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Pre-primitive permutation groups

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ABSTRACT

A transitive permutation group G on a finite set Ω is said to be *pre-primitive* if every G-invariant partition of Ω is the orbit partition of a subgroup of G. It follows that pre-primitivity and quasiprimitivity are logically independent (there are groups satisfying one but not the other) and their conjunction is equivalent to primitivity. Indeed, part of the motivation for studying pre-primitivity is to investigate the gap between primitivity and quasiprimitivity. We investigate the pre-primitivity of various classes of transitive groups including groups with regular normal subgroups, direct and wreath products, and diagonal groups. In the course of this investigation, we describe all G-invariant partitions for various classes of permutation groups G. We also look briefly at conditions similarly related to other pairs of conditions, including transitivity and quasiprimitivity, k-homogeneity and k-transitivity, and primitivity and synchronization.

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1. Introduction

In his pioneering work on permutation groups in his Second Memoir [5], Galois introduced the notion of primitivity, which has occupied the attention of mathematicians ever since. However, Neumann [8] has pointed out that Galois confused two inequivalent conditions for the transitive permutation group G on Ω:

- G preserves no non-trivial partition of Ω (the trivial partitions being the partition into singletons and the partition with a single part);
- every non-trivial normal subgroup of G is transitive.

The first of these conditions is what is now called *primitivity*, while the second is *quasiprimitivity*. Since the orbit partition of a normal subgroup is G-invariant, we see that a primitive group is quasiprimitive; but the converse is false, as we may see by considering the regular representation of a non-abelian simple group.

In order to investigate the gap between these two properties, we make the following definition: The transitive permutation group G on Ω is *pre-primitive* if every G-invariant partition is the orbit partition of a subgroup of G. We can assume that this subgroup is normal:

Proposition 1.1. *If a G-invariant partition is the orbit partition of a subgroup of G, then it is the orbit partition of a normal subgroup.*

Proof. The set of permutations fixing all parts of a G-invariant partition is a normal subgroup of G. □

Theorem 1.2.

(a) *There are permutation groups which are quasiprimitive but not pre-primitive, and permutation groups which are pre-primitive but not quasiprimitive.*
(b) *A permutation group is primitive if and only if it is quasiprimitive and pre-primitive.*

Proof. We establish the second statement first. We have noted that a primitive group is quasiprimitive; it is also pre-primitive, since both trivial partitions are orbit partitions of subgroups (the trivial group and the whole group respectively). Conversely, suppose that G is pre-primitive and quasiprimitive, and let Π be a G-invariant partition. Then Π is the orbit partition of a subgroup H of G. As noted after the definition, we may assume that H is normal in G; now quasiprimitivity shows that H is trivial or transitive, so Π is trivial.

We have seen examples of quasiprimitive groups which are not primitive, and hence not pre-primitive. For the other case, let G be an abelian group which is not of prime order, acting regularly. Then the G-invariant partitions are the coset partitions of subgroups of G, and so G is pre-primitive but not quasiprimitive. □

We give here another general property of pre-primitivity, which it shares with many permutation group properties.

Theorem 1.3. *Pre-primitivity is upward-closed; that is, if G_1 and G_2 are transitive permutation groups on Ω with G_1 pre-primitive and $G_1 \leq G_2$, then G_2 is pre-primitive.*

Proof. With these hypotheses, let Π be a G_2-invariant partition. Then clearly Π is G_1-invariant, so it is the orbit partition of a subgroup H of G_1; and we have $H \leq G_2$. □

We also give a group-theoretical characterisation of pre-primitivity.

Theorem 1.4. *Let G be transitive on Ω, and take $\alpha \in \Omega$. Then G is pre-primitive if and only if every subgroup H containing G_α has the form $H = NG_\alpha$ for some normal subgroup N of G.*

Proof. We observe that every subgroup H containing G_α is the stabiliser of the part containing α of some G-invariant partition Π. If G is pre-primitive, then there is a normal subgroup N of G whose orbits are the parts of Π; then the NG_α-orbit of α is the part of Π containing α, and so $NG_\alpha = H$. Conversely, if $NG_\alpha = H$ for some normal subgroup N of G, then the N-orbit of α is equal to the H-orbit, and so is a part of Π; since N is normal, every N-orbit is a part of Π. Thus G is pre-primitive. □

In the remainder of the paper, we consider various classes of transitive groups, including groups with regular normal subgroups, direct and wreath products of transitive groups, and diagonal groups [2]. We attempt to determine when these groups are pre-primitive; in some cases we succeed, in others we obtain necessary and sufficient conditions which are quite close together. We report the result of computations on the numbers of transitive groups of small degree which are pre-primitive, quasiprimitive and primitive respectively, showing that the first two conditions are approximately statistically independent. In several cases we determine all the G-invariant partitions for certain types of permutation group.

In the last section we consider similar conditions relating to other pairs of permutation group properties, the second being stronger than the first:

- transitivity and quasiprimitivity;
- k-homogeneity and k-transitivity;
- primitivity and synchronization [1,9].

These all turn out to be of less interest. The first case is trivial. In the second, the property we are looking for is Neumann's notion of *generous* $(k-1)$*-transitivity* [7], which has been much studied, especially for $k = 2$. In the third case, we define an appropriate property which we call *pre-synchronization*, but we prove that the only transitive group which is pre-synchronizing but not synchronizing is the Klein group of order 4.

2. Pre-primitive groups of specific types

In this major section we discuss some familiar types of transitive permutation groups with a view to deciding when they are pre-primitive.

2.1. Groups with regular normal subgroups

To help fix the ideas, we first discuss groups acting regularly. If G acts regularly on Ω, then the set Ω is bijective with G and the given action is isomorphic to the action by right multiplication. If the point $\alpha \in \Omega$ corresponds to the identity of G, then:

(a) a partition of G is G-invariant if and only if it is the right coset partition of a subgroup of G;
(b) a partition of G is the orbit partition of a subgroup H of G if and only if it is the left coset partition of H.

For (a), suppose that H is the part of the partition containing the identity. Then for $h_1, h_2 \in H$, multiplication by $h_1^{-1}h_2$ maps h_1 to h_2, so fixes H; thus $1.(h_1^{-1}h_2) \in H$. So H is a subgroup of G. Then, for any $x \in G$, Hx is a part of the partition. So the claim is proved.

For (b), let H be a subgroup of G. Then the orbit of H containing g is the left coset gH. So the claim holds.

It follows that the regular group G is pre-primitive if and only if, for every subgroup H, the left and right coset partitions of H coincide, that is, H is a normal subgroup. We will state this formally as a corollary of the main result of this section.

Theorem 2.1. *Let G be a permutation group on Ω with a regular normal subgroup N. Then G is pre-primitive if and only if every G_α-invariant subgroup H of N is normal in N.*

Proof. Since N is a regular normal subgroup of G, we have $G = NG_\alpha$, and we can identify Ω with N in such a way that N acts by right multiplication and G_α acts by conjugation. Moreover, we can assume that α is the identity element of N (see [11, Theorem 11.2]).

Suppose that G is pre-primitive. Let H be a G_α-invariant subgroup of N. The right coset partition Π of H is N-invariant. It is also G_α-invariant: for if Hn is a right coset of H and $g \in G_\alpha$, then $(Hn)^g = H^g n^g = Hn'$ for some $n' \in N$. Since $NG_\alpha = G$, Π is G-invariant. Since G is pre-primitive, Π is the orbit partition of a normal subgroup K of

G. Now $K \leq G_\alpha(K \cap H)$, so $\alpha H = \alpha K = \alpha(K \cap N)$. Since N s regular, it follows that $K \cap N = H$.

Conversely, assume that every G_α-invariant subgroup of N is normal in N. Choose a G-invariant partition Π. Since Π is N-invariant it is the right coset partition of a subgroup K of N, which is also G_α-invariant, since G_α fixes the part of Π containing the identity. Thus K is normal in N. The normal subgroup of G fixing every part of Π contains K, and Π is the orbit partition of $N \cap K$. □

This result allows us to deal with some special types of permutation groups. First we state and prove our earlier result about regular groups. Recall that a *Dedekind group* is a finite group in which every subgroup is normal. Dedekind [4] showed:

Theorem 2.2. *A finite group G is a Dedekind group if and only if either G is abelian, or $G \cong Q \times A \times B$, where Q is the quaternion group of order 8, A is an elementary abelian 2-group, and B is an abelian group of odd order.*

Corollary 2.3. *The regular action of a finite group G is pre-primitive if and only if G is a Dedekind group.*

Proof. In this case, Theorem 2.1 applies with $N = G$ and $G_\alpha = 1$. Thus every subgroup of N is G_α-invariant; so G is pre-primitive if and only if every subgroup of N is normal. □

The *holomorph* of a group G is the semidirect product of G with $\mathrm{Aut}(G)$. It acts as a permutation group on G, where G acts by right multiplication and $\mathrm{Aut}(G)$ in the natural way.

Corollary 2.4. *For any finite group G, the holomorph of G is pre-primitive.*

Proof. The group G is a regular normal subgroup of its holomorph. A subgroup H of G is $\mathrm{Aut}(G)$-invariant if and only if it is characteristic; and a characteristic subgroup is certainly normal. □

The proof actually shows a stronger result: the semidirect product of G by its inner automorphism group $\mathrm{Inn}(G)$ is pre-primitive.

Finally, since a transitive abelian group is regular, and a direct product of regular groups (in its product action) is regular, we have the following:

Corollary 2.5. *The direct product of transitive abelian groups in its product action is pre-primitive.*

2.2. Direct products

Next we consider various product constructions for transitive groups and ask, is it true that if the factors are pre-primitive, then so is the product? First, the direct product in its product action.

Let G and H be permutation groups on Γ and Δ respectively. Then the direct product $G \times H$ acts coordinatewise on $\Gamma \times \Delta$, by $(\gamma, \delta)(g, h) = (\gamma g, \delta h)$. If G and H are transitive then $G \times H$ is transitive in this action.

It follows from Corollary 2.3 that, if two transitive groups are pre-primitive, then their direct product (in its product action) is pre-primitive if the factors are abelian, but may fail to be pre-primitive in general. (Take two copies of Q_8 acting regularly: Q_8 is a Dedekind group but $Q_8 \times Q_8$ is not.) So it is natural to ask what additional conditions on the factors will guarantee pre-primitivity of the product. To examine these, we look more closely at partitions invariant under $G \times H$.

Suppose that G and H act transitively on Γ and Δ respectively, and let Π be a $(G \times H)$-invariant partition of $\Gamma \times \Delta$. We define two partitions of Γ as follows.

- Let P be a part of Π. Let P_0 be the subset of Γ defined by

$$P_0 = \{\gamma \in \Gamma : (\exists \delta \in \Delta)(\gamma, \delta) \in P\}.$$

  We claim that the sets P_0 arising in this way are pairwise disjoint. For suppose that $\gamma \in P_0 \cap Q_0$, where Q_0 is defined similarly from another part Q of Π; suppose that $(\gamma, \delta_1) \in P$ and $(\gamma, \delta_2) \in Q$. There is an element $h \in H$ mapping δ_1 to δ_2. Then $(1, h)$ maps (γ, δ_1) to (γ, δ_2), and hence maps P to Q, and P_0 to Q_0; but this element acts trivially on Γ, so $P_0 = Q_0$. It follows that the sets P_0 arising in this way form a partition of Γ, which we call the *G-projection partition*.
- Choose a fixed $\delta \in \Delta$, and consider the intersections of the parts of Π with $\Gamma \times \{\delta\}$. These form a partition of $\Gamma \times \{\delta\}$ and so, by ignoring the second factor, we obtain a partition of Γ called the *G-fibre partition*. Now the action of the group $\{1\} \times H$ shows that it is independent of the element $\delta \in \Delta$ chosen.

We note that the G-projection partition and the G-fibre partition are both G-invariant, and the second is a refinement of the first. In a similar way we get H-fibre and H-projection partitions of Δ, both H-invariant.

For a non-trivial example, consider the group $C_8 \times C_8$ acting on $\Gamma \times \Delta$, where each of Γ and Δ is a copy of the integers modulo 8. Take

$$P = \{(0, 0), (0, 4), (4, 0), (4, 4), (2, 2), (2, 6), (6, 2), (6, 6)\}.$$

The images of P under $C_8 \times C_8$ form a partition with eight parts; its projection partition on the first coordinate has two parts consisting of the even and odd elements, while its fibre partition has four parts consisting of the cosets of $\{0, 4\}$.

The next lemma gives some properties of these partitions. First, some definitions.

The partial order of *refinement* is defined on partitions of Ω by the rule that, for partitions Π and Σ, we have $\Pi \preceq \Sigma$ (read Π refines Σ) if every part of Σ is a union of parts of Π.

If Π and Σ are partitions of Γ and Δ respectively, then their *cartesian product* $\Pi \times \Sigma$ is the partition of $\Gamma \times \Delta$ whose parts are all cartesian products of a part of Π and a part of Σ.

Lemma 2.6. *Let G and H be transitive permutation groups on Γ and Δ respectively, and let Π be a G-invariant partition of $\Gamma \times \Delta$.*

(a) *The G-projection and G-fibre partitions of Γ are G-invariant, and the second is a refinement of the first.*
(b) *The number k of parts of the G-fibre partition contained in a part of the G-projection partition is equal to the corresponding number for H.*
(c) *If $k = 1$, then Π is the cartesian product of a G-invariant partition of Γ and an H-invariant partition of Δ.*
(d) *If $k > 1$, then the set of k^2 parts of $F_G \times F_H$ within a part of $P_G \times P_H$ has the structure of a Latin square, where the first and second coordinates define the square grid and the parts of P give the positions of the letters.*

Proof. The first statement is clear from the definition.

For the second, let P_G and F_G be the G-projection and G-fibre partitions of Γ, and P_H and F_H the corresponding partitions of Δ. We claim first that

$$F_G \times F_H \preceq P \preceq P_G \times P_H.$$

The second inequality is clear since, if two elements of $\Gamma \times \Delta$ lie in the same part of P, then their projections onto Γ lie in the same part of P_G, and similarly for Δ and P_H. For the first inequality, suppose that γ_1 and γ_2 lie in the same part of F_G, and δ_1 and δ_2 in the same part of F_H. Then there exists $\delta \in \Delta$ such that (γ_1, δ) and (γ_2, δ) lie in the same part of Π; applying an element of $\{1\} \times H$, we can assume that $\delta = \delta_1$. Similarly, we can assume that (γ_1, δ_1) and (γ_1, δ_2) belong to the same part of Π. Now transitivity gives the result.

Now let A and B be parts of P_G and P_H, and choose a part P of Π which projects onto A and B. Any point of P belongs to $a \times b$ for some parts a, b of F_G and F_H respectively. This induces a bijection between the parts of F_G in A and the parts of F_H in B.

The third part is now clear.

For the final part, note that the parts of $F_G \times F_H$ within $A \times B$ have the structure of a $k \times k$ square grid. Choose a part P of Π within $A \times B$. Then P is a union of parts of $F_G \times F_H$. By the definition of the fibre partition, the first coordinates of pairs in P with given second coordinate form a single part of F_G, so P contains just one part of $F_G \times F_H$

within any row of the grid; and similarly for columns. Since every part of $F_G \times F_H$ is contained in a unique part of P, the result is proved. □

Theorem 2.7. *Let $G \leq \mathrm{Sym}(\Gamma)$ and $H \leq \mathrm{Sym}(\Delta)$ be transitive and let $G \times H$ act component-wise on $\Gamma \times \Delta$. If both G and H are primitive, then $G \times H$ is pre-primitive.*

Proof. Since both G and H are primitive, it follows that given a $G \times H$-invariant partition Π of $\Gamma \times \Delta$ the G and H-fibre partitions and the G and H-projection partitions must be trivial. If $|\Gamma| \neq |\Delta|$, then Π is one of four possibilities: $\{\Gamma \times \Delta\}$, (the partition with a single part), $\{\{(\gamma, \delta)\} \mid \gamma \in \Gamma, \delta \in \Delta\}$, (the partition into singletons), $\{\Gamma \times \{\delta\} \mid \delta \in \Delta\}$, and $\{\{\gamma\} \times \Delta \mid \gamma \in \Gamma\}$. Then Π is the orbit partition of $G \times H$, 1, $G \times 1$, $1 \times H$ respectively, and hence $G \times H$ is pre-primitive.

If $|\Gamma| = |\Delta|$, then there is one additional case to consider, namely the one where the G and H-fibre partitions are the singletons, and the G and H-projection partitions consist of a single part. Each part of P induces a bijection between Γ and Δ, so the stabiliser of a point (γ, δ) fixes every point in the part of P containing (γ, δ), and hence is the identity. So $G \times H$ is regular, whence also G and H are regular. Since they are also primitive, they are cyclic of prime order, whence $G \times H$ is abelian, and hence by Corollary 2.5 it is pre-primitive. □

Theorem 2.8. *Let $G \leq \mathrm{Sym}(\Gamma)$ and $H \leq \mathrm{Sym}(\Delta)$ be transitive and let $G \times H$ act component-wise on $\Gamma \times \Delta$. If G and H are pre-primitive and the sizes of Γ and Δ are coprime, then $G \times H$ is pre-primitive.*

Proof. Let Π be a $G \times H$-invariant partition. We denote the G and H-fibre partitions by Σ_Γ and Σ_Δ respectively, and we let P_G and P_H be the G- and H-projection partitions respectively.

Let k be the number of parts of the G-fibre partition in a part of P_G. Then k divides $|\Gamma|$, since the product of the size of a part of the G-fibre partition times k times the number of parts of P_G is equal to $|\Gamma|$. Similarly k divides Δ. Thus $\Pi = P_G \times P_H$, by Lemma 2.6. Since P_G and P_H are orbit partitions of subgroups G^* and H^* of G and H respectively, Π is the orbit partition of $G^* \times H^*$. □

Corollary 2.9. *Let $G \leq \mathrm{Sym}(\Gamma)$, $H \leq \mathrm{Sym}(\Delta)$ be transitive and regular. The direct product $G \times H$ acting on $\Gamma \times \Delta$ componentwise is pre-primitive if and only if either G and H are abelian, or one is a non-abelian Dedekind group and the other an abelian group whose exponent is not divisible by 4.*

Proof. If G and H both act regulary on Γ and Δ respectively, then $G \times H$ acts regularly on $\Gamma \times \Delta$. Therefore, $G \times H$ is pre-primitive if and only if it is Dedekind, which happens only in the two cases stated, by Dedekind's theorem (Theorem 2.2). □

Theorem 2.10. *Let $G \le \mathrm{Sym}(\Gamma)$ and $H \le \mathrm{Sym}(\Delta)$ be pre-primitive. Suppose that every $G \times H$-invariant partition Π of $\Gamma \times \Delta$ is of one of the following types:*

- *The G-fibre partition induced by Π on Γ is the same as the G-projection partition and the H-fibre partition induced by Π on Δ is the same as the H-projection partition;*
- *The G and H-projection partitions induced by Π on Γ and Δ respectively are the partitions with a single part.*

Then $G \times H$ in its product action is pre-primitive.

Proof. If Π is of the first form, then by Lemma 2.6, $\Pi = P_G \times P_H$. Since G is pre-primitive, there exists a subgroup G^* of G whose orbit partition is P_G, and similarly there exists some $H^* \le H$ whose orbit partition is P_H. Now it is easy to see that the orbit partition of $G^* \times H^*$ is Π.

Now suppose that Π is of the second type and let F_G and F_H be the G- and H-fibre partition induced by Π on Γ and Δ respectively. As in the first part, there are subgroups G^* and H^* of G and H respectively (which can be chosen to be normal, by Proposition 1.1) whose orbit partitions are F_G and F_H respectively; and $F_G \times F_H$ is the orbit partition of $G^* \times H^*$. Moreover, we can take G^* to consist of all elements of G fixing the parts of F_G, and similarly for H^*.

The group $(G \times H)/(G^* \times H^*)$ permutes (faithfully, by the above remark) the parts of $F_G \times F_H$. The argument in the second part of Theorem 2.7 shows that G/G^* and H/H^* are isomorphic and regular. If G/G^* has a non-trivial proper subgroup L/G^*, and M/H^* is the corresponding subgroup of H/H^*, then the inverse image in $G \times H$ of a diagonal subgroup of $L/G^* \times M/H^*$ defines a G-invariant partition not of the form in the theorem. So G/G^* and H/H^* are of prime order p. Then Π is the orbit partition of a subgroup of $G \times H$ whose projection onto $(G \times H)/(G^* \times H^*)$ is a diagonal subgroup of $C_p \times C_p$. □

In the next section, we will show a clean converse of these results: if $G \times H$ is pre-primitive, then both G and H are pre-primitive.

2.3. Wreath products, imprimitive action

By contrast with direct products, wreath products are better behaved. Again let G and H be permutation groups on Γ and Δ respectively. Take $\Omega = \Gamma \times \Delta$, regarded as the disjoint union of copies Γ_δ of Γ indexed by Δ, where $\Gamma_\delta = \{(\gamma, \delta) : \gamma \in \Gamma\}$. The partition of Ω into the sets Γ_δ will be called the *canonical partition*. Now $G \wr H$ is generated by

- the *base group*, the direct product of $|\Delta|$ copies of G indexed by Δ, where copy G_δ with index δ acts on Γ_δ as G acts on Γ and fixes all the other parts of the canonical partition pointwise;

- the *top group*, a copy of H acting on the second coordinate of points in $\Gamma \times \Delta$.

For further use, we note a property of this action.

Lemma 2.11. *Let G and H be permutation groups on Γ and Δ respectively. Then the direct product $G \times H$ in its product action is a subgroup of the wreath product $G \wr H$ in its imprimitive action.*

Proof. The top group is isomorphic to H and acts on the second coordinate. If D is the diagonal subgroup of the base group, consisting of elements with all coordinates equal, then D is isomorphic to G and acts on the first coordinate. Together these subgroups generate $G \times H$ in its product action. $\square$

Proposition 2.12. *Let G and H be transitive permutation groups on Γ and Δ respectively and let $G \wr H$ have its imprimitive action on $\Omega = \Gamma \times \Delta$, with canonical partition Π. If Σ is any $(G \wr H)$-invariant partition, then either $\Sigma \preceq \Pi$ or $\Pi \preceq \Sigma$.*

Proof. Suppose not, and let A be a part of Σ. Then A intersects two parts Γ_δ and $\Gamma_{\delta'}$ of Π but contains neither. Now $\Gamma_\delta \cap A$ is a part of a G-invariant partition of Γ_δ; by transitivity, we can find an element $g \in G_\delta$ which maps this set to a disjoint subset of Γ_δ. Then the element of the base group which acts as g on Γ_δ and the identity outside maps A to a set A' which is neither equal to nor disjoint from A, contradicting the assumption that Σ is $(G \wr H)$-invariant. $\square$

From this we see that, apart from the canonical partition, there are just two types of non-trivial $(G \wr H)$-invariant partitions Σ:

- If $\Sigma \preceq \Pi$, then we take a G-invariant partition Σ_0 of Γ, and copy it to all parts of the canonical partition using the top group. Since Σ is invariant under the top group, the partitions of the parts of Π must correspond in this way.
- If $\Pi \preceq \Sigma$, then we take an H-invariant partition Σ^0 of Δ, and replace each part A by the union of the sets Γ_δ for $\delta \in A$. For the sets of indices δ for which Γ_δ is contained in each part of the partition must form an H-invariant partition of Δ.

From this, we can prove our main result about the imprimitive action of the wreath product:

Theorem 2.13. *Let G, H be transitive groups on Γ and Δ respectively. Then the wreath product $G \wr H$ in its imprimitive action on $\Gamma \times \Delta$ is pre-primitive if and only if both G and H are pre-primitive.*

Proof. Suppose that G and H are pre-primitive, and let Σ be a $G \wr H$-invariant partition of $\Gamma \times \Delta$, different from the canonical partition Π.

- If $\Sigma \preceq \Pi$, then Σ is obtained by copying a G-invariant partition Σ_0 of Γ onto each of the parts Γ_δ. By assumption. Σ_0 is the orbit partition of a subgroup N of G. Then clearly Σ is the orbit partition of N^m, where $m = |\Delta|$, since the δ coordinate of the direct product acts on Γ_δ with orbit partition Σ_δ.
- If $\Pi \preceq \Sigma$, then Σ is obtained by taking the unions of the sets Γ_δ corresponding the points δ in each part of a H-invariant partition Σ^0 of Δ. By assumption, the parts of Σ^0 are the orbits of a subgroup N of H. Then clearly the parts of Σ are the orbits of the subgroup $G^m.N = G \wr N$ of $G \wr H$.

Now suppose conversely that $G \wr H$ is pre-primitive.

- Let Σ_0 be a G-invariant partition of Γ. Then the partition Σ_0 obtained by copying Σ onto each part of the canonical partition Π is $G \wr H$-invariant, and so is the orbit partition of a subgroup of $G \wr H$. The group of all elements fixing this partition induces H on the parts of the canonical partition, and has the form $N \wr H$, where N is the stabiliser of all the parts of Σ_0. So Σ_0 is the orbit partition of the subgroup N of G. Thus, G is pre-primitive.
- Let Σ^0 be a H-invariant partition of Δ. The partition Σ whose parts are unions of parts of Π indexed by elements of a part of Σ^0 is $(G \wr H)$-invariant, and so is the orbit partition of a subgroup of $G \wr H$. Clearly this subgroup has the form $G \wr N$, where N is a subgroup of H whose orbit partition is Σ^0. So H is pre-primitive. □

Here is the promised result for direct products:

Theorem 2.14. *Let $G \leq \mathrm{Sym}(\Gamma)$ and $H \leq \mathrm{Sym}(\Delta)$ be transitive groups. If the direct product $G \times H$ in its product action is pre-primitive then both G and H are pre-primitive.*

Proof. Suppose that $G \times H$ is pre-primitive. Since the direct product is embedded in the wreath product in its imprimitive action (Lemma 2.11), we have that $G \wr H$ in its imprimitive action is pre-primitive by Proposition 1.3; so G and H are pre-primitive, by Theorem 2.13. □

2.4. Wreath products, product action

This is the same group (up to isomorphism) but a different action. As before let G act on Γ and H on Δ. Then $G \wr H$ acts on the Cartesian product of $|\Delta|$ copies of Γ, which we regard as the set of words of length $N = |\Delta|$ over the alphabet Γ. The factor G_δ of the base group acts on the symbols in position δ, fixing the symbols in the other positions; the top group acts by permuting the coordinates.

In this section we examine the product action of the wreath product of two permutation groups G and H. First some preliminary remarks.

To begin, $G \wr H$ is transitive if and only if G is transitive, independent of H (since then the base group G^n is transitive in the product action). It is known that $G \wr H$ is primitive if and only if G is primitive but not cyclic of prime order and H is transitive; a similar result holds for quasiprimitivity [10, Theorem 5.8]. Apart from the first example below, we assume that H is transitive in this section.

Remark. If G is transitive and abelian, then $G \wr H$ is pre-primitive, since the base group is transitive and abelian.

Remark. The quaternion group Q is pre-primitive, but $G = Q \wr C_2$ is not. For G has a regular normal subgroup Q^2; the stabiliser G_α interchanges the two factors. So, if a and b generate Q, then the subgroup generated by (a, a) is G_α-invariant, but is not normal, since $b^{-1}ab = a^{-1}$, and so $(1, b)$ conjugates (a, a) to (a, a^{-1}), which is not in the subgroup generated by (a, a). By Theorem 2.1, G is not pre-primitive.

However, with an extra assumption on G, we do get pre-primitivity of $G \wr H$. Let n be the degree of H, and Γ the set on which G acts. Given a partition Σ of Γ, we denote by Σ^n the partition of Γ^n in which $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ belong to the same part if and only if a_i and b_i belong to the same part of Σ for $i = 1, \dots, n$.

Theorem 2.15. *Let G and H be transitive permutation groups on Γ and Δ respectively. Assume that G is pre-primitive and has the property that the stabiliser of a point fixes no additional points (equivalently, this stabiliser is equal to its normaliser in G). Then $G \wr H$, in the product action, is pre-primitive.*

Proof. Let Π be a non-trivial $(G \wr H)$-invariant partition of Γ^n, where $n = |\Delta|$. We claim that

> Let P be a part of Π, and let $i \in \Delta$. Then P contains two n-tuples which agree in all positions except position i.

To see this, choose $(a_1, \dots, a_n) \in P$. Since Π is non-trivial, we can choose a different element $(b_1, \dots, b_n) \in P$; since H is transitive on the coordinates, we can assume that $b_i \neq a_i$. Now consider the subgroup of the base group which acts as the identity on all positions different from the i-th and acts as G_{a_i} on the i-th position. By assumption, this subgroup fixes P and contains an element mapping b_i to $b'_i \neq b_i$. Then $(b_1, \dots, b_i, \dots, b_n)$ and $(b_1, \dots, b'_i, \dots, b_n)$ are the required n-tuples.

Now P is a block of imprimitivity for $G \wr H$, and so its setwise stabiliser acts transitively on it. Suppose that for one (and hence all) elements of P, there are k other elements of P differing from the chosen one only in the i-th position. The transitivity of H shows that this number is independent of i. Also, the mapping taking one such tuple to another can be chosen to lie in the base group and to stabilise P. Since elements of the base group acting on different coordinates commute, we see that P is the Cartesian product of k-subsets of Γ, one for each coordinate. We show that the partition Π has the form Σ^n for some partition Σ of Γ.

First we look at the parts of Π containing diagonal elements $(a, a, \ldots, a)$ of Ω. Such a part $P(a)$ is invariant under the action of the top group H, which is transitive on the components; so the subsets of the different copies of Γ making up $P(a)$ are all equivalent under the natural bijections between these copies. Hence the sets $P(a) \cap \Gamma_\delta$ for $a \in \Gamma$ form a partition of Γ_δ, and all these parts correspond.

Now take a general element $(a_1, \ldots, a_n)$ of Ω. The unique part P containing it can be mapped to $P(a_1) \cap \Gamma_1$ by an element of the base group fixing Γ_1 pointwise, so $P \cap \Gamma_1 = P(a_1) \cap \Gamma_1$. This shows that there is a unique partition Σ_δ of each set Γ_δ made up of the projections of parts of Π onto Γ_δ; and these parts correspond under the natural maps between different sets Γ_δ. This shows that $\Pi = \Sigma^n$, as claimed.

Now by assumption, G is pre-primitive, so Σ is the orbit partition of a normal subgroup N of G. But now it follows that Π is the orbit partition of the subgroup N^n of the base group of $G \wr H$. Hence $G \wr H$ is pre-primitive. □

Remark. If G is primitive, the extra condition on G in this theorem excludes only the cyclic groups of prime order.

In the other direction, things are simpler:

Theorem 2.16. *If $G \wr H$ in the product action is pre-primitive then G is pre-primitive.*

Proof. Let Σ be a G-invariant partition of Γ. It is easy to see that Σ^n is a $(G \wr H)$-invariant partition of Γ^n. By hypothesis, it is the orbit partition of a normal subgroup N of $G \wr H$. This subgroup clearly contains the top group H, and its intersection with the base group induces on the first coordinate a subgroup of G whose orbit partition is Σ. □

2.5. Diagonal groups

Diagonal groups arose in the celebrated O'Nan–Scott Theorem. However, they form a much larger class of transitive groups, discussed in detail in [2]. Let m be a positive integer and T a group. Then the *diagonal group* $D(T, m)$ is the group of permutations on $\Omega = T^m$ (where we distinguish m-tuples in Ω by putting them in square brackets) generated by the following elements:

(a) elements of T^m, acting by right multiplication;
(b) elements of T, acting simultaneously on all coordinates by left multiplication (that is, $t \in T$ maps $[x_1, \ldots, x_m]$ to $[t^{-1}x_1, \ldots, t^{-1}x_m]$);
(c) automorphisms of T, acting simultaneously on all coordinates;
(d) elements of the symmetric group S_m, permuting the coordinates;

(e) the map

$$[t_1, t_2, \ldots, t_m] \mapsto [t_1^{-1}, t_1^{-1}t_2, \ldots, t_1^{-1}t_m].$$

Note that the permutations of types (d) and (e) generate a group isomorphic to S_{m+1}.

We are going to find a sufficient condition for $D(T, m)$ to be pre-primitive.

Permutations of type (a) constitute a regular subgroup. Types (c), (d) and (e) generate the stabiliser of a point. Type (b) are not actually necessary, since any two of left multiplication, right multiplication and conjugation by a diagonal element generate the third. The regular subgroup T^m is not normal if T is nonabelian, so the results of Section 2.1 do not immediately apply; but we will see that very similar results hold.

Proposition 2.17. *Let $G = D(T, m)$.*

(a) *A G-invariant partition is the right coset partition of a subgroup H of T^m normalised by the elements of types (c), (d) and (e).*
(b) *If any subgroup H of T^m which is normalised by elements of types (c), (d) and (e) is normal in T^m, then G is pre-primitive.*

Proof. (a) Let Π be a G-invariant partition, and let H be the part of Π containing the identity. Since T^m is regular, the arguments of Section 2.1 show that H is a subgroup of T^m and Π is its right coset partition.

If Π is invariant under the point stabiliser, then the same is true for H. In other words, if $(h_1, \ldots, h_m) \in H$, then

- $(h_1^\alpha, \ldots, h_m^\alpha) \in H$ for all $\alpha \in \mathrm{Aut}(T)$;
- $(h_{1\pi}, \ldots, h_{m\pi}) \in H$ for all $\pi \in S_m$;
- $(h_1^{-1}, h_1^{-1}h_2, \ldots, h_1^{-1}h_m) \in H$.

Conversely, suppose that H is invariant under these transformations. Take any right coset of H, say $H(g_1, \ldots, g_m)$.

- Applying an automorphism α of T, we find

$$(h_1g_1, \ldots, h_mg_m)^\alpha = (h_1^\alpha, \ldots h_m^\alpha)(g_1^\alpha, \ldots, g_m^\alpha) \in H(g_1^\alpha, \ldots, g_m^\alpha)$$

since H is invariant under the coordinatewise action of α.
- Applying a permutation π to the subscripts, we find

$$(h_1g_1, \ldots, h_mg_m)^\pi = (h_{1\pi}, \ldots, h_{m\pi})(g_{1\pi}, \ldots, g_{m\pi}) \in H(g_{1\pi}, \ldots, g_{m\pi})$$

since H is invariant under π applied to the subscripts;

- Applying the map ϵ of type (e), we find

$$\begin{aligned}(h_1g_1,\dots,h_mg_m)\epsilon &= (g_1^{-1}h_1^{-1}, g_1^{-1}h_1^{-1}h_2g_2,\dots,g_1^{-1}h_1^{-1}h_mg_m)\\ &= ((h_1^{-1})^{g_1}g_1^{-1}, (h_1^{-1}h_2)^{g_1}g_1^{-1}g_2,\dots,(h_1^{-1}h_m)^{g_1}g_1^{-1}g_m\\ &\in H(g_1,\dots,g_m)\epsilon\end{aligned}$$

  since H is invariant under both conjugation by g_1 and ϵ.

So Π is invariant under these three types of element.

(b) If H is a normal subgroup of T^m, then its right coset partition coincides with its orbit partition. □

Theorem 2.18. *Let T be a finite group and m a positive integer. Suppose that the following property holds:*

> *If K is any characteristic subgroup of T, and L the subgroup of K generated by the $(m+1)$st powers and commutators of elements in K, then every subgroup of K containing L is normal in T.*

Then $G = D(T,m)$ is pre-primitive.

Proof. We examine further the right coset partition of a subgroup H of T^m invariant under the transformations (c), (d) and (e). Let π_i be the projection of H onto the ith direct factor. Since H is invariant under permutations, $\pi_i(H) = K$ (as subgroup of T) is independent of the index i; since H is invariant under automorphisms, K is a characteristic subgroup of T. Thus, $H \le K^m$.

Take $(g_1, g_2,\dots,g_m) \in H$. Then $(g_1^{-1}, g_1^{-1}g_2,\dots,g_1^{-1}g_m) \in H$. It follows by closure that $(g_1^{-2}, g_1^{-1},\dots,g_1^{-1}) \in H$, whence $(g_1^2, g_1,\dots,g_1) \in H$. The same is true with the first and second coordinates swapped; so $(g_1, g_1^{-1}, 1,\dots,1) \in H$.

From this we make two deductions:

- For any $g_1, g_2 \in H$, the elements $(g_1, g_1^{-1}, 1,\dots,1)$, $(g_2, g_2^{-1}, 1,\dots,1)$ and $(g_1g_2, (g_1g_2)^{-1}, 1,\dots,1)$ all belong to H. Multiplying the first two by the inverse of the third and swapping the first two coordinates, we find that $([g_1, g_2], 1,\dots,1) \in H$.
- We know $(g_1^2, g_1,\dots,g_1) \in H$. Multiplying successively by the elements with g_1 in the first coordinate and g_1^{-1} in the ith, for $i = 2,\dots,m$, we get $(g_1^{m+1}, 1,\dots,1) \in H$.

Since this holds with any coordinate replacing the first, we see that $L^m \le H$, where L is the subgroup of K generated by $(m+1)$st powers and commutators. Thus H/L^m is contained in $(K/L)^m$. Note that K/L is an abelian group A of exponent dividing $m+1$.

Let M be the subgroup of A^m consisting of m-tuples for which the product of the coordinates is 1. Then M is generated by elements having one coordinate $a \in K/L$, one coordinate a^{-1}, and the remaining coordinates 1. Since $(g, g^{-1}, 1, \dots, 1) \in H$ for all $g \in K$, we see that H contains the inverse image of M in K^m.

Now any subgroup of $(K/L)^m$ containing A must have the form

$$\{(a_1, \dots, a_m) : a_1 \cdots a_m \in B\}$$

where B is a subgroup of A; and H is normal in T^m if and only if the inverse image of B in T is normal in T. So, with our hypothesis, H is normal in T^m, and G is pre-primitive, by Proposition 2.17(b). □

There are several simpler conditions which guarantee that the hypotheses of this theorem are satisfied.

Corollary 2.19. *Suppose that one of the following holds:*

(a) $|T|$ *is coprime to* $m + 1$*;*
(b) T *is supersoluble;*
(c) T *is a direct product of non-abelian simple groups.*

Then $D(T, m)$ *is pre-primitive.*

Proof. In cases (a) and (c), the subgroup L of K is equal to K, since there is no nontrivial abelian quotient with exponent dividing $m + 1$.

Suppose that T is supersoluble, K is characteristic in T, L the subgroup of K generated by $(m+1)$st powers and commutators, and M a subgroup of K containing L which is not normal in T. Then there is an element $t \in T$ which does not normalise M, and so an element $u \in M/L$ such that $\langle u \rangle$ is not normalised by t. But this means that there is a chief factor of T in M/L containing u and u^t which is not cyclic, contradicting the fact that T is supersoluble. □

It may be that the converse of Theorem 2.18 is true; we have not been able to decide this.

Example. It can be verified by using GAP [6] that $G = D(A_4, 3)$ is not pre-primitive. In this case, with $K = V_4$, $L = 1$, we see that there are subgroups of K which are not normal in $T = A_4$.

2.6. Groups with pre-primitive subgroups

According to *Jordan's theorem*, a primitive group containing a transitive subgroup on a subset Δ of Ω, fixing the points outside Δ, is 2-transitive. We will prove a somewhat similar theorem for pre-primitivity.

Table 1
Numbers of transitive groups, etc.

n	$T(n)$	$P(n)$	$PP(n)$	$QP(n)$	correlation
10	45	9	42	9	0.0133
11	8	8	8	8	0
12	301	6	276	7	0.0014
13	9	9	9	9	0
14	63	4	59	5	−0.0108
15	104	6	102	8	−0.0178
16	1954	22	1833	22	0.0007
17	10	10	10	10	0
18	983	4	900	4	0.0003
19	8	8	8	8	0
20	1117	4	1019	10	−0.0046

Theorem 2.20. *Suppose that G is a transitive permutation group on Ω. Suppose that Δ is a subset of Ω satisfying $|\Delta| > |\Omega|/2$ and H a subgroup of G which fixes every point outside Δ and acts pre-primitively on Δ. Then G is pre-primitive.*

Proof. Suppose that Π is a G-invariant partition. No part of Π can intersect both Δ and $\Omega \setminus \Delta$ non-trivially, since such a part would be fixed by G and therefore would contain the whole of Δ (by transitivity of H). So every part of Π is either contained in or disjoint from Δ.

The parts contained in Δ form a H-invariant partition of Δ, and so by hypothesis form the orbit partition of a subgroup K of H. Now every conjugate of K in G fixes all parts of Π, and so they generate a group whose orbit partition is Π.

Since this holds for every G-invariant partition, G is pre-primitive. □

3. Data on small transitive groups

We remarked earlier that pre-primitivity and quasiprimitivity are logically independent. We might ask whether they are statistically independent, in the sense that if we pick an isomorphism type of transitive permutation group of degree n at random, the events that it is pre-primitive and that it is quasiprimitive are uncorrelated. If $T(n)$, $P(n)$, $QP(n)$ and $PP(n)$ denote the numbers of transitive, primitive, quasiprimitive and pre-primitive groups of degree n up to permutation isomorphism, this is equivalent to asking whether $T(n)P(n) = QP(n)PP(n)$. This equation is true in some cases (for example, if n is prime, then every transitive group of degree n is primitive, so the four numbers are equal). In general, it seems to be roughly true. Table 1, computed from the library in GAP [6], gives the values of the four functions with $10 \leq n \leq 20$, and the correlation coefficient of the properties "pre-primitive" and "quasiprimitive" when a transitive group of degree n is chosen uniformly at random.

It might be useful to have a bound on the correlation coefficient or some evidence about its sign, but we have been unable to do this. The data suggest that most transitive groups are pre-primitive and most quasiprimitive groups are primitive.

4. Degrees for which all transitive groups are pre-primitive

Let

$$\mathcal{S} = \{n \in \mathbb{N} : \text{every transitive group of degree } n \text{ is pre-primitive}\}.$$

Problem. Describe the set $\mathcal{S}$.

We give some context and then give upper and lower bounds for this set.

One could ask in a similar way about the set of natural numbers n for which every pre-primitive group of degree n is primitive. But this is easily seen to be just the set of prime numbers. For, if n is composite, say $n = ab$, then $S_a \wr S_b$ in its imprimitive action is pre-primitive.

What about the set of natural numbers n for which the only primitive groups of degree n are the symmetric and alternating groups? This question has a longer history: for example Mathieu thought about it. But as one of the first applications of the Classification of Finite Simple Groups, the authors of [3] showed that this set contains almost all natural numbers. More precisely, if E is the complementary set (for which non-trivial primitive groups exist), then they showed that

$$|E \cap \{1, \ldots, n\}| = 2\pi(n) + (1 + \sqrt{2})n^{1/2} + O(n^{1/2}/\log n),$$

where $\pi(n)$ is the number of primes in $\{1, \ldots, n\}$.

Clearly our set $\mathcal{S}$ contains all prime numbers, since a transitive group of prime degree is primitive. It also contains all squares of primes:

Proposition 4.1. *A transitive permutation group of degree p^2, where p is prime, is pre-primitive.*

Proof. It suffices to prove the result in the case where the group G is a p-group. For the Sylow p-subgroup of a transitive group G of prime power degree is transitive, and if it is pre-primitive then so is G. So suppose that G is a p-group.

Let Π be any G-invariant partition; it consists of p sets of size p. Let N be the subgroup of G fixing a part of Π. Then $|G : N| = p$ and N fixes all the blocks. But N cannot fix a point, since a point stabiliser has index p^2. So the blocks are orbits of N. Thus G is pre-primitive. $\square$

In the other direction, let $\mathcal{A}$ be the set of natural numbers n for which every group of order n is abelian. This well-studied set consists of all numbers which are not divisible by p^3 (where p is prime), or by pq (where p and q are primes with $q \mid p - 1$) or by p^2q (where p and q are primes with $q \mid p + 1$). (This result is “folklore”, but we have been unable to find a good reference.)

Proposition 4.2. $\mathcal{S} \subseteq \mathcal{A}$.

Proof. A little thought shows that $\mathcal{A}$ is also the set of natural numbers n for which every group of order n is Dedekind; and, if n is not in this set, then a non-Dedekind group acting regularly is not pre-primitive, so $n \notin \mathcal{S}$. □

Strict inequality holds in both cases:

- The groups A_5 and S_5 have transitive imprimitive actions on 15 points (on the cosets of a Sylow 2-subgroup). Since A_5 is simple, this action of A_5 is quasiprimitive, and so not pre-primitive; the same is true for S_5. So $15 \in \mathcal{A} \setminus \mathcal{S}$.
- Computation shows that all transitive groups of degrees 33 and 35 are pre-primitive. So $33, 35 \in \mathcal{S} \setminus (\mathcal{P} \cup \mathcal{P}_2)$, where $\mathcal{P}$ is the set of primes and $\mathcal{P}_2$ the set of primes squared.

A special case of the general problem which may be tractable is to find which products of two distinct primes belong to $\mathcal{S}$. Suppose that p and q are primes with $q < p$. We have seen that, if $q \mid p-1$, then $pq \notin \mathcal{S}$. Further examples can be constructed as follows. Let $p = 2^d + 1 \geq 5$ be a Fermat prime, and let q be a prime divisor of $p - 2$. The group $G = \mathrm{PSL}(2, 2^d)$ has an imprimitive action on pq points (the stabiliser being a subgroup of index q in the Sylow 2-normaliser). Since this group is simple, the action is quasiprimitive, and so not pre-primitive.

5. Related concepts

The guiding principle behind the definition of pre-primitivity was to find a condition logically independent of quasiprimitivity such that its conjunction with quasiprimitivity is equivalent to primitivity.

We could now consider playing the same game with other pairs of properties of permutation groups. We give three examples, and invite readers to consider others.

5.1. From transitivity to quasiprimitivity

We want a property, which we shall call *pre-QP*, which together with transitivity is equivalent to quasiprimitivity. It is clear that such a property can be defined as follows: The permutation group G on Ω is pre-QP if its action on each of its orbits is quasiprimitive (equivalently, every normal subgroup of G acts either transitively or trivially on each G-orbit).

This property does not have such a rich theory as pre-primitivity, so we say no more about it.

5.2. From k-homogeneity to k-transitity

Let k be a positive integer less than $|\Omega|$. A permutation group G on Ω is *k-homogeneous* if its action on the set of k-element subsets of Ω is transitive, and is *k-transitive* if its action on the set of ordered k-tuples of distinct elements of Ω is transitive. Both of these conditions have been intensively studied.

The property that lifts k-homogeneity to k-transitivity is also known, having been first given by Neumann [7]. The permutation group G is *generously* $(k-1)$*-transitive* if the setwise stabiliser in G of any k-set acts on it as the symmetric group S_k. Neumann showed that this condition implies $(k-1)$-transitivity. In the case $k = 2$, it is equivalent to requiring that all the orbitals of G are self-paired. We have nothing more to add here.

5.3. From primitivity to synchronization

The property of synchronization comes from automata theory by way of semigroup theory, and is discussed in [1]. We say that the permutation group G on Ω is *synchronizing* if, for any map $f : \Omega \to \Omega$ which is not a permutation, the monoid $\langle G, f\rangle$ generated by G and f contains an element of rank 1 (that is, one which maps the whole of Ω onto a single point).

For our purposes, the most useful characterisation of this property is due to Neumann [9]. We say that a partition Π of Ω is *section-regular* for the permutation group G on Ω if there exists a subset A of Ω such that Ag is a section (transversal) of Π for all $g \in G$. Now Neumann showed that a permutation group is synchronizing if and only if it has no non-trivial section-regular partition (where the trivial partitions are as described earlier).

From this it is clear that synchronization implies primitivity: for a G-invariant partition Π is section-regular (simply take A to be any transversal to Π). The converse is false, as numerous examples show.

Accordingly, we will say that the transitive permutation group G on Ω is *pre-synchronizing* if every section-regular partition for G is G-invariant.

So our interest is in pre-synchronizing groups which are imprimitive. It turns out that these can be completely classified:

Theorem 5.1. *Let G be a transitive imprimitive permutation group which is pre-synchronizing. Then G is isomorphic to the Klein group, in its regular action of degree* 4.

Proof. Let Π be a non-trivial G-invariant partition, and A a part of Π. Then the images of A under G are the parts of Π. Take Σ to be any partition such that each part of Σ is a transversal for Π. Then Ag is a transversal for Σ, for all $g \in G$; in other words, Σ is section-regular.

Since G is pre-synchronizing, it follows that Σ is G-invariant. Suppose that k is the size of a part B of Σ, and let Σ' be another transversal for P satisfying $|B \cap B'| = k - 1$. Then $B \cap B'$ is a block of imprimitivity contained in B. So $k-1$ divides k, whence $k = 2$.

Now let m be the number of parts of Σ (the size of A). Since Σ is a non-trivial G-invariant partition, we can run the same argument with Π and Σ interchanged to conclude that also $m = 2$, whence G is a permutation group of degree 4. Since it preserves at least two distinct partitions into two sets of size 2, we conclude that G must be the Klein group of order 4. □

Declaration of competing interest

The authors declare no conflict of interest.

Data availability

No data were used in the preparation of this paper apart from the GAP transitive groups library.

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