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Abelian tropical covers

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Abstract

Let $\mathfrak{A}$ be a finite abelian group. In this paper, we classify harmonic $\mathfrak{A}$-covers of a tropical curve Γ (which allow dilation along edges and at vertices) in terms of the cohomology group of a suitably defined sheaf on Γ. We give a realisability criterion for harmonic $\mathfrak{A}$-covers by patching local monodromy data in an extended homology group on Γ. As an explicit example, we work out the case $\mathfrak{A} = \mathbb{Z}/p\mathbb{Z}$ and explain how realisability for such covers is related to the nowhere-zero flow problem from graph theory.

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Introduction

One of the starting points of tropical geometry is the observation that there is a deep analogy between the classical geometry of Riemann surfaces and the geometry of metric graphs, or more generally, (abstract) tropical curves.

Let X be a Riemann surface and let $B \subseteq X$ be a finite set. Ramified covers $X' \to X$ that are branched over B are topological coverings of $X_0 = X \setminus B$, and the Galois correspondence classifies such covers in terms of the fundamental group $\pi_1(X_0, x_0)$ for some base point $x_0 \in X_0$. This beautiful and classical story is explained in many standard textbooks on Riemann surfaces, such as [**CM16, Mir95, Sza09**]. In particular, given a finite group $\mathfrak{G}$, Galois covers with deck group $\mathfrak{G}$ (not necessarily connected) are in one-to-one correspondence with monodromy representations $\pi_1(X_0, x_0) \to \mathfrak{G}$. If $\mathfrak{G} = \mathfrak{A}$ is abelian, the universal coefficient theorem implies that the set of such covers is equal to

$$\mathrm{Hom}\big(\pi_1(X_0, x_0), \mathfrak{A}\big) \simeq \mathrm{Hom}\big(H_1(X_0, \mathbb{Z}), \mathfrak{A}\big) \simeq H^1(X_0, \mathfrak{A}). \tag{1}$$

Replacing H^1 and π_1 with their étale counterparts, this correspondence holds over any algebraically closed field k whose characteristic is zero or relatively prime to $|\mathfrak{A}|$.

The natural tropical analogue of a non-constant holomorphic map of Riemann surfaces is a *finite harmonic morphism* $\Gamma' \to \Gamma$ of metric graphs (or tropical curves), which is a continuous map with finite fibers that pulls back harmonic functions on open subsets of Γ to harmonic functions on their preimages in Γ'. Contrary to the algebraic case, a harmonic morphism need not be a topological covering map (even after finitely many points are removed), as harmonic morphisms allow for *dilation along edges*. Namely, via the natural identification of edges with real intervals, the restriction of a harmonic morphism $\phi : \Gamma' \to \Gamma$ to an edge $e' \subset \Gamma'$ is given by

$$[0, a] \longrightarrow [0, d \cdot a], \qquad x \longmapsto d \cdot x.$$

The coefficient $d \in \mathbb{Z}_{>0}$ is known as the *dilation factor* of ϕ along e'. The behaviour of a harmonic morphism at a vertex $v' \in \Gamma'$ is controlled by another phenomenon that we call *dilation at vertices*, which assigns a dilation factor to each vertex as well (see Section 1 below). We also note that dilation should not be confused with the distinct phenomenon of *ramification* for morphisms of weighted graphs, which we discuss at the end of Section 1.

Dilation phenomena are inherent properties of morphisms of metric graphs, and arise naturally in tropicalisation constructions. For this reason, the fundamental group of a metric graph (specifically, its underlying topological space) cannot be used to classify its harmonic covers, and this classification problem is, to the best of our knowledge, currently open.

Classification of abelian tropical covers.

Our first goal in this paper is to classify *abelian harmonic covers* of a fixed metric graph Γ. Given a finite group $\mathfrak{G}$, a *harmonic $\mathfrak{G}$-cover* of Γ is a harmonic morphism $\phi : \Gamma' \to \Gamma$ together with a fiberwise $\mathfrak{G}$-action, such that the dilation factor of ϕ at a point $p' \in \Gamma'$ is equal to the order of its stabiliser group. If $\mathfrak{G} = \mathfrak{A}$ is abelian, then ϕ admits a convenient cohomological description. Namely, for any $p \in \Gamma$ the stabiliser groups of two points of $\phi^{-1}(p)$ are equal, hence the cover determines a family of subgroups $D(p) \subseteq \mathfrak{A}$ indexed by $p \in \Gamma$, an object which we call the *$\mathfrak{A}$-dilation datum* of the harmonic cover. Choosing a graph model for Γ, the $\mathfrak{A}$-dilation datum D determines (by taking quotients) a sheaf of abelian groups $\mathfrak{A}_D$ on Γ that we call the *codilation sheaf*.

THEOREM A (THEOREM 2·3). *Let Γ be a metric graph or tropical curve, let $\mathfrak{A}$ be a finite abelian group, and let D be an $\mathfrak{A}$-dilation datum on Γ. There is a natural bijection between the sheaf cohomology group $H^1(\Gamma, \mathfrak{A}_D)$ and the set of harmonic $\mathfrak{A}$-covers with $\mathfrak{A}$-dilation datum D.*

We refer to $H^1(\Gamma, \mathfrak{A}_D)$ as the *dilated cohomology group* of Γ with respect to the $\mathfrak{A}$-dilation datum D. One may consider Theorem A as a first step towards a tropical analogue of geometric class field theory.

From algebraic to tropical covers (and back again).

There is a natural tropicalisation procedure that associates to a finite cover $F : X' \to X$ of smooth projective algebraic curves over a non-Archimedean field a harmonic morphism $\phi : \Gamma_{X'} \to \Gamma_X$ between the dual tropical curves. In the literature one may find at least two ways to describe this process: one by restricting the associated map $F^{an} : X'^{an} \to X^{an}$ of Berkovich analytic spaces to the non-Archimedean skeletons, as in [**ABBR15a, ABBR15b**], the other from a moduli-theoretic point of view, as in [**CMR16**], using the moduli space of admissible covers. In Section 3 below we recall the latter approach, paying extra attention to the role of a finite automorphism group $\mathfrak{G}$. In particular, we describe how to associate to a $\mathfrak{G}$-cover $F\colon X' \to X$ of algebraic curves a harmonic $\mathfrak{G}$-cover $\phi\colon \Gamma_{X'} \to \Gamma_X$ of tropical curves.

Describing finite harmonic covers that arise as tropicalisations of finite algebraic covers is a highly non-trivial task, known as the *realisability problem*. We refer the reader to [**Cap14**] and [**CMR16**] for details, including the connection to the still-open *Hurwitz existence problem* from the classical topology of Riemann surfaces (see [**PP06**] for a survey). In the abelian case, however, this problem admits a convenient homological solution, which we describe in Section 4. Given a tropical curve Γ and a finite abelian group $\mathfrak{A}$, we introduce the *extended homology group* $H_1^{\mathrm{ext}}(\Gamma, \mathfrak{A})$ whose elements encode local monodromy data of harmonic $\mathfrak{A}$-covers of Γ. In particular, a class $\eta \in H_1^{\mathrm{ext}}(\Gamma, \mathfrak{A})$ determines an associated $\mathfrak{A}$-dilation datum D_η, and the realisable covers are exactly the ones that have such $\mathfrak{A}$-dilation data:

THEOREM B (THEOREM 4·4). *A harmonic $\mathfrak{A}$-cover $\Gamma' \to \Gamma$ of tropical curves is realisable over a non-Archimedean field of residue characteristic zero or coprime to $|\mathfrak{A}|$ if and only if its $\mathfrak{A}$-dilation datum is associated to a class in the extended homology group $H_1^{ext}(\Gamma, \mathfrak{A})$.*

In Section 5, we specialise to the case of cyclic covers of prime order. It turns out that our realisability criterion is closely related to the so-called nowhere-zero flow problem from graph theory. In particular, Tutte's 5-flow conjecture has an equivalent formulation in terms of the existence of everywhere-dilated $\mathbb{Z}/5\mathbb{Z}$-covers.

We briefly mention how our results may generalise to the case of a non-abelian group $\mathfrak{G}$. A harmonic $\mathfrak{G}$-cover $\Gamma' \to \Gamma$ determines the structure of a *graph of groups* on a model of Γ, and Bass–Serre theory classifies such covers in terms of an appropriately generalised fundamental group [**Ser80, Bas93**]. However, there is no convenient generalisation of the homological realisability criterion, and the difficulties stemming from the Hurwitz existence problem cannot be avoided.

Earlier and related works.

Graphs and tropical curves with group actions have been studied by a number of authors. The simplest example is the case of tropical hyperelliptic curves, which are $\mathbb{Z}/2\mathbb{Z}$-covers of a tree (see [**ABBR15b, BBC17, BN09, Cap14, Cha13, Len17, Pan16**]). Expanding on this, Brandt and Helminck [**BH20**] consider arbitrary cyclic covers of a tree. Helminck [**Hel17**] looks at the tropicalisation of arbitrary abelian covers of algebraic curves from a non-Archimedean perspective, as in [**ABBR15a, ABBR15b**]. Our Section 3 provides a moduli-theoretic approach to the same topic (with possibly non-abelian group) in the spirit of [**CMR16**].

In a different direction, Jensen and Len [**JL18**] consider $\mathbb{Z}/2\mathbb{Z}$-covers of arbitrary tropical curves, and define the tropical Prym variety associated to such a cover. This object is equipped with a canonical polyhedral decomposition, leading to a combinatorial formula for its volume [**GZ23, LZ22**]. A tropical version of Donagi's n-gonal construction is investigated in [**RZ22**]. Applications to algebraic Prym–Brill–Noether theory are studied in [**LU21**] and [**CLRW22**]. See [**Len22**] for a survey on tropical Prym varieties. In a similar vein, Song [**Son19**] considers $\mathfrak{G}$-invariant linear systems with the goal of studying their descent properties to the quotient.

In [**Hel21**] Helminck studies the fundamental group of a metrised curve complex in the sense of Amini and Baker [**AB15**] (which are also crucially used in [**ABBR15a, ABBR15b**]). In his framework he proves a result that amounts to identifying the fundamental group of a metrised curve complex with the étale fundamental group of the generic fiber of its smoothing. Theorem B could have been proved using this framework, but we decided to use the moduli-theoretic approach of [**CMR16**] via $\mathfrak{G}$-admissible covers in the sense of [**ACV03**].

Helminck's result provides a new perspective on an older result of Sadi [**Saï97**], which identifies the étale fundamental group of the generic fiber with the profinite completion of the fundamental group of a suitable graph of groups (in the sense of Bass and Serre [**Bas93, Ser80**]) that encodes the fundamental group of a metrised curve complex. From a moduli-theoretic perspective, a similar observation seems to be inherent in both [**BR11**] and [**Eke95**].

From a moduli-theoretic perspective, studying degenerations of $\mathfrak{G}$-covers of algebraic curves is equivalent to studying the compactification of the moduli space of $\mathfrak{G}$-covers in terms of the moduli space of $\mathfrak{G}$-admissible covers, as constructed in [**ACV03**] and [**BR11**]. In [**BR11**, section 7] the authors have already introduced a graph-theoretic gadget to understand the boundary strata of this moduli space: so-called *modular graphs* with an action of a finite (not necessarily abelian) group $\mathfrak{G}$.

This idea seems to have appeared independently in other works as well: Chiodo and Farkas [**CF17**] study the boundary of the moduli space of level curves, which is equivalent to a component of the moduli space of $\mathfrak{G}$-admissible covers for a cyclic group $\mathfrak{G}$, and look at cyclic covers of an arbitrary graph. Their work has been extended to an arbitrary finite group $\mathfrak{G}$ by Galeotti in [**Gal19a, Gal19b**]. Finally, in [**SvZ20**], Schmitt and van Zelm apply a graph-theoretic approach to the boundary of the moduli space of $\mathfrak{G}$-admissible covers (for an arbitrary finite group $\mathfrak{G}$) to study their pushforward classes in the tautological ring of $\overline{\mathcal{M}}_{g,n}$.

In [**CMR16**] Cavalieri, Markwig and Ranganathan develop a moduli-theoretic approach to the tropicalisation of the moduli space of admissible covers (without a fixed group operation). In [**CMP20**], Caporaso, Melo and Pacini study the tropicalisation of the moduli space of spin curves, which, in view of the results in [**JL18**], is closely related to our story in the case $\mathfrak{G} = \mathbb{Z}/2\mathbb{Z}$.

The problem of classifying covers of a graph with an action of a given group (not necessarily abelian) was studied by Corry in [**Cor11, Cor12, Cor15**]. However, Corry considered a different category of graph morphisms, allowing edge contraction but not dilation. To the best of our knowledge, no author has considered the problem of classifying all covers of a given graph with an action of a fixed group.

1. *Harmonic covers of metric graphs and tropical curves*

In this section, we recall a number of standard definitions concerning graphs, tropical curves, harmonic morphisms and group actions on graphs.

1·1. *Finite graphs and harmonic morphisms*

We use a modified version of Serre's definition of a graph [**Ser80**] that allows for legs, which are a type of extremal edge with no end vertex.

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Definition 1·1. A *graph with legs* G, or simply a *graph*, consists of the following data:

(i) a finite set $X(G)$;

(ii) an idempotent *root map* $r: X(G) \to X(G)$;

(iii) an involution $\iota: X(G) \to X(G)$ whose fixed set contains the image of r.

The set $X(G)$ is the union of the *vertices* $V(G)$ and *half-edges* $H(G)$ of the graph G, where $V(G)$ is the image of r and $H(G) = X(G) \setminus V(G)$ is the complement. The involution ι preserves $H(G)$ and partitions it into orbits of sizes 1 and 2; we call these respectively the *legs* and *edges* of G and denote the corresponding sets by $L(G)$ and $E(G)$. The root map assigns one root vertex to each leg and two root vertices to each edge (each vertex is rooted at itself). A *loop* is an edge whose root vertices coincide. An *orientation* on G is a choice of order (h,h') on each edge $e = \{h, h'\}$ of G and defines *source* and *target* maps $s, t: E(G) \to V(G)$ by $s(e) = r(h)$ and $t(e) = r(h')$. We note that a leg does not have a vertex at its free end and is thus distinct from an extremal edge, and that legs do not require orienting.

Graphs with legs naturally appear in tropical moduli problems, where a leg represents the tropicalisation of a marked point. An extremal edge, on the other hand, represents an irreducible component attached to the rest of the curve at a single node.

The *tangent space* $T_v G$ and *valency* $\mathrm{val}(v)$ of a vertex $v \in V(G)$ are defined by

$$T_v G = \left\{h \in H(G) : r(h) = v\right\} \quad \text{and} \quad \mathrm{val}(v) = |T_v G|,$$

so that a leg is counted once for valency, while a loop is counted twice.

A *morphism* of graphs $f: G' \to G$, is a set map $f: X(G') \to X(G)$ that commutes with the root and involution maps and that sends vertices to vertices, edges to edges, and legs to legs. By abuse of notation, we denote by f the corresponding maps on the vertices, half-edges, edges, and legs. We note that our graph morphisms are *finite* and do not allow edges or legs to contract to vertices. Non-finite morphisms are relevant to tropical geometry, but do not occur as quotients by finite group actions; so we do not consider them.

Let G and G' be graphs. A *harmonic morphism* (f, d_f) consists of a graph morphism $f: G' \to G$ and a *degree* assignment $d_f: X(G') \to \mathbb{Z}_{>0}$ such that $d_f(h'_1) = d_f(h'_2)$ for each edge $e' = \{h'_1, h'_2\} \in E(G')$ (a quantity that we denote by $d_f(e')$), and such that

$$d_f(v') = \sum_{h' \in T_{v'} G' \cap f^{-1}(h)} d_f(h') \tag{2}$$

for every $v' \in V(G')$ and every $h \in T_{f(v)}G$. In particular, the quantity appearing on the right-hand side of (2) does not depend on the choice of $h \in T_{f(v)}G$. The degree d_f is also called the *dilation factor* of f. If G is connected, then the *global degree* of f is defined as

$$\deg(f) = \sum_{v' \in f^{-1}(v)} d_f(v') = \sum_{e' \in f^{-1}(e)} d_f(e') = \sum_{l' \in f^{-1}(l)} d_f(l')$$

for any choice of $v \in V(G)$, $e \in E(G)$ or $l \in L(G)$. A harmonic morphism (f, d_f) is called *free* if $d_f(x) = 1$ for all $x \in X(G)$; a free harmonic morphism is a covering space in the topological sense.

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1·2. *Group quotients and harmonic Galois covers*

An *automorphism* of a graph G is a morphism $f : G \to G$ that has an inverse; such a morphism can be made harmonic by setting $d_f = 1$ everywhere. A priori, a non-trivial automorphism may *flip edges*, in other words exchange the two half-edges making up an edge. Such automorphisms do not give rise to a quotient, however, since we do not allow an edge to map to a leg. Hence we exclude them from consideration.

Definition 1·2. Let G be a graph and $\mathfrak{G}$ a finite group. A $\mathfrak{G}$*-action* on G is a homomorphism from $\mathfrak{G}$ to the automorphism group $\mathrm{Aut}(G)$, such that, for every $g \in \mathfrak{G}$ and every $e = \{h, h'\} \in E(G)$, we have $g(h) \neq h'$ (so that either $g(h) = h$ and $g(h') = h'$, or $g(e) \neq e$).

Given a $\mathfrak{G}$-action on a graph G, we can naturally form the quotient graph $G/\mathfrak{G}$ in such a way that the quotient map $f : G \to G/\mathfrak{G}$ is harmonic of degree $|\mathfrak{G}|$.

Definition 1·3. Let G be a graph and let $\mathfrak{G}$ be a finite group. Given a $\mathfrak{G}$-action on G, we define the *quotient graph* $G/\mathfrak{G}$ by setting $X(G/\mathfrak{G}) = X(G)/\mathfrak{G}$. The root and involution maps on G are $\mathfrak{G}$-invariant and descend to $X(G/\mathfrak{G})$. It is clear that $V(G/\mathfrak{G}) = V(G)/\mathfrak{G}$ and $H(G/\mathfrak{G}) = H(G)/\mathfrak{G}$, and by assumption the $\mathfrak{G}$-action does not identify the two half-edges of any edge of G. Therefore $E(G/\mathfrak{G}) = E(G)/\mathfrak{G}$ and $L(G/\mathfrak{G}) = L(G)/\mathfrak{G}$, and the quotient map

$$f : G \longrightarrow G/\mathfrak{G}$$

is a finite morphism. By the orbit-stabiliser theorem, we can promote f to a harmonic morphism of global degree $\deg(f) = |\mathfrak{G}|$ by setting $d_f(x) = |\mathfrak{G}_x|$, where $\mathfrak{G}_x$ is the stabiliser subgroup of $x \in X(G)$.

We now define a harmonic Galois cover of a graph to be any harmonic morphism obtained in this way.

Definition 1·4. Let G be a graph and let $\mathfrak{G}$ be a finite group of order d. A *harmonic* $\mathfrak{G}$*-cover* of G is a harmonic morphism $f : G' \to G$ of degree d together with a $\mathfrak{G}$-action on G' such that following axioms hold:

(i) the harmonic morphism f is $\mathfrak{G}$-invariant, in other words $f(g(x')) = f(x')$ and $d_f(g(x')) = d_f(x')$ for all $x' \in X(G')$ and all $g \in \mathfrak{G}$;

(ii) for all $x \in X(G)$, the group $\mathfrak{G}$ acts transitively on the fiber $f^{-1}(x)$.

Let $f : G' \to G$ be a harmonic $\mathfrak{G}$-cover, and pick a vertex or half-edge $x \in X(G)$. The group $\mathfrak{G}$ acts transitively on the fiber $f^{-1}(x)$, so we can identify the latter with $\mathfrak{G}/\mathfrak{G}_{x'}$, where $\mathfrak{G}_{x'}$ is the stabiliser of some $x' \in f^{-1}(x)$. On the other hand, for any $x', x'' \in f^{-1}(x)$ we have $|\mathfrak{G}_{x'}| = |\mathfrak{G}_{x''}|$ and $d_f(x') = d_f(x'')$. Since the degrees of f on the fiber $f^{-1}(x)$ add up to $\deg f = |\mathfrak{G}|$, it follows that $d_f(x') = |\mathfrak{G}_{x'}|$ for any $x' \in X(G')$. It follows that f is the quotient morphism $G' \to G'/\mathfrak{G}$.

1·3. *Metric graphs*

Let G be a graph and let $\ell : E(G) \to \mathbb{R}_{>0}$ be an assignment of positive real lengths to the edges of G. The pair (G, ℓ), known as a *model* for G, determines a *metric graph* Γ by gluing

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a closed line segment $[0, \ell(e)]$ for each edge $e \in E(G)$ and an open infinite interval $[0, \infty)$ for each leg $l \in L(G)$ in accordance with the structure of G. We equip Γ with the shortest-path metric. We note that the set of legs does not depend on the choice of model and that the metric graph Γ is compact if and only if G has no legs.

A model (G, ℓ) of a metric graph Γ is called *simple* if G has no loops or multi-edges. Given a simple model (G, ℓ) for Γ, we define the *star cover*

$$\mathcal{U}(G) = \{U_v\}_{v \in V(G)} \cup \{U_l\}_{l \in L(G)}$$

of Γ as follows. For each leg $l \in L(G)$, let $U_l \subset \Gamma$ be the interior of the corresponding infinite segment in Γ. For each vertex $v \in V(G)$, let $U_v \subset \Gamma$ be the union of v and the interiors of all legs and edges incident to v. The distinct U_l have empty intersections, and $U_l \cap U_v = U_l$ if l is rooted at v and is empty otherwise. Finally, for distinct vertices v and w, the intersection $U_v \cap U_w$ is either the open edge connecting v and w, if there is such an edge, or is empty otherwise. Hence each element of $\mathcal{U}(G)$ is contractible, pairwise intersections are open intervals or empty, and all triple intersections are empty, making the star cover convenient for cohomological calculations.

We now define harmonic morphisms and Galois covers of metric graphs. Let Γ and Γ' be metric graphs with models G and G', respectively. Let $f : G' \to G$ be a harmonic morphism of graphs satisfying the condition

$$\ell\big(f(e')\big) = d_f(e')\ell(e') \tag{3}$$

for all $e' \in E(G')$. We define an associated continuous map $\phi : \Gamma' \to \Gamma$ of metric graphs by mapping vertices to vertices, edges to edges, and legs to legs according to f. Along each edge and leg of Γ', the map ϕ is linear with positive integer slope, or *dilation factor*, given by the degree d_f (which we also denote d_ϕ). Condition (3) ensures that ϕ is continuous, and no condition is required along the infinite legs.

A *harmonic morphism of metric graphs* $\phi : \Gamma' \to \Gamma$ is any continuous, piecewise-linear map obtained in this manner, with nonzero integer slopes given by the degree function d_f of a harmonic morphism of graphs $f : G' \to G$ (and thus satisfying the balancing condition (2) at each vertex of Γ'). This definition is equivalent to requiring that ϕ pulls back harmonic functions on Γ to harmonic functions on Γ'. We refer to the datum $(G, G', f : G' \to G, d_f)$ as a *model* for ϕ. We say that ϕ is *free* if f is free, or equivalently, if ϕ is a covering isometry.

We similarly define harmonic Galois covers of metric graphs.

Definition 1·5. Let Γ be a metric graph and let $\mathfrak{G}$ be a finite group of order d. A *harmonic $\mathfrak{G}$-cover* of Γ is a harmonic morphism $\phi \colon \Gamma' \to \Gamma$ of degree d together with an operation of $\mathfrak{G}$ on Γ' by invertible isometries such that following properties hold:

(i) the harmonic cover ϕ is $\mathfrak{G}$-invariant, i.e. $\phi(g(p')) = \phi(p')$ for all $p' \in \Gamma'$ and all $g \in \mathfrak{G}$;

(ii) for all $p \in \Gamma$, the group $\mathfrak{G}$ operates transitively on the fiber $\phi^{-1}(p)$.

It is clear that a harmonic $\mathfrak{G}$-cover $\phi : \Gamma' \to \Gamma$ of metric graphs admits a model $f : G' \to G$ that is a harmonic $\mathfrak{G}$-cover of finite graphs (the models G' and G need to be sufficiently fine to avoid edge-flipping). For any $p' \in \Gamma'$, the degree $d_\phi(p')$ is equal to the order of the stabiliser group $\mathfrak{G}_{p'}$.

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1·4. *Weighted graphs, tropical curves and ramification*

Graphs and metric graphs that arise as tropicalisations of algebraic curves come equipped with an additional vertex weight function that records local genera. These weights allow us to capture the auxiliary phenomenon of *ramification* for harmonic morphisms. We recall the definitions.

A *weighted graph* is a pair (G,g), where G is a finite graph and $g: V(G) \to \mathbb{Z}_{\geq 0}$ is a function, where $g(v)$ is called the *genus* of the vertex v. Similarly, a *tropical curve* (Γ, g) is a metric graph Γ together with a function $g: \Gamma \to \mathbb{Z}_{\geq 0}$ with finite support. When choosing a model (G, ℓ) for a tropical curve (Γ, g), we assume that each point $x \in \Gamma$ with $g(x) > 0$ corresponds to a vertex, and not to an interior point of an edge or a leg, so that (G, g) is a weighted graph. A *harmonic morphism* of tropical curves is a harmonic map of the underlying metric graphs.

Let (G, g) be a weighted graph. We define the *Euler characteristic* $\chi(v)$ of a vertex $v \in V(G)$ as

$$\chi(v) = 2 - 2g(v) - \mathrm{val}(v).$$

Now let $f: G' \to G$ be a harmonic morphism of weighted graphs (G', g') and (G, g). We define the *ramification degree* of f at a vertex $v \in V(G')$ to be the quantity

$$\mathrm{Ram}_f(v') = d_f(v')\chi(f(v')) - \chi(v'). \tag{4}$$

We say that f is *unramified* if it satisfies the *local Riemann–Hurwitz condition* $\mathrm{Ram}_f(v') = 0$ for all $v' \in V(G')$, where we note that, in contrast to the algebraic setting, it is possible for the ramification degree at a vertex to be negative. A harmonic morphism $\phi: \Gamma' \to \Gamma$ of tropical curves is *unramified* if it has an unramified model. Our definition of ramification was introduced in [**UZ19**], and is equivalent to the standard definition found in [**ABBR15a**] or [**CMR16**].

2. *Dilated cohomology and finite harmonic abelian covers*

In this section, we give a cohomological classification of harmonic covers of a given metric graph with abelian structure group. For the remainder of this section, we fix a finite abelian group $\mathfrak{A}$.

Let $\phi: \Gamma' \to \Gamma$ be a harmonic $\mathfrak{A}$-cover. For any point $p \in \Gamma$, the stabiliser subgroups of any two points in the fiber $\phi^{-1}(p)$ are conjugate and hence equal. Therefore this group depends only on p, and we denote it by $D(p) \subseteq \mathfrak{A}$ and call it the *dilation group* of p. Similarly, choosing a finite graph model $f: G' \to G$ of ϕ, we denote by $D(x) \subseteq \mathfrak{A}$ the stabiliser of any element of $f^{-1}(x)$. The groups $D(x)$ fulfil the semicontinuity property $D(h) \subseteq D(v)$ for any half-edge $h \in H(G)$ rooted at a vertex $v \in V(G)$. Furthermore, for any edge $e = \{h, h'\} \in E(G)$ we have $D(h) = D(h')$, and we denote this group by $D(e)$.

This motivates the following definition.

Definition 2·1. An $\mathfrak{A}$*-dilation datum* D on a finite graph G is a choice of a subgroup $D(v) \subseteq \mathfrak{A}$ for every $v \in V(G)$ and $D(h) \subseteq \mathfrak{A}$ for every $h \in H(G)$, such that $D(h) \subset D(v)$ if h is rooted at v and such that $D(h) = D(h') = D(e)$ for any edge $e = \{h, h'\} \in E(G)$. We note that if $e \in E(G)$ is an edge with root vertices $u, v \in V(G)$, then $D(e) \subseteq D(u) \cap D(v)$. An $\mathfrak{A}$*-dilation datum* D on a metric graph Γ is an $\mathfrak{A}$-dilation datum on some model G of Γ, which defines a subgroup $D(p) \in \mathfrak{A}$ for each $p \in \Gamma$.

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An $\mathfrak{A}$-dilation datum on a metric graph Γ together with a choice of simple model naturally gives rise to a dual sheaf of abelian groups.

Definition 2·2. Let D be an $\mathfrak{A}$-dilation datum on a simple model G of a metric graph Γ. We define the *codilation sheaf* $\mathfrak{A}_D$ on Γ as follows. For a vertex $v \in V(G)$, we denote $C(v) = D(v)$. Similarly, for a leg $l \in L(G)$ we denote $C(l) = D(v)$, where $v = r(l)$. Finally, for an edge $e \in E(G)$ with root vertices v and w, we denote $C(e) = D(v) + D(w) \subseteq \mathfrak{A}$. We note that $D(e) \subseteq C(e)$ for any edge $e \in E(G)$ and $D(l) \subseteq C(l)$ for any leg $l \in L(G)$. Now let $\mathcal{U}(G) = \{U_v, U_l\}$ be the star cover of Γ associated to G. The sections of $\mathfrak{A}_D$ over the open cover and the induced intersections are

$$\mathfrak{A}_D(U_v) = \mathfrak{A}/C(v), \quad \mathfrak{A}_D(U_l) = \mathfrak{A}/C(l) \quad \text{and} \quad \mathfrak{A}_D(U_e) = \mathfrak{A}/C(e),$$

where $U_e = U_v \cap U_w$ if e is the (unique) edge between v and w. The restriction maps are induced by the inclusions $D(v) = C(v) \subseteq C(e)$ and $D(v) = C(v) = C(l)$ for an edge e or a leg l rooted at a vertex v. Given a connected open set $U \subseteq \Gamma$, we set $\mathfrak{A}_D(U) = \mathfrak{A}_D(U_v)$ if $v \in U \subseteq U_v$ for some vertex v, while $\mathfrak{A}_D(U) = \mathfrak{A}_D(U_e)$ and $\mathfrak{A}_D(U) = \mathfrak{A}_D(U_l)$ respectively if $U \subseteq U_e$ or $U \subseteq U_l$. For larger open sets, we define the space of sections via the sheaf axioms.

The *dilated cohomology group* of the pair (Γ, D) is the sheaf cohomology group $H^1(\Gamma, \mathfrak{A}_D)$. We note that the sheaf $\mathfrak{A}_D$ depends on the choice of model (see Example 2·4 below), but the group $H^1(\Gamma, \mathfrak{A}_D)$ does not.

We now show that harmonic $\mathfrak{A}$-covers of Γ are in natural bijection with $\mathfrak{A}_D$-torsors. We first recall the definition of torsors over a sheaf of abelian groups, and their description in terms of Čech cocycles. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. We may view $\mathcal{F}$ as a sheaf of $\mathcal{F}$-sets, with each group acting on itself by translation. An $\mathcal{F}$-*torsor* $\mathcal{T}$ on X is a locally trivial sheaf of $\mathcal{F}$-sets, in other words a sheaf of $\mathcal{F}$-sets such that X admits a cover by open sets U with the property that $\mathcal{T}|_U$ and $\mathcal{F}|_U$ are isomorphic as sheaves of $\mathcal{F}$-sets.

It is wellknown that the set of isomorphism classes of $\mathcal{F}$-torsors on X is the sheaf cohomology group $H^1(X, \mathcal{F})$. We explicitly calculate this group for a codilation sheaf $\mathfrak{A}_D$ on a metric graph Γ as a Čech cohomology group. Choose an oriented simple model G for Γ, then the star cover $\mathcal{U}(G) = \{U_v, U_l\}$ is acyclic for $\mathfrak{A}_D$. Let $\mathcal{T}$ be an $\mathfrak{A}_D$-torsor, then we can find trivialisations $g_v : \mathcal{T}|_{U_v} \to \mathfrak{A}_D|_{U_v}$. Each edge $e \in E(G)$ corresponds to a nonempty intersection $U_e = U_{s(e)} \cap U_{t(e)}$, and the composed isomorphism $g^e = g_{t(e)}|_{U_e} \circ (g_{s(e)}|_{U_e})^{-1} : \mathfrak{A}|_{U_e} \to \mathfrak{A}|_{U_e}$ is given by translation by an element of $\mathfrak{A}(U_e) = A/C(e)$, which we also denote by g^e. Hence the $\mathfrak{A}_D$-torsor $\mathcal{T}$ determines a tuple $(g^e)_{e \in E(G)}$, where $g^e \in A/C(e)$. Choosing different trivialisations for $\mathcal{T}$ over the sets U_v determines a different tuple $(\tilde{g}^e)$, and composing the trivialisations produces elements $g^v \in \mathfrak{A}_D(U_v) = A/C(v)$ for $v \in V(G)$ such that $\tilde{g}^e - g^e = g^{t(e)} - g^{s(e)}$ in the common quotient group $\mathfrak{A}/C(e)$. All triple intersections are empty, so the cocycle condition is trivially verified and the tuple (g^e) determines an element of $\check{H}^1\big(\mathcal{U}(G), \mathfrak{A}_D\big) \cong H^1(\Gamma, \mathfrak{A}_D)$, and we can reverse the construction to obtain $\mathcal{T}$ from (g^e).

We now state our main result, which shows that harmonic $\mathfrak{A}$-covers with fixed $\mathfrak{A}$-dilation datum D are classified by the dilated cohomology group $H^1(\Gamma, \mathfrak{A}_D)$.

THEOREM 2·3. *Let Γ be a metric graph and let D be an $\mathfrak{A}$-dilation datum on Γ. There is a natural one-to-one correspondence between $\mathfrak{A}_D$-torsors on Γ and harmonic $\mathfrak{A}$-covers of Γ with associated $\mathfrak{A}$-dilation datum D.*

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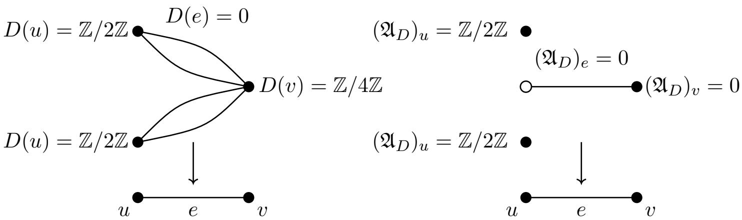
Proof. Choose an oriented simple model G for Γ such that D is defined over G. Let $\phi\colon \Gamma' \to \Gamma$ be a harmonic $\mathfrak{A}$-cover with $\mathfrak{A}$-dilation datum D and let $f: G' \to G$ be a model for ϕ. For any vertex $v \in V(G)$, the fiber $f^{-1}(v)$ is naturally a torsor over $\mathfrak{A}_D(U_v) = \mathfrak{A}/D(v)$. The fiber $f^{-1}(e)$ over an edge e, however, is a torsor over $\mathfrak{A}/D(e)$, not over $\mathfrak{A}_D(U_e) = \mathfrak{A}/C(e)$. The latter group is a quotient of the former, and we replace $f^{-1}(e)$ by its quotient by $C(e)/D(e)$. Similarly, for each leg $l \in L(G)$ we take the quotient of $f^{-1}(l)$ by $C(l)/D(l)$. In this way, we obtain an $\mathfrak{A}_D$-torsor on Γ. We observe that, generally speaking, the espace étalé of this torsor is not Hausdorff, since if $D(v) \subsetneq C(e)$ then the vertex v has more preimages than the adjacent edge e.

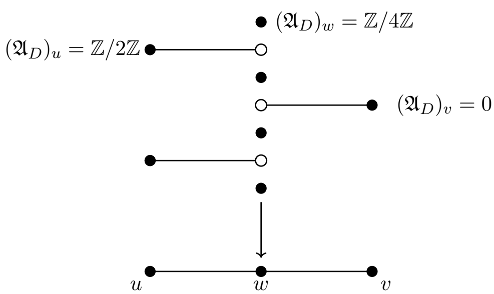
Conversely, let $\mathcal{T}$ be an $\mathfrak{A}_D$-torsor over Γ. We construct a harmonic $\mathfrak{A}$-cover $f: G' \to G$ by resolving the espace étalé of $\mathcal{T}$ in a canonical way. Let $(g^e) \in \check{H}^1(\mathcal{U}(G), \mathfrak{A}_D)$ be a Čech cocycle representing $\mathcal{T}$. We arbitrarily lift each $g^e \in A/C(e)$ to an element $\widetilde{g}^e \in \mathfrak{A}/D(e)$. For each vertex $v \in V(G)$, the fiber $f^{-1}(v)$ is equal to $\mathfrak{A}/D(v)$ as an $\mathfrak{A}$-set. For each edge $e \in V(G)$ with source and target vertices $v = s(e)$ and $w = t(e)$, the fiber $f^{-1}(e)$ is $\mathfrak{A}/D(e)$. The gluing map $f^{-1}(e) \to f^{-1}(v)$ is the natural quotient map $\mathfrak{A}/D(e) \to \mathfrak{A}/D(v)$, while the gluing map $f^{-1}(e) \to f^{-1}(w)$ is translation by $\widetilde{g}^e$ followed by taking the quotient. Finally, for each leg $l \in L(G)$ with root vertex v, we set $f^{-1}(l) = A/D(l)$, and the root map $f^{-1}(l) \to f^{-1}(v)$ is the quotient map $\mathfrak{A}/D(l) \to \mathfrak{A}/D(v)$.

One may now verify that these constructions are inverses of each other, thereby completing the proof.

Example 2·4. In the following picture, on the left, we illustrate a harmonic $\mathbb{Z}/4\mathbb{Z}$-cover, for which the $\mathbb{Z}/4\mathbb{Z}$-dilation datum is given by $D(u) = \mathbb{Z}/2\mathbb{Z}$, $D(v) = \mathbb{Z}/4\mathbb{Z}$, and $D(e) = 0$. In this case we have $C(e) = \mathbb{Z}/4\mathbb{Z}$ and thus $(\mathfrak{A}_D)_u = \mathbb{Z}/2\mathbb{Z}$, $(\mathfrak{A}_D)_v = 0$, and $(\mathfrak{A}_D)_e = 0$. The (non-Hausdorff) espace étalé of the associated $\mathcal{A}_D$-torsor is illustrated on the right.



Consider now a subdivision of the base with extra vertex w. Then the espace étalé of the associated codilation sheaf is given as follows:



We point out that the dilated cohomology group $H^1(\Gamma, \mathfrak{A}_D)$ only depends on the dilation factors at vertices and not on the dilation factors along the edges. The interpretation of a class in $H^1(\Gamma, \mathfrak{A}_D)$ in Proposition 2·3, however, does depend on the dilation along edges. That is,

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different choices of dilation factors would lead to different edge lengths in the corresponding harmonic covers.

We now determine when a harmonic $\mathfrak{A}$-cover $\phi : \Gamma' \to \Gamma$ of tropical curves is unramified. Let $f : G' \to G$ be a model of ϕ, where G' and G are weighted graphs, and let $v' \in V(G')$ be a vertex lying over $v = f(v')$. The number of half-edges $h' \in T_{v'}(G')$ that are rooted at v' and that lie over a given half-edge $h \in T_vG$ is equal to the order of the corresponding quotient $\left|D(v)\big/D(h)\right|$. A short calculation then shows that $\mathrm{Ram}_f(v') = 0$ if and only if

$$g(v') = 1 + \left|D(v)\right|\left(g(v) - 1\right) + \frac{\left|D(v)\right|}{2} \sum_{h \in T_vG} \left[1 - \frac{1}{|D(h)|}\right]. \tag{5}$$

Since $g(v)$ and $g(v')$ are non-negative integers, this condition imposes certain restrictions on the $\mathfrak{A}$-dilation datum of an unramified harmonic $\mathfrak{A}$-cover. As an example, we consider the simplest case of a cyclic cover of prime order.

Example 2·5. Let $\phi : \Gamma' \to \Gamma$ be an unramified harmonic $\mathfrak{A}$-cover of tropical curves with Galois group $\mathfrak{A} = \mathbb{Z}/p\mathbb{Z}$, where $p \geq 2$ is prime, and let $f : G' \to G$ be a model. For any element $x \in X(G)$ we have either $D(x) = \mathbb{Z}/p\mathbb{Z}$ or $D(x) = 1$, and we say that x is *dilated* or *undilated*, respectively. The set of dilated vertices and half-edges forms the *dilation subgraph* $G_{\mathrm{dil}} \subseteq G$.

Now let $v' \in V(G')$ be a vertex mapping to $v = f(v')$. If v is undilated, equation (5) simply reads $g(v') = g(v)$. For a dilated vertex $v \in V(G_{\mathrm{dil}})$, let $d(v) = \left|\{h \in T_vG \mid D(h) = \mathbb{Z}/p\mathbb{Z}\}\right|$ be the valency of v in G_{dil}. Equation (5) then imposes the following conditions on $g(v)$ and $d(v)$:

(i) if $p = 2$, then $d(v) \geq 2$ or $g(v) \geq 1$, and in addition $d(v)$ is even;

(ii) if $p \geq 3$, then $d(v) \geq 2$ or $g(v) \geq 1$.

In other words, the dilation subgraph G_{dil} is *semistable*, and additionally if $p = 2$ then each vertex of G_{dil} has even valency (see [**JL18**, lemma 5·4]).

3. *Moduli of admissible $\mathfrak{G}$-covers and their tropicalisation*

Let $\mathfrak{G}$ be a fixed finite group, which, in this section, does not need to be abelian. In the following, we explain how harmonic $\mathfrak{G}$-covers of weighted graphs and tropical curves naturally arise as tropicalisations of algebraic $\mathfrak{G}$-covers from a moduli-theoretic perspective, expanding on [**ACP15**] and [**CMR16**] (recall that unramified harmonic morphisms of tropical curves are called *tropical admissible covers* in [**CMR16**]). We always work over $\mathrm{Spec}\mathbb{Z}\left[1/|\mathfrak{G}|\right]$ to avoid the wild world of non-tame covers.

3·1. *Compactifying the moduli space of $\mathfrak{G}$-covers*

Let $X \to S$ be a family of smooth projective curves of genus $g \geq 2$ with n marked disjoint sections $s_1, \ldots, s_n \in X(S)$. A $\mathfrak{G}$-*cover* of X is a finite morphism $X' \to X$ together with an operation of $\mathfrak{G}$ on X' over X that is a principal $\mathfrak{G}$-bundle on the complement of the sections, as well as a marking $s'_{ij} \in X'(S)$ of the disjoint preimages of the s_i, indexed by $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$. Denote by $\mathcal{H}_{g,\mathfrak{G}}$ the moduli space of connected $\mathfrak{G}$-covers of smooth curves of genus g (see e.g. [**RW06**] for a construction). There is a good notion of a limit object as X degenerates to a stable curve, as introduced in [**ACV03**].

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Definition 3·1. Let $\mathfrak{G}$ be a finite group and let $X \to S$ be a family of stable curves of genus $g \geq 0$ with n marked disjoint sections $s_1, \ldots, s_n$. Let $\mu = (r_1, \ldots, r_n)$ be an n-tuple of natural numbers that divide $|\mathfrak{G}|$, and denote $k_i = |\mathfrak{G}|/r_i$ for $i = 1, \ldots, n$. An *admissible* $\mathfrak{G}$*-cover* of X consists of a finite morphism $X' \to X$ from a family of stable curves $X' \to S$, an action of $\mathfrak{G}$ on X', and disjoint sections s'_{ij} of X' over S for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$, subject to the following conditions:

(i) the morphism $X' \to X$ is a principal $\mathfrak{G}$-bundle away from the nodes and sections of X;

(ii) the preimage of the set of nodes in X is precisely the set of nodes of X';

(iii) the preimage of a section s_i is precisely given by the sections $s'_{i1}, \ldots, s'_{ik_i}$;

(iv) let p be a node in X and p' a node of X' above p. Then p' is étale-locally given by $x'y' = t$ for a suitable $t \in \mathcal{O}_S$ and p is étale-locally given by $xy = t^r$ for some integer $r \geq 1$ with $(x')^r = x$ and $(y')^r = y$, and the stabiliser of $\mathfrak{G}$ at p' is cyclic of order r and operates via

$$(x', y') \longmapsto (\zeta x', \zeta^{-1} y')$$

for an rth root of unity $\zeta \in \mu_r$;

(v) étale-locally near the sections s_i and s'_{ij}, the morphism $X' \to X$ is given by $\mathcal{O}_S[t_i] \to \mathcal{O}_S[t'_{ij}]$ with $(t'_{ij})^{r_i} = t_i$ for appropriate choices of t_i and t'_{ij}, and the stabiliser of $\mathfrak{G}$ along s_{ij} is cyclic of order r_i and operates via $t'_{ij} \mapsto \zeta t'_{ij}$, for an r_ith root of unity $\zeta \in \mu_{r_i}$.

We emphasise that the $\mathfrak{G}$-action is part of the data; so, in particular, an isomorphism between two admissible $\mathfrak{G}$-covers has to be a $\mathfrak{G}$-equivariant isomorphism. As explained in [**ACV03**], the moduli space $\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$ of admissible $\mathfrak{G}$-covers of stable n-marked curves of genus g is a smooth and proper Deligne–Mumford stack over $\mathrm{Spec}\mathbb{Z}[1/|\mathfrak{G}|]$ that contains the locus $\mathcal{H}_{g,\mathfrak{G}}(\mu)$ of $\mathfrak{G}$-covers of smooth curves of ramification type μ as an open substack. The complement of $\mathcal{H}_{g,\mathfrak{G}}(\mu)$ is a normal crossing divisor.

Remark 3·2. Although closely related, the moduli space $\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$ is actually not quite the same as the one constructed in [**ACV03**]. The quotient

$$\left[\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)/S_{k_1} \times \cdots \times S_{k_n}\right]$$

which forgets about the order of the marked sections on s'_{ij} of X' over S for $i = 1, \ldots, n$ and $j = 1, \ldots, k_i$, is equivalent to a connected component of the moduli space of twisted stable maps to $\mathbf{B}\mathfrak{G}$ in the sense of [**AV02, ACV03**], indexed by ramification profile and decomposition into connected components. Our variant of this moduli space $\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$, with ordered sections on X', has also appeared in [**SvZ20**] and in [**JKK05**] (the latter permitting admissible covers with possibly disconnected domains).

An object in $\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$ is technically not an admissible $\mathfrak{G}$-cover $X' \to X$ but rather a $\mathfrak{G}$-cover $X' \to \mathcal{X}$ of a twisted stable curve $\mathcal{X}$. A *twisted stable curve* $\mathcal{X} \to S$ is a Deligne–Mumford stack $\mathcal{X}$ with sections $s_1, \ldots, s_n \colon S \to \mathcal{X}$ whose coarse moduli space $X \to S$ is a family of stable curves over S with n marked sections (also denoted by $s_1, \ldots, s_n$) such that:

(1) the smooth locus of $\mathcal{X}$ is representable by a scheme;

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(2) the singularities are étale-locally given by $\left[\{x'y'=t\}/\mu_r\right]$ for $t\in\mathcal{O}_S$, where $\zeta\in\mu_r$ acts by $\zeta\cdot(x',y')=(\zeta x',\zeta^{-1}y')$. In this case the singularity in X' is locally given by $xy=t^r$;

(3) the stack $\mathcal{X}$ is a root stack $\left[\sqrt[r_i]{s_i/X}\right]$ along the section s_i for all $i=1,\ldots,n$;

The two notions are naturally equivalent: given an admissible $\mathfrak{G}$-cover $X'\to X$, the associated twisted $\mathfrak{G}$-cover is given by $X'\to[X'/\mathfrak{G}]$. Conversely, given a twisted $\mathfrak{G}$-cover $X'\to\mathcal{X}$ in the corresponding connected component, the composition $X'\to\mathcal{X}\to X$ with the morphism to the coarse moduli space X is an admissible $\mathfrak{G}$-cover. We refer the interested reader to [**BR11**] for an alternative construction.

3·2. *From algebraic to tropical* $\mathfrak{G}$*-covers*

We now explain how to construct unramified harmonic $\mathfrak{G}$-covers of weighted graphs and tropical curves from algebraic $\mathfrak{G}$-covers.

Definition 3·3. Let $F_0: X_0'\to X_0$ be an admissible $\mathfrak{G}$-cover of stable nodal curves over an algebraically closed field k with n smooth distinct marked points on X_0. The *dual harmonic* $\mathfrak{G}$*-cover* $f: G'\to G$ is defined as follows:

(1) the graph G is the dual graph of X_0, namely the irreducible components of X_0 correspond to the vertices of G, the nodes correspond to the edges, and the sections correspond to the legs. Similarly, G' is the dual graph of X_0';

(ii) the vertex weights $g: V(G'_0)\to\mathbb{Z}_{\geq 0}$ and $g: V(G_0)\to\mathbb{Z}_{\geq 0}$ are the genera of the normalisations of the corresponding irreducible components;

(iii) the legs of G_0 are marked l: $\{1,\ldots n\}\simeq L(G_0)$ according to the full order of the marked points;

(iv) the morphism $F_0: X_0'\to X_0$ sends components to components, which defines the morphism $f: V(G')\to V(G)$ on the vertices;

(v) every node $p_{e'}$ of X_0' has a local equation $x'y'=0$, and maps to a node p_e of X_0 with local equation $xy=0$ via $(x')^r=x$ and $(y')^r=y$. This defines the map on the half-edges, and $r=d_f(e')$ gives the dilation factor;

(vi) let u_{ij}' be a uniformiser at s_{ij}' on X_0'. Locally near s_{ij}', the morphism F_0 is given by $u_{ij}'=u_i^{r_i}$ for a choice of uniformiser u_i at s_i. The dilation factor $d_f(l'_{ij})$ along the leg corresponding to s_{ij}' is equal to r_i.

The operation of $\mathfrak{G}$ on X_0' induces an operation of $\mathfrak{G}$ on G' for which the map $f: G'\to G$ is $\mathfrak{G}$-invariant. By Definition 3·1 (iii) and (iv), the stabiliser of every edge e'_i and of every leg l'_{ij} is a cyclic group of order r_i and r_{ij}, respectively. Since $F_0: X_0'\to X_0$ is a principal $\mathfrak{G}$-bundle away from the nodes, the operation of $\mathfrak{G}$ on the fiber over each point in X_0 is transitive and so $f: G'\to G$ is a harmonic $\mathfrak{G}$-cover. Applying the Riemann–Hurwitz formula to the restriction of F_0 to each irreducible component of X_0', we observe that f is unramified.

Definition 3·4. Let X be a smooth projective curve of genus g over a non-Archimedean field K (whose residue characteristic is zero or coprime to $|\mathfrak{G}|$) with n marked points $s_1,\ldots,s_n$ over K. Let $(F: X'\to X, s_{ij}')$ be a $\mathfrak{G}$-cover of X, where $i=1,\ldots,n$ and $j=1,\ldots,k_i$. By the valuative criterion for properness, applied to the stack $\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$, there is a finite extension L of K such that $X'_L \to X_L$ extends to a family of admissible $\mathfrak{G}$-covers $\mathcal{F}: \mathcal{X}' \to \mathcal{X}$ defined over the valuation ring R of L (with marked sections also denoted by s_i and s'_{ij}). The *dual harmonic* $\mathfrak{G}$*-cover* $\phi: \Gamma_{X'} \to \Gamma_X$ is defined as follows:

(1) the graph models of the tropical curves $\Gamma_{X'}$ and Γ_X are the dual graphs $G_{\mathcal{X}'}$ and $G_{\mathcal{X}}$ of the special fibers $\mathcal{X}'_0$ and $\mathcal{X}_0$, respectively;

(2) the edge length function $\ell\colon E(G_{\mathcal{X}}) \to \mathbb{R}_{>0}$ associates to an edge e the positive real number $r \cdot \mathrm{val}(t)$, where the corresponding node of $\mathcal{X}$ is étale-locally given by an equation $xy = t^r$ for $t \in R$. We similarly define the edge length function $\ell\colon E(G_{\mathcal{X}'}) \to \mathbb{R}_{>0}$;

(3) the restriction $\mathcal{F}_0: \mathcal{X}'_0 \to \mathcal{X}_0$ of $\mathcal{F}$ to the special fibers is an admissible $\mathfrak{G}$-cover over k, and the underlying graph model for ϕ is the dual harmonic $\mathfrak{G}$-cover $f: G_{\mathcal{X}'_0} \to G_{\mathcal{X}_0}$ of $\mathcal{F}_0$.

We note that the models $G_{\mathcal{X}'}$ and $G_{\mathcal{X}}$ depend on the choice of extension $\mathcal{F}$, but the tropical curves $\Gamma_{X'}$ and Γ_X do not.

The map $\phi\colon \Gamma_{X'} \to \Gamma_X$ may also seen to be harmonic by [**ABBR15a**, theorem A] upon identifying $\Gamma_{X'}$ and Γ_X with the non-Archimedean skeletons of $(X')^{\mathrm{an}}$ and X^{an}, respectively. The morphism $\phi: \Gamma_{X'} \to \Gamma_X$ is unramified because f is unramified.

3·3. *A modular perspective on tropicalisation*

Following the recipe in [**CMR16**, section 3·2·3] one may construct a tropical moduli space $\mathcal{H}^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)$ as a generalised cone complex that parametrises isomorphism classes of unramified harmonic $\mathfrak{G}$-covers with dilation type μ along the marked legs.

Let us now work over an algebraically closed non-Archimedean field K, whose residue characteristic is either zero or coprime to $|\mathfrak{G}|$. Denote by $\mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)$ the Berkovich analytic space[1] associated to $\mathcal{H}_{g,\mathfrak{G}}(\mu)$. The process described in Section 3·2 above defines a natural *tropicalisation map*

$$\begin{aligned} \mathrm{trop}_{g,\mathfrak{G}}(\mu)\colon \mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu) &\longrightarrow H^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu) \\ \big[X' \to X, s_i, s'_{ij}\big] &\longmapsto \big[(\Gamma_{X'}, g') \to (\Gamma_X, g)\big] \end{aligned}$$

that associates to an admissible $\mathfrak{G}$-cover $X' \to X$ of smooth curves over a non-Archimedean extension L of K an unramified tropical $\mathfrak{G}$-cover $\Gamma_{X'} \to \Gamma_X$ of the dual tropical curve Γ_X of X.

Since the boundary of $\overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$ has normal crossings, the open immersion $\mathcal{H}_{g,\mathfrak{G}}(\mu) \hookrightarrow \overline{\mathcal{H}}_{g,\mathfrak{G}}(\mu)$ is a toroidal embedding in the sense of [**KKMSD73**]. Therefore, as explained in [**ACP15, Thu07, Uli21**], there is a natural strong deformation retraction $\rho_{g,\mathfrak{G}}: \mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu) \to \mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)$ onto a closed subset of $\mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)$ that carries the structure of a generalised cone complex, the *non-Archimedean skeleton* $\Sigma_{g,\mathfrak{G}}(\mu)$ of $\mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)$. Expanding on [**CMR16**, theorem 1 and 4], we have:

[1] We implicitly work with the underlying topological space of the Berkovich analytic stack $\mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)$, as introduced in [**Uli17**, section 3].

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THEOREM 3·5. *The tropicalisation map* $\mathrm{trop}_{g,\mathfrak{G}}(\mu)\colon \mathcal{H}^{an}_{g,\mathfrak{G}}(\mu) \longrightarrow H^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)$ *factors through the retraction to the non-Archimedean skeleton* $\Sigma_{g,\mathfrak{G}}(\mu)$ *of* $\mathcal{H}^{an}_{g,\mathfrak{G}}(\mu)$*, so that the restriction*

$$\mathrm{trop}_{g,\mathfrak{G}}(\mu)\colon\ \Sigma_{g,\mathfrak{G}}(\mu) \longrightarrow H^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)$$

to the skeleton is a finite strict morphism of generalised cone complexes. Moreover, the diagram

$$\begin{array}{ccccc}
\mathcal{H}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu) & & \xrightarrow{\ \mathrm{src}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)\ } & & \mathcal{M}^{\mathrm{an}}_{g',k} \\
& \searrow^{\mathrm{trop}_{g,\mathfrak{G}}(\mu)} & & & \downarrow{\scriptstyle \mathrm{trop}_{g',k}} \\
\downarrow{\scriptstyle \mathrm{tar}^{\mathrm{an}}_{g,\mathfrak{G}}(\mu)} & & H^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu) & \xrightarrow{\ \mathrm{src}^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)\ } & M^{\mathrm{trop}}_{g',k} \\
& & \downarrow{\scriptstyle \mathrm{tar}^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)} & & \\
\mathcal{M}^{\mathrm{an}}_{g,n} & \xrightarrow{\ \mathrm{trop}_{g,n}\ } & M^{\mathrm{trop}}_{g,n} & &
\end{array} \qquad (6)$$

commutes.

In other words, the restriction of $\mathrm{trop}_{g,\mathfrak{G}}(\mu)$ onto a cone in $\Sigma_{g,\mathfrak{G}}(\mu)$ is an isomorphism onto a cone in $H^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)$ and every cone in $H^{\mathrm{trop}}_{g,\mathfrak{G}}(\mu)$ has at most finitely many preimages in $\Sigma_{g,\mathfrak{G}}(\mu)$. Theorem 3·5, in particular, implies that the tropicalisation map $\mathrm{trop}_{g,\mathfrak{G}}(\mu)$ is well-defined, continuous, and proper.

The proof is almost word for word the same as the one of [**CMR16**, theorems 1 and 4]. We need to observe that the construction in [**CMR16**] is compatible with the $\mathfrak{G}$-operation on both the algebraic and the tropical side. Moreover, using [**Uli21**, section 4·5], one can extend the construction of a non-Archimedean skeleton from [**ACP15, Thu07**] to a possibly non-trivially valued base field K. We leave the details to the avid reader, since the statement of Theorem 3·5 is not strictly used in the remainder of this paper.

4. *Realisability of abelian harmonic covers*

In this section, we return to the abelian case and fix a finite abelian group $\mathfrak{A}$. We show that the $\mathfrak{A}$-dilation datum of a harmonic $\mathfrak{A}$-cover $\phi : \Gamma' \to \Gamma$ that is obtained by tropicalizing an algebraic $\mathfrak{A}$-cover has a simple cohomological description. Conversely, we show that any harmonic $\mathfrak{A}$-cover whose $\mathfrak{A}$-dilation datum admits such a description comes from an algebraic $\mathfrak{A}$-cover. This gives us an elementary necessary condition for realisability (see Corollary 4·5), and other similar conditions can be readily found.

We begin by giving the definition of realisability for weighted graphs and for tropical curves.

Definition 4·1. Let k be an algebraically closed field.

(i) an unramified harmonic $\mathfrak{A}$-cover of weighted graphs $f : G' \to G$ is *realisable* over k if there exists an admissible $\mathfrak{A}$-cover $X'_0 \to X_0$ of stable nodal curves over k whose dual harmonic $\mathfrak{A}$-cover is f.

(ii) an unramified harmonic $\mathfrak{A}$-cover of tropical curves $\phi : \Gamma' \to \Gamma$ is *realisable* over k if there exist a non-Archimedean field K whose residue field is k and a Galois $\mathfrak{A}$-cover $F : X' \to X$ of smooth projective curves over K such that ϕ is the tropicalisation of F.

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4·1. *From Galois covers to extended homology*

Let K be a non-Archimedean field with valuation ring R and residue field k, whose characteristic p is either zero or coprime to $|\mathfrak{A}|$. Let $F\colon X' \to X$ be a finite $\mathfrak{A}$-cover of smooth projective curves over K (where X' may be disconnected), which is ramified precisely at n' marked ramification points $p'_1, \ldots, p'_{n'} \in X'$ over a collection of marked branch points $p_1, \ldots, p_n \in X$. Let $\mathcal{F}\colon \mathcal{X}' \to \mathcal{X}$ be an extension of $X' \to X$ to a family of admissible $\mathfrak{A}$-covers over R (where we may have to replace K by a finite extension, as above). Let $\phi\colon \Gamma_{X'} \to \Gamma_X$ be the induced tropical harmonic $\mathfrak{A}$-cover with model $f : G_{\mathcal{X}'} \to G_{\mathcal{X}}$ (which depends on the choice of $\mathcal{F}$ extending F).

Let $v \in V(G_{\mathcal{X}})$ be a vertex, then the smooth locus X_v^* of the irreducible component X_v is a genus $g(v)$ curve over k with $\mathrm{val}(v)$ punctures. The $\mathfrak{A}$-cover $\mathcal{F}^{-1}(X_v^*) \to X_v^*$ is determined by a monodromy representation $m_v : \pi_1^{\acute{e}t}(X_v^*, x_0) \to \mathfrak{A}$. Since $\mathfrak{A}$ is abelian, the choice of base point is irrelevant, and the representation can be recorded by a tuple of elements of $\mathfrak{A}$ in the following way. Let

$$\Pi_{g(v),\mathrm{val}(v)}$$
$$= \big\langle \alpha_1, \ldots, \alpha_{g(v)}, \beta_1 \ldots, \beta_{g(v)}, \gamma_1, \ldots, \gamma_{\mathrm{val}(v)} \ \big|\ [\alpha_1, \beta_1] \cdots [\alpha_{g(v)}, \beta_{g(v)}]\gamma_1 \cdots \gamma_{\mathrm{val}(v)} = 1 \big\rangle$$

be the fundamental group of a genus $g(v)$ Riemann surface with $\mathrm{val}(v)$ punctures, where the γ_j are small loops around the punctures. By a theorem of Grothendieck (see e.g. [**Sza09**, theorem 4·9·1]), the étale fundamental group $\pi_1^{\acute{e}t}(X_v^*, x_0)$ is the profinite completion of $\Pi_{g(v),\mathrm{val}(v)}$ when $p = 0$ and the prime-to-p profinite completion of $\Pi_{g(v),\mathrm{val}(v)}$ when $p > 0$. Since $|\mathfrak{A}|$ is coprime to p, every continuous homomorphism $\pi_1^{\acute{e}t}(X_v^*, x_0) \to \mathfrak{A}$ (where $\mathfrak{A}$ is equipped with the discrete topology) is uniquely determined by a homomorphism $\varphi : \Pi_{g(v),\mathrm{val}(v)} \to \mathfrak{A}$ that factors as

$$\Pi_{g(v),\mathrm{val}(v)} \longrightarrow \pi_1^{\acute{e}t}(X_v^*, x_0) \longrightarrow \mathfrak{A}\,.$$

Hence the monodromy representation $m_v\colon \pi_1^{\acute{e}t}(X_v^*, x_0) \to \mathfrak{A}$ is uniquely determined by the images

$$\xi(v)_i = \varphi(\alpha_i) \in \mathfrak{A} \quad \text{and} \quad \xi(v)_{g(v)+i} = \varphi(\beta_i) \in \mathfrak{A} \quad \text{for} \quad i = 1, \ldots, g(v),$$

of the α_i and β_i, which may be arbitrary, as well as the images

$$\eta(h) = \varphi(\gamma_j) \in \mathfrak{A} \quad \text{for} \quad j = 1, \ldots, \mathrm{val}(v),$$

where $h \in H(G_{\mathcal{X}})$ is the half-edge corresponding to the jth puncture on X_v. We note that $\eta(h)$ acts by multiplication by a primitive rth root of unity in an étale neighbourhood of $p_{h'}$, where $r = d_\phi(h')$. The unique relation in the group $\Pi_{g(v),\mathrm{val}(v)}$ implies that the elements $\eta(h)$ satisfy

$$\sum_{j=1}^{\mathrm{val}(v)} \varphi(\gamma_j) = \sum_{h \in T_v G_{\mathcal{X}}} \eta(h) = 0.$$

Furthermore, for each pair of nodes $e = \{h, h'\}$ we have $\eta(h) + \eta(h') = 0$.

In other words, to an algebraic $\mathfrak{A}$-cover $X' \to X$ we associate the following data on the graph $G_{\mathcal{X}}$:

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(1) an element $\eta(h)\in\mathfrak{A}$ for each $h\in H(G_{\mathcal{X}})$, so that $\eta(h)+\eta(h')=0$ for any edge $e=\{h,h'\}$ in $E(G_{\mathcal{X}})$ and $\sum_{h\in T_vG_{\mathcal{X}}}\eta(h)=0$ for any vertex $v\in V(G_{\mathcal{X}})$;

(2) an element $\xi(v)\in\mathfrak{A}^{2g(v)}$ for every vertex $v\in V(G_{\mathcal{X}})$.

The collection of all $\eta(h)$ is nothing but a class $\eta\in H_1(G_{\mathcal{X}},\mathfrak{A})$ in the simplicial homology of the graph $G_{\mathcal{X}}$ with coefficients in $\mathfrak{A}$. Similarly, each $\mathfrak{A}^{2g(v)}$ can be thought of as the simplicial homology group (with coefficients in $\mathfrak{A}$) of an infinitesimal genus $g(v)$ graph located at the vertex v. This motivates the following definition.

Definition 4·2. Let (G,g) be a weighted graph. The *extended homology group* of G with coefficients in $\mathfrak{A}$ is the finite abelian group

$$H_1^{\text{ext}}(G,\mathfrak{A})=H_1(G,\mathfrak{A})\oplus\bigoplus_{v\in V(G)}\mathfrak{A}^{2g(v)}.$$

Given a harmonic $\mathfrak{A}$-cover $f:G_{\mathcal{X}'}\to G_{\mathcal{X}}$ of weighted graphs that is the tropicalisation of an algebraic $\mathfrak{A}$-cover $F:X'\to X$, the datum $(\eta,\xi)\in H_1^{\text{ext}}(G_{\mathcal{X}},\mathfrak{A})$ defined above is called the $\mathfrak{A}$-*monodromy datum* associated to the cover.

We note that if $|\mathfrak{A}|$ is coprime to p, then the group of continuous homomorphisms from $\pi_1^{ét}(X,x)$ to $\mathfrak{A}$ is naturally identified with $H^1_{ét}(X,\mathfrak{A})$ (see [**Mil13**, example 11·3]), and hence the discussion above provides us with a natural homomorphism

$$H^1_{ét}(X,\mathfrak{A})\longrightarrow H_1^{\text{ext}}(G_{\mathcal{X}},\mathfrak{A}).$$

Note that we pass from cohomology to homology, as $G_{\mathcal{X}}$ is the dual graph of the special fiber $\mathcal{X}_0$.

4·2. *Extended homology, dilation and realisability*

We now observe that the $\mathfrak{A}$-dilation datum of a harmonic $\mathfrak{A}$-cover $f:G_{\mathcal{X}'}\to G_{\mathcal{X}}$ obtained by tropicalising an algebraic $\mathfrak{A}$-cover can be read off from the $\mathfrak{A}$-monodromy datum (η,ξ). Indeed, let $v'\in V(G_{\mathcal{X}'})$ be a vertex mapping to $v=f(v')$. The restricted $\mathfrak{A}$-cover $f^{-1}(X_v^*)\to X_v^*$ corresponds to the monodromy representation $m_v:\pi_1^{ét}(X_v^*,x_0)\to\mathfrak{A}$, so an element of $\mathfrak{A}$ preserves the irreducible component $X^*_{v'}\subseteq f^{-1}(X_v^*)$ (in other words, fixes v') if and only if it is in the image of m_v, which is generated by the images of the α_i, β_i, and γ_j. Similarly, the stabiliser of a node $p_{h'}\in\mathcal{X}'_0$ mapping to $p_h\in\mathcal{X}_0$ is generated by $\eta(h)\in\mathfrak{A}$. Hence we give the following definition.

Definition 4·3. Let (G,g) be a weighted graph, and let $(\eta,\xi)\in H_1^{\text{ext}}(G,\mathfrak{A})$ be an element of the extended homology group of G with coefficients in $\mathfrak{A}$. The *associated* $\mathfrak{A}$-*dilation datum* on G is defined as follows:

(i) for a half-edge $h\in H(G)$, the dilation group $D(h)$ is the cyclic subgroup of $\mathfrak{A}$ generated by the element $\eta(h)$;

(ii) for a vertex $v\in V(G)$, the dilation group $D(v)$ is the subgroup of $\mathfrak{A}$ generated by $\eta(h)$ for all $h\in T_vG$ and by the entries of $\xi(v)$.

We are now ready to state and prove our realisability criterion.

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THEOREM 4·4. *Let k be an algebraically closed field whose characteristic is either zero or is relatively prime to* $|\mathfrak{A}|$.

(i) *an unramified harmonic* $\mathfrak{A}$*-cover* $f : G' \to G$ *of weighted graphs is realisable over k if and only if the* $\mathfrak{A}$*-dilation datum D of f is associated to some element* $(\eta, \xi) \in H_1^{ext}(G, \mathfrak{A})$ *of the extended homology group of G with coefficients in* $\mathfrak{A}$.

(ii) *an unramified harmonic* $\mathfrak{A}$*-cover* $\phi : \Gamma' \to \Gamma$ *of tropical curves is realisable over k if and only if it admits a realisable model* $f : G' \to G$.

Proof. Let $F_0 : X_0' \to X_0$ be an admissible $\mathfrak{A}$-cover of stable nodal curves over k, and let $f : G' \to G$ be the dual cover. The discussion in Section 4.1 produces an $\mathfrak{A}$-monodromy datum $(\eta, \xi) \in H_1^{\mathrm{ext}}(G, \mathfrak{A})$ that defines the $\mathfrak{A}$-dilation datum of f. Similarly, let $F : X' \to X$ be a $\mathfrak{G}$-cover of smooth projective curves over a non-Archimedean field K with residue field k, then the dual cover $\phi : \Gamma_{X'} \to \Gamma_X$ of tropical curves has a realisable model.

Conversely, suppose that the $\mathfrak{A}$-dilation datum D of an unramified harmonic $\mathfrak{A}$-cover $f : G' \to G$ is associated to an element $(\eta, \xi) \in H_1^{\mathrm{ext}}(G, \mathfrak{A})$. We reverse the procedure and construct an admissible $\mathfrak{A}$-cover $F_0 : X_0' \to X_0$ over k tropicalizing to f, as follows. For each vertex $v \in V(G)$, choose a smooth k-curve X_v^* of genus $g(v)$ with $|T_vG|$ punctures. The monodromy element (η, ξ) induces a monodromy representation $m_v : \pi_1^{\acute{e}t}(X_v^*, x_0) \to \mathfrak{A}$ at each $v \in V(G)$ and hence an $\mathfrak{A}$-cover of each X_v^*. We then glue these covers according to the incidence data of the graphs G' and G to obtain an admissible $\mathfrak{A}$-cover $F_0 : X_0' \to X_0$, where the X_v are the irreducible components of X_0. Hence f is realisable.

If $f : G' \to G$ is the underlying model for a harmonic $\mathfrak{A}$-cover $\phi : \Gamma' \to \Gamma$, then the admissible $\mathfrak{A}$-cover $F_0 : X_0' \to X_0$ can be smoothened to a family $\mathcal{F} : \mathcal{X}' \to \mathcal{X}$ by the smoothness of the moduli space of admissible $\mathfrak{A}$-covers over $\mathrm{Spec}\,\mathbb{Z}[1/|\mathfrak{A}|]$ (see [**ACV03**, theorem 3·0·2]). Alternatively, one may also use the smoothening result for harmonic covers of metrised curve complexes from [**ABBR15a**] and observe that their smoothing is compatible with group operations.

This criterion can be used to establish elementary graph-theoretic restrictions on realisable harmonic $\mathfrak{A}$-covers. First, we note that if the dilation group of any half-edge is not cyclic, then the cover is not realisable. Now let G be a weighted graph with a bridge edge $e \in E(G)$. Any $\mathfrak{A}$-monodromy datum $(\eta, \xi) \in H_1^{\mathrm{ext}}(G, \mathfrak{A})$ vanishes on e, so the dilation group of any realisable $\mathfrak{A}$-cover is trivial along e. Hence we have established the following necessary condition for realisability.

COROLLARY 4·5. *Let* $\phi : \Gamma' \to \Gamma$ *be a harmonic* $\mathfrak{A}$*-cover of metric graphs. If any bridge edge of* Γ *is dilated, then f is not realisable.*

In the next section, we show that, for most simple abelian groups, this condition is in fact sufficient. Similarly, if G has a pair of parallel edges e_1 and e_2, then the dilation groups of any realisable $\mathfrak{A}$-cover are equal on e_1 and e_2, since $\eta(e_2) = \pm\eta(e_1)$ for any $(\eta, \xi) \in H_1^{\mathrm{ext}}(G, \mathfrak{A})$.

5. *Cyclic covers and the nowhere-zero flow problem*

In this section, we discuss unramified harmonic $\mathfrak{A}$-covers of a weighted graph (G,g) without legs in the case where $\mathfrak{A} = \mathbb{Z}/p\mathbb{Z}$ is a cyclic group of prime order. We show that the realisability problem for such covers is closely related to a classical problem in graph theory.

Let G be a graph and let $n \geq 2$. A *nowhere-zero n-flow* on G is an element $\eta \in H_1(G, \mathbb{Z}/n\mathbb{Z})$ such that $\eta(e) \neq 0$ for all $e \in E(G)$. The problem is to determine sufficient conditions for the existence of a nowhere-zero n-flow on G. It is clear that G admits a nowhere-zero n-flow for any $n \geq 2$ only if G has no bridges, and that it admits a nowhere-zero 2-flow if and only if $\mathrm{val}(v)$ is even for all $v \in V(G)$. On the other hand, Seymour's 6-flow Theorem [**Sey81**] states that any bridgeless graph admits a nowhere-zero n-flow for $n = 6$, and it easily follows that this holds for any $n \geq 7$ as well. The intermediate cases $n = 3$, 4, and 5, however, are not currently known. Tutte's conjecture states that every bridgeless graph has a nowhere-zero 5-flow [**Tut54**, conjecture II].

Consider an unramified harmonic $\mathbb{Z}/p\mathbb{Z}$-cover $f : G' \to G$ of weighted graphs, where $p \geq 2$ is a prime number. Let $G_{\mathrm{dil}} \subseteq G$ be the dilation subgraph, consisting of those vertices and half-edges whose dilation group is $\mathbb{Z}/p\mathbb{Z}$. We now use the theory of nowhere-zero n-flows to show that the realisability of f is determined by the structure of G_{dil}.

THEOREM 5·1. *Let p be a prime number, and let $f : G' \to G$ be an unramified harmonic $\mathbb{Z}/p\mathbb{Z}$-cover of weighted graphs with no legs. If $p = 2$, then f is realisable. If $p \geq 7$, then f is realisable if and only if the dilation subgraph $G_{dil} \subseteq G$ has no bridges. If Tutte's conjecture holds, the same is true for $p = 5$.*

Proof. We recall the results of Example 2·5. For $v \in V(G_{\mathrm{dil}})$, denote its valency in G_{dil} by $d(v)$. We showed that the dilation subgraph $G_{\mathrm{dil}} \subseteq G$ is semistable: for each $v \in V(G_{\mathrm{dil}})$, either $d(v) \geq 2$ or $g(v) \geq 1$ (in addition, $d(v)$ is even if $p = 2$). Unwrapping the definitions, Theorem 4·4 implies that the cover $f : G' \to G$ is realisable if and only if there exists an element $(\eta, \xi) \in H_1^{\mathrm{ext}}(G, \mathbb{Z}/p\mathbb{Z})$ satisfying the following conditions:

(i) $\eta(e) \neq 0$ if and only if $e \in E(G_{\mathrm{dil}})$;

(ii) if $v \in V(G)$ is a vertex with no adjacent dilated edges, then $v \in V(G_{\mathrm{dil}})$ if and only if $\xi(v) \neq 0$.

The first condition is always satisfied when $p = 2$: setting $\eta(e) = 1$ if and only if $e \in E(G_{\mathrm{dil}})$ gives a cycle $\eta \in H_1(G, \mathbb{Z}/2\mathbb{Z})$, since $d(v)$ is even for all $v \in V(G_{\mathrm{dil}})$ (see [**JL18**, lemma 5·9]). For $p \geq 7$, Seymour's theorem implies that we can find such an η if and only if G_{dil} has no bridges, and Tutte's conjecture implies the same for $p = 5$. The second condition, on the other hand, is trivially satisfied: if $v \in V(G_{\mathrm{dil}})$ is a dilated vertex with $d(v) = 0$, then $g(v) \geq 1$ and hence we can pick $\xi(v)$ to be any nontrivial element of $(\mathbb{Z}/p\mathbb{Z})^{2g(v)}$, and similarly we can set $\xi(v) = 0$ for all $v \in V(G_{\mathrm{dil}}) \backslash V(G)$. This completes the proof.

We note that our results allow us to restate Tutte's conjecture in a purely algebraic form. Let G be a bridgeless graph, and let X_0 be a nodal curve whose dual graph is G (over any algebraically closed field of characteristic not equal to 5). Suppose that X_0 admits an admissible $\mathbb{Z}/5\mathbb{Z}$-cover $X_0' \to X_0$ that is ramified at each node of X_0. The dual harmonic $\mathbb{Z}/5\mathbb{Z}$-cover $f : G' \to G$ has $G_{\mathrm{dil}} = G$, hence the $\mathbb{Z}/5\mathbb{Z}$-monodromy datum (η, ξ) satisfies $\eta(e) \neq 0$ for all $e \in E(G)$. Tutte's conjecture is now equivalent to the following:

CONJECTURE 5·2. *Let k be an algebraically closed field with $\mathrm{char}\, k \neq 5$. Every nodal curve over k with no separating nodes has an admissible $\mathbb{Z}/5\mathbb{Z}$-cover that is ramified at each node.*

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