to $\mathcal{LPT}_{\mathbb{N}}$. Thus \sim is not open with respect to $\mathcal{LPT}_{\mathbb{N}}$ and $(S, \mathcal{LPT}_{\mathbb{N}})$ does not have the left small index property. \square

9.4. Injective functions. In this subsection we discuss injective function monoids (which notably contain all embedding monoids of structures). While most of the monoids we explore in this document only admit one compatible Polish topology, in the case of $\text{Inj}(\mathbb{N})$ we show that there are infinitely many (see Theorems 9.21 and 9.23). Also unlike the previous subsections, we are unable to give nice descriptions of $\mathcal{MF}(\text{Inj}(\mathbb{N}))$ or $\mathcal{FM}(\text{Inj}(\mathbb{N}))$.

Example 9.20 (Topologies on the injective function monoid, Assumed Knowledge: 3.2, 4.13, 9.13). We define the *injective function monoid* $\text{Inj}(\mathbb{N})$ to be the monoid of injective functions from \mathbb{N} to itself with composition of binary relations as the operation. We will consider the following topologies on this semigroup:

- (1) The topology $\mathcal{PT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ generated by the sets of the form

$$U_{x,y} := \left\{ f \in \text{Inj}(\mathbb{N}) \mid (x, y) \in f \right\}$$

for all $x, y \in X$.

- (2) The topology $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ generated by $\mathcal{PT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$, as well as the sets of the form

$$V_x := \left\{ f \in \text{Inj}(\mathbb{N}) \mid x \notin \text{img}(f) \right\}$$

for all $x, y \in X$.

- (3) The *coimage pointwise topology* $\mathcal{CIPT}_{\mathbb{N}}$ generated by $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ as well as the sets of the form

$$W_n := \left\{ f \in \text{Inj}(\mathbb{N}) \mid |\mathbb{N} \setminus \text{img}(f)| = n \right\}$$

for each $n \in \mathbb{N} \cup \{\omega\}$.

Theorem 9.21 (Much Polish, Assumed Knowledge: 4.19, 6.3, 6.4, 9.10, 9.18, 9.20).

There is a collection $\{\mathcal{T}_i \mid i \leq \omega + 1\}$ of Polish topologies compatible with $\text{Inj}(\mathbb{N})$ such that

$$\mathcal{PT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})} \subsetneq \mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})} = \mathcal{T}_0 \subsetneq \mathcal{T}_1 \subsetneq \mathcal{T}_2 \subsetneq \mathcal{T}_3 \dots \subsetneq \mathcal{T}_\omega \subsetneq \mathcal{T}_{\omega+1} = \mathcal{CIPT}_{\mathbb{N}}.$$

Moreover the topology $\mathcal{PT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ is Polish.

Proof. As $\mathcal{PT}_{\mathbb{N}}$ is compatible with $\mathbb{N}^{\mathbb{N}}$ and $\mathcal{IPT}_{\mathbb{N}}$ is compatible with $\mathcal{I}_{\mathbb{N}}$, both of these topologies are compatible with $\text{Inj}(\mathbb{N})$. From Theorems 9.10 and 9.18, the topologies $\mathcal{PT}_{\mathbb{N}}$ and $\mathcal{IPT}_{\mathbb{N}}$ are Polish. As

$$\mathbb{N}^{\mathbb{N}} \setminus \text{Inj}(\mathbb{N}) = \left\{ f \in \mathbb{N}^{\mathbb{N}} \mid \text{there are } x, y \in \mathbb{N} \text{ with } x \neq y \text{ and } (x)f = (y)f \right\},$$

$$\mathcal{I}_{\mathbb{N}} \setminus \text{Inj}(\mathbb{N}) = \left\{ f \in \mathcal{I}_{\mathbb{N}} \mid \text{there is } x \in \mathbb{N} \setminus \text{dom}(f) \right\},$$

the set $\text{Inj}(\mathbb{N})$ is closed in each of these Polish spaces. Thus by Lemma 4.18, the topologies $\mathcal{PT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ and $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ are Polish. As the set $\left\{ f \in \text{Inj}(\mathbb{N}) \mid 0 \notin \text{img}(f) \right\}$ is not open with respect to $\mathcal{PT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$, these topologies are also distinct.

For each $i \leq \omega + 1$, let \mathcal{T}_i be the topology generated by $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ as well as the sets $\left\{ W_j \mid j < i \right\}$ (W_i is as in Example 9.20).

We need to show that these topologies are Polish, compatible with $\text{Inj}(\mathbb{N})$, and distinct. We first show that they are Polish. Suppose for a contradiction that there is $n \in \mathbb{N} \cup \{\omega, \omega + 1\}$ such that \mathcal{T}_n is not Polish, and let n be the smallest such value. There are 3 cases to consider:

- (1) If $n = 0$, then \mathcal{T}_n is defined to be $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ which we have already shown is Polish.
- (2) If $n = \omega$, then \mathcal{T}_ω is the topology generated by the topologies $(\mathcal{T}_i)_{i < \omega}$ which are all Polish. From Lemma 4.19, it follows that \mathcal{T}_ω is Polish as well.
- (3) If $n = m + 1$ for some $m \in \mathbb{N} \cup \{\omega\}$, then \mathcal{T}_m is Polish and

$$\text{Inj}(\mathbb{N}) \setminus W_m = \left(\bigcup_{i < m} W_i \right) \cup \left(\bigcup_{\substack{A \subseteq \mathbb{N} \\ n < |A| < \infty}} \left(\bigcap_{a \in A} \left\{ f \in \text{Inj}(\mathbb{N}) \mid a \notin \text{img}(f) \right\} \right) \right).$$

So W_m is closed with respect to \mathcal{T}_m . As \mathcal{T}_n is generated by \mathcal{T}_m and W_m , it follows from Corollary 4.20 that \mathcal{T}_m is Polish as well.

We next show that they are distinct. It suffices to show that if $n \in \mathbb{N} \cup \{\omega\}$, then W_n is not open with respect to \mathcal{T}_n . Let $f \in W_n$ and suppose that $U \in \mathcal{T}_n$ is such that $f \in U$. As f is not an element

of any of the sets W_m with $m \neq n$, it follows that there is finite $a \subseteq f$ and $B \subseteq \mathbb{N} \setminus \text{img}(a)$ such that

$$U \supseteq \left\{ g \in \text{Inj}(\mathbb{N}) \mid a \subseteq g \text{ and } \text{img}(g) \cap B = \emptyset \right\}$$

If $b : \mathbb{N} \setminus \text{dom}(a) \rightarrow \mathbb{N} \setminus (\text{img}(a) \cup B)$ is any bijection, then $a \cup b \in U \setminus W_m$. In particular $U \neq W_n$. As U is arbitrary, it follows that $W_n \notin \mathcal{T}_n$.

It remains to show that all of these topologies are compatible with $\text{Inj}(\mathbb{N})$. For each $n \in \mathbb{N} \cup \{\omega\}$, let $(\mathbb{N}_n, \mathcal{D}_n)$ be the discrete topological semigroup with universe $\left\{ i \in \mathbb{N} \mid i < n \right\} \cup \{n\}$ and binary operation $+_n$ defined by $i +_n j = \min(n, i + j)$. Define $\phi_n : \text{Inj}(\mathbb{N}) \rightarrow \mathbb{N}_n$ by $(f)\phi_n = \min(n, |\mathbb{N} \setminus \text{img}(f)|)$. If $f, g \in \text{Inj}(\mathbb{N})$, then $\mathbb{N} \setminus \text{img}(fg) = (\mathbb{N} \setminus \text{img}(f))g \cup (\mathbb{N} \setminus \text{img}(g))$, so each map ϕ_n is a semigroup homomorphism. From Proposition 6.3, each of the topologies $(\mathcal{D}_n)\phi_n^{-1}$ is compatible with $\text{Inj}(\mathbb{N})$. Thus for each $i \in \mathbb{N}$, it follows from Lemma 6.4 that the topology generated by $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ and $(\mathcal{D}_n)\phi_n^{-1}$ is compatible with $\text{Inj}(\mathbb{N})$.

As the topology generated by $\mathcal{IPT}_{\mathbb{N}} \upharpoonright_{\text{Inj}(\mathbb{N})}$ and $(\mathcal{D}_n)\phi_n^{-1}$ is the topology \mathcal{T}_{n+1} , it remains only to show that \mathcal{T}_ω is compatible with $\text{Inj}(\mathbb{N})$. As \mathcal{T}_ω is generated by the topologies \mathcal{T}_n for $n \in \mathbb{N}$, this follows from Lemma 6.4. \square

The following property **W** proof is significantly easier than the previous ones we have seen so far, and unlike the previous ones we actually use different term operations for different elements.

Lemma 9.22 (Property **W** for $\text{Inj}(\mathbb{N})$, Assumed Knowledge: 7.5, 9.21).

*The topological semigroup $(\text{Inj}(\mathbb{N}), \mathcal{CIPT}_{\mathbb{N}})$ has property **W** with respect to $\text{Sym}(\mathbb{N})$.*

Proof. It suffices to show that for all $s \in \text{Inj}(\mathbb{N})$, there is some $f_s \in \text{Inj}(\mathbb{N})$ and $h_s \in \text{Sym}(\mathbb{N})$ such that $f_s h_s = s$ and for all $N \in \text{Nbhd}_{\text{Sym}(\mathbb{N})}(h_s)$ we have $f_s N \in \text{Nbhd}_{\text{Inj}(\mathbb{N})}(s)$.

Let $s \in \text{Inj}(\mathbb{N})$ be arbitrary. We define $f_s := s$ and $h_s := 1_{\text{Inj}(\mathbb{N})}$. Let $N \in \text{Nbhd}_{\text{Sym}(\mathbb{N})}(h_s)$ be arbitrary. As $f_s h_s = s 1_{\text{Inj}(\mathbb{N})} = s$, it suffices to show that $f_s N \in \text{Nbhd}_{\text{Inj}(\mathbb{N})}(s)$.

As $N \in \text{Nbhd}_{\text{Sym}(\mathbb{N})}(1_{\text{Inj}(\mathbb{N})})$, there is some finite $A \subseteq \mathbb{N}$ such that

$$N \subseteq \left\{ h \in \text{Sym}(\mathbb{N}) \mid (a)h = a \text{ for all } a \in A \right\}.$$

Let U be the neighbourhood

$$\left\{ g \in \text{Inj}(\mathbb{N}) \mid (\{a\})g^{-1} = (\{a\})f^{-1} \text{ for all } a \in A \text{ and } |\mathbb{N} \setminus \text{img}(g)| = |\mathbb{N} \setminus \text{img}(f)| \right\}$$

of f , and let $g \in U$ be arbitrary. It suffices to show that $g \in f_s N$. We define $h' := f^{-1}g$. Note that h' is a bijection from $\text{img}(f)$ to $\text{img}(g)$ and $fh' = g$.

By the choice of g , we have that $\text{img}(g) \cap A = \text{img}(f) \cap A$ and $|\mathbb{N} \setminus \text{img}(g)| = |\mathbb{N} \setminus \text{img}(f)|$. Thus we can choose a bijection $b : \mathbb{N} \setminus (\text{img}(f) \cup A) \rightarrow \mathbb{N} \setminus (\text{img}(g) \cup A)$. Let $h := h' \cup b \cup \left\{ (a, a) \mid a \in A \setminus \text{img}(f) \right\}$. As h is a union of bijections whose domains and images each partition \mathbb{N} , it follows that $h \in \text{Sym}(\mathbb{N})$. As $h' \subseteq h$, we also have that $fh = fh' = g$.

It therefore suffices to show that $h \in N$. By the definition of A , we need only show that h fixes all elements of A . Let $a \in A$ be arbitrary. If $a \notin \text{img}(f)$, then $(a, a) \in h$ by construction. If $a \in \text{img}(f)$, then $(a)h = (a)h' = (a)f^{-1}g$. As $(\{a\})g^{-1} = (\{a\})f^{-1} = \{(a)f^{-1}\}$, it follows that $(a)h = (a)f^{-1}g = a$ as required. \square

Theorem 9.23 (The topologies of $\text{Inj}(\mathbb{N})$, Assumed Knowledge: 6.13, 6.15, 7.6, 9.21, 9.22).

All homomorphisms from the topological semigroup $(\text{Inj}(\mathbb{N}), \mathcal{CIP}\mathcal{T}_{\mathbb{N}})$ to second countable topological semigroups are continuous and

$$\mathcal{PT} \upharpoonright_{\text{Inj}(\mathbb{N})} = \mathcal{Z}(\text{Inj}(\mathbb{N})) = \mathcal{HM}(\text{Inj}(\mathbb{N})) \subsetneq \mathcal{SCC}(\text{Inj}(\mathbb{N})) = \mathcal{CIP}\mathcal{T}_{\mathbb{N}}.$$

Moreover, the above topologies are Polish.

Proof. From Theorem 9.21, it follows that $\mathcal{PT} \upharpoonright_{\text{Inj}(\mathbb{N})} \subsetneq \mathcal{CIP}\mathcal{T}_{\mathbb{N}}$ and both of these topologies are Polish. As the topologies are Polish it follows also (Proposition 6.15) that

$$\mathcal{Z}(\text{Inj}(\mathbb{N})) = \mathcal{HM}(\text{Inj}(\mathbb{N})) \subseteq \mathcal{PT} \upharpoonright_{\text{Inj}(\mathbb{N})} \subsetneq \mathcal{CIP}\mathcal{T}_{\mathbb{N}} \subseteq \mathcal{SCC}(\text{Inj}(\mathbb{N})).$$

From Corollary 6.13, we know that

$$\mathcal{SCC}(\text{Inj}(\mathbb{N})) \upharpoonright_{\text{Sym}(\mathbb{N})} \subseteq \mathcal{PT} \upharpoonright_{\text{Sym}(\mathbb{N})} = \mathcal{CIP}\mathcal{T}_{\mathbb{N}} \upharpoonright_{\text{Sym}(\mathbb{N})}.$$

Thus from Lemmas 7.6 and 9.22, it follows that $\mathcal{SCC}(\text{Inj}(\mathbb{N})) \subseteq \mathcal{CIP}\mathcal{T}_{\mathbb{N}}$. It remains to show that $\mathcal{PT} \upharpoonright_{\text{Inj}(\mathbb{N})} \subseteq \mathcal{Z}(\text{Inj}(\mathbb{N}))$.

Let $x, y \in \mathbb{N}$ be arbitrary. We need only show that $\{h \in \text{Inj}(\mathbb{N}) \mid (x)h \neq y\}$ is closed with respect to $\mathcal{Z}(\text{Inj}(\mathbb{N}))$. Let $f_0, f_1, g_0, g_1 \in \text{Inj}(\mathbb{N})$ be such that

$$\text{img}(f_0) \cap \text{img}(f_1) = \{x\} \quad \text{and} \quad (n)g_0 = (n)g_1 \iff n \neq y.$$

We then define

$$V_0 := \{h \in \text{Inj}(\mathbb{N}) \mid f_0 h g_0 = f_0 h g_1\}, \quad V_1 := \{h \in \text{Inj}(\mathbb{N}) \mid f_1 h g_0 = f_1 h g_1\}.$$

Note that both of these sets are elementary algebraic and hence closed with respect to $\mathcal{Z}(\text{Inj}(\mathbb{N}))$. From the choice of g_0, g_1 we have

$$f_0 h g_0 = f_0 h g_1 \iff (n)g_0 = (n)g_1 \text{ for all } n \in \text{img}(f_0 h) \iff y \notin \text{img}(f_0 h),$$

$$f_1 h g_0 = f_1 h g_1 \iff (n)g_0 = (n)g_1 \text{ for all } n \in \text{img}(f_1 h) \iff y \notin \text{img}(f_1 h).$$

So $V_0 = \{h \in \text{Inj}(\mathbb{N}) \mid y \notin \text{img}(f_0 h)\}$ and $V_1 = \{h \in \text{Inj}(\mathbb{N}) \mid y \notin \text{img}(f_1 h)\}$. In particular, the set

$$V_0 \cup V_1 = \{h \in \text{Inj}(\mathbb{N}) \mid y \notin \text{img}(f_0 h) \cap \text{img}(f_1 h)\}$$

is closed with respect to $\mathcal{Z}(\text{Inj}(\mathbb{N}))$. It therefore suffices to show that $y \notin \text{img}(f_0 h) \cap \text{img}(f_1 h) \iff (x)h \neq y$. As h is injective, we have

$$\text{img}(f_0 h) \cap \text{img}(f_1 h) = \text{img}(f_0)h \cap \text{img}(f_1)h = (\text{img}(f_0) \cap \text{img}(f_1))h.$$

From the choice of f_0 and f_1 , it follows that

$$\begin{aligned} y \notin \text{img}(f_0 h) \cap \text{img}(f_1 h) &\iff y \notin (\text{img}(f_0) \cap \text{img}(f_1))h \\ &\iff y \notin (\{x\})h \\ &\iff (x)h \neq y \end{aligned}$$

as required. \square

9.5. The Hilbert cube. In this subsection we explore the continuous transformations of the Hilbert cube $[0, 1]^{\mathbb{N}}$ (see Theorem 9.29 and Corollary 9.30). Unlike the previous subsections, the main theorem of this subsection has no automatic continuity component. However if one were to be established for the group $\text{Aut}([0, 1]^{\mathbb{N}})$, we could extend it to $C([0, 1]^{\mathbb{N}})$ similarly to the previous subsections.

Similarly to the proof of Lemma 9.7, we will think of the set of constant elements of a continuous function monoid as a copy of the space we are acting on. This gives us a means of algebraically reconstructing our desired topology (the compact-open topology). With this in mind the following pair of lemmas will be useful (in both this subsection and the Cantor space subsection).

Lemma 9.24 (Closed constants, Assumed Knowledge: 1.6, 6.9).

If S is a semigroup of functions with composition as the operation, then the set C of constant elements of S is closed with respect to $\mathcal{Z}(S)$.

Proof. If $C = \emptyset$, then the result is clear. Otherwise let $c \in C$ be fixed. By the definition of $\mathcal{Z}(S)$, the set

$$RO := \bigcap_{g \in S} \{f \in S \mid gf = f\}$$

is closed with respect to $\mathcal{Z}(S)$. It therefore suffices to show that $RO = C$. As C consists of constant maps, it follows that $C \subseteq RO$. Moreover if $f \in RO$ be arbitrary, then $cf = f$. As c is a constant map, it follows that cf is a constant map as well. Thus $f = cf \in C$ as required. \square

Lemma 9.25 (Containing the compact-open topology, Assumed Knowledge: 3.14, 5.13).

If X and Y are topological spaces, \mathcal{T} is a topology on the set $C(X, Y)$ and the function $a : X \times (C(X, Y), \mathcal{T}) \rightarrow Y$ defined by $(x, f)a = (x)f$ is continuous, then \mathcal{T} contains the compact-open topology (Recall Definition 5.11).

Proof. Let $f \in C(X, Y)$, compact $C \subseteq X$ and open $U \subseteq Y$ be arbitrary such that $(C)f \subseteq U$. It suffices to show that

$$V := \{g \in C(X, Y) \mid (C)g \subseteq U\}$$

is a neighbourhood of f with respect to \mathcal{T} .

As a is continuous, it follows that the set $(U)a^{-1}$ is open in $X \times (C(X, Y), \mathcal{T})$. Thus we can choose an index set I , a collection of open subsets $(U_i)_{i \in I}$ of X , and a collection of open subsets $(W_i)_{i \in I}$ of $(C(X, Y), \mathcal{T})$ such that

$$(U)a^{-1} = \bigcup_{i \in I} U_i \times W_i.$$

As C is compact and $C \times \{f\} \subseteq (U)a^{-1}$, there is a finite set $F \subseteq I$, such that

$$C \times \{f\} \subseteq \bigcup_{i \in F} U_i \times W_i \subseteq (U)a^{-1}.$$

We can assume without loss of generality that $f \in W_i$ for all $i \in F$. Let $W := \bigcap_{i \in F} W_i$, and note that W is a neighbourhood of f with respect to \mathcal{T} . By the definition of W , we have $C \times W \subseteq (U)a^{-1}$. So $f \in W \subseteq V$, and V is a neighbourhood of f as required. \square

The proof of the following lemma involves a somewhat unusual application of Theorem 4.27, in which we show that two spaces are homeomorphic by giving them a group structure.

Lemma 9.26 (Continuous action on $[0, 1]^{\mathbb{N}}$, Assumed Knowledge: 4.18, 4.20, 4.27, 6.15, 9.24).

Suppose that \mathcal{T} is a Polish topology compatible with the semigroup $C([0, 1]^{\mathbb{N}})$. In this case the map $a : [0, 1]^{\mathbb{N}} \times (C([0, 1]^{\mathbb{N}}), \mathcal{T}) \rightarrow [0, 1]^{\mathbb{N}}$ defined by $(x, f)a = (x)f$ is continuous.

Proof. Let C denote the set of constant maps from $[0, 1]^{\mathbb{N}}$ to itself. By Lemma 9.24, the set C is closed with respect to $\mathcal{Z}(C([0, 1]^{\mathbb{N}}))$. So by Proposition 6.15, it follows that C is closed with respect to \mathcal{T} . Thus by Lemma 4.18, the topology $\mathcal{T} \upharpoonright_C$ is Polish.

Let $I : C \rightarrow [0, 1]^{\mathbb{N}}$ be the bijection sending a constant map to the unique point in its image. Note that $a = \langle \pi_0 I^{-1}, \pi_1 \rangle_{C \times C([0, 1]^{\mathbb{N}})} \circ *^{C([0, 1]^{\mathbb{N}})} \circ I$. Thus to show that a is continuous, it suffices to show that $I : C \rightarrow [0, 1]^{\mathbb{N}}$ is a homeomorphism.

Claim: The map I is continuous.

Proof of claim: Let $a, b \in \mathbb{R}$, and $i \in \mathbb{N}$ be arbitrary with $a < b$. It suffices to show that the set

$$U := \left\{ f \in C \mid ((f)I)\pi_i \in (a, b) \right\}$$

is open with respect to $\mathcal{T} \upharpoonright_C$. For each $j \in \mathbb{N} \setminus \{i\}$, define $g_j \in C([0, 1]^{\mathbb{N}}, [0, 1])$ to be the constant map with value 0. We then define $g_i \in C([0, 1]^{\mathbb{N}}, [0, 1])$ to be the map with

$$(f)g_i = \begin{cases} (f)\pi_i & \text{if } (f)\pi_i \in (a, b) \\ a & \text{if } (f)\pi_i \leq a \\ b & \text{if } (f)\pi_i \geq b. \end{cases}$$

The map $g := \langle (g_j)_{j \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}}$ is continuous, and is thus an element of $C([0, 1]^{\mathbb{N}})$. Note that

$$U = \left\{ f \in C \mid ((f)\rho_g)I\pi_i \notin \{a, b\} \right\} = C \setminus (\{a, b\})(I\pi_i)^{-1} \cap \text{img}(\rho_g)\rho_g^{-1}.$$

As the map ρ_g is continuous and the set $(\{a, b\})(I\pi_i)^{-1} \cap \text{img}(\rho_g)$ has size two (and is hence closed), it follows that U is open with respect to $\mathcal{T} \upharpoonright_C$. \diamond

It remains to show that I^{-1} is continuous. For each $i \in \mathbb{N}$, we define $f_i, g_i : [0, 1] \rightarrow [0, 1]$ by

$$(x)f_i = \frac{x}{2} + \frac{1}{4}, \quad \text{and} \quad (x)g_i = \begin{cases} (x)f_i^{-1} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ 1 & \text{if } x \in [\frac{3}{4}, 1] \\ 0 & \text{if } x \in [0, \frac{1}{4}] \end{cases}.$$

We then define $f, g \in C([0, 1]^{\mathbb{N}})$ by $f = \langle (\pi_i f_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}}$ and $g = \langle (\pi_i g_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}}$. In particular $f g = \langle (\pi_i f_i g_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = \langle (\pi_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}}$ is the identity function, and $\text{img}(f) = [\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}} \subseteq (0, 1)^{\mathbb{N}}$. As $I^{-1} = f \circ I^{-1} \upharpoonright_{(0, 1)^{\mathbb{N}}} \circ \rho_g$, it suffices to show that $I \upharpoonright_{(0, 1)^{\mathbb{N}}}$ is continuous.

For each $i \in \mathbb{N}$, let $\phi_i : (0, 1) \rightarrow \mathbb{R}$ be an order preserving homeomorphism. We then define $\phi \in C((0, 1)^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}})$ to be $\langle (\pi_i \phi_i)_{i \in \mathbb{N}} \rangle_{\mathbb{R}^{\mathbb{N}}}$. This allows us to define a group structure on $((0, 1)^{\mathbb{N}})I^{-1}$ by $xy = ((x)I\phi + (y)I\phi)(I\phi)^{-1}$ and $x^{-1} = -(x)I\phi(I\phi)^{-1}$ for all $x, y \in (0, 1)^{\mathbb{N}}$. By definition (and the claim), the map $I\phi$ is a continuous group isomorphism from $((0, 1)^{\mathbb{N}})I^{-1}$ to $\mathbb{R}^{\mathbb{N}}$. As $I^{-1} \upharpoonright_{(0, 1)^{\mathbb{N}}} = \phi(I\phi)^{-1}$ and ϕ is continuous, it suffices to show that $(I\phi)^{-1}$ is continuous. Thus by Theorem 4.27, we need only show that $\mathcal{T} \upharpoonright_{((0, 1)^{\mathbb{N}})I^{-1}}$ is Polish and semicompatible with the group structure on $((0, 1)^{\mathbb{N}})I^{-1}$.

The set $(0, 1)^{\mathbb{N}}$ is a countable intersection of open subsets of $[0, 1]^{\mathbb{N}}$, so by the continuity of I , the set $((0, 1)^{\mathbb{N}})I^{-1}$ is a countable intersection of open subsets of C . As C is Polish, it follows from Corollary 4.20, that $((0, 1)^{\mathbb{N}})I^{-1}$ is Polish.

It remains to show that $\mathcal{T} \upharpoonright_{((0,1)^{\mathbb{N}})I^{-1}}$ is semicompatible with the group structure on $((0,1)^{\mathbb{N}})I^{-1}$. Let $h \in ((0,1)^{\mathbb{N}})I^{-1}$ be arbitrary. As $((0,1)^{\mathbb{N}})I^{-1}$ is commutative, it suffices to show that ρ_h is continuous with respect to $\mathcal{T} \upharpoonright_{((0,1)^{\mathbb{N}})I^{-1}}$ (here ρ_h is right multiplication by h in the group $((0,1)^{\mathbb{N}})I^{-1}$). Let $x \in ((0,1)^{\mathbb{N}})I^{-1}$ be arbitrary. We have

$$(x)\rho_h = ((x)I\phi + (h)I\phi)(I\phi)^{-1} = (((x)I\phi)\rho_{(h)I\phi})(I\phi)^{-1} = (x)I\phi\rho_{(h)I\phi}\phi^{-1}I^{-1}.$$

Let $h^* := \phi\rho_{(h)I\phi}\phi^{-1}$. For each $i \in \mathbb{N}$, define $h_i^* : (0,1) \rightarrow (0,1)$ to be the map $\phi_i\rho_{(h)I\pi_i\phi_i}\phi_i^{-1}$, and note that $h^* = \langle (\pi_i h_i^*)_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}}$. Thus for each $i \in \mathbb{N}$, the map $h'_i := h_i^* \cup \{(0,0), (1,1)\}$ is an order isomorphism (and hence homeomorphism) of $[0,1]$. We can thus define $h' \in C([0,1]^{\mathbb{N}})$ by $h' := \langle (h'_i)_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}}$. As $h^* \subseteq h'$, we have $(x)\rho_h = (x)Ih'I^{-1}$. As $x \in C$ and $h' \in C([0,1]^{\mathbb{N}})$, it follows that

$$(x)\rho_h = I\phi\rho_{(h)I\phi}\phi^{-1}I^{-1} = Ih^*I^{-1} = (x)\rho_{h'}$$

where ρ_h is using the group operation on $((0,1)^{\mathbb{N}})I^{-1}$ and $\rho_{h'}$ is using the semigroup operation on $C([0,1]^{\mathbb{N}})$. As $\rho_{h'}$ is continuous by assumption, it follows that ρ_h is also continuous as required. \square

As in the previous subsections, we need to establish property **W** for $C([0,1]^{\mathbb{N}})$. Most of the work required to establish this is in the proof of the following theorem from [34]. The proof is somewhat lengthy and only tangentially related to the theme of this document so we do not include it here.

Theorem 9.27 (cf. Theorem 5.2.4 of [34], Extending Hilbert homeomorphisms).

Let $d : [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \rightarrow \mathbb{R}$ be defined by

$$(x,y)d = \sum_{n \in \mathbb{N}} \frac{|(x)\pi_n - (y)\pi_n|}{2^{n+1}}.$$

If $\varepsilon > 0$, $C_0, C_1 \subseteq (0,1)^{\mathbb{N}}$ are compact and $h' : C_0 \rightarrow C_1$ is a homeomorphism with

$$\sup_{(x,y) \in h'} (x,y)d < \varepsilon,$$

then there is $h \in \text{Aut}([0,1]^{\mathbb{N}})$ such that $h' \subseteq h$ and $\sup_{(x,y) \in h} (x,y)d < \varepsilon$.

Theorem 9.28 (Property **W** for $C([0,1]^{\mathbb{N}})$, Assumed Knowledge: 1.1, 2.3, 3.11, 5.13, 7.5, 9.27).

The topological semigroup $C([0,1]^{\mathbb{N}})$ has property **W** with respect to $\text{Aut}([0,1]^{\mathbb{N}})$.

Proof. Let d be the metric on $[0,1]^{\mathbb{N}}$ given in Theorem 9.27, and let d_{∞} be the metric on $C([0,1]^{\mathbb{N}})$ given in Theorem 5.12. It suffices to show that for all $s \in C([0,1]^{\mathbb{N}})$, there are $f_s, g_s \in C([0,1]^{\mathbb{N}})$ and $h_s \in \text{Aut}([0,1]^{\mathbb{N}})$ such that $f_s h_s g_s = s$ and for all $N \in \text{Nbhds}_{\text{Aut}([0,1]^{\mathbb{N}})}(h_s)$, we have $f_s N g_s \in \text{Nbhds}_{C([0,1]^{\mathbb{N}})}(s)$.

Let $s \in C([0,1]^{\mathbb{N}})$ be arbitrary. Let $f', g' : [0,1] \rightarrow [0,1]$ be defined by

$$(x)f' = \frac{x}{2} + \frac{1}{4}, \quad \text{and} \quad (x)g' = \begin{cases} (x)f'^{-1} & \text{if } x \in [\frac{1}{4}, \frac{3}{4}] \\ 1 & \text{if } x \in [\frac{3}{4}, 1] \\ 0 & \text{if } x \in [0, \frac{1}{4}] \end{cases}.$$

We define $f_s, g_s \in C([0,1]^{\mathbb{N}})$ by $f_s := \langle (\pi_{\lfloor i/2 \rfloor} f')_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}}$, $g_s := \langle (\pi_{2i} g')_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}}$. Note that $f_s g_s = \langle (\pi_i f' g')_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}} = \langle (\pi_i)_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}}$ is the identity map.

For each $i \in 2\mathbb{N}$, we define $h'_i : \text{img}(f_s) \rightarrow [\frac{1}{4}, \frac{3}{4}]$ by $(x)h'_i = (x)g_s s \pi_{i/2} f'$. For each $i \in 2\mathbb{N} + 1$, we define $h'_i := \pi_i$. We then define $h'_s : \text{img}(f_s) \rightarrow [\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}}$ by $h'_s := \langle (h'_i)_{i \in \mathbb{N}} \rangle_{[0,1]^{\mathbb{N}}}$.

Note that for each $i \in \mathbb{N}$ and $x \in \text{img}(f_s)$, we have $(x)h'_s\pi_{2i+1} = (x)\pi_{2i+1} = (x)\pi_{2i}$, so h_s is injective. Moreover the sets $\text{img}(f_s), \text{img}(h'_s)$ are both contained in $[\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}}$ and as they are continuous images of compact spaces, they are both compact. Thus h'_s is a homeomorphism from $\text{img}(f_s)$ to $\text{img}(h_s)$ (Remark 3.18) and by Theorem 9.27, we can choose $h_s \in \text{Aut}([0, 1]^{\mathbb{N}})$ with $h'_s \subseteq h_s$. We now have

$$\begin{aligned} f_s h_s g_s &= f_s \langle (h'_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} \langle (\pi_{2i} g')_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = f_s \langle (h'_{2i} g')_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} \\ &= f_s g_s \langle (s\pi_i f' g')_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = \langle (s\pi_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = s. \end{aligned}$$

Let $N \in \text{Nbhd}_{\text{Aut}([0, 1]^{\mathbb{N}})}(h_s)$ be arbitrary. It suffices to show that $f_s N g_s \in \text{Nbhd}_{C([0, 1]^{\mathbb{N}})}(s)$. Let $\varepsilon > 0$ be such that for all $h \in \text{Aut}([0, 1]^{\mathbb{N}})$ with $(h_s, h)d_{\infty} < \varepsilon$ we have $h \in N$, and let $k \in C([0, 1]^{\mathbb{N}})$ be arbitrary with $(k, s)d_{\infty} < \varepsilon$. It suffices to show that $k \in f_s N g_s$.

For each $i \in 2\mathbb{N}$, we define $h'_{i,k} : \text{img}(f_s) \rightarrow [\frac{1}{4}, \frac{3}{4}]$ by $(x)h'_{i,k} = (x)g_s k \pi_{i/2} f'$. For each $i \in 2\mathbb{N} + 1$, we define $h'_{i,k} := \pi_i$. We then define $h'_k : \text{img}(f_s) \rightarrow [\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}}$ by $h'_k := \langle (h'_{i,k})_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}}$. As was the case with h'_s , the sets $\text{img}(f), \text{img}(h'_k)$ are both compact subsets of $[\frac{1}{4}, \frac{3}{4}]^{\mathbb{N}}$ and h'_k is a homeomorphism between these sets.

Note that $h_s^{-1}h'_k = \left\{ ((y)h'_s, (y)h'_k) \mid y \in \text{img}(f) \right\}$. For all $y \in \text{img}(f)$, we have

$$\begin{aligned} ((y)h'_s, (y)h'_k)d &= \sum_{n \in \mathbb{N}} \frac{|(y)h'_s \pi_n - (y)h'_k \pi_n|}{2^{n+1}} = \sum_{n \in \mathbb{N}} \frac{|(y)h'_n - (y)h'_{n,k}|}{2^{n+1}} \\ &= \sum_{n \in 2\mathbb{N}} \frac{|(y)h'_n - (y)h'_{n,k}|}{2^{n+1}} = \sum_{n \in 2\mathbb{N}} \frac{|(y)g_s s \pi_{n/2} f' - (y)g_s k \pi_{n/2} f'|}{2^{n+1}} \\ &\leq \sum_{n \in \mathbb{N}} \frac{|(y)g_s s \pi_n f' - (y)g_s k \pi_n f'|}{2^{n+1}} \leq \sum_{n \in \mathbb{N}} \frac{|(y)g_s s \pi_n - (y)g_s k \pi_n|}{2^{n+1}} \\ &= ((y)g_s s, (y)g_s k)d \leq (s, k)d_{\infty} \end{aligned}$$

Thus $\sup_{(x,y) \in h_s^{-1}h'_k} (x, y)d < \varepsilon$. By Theorem 9.27, we can choose $t \in \text{Aut}([0, 1]^{\mathbb{N}})$ such that we have $\sup_{(x,y) \in t} (x, y)d < \varepsilon$ and $h_s^{-1}h'_k \subseteq t$. Let $h_k := h_s t$. Note that

$$\begin{aligned} f_s h_k g_s &= f_s h'_s t g_s = f_s h'_k g_s = f_s \langle (h'_{i,k})_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} \langle (\pi_{2i} g')_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} \\ &= f_s \langle (h'_{2i,k} g')_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = f_s g_s \langle (k\pi_i f' g')_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = \langle (k\pi_i)_{i \in \mathbb{N}} \rangle_{[0, 1]^{\mathbb{N}}} = k. \end{aligned}$$

It therefore suffices to show that $h_k \in N$. By the definition of ε , it therefore suffices to show that $(h_k, h_s)d_{\infty} < \varepsilon$. From the choice of t , we have

$$(h_k, h_s)d_{\infty} = (h_s t, h_s)d_{\infty} = \sup_{x \in [0, 1]^{\mathbb{N}}} (((x)h_s)t, (x)h_s)d = \sup_{x \in [0, 1]^{\mathbb{N}}} ((x)t, x)d < \varepsilon$$

so the result follows. \square

Theorem 9.29 (The Polish topologies of $C([0, 1]^{\mathbb{N}})$, Assumed Knowledge: 4.27, 6.11, 7.6, 9.25, 9.26, 9.28, 5.14).

The compact-open topology is the only Polish topology compatible with the semigroup $C([0, 1]^{\mathbb{N}})$.

Proof. Let \mathcal{T} be a Polish topology compatible with $C([0, 1]^{\mathbb{N}})$, and let \mathcal{T}_{co} be the compact-open topology. As \mathcal{T} is Polish, it follows from Lemmas 9.26 and 9.25 that $\mathcal{T} \supseteq \mathcal{T}_{co}$.

As $\mathcal{T} \supseteq \mathcal{T}_{co}$ and both of them are Polish, it follows from the argument given in Proposition 5.14 that both of $\mathcal{T} \upharpoonright_{\text{Aut}([0, 1]^{\mathbb{N}})}$ and $\mathcal{T}_{co} \upharpoonright_{\text{Aut}([0, 1]^{\mathbb{N}})}$ are Polish.

As $\mathcal{T} \upharpoonright_{\text{Aut}([0,1]^{\mathbb{N}})} \supseteq \mathcal{T}_{co} \upharpoonright_{\text{Aut}([0,1]^{\mathbb{N}})}$, it follows from Theorem 4.27 that $\mathcal{T} \upharpoonright_{\text{Aut}([0,1]^{\mathbb{N}})} = \mathcal{T}_{co} \upharpoonright_{\text{Aut}([0,1]^{\mathbb{N}})}$ (using the identity isomorphism). Again using the identity isomorphism, it follows from Lemma 7.6 that if $C([0,1]^{\mathbb{N}})$ has property **W** with respect to $\text{Aut}([0,1]^{\mathbb{N}})$ then $\mathcal{T} = \mathcal{T}_{co}$. This was shown in Lemma 9.28. \square

Corollary 9.30 (The topologies of the Hilbert clone, Assumed Knowledge: 5.13, 8.2, 8.6, 9.29).

Let \mathcal{C} be the subcategory of topological spaces and continuous functions with the object set $\mathcal{O}_{\mathcal{C}} := \left\{ ([0,1]^{\mathbb{N}})^i \mid i \in \mathbb{N} \right\}$, and the morphism set $\mathcal{M}_{\mathcal{C}}$ consisting of the all continuous maps between these objects. Let $\mathcal{P}_{\mathcal{O}}$ be the discrete topology on $\mathcal{O}_{\mathcal{C}}$ and let $\mathcal{P}_{\mathcal{M}}$ be the topology on $\mathcal{M}_{\mathcal{C}}$ generated by the sets of the form

$$A_{i,j} := C\left(\left([0,1]^{\mathbb{N}}\right)^i, \left([0,1]^{\mathbb{N}}\right)^j\right) \quad \text{and} \quad B(V,U) := \left\{ f \in \mathcal{M}_{\mathcal{C}} \mid (V)f \subseteq U \right\}$$

for all $i, j \in \mathbb{N}$, compact $V \subseteq \left([0,1]^{\mathbb{N}}\right)^i$, and open $U \subseteq \left([0,1]^{\mathbb{N}}\right)^j$.

The triple $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is the a Polish abstract clone, and these are the only Polish topologies compatible with \mathcal{C} .

Proof. We first show that $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is a topological abstract clone. We need to verify the seven conditions in Definition 8.2. The first five conditions are immediate from the definition. We next show that $\circ_{\mathcal{C}}$ is continuous.

As the sets $(A_{n,i} \times A_{i,m})_{i,n,m \in \mathbb{N}}$ are a partition of $\text{Comp}_{\mathcal{C}}$ into open sets, it suffices to show that for all $n, m, i \in \mathbb{N}$, the map $\circ_{\mathcal{C}} \upharpoonright_{A_{n,i} \times A_{i,m}}$ is continuous. For all $j, k \in \mathbb{N}$, the set $(A_{j,k})_{\circ_{\mathcal{C}}} \upharpoonright_{A_{n,i} \times A_{i,m}}^{-1}$ is either empty or $A_{n,i} \times A_{i,m}$ and is thus open. Moreover if $V \subseteq \left([0,1]^{\mathbb{N}}\right)^j$ is compact and $U \subseteq \left([0,1]^{\mathbb{N}}\right)^k$ is open, then $(B(V,U))_{\circ_{\mathcal{C}}} \upharpoonright_{A_{n,i} \times A_{i,m}}^{-1}$ is open by Corollary 5.13.

It remains to show that for all $i, j \in \mathbb{N}$, the bijection $\phi_{i,j} : A_{i,1}^j \rightarrow A_{i,j}$ from Definition 8.2 is continuous. From the definition of the product topology on $\left([0,1]^{\mathbb{N}}\right)^j$, the open sets $U \subseteq \left([0,1]^{\mathbb{N}}\right)^j$ such that $U = \prod_{k \in \{0,1,\dots,j-1\}} (U)\pi_k$, form a basis for $\left([0,1]^{\mathbb{N}}\right)^j$. Thus from Theorem 5.12, the sets $B(V,U)$ where $V \subseteq \left([0,1]^{\mathbb{N}}\right)^i$ is compact and $U \subseteq \left([0,1]^{\mathbb{N}}\right)^j$ is open with $U = \prod_{k \in \{0,1,\dots,j-1\}} (U)\pi_k$, generate the topology on $A_{i,j}$.

If $V \subseteq \left([0,1]^{\mathbb{N}}\right)^i$ is compact, $U \subseteq \left([0,1]^{\mathbb{N}}\right)^j$ is open and $U = \prod_{k \in \{0,1,\dots,j-1\}} (U)\pi_k$, then

$$(B(V,U))\phi_{i,j}^{-1} = \prod_{k \in \{0,1,\dots,j-1\}} B(V, (U)\pi_k)$$

so $\phi_{i,j}$ is indeed continuous as required.

Thus $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is a topological abstract clone. The result now follows from Theorem 9.29 together with Proposition 8.6. \square

9.6. The Cantor space. In this subsection we are concerned with continuous transformations of the Cantor space $2^{\mathbb{N}}$ (see Theorem 9.35 and Corollary 9.36). The arguments follow a similar path to those in the Hilbert cube subsection but we can now make use of Theorem 5.39.

We know of no Cantor space analogue of Theorem 9.27 in the literature so we provide our own (see Lemma 9.31). The proof of Lemma 9.31 is based on the proof of Theorem 5.2.4 in [34] but is significantly shorter due to the fact that $2^{\mathbb{N}}$ can be naturally viewed as a topological group.

Lemma 9.31 (Extending $2^{\mathbb{N}}$ homeomorphisms, Assumed Knowledge: 2.3, 3.18, 4.10, 4.14).

Let $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \rightarrow \mathbb{R}$ be defined by

$$(x, y)d = \inf \left\{ \frac{1}{2^n} \mid n \in \mathbb{N}, x \upharpoonright_n = y \upharpoonright_n \right\}.$$

Let $S := \left\{ x \in 2^{\mathbb{N}} \mid (x)\pi_i = 0 \text{ for all } i \in 2\mathbb{N} + 1 \right\}$. If $\varepsilon > 0$, $C_0, C_1 \subseteq S$ are compact and $h' : C_0 \rightarrow C_1$ is a homeomorphism with $\sup_{(x,y) \in h'} (x, y)d < \varepsilon$, then there is $h \in \text{Aut}(2^{\mathbb{N}})$ such that $h' \subseteq h$ and $\sup_{(x,y) \in h} (x, y)d < \varepsilon$.

Proof. We view $\{0,1\}$ as a discrete topological group with operation addition modulo 2, and we view $2^{\mathbb{N}}$ as a power of this group (we'll denote the operation with $+$). Let $\varepsilon > 0$, $C_0, C_1 \subseteq S$ be closed and $h' : C_0 \rightarrow C_1$ be a homeomorphism with $\sup_{(x,y) \in h'} (x, y)d < \varepsilon$. Let $n \in 2\mathbb{N}$ be such that $\frac{1}{2^n} < \varepsilon$, and define $\psi_0 : 2^{\mathbb{N}} \rightarrow \{0,1\}^{n/2}$, $\psi_1, \psi_2 : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ by

$$(x)\psi_0 = \langle \pi_1, \pi_3, \dots, \pi_{n-1} \rangle_{\{0,1\}^{n/2}}, \quad (x)\psi_1 = \langle (\pi_{2i})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}},$$

$$(x)\psi_2 = \langle (\pi_{n+2i+1})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}}.$$

We then define $\psi := \langle \psi_0, \psi_1, \psi_2 \rangle_{\{0,1\}^{n/2} \times 2^{\mathbb{N}} \times 2^{\mathbb{N}}}$, $C'_0 := C_0\psi_1$ and $C'_1 := C_1\psi_1$ (note that ψ is a homeomorphism). As $C_0, C_1 \subseteq S$, the map ψ_1 is injective on these sets, thus $h'' := \psi_1^{-1}h'\psi_1$ is a homeomorphism from C'_0 to C'_1 .

Let $c : \mathcal{P}(2^{\mathbb{N}}) \setminus \{\emptyset\} \rightarrow 2^{\mathbb{N}}$ be such that $(A)c \in A$ for all $A \in \text{dom}(c)$. For each $i \in \{0,1\}$, let $t_i : 2^{\mathbb{N}} \rightarrow C'_i$ be defined by

$$(x)t_i = \left(\left\{ y \in C'_i \mid (x, y)d \leq (x, z)d \text{ for all } z \in C'_i \right\} \right) c$$

(this is well defined because C_i is compact). Note that for each $i \in \{0,1\}$, $t_i \upharpoonright_{C_i}$ is the identity function, $\text{img}(t_i) = C_i$ and for all $x, y \in 2^{\mathbb{N}}$ we have $((x)t_i, (y)t_i)d \leq (x, y)d$. Let $\phi_0, \phi_1, \phi_2 : \{0,1\}^{n/2} \times \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \rightarrow \{0,1\}^{n/2} \times \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ be defined by

$$(x, y, z)\phi_0 = (x, y, y + z), \quad (x, y, z)\phi_1 = (x, y - (z)t_0 + (z)t_0h'', z),$$

$$(x, y, z)\phi_2 = (x, y, z - (y)t_1h''^{-1}).$$

It is routine to verify that each of these maps is a homeomorphism.

Finally, let $h := \psi\phi_0\phi_1\phi_2\psi^{-1} \in \text{Aut}(2^{\mathbb{N}})$. We now show that h has the required properties. If $x \in C_0$, then

$$\begin{aligned} (x)h &= ((x)\psi_0, (x)\psi_1, 000\dots)\phi_0\phi_1\phi_2\psi^{-1} \\ &= ((x)\psi_0, (x)\psi_1, (x)\psi_1)\phi_1\phi_2\psi^{-1} \\ &= ((x)\psi_0, (x)\psi_1 - (x)\psi_1t_0 + (x)\psi_1t_0h'', (x)\psi_1)\phi_2\psi^{-1} \end{aligned}$$

As $(x)\psi_1 \in C'_0$, it follows that

$$\begin{aligned} (x)h &= ((x)\psi_0, (x)\psi_1 - (x)\psi_1 t_0 + (x)\psi_1 t_0 h'', (x)\psi_1)\phi_2 \psi^{-1} \\ &= ((x)\psi_0, (x)\psi_1 - (x)\psi_1 + (x)\psi_1 h'', (x)\psi_1)\phi_2 \psi^{-1} \\ &= ((x)\psi_0, (x)\psi_1 h'', (x)\psi_1)\phi_2 \psi^{-1} \\ &= ((x)\psi_0, (x)\psi_1 h'', (x)\psi_1 - (x)\psi_1)\psi^{-1} \\ &= ((x)\psi_0, (x)\psi_1 h'', 000\dots)\psi^{-1} = (x)h'. \end{aligned}$$

It remains to show that if $x \in 2^{\mathbb{N}}$ is arbitrary, then $(x, (x)h)d < \varepsilon$. Let $n_\varepsilon \in \mathbb{N}$ be such that if $f, g \in C(2^{\mathbb{N}})$ are arbitrary then $(f, g)d < \varepsilon$ if and only if f, g share a prefix of length n_ε . Let $i < n_\varepsilon$ be arbitrary. We show that $(x)h\pi_i = (x)\pi_i$.

By the definition of h , there is some $z' \in C'_0$ such that

$$((x)h\psi_0, (x)h\psi_1) = ((x)\psi_0, (x)\psi_1 - z' + (z')h'').$$

Let $z \in C_0$ be such that $z' = (z)\psi_1$. Note that $n_\varepsilon \leq n$. Thus if i is odd, then

$$(x)\pi_i = (x)\psi_0 \pi_{(i-1)/2} = (x)h\psi_0 \pi_{(i-1)/2} = (x)h\pi_i.$$

By assumption $(z, (z)h')d < \varepsilon$. So if $i < n_\varepsilon$ is even, then

$$(z')\pi_{i/2} = ((z)\psi_1)\pi_{i/2} = (z)\pi_i = (z)h'\pi_i = (z)h'\psi_1 \pi_{i/2} = (z)\psi_1 h'' \pi_{i/2} = (z')h'' \pi_{i/2}.$$

So $(z')\pi_{i/2} - (z')h'' \pi_{i/2} = 0$ and hence

$$(x)\pi_i = (x)\psi_1 \pi_{i/2} = (x)\psi_1 \pi_{i/2} + (z')\pi_{i/2} - (z')h'' \pi_{i/2} = (x)h\psi_1 \pi_{i/2} = (x)h\pi_i.$$

□

The following proof is very similar to the proof of Lemma 9.28, but adjusted to make use of Lemma 9.31 instead of Theorem 9.27.

Lemma 9.32 (Property **W** for $C(2^{\mathbb{N}})$, Assumed Knowledge: 1.1, 2.3, 3.11, 5.13, 7.5, 9.31).

*The topological semigroup $C(2^{\mathbb{N}})$ with the compact-open topology has property **W** with respect to $\text{Aut}(2^{\mathbb{N}})$.*

Proof. Let d be the metric on $2^{\mathbb{N}}$ given in Theorem 9.31, and let d_∞ be the metric on $C(2^{\mathbb{N}})$ given in Theorem 5.12. It suffices to show that for all $s \in C(2^{\mathbb{N}})$, there are $f_s, g_s \in C(2^{\mathbb{N}})$ and $h_s \in \text{Aut}(2^{\mathbb{N}})$ such that $f_s h_s g_s = s$ and for all $N \in \text{Nbhd}_{\text{Aut}(\{0,1\}^{\mathbb{N}})}(h_s)$, we have $f_s N g_s \in \text{Nbhd}_{C(2^{\mathbb{N}})}(s)$.

Let $s \in C(2^{\mathbb{N}})$ be arbitrary. For each $i \in \mathbb{N}$, we define $f_i : 2^{\mathbb{N}} \rightarrow \{0, 1\}$ by

$$(x)f_i = \begin{cases} 0 & \text{if } i \in 2\mathbb{N} + 1 \\ (x)\pi_{i/4} & \text{if } i \in 4\mathbb{N} \\ (x)\pi_{(i-2)/4} & \text{if } i \in 4\mathbb{N} + 2 \end{cases}.$$

We then define $f_s, g_s \in C(2^{\mathbb{N}})$ by

$$f_s := \langle (f_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} \quad \text{and} \quad g_s := \langle (\pi_{4i})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}}.$$

Note that $f_s g_s = \langle (f_{4i})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = \langle (\pi_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}}$ is the identity map. Moreover, $\text{img}(f_s) \subseteq S$ (where S is as in Theorem 9.31).

For each $i \in 4\mathbb{N}$, we define $h'_i : \text{img}(f_s) \rightarrow \{0, 1\}$ by $(x)h'_i = (x)g_s s \pi_{i/4}$. For each $i \in \mathbb{N} \setminus 4\mathbb{N}$, we define $h'_i := \pi_i$. We then define $h'_s : \text{img}(f_s) \rightarrow S$ by $h'_s := \langle (h'_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}}$.

Note that for each $i \in \mathbb{N}$ and $x \in \text{img}(f_s)$, we have $(x)h'_s \pi_{4i+2} = (x)\pi_{4i+2} = (x)\pi_{4i}$, so h_s is injective. Moreover the sets $\text{img}(f_s), \text{img}(h'_s)$ are both contained in S and they are both compact (see Remark 3.18). Thus h'_s is a homeomorphism from $\text{img}(f_s)$ to $\text{img}(h_s)$ and by Theorem 9.31, we can choose $h_s \in \text{Aut}(2^{\mathbb{N}})$ with $h'_s \subseteq h_s$. We now have

$$\begin{aligned} f_s h_s g_s &= f_s \langle (h'_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} \langle (\pi_{4i})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = f_s \langle (h'_{4i})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} \\ &= f_s g_s \langle (s \pi_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = \langle (s \pi_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = s. \end{aligned}$$

Let $N \in \text{Nbhd}_{\text{Aut}(2^{\mathbb{N}})}(h_s)$ be arbitrary. It suffices to show that $f_s N g_s \in \text{Nbhd}_{C(2^{\mathbb{N}})}(s)$. Let $\varepsilon > 0$ be such that for all $h \in \text{Aut}(2^{\mathbb{N}})$ with $(h_s, h)d_{\infty} < \varepsilon$ we have $h \in N$, and let $k \in C(2^{\mathbb{N}})$ be arbitrary with $(k, s)d_{\infty} < \varepsilon$. It suffices to show that $k \in f_s N g_s$.

For each $i \in 4\mathbb{N}$, we define $h'_{i,k} : \text{img}(f_s) \rightarrow \{0, 1\}$ by $(x)h'_{i,k} = (x)g_s k \pi_{i/4}$. For each $i \in \mathbb{N} \setminus 4\mathbb{N}$, we define $h'_{i,k} := \pi_i$. We then define $h'_k : \text{img}(f_s) \rightarrow 2^{\mathbb{N}}$ by $h'_k := \langle (h'_{i,k})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}}$. As was the case with h'_s , the sets $\text{img}(f_s), \text{img}(h'_k)$ are both compact subsets of S and h'_k is a homeomorphism between these sets.

Note that $h'_s{}^{-1}h'_k = \left\{ ((y)h'_s, (y)h'_k) \mid y \in \text{img}(f) \right\}$. For all $y \in \text{img}(f)$ with $(y)h'_s \neq (y)h'_k$, we have

$$\begin{aligned} ((y)h'_s, (y)h'_k)d &= \inf \left(\left\{ \frac{1}{2^n} \mid n \in \mathbb{N}, (y)h'_s \upharpoonright_n = (y)h'_k \upharpoonright_n \right\} \right) \\ &= \max \left(\left\{ \frac{1}{2^n} \mid n \in \mathbb{N}, (y)h'_s \pi_n \neq (y)h'_k \pi_n \right\} \right) \\ &= \max \left(\left\{ \frac{1}{2^n} \mid n \in 4\mathbb{N}, (y)g_s s \pi_{n/4} \neq (y)g_s k \pi_{n/4} \right\} \right) \\ &= \max \left(\left\{ \frac{1}{2^{4n}} \in \mathbb{N} \mid n \in \mathbb{N}, (y)g_s s \pi_n \neq (y)g_s k \pi_n \right\} \right) \\ &< \max \left(\left\{ \frac{1}{2^n} \in \mathbb{N} \mid n \in \mathbb{N}, (y)g_s s \pi_n \neq (y)g_s k \pi_n \right\} \right) \\ &= ((y)g_s s, (y)g_s k)d \leq (s, k)d_{\infty} < \varepsilon. \end{aligned}$$

Thus $\sup_{(x,y) \in h'_s{}^{-1}h'_k} (x,y)d < \varepsilon$. By Theorem 9.31, we can choose $t \in \text{Aut}(2^{\mathbb{N}})$ such that we have $\sup_{(x,y) \in t} (x,y)d < \varepsilon$ and $h'_s{}^{-1}h'_k \subseteq t$. Let $h_k := h_s t$. Note that

$$\begin{aligned} f_s h_k g_s &= f_s h'_s t g_s = f_s h'_k g_s = f_s \langle (h'_{i,k})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} \langle (\pi_{4i})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = f_s \langle (h'_{4i,k})_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} \\ &= f_s g_s \langle (k \pi_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = \langle (k \pi_i)_{i \in \mathbb{N}} \rangle_{2^{\mathbb{N}}} = k. \end{aligned}$$

It therefore suffices to show that $h_k \in N$. By the definition of ε , it therefore suffices to show that $(h_k, h_s)d_{\infty} < \varepsilon$. From the choice of t , we have

$$(h_k, h_s)d_{\infty} = (h_s t, h_s)d_{\infty} = \sup_{x \in 2^{\mathbb{N}}} (((x)h_s)t, (x)h_s)d = \sup_{x \in 2^{\mathbb{N}}} ((x)t, x)d < \varepsilon$$

so the result follows. \square

Lemma 9.33 (Cantor continuity, Assumed Knowledge: 5.13, 5.14, 5.39, 6.12, 7.6, 9.32).

The topologies $SCC(\text{Aut}(2^{\mathbb{N}}))$ and $SCC(C(2^{\mathbb{N}}))$ are the compact-open topologies on each of these sets. Moreover these topologies are Polish

Proof. Let \mathcal{T}_{co} denote the compact-open topology on $C(2^{\mathbb{N}})$. From Theorem 5.12, this topology is Polish and compatible with $C(2^{\mathbb{N}})$. As \mathcal{T}_{co} is second countable, it follows that $SCC(C(2^{\mathbb{N}})) \supseteq \mathcal{T}_{co}$ and $SCC(\text{Aut}(2^{\mathbb{N}})) \supseteq \mathcal{T}_{co} \upharpoonright_{\text{Aut}(2^{\mathbb{N}})}$.

By Theorem 5.39 every homomorphism from $(\text{Aut}(2^{\mathbb{N}}), \mathcal{T}_{co})$ to a second countable group (and hence inverse semigroup) is continuous. From Proposition 6.12, it follows that every homomorphism from the semigroup obtained by removing the unary operation of $\text{Aut}(2^{\mathbb{N}})$ to a second countable semigroup is continuous.

By Lemma 9.32, $C(2^{\mathbb{N}})$ has property **W** with respect to $\text{Aut}(2^{\mathbb{N}})$. Thus from Lemma 7.6, all homomorphisms from $(C(2^{\mathbb{N}}), \mathcal{T}_{co})$ to second countable topological semigroups are continuous. So from Proposition 6.12 we also have

$$SCC(C(2^{\mathbb{N}})) \subseteq \mathcal{T}_{co} \quad \text{and} \quad SCC(\text{Aut}(2^{\mathbb{N}})) \subseteq \mathcal{T}_{co} \upharpoonright_{\text{Aut}(2^{\mathbb{N}})}.$$

□

The following lemma is the Cantor space analogue of Lemma 9.26. However in this case we can use a different proof with weaker assumptions due to the existence Lemma 9.33.

Lemma 9.34 (Continuous action on $2^{\mathbb{N}}$, Assumed Knowledge: 9.33).

Suppose that \mathcal{T} is a second countable Hausdorff topology compatible with the semigroup $C(2^{\mathbb{N}})$. In this case the map $a : 2^{\mathbb{N}} \times (C(2^{\mathbb{N}}), \mathcal{T}) \rightarrow 2^{\mathbb{N}}$ defined by $(x, f)a = (x)f$ is continuous.

Proof. Let C denote the set of constant maps from $2^{\mathbb{N}}$ to itself and note that $\mathcal{T} \upharpoonright_C$ is second countable and Hausdorff. Let $I : C \rightarrow 2^{\mathbb{N}}$ be the bijection sending a constant map to the unique point in its image. Note that $a = \langle \pi_0 I^{-1}, \pi_1 \rangle_{C \times C(2^{\mathbb{N}})} \circ *^{C(2^{\mathbb{N}})} \circ I$. Thus to show that a is continuous, it suffices to show that $I : C \rightarrow 2^{\mathbb{N}}$ is a homeomorphism. Let \mathcal{C} denote the topology

$$\left\{ U \subseteq C \mid (U)I \text{ is open in } 2^{\mathbb{N}} \right\}$$

on C . We will show that $\mathcal{T} \upharpoonright_C = \mathcal{C}$. If $U \subseteq C$ is open, then $(U)I^{-1} = \left\{ f \in C \mid (2^{\mathbb{N}})f \subseteq U \right\}$. So $\mathcal{C} = \mathcal{T}_{co} \upharpoonright_C$. From Lemma 9.33, it follows that

$$\mathcal{T} \upharpoonright_C \subseteq SCC(C(2^{\mathbb{N}})) \upharpoonright_C = \mathcal{T}_{co} \upharpoonright_C = \mathcal{C}.$$

Note that $\mathcal{T} \upharpoonright_C \subseteq \mathcal{C}$, $\mathcal{T} \upharpoonright_C$ is Hausdorff and \mathcal{C} is compact. Thus from Remark 3.18 we have $\mathcal{T} \upharpoonright_C = \mathcal{C}$ as required. □

Theorem 9.35 (The topologies of $C(2^{\mathbb{N}})$, Assumed Knowledge: 9.25, 9.33, 9.34).

The compact-open topology is the only second countable Hausdorff topology compatible with $C(2^{\mathbb{N}})$. Moreover $SCC(C(2^{\mathbb{N}}))$ is the compact-open topology, and all homomorphisms from $C(2^{\mathbb{N}})$ to second countable topological semigroups are continuous.

Proof. Let \mathcal{T} be an arbitrary second countable Hausdorff topology compatible with $C(2^{\mathbb{N}})$, and let \mathcal{T}_{co} be the compact-open topology. The equality $SCC(C(2^{\mathbb{N}})) = \mathcal{T}_{co}$ follows from Lemma 9.33. It remains

to show that $\mathcal{T} = \mathcal{T}_{co}$. By the definition of $\mathit{SCC}(C(2^{\mathbb{N}}))$, it follows that $\mathcal{T} \subseteq \mathit{SCC}(C(2^{\mathbb{N}})) = \mathcal{T}_{co}$. Moreover, from Lemmas 9.34 and 9.25, we also have $\mathcal{T} \supseteq \mathcal{T}_{co}$. The result follows. \square

Corollary 9.36 (The topologies of the Cantor clone, Assumed Knowledge: 5.13, 8.2, 8.6, 9.30, 9.35). *Let \mathcal{C} be the subcategory of topological spaces and continuous functions with the object set $\mathcal{O}_{\mathcal{C}} := \{(2^{\mathbb{N}})^i \mid i \in \mathbb{N}\}$, and the morphism set $\mathcal{M}_{\mathcal{C}}$ consisting of the all continuous maps between these objects. Let $\mathcal{P}_{\mathcal{O}}$ be the discrete topology on $\mathcal{O}_{\mathcal{C}}$ and let $\mathcal{P}_{\mathcal{M}}$ be the topology on $\mathcal{M}_{\mathcal{C}}$ generated by the sets of the form*

$$A_{i,j} := C((2^{\mathbb{N}})^i, (2^{\mathbb{N}})^j) \quad \text{and} \quad B(V, U) := \left\{ f \in \mathcal{M}_{\mathcal{C}} \mid (V)f \subseteq U \right\}$$

for all $i, j \in \mathbb{N}$, compact $V \subseteq (2^{\mathbb{N}})^i$, and open $U \subseteq (2^{\mathbb{N}})^j$.

The triple $(\mathcal{C}, \mathcal{P}_{\mathcal{O}}, \mathcal{P}_{\mathcal{M}})$ is a Polish topological abstract clone. Moreover if $(\mathcal{T}_{\mathcal{O}}, \mathcal{T}_{\mathcal{M}})$ are compatible with the clone \mathcal{C} then

- (1) If $\mathcal{T}_{\mathcal{M}}$ is Hausdorff and second countable then $\mathcal{T}_{\mathcal{M}} = \mathcal{P}_{\mathcal{M}}$.
- (2) If $\mathcal{T}_{\mathcal{M}}$ is second countable then $\mathcal{T}_{\mathcal{M}} \subseteq \mathcal{P}_{\mathcal{M}}$.

Proof. The proof of this corollary is the same as the proof of Corollary 9.30, but using Theorem 9.35 instead of Theorem 9.29. \square

9.7. Boolean algebras. In this subsection we discuss homomorphisms between boolean algebras. In particular we are interested in B_{∞} the boolean algebra of clopen subsets of the Cantor space. This boolean algebra is of note as up to isomorphism it is the only countably infinite boolean algebra with no atoms (minimal non-zero elements). It is also the Fräissé limit of the class of finite non-trivial boolean algebras. The results of this subsection are corollaries of Cantor space subsection using the Stone Duality Theorem (9.41) and Proposition 8.6.

Definition 9.37 (Boolean algebra signature, Assumed Knowledge: 4.1).

We define the σ_B to be the signature

$$\sigma_B := (\{\wedge, \vee, \neg, \mathbf{0}, \mathbf{1}\}, \emptyset, \{(\wedge, 2), (\vee, 2), (\neg, 1), (\mathbf{0}, 0), (\mathbf{1}, 0)\}).$$

The actual value of the symbols $\wedge, \vee, \neg, \mathbf{0}$ and $\mathbf{1}$ are not very important, but for completeness we'll define

$$\wedge := (01, 14, 04), \quad \vee := (15, 18), \quad \neg := (14, 15, 20),$$

$$\mathbf{0} := (26, 05, 18, 15), \quad \mathbf{1} := (15, 14, 05).$$

Definition 9.38 (Boolean algebra, Assumed Knowledge: 4.2, 9.37).

If \mathbb{B} is σ_B -structure and $a, b \in \mathbb{B}$, then we will often write $a \wedge b, a \vee b, \neg a, \mathbf{0}$ and $\mathbf{1}$ instead of $(a, b) \wedge^{\mathbb{B}}, (a, b) \vee^{\mathbb{B}}, (a) \neg^{\mathbb{B}}, () \mathbf{0}^{\mathbb{B}}$, and $() \mathbf{1}^{\mathbb{B}}$ respectively.

We say a σ_B -structure \mathbb{B} is a *boolean algebra* if for all $a, b, c \in \mathbb{B}$ we have the following:

$$\begin{aligned} a \wedge (b \wedge c) &= (a \wedge b) \wedge c, & a \vee (b \vee c) &= (a \vee b) \vee c, \\ a \wedge b &= b \wedge a, & a \vee b &= b \vee a, \\ a \wedge \mathbf{1} &= a, & a \vee \mathbf{0} &= a, \\ a \vee (b \wedge c) &= (a \vee b) \wedge (a \vee c), & a \wedge (b \vee c) &= (a \wedge b) \vee (a \wedge c), \\ a \vee (a \wedge b) &= a, & a \wedge (a \vee b) &= a, \\ a \vee \neg a &= \mathbf{1}, & a \wedge \neg a &= \mathbf{0}, \\ \neg(a \vee b) &= \neg a \wedge \neg b, & \neg(a \wedge b) &= \neg a \vee \neg b. \end{aligned}$$

It is worth noting that many of the assumptions in the above definitions can be inferred from each other, but the choice of which conditions constitute the definition varies between authors.

From Theorem 9.41, it will follow that every boolean algebra is isomorphic to a boolean algebra of sets (with the operations of intersection, union, complement, empty set, and universe) so these are the key examples to keep in mind.

Example 9.39 (The two element boolean algebra, Assumed Knowledge: 9.38).

If one considers 0 as representing “false” and 1 as representing “true”, then $\{0, 1\}$ is a boolean algebra when given the operations of “and”, “or”, “not”, “false”, and “true”. More concretely if $a, b \in \{0, 1\}$ then

$$a \wedge b := \min(a, b), \quad a \vee b := \max(a, b), \quad \neg a := 1 - a, \quad \mathbf{0} := 0, \quad \text{and} \quad \mathbf{1} := 1$$

Definition 9.40 (Stone spaces, Assumed Knowledge: 3.11, 4.3, 9.39).

We say that a topological space X is a *Stone space* if it is zero-dimensional, Hausdorff and compact. If \mathbb{B} is a boolean algebra then let $\text{SSpace}(\mathbb{B})$ denote the set of homomorphisms from \mathbb{B} to the two element boolean algebra from Example 9.39. We view $\text{SSpace}(\mathbb{B})$ as a topological space equipped with the pointwise topology ($\{0, 1\}$ has the discrete topology).

Theorem 9.41 (cf. Theorem 34 of [14], Stone Duality Theorem, Assumed Knowledge: 3.14, 3.16, 3.12, 3.13, 6.2, 9.40).

If X is a Stone space, then the collection $\text{Clo}(X)$ of clopen subsets of X form a boolean algebra with the following operations. If $A, B \in \text{Clo}(X)$ are arbitrary, then

$$A \vee B = A \cup B, \quad A \wedge B = A \cap B, \quad \neg A = X \setminus A, \quad \mathbf{0} = \emptyset, \quad \mathbf{1} = X.$$

Conversely, if \mathbb{B} is a boolean algebra, then $\text{SSpace}(\mathbb{B})$ (recall Definition 9.40) is the only Stone space X (up to homeomorphism) such that $\mathbb{B} \cong \text{Clo}(X)$. Moreover if X is a Stone space and \mathbb{B} is a boolean algebra, then

$$C(X) \cong \text{End}(\text{Clo}(X))^\dagger \quad \text{and} \quad \text{End}(\mathbb{B}) \cong C(\text{SSpace}(\mathbb{B}))^\dagger \quad (\text{recall Definition 6.2}).$$

Additionally if X, Y are disjoint Stone spaces, then $X \cup Y$ is a Stone space and $\text{Clo}(X \cup Y) \cong \text{Clo}(X) \times \text{Clo}(Y)$.

Theorem 9.42 (The topologies of $\text{End}(B_\infty)$, Assumed Knowledge: 6.3, 9.35, 9.41, 4.13, 5.1).

Let B_∞ denote the boolean algebra $\text{Clo}(2^{\mathbb{N}})$. The pointwise topology $\mathcal{PT}_{B_\infty} \upharpoonright_{\text{End}(B_\infty)}$ (recall Example 4.13) is the only second countable Hausdorff topology compatible with $\text{End}(B_\infty)$. Moreover this

topology is Polish, equal to $\text{SCC}(\text{End}(B_\infty))$, and all homomorphisms from $(\text{End}(B_\infty), \mathcal{PT}_{B_\infty} \upharpoonright_{\text{End}(B_\infty)})$ to second countable semigroups are continuous.

Proof. From Theorem 9.41, we know that $\text{End}(B_\infty) \cong C(2^\mathbb{N})^\dagger$. Thus from Proposition 6.3, there is a (bijective and topological property preserving) correspondence between the topologies compatible with $\text{End}(B_\infty)$, and those compatible with $C(2^\mathbb{N})$. Thus from Theorem 9.35 there is a unique second countable Hausdorff topology compatible with $\text{End}(B_\infty)$, and this topology coincides with $\text{SCC}(\text{End}(B_\infty))$. This topology is also Polish from Lemma 9.33. As $\mathcal{PT}_{B_\infty} \upharpoonright_{\text{End}(B_\infty)}$ is second countable, Hausdorff and compatible with $\text{End}(B_\infty)$, the result follows. \square

Corollary 9.43 (The topologies of the B_∞ polymorphism clone, Assumed Knowledge: 8.2, 8.6, 9.10). *Let B_∞ be a boolean algebra, with universe \mathbb{N} , which is isomorphic to $\text{Clo}(2^\mathbb{N})$. Let \mathcal{C} be the subcategory of boolean algebras and homomorphisms with the object set $\mathcal{O}_\mathcal{C} := \{B_\infty^i \mid i \in \mathbb{N}\}$, and the morphism set $\mathcal{M}_\mathcal{C}$ consisting of the all homomorphisms between these objects. We view $\mathcal{O}_\mathcal{C}$ and $\mathcal{P}_\mathcal{C}$ as subspaces of the topological spaces of the same names given in Theorem 9.11. Let $\mathcal{P}_\mathcal{O}$ and $\mathcal{M}_\mathcal{C}$ be the topologies on these sets respectively.*

The triple $(\mathcal{C}, \mathcal{P}_\mathcal{O}, \mathcal{P}_\mathcal{M})$ is a Polish topological abstract clone. Moreover if $(\mathcal{T}_\mathcal{O}, \mathcal{T}_\mathcal{M})$ are compatible with the clone \mathcal{C} then

- (1) *If $\mathcal{T}_\mathcal{M}$ is Hausdorff and second countable then $\mathcal{T}_\mathcal{M} = \mathcal{P}_\mathcal{M}$.*
- (2) *If $\mathcal{T}_\mathcal{M}$ is second countable then $\mathcal{T}_\mathcal{M} \subseteq \mathcal{P}_\mathcal{M}$.*

Proof. First note that $(\mathcal{C}, \mathcal{P}_\mathcal{O}, \mathcal{P}_\mathcal{M})$ is a topological abstract clone as it is contained in the clone from Theorem 9.11.

Note that $2^\mathbb{N}$ is isomorphic to its disjoint union with itself ($2 \times 2^\mathbb{N} \cong 2^\mathbb{N}$). Thus from Theorem 9.41 it follows that $B_\infty \cong B_\infty^2$. Thus the result follows from Proposition 8.6 together with Theorem 9.42. \square

Part 4. Automorphisms of dV_n

The content of this part is a more detailed version of the Paper [12] by the same author as this document.

The groups dV_n include both the Brin-Thompson groups nV ($= nV_2$), and the Higman-Thompson groups V_n ($= 1V_n$). They also overlap with the class of groups $G_{n,r}$ ($G_{n,1} = 1V_n$). The groups nV [7, 8, 4, 16, 2, 30, 23], as well as automorphisms of Thompson groups [6, 9, 3, 26] have each been well researched in the literature.

The paper [3] of C. Bleak, P. Cameron, Y. Maissel, A. Navas, and F. Olukoya gives a description of the groups $\text{Aut}(G_{n,r})$ and $\text{Out}(G_{n,r})$ using Rubin's Theorem and transducers. Here we give an analogous description of the groups $\text{Aut}(dV_n)$ and $\text{Out}(dV_n)$ (see Theorem 12.18) and use it to give an embedding of $\text{Out}(dV_n)$ into $\text{Out}(V_n) \wr \text{Sym}(d)$ (see Theorem 14.18). Moreover in the case that $n = 2$, our embedding is actually an isomorphism (see Theorem 14.19). This proves a conjecture made by Nathan Barker in 2012: The groups $\text{Out}(dV)$ and $\text{Out}(V) \wr \text{Sym}(d)$ are isomorphic.

The groups nV for $n \geq 2$ have also been described with transducers before but not in the same manner done here (in [2] it is shown that they embed in the rational group). We discuss this alternative transducer viewpoint more in Section 13.

For our representation of dV_n we extend the notion of transducer established by Grigorchuk, Nekrashevich, and Sushchanskii [15], and then identify a class of transducers which is appropriate for representing continuous transformations of n -dimensional Cantor spaces (see Theorem 11.16).

10. HIGHER DIMENSIONAL WORDS

In this section we define the groups of interest and introduce the basic concepts surrounding multi-dimensional words (these will be used heavily throughout this part).

Definition 10.1 (Higher dimensional words, Assumed Knowledge: 3.20, 4.14, 4.28).

If $n \geq 2$, then we define $X_n := \{0, 1, \dots, n-1\}$ and $\mathfrak{C}_n := \{0, 1, \dots, n-1\}^{\mathbb{N}}$. We think of the elements of \mathfrak{C}_n as *infinite words* over the alphabet X_n (see Definition 4.28) and we give them the product topology using the discrete topology on X_n . It follows from Theorem 3.20, that all of these are Cantor spaces. For each $i \in \{0, 1, \dots, d-1\}$ and $x \in X_n$, we define $x_{d,i} \in (X_n^*)^d$ by $(i)x_{d,i} = x$ and $(j)x_{d,i} = \varepsilon$ for all $j \neq i$. Similarly, we define $\varepsilon_d \in (X_n^*)^d$ to be the constant tuple with value ε .

If $d \in \mathbb{N} \setminus \{0\}$, then we extend the prefix relation \leq (from Definition 4.28) on $\{0, 1, \dots, n-1\}^* \cup \mathfrak{C}_n$ to the set $(\{0, 1, \dots, n-1\}^* \cup \mathfrak{C}_n)^d$ in the natural fashion (as in Example 4.15). We also extend the concatenation operation to the domain $(X_n^*)^d \times (X_n^* \cup \mathfrak{C}_n)^d$ coordinatewise. For $w \in (X_n^*)^d$, then we also extend λ_w to the domain $(X_n^* \cup \mathfrak{C}_n)^d$ to be concatenation with w on the left (note that this map is injective).

Note that if $w \in (X_n^*)^d$ then

$$w\mathfrak{C}_n^d = (\mathfrak{C}_n^d)\lambda_w = \left\{ x \in \mathfrak{C}_n^d \mid w \leq x \right\}.$$

It is routine to verify that these sets are clopen, and the collection of all such sets is a basis for \mathfrak{C}_n^d . Such basic open sets will be referred to as *cones*. As the cones form a basis of clopen sets for \mathfrak{C}_n^d and this space is compact, it follows that the clopen subsets of \mathfrak{C}_n^d are precisely the finite unions of cones.

Definition 10.2 (Prefix codes and dV_n , Assumed Knowledge: 5.1, 10.1).

Suppose that $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. We say that a subset $F \subseteq (X_n^*)^d$ is a *complete prefix code* if $\left\{ w\mathfrak{C}_n^d \mid w \in F \right\}$ is a partition of \mathfrak{C}_n^d into clopen sets. Note that as \mathfrak{C}_n^d is compact, all complete prefix codes will be finite.

If F_1, F_2 are complete prefix codes of \mathfrak{C}_n^d , and $\phi : F_1 \rightarrow F_2$ is a bijection, then we define the *prefix exchange map* $f_\phi : \mathfrak{C}_n^d \rightarrow \mathfrak{C}_n^d$ by

$$f_\phi := \bigcup_{w \in F_1} (\lambda_w^{-1} \circ \lambda_{(w)\phi}) \upharpoonright_{w\mathfrak{C}_n^d}.$$

That is, if $w \in F_1$ and $x \in \mathfrak{C}_n^d$, then $(wx)f_\phi = ((w)\phi)(x)$.

It is routine to verify that such prefix exchange maps are always homeomorphisms, and moreover the set of prefix exchange maps is closed under composition and inversion. We define dV_n to be the subgroup of $\text{Aut}(\mathfrak{C}_n^d)$ consisting all the prefix exchange maps. If $n = 2$ or $d = 1$ we will often omit them in the notation dV_n .

Lemma 10.3 (Complete prefix code sizes, Assumed Knowledge: 10.2).

Suppose that $m, d \geq 1$ and $n \geq 2$. There is a complete prefix code for \mathfrak{C}_n^d of cardinality m if and only if $m \in (1 + (n-1)\mathbb{N})$.

Proof. (\Leftarrow) : We show by induction on k , that for all $k \in \mathbb{N}$ there is a complete prefix code of \mathfrak{C}_n^d of size $1 + (n - 1)k$. If $k = 0$, then $1 + (n - 1)k = 1$ and we can choose the complete prefix code $\{\varepsilon_d\}$. Suppose that $k > 0$ and the claim holds for smaller k . Let F be a complete prefix code with $|F| = 1 + (n - 1)(k - 1)$. Choose $w \in F$ and define $F' := (F \setminus \{w\}) \cup \{wx_{d,0} \mid x \in X_n\}$, then F' is a complete prefix code with $|F'| = 1 + (n - 1)k$.

(\Rightarrow) : Let F be a complete prefix code for \mathfrak{C}_n^d . Let $k := \max \{ |(w)\pi_i| \mid w \in F, i \in \{0, 1, \dots, d-1\} \}$. Suppose that there is $w \in F$ and $i \in \{0, 1, \dots, d-1\}$ such that $|(w)\pi_i| \neq k$. It follows that $|(w)\pi_i| < k$. If we then define $F' := (F \setminus \{w\}) \cup \{wx_{d,i} \mid x \in X_n\}$, then $|F'| - |F| = n - 1$ and

$$\max \{ |(w)\pi_i| \mid w \in F \} = \max \{ |(w)\pi_i| \mid w \in F' \}.$$

Thus we can assume without loss of generality that $|(w)\pi_i| = k$ for all $w \in F$ and $i \in \{0, 1, \dots, d-1\}$. As F is a complete prefix code, it follows that $F = (X_n^k)^d$. In particular $|F| = n^{kd} \in 1 + (n - 1)\mathbb{N}$ as required. \square

Remark 10.4 (Scary prefix codes, Assumed Knowledge: 10.3).

In the proof of Lemma 10.3, we construct complete prefix codes of the required sizes by starting at the trivial prefix code $\{\varepsilon_d\}$, and sequentially refining it by replacing one element with n new elements. It can be shown that in the case $d = 1$, all complete prefix codes can be found this way. However this is not true in general. For example consider the following complete prefix code when $n = 2$ and $d = 3$:

$$\{(\varepsilon, 0, 0), (0, 1, \varepsilon), (1, \varepsilon, 1), (0, 0, 1), (1, 1, 0)\}.$$

11. GENERALIZING THE TRANSDUCERS OF GRIGORCHUK, NEKRASHEVICH, AND SUSHCHANSKII

In this section we introduce the type of transducers we will be using throughout this part. The transducers defined by Grigorchuk, Nekrashevich, and Sushchanskii (which we shorten to GNS) in [15], are machines which “read” letters and then “transition” between various “states” and “write” letters accordingly. These machines can then read words by reading each letter in turn.

This can be thought of as assigning to each letter of an alphabet X_n , a transformation of a state set, and a word to write for each state. This assignment is then extended to all elements of X_n^* via its “freeness” property. Our definition has the view of reading/writing elements of a semigroup, but we do not restrict to free monoids.

Definition 11.1 (Transducers, Assumed Knowledge: 1.1, 2.2, 4.5, 5.2).

We say that $T := (Q_T, D_T, R_T, \pi_T, \lambda_T)$ is a transducer if:

- (1) Q_T is a set (called the set of states).
- (2) D_T is a semigroup (called the domain semigroup).
- (3) R_T is a semigroup (called the range semigroup).
- (4) $\pi_T : Q_T \times D_T \rightarrow Q_T$ is an action of D_T on the discrete set Q_T (called the transition function).
- (5) $\lambda_T : Q_T \times D_T \rightarrow R_T$ is a function with the property that for all $q \in Q_T$ and $s, t \in D_T$ we have

$$(q, st)\lambda_T = (q, s)\lambda_T((q, s)\pi_T, t)\lambda_T \text{ (called the output function).}$$

Remark 11.2 (Transducers vs homomorphisms, Assumed Knowledge: 11.1).

Defining a one state transducer with domain D and range R is equivalent to defining a semigroup homomorphism from D to R . In this sense transducers can be thought of as generalisations of semigroup homomorphisms.

Note that our definition of a transducer consists only of a set, two semigroups and some maps between them. It is thus easy and natural to view our transducers as objects of a category. This viewpoint will be key to many of the arguments later in the part.

Definition 11.3 (Transducer homomorphisms, Assumed Knowledge: 2.3, 4.5, 4.16, 11.1).

Let A, B be transducers. We say that ϕ is a *transducer homomorphism* from A to B (written $\phi : A \rightarrow B$), if ϕ is a 3-tuple (ϕ_Q, ϕ_D, ϕ_R) with the following properties:

- (1) $\phi_R : R_A \rightarrow R_B$ is a semigroup homomorphism.
- (2) $\phi_D : D_A \rightarrow D_B$ is a semigroup homomorphism.
- (3) $\phi_Q : Q_A \rightarrow Q_B$ is a function, such that

$$\langle \boldsymbol{\pi}_A, \boldsymbol{\lambda}_A \rangle_{Q_A \times R_A} \langle \pi_0 \phi_Q, \pi_1 \phi_R \rangle_{Q_B \times R_B} = \langle \pi_0 \phi_Q, \pi_1 \phi_D \rangle_{Q_B \times D_B} \langle \boldsymbol{\pi}_B, \boldsymbol{\lambda}_B \rangle_{Q_B \times R_B}$$

(recall Definition 2.3). Or equivalently, for all $q \in Q_A$ and $s \in D_A$ we have

$$(q, s) \boldsymbol{\pi}_A \phi_Q = ((q) \phi_Q, (s) \phi_D) \boldsymbol{\pi}_B \quad \text{and} \quad (q, s) \boldsymbol{\lambda}_A \phi_R = ((q) \phi_Q, (s) \phi_D) \boldsymbol{\lambda}_B.$$

If the maps ϕ_D, ϕ_R are identity maps, then we say that ϕ is *strong*. We compose transducer homomorphisms component-wise. It is routine to verify that transducers and transducer homomorphisms form a category. Moreover transducers and strong transducer homomorphisms form a subcategory of this category.

We say that a transducer homomorphism ϕ is a *quotient map* if each of ϕ_Q, ϕ_D, ϕ_R is surjective (so each of them is a quotient map as in Definition 4.16). We say that $\phi : A \rightarrow B$ is a *transducer isomorphism* if it is an isomorphism in the category of transducers and transducer homomorphisms, or equivalently if each of the maps ϕ_Q, ϕ_D, ϕ_R is a bijection.

We say that two transducers A, B are *isomorphic* (denoted $A \cong B$) if there is a transducer isomorphism $\phi : A \rightarrow B$. Similarly we say that A, B are *strongly isomorphic* (denoted $A \cong_S B$) if there is a strong transducer isomorphism $\phi : A \rightarrow B$.

The following notion of a “minimal transducer” corresponds to the notion of “combining equivalent states” from GNS. The GNS notion of being “reduced” is the notion discussed in Theorem 11.19.

Definition 11.4 (Minimal transducers, Assumed Knowledge: 1.10, 11.3).

If T is a transducer and λ_r injective for all $r \in R_T$, then we define its *minimal transducer* M_T to be $(Q_T / \sim_{M_T}, D_T, R_T, \boldsymbol{\pi}_{M_T}, \boldsymbol{\lambda}_{M_T})$ where $\sim_{M_T}, \boldsymbol{\pi}_{M_T}$ and $\boldsymbol{\lambda}_{M_T}$ are defined by:

- (1) \sim_{M_T} is the equivalence relation

$$\left\{ (p, q) \in Q_T^2 \mid (p, s) \boldsymbol{\lambda}_T = (q, s) \boldsymbol{\lambda}_T \text{ for all } s \in D_T \right\};$$

- (2) If $q \in Q_T, s \in D_T$ then $([q]_{\sim_{M_T}}, s) \boldsymbol{\pi}_{M_T} = [(q, s) \boldsymbol{\pi}_T]_{\sim_{M_T}}$;
- (3) If $q \in Q_T, s \in D_T$ then $([q]_{\sim_{M_T}}, s) \boldsymbol{\lambda}_{M_T} = (q, s) \boldsymbol{\lambda}_T$.

Moreover, we define a strong quotient map $q_T : T \rightarrow M_T$, by $(p) q_T = [p]_{\sim_{M_T}}$. We will justify these definitions in Lemma 11.5.

Lemma 11.5 (Minimal transducers are valid, Assumed Knowledge: 11.4).

The objects defined in Definition 11.4 are well-defined and have the stated properties. Moreover if A is a transducer, λ_r is injective for all $r \in R_A$, and $\phi : A \rightarrow B$ is a strong quotient map, then there is a strong quotient map $\psi : B \rightarrow M_A$ with $q_A = \phi\psi$.

Proof. Let T be as in Definition 11.4. We first need to show that π_{M_T} is well-defined. Let $p, q \in Q_T$ and $s \in D_T$ be such that $p \sim_{M_T} q$. We need to show that $(p, s)\pi_T \sim_{M_T} (q, s)\pi_T$ and $(p, s)\lambda_T = (q, s)\lambda_T$. The equality $(p, s)\lambda_T = (q, s)\lambda_T$ is immediate from the definition of \sim_{M_T} . To show that $(p, s)\pi_T \sim_{M_T} (q, s)\pi_T$, it suffices to show that if $t \in D_T$ is arbitrary then $((p, s)\pi_T, t)\lambda_T = ((q, s)\pi_T, t)\lambda_T$. This follows from

$$\begin{aligned} ((p, s)\pi_T, t)\lambda_T &= (((p, s)\pi_T, t)\lambda_T)\lambda_{(p, s)\lambda_T}^{-1}\lambda_{(p, s)\lambda_T} \\ &= ((p, s)\lambda_T((p, s)\pi_T, t)\lambda_T)\lambda_{(p, s)\lambda_T}^{-1}\lambda_{(p, s)\lambda_T} \\ &= ((p, st)\lambda_T)\lambda_{(p, s)\lambda_T}^{-1}\lambda_{(p, s)\lambda_T} \\ &= ((q, st)\lambda_T)\lambda_{(q, s)\lambda_T}^{-1}\lambda_{(q, s)\lambda_T} \\ &= ((q, s)\lambda_T((q, s)\pi_T, t)\lambda_T)\lambda_{(q, s)\lambda_T}^{-1}\lambda_{(q, s)\lambda_T} \\ &= (((q, s)\pi_T, t)\lambda_T)\lambda_{(q, s)\lambda_T}^{-1}\lambda_{(q, s)\lambda_T} \\ &= ((q, s)\pi_T, t)\lambda_T. \end{aligned}$$

We next need to show that q_T is a quotient map. The maps $q_{T_D}, q_{T_R}, q_{T_Q}$ are all surjective by definition, so we need only show that q_T is a transducer homomorphism. If $q \in Q_T$ and $s \in D_T$, then by definition we have

$$\begin{aligned} (q, s)\pi_T q_{T_Q} &= [(q, s)\pi_T]_{\sim_{M_T}} = ([q]_{\sim_{M_T}}, s)\pi_{M_T} = ((q)q_{T_Q}, (s)q_{T_D})\pi_{M_T}, \\ (q, s)\lambda_T q_{A_R} &= (q, s)\lambda_T = ([q]_{\sim_{M_T}}, s)\lambda_{M_T} = ((q)q_{A_Q}, (s)q_{A_D})\lambda_{M_T} \end{aligned}$$

as required. We define $\psi : B \rightarrow M_A$ by having ψ_D, ψ_R be the identity maps and defining ψ_Q by:

$$((q)\phi_Q)\psi_Q := (q)q_{A_Q}.$$

As q_{A_Q} is surjective, it suffices to show that ψ is a well-defined transducer homomorphism. Note that all of the maps $\phi_D, \phi_R, \psi_D, \psi_R, q_{A_D}$ and q_{A_R} are identity maps so we can ignore them for the purposes of this proof. We first show that ψ is well-defined. Suppose that $q_0, q_1 \in Q_A$ satisfy $(q_0)\phi_Q = (q_1)\phi_Q$. We need to show that $q_0 \sim_{M_T} q_1$. For $s \in D_A$ be arbitrary, we have

$$(q_0, s)\lambda_A = ((q_0)\phi_Q, s)\lambda_B = ((q_1)\phi_Q, s)\lambda_B = (q_1, s)\lambda_A.$$

So indeed $q_0 \sim_{M_T} q_1$. It remains to show that ψ is a homomorphism. Using the fact that ϕ_Q is surjective, let $(p, s) = ((q)\phi_Q, s) \in Q_B \times D_B$ be arbitrary. We need to show that ψ satisfies the third condition for being a transducer homomorphism. We have

$$\begin{aligned} (p, s)\pi_B\psi_Q &= ((q)\phi_Q, s)\pi_B\psi_Q && \text{by the definition of } p \\ &= (q, s)\pi_A\phi_Q\psi_Q && \text{because } \phi \text{ is a homomorphism} \\ &= (((q)\phi_Q)\psi_Q, s)\pi_{M_A} && \text{because } q_A = \phi\psi \text{ is a homomorphism} \\ &= ((p)\psi_Q, s)\pi_{M_A} && \text{by the definition of } p, \end{aligned}$$

and similarly $(p, s)\lambda_B = ((p)\psi_Q, s)\lambda_{M_A}$ as required. \square

Definition 11.6 (Subtransducers, Assumed Knowledge: 1.3, 11.1).

If A, B are transducers, $Q_A \subseteq Q_B, D_A \subseteq D_B, R_A \subseteq R_B, \pi_A \subseteq \pi_B$ and $\lambda_A \subseteq \lambda_B$, then we call A a *subtransducer* of B . We will sometimes identify a subtransducer of a transducer (with full domain and range) with its state set for simplicity.

The transducers and maps from the following definitions, will be our main focus (due to Theorem 11.16).

Definition 11.7 ((d, n) -transducers, Assumed Knowledge: 10.1, 11.1).

If $d, k \in \mathbb{N}$ and $n, m \in \mathbb{N} \setminus \{0, 1\}$, then we define a (d, n, k, m) -transducer to be a transducer T with $(X_n^*)^d$ as its domain, $(X_m^*)^k$ as its range, and such that for all $q \in Q_T$, we have

$$(q, \varepsilon_d)\pi_T = q, \quad \text{and} \quad (q, \varepsilon_d)\lambda_T = \varepsilon_k.$$

We will be primarily concerned with the case that $d = k$ and $n = m$. Thus to simplify notation we define a (d, n) -transducer to be a (d, n, d, n) -transducer.

Definition 11.8 (Reading infinite words, Assumed Knowledge: 11.7).

Suppose that $d, k \in \mathbb{N}$, $n, m \in \mathbb{N} \setminus \{0, 1\}$, T is a (d, n, k, m) -transducer and $q \in Q_T$. If $w \in ((X_n^*)^{\mathbb{N}})^d$ and $j \in \mathbb{N}$, then we denote the unique element of $(X_n^j)^d$, which is a prefix of w , by $w \upharpoonright_j$. Note that if $i \in \{0, 1, \dots, k-1\}$ and $j_0 \leq j_1$, then $(q, w \upharpoonright_{j_0})\lambda_T \pi_i \subseteq (q, w \upharpoonright_{j_1})\lambda_T \pi_i$.

Thus we can define $f_{T,q} : (X_n^{\mathbb{N}})^d \rightarrow (X_m^{\mathbb{N}} \cup X_m^*)^k$ to be the map with

$$(w)f_{T,q}\pi_i = \bigcup_{j \in \mathbb{N}} (q, w \upharpoonright_j)\lambda_T \pi_i$$

for all $i \in \{0, 1, \dots, k-1\}$. In particular $(w)f_{T,q}$ is the smallest element of $(X_m^{\mathbb{N}} \cup X_m^*)^k$ which is greater than $(q, w')\lambda_T$ for all $w' \in (X_n^*)^d$ with $w' \leq w$.

Remark 11.9 (Induced maps are preserved, Assumed Knowledge: 11.4, 11.8).

Let $d, k \in \mathbb{N}$ and $n, m \in \mathbb{N} \setminus \{0, 1\}$. If A is a (d, n, k, m) -transducer, $q \in Q_A$ and $\phi : A \rightarrow B$ is a strong transducer homomorphism, then the maps $s \rightarrow (q, s)\lambda_A$ and $s \rightarrow ((q)\phi, s)\lambda_B$ are equal. Hence $f_{A,q} = f_{B,(q)\phi_Q}$. In particular this is true of the quotient map q_A .

It is routine to verify that the notion of degenerate in the following definition is equivalent to the GNS notion of degenerate when both are applicable.

Definition 11.10 (Degenerate transducers, Assumed Knowledge: 10.1, 11.8).

Let $d, k \in \mathbb{N}$ and $n, m \in \mathbb{N} \setminus \{0, 1\}$. We say that a (d, n, k, m) -transducer T is *degenerate* if there exist $q \in Q_T$, $x \in \mathfrak{C}_n^d$ such that $(x)f_{T,q} \notin \mathfrak{C}_m^k$.

Lemma 11.11 (Continuity of transducers, Assumed Knowledge: 10.1, 11.10).

Let $d, b \in \mathbb{N}$ and $n, l \in \mathbb{N} \setminus \{0, 1\}$. If T is a non-degenerate (d, n, b, l) -transducer and $q \in Q_T$, then for all $m \in \mathbb{N}$ there is $k \in \mathbb{N}$ such that for all $i \in \{0, 1, \dots, b-1\}$ and $w \in (X_n^k)^d$ we have $|(q, w)\lambda_T \pi_i| \geq m$.

Moreover, the map $f_{T,q} : \mathfrak{C}_n^d \rightarrow \mathfrak{C}_l^b$ is continuous.

Proof. Let T be a non-degenerate (d, n, b, l) -transducer and $q \in Q_T$ be arbitrary. To prove the first claim, suppose for a contradiction that there is $m \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ there is $w \in (X_n^k)^d$ and $i \in \{0, 1, \dots, b-1\}$ with $|(q, w)\lambda_T \pi_i| < m$.

Using this assumption we define a sequence $(w_j)_{j \in \mathbb{N}}$ in $(X_n^*)^d$ by induction as follows:

- (1) Define $w_0 := \varepsilon_d$.

- (2) Suppose that w_j is defined, choose $w_{j+1} \in (X_n^{j+1})^d$ such that
- (a) $w_j \leq w_{j+1}$,
 - (b) there is $i \in \{0, 1, \dots, b-1\}$ such that $|(q, w_{j+1})\lambda_T \pi_i| \leq m$, and
 - (c) for all $k \in \mathbb{N} \setminus \{0, 1, \dots, j\}$, there is $w_{j+1,k} \in (X_n^k)^d$ such that $w_{j+1} \leq w_{j+1,k}$ and there is $i \in \{0, 1, \dots, b-1\}$ such that $|(q, w_{j+1,k})\lambda_T \pi_i| \leq m$.

As b is finite, there must be some $i \in \{0, 1, \dots, d-1\}$ such that for infinitely many (and hence all) $j \in \mathbb{N}$ we have $|(q, w_j)\lambda_T \pi_i| \leq m$. Thus if we define $w \in (X_n^{\mathbb{N}})^d$ to be such that for all $i \in \{0, 1, \dots, b-1\}$ we have

$$(w)\pi_i = \bigcup_{j \in \mathbb{N}} (w_j)\pi_i,$$

then $|(w)f_{T,q}\pi_i| \leq m$. This is a contradiction as T was assumed to be non-degenerate.

The topology on $(X_n^{\mathbb{N}})^d$ is the product topology. Thus it suffices to show that if $x \in (X_n^{\mathbb{N}})^d$, $i \in \{0, 1, \dots, n-1\}$ and $j \in \mathbb{N}$ then there is a neighbourhood U of x , in $(X_n^{\mathbb{N}})^d$, such that for all $y \in U$ we have $(j)((i)((y)f_{T,q})) = (j)((i)((x)f_{T,q}))$. By the first part of the lemma, we can choose $k \in \mathbb{N}$ such that for all $a \in \{0, 1, \dots, b-1\}$ and $w \in (X_n^k)^d$ we have $|(q, w)\lambda_T \pi_a| \geq j+1$. Thus defining

$$U := \left\{ y \in (X_n^{\mathbb{N}})^d \mid ((y)\pi_a) \upharpoonright_j = ((x)\pi_a) \upharpoonright_j \text{ for all } a \in \{0, 1, \dots, d-1\} \right\}$$

gives the required neighbourhood of x . \square

We give two distinct ways of multiplying transducers. The first (Definition 11.12) is an extension of the definition given by GNS, and the second (Definition 11.14) is a categorical product object.

Definition 11.12 (Transducer composition, Assumed Knowledge: 11.1).

If A and B are transducers, such that the range of A is contained in the domain of B , then we define their *composite* by

$$AB := (Q_{AB}, D_{AB}, R_{AB}, \pi_{AB}, \lambda_{AB}).$$

Where

- (1) $Q_{AB} := Q_A \times Q_B$, $D_{AB} := D_A$, and $R_{AB} := R_B$.
- (2) $((a, b), s)\pi_{AB} = ((a, s)\pi_A, (b, (a, s)\lambda_A)\pi_B)$.
- (3) $((a, b), s)\lambda_{A,B} = (b, (a, s)\lambda_A)\lambda_B$.

Lemma 11.13 (Composition works as expected, Assumed Knowledge: 11.8, 11.11, 11.12).

If A is a non-degenerate (a, b, c, d) -transducer and B is a non-degenerate (c, d, e, f) transducer, and $(p, q) \in Q_A \times Q_B$, then $f_{A,p}f_{B,q} = f_{AB,(p,q)}$ (in particular, AB is non-degenerate).

Proof. Let $w \in (X_b^{\mathbb{N}})^a$ be arbitrary. From Definition 11.8, it suffices to show that $(w)f_{A,p}f_{B,q}$ is the smallest element of $(X_f^{\mathbb{N}} \cup X_n^*)^e$ which is greater than $((p, q), w')\lambda_{AB}$ for all $w' \in (X_b^*)^a$ with $w' \leq w$.

Let $w' \in (X_b^*)^a$ be arbitrary with $w' \leq w$. We have $(p, w')\lambda_A \leq (w)f_{A,p}$, thus

$$((p, q), w')\lambda_{AB} = (q, (p, w')\lambda_A)\lambda_B \leq ((w)f_{A,p})f_{B,q} = (w)f_{A,p}f_{B,q}.$$

We now have $(w)f_{A,p}f_{B,q} \geq (w)f_{AB,(p,q)}$. To show equality it suffices to show that $(w)f_{AB,(p,q)} \in (X_f^{\mathbb{N}})^e$. Let $m \in \mathbb{N}$ be arbitrary, we show that for all $i \in \{0, 1, \dots, e-1\}$ we have $|(w)f_{AB,(p,q)}\pi_i| \geq m$. By Lemma 11.11, let $k_A, k_B \in \mathbb{N}$ be such that if $w_A \in (X_b^{k_A})^a$, $w_B \in (X_d^{k_B})^c$ then for all valid choices of i we have

$$|(p, w_A)\lambda_A \pi_i| \geq k_B, \quad \text{and} \quad |(p, w_B)\lambda_B \pi_i| \geq m.$$

It follows that if $w' \in (X_b^{kA})^a$, $w' \leq w$ and $i \in \{0, 1, \dots, e-1\}$, then

$$|(w)f_{AB,(p,q)}\pi_i| \geq |((p,q), w')\lambda_{AB}\pi_i| = |(q, (p, w')\lambda_A)\lambda_B\pi_i| \geq m$$

as required. \square

Definition 11.14 (Transducer products, Assumed Knowledge: 2.3, 4.14, 11.3, 11.15).

If $(A)_{i \in I}$ are transducers, then we define their *product* $\prod_{i \in I} A_i$ to be the transducer P where

$$Q_P := \prod_{i \in I} Q_{A_i}, \quad D_P := \prod_{i \in I} D_{A_i}, \quad R_P := \prod_{i \in I} R_{A_i},$$

and for all $(p_i)_{i \in I} \in Q_P$ and $(s_i)_{i \in I} \in D_P$ we have

$$((p_i)_{i \in I}, (s_i)_{i \in I})\pi_P = ((p_i, s_i)\pi_{A_i})_{i \in I},$$

$$((p_i)_{i \in I}, (s_i)_{i \in I})\lambda_P = ((p_i, s_i)\lambda_{A_i})_{i \in I}.$$

We show in Lemma 11.15 that $(P, ((\pi_i, \pi_i, \pi_i))_{i \in I})$ is a product in the category of transducers and transducer homomorphisms (recall Definition 2.3).

Lemma 11.15 (Transducer products are products, Assumed Knowledge: 2.3, 4.14, 11.3, 11.15).

If $(A)_{i \in I}$ are transducers, then $(\prod_{i \in I} A_i, ((\pi_i, \pi_i, \pi_i))_{i \in I})$ is a product in the category of transducers and transducer homomorphisms.

Proof. Let $P := \prod_{i \in I} A_i$. Note that by the definition of P , for all $i \in I$ the triple (π_i, π_i, π_i) is a transducer homomorphism from P to A_i .

Let B be a transducer and for each $i \in I$, let ϕ_i be a transducer homomorphism from B to A_i . We must show that there is a unique transducer homomorphism $\psi : B \rightarrow P$ such that for all $i \in I$, the composition of the transducer homomorphism ψ with the transducer homomorphism (π_i, π_i, π_i) is ϕ_i .

As $(Q_P, (\pi_i)_{i \in I})$, $(D_P, (\pi_i)_{i \in I})$, and $(R_P, (\pi_i)_{i \in I})$ are products in the categories of sets and functions, there exist unique functions $\psi_Q : Q_B \rightarrow Q_P$, $\psi_D : D_B \rightarrow D_P$, and $\psi_R : R_B \rightarrow R_P$ such that

$$\psi_Q \circ \pi_i = \phi_{iQ}, \quad \psi_D \circ \pi_i = \phi_{iD}, \quad \text{and} \quad \psi_R \circ \pi_i = \phi_{iR}$$

for all $i \in I$. It follows that $\psi := (\psi_Q, \psi_D, \psi_R)$ that it is the only triple of functions whose coordinate wise composition with (π_i, π_i, π_i) results in ϕ_i for all $i \in I$. It therefore suffices to show that ψ is a transducer homomorphism from B to P .

As $(D_P, (\pi_i)_{i \in I})$ and $(R_P, (\pi_i)_{i \in I})$ are also products in the category of semigroups and homomorphisms, the maps ψ_D and ψ_R are semigroup homomorphisms. For all $q \in Q_B$ and $s \in D_B$ we have

$$\begin{aligned} (q, s)\pi_B\psi_Q &= ((q, s)\pi_B\psi_{iQ})_{i \in I} = (((q)\psi_{iQ}, (s)\psi_{iD})\pi_{A_i})_{i \in I} \\ &= (((q)\psi_{iQ})_{i \in I}, ((s)\psi_{iD})_{i \in I})\pi_P = ((q)\psi_Q, (s)\psi_D)\pi_P \end{aligned}$$

and similarly

$$\begin{aligned} (q, s)\lambda_B\psi_R &= ((q, s)\lambda_B\psi_{iR})_{i \in I} = (((q)\psi_{iQ}, (s)\psi_{iD})\lambda_{A_i})_{i \in I} \\ &= (((q)\psi_{iQ})_{i \in I}, ((s)\psi_{iD})_{i \in I})\lambda_P = ((q)\psi_Q, (s)\psi_D)\lambda_P \end{aligned}$$

It follows that ψ is a transducer homomorphism as required. \square

Theorem 11.16 (Transducerable = continuous, Assumed Knowledge: 2.3, 10.1, 11.11).

A function $h : \mathfrak{C}_n^d \rightarrow \mathfrak{C}_n^d$ is continuous if and only if there is a non-degenerate (d, n) -transducer T and $q \in Q_T$ such that $h = f_{T,q}$.

Proof. (\Leftarrow): This is immediate from Lemma 11.11.

(\Rightarrow): Let $h : \mathfrak{C}_n^d \rightarrow \mathfrak{C}_n^d$ be a continuous map. We define a (d, n) -transducer T and then proceed to show that T is well-defined, non-degenerate and satisfies $f_{T,\varepsilon_d} = h$.

- (1) Define $Q_T := (X_n^*)^d$, $D_T := (X_n^*)^d$, $R_T := (X_n^*)^d$.
- (2) Define $\pi_T : Q_T \times D_T \rightarrow Q_T$ by $(q, w)\pi_T := qw$.
- (3) For each $i \in \{0, 1, \dots, n-1\}$, define $\lambda_{T,i} : Q_T \times D_T \rightarrow X_n^*$ as follows:
 - (a) If $q = w = \varepsilon_d$, then $(q, w)\lambda_{T,i} = \varepsilon$;
 - (b) if $q = \varepsilon_d$ and $w \neq \varepsilon_d$ and $|(w\mathfrak{C}_n^d)h\pi_i| > 1$, then $(q, w)\lambda_{T,i}$ is defined to be the longest common prefix of all the words in $(w\mathfrak{C}_n^d)h\pi_i$;
 - (c) if $q = \varepsilon_d$, $w \neq \varepsilon_d$, $(w\mathfrak{C}_n^d)h\pi_i = \{x\}$ for some $x \in X_n^{\mathbb{N}}$, and $(q, w')\lambda_{T,i}$ is already defined for all $w' < w$, then $(q, w)\lambda_{T,i}$ is defined to be the prefix of x with length

$$\max \left(\{ |(w)\pi_0|, |(w)\pi_1|, \dots, |(w)\pi_{d-1}| \} \cup \left\{ |(q, w')\lambda_{T,i}| \mid w' < w \right\} \right);$$

- (d) if $q \neq \varepsilon_d$, then $(q, w)\lambda_{T,i}$ is defined to be $((\varepsilon_d, qw)\lambda_{T,i})\lambda_{(\varepsilon_d, q)\lambda_{T,i}}^{-1}$.

We then define $\lambda_T : Q_T \times D_T \rightarrow R_T$ by $\lambda_T = \langle \lambda_{T,0}, \lambda_{T,1}, \dots, \lambda_{T,d-1} \rangle_{(X_n^*)^d}$ (Definition 2.3).

To show that T is well-defined, we first need to show that if $i \in \{0, 1, \dots, d-1\}$, $q, w \in D_T$ and $q \neq \varepsilon_d$, then $(\varepsilon_d, q)\lambda_{T,i}$ is a prefix of $(\varepsilon_d, qw)\lambda_{T,i}$. There are 2 cases to consider:

- (1) If $|(qw\mathfrak{C}_n^d)h\pi_i| \geq 2$, then as $(qw\mathfrak{C}_n^d)h\pi_i \subseteq (q\mathfrak{C}_n^d)h\pi_i$, it follows that $(\varepsilon_d, q)\lambda_{T,i}$ is a common prefix of all the words in $(q\mathfrak{C}_n^d)h\pi_i$. Hence $(\varepsilon_d, q)\lambda_{T,i}$ is a common prefix of all the words in $(qw\mathfrak{C}_n^d)h\pi_i$, so it is a prefix of the longest common prefix $(\varepsilon_d, qw)\lambda_{T,i}$.
- (2) If $(qw\mathfrak{C}_n^d)h\pi_i = \{x\}$ for some $x \in X_n^{\mathbb{N}}$, then (via either 3b or 3c) $(\varepsilon_d, q)\lambda_{T,i}$ is a finite prefix of x . Moreover by definition $(\varepsilon_d, qw)\lambda_{T,i}$ is also a prefix of x with $|(\varepsilon_d, qw)\lambda_{T,i}| \geq |(\varepsilon_d, q)\lambda_{T,i}|$. Thus $(\varepsilon_d, qw)\lambda_{T,i} \geq (\varepsilon_d, q)\lambda_{T,i}$.

We next need to show that T is a transducer. If $q \in Q_T$ and $s, t \in D_T$ are arbitrary, then $(q, st)\pi_T = qst = ((q, s)\pi_T, t)\pi_T$, so π_T is an action of D_T on Q_T . Moreover for all $i \in \{0, 1, \dots, d-1\}$ we have

$$\begin{aligned} ((q, s)\lambda_T(qs, t)\lambda_T)\pi_i &= ((\varepsilon_d, qs)\lambda_T\pi_i)\lambda_{(\varepsilon_d, q)\lambda_{T,\pi_i}}^{-1}((\varepsilon_d, qst)\lambda_T\pi_i)\lambda_{(\varepsilon_d, qs)\lambda_{T,\pi_i}}^{-1} \\ &= (((\varepsilon_d, qst)\lambda_T\pi_i)\lambda_{(\varepsilon_d, qs)\lambda_{T,\pi_i}}^{-1})\lambda_{(\varepsilon_d, qs)\lambda_{T,\pi_i}}\lambda_{(\varepsilon_d, q)\lambda_{T,\pi_i}}^{-1} \\ &= ((\varepsilon_d, qst)\lambda_{T,i})\lambda_{(\varepsilon_d, q)\lambda_{T,i}}^{-1} \\ &= (q, st)\lambda_T\pi_i. \end{aligned}$$

We next show that $h = f_{T,\varepsilon_d}$. Let $i \in \{0, 1, \dots, d-1\}$ be arbitrary. By definition, if $w' \in (X_n^*)^d$ and $w' \leq w \in (X_n^{\mathbb{N}})^d$, then the word $(\varepsilon_d, w')\lambda_{T,i}$ is a prefix of $(w)h\pi_i$. So we need only show that for all $m \in \mathbb{N}$, there is $k_m \in \mathbb{N}$ such that if $w' \in (X_n^{k_m})^d$, then $|(\varepsilon_d, w')\lambda_{T,i}| \geq m$.

For each $v \in (X_n^m)^d$, we have by assumption that $(v\mathfrak{C}_n^d)h^{-1}$ is clopen. So there is some $k_v \in \mathbb{N}$ and $S_v \subseteq (X_n^{k_v})^d$ such that

$$(v\mathfrak{C}_n^d)h^{-1} = \bigcup_{s \in S_v} s\mathfrak{C}_n^d.$$

Let $k_m := \max \left(\left\{ k_v \mid v \in (X_n^m)^d \right\} \cup \{m+1\} \right)$. Thus if $w' \in (X_n^{k_m})^d$, then the longest common prefix of all the elements of the set $((w')\mathfrak{C}_n^d)h\pi_i$ has length at least m . Thus $|(\varepsilon_d, w')\lambda_{T,i}| \geq m$. It follows that indeed $h = f_{T,\varepsilon_d}$.

Finally we show that T is not degenerate. Let $q \in Q_T$ and $w \in (X_n^{\mathbb{N}})^d$ be arbitrary. Suppose for a contradiction that $(w)f_{T,q} \notin (X_n^{\mathbb{N}})^d$. It follows that

$$(qw)h = (w)f_{T,\varepsilon_d} = (\varepsilon_d, q)\lambda_T(w)f_{T,q} \notin (X_n^{\mathbb{N}})^d,$$

this is a contradiction. \square

It will be useful to have concrete transducers representing homeomorphisms of Cantor spaces, we constructed such transducers in the proof of the previous theorem, but these transducers have a rather messy definition. In the case of homeomorphisms this can be made a bit simpler, as we will now see.

Definition 11.17 (Transducing homeomorphisms, Assumed Knowledge: 5.1, 10.1, 11.4, 11.18).

If $h \in \text{Aut}(\mathfrak{C}_n^d)$, then we define T_h to be the (d, n) -transducer with

- (1) $Q_{T_h} := (X_n^*)^d$, $D_{T_h} := (X_n^*)^d$, and $R_{T_h} := (X_n^*)^d$.
- (2) $(s, t)\pi_{T_h} = st$.
- (3) $((s, t)\lambda_{T_h})\pi_i$ is $(b)\lambda_a^{-1}$, where b is the longest common prefix of the words in the set $((st\mathfrak{C}_n^d)h)\pi_i$ and a is the longest common prefix of the words in the set $((s\mathfrak{C}_n^d)h)\pi_i$.

Moreover, we define $M_h := M_{T_h}$ and $e_h := (\varepsilon_d)q_{T_h,Q}$. By Lemma 11.18 we have $f_{M_h, e_h} = h$. Thus we call M_h the *minimal transducer for h* .

Lemma 11.18 (Minimal transducers are nice, Assumed Knowledge: 11.9, 11.16, 11.17).

If $h \in \text{Aut}(\mathfrak{C}_n^d)$ then T_h is a well-defined non-degenerate (d, n) -transducer and $f_{M_h, e_h} = h$. Moreover if $q \in Q_{M_h}$, then $f_{M_h, q}$ is a homeomorphism from \mathfrak{C}_n^d to a clopen subset of \mathfrak{C}_n^d .

Proof. Note that when $h \in \text{Aut}(\mathfrak{C}_n^d)$ is a homeomorphism and $i \in \{0, 1, \dots, d-1\}$, the longest common prefix of the words in $(\mathfrak{C}_n^d)h\pi_i$ is ε . Moreover as h maps open sets to open sets, if $w \in (X_n^*)^d$ then $(w\mathfrak{C}_n^d)h\pi_i$ is an open subset of \mathfrak{C}_n . In particular $|(w\mathfrak{C}_n^d)h\pi_i| \neq 1$. It follows that T_h is identical to the transducer T defined in the proof of Theorem 11.16. Thus $f_{T_h, \varepsilon_d} = h$. By Remark 11.9, it follows that $f_{T_h, \varepsilon_d} = f_{M_h, e_h}$, so in particular $f_{M_h, e_h} = h$.

Let $q \in Q_{M_h}$ be arbitrary and let $w' \in Q_{T_h}$ be such that $(w')q_{T_h, Q} = q$. We next show that $f_{M_h, q}$ is injective. Suppose that $w_0, w_1 \in \mathfrak{C}_n^d$ and $(w_0)f_{M_h, q} = (w_1)f_{M_h, q}$. We have

$$(w'w_0)h = (e_h, w')\lambda_{M_h}(w_0)f_{M_h, q} = (e_h, w')\lambda_{M_h}(w_1)f_{M_h, q} = (w'w_1)h.$$

Thus $w'w_0 = w'w_1$, and hence $w_0 = w_1$ as required.

As $w'\mathfrak{C}_n^d$ is a clopen set, the set $(w'\mathfrak{C}_n^d)h$ is also a clopen set. Moreover if $x \in \mathfrak{C}_n^d$ is arbitrary, then

$$(w'x)h = (e_h, w')\lambda_{M_h}(x)f_{M_h, q}.$$

Thus as h is a homeomorphism, $f_{M_h, q}$ is a homeomorphism onto its image. Moreover every element of $(w'\mathfrak{C}_n^d)h$ is greater than $(e_h, w')\lambda_{M_h}$. So there is some finite $S \subseteq (X_n^*)^d$ such that

$$(w'\mathfrak{C}_n^d)h = \bigcup_{s \in S} (e_h, w')\lambda_{M_h} s \mathfrak{C}_n^d.$$

It follows that $(\mathfrak{C}_n^d)f_{M_h, q} = \bigcup_{s \in S} s\mathfrak{C}_n^d$ and is thus clopen as required. \square

The following theorem establishes that the GNS notion of a reduced transducer (see Proposition 2.8 of [15]) representing a homeomorphism still applies in our context.

Theorem 11.19 (Minimal transducers are unique, Assumed Knowledge: 11.3, 11.4, 11.5, 11.17, 11.18).

Let $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. Suppose that $h \in \text{Aut}(\mathfrak{C}_n^d)$, and (T, q) is a pair such that:

- (1) T is a (d, n) -transducer, $q \in Q_T$ and $f_{T,q} = h$;
- (2) if $p_0, p_1 \in Q_T$ and $p_0 \sim_{M_T} p_1$, then $p_0 = p_1$;
- (3) $Q_T = (q, D_T)\pi_T$;
- (4) if $p \in Q_T$, $w \in R_T$ and $(\mathfrak{C}_n^d)f_{T,p} \subseteq w\mathfrak{C}_n^d$, then $w = \varepsilon_d$ (we call this condition **complete response**).

Then there is a unique strong transducer isomorphism $\phi : T \rightarrow M_h$ with $(q)\phi_Q = e_h$. Moreover if $h \in \text{Aut}(\mathfrak{C}_n^d)$ is arbitrary, then the pair (M_h, e_h) satisfies all of the above conditions.

Proof. Let T be a (d, n) -transducer and $q \in Q_T$ be such that the above conditions are satisfied. If $\phi : T \rightarrow M_h$ is as required then we must have:

- (1) Both ϕ_D, ϕ_R are the identity function on $(X_n^*)^d$;
- (2) if $w \in D_T$ then $((q, w)\pi_T)\phi_Q = ((q)\phi_Q, w)\pi_{M_h} = (e_h, w)\pi_{M_h}$.

As $Q_T = (q, D_T)\pi_T$, it follows that the above two conditions define ϕ . Thus if the required ϕ exists, then it is unique.

Define $\psi_Q : Q_{T_h} \rightarrow Q_T$ by $(p)\psi_Q = (q, p)\pi_T$. As $D_T = Q_{T_h}$, it follows from the second condition on T that ψ_Q is surjective. Let ψ_D, ψ_R be the identity functions on $(X_n^*)^d$. It follows that if $\psi := (\psi_Q, \psi_D, \psi_R)$ is a transducer homomorphism, then ψ is a strong quotient map. We now show that ψ is a transducer homomorphism.

Let $p \in Q_{T_h}$, $s, t \in (X_n^*)^d$ be arbitrary. Then

$$(p, s)\pi_{T_h}\phi_D = (q, (p, s)\pi_{T_h})\pi_T = (q, ps)\pi_T = ((q, p)\pi_T, s)\pi_T = ((p)\phi_Q, s)\pi_T.$$

We need to show that $(p, s)\lambda_{T_h} = ((p)\phi_Q, s)\lambda_T$. For all $i \in \{0, 1, \dots, d-1\}$ we have by definition that $(p, s)\lambda_{T_h}\pi_i = (b)\lambda_a^{-1}$, where b is the longest common prefix of the elements of $(ps\mathfrak{C}_n^d)h\pi_i$ and a is the longest common prefix of the elements of $(p\mathfrak{C}_n^d)h\pi_i$. Define $a' := (q, p)\lambda_T$, $b' := (q, ps)\lambda_T$, as T is a transducer, we have

$$a'((p)\phi_Q, s)\lambda_T = (q, p)\lambda_T((q, p)\pi_T, s)\lambda_T = (q, ps)\lambda_T = b'.$$

Thus $((p)\phi_Q, s)\lambda_T = (b')\lambda_{a'}^{-1}$. To conclude that $(p, s)\lambda_{T_h} = ((p)\phi_Q, s)\lambda_T$, it suffices to show that $a' = a$ and $b' = b$. We show $a' = a$, showing that $b' = b$ is almost identical. As $a' = (q, p)\lambda_T$ and $f_{T,q} = h$, it follows that a' is a prefix of all words in the set $(p\mathfrak{C}_n^d)h\pi_i$. In particular $a' \leq a$. Let $w' \in X_n^*$ be such that $a'w' = a$. For all $x \in \mathfrak{C}_n^d$, we have

$$a'((x)f_{T,(q,p)\pi_T}\pi_i) = (px)f_{T,q}\pi_i = (px)h\pi_i \in a\mathfrak{C}_n = a'w'\mathfrak{C}_n.$$

Thus $(\mathfrak{C}_n^d)f_{T,(q,p)\pi_T}\pi_i \subseteq w'\mathfrak{C}_n$. As T has complete response, it follows that $w' \leq (\varepsilon_d)\pi_i = \varepsilon$ and thus $a = a'$.

We now have that $\psi : T_h \rightarrow T$ is a strong quotient map. By Lemma 11.5, there is a strong quotient map $\phi : T \rightarrow M_h$ with $\psi\phi = q_{T_h}$. From this equality it follows that $(q)\phi = (\varepsilon_d)q_{T_h}Q = e_h$. We next

show that ϕ_Q is injective. Let $p_0, p_1 \in Q_T$ be arbitrary with $(p_0)\phi_Q = (p_1)\phi_Q$, we will show that $p_0 = p_1$. Let $p'_0, p'_1 \in Q_{T_h}$ be such that $(p'_0)\psi_Q = p_0$ and $(p'_1)\psi_Q = p_1$. We have

$$(p'_0)q_{T_h} = (p'_0)\psi_Q\phi_Q = (p_0)\phi_Q = (p_1)\phi_Q = (p'_1)\psi_Q\phi_Q = (p'_1)q_{T_h}.$$

Thus $p'_0 \sim_{M_{T_h}} p'_1$, and as ψ is a transducer homomorphism it follows that $p_0 \sim_{M_T} p_1$. By the second assumption on T , it follows that $p_0 = p_1$. As ψ_Q is injective, it follows that ϕ is a strong transducer isomorphism as required.

Finally we show that M_h and e_h satisfy the required conditions. The first condition is immediate from Lemma 11.18. The second condition follows from the definition of M_h . The third condition follows as $(p)q_{T_h} = (e_h, p)\pi_{M_h}$ for all $p \in Q_{T_h}$. The fourth condition is immediate from the definition on T_h together with Remark 11.9. \square

12. GENERALIZING THE SYNCHRONIZING HOMEOMORPHISMS OF BLEAK, CAMERON, MAISSEL, NAVAS, AND OLUKOYA

We now have a class of transducers for describing continuous maps. We restrict our transducers to those which represent automorphisms of dV_n . As was the case in [3], this is done with a synchronization condition.

Definition 12.1 (Synchronizing transducers, Assumed Knowledge: 1.3, 1.6, 11.7).

Suppose that T is a (d, n) -transducer and $k \in \mathbb{N}$. We say that T is *synchronizing at level k* if for all $p, q \in Q_T$ and $w \in (X_n^*)^d$ with $\min \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq k$, we have

$$(p, w)\pi_T = (q, w)\pi_T.$$

We say that T is *synchronizing* if there is a level at which it is synchronizing and we call the smallest such level the *synchronizing length* of T . In this case we define the function

$$\mathfrak{s}_T := \left\{ (w, q) \in (X_n^*)^d \times Q_T \mid \text{for all } p \in Q_T \text{ we have } (w, p)\pi_T = q \right\}.$$

The map \mathfrak{s}_T is essentially π_T restricted to the part of its domain where the input state is not needed. The image of \mathfrak{s}_T , denoted $\text{Core}(T)$, is called the *core* of T .

Remark 12.2 (Core transducers, Assumed Knowledge: 11.6, 12.1).

Suppose that T is a (d, n) -transducer with synchronizing level k . If $q \in \text{Core}(T)$, $s \in (\{q\})\mathfrak{s}_T^{-1}$ and $w \in D_T$ then

$$(q, w)\pi_T = ((s)\mathfrak{s}_T, w)\pi_T = (sw)\mathfrak{s}_T \in \text{Core}(T).$$

Thus $\text{Core}(T)$ is a subtransducer of T . Moreover $\mathfrak{s}_T \subseteq \mathfrak{s}_{\text{Core}(T)}$ and $\text{Core}(T) = ((X_n^k)^d)\mathfrak{s}_T$, so $\text{Core}(T)$ has at most n^{kd} states and has synchronizing length at most k .

The core of a synchronizing transducer, consists precisely of the state which are “always reachable”, in particular it is the smallest subtransducer which still has the same domain and range as the original transducer.

Definition 12.3 (Identity states, Assumed Knowledge: 11.6, 11.19).

If $I \in \text{Aut}(\mathfrak{C}_n^d)$ is the identity map, then (by Theorem 11.19) M_I is a transducer with one state e_I and output function given by $(e_I, w)\lambda_{M_I} = w$ (for all $w \in (X_n^*)^d$). We call a transducer strongly isomorphic to this transducer an *identity transducer*.

If T is a (d, n) -transducer and $q \in Q_T$ is such that $\{q\}$ is a subtransducer of T isomorphic to an identity transducer, then we call q an *identity state*.

We can now show that, as was the case with $G_{n,r}$ [3], the elements of dV_n are precisely those elements with “trivial” cores.

Proposition 12.4 (dV_n transducers, Assumed Knowledge: 10.2, 11.19, 12.1, 12.2, 12.3).

If $d \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$ and $h \in \text{Aut}(\mathfrak{C}_n^d)$, then $h \in dV_n$ if and only if M_h is a synchronizing transducer whose core is an identity transducer.

Proof. (\Rightarrow): Let $h \in dV_n$, and let F_1, F_2 be complete prefix codes of \mathfrak{C}_n^d and $\phi : F_1 \rightarrow F_2$ be a bijection such that $h = f_\phi$ (as in Definition 10.2). Define

$$k := \max \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\}, w \in F_1 \right\} \right).$$

Let $w \in (X_n^*)^d$ be arbitrary with $\min \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq k$. We show for all $s \in (X_n^*)^d$ that $(w, s)\lambda_{T_h} = s$.

By the definitions of k and F_1 , there is a unique $w' \in F_1$ with $w' \leq w$. It follows that $(e_h, w)\lambda_{M_h} = (\varepsilon_d, w)\lambda_{T_h} = (w)\lambda_{w'}^{-1}\lambda_{(w')\phi}$. So if $s \in (X_n^*)^d$ is arbitrary, then we have

$$(e_h, ws)\lambda_{M_h} = (ws)\lambda_{w'}^{-1}\lambda_{(w')\phi} = (w)\lambda_{w'}^{-1}\lambda_{(w')\phi}s$$

and thus $(w, s)\lambda_{T_h} = s$. It follows that if $v, w \in (X_n^*)^d$ are arbitrary with

$$\min \left(\left\{ |(v)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq k, \quad \min \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq k,$$

then $(v)q_{T_h Q} = (w)q_{T_h Q}$ and moreover this state is an identity state. Thus M_h is a synchronizing of length at most k and the core of M_h consists of a single identity state.

(\Leftarrow): Let $h \in \text{Aut}(\mathfrak{C}_n^d)$ be such that M_h has an identity core. We show that $h \in dV_n$. Let k be the synchronizing length of M_h and let I be the core state of M_h . Define $F_1 := (X_n^k)^d$ and note that F_1 is a complete prefix code for \mathfrak{C}_n^d . Define $\phi : F_1 \rightarrow (X_n^*)^d$ by $(w)\phi = (e_h, w)\lambda_{M_h}$. If $x \in \mathfrak{C}_n^d$ and $w \in (X_n^k)^d$ are arbitrary then

$$(wx)h = (wx)f_{M_h, e_h} = (e_h, w)\lambda_T(x)f_{M_h, I} = (w)\phi x = (wx)\lambda_w^{-1}\lambda_{(w)\phi}.$$

Let $F_2 := \text{img}(\phi)$. As h is a homeomorphism, it follows from the equality $(wx)h = (wx)\lambda_w^{-1}\lambda_{(w)\phi}$ that F_2 is a complete prefix code and ϕ is a bijection between F_1 and F_2 . Thus $h = f_\phi \in dV_n$. \square

Definition 12.5 (Defining transducers with diagrams, Assumed Knowledge: 10.1, 11.7).

As we see in the Example 12.6, it is useful to think about transducers as diagrams consisting of collections of dots (representing states) with arrows between them (representing the transition and output functions). If T is a transducer and X is a generating set for D_T , then we can do this by having an arrow originating from each state for each element of X . We give each arrow from a state q to a state p a label of the form x/s where $x \in X$ and $s \in R_T$. The meaning of this being that

$$(q, x)\pi_T = p \quad \text{and} \quad (q, x)\lambda_T = s.$$

As X generates D_T , the conditions in the definition of a transducer determine the rest of the maps π_T and λ_T . If an element of D_T can be expressed as a product of elements of X in multiple ways then this may result in π_T or λ_T not being well-defined by a particular diagram, so not all diagrams of this type give well-defined transducers.

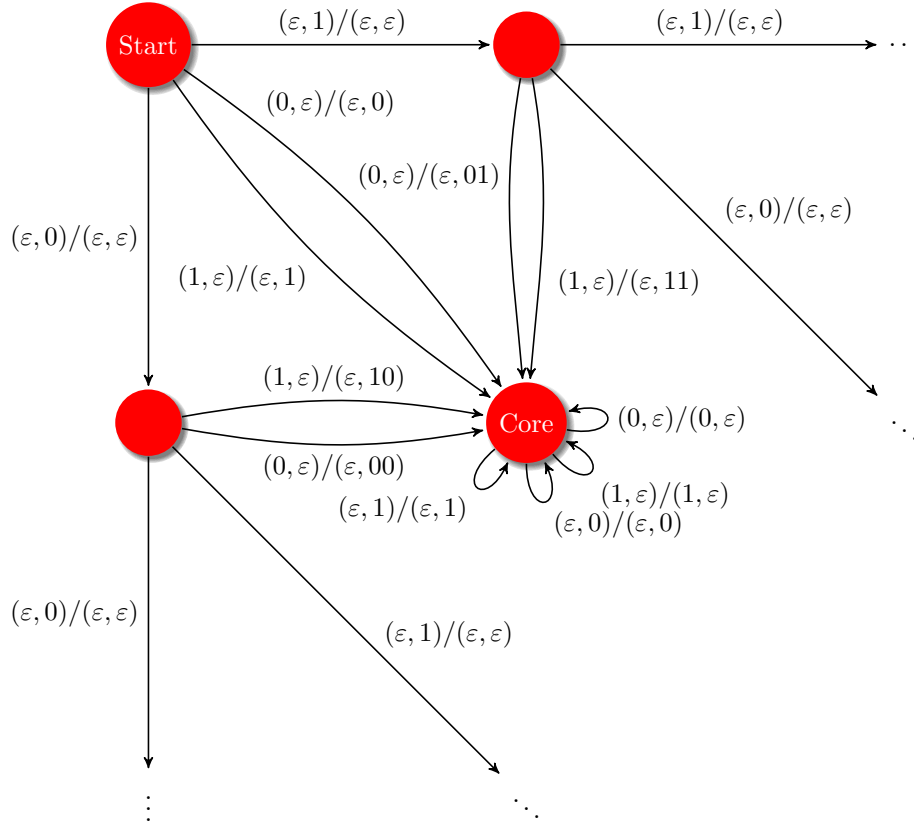


FIGURE 1. Part of a minimal transducer with $(X_2^*)^2$ as domain and range. This represents the baker's map in $2V$ (this transducer is infinite and every state has an edge with the core as its target).

We will be particularly interested in (d, n) -transducers using the generating set

$$\left\{ x_{d,i} \mid x \in X_n, i \in \{0, 1, \dots, d-1\} \right\}$$

(recall Definition 10.1). In the case that $n = 1$ this is particularly easy as each element of X_n^* can be expressed as a product of generators in a unique fashion, thus all diagrams define valid transducers (as was the case in GNS).

Example 12.6 (Baker's map, Assumed Knowledge: 10.2, 11.19, 12.3, 12.4, 12.5).

Unlike for V_n , the minimal transducers for elements of dV_n can sometimes be infinite. For example if we define $F_1 := \{(0, \varepsilon), (1, \varepsilon)\}$, $F_2 := \{(\varepsilon, 0), (\varepsilon, 1)\}$ and $\phi : F_1 \rightarrow F_2$ by

$$(0, \varepsilon)\phi = (\varepsilon, 0), \quad (1, \varepsilon)\phi = (\varepsilon, 1).$$

Then the map $b := f_\phi \in 2V$ (called the *baker's map*) has the property that M_b has infinitely many states.

This can be seen by observing that if $w \in (X_2^*)^2$ and $(w)\pi_0 = \varepsilon$, then $(w, (0, \varepsilon))\lambda_T = (0, w)$. Thus if $v, w \in (X_2^*)^2$ are distinct then $(e_b, v)\pi_{M_b} \neq (e_b, w)\pi_{M_b}$. In Figure 1, we see a diagram of the transducer M_b .

Definition 12.7 (Synchronizing homeomorphisms, Assumed Knowledge: 11.19, 12.1).

We say an element $h \in \text{Aut}(\mathfrak{C}_n^d)$ is *synchronizing* if M_h is synchronizing. We define $d\mathcal{S}_{n,1}$ to be the set of synchronizing elements of $\text{Aut}(\mathfrak{C}_n^d)$.

The subscript 1 in the above definition is due to the overlap between these sets of homeomorphisms and the homeomorphism groups $\mathcal{S}_{n,r}$ from [3] ($\mathcal{S}_{n,1} = 1\mathcal{S}_{n,1}$). Later we will use the notations $d\mathcal{B}_{n,1}$ and $d\mathcal{O}_{n,1}$ for the same reason.

We will need to be able to compute the minimal transducer representative (see Theorem 11.19) of a product of homeomorphisms from the minimal transducers representing the individual homeomorphisms. For this we will need to be able to convert a transducer defining a homeomorphism into one with complete response.

A means of doing this is provided by GNS (see Proposition 2.16 of [15]), we now show that a similar construction applies to our context.

Definition 12.8 (Removing incomplete response, Assumed Knowledge: 11.18, 11.19, 12.9).

Suppose that T is a (d, n) -transducer and for all $q \in Q_T$ the function $f_{T,q}$ is a homeomorphism from \mathfrak{C}_n^d to a clopen subset of \mathfrak{C}_n^d . We define

$$CR(T) := (Q_T, D_T, R_T, \pi_T, \lambda_{CS(T)}).$$

The map $\lambda_{CR(T)} : Q_T \times D_T \rightarrow R_T$ is defined by

$$(q, s)\lambda_{CR(T)}\pi_i = (b)\lambda_a^{-1},$$

where b is the longest common prefix of the elements of $(s\mathfrak{C}_n^d)f_{T,q}\pi_i$ and a is the longest common prefix of the elements of $(\mathfrak{C}_n^d)f_{T,q}\pi_i$. We show in Lemma 12.9 that $CR(T)$ is a well-defined (d, n) -transducer.

Lemma 12.9 (Removing incomplete response works, Assumed Knowledge: 11.18, 11.19, 12.8).

Suppose that T is a (d, n) -transducer and for all $q \in Q_T$ the function $f_{T,q}$ is a homeomorphism from \mathfrak{C}_n^d to a clopen subset of \mathfrak{C}_n^d . The object $CR(T)$ defined in Definition 12.8 is a (d, n) -transducer with complete response (see Theorem 11.19).

Moreover $CR(T)$ is non-degenerate and if $q \in Q_T = Q_{CR(T)}$ and p is the largest common prefix of the elements of $(\mathfrak{C}_n^d)f_{T,q}$, then $f_{CR(T),q} = f_{T,q}\lambda_p^{-1}$.

Proof. We first show that $CR(T)$ is a (d, n) -transducer. We need to show that the image of $\lambda_{CR(T)}$ is contained in R_T . For each $p \in Q_T$, the map $f_{T,p}$ maps open sets to open sets, so if $i \in \{0, 1, \dots, d-1\}$ then $(s\mathfrak{C}_n^d)f_{T,p}\pi_i$ is an open subset of \mathfrak{C}_n^d . Thus $|(s\mathfrak{C}_n^d)f_{T,p}\pi_i| \neq 1$ and so the longest common prefix of the words in $(s\mathfrak{C}_n^d)f_{T,p}\pi_i$ is finite.

Let $p \in Q_{CR(T)}$ and $s, t \in D_{CR(T)}$ be arbitrary. As T is a transducer, the map $\pi_T = \pi_{CR(T)}$ is an action. We need to show that

$$(p, st)\lambda_{CR(T)} = (p, s)\lambda_{CR(T)}((p, s)\pi_{CR(T)}, t)\lambda_{CR(T)}.$$

We have by assumption that $(p, st)\lambda_T = (p, s)\lambda_T((p, s)\pi_T, t)\lambda_T$, and thus if $x \in \mathfrak{C}_n^d$ then

$$(sx)f_{T,p} = (p, s)\lambda_{CR(T)}(x)f_{T,(p,s)\pi_T}$$

so if $i \in \{0, 1, \dots, d-1\}$, then a is a prefix of all the words in the set $(\mathfrak{C}_n^d)f_{T,p}\pi_i$ if and only if $(p, s)\lambda_{CR(T)}\pi_i a$ is a common prefix of all the words in the set $(s\mathfrak{C}_n^d)f_{T,p}\pi_i$. The required condition on $\lambda_{CR(T)}$ follows.

We next show that $CR(T)$ has complete response. Let $p \in Q_{CR(T)}$ and suppose that $w \in (X_n^*)^d$ is such that $(\mathfrak{C}_n^d)f_{CR(T),q} \subseteq w\mathfrak{C}_n^d$. Let $i \in \{0, 1, \dots, d-1\}$ be arbitrary and let a be the longest common prefix of all the words in the set $((\mathfrak{C}_n^d)f_{T,p})\pi_i$. By the definition of $\lambda_{CS(T)}$, $a((w)\pi_i)$ is also a common prefix of all the words in the set $((\mathfrak{C}_n^d)f_{T,p})\pi_i$. Hence $a \leq a((w)\pi_i) \leq a$ and $((w)\pi_i) = \varepsilon$. As i was arbitrary it follows that $w = \varepsilon_d$ as required.

Now let $q \in Q_T$ be arbitrary. By assumption, the set $(\mathfrak{C}_n^d)f_{T,q}$ is open, so for each $i \in \{0, 1, \dots, d-1\}$, the set $(\mathfrak{C}_n^d)f_{T,q}\pi_i$ is open. Thus the element $p \in (X_n^*)^d$ with each $(p)\pi_i$ the longest common prefix of the words in the set $(\mathfrak{C}_n^d)f_{T,q}\pi_i$ is the maximum element of $(X_n^*)^d$ smaller than all the elements of $(\mathfrak{C}_n^d)f_{T,q}$.

For all $w \in \mathfrak{C}_n^d$ and $w' \in (X_n^*)^d$ with $w' \leq w$, we have that $p(q, w')\lambda_{CR(T)}$ is the largest element of $(X_n^*)^d$ which is smaller than all the elements of $(w'\mathfrak{C}_n^d)f_{T,q}$. In particular $p(q, w')\lambda_{CR(T)} \leq (w)f_{T,q}$. As h is continuous, it follows that if $m \in \mathbb{N}$, then there is $k \in \mathbb{N}$ such that for all $w' \in (X_n^k)^d$, there is $v \in (X_n^m)^d$ with $(w'\mathfrak{C}_n^d)f_{T,q} \subseteq v\mathfrak{C}_n^d$. Thus $p(w)f_{CR(T),q} \in \mathfrak{C}_n^d$. As $p(w)f_{CR(T),q}$ is also comparable with $(w)f_{T,q}$, it follows that $(w)f_{T,q} = p(w)f_{CR(T),q}$. As $w \in \mathfrak{C}_n^d$ was arbitrary, the result follows. \square

Lemma 12.10 (Cores from cores, Assumed Knowledge: 11.12, 11.11, 12.1, 12.2).

If A, B are synchronizing, non-degenerate (d, n) -transducers then so is their composite AB . Moreover $\text{Core}(AB)$ is a subtransducer of the composite transducer $\text{Core}(A)\text{Core}(B)$.

Proof. Let k_A, k_B be the synchronizing lengths of A and B respectively. By Remark 12.2, we have $\text{Core}(A)$ is finite. Moreover by Lemma 11.11, for each $q \in \text{Core}(A)$, there is $k_q \in \mathbb{N}$ such that if $w \in (X_n^*)^d$, $\min\left(\left\{(w)\pi_i \mid i \in \{0, 1, \dots, d-1\}\right\}\right) \geq k_q$ and $i \in \{0, 1, \dots, d-1\}$ then $|(q, w)\lambda_A| \geq k_B$.

Let $(p_0, p_1) \in Q_{AB}$ and $w \in D_{AB} = (X_n^*)^d$ be arbitrary such that

$$\min\left(\left\{(w)\pi_i \mid i \in \{0, 1, \dots, d-1\}\right\}\right) \geq k_A + \max\left(\left\{k_q \mid q \in \text{Core}(A)\right\}\right).$$

It suffices to show that $((p_0, p_1), w)\pi_{AB}$ is an element of $\text{Core}(A)\text{Core}(B)$ which does not depend on (p_0, p_1) . Let $w = w_0w_1$ where $w_0 \in (X_n^{k_A})^d$. Then

$$\begin{aligned} ((p_0, p_1), w)\pi_{AB} &= ((p_0, w)\pi_A, (p_1, (p_0, w)\lambda_A)\pi_B) \\ &= ((w)\mathfrak{s}_A, (p_1, (p_0, w)\lambda_A)\pi_B) \\ &= ((w)\mathfrak{s}_A, (p_1, (p_0, w_0)\lambda_A((p_0, w_0)\pi_A, w_1)\lambda_A)\pi_B) \\ &= ((w)\mathfrak{s}_A, ((p_1, (p_0, w_0)\lambda_A)\pi_B, ((p_0, w_0)\pi_A, w_1)\lambda_A)\pi_B) \\ &= ((w)\mathfrak{s}_A, ((p_1, (p_0, w_0)\lambda_A)\pi_B, ((w_0)\mathfrak{s}_A, w_1)\lambda_A)\pi_B) \\ &= ((w)\mathfrak{s}_A, (((w_0)\mathfrak{s}_A, w_1)\lambda_A)\mathfrak{s}_B). \end{aligned}$$

Note that p_0 and p_1 are not mentioned in the final expression above. Moreover this is a state of the composite of $\text{Core}(A)$ and $\text{Core}(B)$. \square

Corollary 12.11 (Synchronizing monoid, Assumed Knowledge: 4.5, 11.18, 11.19, 12.7, 12.9, 12.10).

If $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, then $d\mathcal{S}_{n,1}$ is a monoid (under composition of functions).

Proof. First note that from Definition 12.3, we know that $\text{id}_{\mathfrak{C}_n^d} \in d\mathcal{S}_{n,1}$. Let $f, g \in d\mathcal{S}_{n,1}$ be arbitrary. It suffices to show that $fg \in d\mathcal{S}_{n,1}$. By Lemmas 11.13 and 11.18, we have that $f_{M_f M_g, (e_f, e_g)} = fg$.

Moreover by Lemma 12.10, the transducer $M_f M_g$ is synchronizing. Let M' be the subtransducer of $M_f M_g$ with state set $((e_f, e_g), (X_n^*)^d) \pi_{M_f M_g}$.

By Lemmas 11.13 and 11.18, for all $q \in Q_{M'}$ the function $f_{M',q}$ is a homeomorphism from \mathfrak{C}_n^d to an open subset of \mathfrak{C}_n^d . Thus by Lemma 12.9, $CR(M')$ is a well-defined non-degenerate (d, n) -transducer with complete response such that $f_{CR(M'),(e_f,e_g)} = f_{M',(e_f,e_g)} = fg$. Moreover as $M_f M_g$ was synchronizing, so is M' and hence so is $CR(M')$.

We know that the transducer $M_{CR(M')}$ is synchronizing, and satisfies all of the conditions required by Theorem 11.19. Thus M_{fg} is strongly isomorphic to the synchronizing transducer $M_{CR(M')}$, so M_{fg} is also synchronizing and $fg \in d\mathcal{S}_{n,1}$ as required. \square

Definition 12.12 (Core monoid, Assumed Knowledge: 1.10, 4.5, 11.3, 12.11, 12.13).

For $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, we define $\widetilde{d\mathcal{O}_{n,1}} := \left\{ \text{Core}(M_f) \mid f \in d\mathcal{S}_{n,1} \right\} / \cong_S$ (recall Definitions 1.10 and 11.3) and we define a binary operation on $\widetilde{d\mathcal{O}_{n,1}}$ by

$$[\text{Core}(M_f)]_{\cong_S} [\text{Core}(M_g)]_{\cong_S} = [\text{Core}(M_{fg})]_{\cong_S}.$$

We show in Lemma 12.13 that the above operation is well defined and makes $\widetilde{d\mathcal{O}_{n,1}}$ into a monoid.

Lemma 12.13 (Core monoids work, Assumed Knowledge: 12.10, 12.11, 12.12).

The object $\widetilde{d\mathcal{O}_{n,1}}$ defined in Definition 12.12 is a well-defined monoid, the map $\gamma_{d,n} : d\mathcal{S}_{n,1} \rightarrow \widetilde{d\mathcal{O}_{n,1}}$ defined by $(h)\gamma_{d,n} = [\text{Core}(M_h)]_{\cong_S}$ is a quotient map, and for all $f, g \in d\mathcal{S}_{n,1}$ we have

$$\text{Core}(M_{fg})_{\cong_S} M_{CR(\text{Core}(\text{Core}(M_f) \text{Core}(M_g)))}.$$

Proof. If $\widetilde{d\mathcal{O}_{n,1}}$ is well defined, then the map $\gamma_{d,n}$ is a surjective homomorphism by definition. By Lemma 12.11, the set $d\mathcal{S}_{n,1}$ is a monoid. Thus $\widetilde{d\mathcal{O}_{n,1}}$ is a monoid if and only if the product operation is well-defined.

In the proof of Lemma 12.11, it is shown that if $f, g \in d\mathcal{S}_{n,1}$, then the M_{fg} is strongly isomorphic to the transducer $M_{CR(M')}$ where M' is the subtransducer $((e_f, e_g), (X_n^*)^d) \pi_{M_f M_g}$ of $M_f M_g$. It follows that

$$\text{Core}(M_{fg})_{\cong_S} \text{Core}(M_{CR(M')})_{\cong_S} M_{\text{Core}(CR(M'))} = M_{CR(\text{Core}(M'))}.$$

As the core of a transducer T is contained in every subtransducer of T with the same domain and range, it follows that $\text{Core}(M_{fg})_{\cong_S} M_{CR(\text{Core}(M'))} = M_{CR(\text{Core}(M_f M_g))}$. Thus by Lemma 12.10, we have

$$\text{Core}(M_{fg})_{\cong_S} M_{CR(\text{Core}(\text{Core}(M_f) \text{Core}(M_g)))}.$$

Thus the strong isomorphism type of $\text{Core}(M_{fg})$ is uniquely determined by the isomorphism types of $\text{Core}(M_f)$ and $\text{Core}(M_g)$, and the operation is well-defined as required. \square

Definition 12.14 ($d\mathcal{B}_{n,1}$ and $d\mathcal{O}_n$, Assumed Knowledge: 4.9, 12.12, 12.13).

For $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, we define $d\mathcal{B}_{n,1}$ to be the group of units of $d\mathcal{S}_{n,1}$, and $d\mathcal{O}_{n,1} := (d\mathcal{B}_{n,1})\gamma_{d,n}$.

The groups $d\mathcal{B}_{n,1}$ and $d\mathcal{O}_{n,1}$ are the main purpose of this section, as we will show in Theorem 12.18 these will be respectively the automorphism and outer-automorphism groups of dV_n .

Lemma 12.15 (dV_n is normal, Assumed Knowledge: 5.6, 12.4, 12.13, 12.14).

If $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, then $d\mathcal{V}_n \trianglelefteq d\mathcal{B}_{n,1}$ and $d\mathcal{B}_{n,1}/d\mathcal{V}_n \cong d\mathcal{O}_{n,1}$.

Proof. As $\text{id}_{\mathfrak{C}_n^d}$ is the identity of $d\mathcal{B}_{n,1}$, $(\text{id}_{\mathfrak{C}_n^d})\gamma_{d,n}$ is the identity of $d\mathcal{O}_{n,1}$. Thus if $f \in d\mathcal{B}_{n,1}$, then $(f)\gamma_{d,n}$ is the identity if and only if the core of f is an identity transducer. Thus by Proposition 12.4, $(f)\gamma_{d,n}$ is the identity if and only if $f \in dV_n$. It follows that dV_n is a normal subgroup of $d\mathcal{B}_{n,1}$. Moreover the map $hdV_n \rightarrow (h)\gamma_{d,n}$ is a well-defined isomorphism between $d\mathcal{B}_{n,1}/dV_n$ and $d\mathcal{O}_{n,1}$. \square

Lemma 12.16 (Local synchronization for the normalizer, Assumed Knowledge: 5.6, 11.11, 11.13, 11.18, 11.19, 12.4).

Let $d \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$, $h \in N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n)$ and $s, t \in (X_n^*)^d \setminus \{\varepsilon_d\}$. There exists $K_{h,s,t} \in \mathbb{N}$ such that for all $a \in (X_n^*)^d$ with $\min \left(\left\{ |(a)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq K_{h,s,t}$, we have $(e_h, sa)\pi_{M_h} = (e_h, ta)\pi_{M_h}$.

Proof. For all $x \in (X_n^*)^d$, let $q_x := (e_h, x)\pi_{M_h}$. As s, t are non-trivial, we can find complete prefix codes F_s, F_t for \mathfrak{C}_n^d , such that $s \in F_s, t \in F_t$ and $|F_s| = |F_t|$. Let $\phi : F_s \rightarrow F_t$ be a bijection such that $(s)\phi = t$, and let $f := f_\phi$ (Definition 10.2).

As $h \in N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n)$, we have that $hdV_n = dV_n h$. In particular $fh \in hdV_n$. Let $g \in nV$ such that $fh = hg$. Thus by Lemmas 11.13 and 11.18, we have

$$f_{M_f M_h, (e_f, e_h)} = fh = hg = f_{M_h M_g, (e_h, e_g)}.$$

By Proposition 12.4, each of M_f, M_g has a single identity state as its core. Let I_f, I_g be the core states of M_f and M_g respectively. Note that

$$((e_f, e_h), s)\pi_{M_f M_h} = (I_f, qt), \quad ((e_h, e_g), s)\pi_{M_h M_g} = (q_s, (e_g, (q_h, s)\lambda_{M_h})\pi_{M_g}).$$

By Lemma 11.11 applied to the state q_s of M_h , let $K \in \mathbb{N}$ be such that for all $w \in (X_n^*)^d$ with

$$\min \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq K,$$

we have $\min \left(\left\{ |(q_s, w)\lambda_{M_h}\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right)$ is at least the synchronizing length of M_g .

Let $a \in (X_n^*)^d$ be arbitrary such that $\min \left(\left\{ |(a)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq K$. It suffices to show that $q_{ta} = q_{sa}$. From the choice of K , we obtain

$$((e_f, e_h), sa)\pi_{M_f M_h} = (I_f, q_{ta}), \quad ((e_h, e_g), sa)\pi_{M_h M_g} = (q_{sa}, I_g).$$

Thus, for all $v \in (X_n^{\mathbb{N}})^d$ we have

$$\begin{aligned} ((e_f, e_h), sa)\lambda_{M_f M_h}(v)f_{M_h, q_{ta}} &= ((e_f, e_h), sa)\lambda_{M_f M_h}(v)f_{M_f M_h, (I_f, q_{ta})} \\ &= (sav)fh \\ &= (sav)hg \\ &= ((e_h, e_g), sa)\lambda_{M_h M_g}(v)f_{M_h M_g, (q_{sa}, I_g)} \\ &= ((e_h, e_g), sa)\lambda_{M_h M_g}(v)f_{M_h, q_{sa}}. \end{aligned}$$

From the above equation we have for all $i \in \{0, 1, \dots, d-1\}$, that the words $((e_f, e_h), sa)\lambda_{M_f M_h}\pi_i$ and $((e_h, e_g), sa)\lambda_{M_h M_g}\pi_i$ are comparable.

If there was $i \in \{0, 1, \dots, d-1\}$ such that $((e_f, e_h), sa)\lambda_{M_f M_h} \pi_i \neq ((e_h, e_g), sa)\lambda_{M_h M_g} \pi_i$, it would follow that either the map $f_{M_h, q_{ta}}$ or $f_{M_h, q_{sa}}$ has its image contained in a proper cone. By Theorem 11.19, the transducer M_h has complete response, so this cannot occur and we must have $((e_f, e_h), sa)\lambda_{M_f M_h} = ((e_h, e_g), sa)\lambda_{M_h M_g}$.

From the equality

$$((e_f, e_h), sa)\lambda_{M_f M_h}(v)f_{M_h, q_{ta}} = ((e_h, e_g), sa)\lambda_{M_h M_g}(v)f_{M_h, q_{sa}}$$

it follows that $f_{M_h, q_{ta}} = f_{M_h, q_{sa}}$. Thus, as M_h has complete response, we have $q_{ta} \sim_{M_h} q_{sa}$. Finally, as M_h is minimal, it follows that $q_{ta} = q_{sa}$ as required. \square

Lemma 12.17 (Synchronizing normalizer, Assumed Knowledge: 12.14, 12.16).

For $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, the group $N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n)$ is contained in $d\mathcal{B}_{n,1}$.

Proof. Let $h \in N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n)$ be arbitrary. We show that $h \in d\mathcal{S}_n$, as h is arbitrary we then also conclude that $h^{-1} \in d\mathcal{S}_n$ and so h must be a unit of $d\mathcal{S}_n$.

Thus we need only show that M_h is a synchronizing transducer. We define

$$K := \max \left(\left\{ K_{h, x_{d,i}, y_{d,j}, z_{d,k}} \mid x, y, z \in X_n, i, j, k \in \{0, 1, \dots, d-1\} \right\} \right) + 1$$

where $x_{d,i}, y_{d,j}, z_{d,k}$ are as in Definition 10.1, and $K_{h, x_{d,i}, y_{d,j}, z_{d,k}}$ is as in Lemma 12.16. Let $w \in (X_n^*)^d$ be arbitrary, such that

$$\min \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) \geq K.$$

It suffices to show for all $q \in Q_{M_h}$, we have $(q, w)\pi_{M_h} = (e_h, w)\pi_{M_h}$. Let $q \in Q_{M_h}$ be arbitrary and let $v \in (X_n^*)^d$ be such that $(e_h, v)\pi_{M_h} = q$. Let $w = w_0 w'$, and $v = v_0 v_1 \dots v_l$ where

$$v_0, v_1, \dots, v_l, w_0 \in \left\{ x_{d,i} \mid x \in X_n, i \in \{0, 1, \dots, d-1\} \right\}.$$

By the choice of w , we obtain:

$$\begin{aligned} (q, w)\pi_{M_h} &= (e_h, vw)\pi_{M_h} = (e_h, v_0 v_1 v_2 \dots v_l w_0 w')\pi_{M_h} \\ &= (e_h, (v_0 v_1)(v_2 \dots v_l w_0 w'))\pi_{M_h} = (e_h, (v_1)(v_2 \dots v_l w_0 w'))\pi_{M_h} \\ &= (e_h, (v_1 v_2)(v_3 \dots v_l w_0 w'))\pi_{M_h} = (e_h, (v_2)(v_3 \dots v_l w_0 w'))\pi_{M_h} \\ &\vdots \\ &= (e_h, (v_l w_0 w')\pi_{M_h} = (e_h, w_0 w')\pi_{M_h} = (e_h, w)\pi_{M_h} \end{aligned}$$

as required. \square

We can now tie everything together and use Rubin's Theorem to connect our groups to the automorphism groups of dV_n .

Theorem 12.18 (Automorphisms characterised, Assumed Knowledge: 12.15, 12.17).

Let $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. The groups $N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n)$ and $d\mathcal{B}_{n,1}$ are equal, countably infinite and isomorphic to $\text{Aut}(dV_n)$. Moreover $\text{Out}(dV_n) \cong d\mathcal{O}_{n,1}$.

Proof. We show the inclusions

$$N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n) \subseteq d\mathcal{B}_{n,1} \subseteq N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n)$$

in Lemmas 12.17 and 12.15 respectively. Thus $N_{\text{Aut}(\mathfrak{C}_n^d)}(dV_n) = d\mathcal{B}_{n,1}$.

As there is a countably infinite number of complete prefix codes for \mathfrak{C}_n^d , the group dV_n is countably infinite. By Theorem 12.15, we have

$$|d\mathcal{B}_{n,1}| = |dV_n||d\mathcal{B}_{n,1}/dV_n| = |dV_n||d\mathcal{O}_{n,1}|.$$

An element of $d\mathcal{O}_{n,1}$ is a strong isomorphism class of core transducers all of which are finite. Moreover the monoid $(X_n^*)^d$ is finitely generated and by the definition of a transducer, the output and transition maps are determined by their actions on the generators of $(X_n^*)^d$. Hence for each $k \in \mathbb{N}$ there are only finitely many (d, n) -transducer with k states up to strong isomorphism. Thus the group $d\mathcal{O}_{n,1}$ is countable and so

$$|d\mathcal{B}_{n,1}| = |dV_n||d\mathcal{O}_{n,1}| = |\mathbb{N}||d\mathcal{O}_{n,1}| = |\mathbb{N}|.$$

By Rubin's Theorem (Corollary 5.50) and Lemma 12.15, it suffices to show that the standard action $a : \mathfrak{C}_n^d \times dV_n \rightarrow \mathfrak{C}_n^d$ given by $(p, f)a = (p)f$ is locally moving (Definition 5.41).

Let $x \in \mathfrak{C}_n^d$ be arbitrary and let $w\mathfrak{C}_n^d$ be a basic open neighbourhood of x . Let $u_x, v \in (X_n^*)^d$ be arbitrary such that $w < v$ and $w < u_x < x$. It suffices to show that $(x)(dV_n)_{w\mathfrak{C}_n^d, a} \cap v\mathfrak{C}_n^d \neq \emptyset$.

Let $S \subseteq (X_n^*)^d$ be such that $\left\{s\mathfrak{C}_n^d \mid s \in S\right\}$ is a partition of $\mathfrak{C}_n^d \setminus w\mathfrak{C}_n^d$ into clopen sets. Also choose $F_{u_x}, F_v \subseteq (X_n^*)^d$ to be such that $|F_{u_x}| = |F_v|$, $u_x \in F_{u_x}, v \in F_v$ and $\left\{s\mathfrak{C}_n^d \mid s \in F_{u_x}\right\}, \left\{s\mathfrak{C}_n^d \mid s \in F_v\right\}$, are both partitions of $w\mathfrak{C}_n^d$ into clopen sets. Let $\phi' : F_{u_x} \rightarrow F_v$ be any bijection with $(u_x)\phi' = v$, and let $\phi := \phi' \cup \left\{(s, s) \mid s \in S\right\}$. It follows that the map f_ϕ is an element of $(dV_n)_{w\mathfrak{C}_n^d, a}$ and satisfies $(x)f_\phi \in v\mathfrak{C}_n^d$ as required. \square

13. RATIONALITY

In GNS [15], the notion of a rational homomorphism is introduced (GNS also proves Proposition 13.2).

Definition 13.1 (Rational transformations, Assumed Knowledge: 2.2, 5.1, 11.7, 11.8, 11.13).

If say that $f : \mathfrak{C}_n \rightarrow \mathfrak{C}_m$ is *rational* if $f = f_{T,q}$ for some $(1, n, 1, m)$ -transducer T with finitely many states and some $q \in Q_T$. By Lemma 11.13 it follows that the set of objects

$$\left\{\mathfrak{C}_n \mid n \in \mathbb{N} \setminus \{0, 1\}\right\}$$

together with the set of morphisms

$$\left\{f : \mathfrak{C}_n \rightarrow \mathfrak{C}_m \mid f \text{ is rational and } n, m \in \mathbb{N} \setminus \{0, 1\}\right\}$$

form a small category. We call this the category of rational transformations and denote it by \mathcal{R} . We denote the endomorphism monoid and automorphism group of \mathfrak{C}_n in this category by $\widetilde{\mathcal{R}}_n$ and \mathcal{R}_n respectively.

Proposition 13.2 (cf [15], Rational isomorphisms, Assumed Knowledge: 12.5, 13.1).

For all $n, m \in \mathbb{N} \setminus \{0, 1\}$, the objects $\mathfrak{C}_n, \mathfrak{C}_m$ are isomorphic in the category \mathcal{R} , thus

$$\widetilde{\mathcal{R}}_n \cong \widetilde{\mathcal{R}}_m, \quad \text{and} \quad \mathcal{R}_n \cong \mathcal{R}_m.$$

Proof. Let $n \in \mathbb{N} \setminus \{0, 1, 2\}$ be arbitrary. We give a rational isomorphism from $\mathfrak{C}_2 \rightarrow \mathfrak{C}_n$. We define $F_1 := \{1, 01, 001, \dots, 0^{n-1}1\}$, $F_2 := X_n$ and $\phi : F_1 \rightarrow F_2$ by $((0)^i 1)\phi = i$ for all $i \in \{0, 1, \dots, n-1\}$. We then define $f_\phi : \mathfrak{C}_n \rightarrow \mathfrak{C}_n$ and $f_{\phi^{-1}} : \mathfrak{C}_n \rightarrow \mathfrak{C}_2$ as in Figure 2 (recall Definition 12.5).

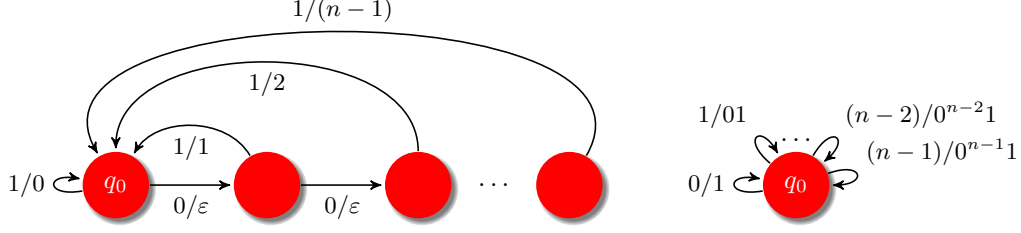


FIGURE 2. We define $f_\phi := f_{T,q_0}$ where T is the left transducer, and $f_{\phi^{-1}} := f_{T,q_0}$ where T is the right transducer

Note that

$$(wx)f_\phi = (w)\phi(x)f_\phi, \quad \text{and} \quad (vy)f_{\phi^{-1}} = (v)\phi^{-1}(y)f_{\phi^{-1}}$$

where $w \in F_1, v \in F_2, x \in \mathfrak{C}_2, y \in \mathfrak{C}_n$ are arbitrary. In particular we have that $f_\phi, f_{\phi^{-1}}$ are inverses of each other. As these maps are rational, the result follows. \square

If $k, n \in \mathbb{N}$, then there are only finitely many words over the alphabet X_n with length at most k . Thus from Remark 12.2, it follows that $1\mathcal{B}_{n,1} \leq \mathcal{R}_n$. However if $d \geq 2$, then there are infinitely many elements w of $(X_n^*)^d$ with

$$\min \left(\left\{ |(w)\pi_i| \mid i \in \{0, 1, \dots, d-1\} \right\} \right) < k.$$

Thus this reasoning does not follow in the case of $d\mathcal{B}_{n,1}$. In Figure 1 we see that the bakers map cannot be defined by a finite $(2, 2)$ -transducer.

In [2] Theorem 5.2, it is shown that $2V$ can be embedded in the rational group. They embed $2V$ by conjugating it by the map f defined the transducer in Figure 5. As $\mathcal{B}_{2,1} \leq \mathcal{R}_2$ and $f2Vf^{-1} \leq \mathcal{R}_4$, one might expect that $f2\mathcal{B}_{2,1}f^{-1} \leq \mathcal{R}_4$. We show in Proposition 13.4 that this is not the case.

Example 13.3 ($0 \leftrightarrow 00$ transducer, Assumed Knowledge: 11.7, 11.8, 12.1, 12.5).

Suppose T is the left $(1, 2)$ -transducer from Figure 3 (recall Definition 12.5). The transducer T is synchronizing with $\text{Core}(T) = T$. Moreover if $n_0 \in \mathbb{N}$, $n_1, \dots, m_0, m_1, \dots \in (\mathbb{N} \setminus \{0\}) \cup \{\infty\}$ and $s : \mathbb{N} \cup \{\infty\} \rightarrow \mathbb{N} \cup \{\infty\}$ is the permutation with $(1)s = 2, (2)s = 1, (n)s = n$ for $n \notin \{1, 2\}$, then

$$(0^{n_0} 1^{m_0} 0^{n_1} \dots) f_{T,q_0} = 0^{(n_0)s} 1^{m_0} 0^{(n_1)s} \dots$$

In particular f_{T,q_0} is a homeomorphism.

Proof. By the definition of T , it follows that for all $q \in Q_T$ we have

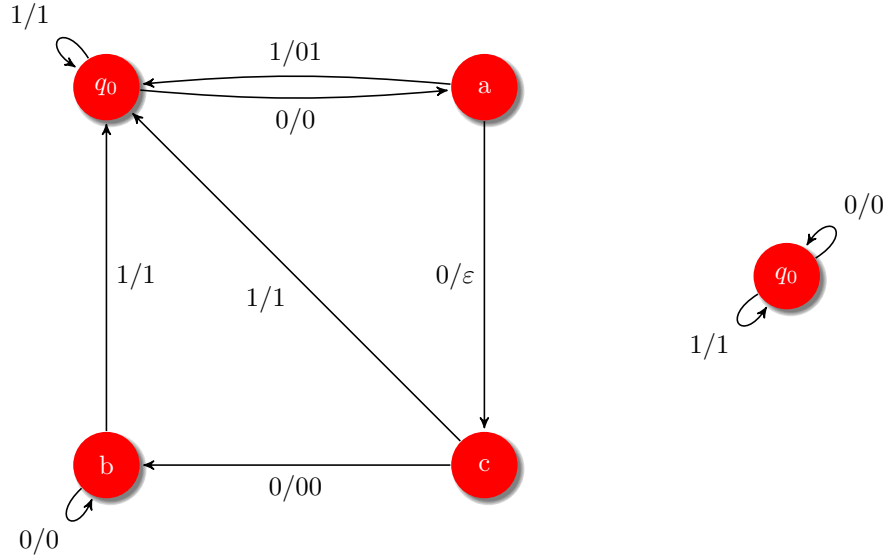
$$(q, 000)\pi_T = b, \quad (q, 100)\pi_T = c, \quad (q, 10)\pi_T = a, \quad (q, 1)\pi_T = q_0$$

In particular if T is synchronizing at level 3. By definition we have

$$(0^{n_0} 1^{m_0} 0^{n_1} \dots) f_{T,q_0} = (q_0, 0^{n_0} 1^{m_0}) \lambda_T (0^{n_1} 1^{m_1} 0^{n_2} \dots) f_{T,(q_0, 0^{n_0} 1^{m_0}) \pi_T}.$$

Moreover, as $m_0 \geq 1$ and $(q, 1)\pi_T = q_0$ for all $q \in Q_T$, we have $(q_0, 0^{n_0} 1^{m_0}) \pi_T = q_0$. Thus

$$(0^{n_0} 1^{m_0} 0^{n_1} \dots) f_{T,q_0} = (q_0, 0^{n_0} 1^{m_0}) \lambda_T (0^{n_1} 1^{m_1} 0^{n_2} \dots) f_{T,q_0}.$$


 FIGURE 3. Two transducers with domain and range X_2^*

By repeatedly applying the above equality, it suffices to show that $(q_0, 0^{n_0}1^{m_0})\lambda_T = 0^{(n_0)s}1^{m_0}$. If $n \in \mathbb{N}$ is arbitrary, then by definition

$$(q_0, 0^11^{m_0})\lambda_T = (0)(01)(1)^{m_0-1} = 0^21^{m_0},$$

$$(q_0, 0^21^{m_0})\lambda_T = (0)() (1)(1)^{m_0-1} = 0^11^{m_0},$$

$$(q_0, 0^{3+n}1^{m_0})\lambda_T = (0)() (00)(0)^n(1)(1)^{m_0-1} = 0^{3+n}1^{m_0},$$

as required. \square

Proposition 13.4 (Rationality problems in higher dimensions, Assumed Knowledge: 12.5, 12.4, 11.13, 11.17, 4.28, 13.3, 11.19).

Let $f : \mathfrak{C}_4 \rightarrow \mathfrak{C}_2^2$ be the homeomorphism defined by the transducer in Figure 5 (recall Definition 12.5). This transducer has finitely many states but f^{-1} cannot be defined by a $(2, 2, 1, 2)$ -transducer with finitely many states. Moreover we have

$$f2Vf^{-1} \subseteq \mathcal{R}_4 \quad \text{but} \quad f2\mathcal{B}_{2,1}f^{-1} \not\subseteq \mathcal{R}_4.$$

Proof. Let T be the transducer in Figure 5. Note that if $w \in (X_2^*)^2$, then $w \in (Q_T \times D_T)\lambda_T$ if and only if $|(w)\pi_0| = |(w)\pi_1|$. Let D be the $(2, 2, 1, 4)$ -transducer defined by

- (1) $Q_D := (X_2^* \times \{\varepsilon\}) \cup (\{\varepsilon\} \times X_2^*)$.
- (2) $(q, s)\pi_T = p$, where $p \in Q_D$ is such that there is $b \in (X_2^*)^2$ with $|(b)\pi_0| = |(b)\pi_1|$ such that $qs = bp$.
- (3) $(q, s)\lambda_T = w$, where $w \in X_4^*$ is such that there is $p \in Q_D$ is with $qs = (q_0, w)\lambda_T p$.

Note that D is well-defined. Moreover for all $p \in Q_D$ and $x \in \mathfrak{C}_2^2$ we have $(x)f_{D,p} = (px)f^{-1}$. In particular $f_{D,\varepsilon_2} = f^{-1}$.

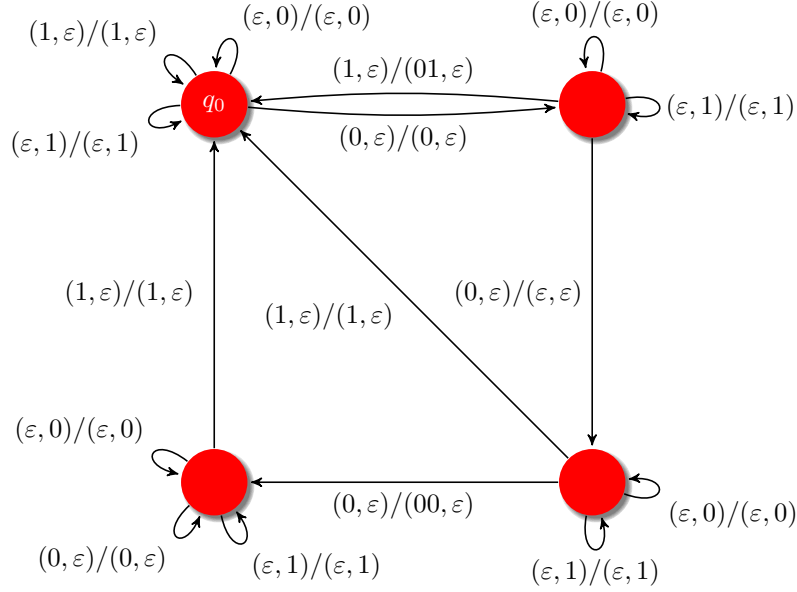


FIGURE 4. The transducer obtained by taking the categorical product of the transducers in Figure 3

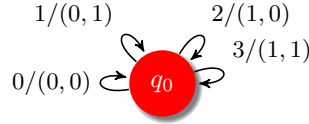


FIGURE 5. A single state $(1, 4, 2, 2)$ -transducer defining a homeomorphism from \mathfrak{C}_4 to \mathfrak{C}_2^2 (analogous to those in Figure 2)

We now show that $f2Vf^{-1} \subseteq \mathcal{R}_4$. Let $g \in 2V$ be arbitrary. By Theorem 12.4, the transducer M_g is synchronizing and has a single identity state as its core. Let k be the synchronizing length of M_g . Thus by Lemma 11.13, we have

$$fgf^{-1} = f_{TM_g D, (q_0, e_g, \varepsilon_2)}.$$

We define $P := TM_g D$ and $e := (q_0, e_g, \varepsilon_2)$. As D has infinitely many states, so does P . However if $w \in X_4^*$ is arbitrary with $|w| \geq k$, then

$$\left| |((q_0, e_g), w)\lambda_{TM_g \pi_0}| - |((q_0, e_g), w)\lambda_{TM_g \pi_1}| \right|$$

is equal to

$$\left| |((q_0, e_g), w \upharpoonright_k)\lambda_{TM_g \pi_0}| - |((q_0, e_g), w \upharpoonright_k)\lambda_{TM_g \pi_1}| \right|.$$

Thus the subtransducer $S := (e, X_4^*)\pi_P$ of P has finitely many states and satisfies $f_{S,e} = fgf^{-1}$, so fgf^{-1} is rational as required.

We next show that $f2\mathcal{B}_{2,1}f^{-1} \not\subseteq \mathcal{R}_2$. Let $b := f_{B,q_0}$, where B is the transducer from Figure 4. By Example 13.3, we know that $b \in 2\mathcal{B}_{2,1}$. We show that $fbf^{-1} \notin \mathcal{R}_4$.

By definition, for all $n \in 3\mathbb{N}$ we have

$$\begin{aligned} (q_0, (02)^n)\lambda_T &= ((01)^n, (00)^n) \\ (q_0, ((01)^n, (00)^n))\lambda_B &= ((001)^n, (00)^n) = ((001)^n, (000)^{2n/3}) \\ (\varepsilon_2, ((001)^n, (000)^{2n/3}))\lambda_D &= (\varepsilon_2, ((001)^{2n/3}, (000)^{2n/3}))\lambda_D = (002)^{2n/3}. \end{aligned}$$

Thus $((q_0, q_0, \varepsilon_2), (02)^n)\lambda_{TBD} = (002)^{2n/3}$. Moreover we have

$$((02)^n \mathfrak{C}_4) f b f^{-1} = (((001)^n, (000)^{2n/3}) \mathfrak{C}_2^2) f^{-1} = (002)^{2n/3} (((001)^{n/3}, \varepsilon) \mathfrak{C}_2^2) f^{-1}.$$

For all $n \in 3\mathbb{N}$, let $p_n := ((q_0, q_0, \varepsilon_2), (02)^n)\pi_{TBD}$. From the above equalities it follows that

$$(\mathfrak{C}_4) f_{TBD, p_n} = (((001)^{n/3}, \varepsilon) \mathfrak{C}_2^2) f^{-1}.$$

In particular, for $n \neq m$ we have $f_{TBD, p_n} \neq f_{TBD, p_m}$ and TBD has complete response from these states. Consider the subtransducer $S := ((q_0, q_0, \varepsilon_2), X_4^*)\pi_{TBD}$ of TBD .

By Theorem 11.19, it follows that if S' is any $(1, 4)$ -transducer such that there is $q \in Q_{S'}$ with $(q, X_4^*)\pi_{S'} = Q_{S'}$ and $f_{S', q} = f b f^{-1}$, then $M_{f b f^{-1}} \cong_S M_{CR(S')}$. Thus from the above observations, it follows that S' must be infinite and so $f b f^{-1}$ is not rational.

It remains to show that f^{-1} cannot be defined by a transducer with finitely many states. Suppose for a contradiction that D' is a finite state transducer defining f^{-1} . It follows that TBD' is a transducer with finitely many states with a state defining $f b f^{-1}$. As $f b f^{-1}$ is not rational, this is a contradiction. \square

14. A CLOSER LOOK AT THE GROUPS $d\mathcal{O}_{n,1}$

In this section we investigate the core transducers used to describe the outer-automorphisms of dV_n in the previous section. We show that an element of $d\mathcal{O}_{n,1}$ can be “decomposed” into d elements of $1\mathcal{O}_{n,1}$ and use this to show that $\text{Out}(dV)$ is a wreath product (Theorem 14.19).

Definition 14.1 (Semidirect products, Assumed Knowledge: 5.2).

If G, H are groups and $a : G \times H \rightarrow G$ is an action of H on G , then we define the semidirect product $G \rtimes_a H$ to be the group with the underlying set $G \times H$ and operations:

- (1) If $(g_0, h_0), (g_1, h_1) \in G \rtimes_a H$, then

$$(g_0, h_0)(g_1, h_1) = (g_0(g_1, h_0^{-1})a, h_0 h_1),$$

- (2) If $(g, h) \in G \rtimes_a H$, then

$$(g, h)^{-1} = ((g^{-1}, h)a, h^{-1}).$$

It is routine to verify that this defines a group.

Definition 14.2 (Wreath products, Assumed Knowledge: 14.1).

If G, H are groups, and $a : X \times H \rightarrow X$ is an action of H on a set X , then we define the wreath product $G \wr_a H$ to be the group $G^X \rtimes_{a'} H$ where $a' : G^X \times H \rightarrow G^X$ is defined by

$$((g_x)_{x \in X}, h)a' = (g_{(x, h)a})_{x \in X}.$$

We start by identifying an action of the semigroup $\widehat{d\mathcal{O}_{n,1}}$ on the set $\{0, 1, \dots, d-1\}$.

Definition 14.3 (Restricted coordinates, Assumed Knowledge: 10.1).

For $d \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$ and $S \subseteq \{0, 1, \dots, d-1\}$, we define

$$F_{d,n,S} := \left\{ w \in (X_n^*)^d \mid (w)\pi_i = \varepsilon \text{ for all } i \in \{0, 1, \dots, d-1\} \setminus S \right\}.$$

So $F_{d,n,S}$ is the submonoid of $(X_n^*)^d$ consisting of those elements only allowed to be non-trivial in the S coordinates.

Lemma 14.4 ($\widetilde{d\mathcal{O}_{n,1}}$ acts on coordinates independently, Assumed Knowledge: 1.3, 1.6, 11.11, 11.18, 12.12, 14.3).

Let $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. If $T = \text{Core}(M_f)$ for some $f \in d\mathcal{S}_{n,1}$ then

$$\psi_T := \left\{ (i, j) \in \{0, 1, \dots, d-1\}^2 \mid (Q_T \times F_{d,n,\{i\}})\lambda_T \subseteq F_{d,n,\{j\}} \right\}$$

is a function with domain $\{0, 1, \dots, d-1\}$.

Proof. It suffices to show for all $i \in \{0, 1, \dots, d-1\}$, that the set $(\{i\})\psi_T$ (recall Definition 1.6) is a singleton. Let $i \in \{0, 1, \dots, d-1\}$ be arbitrary, we start by showing that $(\{i\})\psi_T \neq \emptyset$. Recall that for all $q \in Q_T$, the map $f_{T,q}$ is necessarily injective (Lemma 11.18).

Suppose for a contradiction that there is $i, \alpha, \beta \in \{0, 1, \dots, d-1\}$ and $(q_0, l_0), (q_1, l_1) \in Q_T \times F_{d,n,\{i\}}$, such that

$$\alpha \neq \beta, \quad |(q_0, l_0)\lambda_T \pi_\alpha| > 0, \quad \text{and} \quad |(q_1, l_1)\lambda_T \pi_\beta| > 0.$$

We may assume without loss of generality that $\alpha = 0$ and $\beta = 1$.

Let $p_0 \in Q_T$ be fixed. The transducer T is non-degenerate (as it is a subtransducer of M_f) so by Lemma 11.11, there is $s \in (X_n^*)^d$ such that $|(p_0, s)\lambda_T \pi_j| \geq 1$ for all $j \in \{0, 1, \dots, d-1\}$. For each $j \in \{0, 1, \dots, d-1\}$ define $s_j \in F_{d,n,\{j\}}$ such that $s = s_0 s_1 s_2 \dots s_j$. Moreover for each $j \in \{1, 2, 3, \dots, d-1\}$, define $p_j := (p_0, s_0 s_1 \dots s_{j-1})\pi_T$. As T is a transducer, it follows that

$$(p_0, s)\lambda_T = (p_0, s_0 s_1 s_2 \dots s_j)\lambda_T = (p_0, s_0)\lambda_T (p_1, s_1)\lambda_T \dots (p_{d-1}, s_{d-1})\lambda_T.$$

Thus for all $j \in \{0, 1, 2, 3, \dots, d-1\} \setminus \{0, 1\}$ we can choose some, $q_j \in Q_T$, $i_j \in \{0, 1, \dots, d-1\}$ and $l_j \in F_{d,n,\{i_j\}}$ such that $|(q_j, l_j)\lambda_T \pi_{i_j}| > 0$. We also define both i_0 and i_1 to be i .

For all $j \in \{0, 1, \dots, d-1\}$, we now have $|(q_j, l_j)\lambda_T \pi_{i_j}| > 0$. Moreover $i_0 = i_1$, so we can fix some $b \in \{0, 1, \dots, d-1\} \setminus \{i_0, i_1, \dots, i_{d-1}\}$. We will use this observation to contradict injectivity.

For each $j \in \{0, 1, \dots, d-1\}$, choose some $w_j \in (\{q_j\})\mathfrak{s}_T^{-1}$. Now if $p \in Q_T$ is arbitrary, then $|(p, w_j l_j)\lambda_T \pi_{i_j}| \geq |(q_j, l_j)\lambda_T \pi_{i_j}| \geq 1$. For each $j \in \{0, 1, \dots, d-1\}$, define $w_{j,b} \in F_{d,n,\{b\}}$ and $w'_j \in F_{d,n,\{0,1,\dots,d-1\} \setminus \{b\}}$ such that $w_j = w_{j,b} w'_j$.

For each $m \in \mathbb{N}$, we define

$$t_m := w_{0,b} w_{1,b} \dots w_{d-1,b} (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^m.$$

For each $j \in \{0, 1, \dots, n-1\}$, the element $w_{j,b} \in F_{d,n,\{b\}}$ commutes with the product

$$w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1} \in F_{d,n,\{0,1,\dots,d-1\} \setminus \{b\}}.$$

Thus for all $m \in \mathbb{N}$ and $j \in \{0, 1, \dots, d-1\}$, we can find $u_{j,m}, v_{j,m} \in (X_n^*)^d$ such that

$$t_m = u_{j,m} w_{j,b} (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^m v_{j,m}.$$

Moreover, $w_{j,b}$ commutes with all elements of $F_{d,n,\{0,1,\dots,d-1\}\setminus\{b\}}$ and so if $q \in Q_T, j \in \{0, 1, \dots, d-1\}, m \in \mathbb{N}$ are arbitrary, then

$$\begin{aligned}
|(q, t_m)\lambda_T \pi_j| &= |(q, u_{j,m} w_{j,b} (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^m v_{j,m})\lambda_T \pi_j| \\
&\geq |((q, u_{j,m} w_{j,b})\pi_T, (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^m)\lambda_T \pi_j| \\
&\geq |((w_j)\mathfrak{s}_T, l_j w'_{j+1} l_{j+1} \dots w'_{d-1} l_{d-1} (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^{m-1})\lambda_T \pi_j| \\
&\geq 1 + |((w_{j,b} w'_j l_j \dots w'_{d-1} l_{d-1})\mathfrak{s}_T, (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^{m-1})\lambda_T \pi_j| \\
&= 1 + |((w'_j l_j \dots w'_{d-1} l_{d-1} w_{j,b})\mathfrak{s}_T, (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^{m-1})\lambda_T \pi_j| \\
&\geq 2 + |((w'_j l_j \dots w'_{d-1} l_{d-1} w_{j,b})\mathfrak{s}_T, (w'_0 l_0 w'_1 l_1 \dots w'_{d-1} l_{d-1})^{m-2})\lambda_T \pi_j| \\
&\vdots \\
&\geq m.
\end{aligned}$$

Thus all elements of $(X_n^{\mathbb{N}})^d$ which have all of the elements of t_m as prefixes have the same image under $f_{T,q}$ (for any choice of q). This is a contradiction as there are infinitely many such elements and $f_{T,q}$ is injective.

It remains to show that $|(\{i\})\psi_T| \leq 1$. Suppose for a contradiction that $\alpha, \beta \in (\{i\})\psi_T$ and $\alpha \neq \beta$. It follows that

$$(Q_T \times F_{d,n,\{i\}})\lambda_T \subseteq F_{d,n,\{\alpha\}} \cap F_{d,n,\{\beta\}} = \{\varepsilon_d\}.$$

For each $j \in \{0, 1, \dots, d-1\} \setminus \{i\}$, by the first part of the proof, we can choose some $j' \in (\{j\})\psi_T$. Let $b \in \{0, 1, \dots, d-1\} \setminus \{j' \mid j \neq i\}$. It follows that for all $q \in Q_T$ and $w \in (X_n^*)^d$, we have $|(q, w)\lambda_T \pi_b| = 0$. This contradicts Lemma 11.11. \square

Definition 14.5 ($d\mathcal{O}_n$ coordinate map, Assumed Knowledge: 1.4, 14.4).

We define a map $\Psi : d\mathcal{O}_n \rightarrow \{0, 1, \dots, d-1\}^{\{0,1,\dots,d-1\}}$ by

$$([T]_{\cong_S})\Psi = \psi_T,$$

where ψ_T is as in Lemma 14.4. Note that the property defining ψ_T is invariant under strong transducer isomorphisms, so this is a well-defined map.

Theorem 14.6 ($d\mathcal{O}_{n,1}$ is semidirect, Assumed Knowledge: 2.3, 5.10, 11.19, 12.13, 14.4, 14.5, 14.1).

The map $\Psi : d\mathcal{O}_n \rightarrow \{0, 1, \dots, d-1\}^{\{0,1,\dots,d-1\}}$ (Definition 14.5) is a semigroup homomorphism. Moreover, the group $d\mathcal{O}_{n,1}$ is isomorphic to $d\mathcal{K}_{n,1} \rtimes_a \text{Sym}(d)$, where

$$d\mathcal{K}_{n,1} := \left\{ S \in d\mathcal{O}_{n,1} \mid (S)\Psi = id_{\{0,1,\dots,d-1\}} \right\},$$

$a : d\mathcal{K}_{n,1} \times \text{Sym}(d) \rightarrow d\mathcal{K}_{n,1}$ is defined by

$$([A]_{\cong_S}, g)a = [\text{Core}(M_{\mathbf{g}})]_{\cong_S}^{-1} [\text{Core}(M_f)]_{\cong_S} [\text{Core}(M_{\mathbf{g}})]_{\cong_S}$$

and $\mathbf{g} := \langle (\pi_{(i)g})_{i \in \{0,1,\dots,d-1\}} \rangle_{\mathfrak{C}_n^d}$ (recall Definition 2.3).

Proof. If $[A]_{\cong_S}, [B]_{\cong_S} \in d\mathcal{O}_{n,1}$ are arbitrary, then by Lemma 12.13, we have

$$[A]_{\cong_S} [B]_{\cong_S} = [M_{CR(\text{Core}(AB))}]_{\cong_S}.$$

If $i \in \{0, 1, \dots, d-1\}$ then $(Q_A \times F_{d,n,\{i\}})\lambda_A \subseteq F_{d,n,\{(i)\psi_A\}}$ and $(Q_B \times F_{d,n,\{(i)\psi_A\}})\lambda_B \subseteq F_{d,n,\{(i)\psi_A\psi_B\}}$. Thus

$$(Q_{AB} \times F_{d,n,\{i\}})\lambda_{AB} = (Q_B \times (Q_A \times F_{d,n,\{i\}})\lambda_A)\lambda_B \subseteq F_{d,n,\{(i)\psi_A\psi_B\}}.$$

It follows that

$$(Q_{CR(\text{Core}(AB))} \times F_{d,n,\{i\}})\lambda_{CR(\text{Core}(AB))} \subseteq F_{d,n,\{(i)\psi_A\psi_B\}}$$

as well, and hence the map Ψ is a homomorphism. If T is an identity transducer (Definition 12.3), then ψ_T is the identity map on $\{0, 1, \dots, d-1\}$. Therefor, as $d\mathcal{O}_{n,1}$ is a group, we must have that Ψ is a group homomorphism from $d\mathcal{O}_{n,1}$ to $\text{Sym}(d)$. It follows that the set $d\mathcal{K}_{n,1} = (\{\text{id}_{\{0,1,\dots,d-1\}}\})\Psi^{-1}$ is a group.

For each $g \in \text{Sym}(d)$, let T_g be the (d, n) -transducer with state set $\{0\}$ and λ_{T_g} defined by

$$(0, w)\lambda_{T_g} = ((w)\pi_{(0)g}, (w)\pi_{(1)g}, \dots, (w)\pi_{(d-1)g}),$$

for all $w \in (X_n^*)^d$. For all $g \in \text{Sym}(d)$, we have that $f_{T_g,0} = \mathbf{g}$. Thus by Lemma 11.19, we have $\text{Core}(T_g) = T_g \cong_S M_{\mathbf{g}} = \text{Core}(M_{\mathbf{g}})$ for all $g \in \text{Sym}(d)$. In particular we have that $\mathbf{g} \in d\mathcal{B}_{n,1}$ for all $g \in \text{Sym}(d)$.

Define $\Psi' : \text{Sym}(d) \rightarrow d\mathcal{O}_{n,1}$ by

$$(g)\Psi' = [\text{Core}(M_{\mathbf{g}})]_{\cong_S}.$$

As each of the maps $g \rightarrow \mathbf{g}$ and $\mathbf{g} \rightarrow [\text{Core}(M_{\mathbf{g}})]_{\cong_S}$ are homomorphisms, the map Ψ' is a homomorphism. Moreover as $\text{Core}(M_{\mathbf{g}}) = M_{\mathbf{g}} \cong_S T_g$, it follows that $\Psi'\Psi$ is the identity map on $\text{Sym}(d)$.

We define maps $\phi : d\mathcal{K}_{n,1} \rtimes_a \text{Sym}(d) \rightarrow d\mathcal{O}_n$ and $\phi' : d\mathcal{O}_n \rightarrow d\mathcal{K}_{n,1} \rtimes_a \text{Sym}(d)$ by

$$(S, g)\phi = S(g)\Psi', \quad \text{and} \quad (S)\phi' = (S((S)\Psi\Psi')^{-1}, (S)\Psi).$$

Note that ϕ' is well defined as for all $S \in d\mathcal{O}_n$, we have

$$(S((S)\Psi\Psi')^{-1})\Psi = (S(((S)\Psi)^{-1})\Psi')\Psi = (S)\Psi((S)\Psi)^{-1} = \text{id}_{\{0,1,\dots,d-1\}}.$$

By construction, the maps $\phi\phi'$ and $\phi'\phi$ are both identity maps, thus ϕ is a bijection. It now suffices to show that ϕ is a homomorphism. If $(S_0, g_0), (S_1, g_1) \in d\mathcal{K}_{n,1} \rtimes_a \text{Sym}(d)$, then we have

$$\begin{aligned} (S_0, g_0)\phi(S_1, g_1)\phi &= S_0(g_0)\Psi'S_1(g_1)\Psi' \\ &= S_0(g_0)\Psi'S_1((g_0)\Psi')^{-1}(g_0g_1)\Psi' \\ &= S_0(S_1, g_0^{-1})a(g_0g_1)\Psi' \\ &= (S_0(S_1, g_0^{-1})a, g_0g_1)\phi \\ &= ((S_0, g_0)(S_1, g_1))\phi. \end{aligned}$$

Thus the result follows. \square

We have that $d\mathcal{O}_{n,1}$ is a semidirect product of $\text{Sym}(d)$ and the group $d\mathcal{K}_{n,1}$. We next work towards decomposing the transducers in the group $d\mathcal{K}_{n,1}$ (see Lemma 14.10). We then use our description of these transducers to embed $d\mathcal{K}_{n,1}$ into the group $d\mathcal{P}_{n,n-1}$ (Theorem 14.12).

The group $d\mathcal{P}_{n,n-1}$ is based on the group $\mathcal{O}_{n,n-1}$ from [3]. It is shown that [3], that $\mathcal{O}_{n,n-1}$ is a group isomorphic to $\text{Out}(G_{n,n-1})$ (hence the name). The insistence that $Q_T \subseteq \mathbb{N}$ in the next definition is arbitrary, and done only to avoid proper classes.

Definition 14.7 ($\mathcal{O}_{n,n-1}$, Assumed Knowledge: 1.10, 4.9, 11.10, 11.19, 12.1, 12.13, 14.8).

We say that a transducer T is an $\widetilde{\mathcal{O}_{n,n-1}}$ transducer if

- (1) $Q_T \subseteq \mathbb{N}$, and $D_T = R_T = X_n^*$.
- (2) $T \cong_S M_T$.
- (3) T has complete response.
- (4) T is synchronizing and $\text{Core}(T) = T$.
- (5) For all $q \in Q_T$, the map $f_{T,q}$ is a homeomorphism from \mathfrak{C}_n^d to an open subset of \mathfrak{C}_n^d .

We also define $\widetilde{\mathcal{O}_{n,n-1}}$ to be the set $\left\{ T \mid T \text{ is an } \widetilde{\mathcal{O}_{n,n-1}} \text{ transducer} \right\} / \cong_S$, with multiplication given by $[A]_{\cong_S} [B]_{\cong_S} = [C]_{\cong_S}$, where C is strongly isomorphic to $M_{CR(\text{Core}(AB))}$ (we show in Lemma 14.8 that this operation is well-defined and makes $\widetilde{\mathcal{O}_{n,n-1}}$ a monoid). We then define $\mathcal{O}_{n,n-1}$ to be the group of units of $\widetilde{\mathcal{O}_{n,n-1}}$, and we say that an $\widetilde{\mathcal{O}_{n,n-1}}$ transducer T is *invertible* if $[T]_{\cong_S} \in \mathcal{O}_{n,n-1}$.

Lemma 14.8 ($\mathcal{O}_{n,n-1}$ works as expected, Assumed Knowledge: 11.13, 11.9, 12.3, 12.9, 14.7).

The operation given in Definition 14.7, makes $\widetilde{\mathcal{O}_{n,n-1}}$ into a well-defined monoid. Moreover, if A, B are synchronizing $(1, n)$ -transducers with $A \cong_S M_{CR(A)}$, $B \cong_S M_{CR(B)}$ and there is $a \in Q_A, b \in Q_B$ such that $f_{A,a} = f_{B,b}$, then $\text{Core}(A) \cong_S \text{Core}(B)$.

Proof. Let A, B be arbitrary $\widetilde{\mathcal{O}_{n,n-1}}$ transducers. For remainder of this proof (to avoid nested subscripts) if T is a $(1, n)$ -transducer, then we will denote M_T by $M(T)$ instead.

We first need to show that $M(CR(\text{Core}(AB)))$ is strongly isomorphic to an $\widetilde{\mathcal{O}_{n,n-1}}$ transducer. All states of A and B define homeomorphisms from \mathfrak{C}_n to open subsets of \mathfrak{C}_n , thus it follows from Lemma 11.13 that all the states of AB define homeomorphisms from \mathfrak{C}_n to open subsets of \mathfrak{C}_n . Therefore by Remark 11.9 and Lemma 12.9, all states of $M(CR(\text{Core}(AB)))$ also define homeomorphisms from \mathfrak{C}_n^d to open subsets of \mathfrak{C}_n^d .

By construction $M(CR(\text{Core}(AB)))$ is its own minimal transducer, has complete response, is its own core, and has the required domain and range. Thus the operation of $\widetilde{\mathcal{O}_{n,n-1}}$ is well defined.

Claim: Suppose that S, T are arbitrary synchronizing $(1, n)$ -transducers such that $S \cong_S M(CR(S))$ and $T \cong_S M(CR(T))$. If there are $q_s \in Q_S$ and $q_t \in Q_T$ such that $f_{T,q_t} = f_{S,q_s}$, then $\text{Core}(T) \cong_S \text{Core}(S)$.

Proof of Claim: If $q \in Q_T$, then for all $v, w \in X_n^*, x \in \mathfrak{C}_n$ we have

- (1) $(q, w)\lambda_T$ is determined by the sets $(\mathfrak{C}_n)f_{T,q}$ and $(w\mathfrak{C}_n)f_{T,q}$. Thus $(q, w)\lambda_T$ is determined by $f_{T,q}$.
- (2) $(wx)f_{T,q} = (q, w)\lambda_T(x)f_{T,(q,w)\pi_T}$ and

$$(q, v)\pi_T = (q, w)\pi_T \iff f_{T,(q,w)\pi_T} = f_{T,(q,v)\pi_T},$$

thus $f_{T,q}$ determines whether or not $(q, v)\pi_T = (q, w)\pi_T$.

- (3) $\text{Core}(T) \subseteq (q, X_n^*)\pi_T$.

Combining the above observations gives the result. \diamond

Identity transducers with the appropriate domain, range, and state sets (Definition 12.3) are $\widetilde{\mathcal{O}_{n,n-1}}$ transducers, so $\widetilde{\mathcal{O}_{n,n-1}}$ has an identity.

It remains to show that our binary operation is associative. Let A, B, C be arbitrary $\widetilde{\mathcal{O}}_{n,n-1}$ transducers, we need to show that

$$M(CR(\text{Core}(M(CR(\text{Core}(AB)))C))) \cong_S M(CR(\text{Core}(AM(CR(\text{Core}(BC)))))).$$

By the same argument showing that multiplication in $\widetilde{\mathcal{O}}_{n,n-1}$ is well-defined, the transducer $T := M(CR(\text{Core}(ABC)))$ is an $\widetilde{\mathcal{O}}_{n,n-1}$ transducer.

Note that the Core operator commutes with the M and CR operators, so $T = \text{Core}(M(CR(ABC)))$. Let $(a, b, c) \in Q_{ABC}$ be arbitrary. By Remark 11.9 and Lemma 12.9, there is some $q \in Q_{M(CR(ABC))}$ and $p \in X_n^*$ such that

$$f_{M(CR(ABC)),q} = f_{A,a}f_{B,b}f_{C,c}\lambda_p^{-1}.$$

Let $(a, b, c) \in Q_A \times Q_B \times Q_C$ be arbitrary such that $(a, b) \in \text{Core}(AB)$. Again by Remark 11.9 and Lemma 12.9, there is $q \in Q_{M(CR(\text{Core}(AB)))}$ and $p \in X_n^*$ such that

$$f_{M(CR(AB))C,(q,c)} = f_{A,a}f_{B,b}f_{C,c}\lambda_p^{-1},$$

and hence there is $q \in Q_{M(CR(M(CR(\text{Core}(AB)))C))}$ and $p \in X_n^*$ such that

$$f_{M(CR(M(CR(AB))C)),q} = f_{A,a}f_{B,b}f_{C,c}\lambda_p^{-1}.$$

By commuting the Core operator again we have

$$M(CR(\text{Core}(M(CR(\text{Core}(AB)))C))) = \text{Core}(M(CR(M(CR(\text{Core}(AB)))C))).$$

Thus, the transducers T and $M(CR(\text{Core}(M(CR(\text{Core}(AB)))C)))$ are the cores of transducers which have states defining a common map (as they have complete response, the choices of p above must coincide). By the Claim, this implies that

$$T \cong_S M(CR(\text{Core}(M(CR(\text{Core}(AB)))C))).$$

By a symmetric argument, we also have

$$T \cong_S M(CR(\text{Core}(AM(CR(\text{Core}(BC)))))).$$

so the result follows. \square

Definition 14.9 ($d\mathcal{P}_{n,n-1}$, Assumed Knowledge: 11.14, 14.7, 14.10).

We say that a transducer T is a $d\mathcal{P}_{n,n-1}$ transducer if $T = \prod_{i \in \{0,1,\dots,d-1\}} T_i$ where T_0, T_1, \dots, T_{d-1} are $\widetilde{\mathcal{O}}_{n,n-1}$ transducers. We also define $d\widetilde{\mathcal{P}}_{n,n-1}$ to be the set $\left\{ T \mid T \text{ is an } d\mathcal{P}_{n,n-1} \text{ transducer} \right\} / \cong_S$, with multiplication given by $[A]_{\cong_S} [B]_{\cong_S} = [C]_{\cong_S}$, where C is strongly isomorphic to $M_{CR(\text{Core}(AB))}$ (we show in Lemma 14.10 that this object is well defined).

Lemma 14.10 (Product decomposition, Assumed Knowledge: 1.10, 2.3, 11.5, 11.18, 14.6, 14.7, 14.9). *If $[T]_{\cong_S} \in d\mathcal{K}_{n,1}$ then T is strongly isomorphic to a $d\mathcal{P}_{n,n-1}$ transducer. Moreover if A, B are $d\mathcal{P}_{n,n-1}$ transducers then $T := M_{CR(\text{Core}(AB))}$ is strongly isomorphic to a $d\mathcal{P}_{n,n-1}$ transducer and hence the object $d\widetilde{\mathcal{P}}_{n,n-1}$ of Definition 14.9 is well-defined.*

Proof. Let T be a transducer such that either $[T]_{\cong_S} \in d\mathcal{K}_{n,1}$ or $T = M_{CR(\text{Core}(AB))}$ for some $d\mathcal{P}_{n,n-1}$ transducers A and B . For each $i \in \{0, 1, \dots, d-1\}$, let

$$\sim_i := \left\{ (p, q) \in Q_T^2 \mid \text{there is } w \in F_{d,n,\{i\}} \text{ with } (p, w)\pi_T = q \right\}.$$

Claim 1: For all $i \in \{0, 1, \dots, d-1\}$, the binary relation \sim_i is an equivalence relation on Q_T . Moreover if $p \in Q_T$, then there is $w_p \in F_{d,n,\{i\}}$ such that for all $q \in [p]_{\sim_i}$ we have $(q, w_p)\pi_T = p$.

Proof of Claim:

- (1) Reflexive: If $q \in Q_T$, then $(q, \varepsilon_d)\pi_T = q$ and $\varepsilon_d \in F_{d,n,\{i\}}$. Thus $q \sim_i q$.
- (2) Symmetric: Suppose that $p, q \in Q_T$ are such that $(p, q) \in \sim_i$. Let $w \in (\{p\})\mathfrak{s}_T^{-1}$ and $v \in F_{d,n,\{i\}}$ be such that $(p, v)\pi_T = q$. Let $w = w_p w'$ where $w_p \in F_{d,n,\{i\}}$ and $w' \in F_{d,n,\{0,1,\dots,d-1\} \setminus \{i\}}$. We have

$$\begin{aligned} (q, w_p)\pi_T &= ((p, v)\pi_T, w_p)\pi_T = ((p, w)\pi_T, v w_p)\pi_T = (p, v w w_p)\pi_T \\ &= (p, w_p w' v w_p)\pi_T = (p, w_p v w' w_p)\pi_T = (p, w_p v w)\pi_T \\ &= ((p, w_p v)\pi_T, w)\pi_T = (w)\mathfrak{s}_T = p. \end{aligned}$$

Thus $q \sim_i p$.

- (3) Transitive: Suppose that $q_0 \sim_i q_1$ and $q_1 \sim_i q_2$. Thus there are $v_0, v_1 \in F_{d,n,\{i\}}$ with $(q_0, v_0)\pi_T = q_1$ and $(q_1, v_1)\pi_T = q_2$. In particular

$$(q_0, v_0 v_1)\pi_T = ((q_0, v_0)\pi_T, v_1)\pi_T = (q_1, v_1)\pi_T = q_2.$$

So $q_0 \sim_i q_2$.

. If we choose w_p , as we did when showing \sim_i is symmetric, then the second part of the claim follows from the same sequence of equalities. \diamond

If $q \in Q_T$, $i \in \{0, 1, \dots, d-1\}$, and we restrict the domain, range, and state set of T to $F_{d,n,\{i\}}$, $F_{d,n,\{i\}}$ and $[q]_{\sim_i}$ respectively, then we obtain a subtransducer of T (in the case that $[T]_{\cong_S} \in d\mathcal{K}_{n,1}$ this is only true because $([T]_{\cong_S})\Psi$ is trivial).

We denote by $S_{q,i}$, the $(1, n)$ -transducer isomorphic to this transducer obtained by replacing the monoid $F_{d,n,\{i\}}$ with X_n^* in the natural fashion.

Claim 2: If $q \in Q_T$ and $w \in (X_n^*)^d$, then

$$(q, w)\lambda_T = ((q, (w)\pi_0)\lambda_{S_{q,0}}, (q, (w)\pi_1)\lambda_{S_{q,1}}, \dots, (q, (w)\pi_{d-1})\lambda_{S_{q,d-1}}).$$

Proof of Claim: Let $i \in \{0, 1, \dots, d-1\}$ be arbitrary. We show that $(q, w)\lambda_T \pi_i = (q, (w)\pi_i)\lambda_{S_{q,i}}$. Let $w = w_i w'$, where $w_i \in F_{d,n,\{i\}}$ and $w' \in F_{d,n,\{0,1,\dots,d-1\} \setminus \{i\}}$. We have that

$$\begin{aligned} (q, w)\lambda_T \pi_i &= (q, w_i w')\lambda_T \pi_i = ((q, w_i)\lambda_T ((q, w_i)\pi_T, w')\lambda_T)\pi_i \\ &= (q, w_i)\lambda_T \pi_i ((q, w_i)\pi_T, w')\lambda_T \pi_i. \end{aligned}$$

By the choice of T , we have $((q, w_i)\pi_T, w')\lambda_T \pi_i = \varepsilon$. Thus

$$(q, w)\lambda_T \pi_i = (q, w_i)\lambda_T \pi_i = (q, (w)\pi_i)\lambda_{S_{q,i}}$$

as required. \diamond

Claim 3: If $q \in Q_T$ and $i \in \{0, 1, \dots, d-1\}$, then $S_{q,i}$ is strongly isomorphic to an $\widetilde{\mathcal{O}}_{n,n-1}$ transducer.

Proof of Claim: The transducer $S_{q,i}$ is synchronizing and is its own core by Claim 1. Suppose that $p \in [q]_{\sim_i}$ and $w \in X_n^*$ is such that $(\mathfrak{C}_n)f_{S_{q,i},p} \subseteq w\mathfrak{C}_n$. It follows from Claim 2 that

$$(\mathfrak{C}_n^d)f_{T,p} \subseteq w'\mathfrak{C}_n^d$$

where $(w')\pi_i = w$ and $(w')\pi_j = \varepsilon$ for $j \neq i$. As T has complete response, it follows that $w = \varepsilon$. Thus $S_{q,i}$ has complete response.

If $[T]_{\cong_S} \in d\mathcal{K}_{n,1}$, then (by Remark 11.18) for all $q \in Q_T$, the function $f_{T,q}$ is injective with open image. If $T = M_{CR(\text{Core}(AB))}$ for some $\widetilde{d\mathcal{P}_{n,n-1}}$ transducers A, B , and $p \in Q_T$ is arbitrary, then (by Lemmas 11.5, 11.13 and 12.9) there are $p_a \in Q_A$, $p_b \in Q_B$ and $w \in (X_n^*)^d$ such that $f_{T,p} = f_{A,p_a} f_{B,p_b} \lambda_w^{-1}$. So again for all $q \in Q_T$, the function $f_{T,q}$ is injective with open image.

By Claim 2, it follows that if $p \in [q]_{\sim_i}$, then

$$f_{T,p} = \langle \pi_0 f_{S_{q,0,p}}, \pi_1 f_{S_{q,1,p}}, \dots, \pi_{d-1} f_{S_{q,d-1,p}} \rangle \mathfrak{C}_n^d.$$

In particular $f_{S_{q,i,p}}$ is continuous, injective, and has image $(\mathfrak{C}_n^d) f_{T,p} \pi_i$ (which is open as it is a projection of an open set). By Remark 3.18, $f_{S_{q,i,p}}^{-1}$ is also continuous.

It remains to show that $S_{q,i} \cong_S M_{S_{q,i}}$. Suppose that $q_0, q_1 \in Q_{S_{q,i}}$ are such that $(p_0, v)\lambda_{S_{q,i}} = (p_1, v)\lambda_{S_{q,i}}$ for all $v \in X_n^*$. As $T \cong_S M_T$, it suffices to show that $(p_0, w)\lambda_T = (p_1, w)\lambda_T$ for all $w \in (X_n^*)^d$.

Let $j \in \{0, 1, \dots, d-1\}$, and $w \in F_{d,n,\{j\}}$ be arbitrary. By Claim 2, it suffices to show that $(p_0, w)\lambda_T \pi_j = (p_1, w)\lambda_T \pi_j$. If $j = i$ then this follows by the assumption on p_0, p_1 . Otherwise let $s \in F_{d,n,\{i\}}$ be such that $(p_0, s)\pi_T = p_1$. Then

$$\begin{aligned} (p_0, w)\lambda_T \pi_j &= (p_0, w)\lambda_T \pi_j ((p_0, w)\pi_T, s)\lambda_T \pi_j = (p_0, ws)\lambda_T \pi_j = (p_0, sw)\lambda_T \pi_j \\ &= ((p_0, s)\lambda_T (p_1, w)\lambda_T) \pi_j = (p_1, w)\lambda_T \pi_j \end{aligned}$$

as required. \diamond

Claim 4: For all $w \in F_{d,n,\{0,1,\dots,d-1\} \setminus \{i\}}$ and $i \in \{0, 1, \dots, d-1\}$, the map $p \mapsto (p, w)\pi_T$ defines a strong transducer isomorphism from $S_{q,i}$ to $S_{(q,w)\pi_T,i}$.

Proof of Claim: It follows from the fact that all elements of $F_{d,n,\{0,1,\dots,d-1\} \setminus \{i\}}$ commute with all elements of $F_{d,n,\{i\}}$, that this map is a homomorphism.

It remains to show that the map is invertable. For all $p \in [q]_{\sim_i}$, let $w_p \in F_{d,n,\{i\}}$ be as in Claim 1. Let $w' \in (\{q\})\bar{s}_T^{-1}$ be arbitrary, and let $w'' \in F_{d,n,\{0,1,\dots,d-1\} \setminus \{i\}}$ be such that $w \in w'' F_{d,n,\{i\}}$. Note that for all $p \in Q_T$, we have $q \sim_i (p, w'')\pi_T$. It follows that for all $p \in [q]_{\sim_i}$, we have

$$\begin{aligned} ((p, w)\pi_T, w'')\pi_T &= (((p, w_p)\pi_T, w)\pi_T, w'')\pi_T = (p, w_p w w'')\pi_T \\ &= (p, w w'' w_p)\pi_T = ((p, w)\pi_T, w'' w_p)\pi_T = p \end{aligned}$$

So the maps $p \mapsto (p, w)\pi_T$ and $p \mapsto (p, w'')\pi_T$ are mutually inverse homomorphisms as required. \diamond

By Claim 4, for each $i \in \{0, 1, \dots, d-1\}$, let T_i be an $\widetilde{\mathcal{O}_{n,n-1}}$ transducer isomorphic to $S_{q,i}$ for all $q \in Q_T$. Moreover for each $q \in Q_T$ and $i \in \{0, 1, \dots, d-1\}$ let $\phi_{q,i} : S_{q,i} \rightarrow T_i$ be a strong transducer isomorphism. For all $i \in \{0, 1, \dots, d-1\}$, we define $\phi_{iD} = \phi_{iR}$ to be the map $\pi_i : (X_n^*)^d \rightarrow (X_n^*)$ and ϕ_{iQ} to be the union of the maps $\phi_{q,iQ}$ over all $q \in Q_T$.

Claim 5: For all $i \in \{0, 1, \dots, d-1\}$, the triple $\phi_i := (\phi_{iQ}, \phi_{iD}, \phi_{iR})$ is a transducer homomorphism from T to T_i .

Proof of Claim: If $p, q \in Q_T$ and $p \sim_i q$ and $w \in \text{dom}(\mathfrak{s}_{S_{q,i}})$, then

$$((w)\mathfrak{s}_{S_{q,i}})\phi_{q,iQ} = (w)\mathfrak{s}_{T_i} = ((w)\mathfrak{s}_{S_{q,i}})\phi_{p,iQ}$$

so $\phi_{q,i} = \phi_{p,i}$. In particular ϕ_{iQ} is a union of functions with disjoint domains and is hence a function.

We next show that ϕ_i is a homomorphism. Let $q \in Q_T$ and $w \in D_T$ be arbitrary. If $w_q \in (\{q\})\mathfrak{s}_T^{-1}$, then

$$(q, w)\pi_T \phi_{iQ} = ((w_q w)\pi_i)\mathfrak{s}_{T_i} = (((w_q)\pi_i)\mathfrak{s}_{T_i}, (w)\pi_i)\pi_{T_i} = ((q)\phi_{iQ}, (w)\pi_i)\pi_{T_i}.$$

The condition on the output functions follows from Claim 2. \diamond d

Let $P := \prod_{i \in \{0,1,\dots,d-1\}} T_i$ (Definition 11.14), and $\phi := \langle \phi_0, \phi_1, \dots, \phi_{d-1} \rangle_P$ (Definition 2.3). We need to show that ϕ is a strong transducer isomorphism. We have $\phi_D = \phi_R = \langle \pi_0, \pi_1, \dots, \pi_{d-1} \rangle_{(X_n^*)^d} = \text{id}_{(X_n^*)^d}$ by definition, so we need only show that ϕ_Q is a bijection.

Let $(q_0, q_1, \dots, q_{d-1}) \in Q_P$ be arbitrary. For each $i \in \{0, 1, \dots, d-1\}$, let $w_i \in (\{q_i\})\mathfrak{s}_{T_i}^{-1}$. As ϕ is a homomorphism, it follows that

$$(q, (w_0, w_1, \dots, w_{d-1}))\pi_T \phi_Q = (q_0, q_1, \dots, q_{d-1})$$

where $q \in Q_T$ is arbitrary. In particular ϕ is surjective.

By Lemma 11.5, it follows that there is a strong transducer homomorphism ψ such that $\phi\psi = q_T$. As $T \cong_S M_T$, the quotient map q_T is bijective, thus so is ϕ as required. \square

Definition 14.11 ($d\mathcal{P}_{n,n-1}$, Assumed Knowledge: 14.9, 14.10).

We will show in Theorem 14.12 that $d\widetilde{\mathcal{P}}_{n,n-1}$ is a monoid. We define $d\mathcal{P}_{n,n-1}$ to be the group of units of $d\widetilde{\mathcal{P}}_{n,n-1}$. Moreover we will say that a $d\widetilde{\mathcal{P}}_{n,n-1}$ transducer T is *invertable* if $[T]_{\cong_S} \in d\mathcal{P}_{n,n-1}$.

Theorem 14.12 ($d\mathcal{P}_{n,n-1}$ embedding, Assumed Knowledge: 11.9, 11.13, 12.9, 12.13, 14.8, 14.10, 14.11).

The function $\phi : \widetilde{\mathcal{O}}_{n,n-1}^d \rightarrow d\widetilde{\mathcal{P}}_{n,n-1}$ defined by $([T_0]_{\cong_S}, [T_1]_{\cong_S}, \dots, [T_{d-1}]_{\cong_S})\phi = \prod_{i \in \{0,1,\dots,d-1\}} T_i]_{\cong_S}$ is a well-defined monoid isomorphism. Moreover there is an embedding $\Phi : d\mathcal{K}_{n,1} \rightarrow d\mathcal{P}_{n,n-1}$ such that whenever $([T]_{\cong_S})\Phi = [P]_{\cong_S}$ we have $P \cong_S T$.

Proof. Note that if $A_0, A_1, \dots, A_{d-1}, B_0, B_1, \dots, B_{d-1}$ are $\widetilde{\mathcal{O}}_{n,n-1}$ transducers and

$$\prod_{i \in \{0,1,\dots,d-1\}} A_i \cong_S \prod_{i \in \{0,1,\dots,d-1\}} B_i,$$

then for all $i \in \{0, 1, \dots, d-1\}$ we have $A_i \cong_S B_i$ (this is shown during the proof of Lemma 14.10).

Thus ϕ is a well-defined bijection.

Let $[A]_{\cong_S}, [B]_{\cong_S} \in d\widetilde{\mathcal{P}}_{n,n-1}$ be arbitrary, and suppose that $[A]_{\cong_S}[B]_{\cong_S} = [C]_{\cong_S}$

$$([A]_{\cong_S})\phi^{-1} = ([A_0]_{\cong_S}, [A_1]_{\cong_S}, \dots, [A_{d-1}]_{\cong_S}),$$

$$([B]_{\cong_S})\phi^{-1} = ([B_0]_{\cong_S}, [B_1]_{\cong_S}, \dots, [B_{d-1}]_{\cong_S}),$$

$$([C]_{\cong_S})\phi^{-1} = ([C_0]_{\cong_S}, [C_1]_{\cong_S}, \dots, [C_{d-1}]_{\cong_S}).$$

Also let $\alpha : A \rightarrow \prod_{i \in \{0,1,\dots,d-1\}} A_i$, $\beta : B \rightarrow \prod_{i \in \{0,1,\dots,d-1\}} B_i$, $\gamma : C \rightarrow \prod_{i \in \{0,1,\dots,d-1\}} C_i$ be transducer isomorphisms.

Let $i \in \{0, 1, \dots, d-1\}$ be arbitrary. To conclude that ϕ is an isomorphism (and hence $d\widetilde{\mathcal{P}}_{n,n-1}$ is a monoid) we need only show that $M_{CR(\text{Core}(A_i B_i))} \cong_S C_i$. Let $q \in Q_C$ be arbitrary. By Remark 11.9, Lemma 11.13 and Lemma 12.9, there are $q_a \in Q_A$, $q_b \in Q_B$ and $w \in (X_n^*)^d$ such that

$$f_{C,q} = f_{A,q_a} f_{B,q_b} \lambda_w^{-1}.$$

Moreover (as α, β, γ are strong transducer isomorphisms) we have

$$\begin{aligned} f_{A,q} &= \langle \pi_0 f_{A_0,(q)\alpha_Q \pi_0}, \pi_1 f_{A_1,(q)\alpha_Q \pi_1}, \dots, \pi_{d-1} f_{A_{d-1},(q)\alpha_Q \pi_{d-1}} \rangle \mathfrak{C}_n^d, \\ f_{B,q} &= \langle \pi_0 f_{B_0,(q)\beta_Q \pi_0}, \pi_1 f_{B_1,(q)\beta_Q \pi_1}, \dots, \pi_{d-1} f_{B_{d-1},(q)\beta_Q \pi_{d-1}} \rangle \mathfrak{C}_n^d, \\ f_{C,q} &= \langle \pi_0 f_{C_0,(q)\gamma_Q \pi_0}, \pi_1 f_{C_1,(q)\gamma_Q \pi_1}, \dots, \pi_{d-1} f_{C_{d-1},(q)\gamma_Q \pi_{d-1}} \rangle \mathfrak{C}_n^d. \end{aligned}$$

So in particular we have $f_{C_i,(q)\gamma_Q \pi_i} = f_{A_i,(q_a)\gamma_Q \pi_i} f_{B_i,(q_b)\gamma_Q \pi_i} \lambda_{(w)\pi_i}^{-1}$. Let $P_i := M_{CR(A_i B_i)}$. As the Core operator commutes with the CR and M operators, we have that $\text{Core}(P_i) = M_{CR(\text{Core}(A_i B_i))}$.

By Remark 11.9 and Lemma 11.13, there is a state $p \in Q_{P_i}$ and a word $v \in X_n^*$ such that

$$f_{P_i,p} = f_{A_i,(q_a)\gamma_Q \pi_i} f_{B_i,(q_b)\gamma_Q \pi_i} \lambda_v^{-1}.$$

As P_i and C_i both have complete response, it follows that $f_{P_i,p} = f_{C_i,(q)\gamma_Q \pi_i}$. Thus by Lemma 14.8, we have

$$C_i = \text{Core}(C_i) \cong_S \text{Core}(P_i) = M_{CR(\text{Core}(A_i B_i))}$$

as required.

We now prove the final part of the theorem. By Lemma 14.10, there is a unique map $\Phi : \mathcal{K}_{n,1} \rightarrow d\widetilde{\mathcal{O}}_{n,n-1}$ such that the required isomorphism type condition holds. Moreover, because Φ satisfies this condition, Φ must be injective. The fact that Φ is a homomorphism follows from Lemma 12.13.

Note that Φ is a homomorphism, $d\mathcal{K}_{n,1}$ is a group, and Φ maps the identity of $d\mathcal{K}_{n,1}$ to the identity of $d\widetilde{\mathcal{P}}_{n,n-1}$. Thus $(d\mathcal{K}_{n,1})\Phi$ is contained in the group of units of $d\widetilde{\mathcal{P}}_{n,n-1}$. So Φ is indeed an map from $d\mathcal{K}_{n,1}$ to $d\widetilde{\mathcal{P}}_{n,n-1}$ as required. \square

Theorem 14.13 (Wreath embedding, Assumed Knowledge: 14.6, 14.12, 14.2).

The group $\text{Out}(dV_n)$ embeds in the group $\mathcal{O}_{n,n-1} \wr_a \text{Sym}(d)$ where a is the usual action of $\text{Sym}(d)$ on $\{0, 1, \dots, d-1\}$.

Proof. This follows from Theorems 14.6 and 14.12. \square

Theorem 14.13, gives us embeddings of both $\text{Out}(dV_n)$ and $\text{Out}(V_n) \wr_a \text{Sym}(d)$ into the group $\mathcal{O}_{n,n-1} \wr_a \text{Sym}(d)$. Thus to compare these groups we need to establish a useful description of the images of these embeddings. To do this we use the notion of *signatures* from [27] Definition 7.6. Each clopen subset of \mathfrak{C}_n^d can be expressed as a finite union of cones (Definition 10.1). Thus any homeomorphism of \mathfrak{C}_n^d , must map every cone to a finite collection of cones, the signature map measures the degree to which cones are split.

Definition 14.14 (Clopen set signatures, Assumed Knowledge: 1.1, 5.6, 11.14, 14.7, 14.12).

For each $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, we define a map $\overline{\text{ssig}}_{d,n} : \left\{ S \subseteq \mathfrak{C}_n^d \mid S \text{ is clopen} \right\} \rightarrow \mathbb{Z}/(n-1)\mathbb{Z}$ by $(S)\overline{\text{ssig}}_{d,n} = m + (n-1)\mathbb{Z}$ where m is such that there are pairwise disjoint $w_0 \mathfrak{C}_n^d, w_1 \mathfrak{C}_n^d, \dots, w_{m-1} \mathfrak{C}_n^d$ with

$$(w \mathfrak{C}_n) f_{T,q} = \bigcup_{i \in \{0, 1, \dots, m-1\}} w_i \mathfrak{C}_n^d.$$

We then define a map $\overline{\text{sig}}_{d,n} : \widetilde{d\mathcal{P}}_{n,n-1} \rightarrow \mathbb{Z}/(n-1)\mathbb{Z}$ by

$$([P]_{\cong_S})\overline{\text{sig}}_{d,n} = ((w\mathfrak{C}_n)f_{P,q})\overline{\text{ssig}}_{d,n}$$

where $w \in (X_n^*)^d$, $q \in Q_P$ are arbitrary. We show in Lemma 14.15 that these maps are well-defined.

Lemma 14.15 (Signatures work, Assumed Knowledge: 11.9, 11.13, 12.9, 14.14).

Let $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. The map $\overline{\text{sig}}_{d,n} : \widetilde{d\mathcal{P}}_{n,n-1} \rightarrow \mathbb{Z}/(n-1)\mathbb{Z}$ of Definition 14.14 is well-defined, moreover if we view $\mathbb{Z}/(n-1)\mathbb{Z}$ as a semigroup with multiplication of integers as the operation, then $\overline{\text{sig}}_{d,n}$ is a semigroup homomorphism.

Proof. We first show that $\overline{\text{ssig}}_{d,n}$ is well defined. Let S be a clopen subset of \mathfrak{C}_n^d . Suppose that $F_1, F_2 \subseteq (X_n^*)^d$ are such that both collections of sets $\{v\mathfrak{C}_n^d \mid v \in F_1\}$ and $\{v\mathfrak{C}_n^d \mid v \in F_2\}$ are individually pairwise disjoint, and

$$S = \bigcup_{v \in F_1} v\mathfrak{C}_n^d = \bigcup_{v \in F_2} v\mathfrak{C}_n^d.$$

As S is compact, it follows that both F_1 and F_2 are finite. We define

$$k := \max \left(\left\{ |(v)\pi_i| \mid v \in F_1 \cup F_2, i \in \{0, 1, \dots, d-1\} \right\} \right),$$

and define $F_3 := \{v \in (X_n^k)^d \mid v\mathfrak{C}_n^d \subseteq S\}$. For all $v \in F_1 \cup F_2$, we have $v\mathfrak{C}_n^d = \bigcup \{u\mathfrak{C}_n^d \mid u \in F_3, v \leq u\}$. Thus by Lemma 10.3, we have

$$\sum_{v \in F_1} 1 + (n-1)\mathbb{Z} = \sum_{v \in F_1} \left| \{u \in F_3 \mid v \leq u\} \right| + (n-1)\mathbb{Z} = |F_3| + (n-1)\mathbb{Z}.$$

In particular $|F_1| + (n-1)\mathbb{Z} = |F_3| + (n-1)\mathbb{Z}$ and by a symmetric argument, we have $|F_2| + (n-1)\mathbb{Z} = |F_3| + (n-1)\mathbb{Z} = |F_1| + (n-1)\mathbb{Z}$. Thus the map $\overline{\text{ssig}}_{d,n}$ is well-defined.

Let $[P]_{\cong_S} \in \widetilde{d\mathcal{P}}_{n,n-1}$, $w \in (X_n^*)^d$ and $q \in Q_P$ be arbitrary. Let k be the synchronizing length of P . To conclude that $\overline{\text{sig}}_{d,n}$ is well-defined, we need only show that $((w\mathfrak{C}_n^d)f_{P,q})\overline{\text{ssig}}_{d,n}$ is independent of w and q . This can be seen as follows:

$$\begin{aligned} ((w\mathfrak{C}_n^d)f_{P,q})\overline{\text{ssig}}_{d,n} &= \left(\left(\bigcup_{v \in (X_n^k)^d} wv\mathfrak{C}_n^d \right) f_{P,q} \right) \overline{\text{ssig}}_{d,n} = \left(\bigcup_{v \in (X_n^k)^d} (wv\mathfrak{C}_n^d) f_{P,q} \right) \overline{\text{ssig}}_{d,n} \\ &= \sum_{v \in (X_n^k)^d} ((wv\mathfrak{C}_n^d) f_{P,q}) \overline{\text{ssig}}_{d,n} = \sum_{v \in (X_n^k)^d} ((q, wv)\lambda_P(\mathfrak{C}_n^d) f_{P,(wv)\mathfrak{s}_P}) \overline{\text{ssig}}_{d,n} \\ &= \sum_{v \in (X_n^k)^d} ((\mathfrak{C}_n^d) f_{P,(wv)\mathfrak{s}_P}) \overline{\text{ssig}}_{d,n} = \sum_{v \in (X_n^k)^d} ((\mathfrak{C}_n^d) f_{P,(v)\mathfrak{s}_P}) \overline{\text{ssig}}_{d,n}. \end{aligned}$$

We now need to show that $\overline{\text{sig}}_{d,n}$ is a semigroup homomorphism. Let $A, B, C \in \widetilde{d\mathcal{P}}_{n,n-1}$ be such that $[A]_{\cong_S}[B]_{\cong_S} = [C]_{\cong_S}$. Let $q \in Q_C$, by Remark 11.9, Lemma 11.13 and 12.9, there is $(q_a, q_b) \in Q_A \times Q_B$ and $w \in X_n^*$ such that $f_{C,q} = f_{A,q_a}f_{B,q_b}\lambda_w^{-1}$. Moreover, let $F \subseteq (X_n^*)^d$ be arbitrary such that $(\mathfrak{C}_n)f_{A,q_a} = \bigcup_{v \in F} v\mathfrak{C}_n$ and such that the sets of the collection $\{v\mathfrak{C}_n \mid v \in F\}$ are pairwise disjoint.

We now have

$$\begin{aligned}
([C]_{\cong_S})\overline{\text{sig}_{d,n}} &= ((\mathfrak{C}_n)f_{C,q})\overline{\text{ssig}_{d,n}} = ((\mathfrak{C}_n)f_{A,q_a}f_{B,q_b}\lambda_w^{-1})\overline{\text{ssig}_{d,n}} = ((\mathfrak{C}_n)f_{A,q_a}f_{B,q_b})\overline{\text{ssig}_{d,n}} \\
&= \left(\left(\bigcup_{v \in F} v\mathfrak{C}_n \right) f_{B,q_b} \right) \overline{\text{ssig}_{d,n}} = \left(\bigcup_{v \in F} (v\mathfrak{C}_n) f_{B,q_b} \right) \overline{\text{ssig}_{d,n}} \\
&= \sum_{v \in F} ((v\mathfrak{C}_n) f_{B,q_b}) \overline{\text{ssig}_{d,n}} = \sum_{v \in F} ([B]_{\cong_S}) \overline{\text{sig}_{d,n}} = ([B]_{\cong_S}) \overline{\text{sig}_{d,n}} |F| \\
&= ([A]_{\cong_S}) \overline{\text{sig}_{d,n}} ([B]_{\cong_S}) \overline{\text{sig}_{d,n}}.
\end{aligned}$$

□

Lemma 14.16 (Higher dimensional signatures from decomposition, Assumed Knowledge: 10.1, 14.12, 14.14, 14.15).

Let $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. Let ϕ be as in Theorem 14.12. If $[T]_{\cong_S} \in \widetilde{d\mathcal{P}_{n,n-1}}$, then

$$([T]_{\cong_S})\overline{\text{sig}_{d,n}} = \prod_{i \in \{0,1,\dots,d-1\}} (([T]_{\cong_S})\phi^{-1}\pi_i)\overline{\text{sig}_{1,n}}$$

Proof. Let T_0, T_1, \dots, T_{d-1} be arbitrary $\widetilde{\mathcal{O}_{n,n-1}}$ transducers, and let $P := \prod_{i \in \{0,1,\dots,d-1\}} T_i$. For each $i \in \{0, 1, \dots, d-1\}$, let $q_i \in Q_{T_i}$, and $F_i \subseteq X_n^*$ be such that the elements of F_i are pairwise incomparable and

$$(\mathfrak{C}_n)f_{T_i,q_i} = \bigcup_{w \in F_i} w\mathfrak{C}_n.$$

We now have

$$\begin{aligned}
((\mathfrak{C}_n^d)f_{P,(q_0,q_1,\dots,q_{d-1})})\overline{\text{ssig}_{d,n}} &= \left(\prod_{i \in \{0,1,\dots,d-1\}} (\mathfrak{C}_n)f_{T_i,q_i} \right) \overline{\text{ssig}_{d,n}} = \left(\prod_{i \in \{0,1,\dots,d-1\}} \bigcup_{w \in F_i} w\mathfrak{C}_n \right) \overline{\text{ssig}_{d,n}} \\
&= \left(\bigcup_{w \in F_0 \times F_1 \times \dots \times F_{d-1}} w\mathfrak{C}_n^d \right) \overline{\text{ssig}_{d,n}} = \prod_{i \in \{0,1,\dots,d-1\}} |F_i| + (n-1)\mathbb{Z}.
\end{aligned}$$

By definition, $((\mathfrak{C}_n^d)f_{P,(q_0,q_1,\dots,q_{d-1})})\overline{\text{ssig}_{d,n}} = ([P]_{\cong_S})\overline{\text{sig}_{d,n}}$ and for each $i \in \{0, 1, \dots, d-1\}$ we have $|F_i| + (n-1)\mathbb{Z} = ([T_i]_{\cong_S})\overline{\text{sig}_{1,n}}$. Thus the result follows. □

In [27] Proposition 7.7, the notion of signature is used to describe $\mathcal{O}_{n,r}$ as subgroup of $\mathcal{O}_{n,n-1}$. We can now do the same with $d\mathcal{K}_{n,1}$ and $d\mathcal{P}_{n,n-1}$.

Lemma 14.17 (Identifying $d\mathcal{K}_{n,1}$ with signatures, Assumed Knowledge: 10.3, 11.11, 11.19, 11.9, 12.9, 14.12, 14.15).

Let $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$. If $\Phi : d\mathcal{K}_{n,1} \rightarrow d\mathcal{P}_{n,n-1}$ is as in Theorem 14.12, then

$$(d\mathcal{K}_{n,1})\Phi = (\{1 + (n-1)\mathbb{Z}\})\overline{\text{sig}_{d,n}}^{-1} \cap d\mathcal{P}_{n,n-1}.$$

Proof. (\subseteq) : Let $[T]_{\cong_S} \in d\mathcal{K}_{n,1}$ be arbitrary, and let $f \in d\mathcal{B}_{n,1}$ be such that $T = \text{Core}(M_f)$. Let k be the synchronizing length of M_f . It follows that

$$\begin{aligned} 1 + (n-1)\mathbb{Z} &= (\mathfrak{C}_n^d) \overline{\text{ssig}_{d,n}} = ((\mathfrak{C}_n^d)f) \overline{\text{ssig}_{d,n}} = \left(\bigcup_{w \in (X_n^k)^d} (w\mathfrak{C}_n^d)f \right) \overline{\text{ssig}_{d,n}} \\ &= \left(\bigcup_{w \in (X_n^k)^d} (e_f, w) \lambda_{M_f}(\mathfrak{C}_n^d) f_{T, (w)\mathfrak{s}_T} \right) \overline{\text{ssig}_{d,n}} \\ &= \sum_{w \in (X_n^k)^d} ((e_f, w) \lambda_{M_f}(\mathfrak{C}_n^d) f_{T, (w)\mathfrak{s}_T}) \overline{\text{ssig}_{d,n}} = \sum_{w \in (X_n^k)^d} ((\mathfrak{C}_n^d) f_{T, (w)\mathfrak{s}_T}) \overline{\text{ssig}_{d,n}} \\ &= \sum_{w \in (X_n^k)^d} (([T]_{\cong_S}) \Phi) \overline{\text{sig}_{d,n}} = n^{kd} (([T]_{\cong_S}) \Phi) \overline{\text{sig}_{d,n}} = (([T]_{\cong_S}) \Phi) \overline{\text{sig}_{d,n}} \end{aligned}$$

as required.

(\supseteq) : Let $[P]_{\cong_S} \in (\{1 + (n-1)\mathbb{Z}\}) \overline{\text{sig}_{d,n}^{-1}} \cap d\mathcal{P}_{n,n-1}$, and $q \in Q_P$ be arbitrary. Then let $(a\mathfrak{C}_n^d)_{a \in A}$ be disjoint cones with union $(\mathfrak{C}_n^d) f_{P,q}$. By the choice of P , we have $|A| \in 1 + (n-1)\mathbb{Z}$. By Lemma 11.11, we can choose $k \in \mathbb{N}$ greater than $|A|$, such that for all $w \in (X_n^k)^d$ and $i \in \{0, 1, \dots, d-1\}$, we have

$$|(q, w) \lambda_P \pi_i| \geq \max \left\{ |(a)\pi_j| \mid a \in A, j \in \{0, 1, \dots, d-1\} \right\}.$$

For all $w \in (X_n^k)^d$ let $a_w \in A$ be the unique element such that $a_w \leq (q, w) \lambda_P$. By the definition of A , for all $a \in A$ we now have

$$\bigcup_{\substack{w \in (X_n^k)^d \\ a_w = a}} (w\mathfrak{C}_n^d) f_{P,q} = a\mathfrak{C}_n^d.$$

In particular

$$\left\{ ((q, w) \lambda_P \lambda_{a_w}^{-1}) (\mathfrak{C}_n^d) f_{P, (q, w)\pi_T} \mid w \in (X_n^k)^d \text{ satisfies } a_w = a \right\}$$

is a partition of \mathfrak{C}_n^d .

As $|A| \in 1 + (n-1)\mathbb{N}$, we have by Lemma 10.3 that there is a complete prefix code B for \mathfrak{C}_n^d with $|B| = |A|$. Let $\phi : A \rightarrow B$ be a bijection. It follows that

$$\left\{ (a_w) \phi \left(((q, w) \lambda_P \lambda_{a_w}^{-1}) (\mathfrak{C}_n^d) f_{P, (q, w)\pi_T} \right) \mid w \in (X_n^k)^d \right\}$$

is a partition of \mathfrak{C}_n^d . We define a (d, n) -transducer D as follows:

- (1) $N := \left\{ w \in (X_n^*)^d \mid \text{there is } i \in \{0, 1, \dots, d-1\} \text{ with } (w)\pi_i < k \right\}$ (we assume without loss of generality that $N \cap Q_P = \emptyset$).
- (2) $Q_D := Q_P \cup N$.
- (3) If $p \in N$, $w \in (X_n^*)^d$ and $pw \in N$, then $(p, w)\pi_D = pw$ and $(p, w)\lambda_D = \varepsilon_d$.
- (4) If $p \in N$, $w \in (X_n^*)^d$ and $pw = bt$ where $b \in (X_n^k)^d$ and $t \in (X_n^*)^d$, then

$$(p, w)\pi_D := ((q, w)\pi_P, t)\pi_P, \quad (p, w)\lambda_D := (a_w)\phi \left(((q, w) \lambda_P \lambda_{a_w}^{-1}) ((q, w)\pi_P, t)\lambda_P \right).$$

- (5) If $p \in Q_P$ and $w \in (X_n^*)^d$ then

$$(p, w)\pi_D := (p, w)\pi_P, \quad (p, w)\lambda_D := (p, w)\lambda_P.$$

It is routine to verify that D is a (d, n) -transducer. By definition, the transducer D is synchronizing with synchronizing length at most k plus the synchronizing length of P . Moreover $\text{Core}(D) = P$. We now define $f := f_{D, \varepsilon_d}$

If $w \in (X_n^k)^d$, $x \in \mathfrak{C}_n^d$, then

$$(wx)f = (a_w)\phi((q, w)\lambda_P\lambda_{a_w}^{-1})(x)f_{P, (q, w)\pi_P},$$

as each of the maps $f_{P, (q, w)\pi_P}$ is a homeomorphism to an open subset of \mathfrak{C}_n^d , it follows that $f \in \text{Aut}(\mathfrak{C}_n^d)$.

By Remark 11.9 and Lemma 12.9, together with Theorem 11.19, we see that $M_{CR(D)} \cong_S M_f$, and $\text{Core}(M_{CR(D)}) \cong_S P$. Thus if $f \in d\mathcal{B}_{n,1}$, then $[M_f]_{\cong_S} \in d\mathcal{O}_{n,1}$ and $([M_f]_{\cong_S})\Phi = [P]_{\cong_S}$ as required.

We have already shown that $f \in d\mathcal{S}_{n,1}$, thus we need only show that $f^{-1} \in d\mathcal{S}_{n,1}$. By assumption, the transducer P is invertable, so let $P^{-1} \in [P]_{\cong_S}^{-1}$ be arbitrary. By the same construction given for P , let $g \in d\mathcal{S}_{n,1}$ be such that $\text{Core}(M_g) \cong_S P^{-1}$.

We thus have $[\text{Core}(M_{fg})]_{\cong_S} = [\text{Core}(M_f)]_{\cong_S}[\text{Core}(M_g)]_{\cong_S} = 1_{d\mathcal{O}_{n,1}}$. So by Lemma 12.4, we have $fg \in dV_n$ and in particular $(fg)^{-1} \in d\mathcal{S}_{n,1}$. Therefore $f^{-1} = g(fg)^{-1} \in d\mathcal{S}_{n,1}$ as well. \square

Theorem 14.18 (Connecting $\text{Out}(V_n)$ with $\text{Out}(dV_n)$, Assumed Knowledge: 12.18, 14.6, 14.12, 14.16, 14.17).

For all $d \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{N} \setminus \{0, 1\}$, we have

$$\text{Out}(dV_n) \cong \left\{ (\mathbf{T}, f) \in \mathcal{O}_{n, n-1} \lambda_a \text{Sym}(d) \mid (\mathbf{T})\overline{\phi \text{sig}_{d,n}} = 1 + (n-1)\mathbb{Z} \right\},$$

Moreover, the group $\text{Out}(V_n) \lambda_a \text{Sym}(d)$ embeds in the group $\text{Out}(dV_n)$. Here a is the standard action of $\text{Sym}(d)$ on $\{0, 1, \dots, d-1\}$, and ϕ is as in Theorem 14.12. $\text{Out}(dV_n)$.

Proof. By Theorem 12.18 we have $\text{Out}(dV_n) \cong d\mathcal{O}_{n,1}$. Thus from Theorem 14.6 and Lemma 14.17, we have

$$\text{Out}(dV_n) \cong \left\{ (\mathbf{T}, f) \in \mathcal{O}_{n, n-1} \lambda_a \text{Sym}(d) \mid (\mathbf{T})\overline{\phi \text{sig}_{d,n}} = 1 + (n-1)\mathbb{Z} \right\}.$$

In particular $\text{Out}(V_n) \cong \left\{ (\mathbf{T}, f) \in \mathcal{O}_{n, n-1} \mid (\mathbf{T})\overline{\phi \text{sig}_{1,n}} = 1 + (n-1)\mathbb{Z} \right\}$. The result follows from Lemma 14.16. \square

Theorem 14.19 (Connecting $\text{Out}(V)$ with $\text{Out}(dV)$, Assumed Knowledge: 14.18).

For all $d \in \mathbb{N} \setminus \{0\}$, we have $\text{Out}(dV) \cong \text{Out}(V) \lambda_a \text{Sym}(d)$ (where a is the standard action of $\text{Sym}(d)$ on $\{0, 1, \dots, d-1\}$).

Proof. Note that $|\mathbb{Z}/(2-1)\mathbb{Z}| = 1$, thus by Theorem 14.18, we have $\text{Out}(V) \cong \mathcal{O}_{n, n-1}$ and $\text{Out}(dV) \cong \mathcal{O}_{n, n-1} \lambda_a \text{Sym}(d)$. \square

REFERENCES

- [1] Mike Behrisch, John K. Truss, and Edith Vargas-García. Reconstructing the topology on monoids and polymorphism clones of the rationals. *Studia Logica*, 105(1):65–91, 2017.
- [2] James Belk and Collin Bleak. Some undecidability results for asynchronous transducers and the Brin-Thompson group $2V$. *Transactions of the American Mathematical Society*, 369(5):3157–3172, Dec 2016.
- [3] Collin Bleak, Peter Cameron, Yonah Maissel, Andrés Navas, and Feyishayo Olukoya. The further chameleon groups of Richard Thompson and Graham Higman: Automorphisms via dynamics for the Higman groups $G_{n,r}$, 2016.
- [4] Collin Bleak and Daniel Lanoue. A family of non-isomorphism results. *Geometriae Dedicata*, 146(1):21–26, Jun 2010.

- [5] Manuel Bodirsky, Michael Pinsker, and András Pongrácz. Reconstructing the topology of clones. *Trans. Amer. Math. Soc.*, 369(5):3707–3740, 2017.
- [6] Matthew G. Brin. The chameleon groups of Richard J. Thompson: automorphisms and dynamics. *Inst. Hautes Études Sci. Publ. Math.*, (84):5–33 (1997), 1996.
- [7] Matthew G. Brin. Higher dimensional Thompson groups. *Geometriae Dedicata*, 108(1):163–192, Oct 2004.
- [8] Matthew G. Brin. Presentations of higher dimensional Thompson groups. *Journal of Algebra*, 284(2):520–558, Feb 2005.
- [9] Matthew G. Brin and Fernando Guzmán. Automorphisms of generalized Thompson groups. *Journal of Algebra*, 203(1):285–348, May 1998.
- [10] John D. Dixon, Peter M. Neumann, and Simon Thomas. Subgroups of small index in infinite symmetric groups. *Bull. London Math. Soc.*, 18(6):580–586, 1986.
- [11] L. Elliott, J. Jonušas, Z. Mesyan, J. D. Mitchell, M. Morayne, and Y. Péresse. Automatic continuity, unique Polish topologies, and Zariski topologies on monoids and clones, 2021.
- [12] Luke Elliott. A description of $\text{aut}(\text{dvn})$ and $\text{out}(\text{dvn})$ using transducers, 2020.
- [13] David M. Evans. Subgroups of small index in infinite general linear groups. *Bulletin of the London Mathematical Society*, 18(6):587–590, 1986.
- [14] Steven Givant and Paul Halmos. *Introduction to Boolean algebras*. Undergraduate Texts in Mathematics. Springer, New York, 2009.
- [15] R. I. Grigorchuk, V. V. Nekrashevich, and V. I. Sushchanskiĭ. Automata, dynamical systems, and groups. *Proc. Steklov Inst. Math*, 231:128–203, 2000.
- [16] Johanna Hennig and Francesco Matucci. Presentations for the higher dimensional Thompson’s groups nV , 2011.
- [17] Bernhard Herwig. Extending partial isomorphisms for the small index property of many ω -categorical structures. *Israel Journal of Mathematics*, 107(1):93–123, December 1998.
- [18] Wilfrid Hodges, Ian Hodkinson, Daniel Lascar, and Saharon Shelah. The Small Index Property for ω -Stable (ω -Categorical Structures and for the Random Graph). *Journal of the London Mathematical Society*, s2-48(2):204–218, 1993.
- [19] Trevor Irwin and Slawomir Solecki. Projective Fraïssé limits and the pseudo-arc. *Transactions of the American Mathematical Society*, 358, 07 2006.
- [20] Alexander Kechris. *Classical descriptive set theory*, volume 156. Springer Science & Business Media, 2012.
- [21] Alexander S Kechris and Christian Rosendal. Turbulence, amalgamation, and generic automorphisms of homogeneous structures. *Proceedings of the London Mathematical Society*, 94(2):302–350, 2007.
- [22] Aleksandra Kwiatkowska. The group of homeomorphisms of the Cantor set has ample generics. *Bulletin of the London Mathematical Society*, 44(6):1132–1146, Apr 2012.
- [23] Mark V Lawson and Alina Vdovina. Higher dimensional generalizations of the Thompson groups, 2019.
- [24] Kathryn Mann. Automatic continuity for homeomorphism groups and applications. *Geometry & Topology*, 20(5):3033–3056, October 2016.
- [25] James R Munkres. *Topology*. Pearson India Education Services Pvt. Ltd., 2016.
- [26] Feyishayo Olukoya. Automorphisms of the generalised Thompson’s group $T_{n,r}$. In Preparation, 2018.
- [27] Feyishayo Olukoya. Automorphisms of the generalised Thompson’s group $T_{n,r}$, 2019.
- [28] Gianluca Paolini and Saharon Shelah. Automorphism groups of countable stable structures. *Fundamenta Mathematicae*, 248(3):301–307, 2020.
- [29] Christian Pech and Maja Pech. Polymorphism clones of homogeneous structures: gate coverings and automatic homeomorphicity. *Algebra universalis*, 79(2):35, 2018.
- [30] Martyn Quick. Permutation-based presentations for Brin’s higher-dimensional Thompson groups nV , 2019.
- [31] Matatyahu Rubin. On the reconstruction of topological spaces from their groups of homeomorphisms. *Transactions of the American Mathematical Society*, 312(2):487–538, 1989.
- [32] J. K. Truss. Infinite permutation groups II. Subgroups of small index. *Journal of Algebra*, 120(2):494–515, February 1989.
- [33] Todor Tsankov. Automatic continuity for the unitary group. *Proc. Amer. Math. Soc.*, 141(10):3673–3680, 2013.

- [34] Jan van Mill. *The infinite-dimensional topology of function spaces*, volume 64 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 2001.
- [35] Jan van Mill. A note on the Effros theorem. *The American Mathematical Monthly*, 111(9):801–806, 2004.
- [36] Itai Yaacov, Alexander Berenstein, and Julien Melleray. Polish topometric groups. *Trans. Amer. Math. Soc.*, 365, 07 2010.