

PRESENTATIONS AND EFFICIENCY OF SEMIGROUPS

Hayrullah Ayik

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AND
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Hayrullah Ayık

Ph. D. Thesis
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1998



to
Gonca G ng r
and
the memory of
my mother Ay e Sittika ( zden) Ayık,
my sister Nigar (Menev e)   o lu and
my brother Kadir Menev e who died on 6th of June 1997.

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Declaration


I, Hayrullah Ayık, hereby certify that this thesis has been written by me, that it is the record of work carried out by me and that it has not been submitted in any previous application for a higher degree.


Signature . . .  . . . Name Hayrullah Ayık Date 6/7/98

I was admitted as a candidate for the degree of Doctor of Philosophy in September, 1995; the higher study for which this is a record was carried out in the University of St Andrews between 1995 and 1998.


Signature . . .  . . . Name Hayrullah Ayık Date 6/7/98

We hereby certify that the candidate has fulfilled the conditions of the Resolution and Regulations appropriate for the degree of Doctor of Philosophy in the University of St Andrews and that the candidate is qualified to submit this thesis in application for the degree of Doctor of Philosophy.

Signature  . . . Name Colin C. Campbell Date 6/7/98

Signature  . . . Name John J. O'Connor Date 6/7/98

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Date 6/7/98

Preface

In the last two years one of the most significant things that I have learned is how important identity elements are in algebra. Without an identity, especially in combinatorial semigroup theory, everything becomes very much harder. It is well-known that the additive semigroup $\mathbb{N} \times \mathbb{N}$ is not finitely generated. This kind of example shows that finite generation and finite presentability are more interesting problems in combinatorial semigroup theory. In Chapters 2, 3 and 4 we investigate finite generation and finite presentability together with some other finiteness conditions for semigroup constructions, namely 0-direct unions of semigroups and Rees matrix semigroups.

For a mathematician, it is nice to see some connection between two different mathematical disciplines. I. Schur constructed the Schur multiplier $M(G)$ of a finite group G directly from the multiplication table of G and proved that $\text{rank}(M(G)) \leq \text{def}(G)$ in [60]. The Schur multiplier of a finite group G can also be considered as the second homology group $H_2(G, \mathbb{Z})$. C. Squier constructed a free resolution of \mathbb{Z} over the monoid ring $\mathbb{Z}M$ of a monoid M which has a monoid presentation $\langle A \mid R \rangle$ such that R is a reduced complete rewriting system over A in 1987 and S. J. Pride proved in 1997 that, for such a monoid M , $\text{rank}(H_2(M, \mathbb{Z})) \leq \text{def}(M)$ by using a resolution similar to Squier's. In Chapters 5, 6, 7 and 8, we investigate the efficiency of semigroups and the efficiency of monoids and of groups.

Most of the results presented in this thesis have been submitted for publication

by H. Ayık and N. Ruškuc; and by H. Ayık, C. M. Campbell, J. J. O'Connor and N. Ruškuc. One of them [1] has been accepted for publication. However, this thesis is not a compilation of these submitted papers. The reason for this is that the papers have been submitted while the research has been in progress. In addition, this thesis contains more general results and some results have not yet been submitted for publication.

Hayrullah Ayık

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Hayrullah Ayık

Abstract

In this thesis we consider in detail the following two problems for semigroups:

- (i) When are semigroups finitely generated and presented?
- (ii) Which families of semigroups can be efficiently presented?

We also consider some other finiteness conditions for semigroups, homology of semigroups and wreath product of groups.

In Chapter 2 we investigate finite presentability and some other finiteness conditions for the 0-direct union of semigroups with zero. In Chapter 3 we investigate finite generation and presentability of Rees matrix semigroups over semigroups. We find necessary and sufficient conditions for finite generation and presentability. In Chapter 4 we investigate some other finiteness conditions for Rees matrix semigroups.

In Chapter 5 we consider groups as semigroups and investigate their semigroup efficiency. In Chapter 6 we look at “proper” semigroups, that is semigroups that are not groups. We first give examples of efficient and inefficient “proper” semigroups by computing their homology and finding their minimal presentations. In Chapter 7 we compute the second homology of finite simple semigroups and find a “small” presentation for them. If that “small” presentation has a special relation, we prove that finite simple semigroups are efficient. Finally, in Chapter 8, we investigate the efficiency of wreath products of finite groups as groups and as semigroups. We give more examples of efficient groups and inefficient groups.

Chapter 1

Introduction

In this chapter we introduce certain basic definitions and results about semigroup and group presentations, finiteness conditions for semigroups and homology groups.

1.1 Semigroups

A *semigroup* (S, \circ) is defined as a non-empty set S on which an associative binary operation \circ is defined. However we usually write xy instead of $x \circ y$ ($x, y \in S$). If a semigroup S contains an element 1 such that $1x = x1 = x$ for all $x \in S$, we say that 1 is an *identity*, and that S is a *monoid*. We may always adjoin an identity to S (even when S already has one). We now define S^1 to be the monoid obtained from S by adjoining an identity, that is $S^1 = S \cup \{1\}$. Note that if S had an identity, it is no longer an identity in S^1 .

If a semigroup S contains an element z such that $zx = z$ for all $x \in S$, we say that z is a *left zero element*. Similarly, we define a *right zero element*. If an element 0 of a semigroup is both a left and right zero, we say that 0 is a *zero element*. Also we define S^0 as the semigroup obtained from S by adjoining a zero if necessary, that is $S^0 = S \cup \{0\}$ if $0 \notin S$, and $S^0 = S$ otherwise. Note that the

definition of S^0 is the same as the one in [29], but the definition of S^1 is different from the one in [29].

Both identity and zero elements are unique if they exist. However, left or right zero elements need not be unique. If a semigroup consists of left zero elements only, then it is called a *left zero semigroup*, and we denote it by L_m where m is the size of the semigroup. Similarly, we define *right zero semigroups*, and we denote them by R_n .

If a semigroup S contains an element e such that $e^2 = e$, we say that e is an *idempotent*. If S consists of idempotents only, we say that S is a *band*. A *commutative semigroup* S is a semigroup such that $xy = yx$ for all $x, y \in S$. If S is both a band and commutative, we say that S is a *semilattice*.

Let A be a set and let \mathcal{SL}_A be the set of all non-empty subsets of A . Consider the set-theoretical union \cup as a binary operation. With respect to this operation, it is clear that \mathcal{SL}_A is a semilattice, and it is called the *free semilattice over A* .

Let $\mathcal{Z}_n = \{x_0, x_1, \dots, x_{n-1}\}$ $n \geq 2$. Define a binary operation by $x_i x_j = x_0$ for all $x_i, x_j \in \mathcal{Z}_n$. It is clear that \mathcal{Z}_n is a semigroup with respect to this operation, which is commutative but not a band, and it is called the *zero semigroup* of order n .

Let $R_{m,n} = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Define a binary operation by $(i, j)(k, l) = (i, l)$ for all $(i, j), (k, l) \in R_{m,n}$. With respect to this operation, $R_{m,n}$ is a semigroup, which is a band but not commutative, and it is called a *rectangular band*. Note that if $m = 1$, then $R_{1,n}$ is a right zero semigroup, and if $n = 1$, then $R_{m,1}$ is a left zero semigroup.

1.2 Semigroup presentations

Let S be a semigroup and let T be a subset of S . If $T^2 \subseteq T$, that is $tt' \in T$ for all $t, t' \in T$, we say that T is a *subsemigroup* of S . Let X be a subset of S . The smallest subsemigroup of S which contains X is called the *subsemigroup*

generated by X , and is denoted by $\langle X \rangle$.

We say that a semigroup is *finitely generated* if there exists a finite subset X of S such that $S = \langle X \rangle$. The *rank* of S is denoted by $\text{rank}(S)$ and is defined by $\text{rank}(S) = \min \{ |X| \mid X \text{ is a generating set for } S \}$.

An element x of a semigroup S is called *indecomposable* if $x \neq yz$ for all $y, z \in S$, that is $x \in S \setminus S^2$. Notice that all the indecomposable elements of a semigroup S are contained in every generating set of S .

Consider the left zero semigroup L_m of order m . Since $xy = x$ for all $x, y \in L_m$, it follows that the smallest generating set of L_m is itself. Therefore $\text{rank}(L_m) = |L_m| = m$. Now consider the zero semigroup $\mathcal{Z}_n = \{ x_0, x_1, \dots, x_{n-1} \}$ $n \geq 2$. Since $x_i x_j = x_0$ for all $x_i, x_j \in \mathcal{Z}_n$, it follows that the smallest generating set of \mathcal{Z}_n is $\{ x_1, \dots, x_{n-1} \}$. Therefore $\text{rank}(\mathcal{Z}_n) = n - 1$. Next we consider the free semilattice \mathcal{SL}_A over a set A . Since, for each $a \in A$, $\cup_{i \in I} X_i = \{a\}$ implies $X_i = \{a\}$ for all $i \in I$, and since every element of \mathcal{SL}_A is a union of some $\{a\}$ ($a \in A$), it follows that $\{ \{a\} \mid a \in A \}$ is the smallest generating set of \mathcal{SL}_A . Therefore $\text{rank}(\mathcal{SL}_A) = |A|$.

Let A be an “alphabet”. Then A^+ is defined to be the set of all non-empty finite words in the alphabet. A binary operation is defined on A^+ by *juxtaposition*: $(a_1 \cdots a_n)(b_1 \cdots b_m) = a_1 \cdots a_n b_1 \cdots b_m$ for all $a_1 \cdots a_n, b_1 \cdots b_m \in A^+$. With respect to this operation, A^+ is a semigroup, and it is called the *free semigroup* on A . Similarly, we define the *free monoid* $A^* = A^+ \cup \{\epsilon\}$, where ϵ is the empty word.

Theorem 1.1 *Let A an alphabet, let S be a semigroup, and let $f : A \longrightarrow S$ be any mapping. Then there exists a unique homomorphism $\phi : A^+ \longrightarrow S$ such that $\phi(a) = f(a)$ for all $a \in A$. ■*

Theorem 1.2 *Every semigroup is a homomorphic image of a free semigroup. ■*

The proofs of Theorems 1.1 and 1.2 are standard and can be found in [42].

Let S be a semigroup and let $\rho \subseteq S \times S$ be an equivalence relation on S . Then ρ is called a *congruence* if $(xz, yt) \in \rho$ for all $(x, y), (z, t) \in \rho$. For convenience, we denote the equivalence class containing x by ρx and we denote the set of the equivalence classes by S/ρ .

If ρ is a congruence on a semigroup S , then we define a binary operation on the quotient set S/ρ by $(\rho x)(\rho y) = \rho(xy)$. With respect to this operation, S/ρ is a semigroup.

A *semigroup (monoid) presentation* is an ordered pair $\langle A \mid R \rangle$, where A is an alphabet and $R \subseteq A^+ \times A^+$ (respectively, $R \subseteq A^* \times A^*$) is a set of pairs of words. An element (r, s) of R is called a (defining) *relation*, and is usually written $r = s$ instead of (r, s) . The semigroup (monoid) *defined by* $\langle A \mid R \rangle$ is the quotient semigroup A^+/ρ (respectively, A^*/ρ) where ρ is the smallest congruence on A^+ (respectively, A^*) containing R . Let S be the semigroup defined by $\langle A \mid R \rangle$. Then, for two words w_1, w_2 in A^+ , we write $w_1 \equiv w_2$ if they are identical words, and we write $w_1 = w_2$ if they represent the same element of the semigroup S , that is $(w_1, w_2) \in \rho$ and we say that $w_1 = w_2$ *holds in* S . If a semigroup S can be defined by $\langle A \mid R \rangle$ with both A and R finite then the presentation is said to be a *finite presentation* and S is said to be *finitely presented*.

Theorem 1.3 *Let $\mathcal{P} = \langle A \mid R \rangle$ be a semigroup presentation, and let S be the semigroup defined by \mathcal{P} . For a semigroup T generated by X , and an onto map $f : A \longrightarrow X$, define $\phi : A^+ \longrightarrow T$ to be the unique homomorphism extension of f . If $\phi(r) = \phi(s)$ for all $(r, s) \in R$, then T is a homomorphic image of S . ■*

For a proof, see [55, Proposition 2.1].

Let $\mathcal{P} = \langle A \mid R \rangle$ be a semigroup presentation and let S be the semigroup defined by \mathcal{P} . For $w_1, w_2 \in A^+$, we say that w_2 is *obtained from* w_1 by one application of one relation from R if there exist $u, v \in A^*$ and (r, s) or $(s, r) \in R$ such that $w_1 \equiv urv$ and $w_2 \equiv usv$. We say that w_2 is *deduced from* w_1 if there

exists a finite sequence

$$w_1 \equiv \alpha_1 \rightarrow \alpha_2 \rightarrow \cdots \rightarrow \alpha_{k-1} \rightarrow \alpha_k \equiv w_2$$

of words from A^+ such that α_{i+1} is obtained from α_i by one application of one relation from R . In this case, we may also say that $w_1 = w_2$ is a *consequence* of R .

Proposition 1.4 *Let $\mathcal{P} = \langle A \mid R \rangle$ be a semigroup presentation, let S be the semigroup defined by \mathcal{P} , and let w_1 and w_2 be two words in A^+ . Then $w_1 = w_2$ holds in S if and only if w_2 is deduced from w_1 . ■*

For a proof, see [29, Proposition 1.5.9]. Next we give a stronger result which will be useful in Chapters 2 and 3.

Proposition 1.5 *Let S be a semigroup generated by a set A , and let $R \subseteq A^+ \times A^+$. Then $\mathcal{P} = \langle A \mid R \rangle$ is a presentation for S if and only if the following two conditions are satisfied:*

- (i) *S satisfies all the relations of R and;*
- (ii) *if $u, v \in A^+$ are any two words such that $u = v$ holds in S then $u = v$ is a consequence of R . ■*

For a proof, see [55, Proposition 2.3].

Therefore, we may also say that the relation $u = v$ holds in S if $u = v$ is a consequence of R where $\langle A \mid R \rangle$ is a presentation for S .

Proposition 1.6 *Let S be a semigroup generated by a set A , let $R \subseteq A^+ \times A^+$ and let $W \subseteq A^+$. If the following three conditions are satisfied:*

- (i) *S satisfies all the relations of R ;*
- (ii) *for each word $w \in A^+$, there exists $\bar{w} \in W$ such that $w = \bar{w}$ is a consequence of R and;*
- (iii) *$|W| \leq |S|$,*

then $\langle A \mid R \rangle$ is a presentation for S . ■

For a proof, see [55, Proposition 2.2].

Now we apply the previous proposition to find a presentation for the free semilattice \mathcal{SL}_A over a finite set A .

Example 1.7 *The presentation*

$$\mathcal{P} = \langle a_1, \dots, a_n \mid a_i^2 = a_i \ (1 \leq i \leq n), \ a_j a_i = a_i a_j \ (1 \leq i < j \leq n) \rangle$$

defines the free semilattice \mathcal{SL}_A over $A = \{a_1, \dots, a_n\}$.

Proof As we mentioned at the beginning of this section, it is clear that A is a generating set for \mathcal{SL}_A . Moreover, since \mathcal{SL}_A is a commutative semigroup of idempotents, \mathcal{SL}_A satisfies all the relations in \mathcal{P} .

Let

$$W = \{ a_1^{\varepsilon_1} \cdots a_n^{\varepsilon_n} \in A^+ \mid \varepsilon_i \in \{0, 1\} \ (1 \leq i \leq n) \}.$$

Now let w be any word in A^+ . First applying suitable relations of the form $a_j a_i = a_i a_j$ yields a word w' of the form $a_1^{\lambda_1} \cdots a_n^{\lambda_n}$ where $\lambda_i \geq 0$ ($1 \leq i \leq n$). Then applying relations of the form $a_i^2 = a_i$ as much as possible yields a word \bar{w} from W .

Since $|W| = 2^n - 1$, it follows from the previous proposition that \mathcal{P} defines \mathcal{SL}_A , as required. ■

The monoid defined by a monoid presentation $\langle A \mid R \rangle$ may be defined by the semigroup presentation

$$\langle A \cup \{e\} \mid \bar{R} \cup \{e^2 = e, \ ae = a, \ ea = a \mid a \in A\} \rangle$$

where \bar{R} is obtained from R by replacing every occurrence of the empty-word by e . (We assume that $e \notin A$.)

We may consider a semigroup presentation $\mathcal{P} = \langle A \mid R \rangle$ as a monoid presentation. If \mathcal{P} defines a semigroup S , then \mathcal{P} , as a monoid presentation, defines the monoid $S^1 = S \cup \{1\}$.

1.3 Group presentations

A group presentation is a ordered pair $\langle A \mid R \rangle$ where A is an alphabet, R is a subset of $(A \cup A^{-1})^* \times (A \cup A^{-1})^*$ and $A^{-1} = \{ a^{-1} \mid a \in A \}$. The group defined by $\langle A \mid R \rangle$ is the quotient group $F(A)/N$ where $F(A)$ is the free group on A and N is the smallest normal subgroup containing $\{ rs^{-1} \mid (r, s) \in R \}$. Note that we will use relations instead of relators to define group presentations.

Observe that the free group $F(A)$ over A may be defined by the monoid presentation

$$\langle A \cup A^{-1} \mid aa^{-1} = \epsilon, a^{-1}a = \epsilon (a \in A) \rangle.$$

Moreover, the group defined by $\langle A \mid R \rangle$ may be defined as a monoid by the monoid presentation

$$\langle A \cup A^{-1} \mid R \cup \{ aa^{-1} = \epsilon, a^{-1}a = \epsilon \mid a \in A \} \rangle.$$

Theorem 1.8 *Let $\mathcal{P} = \langle A \mid R \rangle$ be a semigroup presentation, let S be the semigroup defined by \mathcal{P} , and let G be the group defined by \mathcal{P} when we consider \mathcal{P} as a group presentation.*

- (i) *If S is finite, then G is a homomorphic image of S .*
- (ii) *If S is a group, then G is isomorphic to S .* ■

For more general results on the relation between S and G , see [17] and [51].

Proposition 1.9 *If G is the group defined by the group presentation $\langle A \mid R \rangle$ and if H is the group defined by the group presentation $\langle A \mid Q \rangle$ where $R \subseteq Q$, then H is a homomorphic image of G .* ■

For a proof, see [36, Proposition 4.2].

Proposition 1.10 *Let G be the group defined by the group presentation $\langle A \mid R \rangle$ and let H be the group defined by the group presentation $\langle B \mid Q \rangle$. Then the direct product $G \times H$ of groups G and H can be defined by the group presentation $\langle A \cup B \mid R \cup Q \cup C \rangle$ where $C = \{ ab = ba \mid a \in A, b \in B \}$.* ■

See for a proof [36, Proposition 4.4].

1.4 Tietze transformations

Another method for finding presentations for a semigroup S is to apply Tietze transformations. Neumann used this method in [47] to find a presentation for left zero semigroups.

There are four basic types of transformation which are called *elementary Tietze transformations*:

(T1) If $r = s$ holds in the semigroup defined by $\langle A \mid R \rangle$, then transform

$$\langle A \mid R \rangle \text{ to } \langle A \mid R \cup \{ r = s \} \rangle \quad (\text{“adding a new relation”}).$$

(T2) If $r = s$ holds in the semigroup defined by $\langle A \mid R \rangle$, then transform

$$\langle A \mid R \cup \{ r = s \} \rangle \text{ to } \langle A \mid R \rangle \quad (\text{“removing a relation”}).$$

(T3) If $b \notin A$ and $w \in A^+$, then transform

$$\langle A \mid R \rangle \text{ to } \langle A \cup \{b\} \mid R \cup \{ w = b \} \rangle \quad (\text{“adding a new generator”}).$$

(T4) If $b \in A$ and $w = b \in R$ such that $w \in (A \setminus \{b\})^+$, then transform

$$\langle A \mid R \rangle \text{ to } \langle A \setminus \{b\} \mid \bar{R} \rangle \quad (\text{“removing a generator”})$$

where \bar{R} is obtained from $R \setminus \{ w = b \}$ by replacing b by w for every occurrence of b .

These are analogous to Tietze transformations in the theory of groups (see, for example, [36]).

Theorem 1.11 *Two finite semigroup (respectively, group) presentations define the same semigroup (respectively, group) if and only if one can be obtained from the other one by applying a finite number of applications of elementary Tietze transformations.* ■

For the proof of the semigroup case, see [55, Proposition 2.5], and of the group case, see [36, Proposition 4.5 and 4.6].

1.5 Finiteness conditions for semigroups

Let \mathcal{P} be a property of semigroups. We say that \mathcal{P} is a *finiteness condition* if every finite semigroup has the property \mathcal{P} . Since every finite semigroup is finitely generated, finite generation is a finiteness condition.

Let S be a semigroup and let T be a subsemigroup of S . We say that S is a *small extension* of T if the set $S \setminus T$ is finite. Moreover, the cardinal of the set $S \setminus T$ is called the *index* of T in S , and is denoted by $[S : T]$.

Theorem 1.12 *Let S be a semigroup and let T be a subsemigroup of S with finite index. Then S is finitely generated if and only if T is finitely generated. ■*

This was first proved by Jura (see [37]), and reproved in [15] (see also [56, Theorem 1.1]).

For a semigroup S , we may consider the set S as a generating set for itself and the multiplication table (Cayley table) as a set of relations, and so if a semigroup is finite, then it is finitely presented. Therefore, finite presentability is a finiteness condition.

Theorem 1.13 *Let S be a semigroup and let T be a subsemigroup of S with finite index. Then S is finitely presented if and only if T is finitely presented. ■*

For the proof, see [56, Theorem 1.3].

Let a be an element of a semigroup S , and consider the monogenic subsemigroup $\langle a \rangle = \{a, a^2, a^3, \dots\}$ generated by a . If there are some repetitions in the list a, a^2, a^3, \dots , that is there are integers $m \neq n$ such that $a^m = a^n$, then the monogenic subsemigroup $\langle a \rangle$ is finite. We say that a semigroup is *periodic* if all

monogenic subsemigroups are finite. Since finite semigroups are periodic, being periodic is a finiteness condition.

Let A be an infinite set. Since \mathcal{SL}_A consists only of idempotents, it follows that \mathcal{SL}_A is periodic. The (left/right) zero semigroups and rectangular bands are periodic but not necessarily finite.

Let \mathcal{P} be a class of semigroups. Then a semigroup S is said to be *locally \mathcal{P}* if every finitely generated subsemigroup of S belongs to \mathcal{P} . Since every finite semigroup is locally finite, local finiteness is a finiteness condition. Moreover, since every finite semigroup is locally finitely presented, local finite presentability is a finiteness condition.

Note that, since finite semigroups are finitely presented, locally finite semigroups are locally finitely presented.

Example 1.14 *Let A be any non-empty set. Then the free semilattice \mathcal{SL}_A over A is locally finite and locally finitely presented.*

Proof If B a finite subset of \mathcal{SL}_A , then it is clear that $\langle B \rangle$ is isomorphic to a subsemigroup of \mathcal{SL}_B which is finite. Therefore $\langle B \rangle$ is finite, and so $\langle B \rangle$ is finitely presented, as required. ■

Example 1.15 *Rectangular bands are locally finite and locally finitely presented.*

Proof Let X be a finite subset of a rectangular band $R = \{(i, j) \mid i \in I, j \in J\}$. Then define

$$I' = \{i \in I \mid \text{there exists } j \in J \text{ such that } (i, j) \in X\} \text{ and}$$

$$J' = \{j \in J \mid \text{there exists } i \in I \text{ such that } (i, j) \in X\}.$$

Since I' and J' are finite, the subsemigroup $R' = \{(i, j) \mid i \in I', j \in J'\}$ of R , which is also a rectangular band, is finite. Moreover, R' contains X , in fact $\langle X \rangle = R'$. Therefore, rectangular band are locally finite, and so locally finitely presented. ■

A semigroup S is said to be *residually finite* if, for any two elements $s \neq t$ of S , there exists a finite semigroup S' and a homomorphism ψ from S onto S' such that $\psi(s) \neq \psi(t)$. Since every finite semigroup is residually finite (by considering the identity homomorphism from S to itself), residual finiteness is also a finiteness condition.

Let S and T be two semigroups and let $\phi : S \longrightarrow T$ be a homomorphism. Since

$$\ker \phi = \{ (s, t) \in S \times S \mid \phi(s) = \phi(t) \}$$

is a congruence on S , we have the following result:

Proposition 1.16 *A semigroup S is residually finite if and only if, for any two distinct elements $s, t \in S$, there exists a congruence ρ on S such that $(s, t) \notin \rho$ and ρ has finitely many equivalence classes.* ■

A semigroup S is *hopfian* if every onto endomorphism of S is an automorphism. Since every finite semigroup is hopfian, hopficity is also a finiteness condition.

Let S be a semigroup and let A be a generating set for S . We say that S has a *soluble word problem* with respect to A if there exists an algorithm which, for any two words $u, v \in A^+$, decides in a finite numbers of steps, whether the relation $u = v$ holds in S or not. If S is finitely generated then it is a well-known fact that the solubility of the word problem does not depend on the choice of the generating set for S . Therefore solubility of the word problem is a property of finitely generated semigroups. Moreover, since every finite semigroup has a soluble word problem (see [7] or [53]), having a soluble word problem is a finiteness condition.

Theorem 1.17 *Let S be a semigroup and let T be a subsemigroup of S with finite index. Then*

- (i) *S is periodic if and only if T is periodic;*

- (ii) S is locally finite if and only if T is locally finite;
- (iii) S is locally finite presented if and only if T is locally finite presented;
- (iv) S is residually finite if and only if T is residually finite.
- (v) In addition, if S is finitely generated, then S has a soluble word problem if and only if T has a soluble word problem. ■

For the proof of Theorem 1.17(iv) see [58, Corollary 4.6], and for the other proofs of Theorem 1.17 see [56, Theorem 5.1].

A non-empty subset I of a semigroup S is called a *left ideal* if $SI \subseteq I$ and is called a *right ideal* if $IS \subseteq I$. If I is both a left and a right ideal, we say that I is a (two-sided) *ideal*. A (one or two-sided) ideal I such that $I \neq S$ and $I \neq \{0\}$ (if S has a zero) is called *proper*. A (right/left) ideal I of a semigroup S is said to be *minimal* if I does not contain other (right/left) ideals of S . Clearly, having a minimal (right/left) ideal and having finitely many (right/left) ideals are finiteness conditions. If we define the index of an ideal as its index as a subsemigroup, having finite index is also a finiteness condition.

Note that not every semigroup has a minimal ideal. For example, the semigroup $(\mathbb{N}, +)$ has no minimal ideal (see [42]). If a minimal ideal exists then it is unique (see [42]). However minimal right and left ideals are not necessarily unique. For example, for each $i_0 \in I$ the set $\{(i_0, j) \mid j \in J\}$ is a minimal right ideal of the rectangular band $R = \{(i, j) \mid i \in I, j \in J\}$.

Next we give some results from [56].

Theorem 1.18 *Let S be a semigroup, and let T be a subsemigroup of S with finite index.*

- (i) *If T has a minimal right ideal then S also has a minimal right ideal.*
- (ii) *If every right ideal of S has finite index then every right ideal of T has finite index.*
- (iii) *S has finitely many right ideals if and only if T has finitely many right ideals.* ■

For the proof, see [56, Theorem 10.3], [56, Theorem 10.5] and [56, Theorem 10.4], respectively.

The converses of Theorem 1.18(i) and 1.18(ii) are not necessarily true. For this, consider the semigroup $(\mathbb{N}, +)$ has no minimal right ideal but $(\mathbb{N} \cup \{0\}, +)$ has one, namely $\{0\}$. Let T be an infinite group. Then T has only one right ideal, which is itself. However, the right ideal $\{0\}$ of the semigroup $T \cup \{0\}$ has index $|T|$ which is infinite.

1.6 Homology groups of monoids

Let R be a ring with an identity 1 and let A be an additive group. We say that A is a (left) R -module if there is an action $\circ : R \times A \longrightarrow A$ denoted by $r \circ a \mapsto ra$ such that the following are satisfied:

$$\begin{aligned} \text{(i)} \quad r(a_1 + a_2) &= ra_1 + ra_2 & \text{(ii)} \quad (r_1 + r_2)a &= r_1a + r_2a \\ \text{(iii)} \quad (r_1r_2)a &= r_1(r_2a) & \text{(iv)} \quad 1a &= a \end{aligned}$$

for all $a, a_1, a_2 \in A$ and all $r, r_1, r_2 \in R$. Similarly, we define a right R -module.

Let M be a monoid written multiplicatively and let \mathbb{Z} denote the integer numbers. The *integer monoid ring* $\mathbb{Z}M$ of M is defined as follows. Its underlying abelian group is the free abelian group on the set of elements of M as basis; the product of two basis elements is given by the product in M . Thus the elements of the monoid ring $\mathbb{Z}M$ are sums $\sum_{x \in M} n_x x$, where n_x are integers which are all zero except finitely many. The multiplication is given by

$$\left(\sum_{x \in M} n_x x \right) \left(\sum_{y \in M} m_y y \right) = \sum_{x, y \in M} (n_x m_y) xy.$$

The definition of a monoid ring is similar to a group ring (see, for example, [28]). If we consider this multiplication as an action on itself, $\mathbb{Z}M$ becomes a (left/right) $\mathbb{Z}M$ -module.

Let M be a monoid. Then define an action by $xn = n$ for each $x \in M$ and $n \in \mathbb{Z}$. With respect to this action, \mathbb{Z} is a $\mathbb{Z}M$ -module, and it is called the *trivial left $\mathbb{Z}M$ -module*. Similarly, we define the *trivial right $\mathbb{Z}M$ -module*, \mathbb{Z} .

Let A and B be two R -module. An R -map $f : A \rightarrow B$ satisfies $f(x + y) = f(x) + f(y)$ and $f(rx) = r(f(x))$ for all $x, y \in A$ and $r \in R$. For example, consider the monoid ring $\mathbb{Z}M$ of a monoid M and consider the $\mathbb{Z}M$ -modules: $\mathbb{Z}M$ and the trivial left module \mathbb{Z} . Then define a map $\varepsilon : \mathbb{Z}M \rightarrow \mathbb{Z}$ by $\sum_{x \in M} n_x x \mapsto \sum_{x \in M} n_x$. It is an $\mathbb{Z}M$ -map (also a ring map) and is called the *augmentation map*.

Let $f : A \rightarrow B$ be an R -map. Then we define

$$\ker f = \{ x \in A \mid f(x) = 0_B \} \text{ and } \operatorname{im} f = \{ y \in B \mid \exists x \in A \text{ with } f(x) = y \}$$

where 0_B is the identity element of the additive group B .

We say that two R -maps $A \xrightarrow{f} B \xrightarrow{g} C$ are *exact* at B if $\ker g = \operatorname{im} f$. The sequence

$$\cdots \rightarrow B_{n+1} \xrightarrow{f_{n+1}} B_n \xrightarrow{f_n} B_{n-1} \cdots$$

of R -maps and R -modules is called a *chain complex* if $\operatorname{im} f_{n+1} \subseteq \ker f_n$ for all n and an *exact sequence* if $\operatorname{im} f_{n+1} = \ker f_n$ for all n .

Proposition 1.19 *The sequence*

$$\mathbf{B} = \cdots \rightarrow B_{n+1} \xrightarrow{f_{n+1}} B_n \xrightarrow{f_n} B_{n-1} \cdots$$

of R -maps and R -modules is exact if and only if, for all n and all $x_n \in B_n$, $f_{n-1}(f_n(x_n)) = 0_{n-2}$ where 0_{n-2} is the identity element of B_{n-2} ; and there exists a sequence of homomorphisms $g_n : B_n \rightarrow B_{n+1}$ such that $f_{n+1}g_n + g_{n-1}f_n = I_{B_n}$ where I_{B_n} is the identity map on B_n . ■

The proof is standard and can be found in any homological algebra book as a simple exercise (see, for example, [52]). We call the sequence of maps $g_n : B_n \rightarrow B_{n+1}$ a *contracting homotopy* for \mathbf{B} .

Let A be a R -module and let X be a subset of A . The smallest submodule of A containing X , which is an R -module with respect to the same action on A , is called the *module generated by X* , and is denoted by $\langle X \rangle$. We say that a R -module F is *free* if there exists a subset X of F such that $F = \langle X \rangle$ and $\sum_{x \in X} n_x x = 0$ implies $n_x = 0$ for all $x \in X$. In this case, X is called a *basis* for the free module F . A *projective R -module* is a summand of a free R -module.

A *free (projective) resolution* of an R -module A is an exact sequence

$$\cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0$$

where each F_n is a free (projective) R -module ($n \geq 0$).

Let M a monoid. Then the (left) *bar resolution* of the (left) trivial $\mathbb{Z}M$ -module \mathbb{Z} is an exact sequence

$$\cdots \longrightarrow B_n \xrightarrow{\partial_n} B_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} B_1 \xrightarrow{\partial_1} B_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

in which each B_n is a (left) free $\mathbb{Z}M$ -module on the set of formal symbols $[x_1|x_2|\dots|x_n]$ with $x_i \in M$ and $x_i \neq 1_M$ (where 1_M is the identity of M) and B_0 is a (left) free $\mathbb{Z}M$ -module on the unique symbol $[\]$, \mathbb{Z} is a trivial $\mathbb{Z}M$ -module, ε is the augmentation map and the map $\partial_n : B_n \longrightarrow B_{n-1}$ is defined by

$$\begin{aligned} \partial_n([x_1|\dots|x_n]) &= x_1[x_2|\dots|x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1|\dots|x_i x_{i+1}|\dots|x_n] \\ &\quad + (-1)^n [x_1|\dots|x_{n-1}]. \end{aligned}$$

Remark. In order that ∂_n be defined, we require that $[x_1|\dots|x_n] = 0$ whenever some $x_i = 1$.

The (left) *standard resolution* of a (left) trivial $\mathbb{Z}M$ -module \mathbb{Z} is an exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{\delta_n} P_{n-1} \xrightarrow{\delta_{n-1}} \cdots \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

in which every P_n is a (left) free $\mathbb{Z}M$ -module on the set of formal symbols $[x_1|x_2|\dots|x_n]$ with $x_i \in M$ (x_i 's are allowed to be 1_M), \mathbb{Z} is a trivial $\mathbb{Z}M$ -module,

ε is the augmentation map and the map $\delta_n : P_n \longrightarrow P_{n-1}$ is defined, as ∂_n , by

$$\begin{aligned} \delta_n([x_1 | \dots | x_n]) &= x_1[x_2 | \dots | x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1 | \dots | x_i x_{i+1} | \dots | x_n] \\ &\quad + (-1)^n [x_1 | \dots | x_{n-1}]. \end{aligned}$$

As shown in [52, Theorems 10.19 and 10.23] for groups, it is similarly shown that both the bar and standard resolutions are, indeed, resolutions of a (left) trivial $\mathbb{Z}M$ -module \mathbb{Z} . Note that the bar resolutions and some other resolutions for monoids were also considered and compared in [23].

Let A be a right R -module, let B be a left R -module, let F be the free abelian group on the Cartesian product of sets $A \times B$ and let N be the subgroup of F generated by the set

$$\begin{aligned} \{ (x + x', y) - (x, y) - (x', y), (x, y + y') - (x, y) - (x, y'), (xr, y) - (x, ry) | \\ x, x' \in A, y, y' \in B, r \in R \}. \end{aligned}$$

Then the *tensor product*, $A \otimes_R B$ of R -modules A and B is the quotient group F/N . We usually use $A \otimes B$ instead of $A \otimes_R B$ and we denote the coset $(x, y) + N$ by $x \otimes y$.

Theorem 1.20 *Let $f : A \longrightarrow B$ be an R -map of right R -modules and $g : C \longrightarrow D$ be an R -map of left R -modules. Then there is a unique (group) homomorphism $A \otimes C \longrightarrow B \otimes D$ with $x \otimes y \mapsto f(x) \otimes g(y)$. ■*

For a proof, see [52, Theorem 1.5] or [31, Corollary 5.3].

The homomorphism $A \otimes C \longrightarrow B \otimes D$ sending $x \otimes y \mapsto f(x) \otimes g(y)$ is denoted by $f \otimes g$.

Proposition 1.21 *Let M be a monoid and let F be a free left $\mathbb{Z}M$ -module on the basis X . If \mathbb{Z} is the right trivial $\mathbb{Z}M$ -module, then the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}M} F$ is isomorphic to the free abelian group on the basis X . ■*

The result is a special case of Corollary 5.13 in [31] and Proposition 3.4 in [24].

Let M be a monoid and let

$$\cdots \longrightarrow F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

be a left projective (free) resolution of the trivial left $\mathbb{Z}M$ -module \mathbb{Z} . Then apply the tensor product $\mathbb{Z} \otimes_{\mathbb{Z}M} -$, where \mathbb{Z} is the right trivial $\mathbb{Z}M$ -module, to this resolution. Then we have the chain complex

$$\begin{aligned} \cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}M} F_n \xrightarrow{1 \otimes f_n} \mathbb{Z} \otimes_{\mathbb{Z}M} F_{n-1} \xrightarrow{1 \otimes f_{n-1}} \cdots \xrightarrow{1 \otimes f_2} \mathbb{Z} \otimes_{\mathbb{Z}M} F_1 \xrightarrow{1 \otimes f_1} \\ \mathbb{Z} \otimes_{\mathbb{Z}M} F_0 \xrightarrow{1 \otimes \varepsilon} \mathbb{Z} \otimes_{\mathbb{Z}M} \mathbb{Z} \longrightarrow 0 \end{aligned} \quad (1)$$

of abelian groups where 1 is the identity R -map of the right trivial module \mathbb{Z} (since $\text{im}(1 \otimes f_{n+1}) \subseteq \ker(1 \otimes f_n)$). Therefore, we define the n th *left integral homology group* of a monoid M , $H_n(M)$, to be

$$H_n(M) = \ker(1 \otimes f_n) / \text{im}(1 \otimes f_{n+1}).$$

We usually say the n th homology of M for $H_n(M)$. The n th homology of groups is defined similarly.

As in the group case (see Chapter II Section 3 in [11]), if the chain complex in (1) is obtained from the standard resolution, then, by Proposition 1.21 and the action on the right trivial module \mathbb{Z} , we may consider the chain complex as follows:

$$\cdots \longrightarrow \bar{P}_n \xrightarrow{\bar{\delta}_n} \bar{P}_{n-1} \xrightarrow{\bar{\delta}_{n-1}} \cdots \xrightarrow{\bar{\delta}_2} \bar{P}_1 \xrightarrow{\bar{\delta}_1} \bar{P}_0 \longrightarrow 0 \quad (2)$$

where $\bar{P}_n \cong \mathbb{Z} \otimes_{\mathbb{Z}M} P_n$ is the free abelian group on all $[x_1 | \dots | x_n]$ with $x_i \in M$, and the group homomorphism $\bar{\delta}_n = 1 \otimes_{\mathbb{Z}M} \delta_n$ is given by

$$\begin{aligned} \bar{\delta}_n([x_1 | \dots | x_n]) &= [x_2 | \dots | x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1 | \dots | x_i x_{i+1} | \dots | x_n] \\ &\quad + (-1)^n [x_1 | \dots | x_{n-1}]. \end{aligned}$$

Thus, the n th homology of M is $H_n(M) = \ker(\bar{\delta}_n) / \text{im}(\bar{\delta}_{n+1})$.

Proposition 1.22 *Let M be a monoid with identity 1_M and let M^1 be the monoid obtained from M by adjoining an identity 1 . Then $H_2(M) = H_2(M^1)$.*

Proof We consider the relevant part of the standard resolution of \mathbb{Z}

$$P_3 \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1,$$

where P_3 , P_2 and P_1 are the free $\mathbb{Z}M^1$ -modules on the set of formal symbols $[x|y|z]$, $[x|y]$ and $[x]$ ($x, y, z \in M^1$) respectively, and δ_3 and δ_2 are given by

$$\begin{aligned}\delta_3([x|y|z]) &= x[y|z] - [xy|z] + [x|yz] - [x|y], \\ \delta_2([x|y]) &= x[y] - [xy] + [x].\end{aligned}$$

As explained in (2), applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}M^1} -$, where \mathbb{Z} is a trivial right $\mathbb{Z}M^1$ -module yields the chain complex

$$\bar{P}_3 \xrightarrow{\bar{\delta}_3} \bar{P}_2 \xrightarrow{\bar{\delta}_2} \bar{P}_1$$

where \bar{P}_3 , \bar{P}_2 and \bar{P}_1 are the free abelian groups on the set of all formal symbols $[x|y|z]$, $[x|y]$ and $[x]$ ($x, y, z \in M^1$) respectively, and the group homomorphisms $\bar{\delta}_3$ and $\bar{\delta}_2$ are given by

$$\begin{aligned}\bar{\delta}_3([x|y|z]) &= [y|z] - [xy|z] + [x|yz] - [x|y], \\ \bar{\delta}_2([x|y]) &= [y] - [xy] + [x].\end{aligned}$$

We find a basis set for $\ker(\bar{\delta}_2)$. Each $\alpha \in \bar{P}_2$ has the form;

$$\alpha = \alpha(1, 1)[1|1] + \sum_{x \in M} (\alpha(1, x)[1|x] + \alpha(x, 1)[x|1]) + \sum_{x, y \in M} \alpha(x, y)[x|y]$$

where each $\alpha(x, y)$ is integer. It follows that $\alpha \in \ker(\bar{\delta}_2)$ if and only if

$$0 = \bar{\delta}_2(\alpha) = \left(\alpha(1, 1) + \sum_{x \in M} (\alpha(1, x) + \alpha(x, 1)) \right) [1|1] + \bar{\delta}_2 \left(\sum_{x, y \in M} \alpha(x, y)[x|y] \right)$$

or, equivalently, if and only if

$$\alpha(1, 1) = - \left(\sum_{x \in M} (\alpha(1, x) + \alpha(x, 1)) \right) \quad \text{and} \quad \bar{\delta}_2 \left(\sum_{x, y \in M} \alpha(x, y)[x|y] \right) = 0.$$

The equation $\bar{\delta}_2(\sum_{x,y \in M} \alpha(x,y)[x|y]) = 0$ gives a basis for $\ker(\bar{\delta}_2^M)$ where $H_2(M) = \ker(\bar{\delta}_2^M)/\text{im}(\bar{\delta}_3^M)$ by considering the standard resolution of the trivial $\mathbb{Z}M$ -module \mathbb{Z} . Say $\{U_i \mid i \in I\}$ is a basis for $\ker(\bar{\delta}_2^M)$. From the equation

$$\alpha(1,1) = -\left(\sum_{x \in M} (\alpha(1,x) + \alpha(x,1))\right),$$

we obtain the generators: $U_{1,x} = [1|x] - [1|1]$ and $U_{x,1} = [x|1] - [1|1]$ for each $x \in X$. Therefore the set

$$Z = \{U_i, U_{1,x}, U_{x,1} \mid i \in I, x \in M\}$$

is a basis for $\ker(\bar{\delta}_2)$.

Now we find a generating set for $\text{im}(\bar{\delta}_3)$. First consider the image of the generators of \bar{P}_3 which contains 1 under $\bar{\delta}_3$. They are:

$$\begin{aligned} \bar{\delta}_3([1|1|1]) &= 0, \\ \bar{\delta}_3([1|1|x]) &= [1|x] - [1|x] + [1|x] - [1|1] = [1|x] - [1|1] = V_{1,x}, \\ \bar{\delta}_3([1|x|1]) &= [x|1] - [x|1] + [1|x] - [1|x] = 0, \\ \bar{\delta}_3([x|1|1]) &= [1|1] - [x|1] + [x|1] - [x|1] = [1|1] - [x|1] = V_{x,1}, \\ \bar{\delta}_3([x|y|1]) &= [y|1] - [xy|1] + [x|y] - [x|y] = [y|1] - [xy|1] = V_{1,x,y}, \\ \bar{\delta}_3([x|1|y]) &= [1|y] - [x|y] + [x|y] - [x|1] = [1|y] - [x|1] = V_{2,x,y}, \\ \bar{\delta}_3([1|x|y]) &= [x|y] - [x|y] + [1|xy] - [1|x] = [1|xy] - [1|x] = V_{3,x,y} \end{aligned}$$

where $x, y \in M$. Therefore, since $V_{1,x,y} = V_{xy,1} - V_{y,1}$, $V_{2,x,y} = V_{x,1} + V_{i,y}$ and $V_{3,x,y} = V_{1,xy} - V_{1,x}$, it follows that if $\{V_j \mid j \in J\}$ is a generating set for $\text{im}(\bar{\delta}_3^M)$ then the set $B = \{V_j, V_{1,x}, V_{x,1} \mid j \in J, x \in M\}$ is a generating set for $\text{im}(\bar{\delta}_3)$.

Since $U_{1,x} = V_{1,x}$ and $U_{x,1} = -V_{x,1}$, it follows that $H_2(M) = H_2(M^1)$, as required. ■

More generally, since the bar resolution of the trivial $\mathbb{Z}M^1$ -module \mathbb{Z} is also the standard resolution of the trivial $\mathbb{Z}M$ -module \mathbb{Z} and since the homology does not depend on the choice of resolution (see [52]), we have the following result:

Proposition 1.23 *Let M be a monoid. Then $H_n(M) = H_n(M^1)$ for $n \geq 0$.* ■

1.7 Efficiency of semigroups

The *deficiency* of a finite semigroup (monoid or group) presentation $\mathcal{P} = \langle A \mid R \rangle$ is defined to be $|R| - |A|$, and is denoted by $\text{def}(\mathcal{P})$. The *semigroup deficiency* of a finitely presented semigroup S , $\text{def}_S(S)$, is given by

$$\text{def}_S(S) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite semigroup presentation for } S \}.$$

The *monoid deficiency* of a finitely presented monoid M is given by

$$\text{def}_M(M) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite monoid presentation for } M \}.$$

The *group deficiency* of a finitely presented group G is given by

$$\text{def}_G(G) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite group presentation for } G \}.$$

Therefore a finitely presented group G has three deficiencies, namely $\text{def}_G(G)$, $\text{def}_M(G)$ and $\text{def}_S(G)$; and a finitely presented monoid M has two deficiencies, namely $\text{def}_M(M)$ and $\text{def}_S(M)$.

Let \mathcal{P} be a finite semigroup presentation for a finitely presented semigroup S . We say that \mathcal{P} is a *minimal semigroup presentation* if $\text{def}(\mathcal{P}) = \text{def}_S(S)$. Similarly, we define a *minimal monoid presentation* for a monoid and a *minimal group presentation* for a group.

It is a well-known fact that if a group presentation $\langle A \mid R \rangle$ defines a finite group G , then $|R| - |A| \geq 0$ (see for example [43]). Moreover, if $\langle A \mid R \rangle$ defines a finite semigroup S , then, since the group G defined by the presentation (as a group presentation) is a homomorphic image of S (see Proposition 1.8(i)), G is finite, and so we have $|R| - |A| \geq 0$.

After Schur's study in [60], it is a well-known fact that if a group presentation $\langle A \mid R \rangle$ defines a finite group G , then we have

$$|R| - |A| \geq \text{rank}(H_2(G))$$

(see, for example [52, Corollary 10.17]).

We say that a (finite) group G is *efficient* if G has a group presentation $\langle A | R \rangle$ such that $|R| - |A| = \text{rank}(H_2(G))$ and that it is *inefficient* otherwise. A group presentation $\langle A | R \rangle$ of G such that $|R| - |A| = \text{rank}(H_2(G))$ is called an *efficient group presentation*. Many families of groups are known to be efficient groups (see [6], [8], [13], [20], etc.). The first examples of inefficient groups were given by Swan in [64]. More examples of inefficient groups can be found in [33], [41] and [50].

Recently, Steve Pride showed that if $\langle A | R \rangle$ is a finite monoid presentation for a finite monoid M , then

$$|R| - |A| \geq \text{rank}(H_2(M)).$$

Since a semigroup presentation for a semigroup S can be considered as a monoid presentation for the monoid S^1 , it follows that if $\langle A | R \rangle$ is a finite semigroup presentation for a finite semigroup S , then

$$|R| - |A| \geq \text{rank}(H_2(S^1)).$$

We call a finite semigroup (monoid) S *efficient* as a semigroup (monoid) if S has a semigroup (monoid) presentation $\langle A | R \rangle$ such that $|R| - |A| = \text{rank}(H_2(S^1))$ and *inefficient* otherwise. A semigroup (monoid) presentation $\langle A | R \rangle$ of S such that $|R| - |A| = \text{rank}(H_2(S^1))$ is called an *efficient semigroup (monoid) presentation*.

Therefore there are three potentially different notions of efficiency for a group and two for a monoid.

Note that if a semigroup (monoid or group) S is efficient, then a semigroup (monoid or group) presentation is an efficient presentation for S if and only if it is a minimal presentation for S .

1.8 Finite presentability

One of the most important research fields in combinatorial group theory is the study of subgroups of finite index in finitely presented groups. The most important result in this field is the Reidemeister-Schreier Theorem about the method for determining presentations for subgroups of finite index in finitely presented groups. As a consequence of this, subgroups of finite index in finitely presented groups are finitely presented.

An analogous theory for semigroups has been one of the major research fields in combinatorial semigroup theory. The following questions for a finitely presented semigroup S were considered.

- (i) Is every subsemigroup (one sided ideal or two sided ideal) of S finitely generated?
- (ii) Is every subsemigroup (one sided ideal or two sided ideal) of S which is finitely generated (as a semigroup) finitely presented?
- (iii) Is every subsemigroup (one sided ideal or two sided ideal) of finite index in S finitely presented?

The answer of the first question is, in general, negative. To see this, consider the free semigroup F on two generators $\{a, b\}$ and the set I of all words containing both a and b . I is obviously a subsemigroup (one-sided ideal, two-sided ideal) of F . However, the words ab^i ($i \geq 1$), are indecomposable in I , and so I is not finitely generated. (Note that this example was given in [18, Example 3.4].) It is obvious that I has infinite index in S (by considering $\{a^i \mid i \geq 1\} \subset S \setminus I$). We change the first question as below:

- (i') Is every subsemigroup (one sided ideal or two sided ideal) of finite index in S finitely generated?

The answer of this question is, in general, yes. It was first considered and answered by Jura (see [37]). Let S be finitely generated semigroup and let T be a subsemigroup of finite index in S . In [15] (see also [56, Theorem 1.1]) it is shown

that if the set A generates S , then the set

$$X = \{ s_1 a s_2 \mid s_1, s_2 \in {}^1S \setminus T, a \in A, s_1 a, s_1 a s_2 \in T \},$$

where 1S is the monoid obtained from S by adjoining an identity 1 if necessary, generates T . (Note that the monoids 1S and S^1 may not be the same.) Hence T is finitely generated as well.

The answer of the second question is, in general, negative. To see this for the case of subsemigroups, consider the free semigroup F on the generators $\{a, b, c\}$, and the subsemigroup I generated by $X = \{ba, ba^2, a^3, ac, a^2c\}$. I is obviously finitely generated, but it is shown (see [18, Example 4.5]) that I is not finitely presented. Note that, if a semigroup is finitely presented with respect to one generating set, then it is finitely presented with respect to any finite generating set (by considering Tietze transformations). Let $\{v, w, x, y, z\}$ be an alphabet in one to one correspondence with X . In [18] it is shown that the relations $vx^n z = wx^n y$ ($n \geq 0$) hold in I and any set of defining relations for I must include all the relations $vx^n z = wx^n y$ ($n \geq 0$).

For the case of ideals, we have the following result:

Theorem 1.24 *Let S be the semigroup defined by the presentation*

$$\langle a, x, y \mid ax = xy, xa = yx, ay = xy^2, ya = yx \rangle,$$

and let T be the subsemigroup of S generated by $\{x, y\}$. Then T is a two-sided ideal of S , but T is not finitely presented. ■

For a proof, see [19, Theorem 3.1].

To prove the above theorem and some of the following theorems the authors used a method based on the idea of a (Reidemeister-Schreier) rewriting mappings. The method was developed for finding presentations for subgroups of finite index in finitely presented groups (for more details, see [45]). A general theory of Reidemeister-Schreier type rewriting for semigroups has been developed in [16].

Since rewriting mappings are also important in this thesis, we define rewriting mappings here.

Let $\mathcal{P} = \langle A \mid R \rangle$ be an arbitrary presentation, let S be the semigroup defined by \mathcal{P} , and let

$$X = \{ w_i \mid i \in I \} \subseteq A^+$$

be any set of words. If we consider X as a subset of S , it generates a subsemigroup of S . Let T denote the subsemigroup of S generated by X . We first introduce a new alphabet

$$B = \{ b_i \mid i \in I \}$$

in one to one correspondence with X . The *representation mapping* is the unique homomorphism $\Psi : B^+ \longrightarrow A^+$ extending the mapping $b_i \mapsto w_i$, $i \in I$. For a subset T of S , we define the set $L(A, T)$ to be

$$L(A, T) = \{ w \in A^+ \mid w \text{ represents an element of } T \}.$$

A *rewriting mapping* is a mapping $\Phi : L(A, T) \longrightarrow B^+$ such that the relation

$$\Psi(\Phi(w)) = w$$

holds in S for all $w \in L(A, T)$. Note that a rewriting mapping always exists (see [19]).

By using the method of rewriting mappings, the following result was obtained:

Theorem 1.25 *Let S be a finitely presented semigroup such that S has finitely many minimal left ideals and finitely many minimal right ideals. Suppose we are given a word representing an element of some minimal left ideal L and of some minimal right ideal R . Then the group $R \cap L$ is finitely presented. There exists an effective algorithm to determine a presentation for $R \cap L$ and a presentation for the minimal two-sided ideal of S . ■*

For a proof, see [16, Theorem 6.1]. It is worth to note that the rewriting mapping which was introduced in [16] is a homomorphism.

In [15, Theorem 2.1] a presentation which is not finite is given for subsemigroups. When a subsemigroup is an ideal with finite index, we have the following result from [15, Theorem 4.1] and [18, Theorem 2.1]:

Theorem 1.26 *Let S be a semigroup, and let I be a two-sided ideal of finite index in S . Then S is finitely presented if and only if I is finitely presented. ■*

Moreover, from [18, Theorem 5.1], we have the following result for one-sided ideals:

Theorem 1.27 *Let S be a semigroup, and let T be a right (left) ideal of finite index in S . If S is finitely presented, then T is finitely presented. ■*

A complete answer for the last question concerning subsemigroups had been open until the study of N. Ruškuc was published which we stated in Theorem 1.13 in Section 1.5.

Although all the answers of three questions has been given, finitely presentability of an arbitrary subsemigroup T (not necessarily of finite index) of finitely presented semigroup S is still an open problem. Before we give some results related to this problem, we define co-index.

Let S be a monoid, and let X be a non-empty subset of S . For $s \in S$, we say that Xs is a (*right*) *coset* if there exists $t \in S$ such that $Xst = X$. The number of disjoint (*right*) cosets of X in S is said to be (*right*) *co-index*. (See for more details [57].)

Theorem 1.28 *A subgroup of finite co-index in a finitely presented monoid is finitely presented as a monoid, and so as a group. ■*

For a proof, see [57, Corollary 2.11]. It is obvious that the above result can be extended to subgroups of semigroups.

Is it always necessary to have finite (co-)index? The answer of this question was given in [72].

Theorem 1.29 *Let S be a monoid with a finite presentation $\langle A \mid R \rangle$ such that $R \subset A^+ \times \{\epsilon\}$ (or equivalently, $R \subset \{\epsilon\} \times A^+$) where ϵ is the empty-word. Then the group of units $U(S)$, consisting of all invertible elements of S , is finitely presented.* ■

Note that the co-index of $U(S)$ is either 1 or infinite (see [42, Proposition 2.3.7] or [57]). Also note that the above result can not be generalised to an arbitrary finitely presented monoid (see [57, Proposition 3.3]).

A monoid S is said to be regular if for every $s \in S$ there exists $t \in S$ such that $sts = s$.

Theorem 1.30 *Let S be a regular monoid with finitely many right and left ideals. Then S is finitely presented if and only if all maximal subgroups of S are finitely presented.* ■

For a proof, see [57, Theorem 4.1]. The most important result in [57] for this thesis is the one concerning ideal extension.

Let T and U be semigroups. An *ideal extension of I by T* is an semigroup S such that I is an ideal of S and the Rees quotient S/I is isomorphic to T .

Proposition 1.31 *An ideal extension of a finitely presented semigroup by another finitely presented semigroup is finitely presented.* ■

For a proof, see [57, Proposition 4.4].

As finite presentability of subsemigroups, finite presentability of various semigroup constructions of semigroups is also an important research field in combinatorial semigroup theory. It is well-known that the direct product of two groups (monoids) defined by the group (monoid) presentations $\langle A \mid R \rangle$ and $\langle B \mid Q \rangle$ may be defined by the following group (monoid) presentation

$$\langle A, B \mid R, Q, ab = ba \ (a \in A, b \in B) \rangle.$$

For a proof for groups, see [36, Proposition 4.4], and a proof for monoids is similar.

Therefore the direct product of finitely presented groups (monoids) is finitely presented. This is, in general, not true for direct products of finitely presented semigroups. It is well-known that the direct product $\mathbb{N} \times \mathbb{N}$ of the additive group $\mathbb{N} = \{1, 2, 3, \dots\}$ is not finitely presented even not finitely generated. Finite generation and finite presentability of direct products of semigroups have been considered in [49].

Theorem 1.32 *Let S and T be two infinite semigroups. Then $S \times T$ is finitely generated if and only if both S and T are finitely generated and $S^2 = S$ and $T^2 = T$.* ■

We define $S^2 = \{ss' \mid s, s' \in S\}$. For a proof, see [49, Theorem 2.1].

Let $\mathcal{P} = \langle A \mid R \rangle$ be a semigroup presentation, let S be the semigroup defined by it, and let $w_1, w_2 \in A^+$ be arbitrary words. The pair (w_1, w_2) is called a *critical pair* (for S with respect to \mathcal{P}) if the following conditions are satisfied:

- (i) the relation $w_1 = w_2$ holds in S ;
- (ii) for every sequence $w_1 \equiv \alpha_1, \alpha_2, \dots, \alpha_k \equiv w_2$, where α_{i+1} is obtained from α_i by applying one relation from R , there exists j such that $|\alpha_j| \leq \min(|w_1|, |w_2|)$.

Let S be a semigroup with a finite generating set A . We say that S is *stable* (with respect to A) if there exists a finite presentation $\mathcal{P} = \langle A \mid R \rangle$ defining S in terms of A , with respect to which S has no critical pairs.

Stability is invariant under the change of the finite generating set (see [49, Proposition 3.4]).

Theorem 1.33 *Let S and T be two infinite semigroups. Then $S \times T$ is finitely presented if and only if the following conditions are satisfied:*

- (i) $S^2 = S$ and $T^2 = T$;
- (ii) S and T are finitely presented and stable.

■

For a proof, see [49, Theorem 4.1].

As we have seen finite presentability of direct products of semigroups depends on more conditions than finitely presentability of direct products of groups or monoids. These kinds conditions make finitely presentability of semigroup constructions more interesting problems in combinatorial semigroup theory. It is still not known What are necessary and sufficient conditions for finite presentability of wreath products, semidirect products, the Schützenberger products of semigroups and regular semigroups. Some of these problems have been considered for monoids in [30].

Let S and T be monoids. The direct product of $|T|$ copies of S is denoted by $S^{|T|}$. One may consider $S^{|T|}$ as the set of all functions having finite support, that is the images of them are finite, from T into S . The *restricted wreath product* of the monoid S by the monoid T , denoted by $S \wr T$, is the set $S^{|T|} \times T$ with the multiplication

$$(f, t)(g, t') = (fg^t, tt')$$

where $g^t : T \rightarrow S$ is defined by $g^t(x) = g(xt)$ ($x \in T$).

Theorem 1.34 *Let S be a finitely presented monoid, and T be a finite monoid. Then the restricted wreath product $S \wr T$ is finitely presented. In particular if $S \cong \langle A \mid R \rangle$ and $T \cong \langle B \mid Q \rangle$, then*

$$S \wr T \cong \langle A_t (t \in T), B \mid R_t (t \in T), Q, a_t a'_u = a'_u a_t (a, a' \in A, t, u \in T), \\ ba_t = \left(\prod_{c \in \{y \in T \mid yb=t\}} a_c \right) b (a \in A, b \in B, t \in T) \rangle$$

where $A_t = \{ a_t \mid a \in A \}$ and R_t is obtained from R by replacing a by a_t . ■

For a proof, see [30, Theorem 2.2].

The Schützenberger products of the monoids S and T , denoted by $S \diamond T$, is the set $S \times \mathcal{P}(S \times T) \times T$ where $\mathcal{P}(S \times T)$ denotes the set of all subsets of $S \times T$ with multiplication

$$(s_1, P_1, t_1)(s_2, P_2, t_2) = (s_1 s_2, P_1 t_2 \cup s_1 P_2, t_1 t_2)$$

where $P_1 t_2 = \{ (s, tt_2) \mid (s, t) \in P_1 \}$ and $s_1 P_2 = \{ (s_1 s, t) \mid (s, t) \in P_2 \}$.

Theorem 1.35 *Let S and T be the monoids defined by the monoid presentations $\langle A \mid R \rangle$ and $\langle B \mid Q \rangle$, respectively. Then the monoid presentation*

$$\begin{aligned} &\langle A, B, c_{s,t} (s \in S, t \in T) \mid R, Q, ab = ba (a \in A, b \in B), \\ &\quad c_{s,t}^2 = c_{s,t}, c_{s,t} c_{u,v} = c_{u,v} c_{s,t} (s, u \in S, t, v \in T), \\ &\quad ac_{s,t} = c_{as,t} a, c_{s,t} b = bc_{s,tb} (s \in S, t \in T, a \in A, b \in B) \rangle \end{aligned}$$

defines the Schützenberger products $S \diamond T$. ■

For a proof, see [30, Theorem 3.2]. Note that if the sets $A \cup \{1_S\}$ and $B \cup \{1_T\}$ are indecomposable and either S or T is infinite, then the above presentation is infinite. Therefore, there are still some open questions for finite presentability of Schützenberger products of monoids.

The most important result in [30] for this thesis is the one concerning Rees matrix semigroups. The authors proved the following result for Rees matrix semigroups (over monoids) with zero, the same proof still remains valid for Rees matrix semigroups (over monoids).

Let S be a monoid, let I and J be index sets, and let $P = (p_{ji})$ be a $J \times I$ matrix with entries from S . The *Rees matrix semigroup* $\mathcal{M}[S; I, J; P]$ is the set

$$I \times S \times J = \{ (i, s, j) \mid i \in I, s \in S, j \in J \}$$

with the multiplication

$$(i, s, j)(k, t, l) = (i, sp_{jkt}, l).$$

We will mention the importance of this construction in the chapters which concern Rees matrix semigroups. Now we give a presentation for Rees matrix semigroups which is useful throughout this thesis.

Theorem 1.36 *Let S be a monoid, let $\langle A \mid R \rangle$ be a semigroup presentation for S , let $p_{11} = 1_S$ where 1 is a common element of I and J , and let $e \in A^+$ be a word representing the identity element 1_S of S . Then the monoid presentation*

$$\begin{aligned} &\langle A, y_i (i \in I - \{1\}), z_j (j \in J - \{1\}) \mid R, y_i e = y_i, e y_i = p_{1i}, \\ &z_j e = p_{j1}, e z_j = z_j, z_j y_i = p_{ji} (i \in I - \{1\}, j \in J - \{1\}) \rangle \end{aligned}$$

defines the Rees matrix semigroup $\mathcal{M}[S; I, J; P]$. ■

For a proof, see [30, Theorem 6.2]. Note that each element p_{ji} is also considered as a word (representing p_{ji}) in the above presentation. Also note that if S is finitely presented, and if both I and J is finite, then the Rees matrix semigroup $\mathcal{M}[S; I, J; P]$ is finitely presented.

1.9 The Schur multiplier and efficiency of finite groups

Although the Schur multiplier was first introduced in the important paper [59] about fractional linear substitutions, it became very important in combinatorial group theory after the study of Schur in [60]. In [59] Schur constructed the Schur multiplier $M(G)$ of a finite group from the multiplication table (a presentation) of G .

Let $\langle A \mid R \rangle$ be a finite (group) presentation, where R is a set of “relators”, for a finite group G . Let F be the free group on A , N be the normal closure of R in F , let F' denote the commutator subgroup of F , and let $[F, N]$ denote the subgroup generated by the set $\{u^{-1}v^{-1}uv \mid u \in F, v \in N\}$. Then *Schur multiplier* of G is defined by

$$M(G) = \frac{F' \cap N}{[F, N]}.$$

In [60], for a finite group G , Schur proved the followings:

(i) $M(G)$ is an invariant, that is it does not depend on the finite presentation of G .

(ii) $M(G)$ is a finite abelian group generated by $\text{def}_G(G)$ elements.

As a consequence of (ii), if $\langle A | R \rangle$ is a finite presentation for G , then we have

$$|R| - |A| \geq \text{rank}(M(G)).$$

After this result this is a natural problem which asks when the rank of $M(G)$ is equal to $\text{def}_G(G)$ that is, which groups are efficient. Since then, many results about efficiency of finite groups have been published. In this section we mention some of these results which will be useful for this thesis. Before this it is worth commenting on inefficient groups.

The first examples of inefficient groups were given by Swan. In [64] Swan gave a class of finite groups with trivial multiplier but non-zero deficiency. More examples of inefficient groups can be found in [41, 50, 33] and for a survey article, see [70]. Here we give a result from [50]. (Therefore we see one of the methods to show that a group is efficient.)

Theorem 1.37 *The group presentation*

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = (abac)^n = 1 \rangle$$

is a symmetric presentation of a finite group G_n which is inefficient when $(n, 6) = 3$. ■

We denote the greatest common divisor of two integers m and n by (m, n) . For a proof and more details on symmetric presentations, see [50]. To prove the above theorem the authors first computed the Schur multiplier of G_n directly from its presentation showing it to be trivial when n is odd. By using the following lemma, they proved that $\text{def}_G(G_n) = 1$ when $(n, 6) = 3$.

Lemma 1.38 *If G is a finite group with a subgroup H of index k and $M(H)$ has rank r , then we have*

$$\text{def}_G(G) \geq \frac{r+1}{k} - 1.$$

■

This follows from the Reidemeister-Schreier theorem (see [45]).

Since G_n is a quotient group of a Coxeter group, namely

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^3 = (ca)^3 = 1 \rangle,$$

it is worth noticing that the efficiency of Coxeter groups has been studied in [6].

When Eilenberg-MacLane defined cohomology of groups, they did not see the connection between the Schur multiplier of a group G and its second cohomology $H^2(G, \mathbb{C}^*)$ where \mathbb{C}^* is the multiplicative group of non zero complex numbers. The second cohomology of a group $H^2(G, \mathbb{C}^*)$, with trivial G -module structure for \mathbb{C}^* and the Schur multiplier $M(G)$ are isomorphic. This was first noticed by MacLane (see [44], page 163). After this, the theory of (co)homology became one of the most important tools for combinatorial group theory. (Note that the second integral homology $H_2(G, \mathbb{Z})$ is also isomorphic to the Schur multiplier $M(G)$ (see, for example [40, Theorem 2.7.3].)

Now we state the result known as the Schur-Künneth formula:

Theorem 1.39 *If G and H are any finite groups, then the Schur multiplier of the direct product $G \times H$ is*

$$M(G \times H) \cong M(G) \times M(H) \times (G/G' \otimes_{\mathbb{Z}} H/H')$$

where the additive groups G/G' and H/H' are considered as \mathbb{Z} -modules with the action $nx = x + \dots + x$ where $n \in \mathbb{Z}$ and $x \in G/G'$ (H/H'). ■

The first proof of the theorem above given by Schur directly from a multiplication table in [60]. Then an easier proof was given by Wiegold by using homological arguments in [69].

One of the most interesting classes of groups is the class of groups with trivial Schur multiplier. (Notice that D. L. Johnson uses the name “interesting groups” for this class of groups.) Among the interesting groups are the metacyclic groups. A group G is *metacyclic* if it has a normal subgroup H such that both H and G/H are cyclic. The efficiency of metacyclic groups has been proved in [68] (see also [36]).

Proposition 1.40 *Let G be a finite metacyclic group, and let H be a its normal cyclic subgroup of order m with index n . Then the presentation*

$$\mathcal{P}_{m,r,n,s} = \langle a, b \mid a^m = 1, b^{-1}ab = a^r, b^n = a^s \rangle,$$

where $r, s \leq m$, $r^n \equiv 1 \pmod{m}$ and $rs \equiv s \pmod{m}$, defines G . ■

For a proof and more details about metacyclic groups, see [36, Proposition 7.1].

To talk about the efficiency of metacyclic groups we give the Schur multiplier of metacyclic groups.

Theorem 1.41 *With the above notation, the Schur multiplier of the metacyclic group given by the presentation $\mathcal{P}_{m,r,n,s}$ is the cyclic group C_t where*

$$t = \frac{(r-1, m)(1 + r + \cdots + r^{n-1}, s)}{m}.$$

■

For a proof, see [40, Theorem 2.11.3]. Therefore, if $t \neq 0$, then $\mathcal{P}_{m,r,n,s}$ is an efficient presentation.

Theorem 1.42 *Let G be a finite metacyclic group defined by*

$$\mathcal{P}_{m,r,n,s} = \langle a, b \mid a^m = 1, b^{-1}ab = a^r, b^n = a^s \rangle,$$

where $r, s \leq m$, $r^n \equiv 1 \pmod{m}$ and $rs \equiv s \pmod{m}$. With above notation, if $t = 0$, then The presentation

$$\langle a, b \mid b^n = a^s, b^{-1}x^k b x^{-k} = x^{(m, r-1)} \rangle,$$

where $k = u + ws$ and where $(m, r - 1) = u(r - 1) + vm$ and w is the largest factor of m coprime to u , defines G . ■

Therefore we deduce that finite metacyclic groups are efficient. Since the dihedral group D_{2n} of order $2n$ is a metacyclic group, we deduce that D_{2n} is also efficient.

Since the projective linear group $PSL(2, p)$ is also considered in this thesis, we give some result concerning the efficiency of $PSL(2, p)$ and of some related groups.

A presentation of the group $PSL(2, p)$, where p is a prime may be found in [25]. A nice presentation for $PSL(2, p)$ for an odd prime p

$$\langle a, b \mid a^p = 1, b^2 = 1, (ab)^3 = 1, (a^2ba^{(p+1)/2}b)^3 = 1 \rangle$$

was given in [9].

Theorem 1.43 *The Schur multiplier of the projective linear group $PSL(2, p)$ for an odd prime p is the cyclic group C_2 of order 2.* ■

For a proof, see [40, Theorem 7.1.1].

Therefore, the above presentation is not an efficient presentation. But this presentation was used to find some efficient presentations by Zassenhaus in [71] and by Sunday in [63]. Although Zassenhaus's efficient presentation may be the first efficient presentation for $PSL(2, p)$ with odd prime p , we give Sunday's presentation here since it is important in this thesis.

Theorem 1.44 *The projective linear group $PSL(2, p)$ for an odd prime p can be defined by the following presentation*

$$\langle a, b \mid a^p = 1, b^2 = (ab)^3, (a^{(p+1)/2}ba^4b)^2 = 1 \rangle.$$

In particular, $PSL(2, p)$ with odd prime p is efficient. ■

Note that it is well-known that as $PSL(2, 2) \cong D_6$ we have already mentioned its efficiency.

Since the Schur multiplier of special linear group $SL(2, p)$ is trivial (see [32, Theorem 25.5]), it is an interesting group and since $SL(2, p)$ is an important family of groups related to $PSL(2, p)$, we give an efficient presentation from [13]. In [13] the authors proved the presentation

$$\langle a, b \mid a^2 = 1, (ab)^3 = 1, (ab^4ab^{(p+1)/2})^2b^p = 1 \rangle$$

defines $PSL(2, p)$ with p odd prime. Then, by using the fact that $SL(2, p)$ is the covering group for $PSL(2, p)$, they obtained the following result:

Theorem 1.45 *The presentation*

$$\langle a, b \mid a^2 = (ab)^3, (ab^4ab^{(p+1)/2})^2b^pa^{2k} = 1 \rangle,$$

where k is the integer part of $p/3$, defines $SL(2, p)$ for an odd prime p , and so $SL(2, p)$ is efficient. ■

It is also worth noticing that the smallest simple non-abelian group $SL(2, 8)$ with trivial Schur multiplier is efficient which was shown in the same paper [13].

To see an application of the Schur-Künneth formula in combinatorial group theory we give some results concerning the efficiency of direct product of certain groups.

The efficiency of the direct product $PSL(2, p) \times PSL(2, p)$ with p prime was considered in [20]. It is shown that $PSL(2, p) \times PSL(2, p)$ is efficient for all primes p . The efficiency of $PSL(2, p)^3$ was considered and it is proved that they are efficient in [12]. The efficiency of $PSL(2, p)^n$ for an odd prime $p \neq 5$ and $n \geq 4$ is still an open problem.

In fact it is an open problem whether there exists a simple non-abelian group G such that G^n ($n \geq 1$) is efficient. However there is a family of non-simple groups, namely D_{2n} , such that the direct powers of them are efficient.

In [21], since the efficiency of the direct powers of D_{2n}^m with n even are obvious, the authors considered the case when n is odd. They first proved the following:

Lemma 1.46 *The following*

$$\langle a, b \mid a^{2n} = 1, (a^n b)^2 = 1, b^{2n} = (ab)^2 \rangle$$

is an efficient presentation for D_{2n}^2 with n odd. ■

They also proved the following:

Lemma 1.47 *The following*

$$\langle a, b, c \mid a^2 = 1, (ac)^2 = 1, (ac^n)^{2n} = 1, ab = ba, b^{n-1}c = cb^{n+1}, b^{-1}c^{1-n}b = c^{n+1} \rangle$$

is an efficient presentation for D_{2n}^3 with n odd. ■

Then, by induction, they proved that the direct powers of D_{2n}^m with n odd are efficient.

The efficiency of direct powers of imperfect groups was considered in [22]. It was proved that, for an imperfect group G , there exists a positive integer N such that, for $n \geq N$, the direct powers G^n is efficient. In particular, it was proved that A_4^n ($n \geq 1$), where A_4 is the alternating group of degree 4, is efficient.

An analogous definition of the Schur multiplier of groups has not been made for semigroups or monoids. However the n th homology of monoids have been defined, and some results has been obtained. For example, in [27], the second and third integer homology of free Burnside monoids was computed.

The *free Burnside monoid* $M_{r,m,d}$ is the monoid defined by the presentation

$$\langle A \mid w^{m+d} = w^m \ (w \in A^*) \rangle$$

where $|A| = r$. In [27] it is shown that if $m \geq 3$ and $d \geq 1$, then $H_2(M_{r,m,d})$ is a free abelian group of infinite rank and $H_3(M_{r,m,d})$ is a direct sum of infinitely many copies of \mathbb{Z}_d . As a consequence of this, the free Burnside monoid $M_{r,m,d}$ with $m \geq 3$ and $d \geq 1$, is not finitely presented.

1.10 A computational tool

In 1967 B. H. Neumann introduced an enumeration procedure for finitely presented semigroups in [47], but he did not prove the validity of the procedure. A proof of Neumann's enumeration procedure was given by A. Jura in [38]. This procedure is also known as the Todd-Coxeter enumeration procedure for semigroups since the procedure is analogous to the Todd-Coxeter enumeration procedure for groups in [65].

The first machine (PASCAL version) implementation of the Todd-Coxeter enumeration procedure running in St Andrews was given by E. F. Robertson and Y. Ünlü in [51]. An improved *C* version is due to T. G. Walker which is called *SEMI*; see [67]. Two modifications of the Todd-Coxeter enumeration procedure for enumerating minimal one-sided ideals and idempotents of the minimal two sided ideal are described in [14], [15] and [16]. (See [55] for a short history of the Todd-Coxeter enumeration procedure.) The last improved version of *SEMI* determines idempotents, provides a multiplication table, decides if semigroup is a group, finds \mathcal{D} -classes, \mathcal{H} -classes, etc. This version is now running at the University of St Andrews.

Chapter 2

Finiteness Conditions for the 0-direct Unions of Semigroups

Finiteness conditions of various semigroup constructions have been studied in many articles. For example, finite generation and finite presentability of completely (0-)simple semigroups in [30] and [15], of direct products of semigroups in [49], of subsemigroups, ideals and small extension of semigroups in [37], [15] and [56]. Some other finiteness conditions have been studied in [58] and [56].

It is a well-known fact that a semigroup S without zero is completely simple if and only if S is regular and every idempotent is primitive (see [29, Theorem 3.3.3]). It is also well-known that this result cannot be generalised for a semigroup with zero. However a semigroup S with a zero is regular and every non-zero idempotent of S is primitive if and only if S is a 0-direct union of completely 0-simple semigroups (see [29, Theorem 3.3.3] or [62, 66]). A semigroup S is said to be a *0-direct union of completely 0-simple semigroups* if there exists a family of completely 0-simple semigroups S_i ($i \in I$) such that

$$S = \bigcup_{i \in I} S_i, \quad S_i \cap S_j = S_i S_j = \{ 0 \} \quad (i \neq j \in I).$$

Since the 0-direct union of arbitrary semigroups with zero is a more general semigroup construction than the 0-direct union of completely 0-simple semi-

groups, in this chapter, we study some finiteness conditions of the 0-direct union of any two semigroups with zero.

Throughout this chapter, we fix S_1 and S_2 to be any two semigroups with zero, and T_0 to be the 0-direct union of them, that is

$$T_0 = S_1 \cup S_2, \quad S_1 \cap S_2 = S_1 S_2 = S_2 S_1 = \{0\}.$$

2.1 Presentations

First we consider the finite generation of the 0-direct union of semigroups with zero. In the process we also construct certain natural generating sets for T_0 , S_1 and S_2 where T_0 is the 0-direct union of semigroups S_1 and S_2 with zero. Thus we prepare the ground for the presentations.

Theorem 2.1 *Let T_0 be the 0-direct union of semigroups S_1 and S_2 . Then T_0 is finitely generated if and only if both S_1 and S_2 are finitely generated.*

Proof (\Rightarrow) Let X be a generating set for T_0 . Then we show that the sets

$$X_i = (X \cap S_i) \cup \{0\} \quad (i = 1, 2)$$

generate S_i for $i = 1, 2$. For arbitrary $s \in S_i$, since $s \in T_0 = \langle X \rangle$, we have $s = x_1 \cdots x_n$ where $x_1, \dots, x_n \in X$. Since $X_1 X_2 = X_2 X_1 = \{0\}$, it follows that either $s = 0$ or $x_1, \dots, x_n \in X_i$. Therefore X_i generates S_i ($i = 1, 2$). In particular if X is finite, then both X_1 and X_2 are finite.

(\Leftarrow) It is clear that if $S_1 = \langle Y_1 \rangle$ and $S_2 = \langle Y_2 \rangle$, then $T_0 = \langle Y \rangle$ where $Y = Y_1 \cup Y_2$. In particular, Y is finite when both Y_1 and Y_2 are finite. ■

We now construct presentations. First note that if $\langle A \mid R \rangle$ is a presentation for a semigroup S with zero, then there exists a word w in A^+ which represents the zero element of S . By adding a new generator $z = w$ into the presentation, we have a new presentation $\langle A, z \mid R, w = z \rangle$ for S . The new presentation remains

finite if A and R are both finite. Therefore we consider all finite presentations for a semigroup S with zero of the form $\langle A, z \mid R \rangle$ where z represents the zero element of S . The purpose of this is that the zero element of the 0-direct union is represented by the same generator in all the presentations for S_1 , S_2 and T_0 .

Theorem 2.2 *Let T_0 be the 0-direct union of semigroups S_1 and S_2 . Then T_0 is finitely presented if and only if both S_1 and S_2 are finitely presented.*

Proof (\Leftarrow) Let $\mathcal{P}_1 = \langle A_1, z \mid R_1 \rangle$ and $\mathcal{P}_2 = \langle A_2, z \mid R_2 \rangle$ be presentations in terms of $Y_1 \cup \{0\} \subseteq S_1$ and $Y_2 \cup \{0\} \subseteq S_2$ for S_1 and S_2 , respectively. Then we construct a presentation for T_0 in terms of $Y = Y_1 \cup Y_2 \cup \{0\}$, since, by the previous theorem, Y generates T_0 . Since each relation from R_1 , R_2 and Z where

$$Z = \{ a_1 a_2 = z, a_2 a_1 = z \mid a_1 \in A_1, a_2 \in A_2 \}$$

holds in T_0 , it follows that T_0 is a homomorphic image of the semigroup S defined by the following presentation

$$\mathcal{P}_T = \langle A_1, A_2, z \mid R_1, R_2, Z \rangle.$$

To prove that $T_0 \cong S$, we show that an arbitrary relation $w_1 = w_2$, ($w_1, w_2 \in A^+$) where $A = A_1 \cup A_2 \cup \{z\}$, which holds in T_0 , is a consequence of the relations from \mathcal{P}_T .

Let $W = (A_1 \cup \{z\})^+ \cup (A_2 \cup \{z\})^+$. Then there exists a mapping $\phi : A^+ \longrightarrow W$ such that, for $w \in A^+$, $w = \phi(w)$ is a consequence of the relations from Z . Indeed, if $w \in W$, then define $\phi(w) \equiv w$. If $w \notin W$, then w contains subwords of the forms $a_1 a_2$ or $a_2 a_1$ where $a_1 \in A_1$ and $a_2 \in A_2$. Then we systematically apply relations from Z to eliminate all the subwords of the forms $a_1 a_2$ or $a_2 a_1$ until we obtain a word \bar{w} from W . Then define $\phi(w) \equiv \bar{w}$.

Since the relation $w_1 = w_2$ ($w_1, w_2 \in A^+$) holds in T_0 , it follows that $\phi(w_1) = \phi(w_2)$ holds in T_0 . It remains to prove that the relation $\phi(w_1) = \phi(w_2)$ is a consequence of the relations from $R_1 \cup R_2$. Indeed, either $\phi(w_1)$ and $\phi(w_2)$ are

in the same free semigroup $(A_i \cup \{z\})^+$ ($i = 1, 2$), or $\phi(w_1) \in (A_i \cup \{z\})^+$ and $\phi(w_2) \in (A_j \cup \{z\})^+$ ($i \neq j$). In the first case, that is $\phi(w_1)$ and $\phi(w_2)$ are in the same free semigroup $(A_i \cup \{z\})^+$ ($i = 1, 2$), then since the relation $\phi(w_1) = \phi(w_2)$ holds in S_i ($i = 1, 2$), it follows that the relation $\phi(w_1) = \phi(w_2)$ is a consequence of the relations from R_i ($i = 1, 2$). In the second case, since $\phi(w_1)$ represents an element in S_i and $\phi(w_2)$ represents an element in S_j ($i \neq j$), and since $S_i \cap S_j = \{0\}$, it follows that both $\phi(w_1)$ and $\phi(w_2)$ represent the zero. Therefore, the relation $\phi(w_i) = z$ holds ($i = 1, 2$). Since the relation $\phi(w_1) = z$ ($i = 1, 2$) is a consequence of the relations from R_i and $\phi(w_1) = z$ ($i = 1, 2$) is a consequence of the relations from R_j ($i \neq j$), it follows that the relation $\phi(w_1) = \phi(w_2)$ is a consequence of the relations from $R_1 \cup R_2$. Thus the first part of the proof is now complete.

(\Rightarrow) Let $\mathcal{P} = \langle B, z \mid Q \rangle$ be a presentation for T_0 in terms of X . Then define

$$B_i = \{b \in B \mid \pi(b) \in S_i\} \cup \{z\} \quad (i = 1, 2)$$

where π is the natural projection from $(B \cup \{z\})^+$ onto T_0 . Then it is clear that $\pi(B_i) = X_i = X \cap S_i$ which generates S_i by the previous theorem.

Since $\pi(B_i) = X_i$, and $\langle X_i \rangle = S_i$ ($i = 1, 2$), it follows that the relations $b_1 b_2 = z$ and $b_2 b_1 = z$ ($b_1 \in B_1, b_2 \in B_2$) hold in T_0 . By adding these relations into \mathcal{P} we have the following presentation

$$\langle B, z \mid Q, b_2 b_1 = z, b_1 b_2 = z (b_1 \in B_1, b_2 \in B_2) \rangle$$

for T_0 .

Assume that there are some relations $r = s$ in Q such that r or $s \in (B \cup \{z\})^+ \setminus (B_1^+ \cup B_2^+)$. If this happens, then it is clear that the relations $r = z$ and $s = z$ hold in T_0 . Since $r = s$ is a consequence of $r = z$ and $s = z$, the following presentation

$$\langle B, z \mid \bar{Q}, b_2 b_1 = z, b_1 b_2 = z (b_1 \in B_1, b_2 \in B_2) \rangle,$$

where \bar{Q} is obtained from Q by replacing the relation $r = s$ by the relations $r = z$ and $s = z$ when r or $s \in (B \cup \{z\})^+ \setminus (B_1^+ \cup B_2^+)$, defines T_0 .

Next define

$$Q_i = \{ (r = s) \in \bar{Q} \mid r, s \in B_i^+ \} \quad (i = 1, 2)$$

and

$$Z_i = \{ bz = z, zb = z, \mid b \in B_i \} \quad (i = 1, 2).$$

Notice that the relations in $\bar{Q} \setminus (Q_1 \cup Q_2)$ are redundant relations since they are all consequences of the relations of the forms $b_1 b_2 = z$ and $b_2 b_1 = z$ ($b_1 \in B_1$, $b_2 \in B_2$). Therefore the following presentation

$$\mathcal{P}' = \langle B, z \mid Q_1, Q_2, b_1 b_2 = z, b_2 b_1 = z \ (b_1 \in B_1, b_2 \in B_2) \rangle$$

defines T_0 . Moreover, we show that the presentation

$$\mathcal{P}_i = \langle B_i \mid Q_i, Z_i \rangle$$

defines the semigroup S_i ($i = 1, 2$). Since $\pi(B_i) = X_i$ generates S_i ($i = 1, 2$) and all the relations in Q_i and Z_i hold in S_i ($i = 1, 2$), it follows that S_i ($i = 1, 2$) is a homomorphic image of the semigroup defined by \mathcal{P}_i . Note that since $z \in B_i$ for each $i = 1, 2$, it follows that $Z_i \subseteq \{ b_1 b_2 = z, b_2 b_1 = z \mid b_1 \in B_1, b_2 \in B_2 \}$.

Next we show that the relation $w_1 = w_2$ ($w_1, w_2 \in B_i^+$) which holds in S_i is a consequence of the relations in Q_i and Z_i .

Let the relation $w_1 = w_2$ ($w_1, w_2 \in B_i^+$) hold in S_i . Then it holds in T_0 , and so there exists a sequence $w_1 \equiv \alpha_1, \alpha_2, \dots, \alpha_k \equiv w_2$ of words such that α_{j+1} is obtained from α_j by one application of one relation from the relations of \mathcal{P}' , that is there exist $(r_j = s_j)$ in \mathcal{P}' and $u_j, v_j \in B^*$ ($1 \leq j \leq k-1$) such that

$$\alpha_j \equiv u_j r_j v_j \text{ and } \alpha_{j+1} \equiv u_j s_j v_j.$$

If all the relations $r_j = s_j$ ($1 \leq j \leq k-1$) are in $Q_i \cup Z_i$, then the relation $w_1 = w_2$ is clearly a consequence of $Q_i \cup Z_i$. If they are not all in Q_i , then there exists

$1 \leq q < k$ such that $(r_j = s_j) \in Q_i \cup Z_i$ for $j < q$ and $(r_q = s_q) \notin Q_i \cup Z_i$. This is possible when $r_q \equiv z$ and $s_q \notin B_i^+$. Since $\alpha_q \equiv u_q z v_q \in B_i^+$, it is clear that $\alpha_q = z$ is a consequence of the relations from Z_i . Therefore we have a sequence $w_1 \equiv \alpha_1, \dots, \alpha_{q-1}, \alpha_q \equiv \beta_1, \dots, \beta_n \equiv z$ where $\beta_{\lambda+1}$ is obtained from β_λ ($1 \leq \lambda \leq n-1$) by applying one relation from Z_i . Similarly, by considering $w_2 \in B_i^+$, there exists $q < p \leq k$ such that $(r_j = s_j) \in Q_i \cup Z_i$ for $j > p$ and $(r_p = s_p) \notin Q_i \cup Z_i$. Similarly we have a sequence $z \equiv \gamma_1, \dots, \gamma_m, \alpha_{p+1}, \dots, \alpha_k \equiv w_2$ such that γ_μ is obtained from $\gamma_{\mu+1}$ by applying one relation from Z_i ($1 \leq \mu \leq m-1$). Thus we have the following sequence

$$w_1 \equiv \alpha_1, \dots, \alpha_{q-1}, \beta_1, \dots, \beta_n, z, \gamma_1, \dots, \gamma_m, \alpha_{p+1}, \dots, \alpha_k \equiv w_2$$

which is a consequence of the relations from $Q_i \cup Z_i$. Therefore the presentations $\mathcal{P}_i = \langle B_i \mid Q_i, Z_i \rangle$ define the semigroups S_i ($i = 1, 2$). In particular if \mathcal{P} is a finite presentation then \mathcal{P}' is finite, and so each \mathcal{P}_i is a finite presentation for S_i ($i = 1, 2$), as required. ■

Note that it is an immediate consequence of the proof of Theorem 2.1 that $\text{rank}(S_1) + \text{rank}(S_2) - 2 \leq \text{rank}(T_0) \leq \text{rank}(S_1) + \text{rank}(S_2)$. Note also that, by the previous theorem, we have the following corollary:

Corollary 2.3 *If S_i is defined by the presentation $\langle A_i \mid R_i \rangle$ for $i = 1, 2$. Then the presentation*

$$\langle A_1, A_2, \mid R_1, R_2, w_1 = w_2, a_1 a_2 = w_1, a_2 a_1 = w_1 (a_1 \in A_1, a_2 \in A_2) \rangle,$$

where $w_i \in A_i^+$ represents the zero element of S_i ($i = 1, 2$), defines the 0-direct union of S_1 and S_2 .

2.2 Finiteness conditions

In this section, we study some finiteness conditions, namely being periodic, locally finite, locally finitely presented, residually finite and hopfian, and having soluble

word problem for 0-direct unions of semigroups with zero. Before we state and prove our main results on finiteness conditions, we first give the following lemma which will prove useful for the remaining part of this section.

Lemma 2.4 *Let S and T be two semigroups, and let $\phi : S \longrightarrow T$ be an onto homomorphism. If S contains a zero 0_S , then T contains a zero 0_T , and moreover $\phi(0_S) = 0_T$.*

Proof Let $t \in T$ be arbitrary. Since ϕ is onto, there is some $s \in S$ such that $\phi(s) = t$. It follows that

$$t\phi(0_S) = \phi(s)\phi(0_S) = \phi(s0_S) = \phi(0_S),$$

and so $\phi(0_S)$ is a right zero for T . Similarly, it is shown that $\phi(0_S)$ is a left zero, and so $\phi(0_S)$ is a zero element for T . By the uniqueness of the zero element, we have $\phi(0_S) = 0_T$, as required. ■

Next we state and prove our main results on the first four finiteness conditions listed above.

Theorem 2.5 *Let T_0 be the 0-direct union of semigroups S_1 and S_2 . Then*

- (i) T_0 is periodic if and only if both S_1 and S_2 are periodic;
- (ii) T_0 is locally finite if and only if both S_1 and S_2 are locally finite;
- (iii) T_0 is locally finitely presented if and only if both S_1 and S_2 are locally finitely presented;
- (iv) T_0 is residually finite if and only if both S_1 and S_2 are residually finite.

Proof (i) The proof is clear.

(ii) (\Rightarrow) Every subsemigroup of a locally finite semigroup is itself locally finite.

(\Leftarrow) Let U be any finitely generated subsemigroup of T_0 . Then define $U_i = U \cap S_i$ ($i = 1, 2$). It is clear that U_i^0 ($i = 1, 2$), where U_i^0 is obtained from U_i by adjoining the zero element of T_0 if necessary, is a subsemigroup of S_i ($i = 1, 2$).

It is also clear that U^0 , which is obtained from U by adjoining the zero element of T_0 if necessary, is finitely generated and it is the 0-direct union of U_1^0 and U_2^0 . Since U_1^0 and U_2^0 are finitely generated by Theorem 2.1, it follows that both U_1^0 and U_2^0 are finite. Thus U^0 is finite, and so is U .

(iii) (\Leftarrow) Every subsemigroup of a locally finitely presented semigroup is itself locally finitely presented.

(\Rightarrow) Let U be any finitely generated subsemigroup of T_0 . Then define $U_i = U \cap S_i$ ($i = 1, 2$). With the above notation, both U_1^0 and U_2^0 are finitely generated. Therefore they are finitely presented. It follows, from Theorem 2.2, that U^0 is finitely presented. Since U^0 is a small extension of U , it follows from [56, Theorem 1.3] that U is finitely presented.

(iv) (\Leftarrow) Every subsemigroup of a residually finite semigroup is itself residually finite.

(\Rightarrow) Let s and t be two different elements in T_0 . If they are not in the same subsemigroup S_i ($i = 1, 2$) of T_0 , then both of them are distinct from the zero element. Assume that $s \in S_1$ and $t \in S_2$. Since S_2 is residually finite, there exists a finite semigroup S and a homomorphism ψ from S_2 onto S such that $\psi(t) \neq \psi(0)$. Since, by Lemma 2.4, S has a zero such that $\psi(0) = 0$, we define $\phi : T_0 \longrightarrow S$ by

$$\phi(w) = \begin{cases} \psi(w) & \text{if } w \in S_2 \\ 0 & \text{if } w \in S_1. \end{cases}$$

Since ψ is onto, it is clear that ϕ is well-defined. Thus it follows that $\phi : T_0 \longrightarrow S$ is an onto homomorphism such that $\phi(s) = \psi(0) \neq \psi(t) = \phi(t)$.

If s, t are both in the same subsemigroup S_1 or S_2 , say $s, t \in S_2$, then there exists a finite semigroup S' and a homomorphism σ from S_2 onto S' such that $\sigma(s) \neq \sigma(t)$. Since, by Lemma 2.4, S has a zero such that $\sigma(0) = 0$, similarly, we

define $\varphi : T_0 \longrightarrow S'$ by

$$\varphi(w) = \begin{cases} \sigma(w) & \text{if } w \in S_2 \\ 0 & \text{if } w \in S_1. \end{cases}$$

It is also clear that φ is a well-defined onto homomorphism from T_0 onto S' such that $\varphi(s) \neq \varphi(t)$, as required. ■

We now study the word problem for the 0-direct union of semigroups.

Theorem 2.6 *Let T_0 be the 0-direct union of semigroups S_1 and S_2 , and let T_0 be finitely generated. Then T_0 has soluble word problem if and only if both S_1 and S_2 have soluble word problems.*

Proof (\Rightarrow) Let T_0 have soluble word problem. Since both S_1 and S_2 are finitely generated subsemigroups of the finitely generated semigroup T_0 by Theorem 2.1, it follows that there exist two finite generating sets Y_1 and Y_2 for S_1 and S_2 respectively. Since $Y = Y_1 \cup Y_2$ is a finite generating for T_0 , there exists an algorithm which for any two words $w_1, w_2 \in Y^+$ decides whether $w_1 = w_2$ holds in T_0 or not. In particular, for any two words $w_1, w_2 \in Y_i^+$, the algorithm decides whether $w_1 = w_2$ holds in S_i or not, and hence both S_1 and S_2 have soluble word problem.

(\Leftarrow) Let both S_1 and S_2 have soluble word problem, and let X be a finite generating set for T_0 . Then, as in the proof of Theorem 2.1,

$$X_i = (X \cap S_i) \cup \{0\}$$

are finite generating sets for S_i for $(i = 1, 2)$. Note that the set

$$Z = \{x_1x_2 = 0, x_2x_1 = 0 \mid x_1 \in X_1, x_2 \in X_2\}$$

of relations, which hold in T_0 , is finite.

For any two words $w_1, w_2 \in X^+$, one can use some relations from Z to obtain words $w'_1, w'_2 \in X^+$ such that $w_i = w'_i$ holds in T_0 and that either $w'_i \in X_1^+$

$(i = 1, 2)$ or $w'_i \in X_2^+$ ($i = 1, 2$). If w'_1 and w'_2 are not in the same free semigroup X_i^+ ($i = 1, 2$), then it is immediate that the relation $w'_1 = w'_2$ does not hold in T_0 . If both w'_1 and w'_2 are in the same free semigroup X_i^+ ($i = 1, 2$), then $w'_1 = w'_2$ is decidable since both S_1 and S_2 have soluble word problem. Therefore T_0 has a soluble word problem, as required. ■

We finish this section with the concept of hopfian semigroups.

Theorem 2.7 *Let T_0 be the 0-direct union of semigroups S_1 and S_2 . If T_0 is hopfian then both S_1 and S_2 are hopfian.*

Proof Let T_0 be hopfian and let $\psi_i : S_i \longrightarrow S_i$ be an onto endomorphism ($i = 1, 2$). Then define $\phi_i : T_0 \longrightarrow T_0$ by

$$\phi_i(w) = \begin{cases} \psi_i(w) & \text{if } w \in S_i, \\ w & \text{otherwise.} \end{cases}$$

Since ψ_i is onto, it follows from Lemma 2.4 that $\psi_i(0) = 0$, and hence ϕ_i is a well-defined onto endomorphism of T_0 . Since T_0 is hopfian, it follows that ϕ_i is an automorphism, that is $\ker(\phi_i) = \{ (w, w) \mid w \in T_0 \} = \Delta_T$.

Since

$$\ker(\psi_i) \cup \Delta_{S_j} = \Delta_T = \Delta_{S_1} \cup \Delta_{S_2}$$

where $\Delta_{S_i} = \{ (s, s) \mid s \in S_i \}$ ($i, j = 1, 2$) and $(i \neq j \in \{1, 2\})$, it follows that $\ker(\psi_i) = \Delta_{S_i}$. Therefore ψ_i is an automorphism of S_i , as required. ■

Open problem. Let T_0 be the 0-direct union of semigroups S_1 and S_2 . Is it true that if both S_1 and S_2 are hopfian, then T_0 is hopfian?

2.3 Subsemigroups and ideals

Let U be a subsemigroup (one or two-sided ideal) of T_0 . Then define $U_i = U \cap S_i$ for $(i = 1, 2)$. If U_i is not empty then it is easy to see that U_i is a subsemigroup of

S_i ($i = 1, 2$). (Note that if U is a one or two-sided ideal than U_i ($i = 1, 2$) is not empty since $0 \in U_i$ ($i = 1, 2$).) Now assume that both U_1 and U_2 are non-empty. Then, for $s_1 \in U_1$ and $s_2 \in U_2$, $s_1 s_2 = 0 \in U$. Therefore $0 \in U_i$ ($i = 1, 2$), and so U is the 0-direct union of U_1 and U_2 .

Notice also that S_i ($i = 1, 2$) is a subsemigroup (one or two-sided ideal) of T_0 .

Theorem 2.8 *Let T_0 be the 0-direct union of the semigroups S_1 and S_2 .*

(i) *T_0 has finitely many subsemigroups (one or two-sided ideals) if and only if both S_1 and S_2 have finitely many subsemigroups (one or two-sided ideals).*

(ii) *Every subsemigroup (one or two-sided ideal) of T_0 has finite index if and only if both S_1 and S_2 are finite, or equivalently, if and only if T_0 is finite.*

Proof First we prove the result concerning subsemigroups.

(i) (\Rightarrow) Since every subsemigroup of S_i ($i = 1, 2$) is a subsemigroup of T_0 , it follows that S_i ($i = 1, 2$) has finitely many subsemigroups when T_0 has finitely many subsemigroups.

(\Leftarrow) Let U be a subsemigroup of T_0 . Then define $U_i = U \cap S_i$ for ($i = 1, 2$). It is clear that if one of them is empty, say U_i is empty, then $U = U_j$ ($i \neq j$) and $U = U_j$ is a subsemigroup of S_j . If both U_1 and U_2 are non-empty, then U is the 0-direct union of the subsemigroups U_1 and U_2 of S_1 and S_2 , respectively. Hence we deduce that the number of subsemigroups of T_0 , n , is smaller than $n_1 + n_2 + n_1 n_2$ where n_i denotes the number of subsemigroups of S_i ($i = 1, 2$). Therefore n is finite when both n_1 and n_2 are finite.

(ii) (\Rightarrow) Let every subsemigroup of T_0 have finite index. Since S_i is a subsemigroup of T_0 , it follows that $T_0 \setminus S_i = S_j \setminus \{0\}$ is finite ($j \neq i$). Therefore S_j is finite, and so T_0 is finite.

(\Leftarrow) The converse is clear.

Next we prove the result concerning left ideals. The cases of right and two-sided ideals are proved similarly.

(i) (\Rightarrow) Let I_i be a left ideal of S_i ($i = 1, 2$). Then it is clear that $0 \in I_i$ and I_i is a left ideal of T_0 . It follows that S_i ($i = 1, 2$) has finitely many left ideals when T_0 has finitely many left ideals.

(\Leftarrow) Let J be a left ideal of T_0 . Then define $J_i = J \cap S_i$ for ($i = 1, 2$). Since $0 \in J_1 \cap J_2$, both of J_1 and J_2 are non-empty. Hence J is the 0-direct union of the left ideals J_1 and J_2 of S_1 and S_2 , respectively. Hence we deduce that the number of left ideals of T_0 , m , is smaller than $m_1 m_2$ where m_i denotes the number of left ideals of S_i ($i = 1, 2$). Therefore m is finite when both m_1 and m_2 are finite.

(ii) (\Rightarrow) Let every left ideal of T_0 have finite index. Since S_i is a left ideal of T_0 , it follows that $T_0 \setminus S_i = S_j \setminus \{0\}$ is finite ($j \neq i$). Therefore S_j is finite, and so T_0 is finite.

(\Leftarrow) The converse is clear. ■

Note that all the results which have been obtained for the 0-direct union of two semigroups can be generalised for a 0-direct union of finitely many semigroups with zero. For example we have the following immediate corollary.

Corollary 2.9 *A regular semigroup in which every non-zero idempotent is primitive is finitely generated (respectively, presented) if and only if it is a 0-direct union of finitely many completely 0-simple semigroups which are finitely generated (respectively, presented).* ■

Chapter 3

Generators and Relations of Rees Matrix Semigroups

Rees matrix semigroups are one of the most important semigroup constructions, with numerous applications. Especially, they are very important for the structure theory of simple semigroups (see [48] or [29]). In this chapter we give necessary and sufficient conditions for a Rees matrix semigroup over a semigroup to be finitely generated or finitely presented (for a survey for Rees matrix semigroups over semigroups, see [46]).

Let S be a semigroup, let I and J be two index sets and let $P = (p_{ji})_{j \in J, i \in I}$ be a $J \times I$ matrix with entries from S . The set

$$I \times S \times J = \{ (i, s, j) \mid i \in I, s \in S, j \in J \}$$

with multiplication defined by

$$(i, s, j)(k, t, l) = (i, sp_{jk}t, l)$$

is a semigroup. This semigroup is called a *Rees matrix semigroup*, and is denoted by $\mathcal{M}[S; I, J; P]$.

If S is a group, then T is a completely simple semigroup, and, conversely, every completely simple semigroup can be obtained in this way; see [48] or [29]. There

is a similar construction for completely 0-simple semigroups; this is considered in Section 4.

The results of this chapter will appear in Proceedings of the Edinburgh Mathematical Society (see [5]).

3.1 Generators

The purpose of this section is twofold. In it we prove a necessary and sufficient condition for a Rees matrix semigroup to be finitely generated. In the process we also construct certain natural generating sets for S and T , thus preparing the ground for the material on presentations considered in Sections 2 and 3.

Proposition 3.1 *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup, and let U be the ideal of S generated by the entries of P . If T is finitely generated then I , J and $S \setminus U$ are finite sets.*

Proof Observe that, for any $(i, s, j), (k, t, l) \in T$, we have $(i, s, j)(k, t, l) = (i, sp_{jk}t, l)$ and $sp_{jk}t \in U$. Therefore, if $S \neq U$, then every element of the set $I \times (S \setminus U) \times J$ is indecomposable (i.e. not equal to the product of two elements from T), and hence belongs to every generating set of T . If T is finitely generated, then all I , J and $S \setminus U$ must be finite. If $S = U$, we show that I and J are finite. First we fix $s_0 \in S$ and $j_0 \in J$. Then, for each $i \in I$, we have

$$(i, s_0, j_0) = (i_1, s_1, j_1) \cdots (i_n, s_n, j_n)$$

where $(i_1, s_1, j_1), \dots, (i_n, s_n, j_n)$ belong to a finite generating set X for T . Since i_1 must be equal to i , it follows that I is finite. Similarly, it is shown that J is finite. ■

Next we describe a generating set for S , given a generating set for T .

Proposition 3.2 *If X is a generating set for a Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$, then the set*

$$Y = \{ s \in S \mid (i, s, j) \in X \text{ for some } i \in I, j \in J \} \cup \{ p_{ji} \mid j \in J, i \in I \}$$

generates S .

Proof Let $s \in S$ be an arbitrary element. By taking arbitrary $i \in I$, $j \in J$ and decomposing

$$(i, s, j) = (i_1, s_1, j_1) \cdots (i_m, s_m, j_m) = (i_1, s_1 p_{j_1 i_2} s_2 \cdots p_{j_{m-1} i_m} s_m, j_m)$$

into a products of generators $(i_1, s_1, j_1), \dots, (i_m, s_m, j_m) \in X$, we conclude that $s = s_1 p_{j_1 i_2} s_2 \cdots p_{j_{m-1} i_m} s_m \in \langle Y \rangle$, as required. ■

Now we construct a natural generating set for T , assuming that S is a monoid.

Proposition 3.3 *Let S be a monoid and let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. Denote by U the ideal of S generated by the entries of P , and let Z be a set generating U as a semigroup. Write an arbitrary element $z \in Z$ as $s(z)p_{j(z)i(z)}s'(z)$ with $s(z), s'(z) \in S$, $j(z) \in J$ and $i(z) \in I$, and let*

$$H = \{ s(z), s'(z) \in S \mid z \in Z \} \cup \{ 1 \}.$$

Then the set

$$X = I \times (H^2 \cup S \setminus U) \times J,$$

where $H^2 = \{ hh' \mid h, h' \in H \}$, generates T .

Proof Take an arbitrary element $(i, s, j) \in T$. If $s \notin U$, then $(i, s, j) \in X$. Assume that $s \in U$, say $s = z_1 \cdots z_m$, where $z_1, \dots, z_m \in Z$. Then we have

$$\begin{aligned} (i, s, j) &= (i, z_1 \cdots z_m, j) \\ &= (i, s(z_1)p_{j_1 i_1} s'(z_1) \cdots s(z_m)p_{j_m i_m} s'(z_m), j) \\ &= (i, s(z_1), j_1)(i_1, s'(z_1)s(z_2), j_2) \cdots (i_m, s'(z_m), j) \in \langle X \rangle, \end{aligned}$$

completing the proof. ■

Remark If S is not a monoid, one can still construct a generating set for T along the same lines, replacing H^2 by the set

$$\{ s(z), s'(z) \mid z \in Z \} \cup \{ s'(z)s(z) \mid z \in Z \}.$$

However, it is the generating set given in Proposition 3.3 which will prove useful in Section 3. Alternatively, when S is not a monoid one may note that $\mathcal{M}[S; I, J; P]$ is a subsemigroup of $\mathcal{M}[S^1; I, J; P]$ (where, as usual, S^1 denotes the monoid obtained from S by adjoining an identity element), and then use the methods from [37, 15, 18] or [56] to obtain a generating set for $\mathcal{M}[S; I, J; P]$. This idea is used in the following:

Theorem 3.4 *Let S be a semigroup, let I and J be index sets, let $P = (p_{ji})_{j \in J, i \in I}$ be a $J \times I$ matrix with entries from S , and let U be the ideal of S generated by the set $\{ p_{ji} \mid j \in J, i \in I \}$ of all entries of P . Then the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is finitely generated if and only if the following three conditions are satisfied:*

- (i) *both I and J are finite;*
- (ii) *S is finitely generated and;*
- (iii) *the set $S \setminus U$ is finite.*

In particular, if S is a group, then T is finitely generated if and only if S is finitely generated and both I and J are finite.

Proof (\Rightarrow) The result follows from Propositions 3.1 and 3.2.

(\Leftarrow) Let S be finitely generated, and let I, J and $S \setminus U$ be finite. It follows by Theorem 1.12 that U is finitely generated as a semigroup. Therefore, by Proposition 3.3, $T' = \mathcal{M}[S^1; I, J; P]$ is also finitely generated. Finally, note that $T' \setminus T \subseteq I \times \{1\} \times J$ is finite, so that T is finitely generated by Theorem 1.12. ■

3.2 Presentations (1)

In this section we construct a presentation for a semigroup S , starting from a presentation for a Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$. This presentation for S is finite whenever the starting presentation for T is finite.

Let $H = \{(i(k), s(k), j(k)) \mid k \in K\}$ be a generating set for $T = \mathcal{M}[S; I, J; P]$. If we define $Y = \{s(k) \mid k \in K\}$, then it is clear that the set $X = I \times Y \times J$ contains H , and so generates T . Moreover, it is clear that if X is finite then Y , and so H , is finite. Conversely, if H is finite then, by Proposition 3.1, both I and J are finite, and so X is finite.

Take an alphabet

$$A = \{a(i, y, j) \mid i \in I, y \in Y, j \in J\}$$

in one-one correspondence with X . Let $\langle A \mid R \rangle$ be a presentation for T in terms of X , and let

$$\pi_T : A^+ \rightarrow T, \quad a(i, y, j) \mapsto (i, y, j), \quad (1)$$

be the natural projection. By Proposition 3.2, the set

$$Y \cup \{p_{ji} \mid j \in J, i \in I\}$$

generates S . Take a new alphabet

$$C = \{c(y) \mid y \in Y\} \cup \{d(j, i) \mid j \in J, i \in I\}$$

and let

$$\pi_S : C^+ \rightarrow S, \quad c(y) \mapsto y, \quad d(j, i) \mapsto p_{ji} \quad (2)$$

be the natural projection.

Next we define a mapping $\psi : A^+ \rightarrow C^+$ by

$$\psi(a(i_1, y_1, j_1) \cdots a(i_m, y_m, j_m)) = c(y_1)d(j_1, i_2)c(y_2) \cdots d(j_{m-1}, i_m)c(y_m), \quad (3)$$

where $i_1, \dots, i_m \in I$, $y_1, \dots, y_m \in Y$, and $j_1, \dots, j_m \in J$. (Intuitively, ψ rewrites a word $w \in A^+$ into a word from C^+ which represents the middle component of the element $\pi_T(w) \in T$.)

For a word $w \equiv a(i_1, y_1, j_1) \cdots a(i_m, y_m, j_m) \in A^+$, define

$$\lambda(w) = i_1 \text{ and } \rho(w) = j_m.$$

With this notation, the above definition of ψ has the following immediate consequence

$$\psi(w_1 w_2) \equiv \psi(w_1) d(\rho(w_1), \lambda(w_2)) \psi(w_2) \quad (4)$$

for all $w_1, w_2 \in A^+$. Notice that if $w_1 = w_2$ holds in T , then $\lambda(w_1) \equiv \lambda(w_2)$ and $\rho(w_1) \equiv \rho(w_2)$.

If we let $W = \text{im}(\psi)$, then we have the following:

Lemma 3.5 *For all $y, y' \in Y$, $i \in I$ and $j \in J$, there exist words $\zeta(y, y')$, $\eta(j, i) \in W$ such that*

$$c(y)c(y') = \zeta(y, y') \quad (5)$$

$$d(j, i) = \eta(j, i) \quad (6)$$

hold in S .

Proof Let $w \in C^+$, $i_0 \in I$, $j_0 \in J$ be arbitrary, and consider the element $(i_0, \pi_S(w), j_0) \in T$. If we write

$$(i_0, \pi_S(w), j_0) = (i_1, y_1, j_1) \cdots (i_m, y_m, j_m)$$

a product of generators from X , we conclude that

$$\pi_S(w) = y_1 p_{j_1 i_2} y_2 \cdots p_{j_{m-1} i_m} y_m.$$

For $w' \equiv c(y_1) d(j_1, i_2) c(y_2) \cdots d(j_{m-1}, i_m) c(y_m)$, we now have $w' \in W$, and the relation $w = w'$ holds in S . By putting in the above argument $w \equiv c(y)c(y')$ and $w \equiv d(j, i)$ respectively, we complete the proof. ■

For the remainder of this section, we consider the words $\zeta(y, y')$, $\eta(j, i)$ to be fixed as given in Lemma 3.5.

Lemma 3.6 *There exists a mapping $\sigma : C^+ \rightarrow W$ such that the relation $w = \sigma(w)$ is a consequence of the relations (5) and (6).*

Proof If $w \in W$, then define $\sigma(w) = w$. If $w \notin W$ then first apply (6) to obtain a word $c(y_1)w_1c(y_2)$ containing no subword of the form $d(j_1, i_1)d(j_2, i_2)$, and then systematically apply (5) to eliminate all the subwords of the form $c(y)c(y')$. ■

Intuitively, σ rewrites an arbitrary word from C^+ into a corresponding word in the image of ψ . We now use σ to define a mapping $\phi : C^+ \rightarrow A^+$, which will act as a kind of inverse to ψ , as follows:

$$\phi(w) = a(i_0, y_1, j_1)a(i_2, y_2, j_2) \cdots a(i_m, y_m, j_0) \quad (7)$$

where $i_0 \in I$ and $j_0 \in J$ are fixed and where

$$\sigma(w) \equiv c(y_1)d(j_1, i_2)c(y_2) \cdots d(j_{m-1}, i_m)c(y_m). \quad (8)$$

Finally, we let

$$\mu : T \rightarrow S, \quad (i, s, j) \mapsto s$$

be the second projection. In the following lemma, we establish certain connections between π_S , π_T , ϕ , ψ and μ .

Lemma 3.7 (i) *For any word $w \in C^+$, we have $\mu\pi_T\phi(w) = \pi_S(w)$.*

(ii) *For any word $w \in A^+$, we have $\mu\pi_T(w) = \pi_S\psi(w)$.*

Proof (i) If $\sigma(w) \equiv c(y_1)d(j_1, i_2)c(y_2) \cdots d(j_{m-1}, i_m)c(y_m)$, then

$$\mu\pi_T\phi(w) = \mu\pi_T(a(i_0, y_1, j_1)a(i_2, y_2, j_2) \cdots a(i_m, y_m, j_0)) \quad (7)$$

$$= \mu((i_0, y_1, j_1)(i_2, y_2, j_2) \cdots (i_m, y_m, j_0)) \quad (1)$$

$$= \mu((i_0, y_1 p_{j_1 i_2} y_2 \cdots p_{j_{m-1} i_m} y_m, j_0))$$

$$= y_1 p_{j_1 i_2} y_2 \cdots p_{j_{m-1} i_m} y_m$$

$$= \pi_S(\sigma(w)) \quad (2)$$

$$= \pi_S(w). \quad (\text{Lemma 3.6})$$

(ii) If $w \equiv a(i_1, y_1, j_1)a(i_2, y_2, j_2) \cdots a(i_m, y_m, j_m)$, then

$$\mu\pi_T(w) = \mu((i_1, y_1, j_1)(i_2, y_2, j_2) \cdots (i_m, y_m, j_m)) \quad (1)$$

$$= \mu((i_1, y_1 p_{j_1 i_2} y_2 \cdots p_{j_{m-1} i_m} y_m, j_m))$$

$$= y_1 p_{j_1 i_2} y_2 \cdots p_{j_{m-1} i_m} y_m$$

$$= \pi_S(c(y_1)d(j_1, i_2)c(y_2) \cdots d(j_{m-1}, i_m)c(y_m)) \quad (2)$$

$$= \pi_S\psi(w), \quad (3)$$

as required. ■

Now we can state and prove the main result of this section.

Theorem 3.8 *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup and let $\langle A | R \rangle$ be a presentation for T in terms of a generating set of the form $I \times Y \times J$, with $Y \subseteq S$. With the above notation, S is defined by the presentation;*

$$\langle C \mid \psi(u) = \psi(v) \quad ((u = v) \in R) \quad (9)$$

$$c(y)c(y') = \zeta(y, y') \quad (y, y' \in Y) \quad (10)$$

$$d(j, i) = \eta(j, i) \quad (j \in J, i \in I) \quad (11)$$

in terms of the generating set $Y \cup \{p_{ji} \mid j \in J, i \in I\}$.

Proof Since the relation $u = v$ holds in T , it follows that $\pi_T(u) = \pi_T(v)$, and so, by Lemma 3.7 (ii), we have

$$\pi_S\psi(u) = \mu\pi_T(u) = \mu\pi_T(v) = \pi_S\psi(v).$$

Thus, all relations (9) hold in S . That all relations (10) and (11) hold in S follows from Lemma 3.5.

To complete the proof of the theorem, we show that an arbitrary relation $w_1 = w_2$ ($w_1, w_2 \in C^+$) which holds in S is a consequence of (9), (10) and (11). We do this in three steps.

Step 1: The relation $\phi(w_1) = \phi(w_2)$ holds in T . Indeed, by Lemma 3.7 (i), (1) and (7), we have

$$\begin{aligned} \pi_T \phi(w_1) &= (i_0, \mu \pi_T \phi(w_1), j_0) = (i_0, \pi_S(w_1), j_0) \\ &= (i_0, \pi_S(w_2), j_0) = (i_0, \mu \pi_T \phi(w_2), j_0) \\ &= \pi_T(\phi(w_2)). \end{aligned}$$

Step 2: The relation $\psi\phi(w_1) = \psi\phi(w_2)$ is a consequence of (9). From Step 1, we know that $\phi(w_2)$ can be obtained from $\phi(w_1)$ by applying relations from R . Without loss of generality, we can assume that $\phi(w_2)$ can be obtained from $\phi(w_1)$ by one application of one relation $(u = v) \in R$, i.e. that

$$\phi(w_1) \equiv \alpha u \beta \text{ and } \phi(w_2) \equiv \alpha v \beta$$

for some $\alpha, \beta \in A^*$. If both α and β are non-empty, then we have

$$\begin{aligned} \psi\phi(w_1) &\equiv \psi(\alpha u \beta) \\ &\equiv \psi(\alpha) d(\rho(\alpha), \lambda(u)) \psi(u) d(\rho(u), \lambda(\beta)) \psi(\beta) \quad (\text{by (4)}) \\ &= \psi(\alpha) d(\rho(\alpha), \lambda(u)) \psi(v) d(\rho(u), \lambda(\beta)) \psi(\beta) \quad (\text{by relation (9)}) \\ &\equiv \psi(\alpha) d(\rho(\alpha), \lambda(v)) \psi(v) d(\rho(v), \lambda(\beta)) \psi(\beta) \quad (\text{since } u = v \text{ in } T) \\ &\equiv \psi(\alpha v \beta) \equiv \psi\phi(w_2). \end{aligned}$$

If α is empty but β is non-empty, then we have

$$\begin{aligned}
 \psi\phi(w_1) &\equiv \psi(u\beta) \\
 &\equiv \psi(u)d(\rho(u), \lambda(\beta))\psi(\beta) \quad (\text{by (4)}) \\
 &= \psi(v)d(\rho(u), \lambda(\beta))\psi(\beta) \quad (\text{by relation (9)}) \\
 &\equiv \psi(v)d(\rho(v), \lambda(\beta))\psi(\beta) \quad (\text{since } u = v \text{ in } T) \\
 &\equiv \psi(v\beta) \equiv \psi\phi(w_2).
 \end{aligned}$$

The cases where β is empty or where both α and β are empty are treated similarly.

Step 3: The relation $\psi\phi(w_k) = w_k$ ($k = 1, 2$) is a consequence of (10) and (11).

Indeed, by (3), (7), (8) and Lemma 3.6, we have

$$\psi(\phi(w_k)) \equiv \sigma(w_k) = w_k \quad (k = 1, 2),$$

as a consequence of (10) and (11).

The proof of the theorem is now complete. ■

Corollary 3.9 *If $T = \mathcal{M}[S; I, J; P]$ is finitely presented, then so is S .*

Proof As explained at the beginning of this section, if T is finitely generated, then it has a finite generating set of the form $I \times Y \times J$. Moreover, if T is finitely presented, it can be defined by a finite presentation $\langle A \mid R \rangle$ in terms of this generating set. An application of the previous theorem to $\langle A \mid R \rangle$ yields a finite presentation for S . ■

3.3 Presentations (2)

Now we find a presentation for a Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$, given a presentation for the ideal U of S generated by the entries of P . We do this in the case where S is a monoid. Then we use the main result of [56] to extend this presentation to the case where S is an arbitrary semigroup.

So let S be a monoid, let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup and let U be the ideal generated by the set $\{ p_{ji} \mid i \in I, j \in J \}$ of all entries of P . Let $Z \subseteq U$ be any set generating U as a semigroup. As in Proposition 3.3, write an arbitrary element $z \in Z$ as

$$z = s(z)p_{j(z)i(z)}s'(z)$$

with $s(z), s'(z) \in S$, $i(z) \in I$, $j(z) \in J$ and let

$$H = \{ s(z), s'(z) \mid z \in Z \} \cup \{ 1 \}.$$

Then, clearly, the set

$$Y = \{ hp_{ji}h' \mid h, h' \in H, j \in J, i \in I \}$$

contains Z and hence generates U as a semigroup. Moreover, Y is finite, provided that Z , I and J are all finite.

Now let

$$C = \{ c(h, j, i, h') \mid h, h' \in H, j \in J, i \in I \}$$

be a new alphabet representing elements of Y , and let $\langle C \mid R \rangle$ be a presentation for U . For technical reasons, we also introduce an alphabet

$$D = \{ d(s) \mid s \in S \setminus U \}$$

representing the elements of $S \setminus U$. It is obvious that the set $Y \cup S \setminus U$ generates S , and so the natural homomorphism

$$\pi_S : (C \cup D)^+ \rightarrow S, \quad c(h, j, i, h') \mapsto hp_{ji}h', \quad d(s) \mapsto s \quad (12)$$

is onto. By Proposition 3.3, the set

$$X = I \times (H^2 \cup S \setminus U) \times J$$

generates T . Let

$$A = \{ a(i, h', h, j) \mid i \in I, h, h' \in H, j \in J \}$$

and

$$B = \{ b(i, s, j) \mid i \in I, s \in S \setminus U, j \in J \}$$

be two alphabets, and let

$$\pi_T : (A \cup B)^+ \rightarrow T, \quad a(i, h', h, j) \mapsto (i, h'h, j) \quad b(i, s, j) \mapsto (i, s, j) \quad (13)$$

be the natural projection.

Next, we define a mapping $\phi : (I \times H \times C^+ \times H \times J) \rightarrow A^+$ by

$$\phi(i, h', w, h, j) = a(i, h', h_1, j_1) a(i_1, h'_1, h_2, j_2) \cdots a(i_m, h'_m, h, j) \quad (14)$$

where $w \equiv c(h_1, j_1, i_1, h'_1) \cdots c(h_m, j_m, i_m, h'_m) \in C^+$. Intuitively, $\phi(i, h', w, h, j)$ is a word in A^+ representing the element $(i, h'\pi_S(w)h, j) \in T$. Immediately, from the above definition of ϕ , it follows that

$$\phi(i, h', w_1 c(h_1, j_1, i_1, h'_1) w_2, h, j) \equiv \phi(i, h', w_1, h_1, j_1) \phi(i_1, h'_1, w_2, h, j) \quad (15)$$

for all $i, i_1 \in I, j, j_1 \in J, h, h', h_1, h'_1 \in H$ and all $w_1, w_2 \in C^*$, where we introduce the convention that $\phi(i, h', \epsilon, h, j) \equiv a(i, h', h, j)$.

We also need a mapping $(A \cup B)^+ \rightarrow (C \cup D)^+$, which rewrites a word $w \in (A \cup B)^+$ into a word representing the middle component of $\pi_T(w)$. To this end, we let

$$W = \{ \phi(i, 1, w, 1, j) \mid i \in I, w \in A^+, j \in J \},$$

and then establish certain relations allowing us to transform words from $(A \cup B)^+$ into W .

Lemma 3.10 *For arbitrary $i, i', i'' \in I, j, j', j'' \in J, h, h' \in H$ and $s', s'' \in S \setminus U$, there exist words $\zeta(i, h', h, j) \in W \cup B, \eta(i', i'', j', j'', s', s''), \theta(i, i', j, j', h, h', s'), \lambda(i, i', j, j', h, h', s') \in W$ such that*

$$a(i, h', h, j) = \zeta(i, h', h, j), \quad (16)$$

$$b(i', s', j') b(i'', s'', j'') = \eta(i', i'', j', j'', s', s''), \quad (17)$$

$$b(i', s', j') a(i, h', h, j) = \theta(i, i', j, j', h, h', s'), \quad (18)$$

$$a(i, h', h, j) b(i', s', j') = \lambda(i, i', j, j', h, h', s') \quad (19)$$

hold in T .

Proof Let $w \in (A \cup B)^+$ be arbitrary, and write $\pi_T(w) = (i_0, s, j_0)$. If $s \in S \setminus U$ then define $w' \equiv b(i_0, s, j_0)$. Otherwise, if $s \in U$, then we can write

$$s = h_1 p_{j_1 i_1} h'_1 \cdots h_m p_{j_m i_m} h'_m$$

a product of generators from Y , and then define

$$w' \equiv a(i_0, 1, h_1, j_1) a(i_1, h'_1, h_2, j_2) \cdots a(i_m, h'_m, 1, j_0) \in W.$$

With this choice we have $w' \in W \cup B$, and the relation $w = w'$ holds in T . The proof of the lemma is completed by letting $w \equiv a(i, h', h, j)$, $w \equiv b(i', s', j') b(i'', s'', j'')$, $w \equiv b(i', s', j') a(i, h', h, j)$ and $w \equiv a(i, h', h, j) b(i', s', j')$ respectively, and noting that in the last three cases we cannot have $w' \in B$. ■

For the remainder of this section, we consider the words ζ , η , θ and λ to be fixed.

Lemma 3.11 *There exists a mapping $\sigma : (A \cup B)^+ \rightarrow W \cup B$ such that the relation $w = \sigma(w)$ is a consequence of relations (16)–(19) for every word $w \in (A \cup B)^+$.*

Proof Let $w \in (A \cup B)^+$. First replace each $a(i, h', h, j)$ in w by the corresponding $\zeta(i, h', h, j)$. If the resulting word is $b(i', s', j')$ define $\sigma(w) = b(i', s', j')$. Otherwise, use (17), (18) and (19) to systematically eliminate all symbols $b(i', s', j')$, and define $\sigma(w)$ to be the resulting word. ■

Now we define the required mapping $\psi : (A \cup B)^+ \rightarrow (C \cup D)^+$ as follows:

$$\psi(w) = \begin{cases} d(s) & \text{if } \sigma(w) \equiv b(i, s, j), \\ c(h_1, j_1, i_1, h'_1) \cdots c(h_m, j_m, i_m, h'_m) & \\ & \text{if } \sigma(w) \equiv a(i, 1, h_1, j_1) a(i_1, h'_1, h_2, j_2) \cdots a(i_m, h'_m, 1, j). \end{cases} \quad (20)$$

As before, we also let

$$\mu : T \rightarrow S, \quad (i, s, j) \mapsto s,$$

be the second projection.

Lemma 3.12 (i) For all $w \in (A \cup B)^+$, we have $\pi_S \psi(w) = \mu \pi_T(w)$.

(ii) For all $w \in C^+$, $i \in I$, $j \in J$, $h, h' \in H$, we have $\mu \pi_T \phi(i, h', w, h, j) = h' \pi_S(w) h$.

Proof (i) If $\sigma(w) \equiv b(i, s, j)$, then

$$\pi_S \psi(w) = \pi_S(d(s)) = s = \mu((i, s, j)) = \mu \pi_T(b(i, s, j)) \equiv \mu \pi_T \sigma(w) = \mu \pi_T(w)$$

by (20), (12), (13) and Lemma 3.11, while if

$$\sigma(w) \equiv a(i, 1, h_1, j_1) a(i_1, h'_1, h_2, j_2) \cdots a(i_m, h'_m, 1, j) \in W,$$

then it follows from (20), (12), (13) and Lemma 3.11, respectively that

$$\begin{aligned} \pi_S \psi(w) &= \pi_S(c(h_1, j_1, i_1, h'_1) \cdots c(h_m, j_m, i_m, h'_m)) \\ &= h_1 p_{j_1 i_1} h'_1 \cdots h_m p_{j_m i_m} h'_m \\ &= \mu(i, h_1 p_{j_1 i_1} h'_1 \cdots h_m p_{j_m i_m} h'_m, j) \\ &= \mu((i, 1, h_1, j_1)(i_1, h'_1, h_2, j_2) \cdots (i_m, h'_m, 1, j)) \\ &= \mu \pi_T(a(i, 1, h_1, j_1) a(i_1, h'_1, h_2, j_2) \cdots a(i_m, h'_m, 1, j)) \\ &\equiv \mu \pi_T \sigma(w) = \mu \pi_T(w), \end{aligned}$$

as required.

(ii) If $w \equiv c(h_1, j_1, i_1, h'_1) \cdots c(h_m, j_m, i_m, h'_m) \in C^+$, then it follows from (14), (13) and (12), respectively that

$$\begin{aligned} \mu \pi_T \phi(i, h', w, h, j) &= \mu \pi_T(a(i, h', h_1, j_1) a(i_1, h'_1, h_2, j_2) \cdots a(i_m, h'_m, h, j)) \\ &= \mu((i, h', h_1, j_1)(i_1, h'_1, h_2, j_2) \cdots (i_m, h'_m, h, j)) \\ &= \mu(i, h'(h_1 p_{j_1 i_1} h'_1) \cdots (h_m p_{j_m i_m} h'_m) h, j) \\ &= h'(h_1 p_{j_1 i_1} h'_1) \cdots (h_m p_{j_m i_m} h'_m) h \\ &= h' \pi_S(w) h, \end{aligned}$$

as required. ■

Next we give a presentation for a Rees matrix semigroup over a semigroup.

Theorem 3.13 *Let S be a monoid, let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup, and let U be the ideal of S generated by $\{p_{ji} \mid j \in J, i \in I\}$. If $\langle C \mid R \rangle$ is a presentation for U in terms of a generating set $Y = \{hp_{ji}h' \mid h, h' \in H, j \in J, i \in I\}$ with $1 \in H \subseteq S$, then, with the above notation the presentation with generators $A \cup B$ and relations*

$$\phi(i, h', u, h, j) = \phi(i, h', v, h, j), \quad (21)$$

$$a(i, h', h, j) = \zeta(i, h', h, j), \quad (22)$$

$$b(i', s', j')b(i'', s'', j'') = \eta(i', i'', j', j'', s', s''), \quad (23)$$

$$b(i', s', j')a(i, h', h, j) = \theta(i, i', j, j', h, h', s'), \quad (24)$$

$$a(i, h', h, j)b(i', s', j') = \lambda(i, i', j, j', h, h', s'), \quad (25)$$

where $(u = v) \in R$, $i, i', i'' \in I$, $j, j', j'' \in J$, $h, h' \in H$, $s', s'' \in S \setminus U$, defines T in terms of the generating set $X = I \times (H^2 \cup S \setminus U) \times J$.

Proof Note that by (13), (14) and Lemma 3.12 (ii), we have

$$\begin{aligned} \pi_T \phi(i, h', u, h, j) &= (i, \mu \pi_T \phi(i, h', u, h, j), j) = (i, h' \pi_S(u) h, j) \\ &= (i, h' \pi_S(v) h, j) = (i, \mu \pi_T \phi(i, h', v, h, j), j) \\ &= \pi_T \phi(i, h', v, h, j), \end{aligned}$$

and thus all relations (21) hold in T . That all the other relations (22)–(25) hold in T was proved in Lemma 3.10.

To complete the proof of the theorem, we show that any relation $w_1 = w_2$ ($w_1, w_2 \in (A \cup B)^+$) holding in T is a consequence of the relations (21)–(25). Recall that $\sigma(w_1), \sigma(w_2) \in W \cup B$. Note that the words from W represent decomposable elements of T , while the letters from B represent indecomposable

elements of T . Therefore we have $\sigma(w_1) \in B$ if and only if $\sigma(w_2) \in B$. Also note that distinct letters from B represent distinct elements of T by (13). Thus, if $\sigma(w_1), \sigma(w_2) \in B$, then we must have $\sigma(w_1) \equiv \sigma(w_2)$, and then we have $w_1 = \sigma(w_1) \equiv \sigma(w_2) = w_2$ as a consequence of the relations (22)–(25) by Lemma 3.10.

For the remainder of this proof, we consider the case $\sigma(w_1), \sigma(w_2) \in W$. We proceed in three steps.

Step 1: The relation $\psi(w_1) = \psi(w_2)$ holds in S . Indeed, by Lemma 3.12 (i), we have

$$\pi_S \psi(w_1) = \mu \pi_T(w_1) = \mu \pi_T(w_2) = \pi_S \psi(w_2).$$

Step 2: The relation $\phi(i, 1, \psi(w_1), 1, j) = \phi(i, 1, \psi(w_2), 1, j)$ is a consequence of the relations (21). From Step 1, it follows that $\psi(w_2)$ can be obtained from $\psi(w_1)$ by applying relations from R . Without loss of generality, we may assume that it can be obtained by one application of one relation from R , say

$$\psi(w_1) \equiv \alpha u \beta \quad \text{and} \quad \psi(w_2) \equiv \alpha v \beta,$$

where $\alpha, \beta \in C^*$, $(u = v) \in R$. If both α and β are non-empty, then we can write $\alpha \equiv \alpha_1 c(h_1, j_1, i_1, h'_1)$ and $\beta \equiv c(h_2, j_2, i_2, h'_2) \beta_1$, and then we have

$$\begin{aligned} \phi(i, 1, \psi(w_1), 1, j) &\equiv \phi(i, 1, \alpha_1 c(h_1, j_1, i_1, h'_1) u c(h_2, j_2, i_2, h'_2) \beta_1, 1, j) \\ &\equiv \phi(i, 1, \alpha_1, h_1, j_1) \phi(i_1, h'_1, u, h_2, j_2) \phi(i_2, h'_2, \beta_1, 1, j) \\ &= \phi(i, 1, \alpha_1, h_1, j_1) \phi(i_1, h'_1, v, h_2, j_2) \phi(i_2, h'_2, \beta_1, 1, j) \\ &\equiv \phi(i, 1, \alpha_1 c(h_1, j_1, i_1, h'_1) v c(h_2, j_2, i_2, h'_2) \beta_1, 1, j) \\ &\equiv \phi(i, 1, \psi(w_2), 1, j), \end{aligned}$$

by using (15) and (21). If α is empty but β is non-empty, then we can write

$\beta \equiv c(h_2, j_2, i_2, h'_2)\beta_1$, and then we have

$$\begin{aligned}
 \phi(i, 1, \psi(w_1), 1, j) &\equiv \phi(i, 1, uc(h_2, j_2, i_2, h'_2)\beta_1, 1, j) \\
 &\equiv \phi(i_1, 1, u, h_2, j_2)\phi(i_2, h'_2, \beta_1, 1, j) \\
 &= \phi(i_1, 1, v, h_2, j_2)\phi(i_2, h'_2, \beta_1, 1, j) \\
 &\equiv \phi(i, 1, vc(h_2, j_2, i_2, h'_2)\beta_1, 1, j) \equiv \phi(i, 1, \psi(w_2), 1, j),
 \end{aligned}$$

by using (15) and (21). The cases where β is empty or where both α and β are empty are treated similarly.

Step 3: The relations $w_k = \phi(i, 1, \psi(w_k), 1, j)$, $k = 1, 2$, are consequences of (22)–(25). To prove this, it is enough to note that, by (14) and (20), we have $\phi(i, 1, \psi(w_k), 1, j) = \sigma(w_k)$. Then apply Lemma 3.11. The proof of the theorem is now complete. ■

Theorem 3.14 *Let S be a semigroup, let I and J be index sets, let $P = (p_{ji})$ be a $J \times I$ matrix with entries from S , and let U be the ideal of S generated by the set $\{p_{ji} \mid j \in J, i \in I\}$ of all entries of P . Then the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is finitely presented if and only if the following three conditions are satisfied:*

- (i) *both I and J are finite;*
- (ii) *S is finitely presented and;*
- (iii) *the set $S \setminus U$ is finite.*

In particular, if S is a group, then T is finitely presented if and only if S is finitely presented and both I and J are finite.

Proof (\Leftarrow) The result follows from Corollary 3.9.

(\Rightarrow) If S finitely presented then so is U by Theorem 1.13. As explained at the beginning of this section, U can be generated (as a semigroup) by a finite set $Y = \{hp_{ji}h' \mid h, h' \in H, j \in J, i \in I\}$ where $1 \in H \subseteq S^1$. Moreover, U can be presented by a finite presentation $\langle C \mid R \rangle$ in terms of Y . From the previous

theorem, it follows that $T' = \mathcal{M}[S^1; I, J; P]$ is finitely presented. Finally, note that $T' \setminus T \subseteq I \times \{1\} \times J$ is finite, and hence T is finitely presented by Theorem 1.13. ■

3.4 Rees matrix semigroups with zero

One common variant of the Rees matrix construction is as follows. Let S be a semigroup with zero, and let $T' = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. The set $I \times \{0\} \times J$ is an ideal of T' . Hence one can form the Rees quotient $T'/(I \times \{0\} \times J)$ to obtain a new semigroup. This semigroup is called a *Rees matrix semigroup with zero*, and is denoted by $T = \mathcal{M}^0[S; I, J; P]$. It is well known that if $S = G^0$ is a group with a zero adjoined, and if P is *regular*, in the sense that no row or column of P consists entirely of zeros, then T is a completely 0-simple semigroup, and it is also well-known that all completely 0-simple semigroups can be obtained in this way (see [48] or [29]).

Our main results of this chapter remain valid for this new construction, that is we have the following result for a Rees matrix semigroup with zero.

Theorem 3.15 *Let S be a semigroup with zero, let I and J be index sets, let $P = (p_{ji})_{j \in J, i \in I}$ be a $J \times I$ matrix with entries from S , and let U be the ideal of S generated by the set $\{p_{ji} \mid j \in J, i \in I\}$ of all entries of P . Then the Rees matrix semigroup $T = \mathcal{M}^0[S; I, J; P]$ is finitely generated (respectively finitely presented) if and only if the following three conditions are satisfied:*

- (i) *both I and J are finite;*
- (ii) *S is finitely generated (respectively, finitely presented) and;*
- (iii) *the set $S \setminus U$ is finite.*

In particular, if $S = G^0$ where G is a group and P is regular, then T is finitely generated (respectively, finitely presented) if and only if S is finitely generated (respectively, finitely presented) and both I and J are finite.

Proof If we let $T' = \mathcal{M}[S; I, J; P]$, then we can think of T as being T' with all the elements of $I \times \{0\} \times J$ being equal (and denoted by 0).

(\Rightarrow) Assume that T is finitely generated. As in Proposition 3.1, we can prove that I and J are finite. Therefore the ideal $I \times \{0\} \times J$ is finite and so T' is finitely generated as well. It follows from Propositions 3.1 and 3.2 that S is finitely generated and that $S \setminus U$ is finite. Moreover, if T is finitely presented, then so is T' (as an ideal extension of a finite semigroup by a finitely presented semigroup (see Proposition 1.31)). It follows from Corollary 3.9 that S is finitely presented.

(\Leftarrow) If S is finitely generated and all I , J and $S \setminus U$ are finite, then, by Theorem 3.4, T' is finitely generated. Since T is a quotient of T' , it follows that T is finitely generated as well. Moreover, if S is finitely presented then so is T' by Theorem 3.14. Since the ideal $I \times \{0\} \times J$ is finite, it follows that $T = T'/(I \times \{0\} \times J)$ is also finitely presented. ■

3.5 Remarks

Finite presentability of Rees matrix semigroups has already been investigated in certain special cases. Thus, Howie and Ruškuc in [30] prove the converse part of Theorems 3.14 and 3.15 in the case where S is a monoid and P contains at least one invertible entry. Also, an immediate application of the Reidemeister-Schreier type rewriting technique developed in [15] proves the direct part of Theorem 3.15 in the completely (0-)simple semigroup case. Finite generation of Rees matrix semigroups (in the completely (0-)simple case) has been considered in [26] and [54].

This chapter (as Chapter 2) is part of wider research into finite presentability (and other finiteness conditions) of various semigroup constructions; see [19], [30], [49], [56] and [57]. A common feature in all these results is that of a rewriting mapping (mappings ψ and ϕ in Sections 2 and 3). It is interesting to note that,

unlike in other constructions considered so far, no rewriting mapping defined in this chapter is a homomorphism. This is because, in general, S is neither a subsemigroup nor a homomorphic image of $T = \mathcal{M}[S; I, J; P]$.

Chapter 4

Finiteness Conditions for Rees Matrix Semigroups

In this chapter we investigate some finiteness conditions of Rees matrix semigroups. In particular we consider periodicity, local finiteness, residual finiteness, and having soluble word problem, finitely many ideals, minimal ideals and finite index. We consider a Rees matrix semigroup on a semigroup instead of a group (as in Chapter 3).

The results obtained in [56] and [58] prove useful throughout this chapter.

4.1 Periodicity

We start with a technical lemma.

Lemma 4.1 *Let X be a non-empty subset of a semigroup S . Then the following are equivalent*

- (i) XS^1 is periodic, (ii) S^1X is periodic, (iii) S^1XS^1 is periodic.

Proof (i) \Rightarrow (ii): For an arbitrary element $sx \in S^1X$ ($s \in S^1$, $x \in X$), consider $xs \in XS^1$ so that there exist two positive integers $m \neq n$ such that $(xs)^m = (xs)^n$.

It follows that

$$(sx)^{m+1} = s(xs)^m x = s(xs)^n x = (sx)^{n+1}.$$

(ii) \Rightarrow (iii): For an arbitrary element $sxt \in S^1 X S^1$ ($s, t \in S^1$, $x \in X$), consider $tsx \in S^1 X$ so that there exist two positive integers $m \neq n$ such that $(tsx)^m = (tsx)^n$. It follows that

$$(sxt)^{m+1} = sx(tx)^m t = sx(tx)^n t = (sxt)^{n+1}.$$

(iii) \Rightarrow (i): It is clear. ■

Next we have the following result.

Theorem 4.2 *If a semigroup S is periodic then the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is periodic.*

Proof For an arbitrary element (i, s, j) of T , consider $sp_{ji} \in S$ so that there exist two positive integers $m \neq n$ such that $(sp_{ji})^m = (sp_{ji})^n$. It follows that

$$(i, s, j)^{m+1} = (i, (sp_{ji})^m s, j) = (i, (sp_{ji})^n s, j) = (i, s, j)^{n+1}.$$

Thus T is periodic as well. ■

Let S be a semigroup, $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup and let U be the ideal of S generated by all the entries of the matrix $P = (p_{ji})$. Then we have the following result.

Theorem 4.3 *The Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is periodic if and only if the ideal U of S is periodic.*

Proof (\Rightarrow): Let V be the left ideal of S generated by the entries of P . It is clear that an arbitrary element of V has the form sp_{ji} ($s \in S^1$). For $sp_{ji} \in V$, consider $(i, sp_{ji}s, j) \in T$ so that there exist two positive integers $m \neq n$ such that

$$(i, sp_{ji}s, j)^m = (i, (sp_{ji})^{2m-1} s, j) = (i, (sp_{ji})^{2n-1} s, j) = (i, sp_{ji}s, j)^n.$$

Thus we have $(sp_{ji})^{2m-1}s = (sp_{ji})^{2n-1}s$, and so

$$(sp_{ji})^{2m} = (sp_{ji})^{2n}.$$

Therefore, V is periodic. It follows, from Lemma 4.1, that U is periodic.

(\Leftarrow): Let $(i, s, j) \in T$ be arbitrary. Consider $sp_{ji} \in U$ (since $p_{ji} \in U$ and U is an ideal) so that there exist two positive integers $m_1 \neq n_1$ such that $(sp_{ji})^{m_1} = (sp_{ji})^{n_1}$. It follows that

$$(i, s, j)^{m_1+1} = (i, (sp_{ji})^{m_1}s, j) = (i, (sp_{ji})^{n_1}s, j) = (i, s, j)^{n_1+1},$$

and so T is periodic. ■

Note that if the ideal U has finite index in S , that is $S \setminus U$ is finite, then S becomes a small extension of U . It follows by Theorem 1.17(i) that U is periodic if and only if S is periodic. Therefore, from Theorem 4.2 and 4.3, we have the following immediate result.

Corollary 4.4 *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup and let U be the ideal of S generated by all the entries of P . If U has finite index in S , then T is periodic if and only if S is periodic.* ■

4.2 Local finiteness

It is obvious that a semigroup S is locally finite if and only if S^1 is locally finite.

Next we give a less obvious similar result for Rees matrix semigroups.

Lemma 4.5 *Let S be a semigroup without an identity. Then the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is locally finite if and only if the Rees matrix semigroup $T' = \mathcal{M}[S^1; I, J; P]$ is locally finite.*

Proof (\Leftarrow): Let T' be locally finite. Since every subsemigroup of a locally finite semigroup is locally finite and since T is a subsemigroup of T' , it follows that T is also locally finite.

(\Rightarrow): Let T be locally finite and let X be a non-empty finite subset of T' . Then define $Y = X \cap T$ and $Z = X \setminus Y$. If Z is empty then $X = Y \subseteq T$, and so the subsemigroup generated by X is finite. If Z is not empty, then note that $\langle X \rangle = \langle W \rangle \cup Z$ where $W = Y \cup YZ \cup ZY \cup Z^2$. Since $W \subseteq T$ is finite, it follows from the local finiteness of T that $\langle W \rangle$ is finite. Therefore T' is locally finite. ■

Theorem 4.6 *The Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ is locally finite if and only if the ideal U of S generated by the entries of P is locally finite.*

Proof (\Rightarrow): Let $T = \mathcal{M}[S; I, J; P]$ be locally finite and let X be a finite subset of U . Since each $x \in X$ has the form $z = sp_{ji}s'$ ($s, s' \in S^1$), we may take

$$X = \{ s_k p_{j_k i_k} s'_k \mid s_k, s'_k \in S^1, 1 \leq k \leq n \}.$$

Then define

$$I_1 = \{ i_k \mid 1 \leq k \leq n \}, \quad Y = \{ s_k, s'_k, s'_k s_l \in S^1 \mid 1 \leq k, l \leq n \}$$

and

$$J_1 = \{ j_k \mid 1 \leq k \leq n \}.$$

Since $I_1 \times Y \times J_1$ is a finite subset of $T' = \mathcal{M}[S^1; I, J; P]$, it follows, from the previous lemma, that $\langle I_1 \times Y \times J_1 \rangle$ is finite. If $(i, s, j) \in I_1 \times \langle X \rangle \times J_1$, then we have

$$\begin{aligned} (i, s, j) &= (i, s_{k_1} p_{j_{k_1} i_{k_1}} s'_{k_1} \cdots s_{k_q} p_{j_{k_q} i_{k_q}} s'_{k_q}, j) \\ &= (i, s_{k_1}, j_{k_1})(i_{k_1}, s'_{k_1} s_{k_2}, j_{k_2}) \cdots (i_{k_q}, s'_{k_q}, j) \end{aligned}$$

so that $I_1 \times \langle X \rangle \times J_1 \subseteq \langle I_1 \times Y \times J_1 \rangle$. It follows that $\langle X \rangle$ is finite, and so U is locally finite.

(\Leftarrow): Let U be locally finite and let $Y = \{ (i_k, s_k, j_k) \mid 1 \leq k \leq m \}$ be a finite subset of $T = \mathcal{M}[S; I, J; P]$. Then define

$$I_2 = \{ i_k \mid 1 \leq k \leq m \}, \quad Z = \{ s_k p_{j_k i_l}, s_k p_{j_k i_l} s_l \mid 1 \leq k, l \leq m \}$$

and

$$J_2 = \{ j_k \mid 1 \leq k \leq m \}.$$

Since Z is a finite subset of U , it follows that $\langle Z \rangle$ is finite.

Next observe that since, for $(i, s, j) \in \langle Y \rangle \setminus Y$,

$$\begin{aligned} (i, s, j) &= (i_{k_1}, s_{k_1}, j_{k_1}) \cdots (i_{k_q}, s_{k_q}, j_{k_q}) \\ &= (i_{k_1}, s_{k_1} p_{j_{k_1} i_{k_2}} s_{k_2} \cdots s_{k_{q-1}} p_{j_{k_{q-1}} i_{k_q}} s_{k_q}, j_{k_q}), \end{aligned}$$

where $q \geq 2$ and $(i_{k_1}, s_{k_1}, j_{k_1}), \dots, (i_{k_q}, s_{k_q}, j_{k_q}) \in Y$, it follows that $(i, s, j) \in I_2 \times \langle Z \rangle \times J_2$. Therefore $\langle Y \rangle \subseteq Y \cup (I_2 \times \langle Z \rangle \times J_2)$. Since $I_2 \times \langle Z \rangle \times J_2$ is finite, it follows that $\langle Y \rangle$ is finite, as required. ■

Note that if $S \setminus U$ is finite then, by Theorem 1.17(ii), we have the following immediate corollary.

Corollary 4.7 *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup and let the ideal U of S generated by all the entries of P have finite index in S . Then T is locally finite if and only if S is locally finite.* ■

4.3 Residual finiteness

Let I and J be the index sets of the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$. Define

$$\eta_I = \{ (i, i') \in I \times I \mid p_{ji} = p_{ji'} \text{ for each } j \in J \}$$

and

$$\eta_J = \{ (j, j') \in J \times J \mid p_{ji} = p_{ji'} \text{ for each } i \in I \}.$$

It is clear that η_I and η_J are equivalences on I and J , respectively. We say that η_I (η_J) has finite index in I (J) if the number of equivalence classes is finite. We denote a subset of I which contains one and only one representative from each of the equivalence classes by I_η (and similarly J_η). With this notation we have the following result:

Lemma 4.8 *Let S be a semigroup and let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. If S is residually finite, and both the equivalence relations η_I and η_J have finite index, then T is a residually finite semigroup.*

Proof For $(i_1, s_1, j_1) \neq (i_2, s_2, j_2) \in T$, first assume that $i_1 \neq i_2$. Then consider the rectangular band

$$R = \{ (i_1, j_1), (i_2, j_1) \},$$

and the mapping $\phi : T \rightarrow R$ defined by

$$\psi(i, s, j) = \begin{cases} (i_1, j_1) & \text{if } i = i_1 \\ (i_2, j_1) & \text{if } i \neq i_1. \end{cases}$$

It is clear that ϕ is an onto homomorphism such that $\phi(i_1, s_1, j_1) \neq \phi(i_2, s_2, j_2)$.

If $j_1 \neq j_2$, then it may be checked similarly. Now we check the case where $i_1 = i_2$ and $j_1 = j_2$, but $s_1 \neq s_2$. Since S is residually finite, there is an onto homomorphism $\psi : S \rightarrow S'$ such that S' is a finite semigroup and $\psi(s_1) \neq \psi(s_2)$. Since both I_η and J_η are finite sets, the Rees matrix semigroup $T' = \mathcal{M}[S'; I_\eta, J_\eta; Q]$, where $Q = (q_{j'i'})_{I_\eta \times J_\eta}$ and $q_{j'i'} = \psi(p_{ji})$, where $(i, i') \in \eta_I$ and $(j, j') \in \eta_J$, is a finite semigroup.

Then define $\theta : T \rightarrow T'$ by $\theta(i, s, j) = (i', \psi(s), j')$ where $(i, i') \in \eta_I$ and $(j, j') \in \eta_J$. Since

$$\begin{aligned} \theta((i, s, j)(k, t, l)) &= \theta(i, sp_{jk}t, l) = (i', \psi(sp_{jk}t), l') \\ &= (i', \psi(s)\psi(p_{jk})\psi(t), l') \\ &= (i', \psi(s)q_{j'k'}\psi(t), l') = (i', \psi(s), j')(k', \psi(t), l') \\ &= \theta(i, s, j)\theta(k, t, l), \end{aligned}$$

θ is a homomorphism. It is clear that θ is onto and

$$\theta(i_1, s_1, j_1) \neq \theta(i_2, s_2, j_2).$$

Therefore, T is residually finite. ■

Since a rectangular band R is isomorphic to a Rees matrix semigroup, namely $R \cong \mathcal{M}[\{1\}; I, J; P_1]$ where $P_1 = (p_{ji})$ with $p_{ji} = 1$ for all $i \in I, j \in J$, we have the following immediate corollary.

Corollary 4.9 *All rectangular bands are residually finite.* ■

In general, the converse of Lemma 4.8 may not be true. For this we show that there is a residually finite Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ such that η_I and η_J have finite indices, but S is not residually finite.

Let S be a non-residually finite semigroup. Then consider the Rees matrix semigroup $T_0 = \mathcal{M}[S^0; I, J; P_0]$ where S^0 is obtained from S by adding a zero if necessary, and $P_0 = (p_{ji})$ with $p_{ji} = 0$ for all $i \in I, j \in J$. Now we show that T_0 is residually finite. For $w_1 = (i_1, s_1, j_1) \neq (i_2, s_2, j_2) = w_2 \in T_0$, if $i_1 \neq i_2$ or $j_1 \neq j_2$, it may be checked as in the previous lemma. Assume that $i_1 = i_2$ and $j_1 = j_2$, but $s_1 \neq s_2$, and assume that $s_1 \neq 0$. Then consider the zero semigroup $\mathcal{Z}_2 = \{0, x\}$ of order 2 and consider the mapping $\psi_{w_1} : T_0 \rightarrow \mathcal{Z}_2$ defined by

$$\psi_{w_1}(i, s, j) = \begin{cases} x & \text{if } s = s_1 \\ 0 & \text{if } s \neq s_1 \end{cases}$$

for all $w = (i, s, j) \in T_0$. Since, for any $w, w' \in T_0$, the middle term of ww' is always zero, it follows that

$$\psi_{w_1}(ww') = 0 = \psi_{w_1}(w)\psi_{w_1}(w'),$$

and so ψ_{w_1} is a homomorphism. It is clear that ψ_{w_1} is onto and $\psi_{w_1}(w_1) = x \neq 0 = \psi_{w_1}(w_2)$. Therefore T_0 is residually finite. However, since S^0 is a small extension of S , it follows from [58, Corollary 4.6] that S^0 is not residually finite.

We have given an example which shows that if $T = \mathcal{M}[S; I, J; P]$ is residually finite then S may not be residually finite. However if S is group then we have the following result.

Lemma 4.10 *Let G be a group and let $T = \mathcal{M}[G; I, J; P]$ be a Rees matrix semigroup. If T is a residually finite semigroup, then G is a residually finite group.*

Proof By [29, Theorem 3.4.2], we first may assume that $1 \in I \cap J$ and that $p_{1i} = p_{j1} = 1$ for all $i \in I$ and $j \in J$.

It is a well-known fact that $G' = \{\psi(1, s, 1) \mid s \in G\}$ is a subgroup of T . Thus G' is residually finite. Since G' is isomorphic to G , it follows that G is residually finite, as required. ■

Corollary 4.11 *Let G be a group and let I and J be finite index sets. Then the Rees matrix semigroup $T = \mathcal{M}[G; I, J; P]$ is residually finite if and only if G is residually finite.*

Proof The result follows from Lemmas 4.10 and 4.8. ■

Next we give an example for the case S is a residually finite group, but T is not residually finite. First, from Proposition 1.16, we may redefine that a semigroup S is *residually finite* if, for each pair $s_1 \neq s_2 \in S$, there exists a congruence ρ with finite index (which means ρ has finitely many equivalence classes) in S such that $(s_1, s_2) \notin \rho$.

Consider the Rees matrix semigroup $T = \mathcal{M}[C_2; N, N; P]$ where $C_2 = \{a, a^2\}$ ($a^3 = a$) is the cyclic group of order 2, N is the natural numbers and the matrix $P = (p_{ji})_{N \times N}$ where

$$p_{ji} = \begin{cases} a & \text{if } j > i \\ a^2 & \text{if } j \leq i. \end{cases}$$

Assume that T is residually finite. Then, for $(1, a, 1), (1, a^2, 1) \in T$, there is a proper congruence ρ with finite index such that $((1, a, 1), (1, a^2, 1)) \notin \rho$. Since ρ has finite index, for $j < l$, either $((i, a, j), (k, a, l)) \in \rho$ or $((i, a, j), (k, a^2, l)) \in \rho$.

If $((i, a, j), (k, a, l)) \in \rho$, then choose $i_0 \in I$ such that $p_{ji_0} = a^2$ and $p_{li_0} = a$. Since $p_{1i} = p_{1k} = a^2$, and $a^3 = a$, it follows that

$$(1, a^2, 1)(i, a, j)(i_0, a^2, 1) = (1, a^2 p_{1i} a p_{ji_0} a^2, 1) = (1, a^2 a^2 a a^2 a^2, 1) = (1, a, 1)$$

and

$$(1, a^2, 1)(k, a, l)(i_0, a^2, 1) = (1, a^2 p_{1k} a p_{li_0} a^2, 1) = (1, a^2 a^2 a a a^2, 1) = (1, a^2, 1),$$

and so $((1, a, 1), (1, a^2, 1)) \in \rho$ which is a contradiction to the choice of ρ .

If $((i, a, j), (k, a^2, l)) \in \rho$, then choose $i_0 \in I$ such that $p_{ji_0} = a^2$ and $p_{li_0} = a^2$. It may be shown similarly that $((1, a, 1), (1, a^2, 1)) \in \rho$ which is a contradiction. Therefore, T is not residually finite although C_2 is residually finite.

4.4 The word problem

Recall that a semigroup S is said to have a *soluble word problem* with respect to a generating set A if there exists an algorithm which, for any two words $u, v \in A^+$, decides whether the relation $u = v$ holds in S or not (in finite steps). It is a well-known fact that for a finitely generated semigroup S the solubility of the word problem does not depend upon the choice of the finite generating set for S .

In this section we assume that $T = \mathcal{M}[S; I, J; P]$ is finitely generated, and so, by Propositions 3.1 and 3.2, all I, J and $S \setminus U$ (U is the ideal of S generated by the entries of P) are finite, and S is finitely generated.

Theorem 4.12 *Let S be a semigroup, and let $T = \mathcal{M}[S; I, J; P]$ be a finitely generated Rees matrix semigroup. Then T has a soluble word problem if and only if S has a soluble word problem.*

Proof (\Rightarrow): Let T have a soluble word problem, and let X be a finite generating set for T . Now we show that S has a soluble word problem with respect to the

generating set

$$Y = \{ s \in S \mid (i, s, j) \in X \} \cup \{ p_{ji} \in P \mid i \in I, j \in J \}.$$

Let $u, v \in Y^+$ be any two arbitrary words. Then, from Lemma 3.6, there are two words

$$u' \equiv s_1 p_{j_1 i_2} s_2 p_{j_2 i_3} \cdots p_{j_{m-1} i_m} s_m \text{ and } v' \equiv z_1 p_{l_1 k_2} z_2 p_{l_2 k_3} \cdots p_{l_{n-1} k_n} z_n$$

in Y^+ such that $u = u'$ and $v = v'$ hold in S . Moreover, we can construct u' and v' from u and v , respectively in finite steps.

Now we define $X' = I \times Y \times J$. Since X' is a finite generating set for T , and since T has a soluble word problem, it follows that, for fixed $i_0 \in I$ and $j_0 \in J$, we can decide whether the relation

$$(i_0, s_1, j_1)(i_2, s_2, j_3) \cdots (i_m, s_m, j_0) = (i_0, z_1, l_1)(k_2, z_2, l_3) \cdots (k_n, z_n, j_0)$$

holds in T , that is we can decide whether the relation

$$(i_0, s_1 p_{j_1 i_2} s_2 p_{j_2 i_3} \cdots p_{j_{m-1} i_m} s_m, j_0) = (i_0, z_1 p_{l_1 k_2} z_2 p_{l_2 k_3} \cdots p_{l_{n-1} k_n} z_n, j_0)$$

holds in T . Therefore we can decide whether the relation $u' = v'$ holds in S , and so $u = v$. Therefore, since Y is finite, S has a soluble word problem.

(\Leftarrow): Let S have a soluble word problem. Since $S \setminus U$ is finite, it follows by Theorem 12 that U is finitely generated and, by Theorem 17(v), that U has a soluble word problem. Let Z be a finite generating set for U as a semigroup. Recall that every $z \in Z$ has the form $z = s(z)p_{ji}s'(z)$ where $s(z), s'(z) \in S^1$. Then we take

$$H = \{ s(z), s'(z) \in S^1 \mid s(z)p_{ji}s'(z) \in Z \} \cup \{1\}$$

so that, by Proposition 3.3, the finite set

$$X = I \times (H^2 \cup (S \setminus U) \times J$$

generates $T' = \mathcal{M}[S^1; I, J; P]$. Now we prove that T' has soluble word problem with respect to X . Let $u, v \in X^+$ be two arbitrary words. By applying finitely many relations from the presentation given in Theorem 3.13, we determine effective words

$$u_1 \equiv (i_0, s, j_0) \text{ or } u_1 \equiv (i_0, h_1, j_1)(i_1, h'_1 h_2, j_2) \cdots (i_{m-1}, h'_{m-1} h_m, j_m)(i_m, h'_m, j_0)$$

and

$$v_1 \equiv (k_0, s', l_0) \text{ or } v_1 \equiv (k_0, g_1, l_1)(k_1, g'_1 g_2, l_2) \cdots (k_{n-1}, g'_{n-1} g_n, l_n)(k_n, g'_n, l_0),$$

where $s, s' \in S \setminus U$, $h, h', g, g' \in H$, such that the relations $u = u_1$ and $v = v_1$ hold in T' . If $u_1 \equiv (i_0, s, j_0)$ or $v_1 \equiv (k_0, s', l_0)$, then u_1 or v_1 represents an element in $I \times (S \setminus U) \times J$ which is indecomposable, and hence $u = v$ holds in T' if and only if $u_1 \equiv v_1$. Otherwise $u = v$ holds in T' if and only if $i_0 = k_0$, $j_0 = l_0$ and $u' = v'$ holds in U where

$$\begin{aligned} u' &\equiv h_1 p_{j_1 i_1} h'_1 h_2 p_{j_2 i_2} \cdots h'_{m-1} h_m p_{j_m i_m} h'_m \\ v' &\equiv g_1 p_{l_1 k_1} g'_1 g_2 p_{l_2 k_2} \cdots g'_{n-1} g_n p_{l_n k_n} g'_n \in Z^+. \end{aligned}$$

Since we can decide whether $u' = v'$ holds in U and since I and J are both finite, it follows that we can decide whether $u = v$ holds in T' . Therefore, since X is finite, T' has a soluble word problem.

Since T is a subgroup of T' and $|T' \setminus T| \leq |I| \times |J|$ is finite, it follows by Theorem 1.17(v) that T has a soluble word problem as well. ■

4.5 Ideals

Let V be a two-sided ideal of S . Then it is easy to show that $I \times V \times J$ is a two-sided ideal of $T = \mathcal{M}[S; I, J; P]$ and we denote this two-sided ideal by V_T . With this notation, we have the following result.

Lemma 4.13 *If the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ has finitely many two-sided ideals, then S has finitely many two-sided ideals.*

Proof Let A denote the set of all two-sided ideals of S and let B denote the set of all two-sided ideals of T . Then, define the map $\Psi : A \longrightarrow B$ by $\Psi(V) = V_T$. Since Ψ is an injection from A to B , it follows that $|A| \leq |B|$, as required. ■

The converse of the statement in the previous lemma, in general, is not true. For a counterexample, we consider the Rees matrix semigroup $T' = \mathcal{M}[S; I, J; P_0]$ where S is any infinite 0-simple semigroup and $P_0 = (p_{ji})$ with $p_{ji} = 0$ for all $i \in I, j \in J$ (I and J may be finite). Then define, for each $s \in S$,

$$V(s) = I \times \{0, s\} \times J.$$

It is clear that, for each $s \in S$, $V(s)$ is an ideal of T' , and so T' has at least $|S|$ many ideals, that is T' has infinitely many ideals. However S has only two ideals.

Although this is a counterexample, we have the following result for monoids under certain assumptions.

Theorem 4.14 *Let M be a monoid and let $T = \mathcal{M}[M; I, J; P]$ be a Rees matrix semigroup. If there exists $(j_0, i_0) \in J \times I$ such that, for each $j \in J$ and $i \in I$, p_{ji_0} and p_{j_0i} are invertible (units), then T has finitely many ideals if and only if M has finitely many ideals.*

Proof The direct part of the theorem follows from Lemma 4.13. We prove the converse part of the theorem.

Let M have finitely many ideals. Let U be an ideal of T . Define

$$U_M = \{s \in M \mid (i_s, s, j_s) \in U \text{ for some } i_s \in I, j_s \in J\}.$$

Let $i \in I, j \in J$ and $s \in U_M$ be arbitrary. Denote the inverse of $p_{j_0i_s}$ and $p_{j_s i_0}$ by q_{i_s} and q_{j_s} , respectively. Then observe that we have

$$(i, q_{j_s}, j_0)(i_s, s, j_s)(i_0, q_{i_s}, j) = (i, q_{i_s} p_{j_0 i_s} s p_{j_s i_0} q_{j_s}, j) = (i, s, j)$$

so that $I \times U_M \times J \subseteq U$, and so we have

$$U = I \times U_M \times J.$$

Let A denote the set of all two-sided ideals of M and let B denote the set of all two-sided ideals of T . Then, define the map $\Phi : B \rightarrow A$ by $\Phi(U) = U_M$. Since Φ is an injection from B to A , it follows that $|B| \leq |A|$. In fact, we have from Lemma 4.13 that $|A| = |B|$, as required. ■

Let V be a right (left) ideal of S . Then it is easy to show that, for fixed $i_0 \in I$ ($j_0 \in J$),

$$V_T^r = \{i_0\} \times V \times J \quad (V_T^l = I \times V \times \{j_0\})$$

is a right (left) ideal of $T = \mathcal{M}[S; I, J; P]$. Similarly, with this notation, we have the following result.

Lemma 4.15 *If the Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$ has finitely many right (left) ideals, then S has finitely many right (left) ideals.*

Proof The proof is similar to the proof of Lemma 4.13. ■

Next we investigate the minimal ideals of a Rees matrix semigroup $T = \mathcal{M}[S; I, J; P]$.

Theorem 4.16 *Let S be a semigroup and let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. Then S has a minimal two-sided ideal if and only if T has a minimal two-sided ideal.*

Proof (\Rightarrow): Let V be a minimal ideal of S . Then consider the ideal $V_T = I \times V \times J$ of T . We claim that V_T is a minimal ideal.

Assume that there is an ideal $W \subseteq V_T$ of T . Then take

$$X = \{ wp_{ji}wp_{ji}w \mid (i, w, j) \in W \},$$

and let U be the ideal of S generated by X . Now consider the ideal $U_T = I \times U \times J$ of T so that if $(i, u, j) \in U_T$, then

$$(i, u, j) = (i, swp_{lk}wp_{lk}ws', j),$$

where $s, s' \in {}^1S$ (in the monoid obtained from S by adjoining an identity if necessary) and $wp_{lk}wp_{lk}w \in X$ so that we have

$$(i, u, j) = (i, sw, l)(k, w, l)(k, ws', j) \in W$$

since $(k, w, l) \in W$ and W is an ideal. Therefore, we have $U_T \subseteq W \subseteq V_T$.

On the other hand, since $X \subseteq V$, it follows from the minimality of V that $V = U$ so that $U_T = V_T$. Thus $W = V_T$ so that V_T is a minimal ideal of T .

(\Leftarrow): Let W be a minimal ideal of T . Then we show that the ideal U of S generated by $X = \{ wp_{ji}wp_{ji}w \mid (i, w, j) \in W \}$ is a minimal ideal of S . For this, consider the ideal $U_T = I \times U \times J$ of T . Similarly, it is shown that $U_T \subseteq W$. It follows from the minimality of W that $U_T = W$. If there is an ideal V of S such that V is a proper subset of U , then the ideal $V_T = I \times V \times J$ of T would be a proper subset of $U_T = W$ which is a contradiction to the minimality of W . Therefore U must be a minimal ideal of S , as required. ■

Theorem 4.17 *Let S be a semigroup and let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. Then S has a minimal right (left) ideal if and only if T has a minimal right (left) ideal.*

Proof (\Rightarrow): Let V be a minimal right ideal of S . Then, for fixed $i_0 \in I$, consider the right ideal $V_T^r = \{i_0\} \times V \times J$ of T . We claim that V_T^r is a minimal right ideal of T .

Assume that there is a right ideal $W \subseteq V_T^r$ of T . Then take

$$X = \{ wp_{j,i_0}w \mid (i_0, w, j) \in W \},$$

and let U be the right ideal of S generated by X . Now consider the right ideal $U_T^r = \{i_0\} \times U \times J$ of T , and so, if $(i_0, u, j) \in U_T$, then

$$(i_0, u, j) = (i_0, wp_{l,i_0}ws, j)$$

where $s \in {}^1S$ and $wp_{l,i_0}w \in X$ so that we have

$$(i_0, u, j) = (i_0, w, l)(i_0, ws, j) \in W$$

since $(i_0, w, l) \in W$ and W is a right ideal. Therefore, we have $U_T^r \subseteq W \subseteq V_T^r$.

On the other hand, since $X \subseteq V$, it follows from the minimality of V that $V = U$ so that $U_T^r = V_T^r$. Thus $W = V_T^r$ so that V_T^r is a minimal right ideal of T .

(\Leftarrow): Let W be a minimal right ideal of T . Then we show that the right ideal U of S generated by $X = \{wp_{ji_0}w \mid (i_0, w, j) \in W\}$ (for fixed $i_0 \in I$ such that $(i_0, w_0, j_0) \in W$) is a minimal right ideal of S . For this, consider the right ideal $U_T^r = \{i_0\} \times U \times J$ of T . It is, similarly, shown that $U_T \subseteq W$. It follows from the minimality of W that $U_T^r = W$. If there is any right ideal V of S such that V is a proper subset of U , then the right ideal $V_T^r = \{i_0\} \times V \times J$ of T would be a proper subset of $U_T^r = W$ which is a contradiction to the minimality of W . Therefore, U must be a minimal right ideal of S , as required. ■

Notice that, for each minimal right (left) ideal of S , we can construct at least $|I|$ ($|J|$) many minimal right (left) ideals of T .

Recall that, for a semigroup T and its subsemigroup S , the index of S in T is the number $|T \setminus S|$. If the index is finite then we say that S has finite index in T . With this concept, we have the following result for Rees matrix semigroups.

Theorem 4.18 *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. Then every two-sided ideal of T has finite index if and only if every two-sided ideal of S has finite index and both I and J are finite.*

Proof (\Rightarrow): Let V be any ideal of S . Then, since the ideal $V_T = I \times V \times J$ of T has finite index, $I \times (S \setminus V) \times J$ is finite so that I , $S \setminus V$ and J are all finite.

(\Leftarrow): Let W be any ideal of T . Then consider the ideal U of S generated by $\{wp_{ji}wp_{ji}w \mid (i, w, j) \in W\}$. As in the previous proof, it is shown that $U_T = (I \times U \times J) \subseteq W$. It follows that

$$|T \setminus W| \leq |T \setminus U_T| = |I \times (S \setminus U) \times J| < \infty$$

since $|I|$, $|S \setminus U|$ and $|J|$ are all finite. Therefore W has finite index in T , as required. ■

It is similarly shown that:

Theorem 4.19 *Let $T = \mathcal{M}[S; I, J; P]$ be a Rees matrix semigroup. Then every right (left) ideal of T has finite index if and only if every right (left) ideal of S has finite index and J (I) is finite.* ■

Chapter 5

Semigroup Efficiency of Groups

The aim of this chapter is to investigate the efficiency of groups as monoids and as semigroups. We show that any efficient group is efficient as a monoid and further that certain efficient groups are efficient as semigroups.

Recall that the deficiency of a finite semigroup (monoid or group) presentation $\mathcal{P} = \langle A | R \rangle$ is $\text{def}(\mathcal{P}) = |R| - |A|$, the semigroup deficiency of a finitely presented semigroup S is given by

$$\text{def}_S(S) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite semigroup presentation for } S \},$$

the monoid deficiency of a finitely presented monoid M is given by

$$\text{def}_M(M) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite monoid presentation for } M \}$$

and the group deficiency of a finitely presented group G is given by

$$\text{def}_G(G) = \min\{ \text{def}(\mathcal{P}) \mid \mathcal{P} \text{ is a finite group presentation for } G \}.$$

Since every semigroup (monoid) presentation for a group G is also a group presentation by Theorem 1.8(ii), we have

$$\text{def}_S(G) \geq \text{def}_G(G) \quad \text{and} \quad \text{def}_M(G) \geq \text{def}_G(G).$$

In the first section we prove that

$$\text{def}_M(G) = \text{def}_G(G).$$

In the second section we show that $\text{def}_M(M) = \text{def}_S(M)$ is not true in general. In the remaining sections we prove that, for certain classes of efficient groups,

$$\text{def}_S(G) = \text{def}_G(G).$$

We say that a finite monoid (group) M is *efficient as a semigroup* if M has a semigroup presentation $\mathcal{P} = \langle A \mid R \rangle$ such that $\text{def}(\mathcal{P}) = \text{rank}(H_2(M))$. We call a finite group G *efficient as a monoid* if G has a monoid presentation $\mathcal{P} = \langle A \mid R \rangle$ such that $\text{def}(\mathcal{P}) = \text{rank}(H_2(G))$.

The most of the results of this chapter have been submitted for publication by H. Ayık, C. M. Campbell, J. J. O'Connor and N. Ruškuc (see [3]).

5.1 Efficiency of groups as monoids

If a group G is efficient as a semigroup then it is clear that it is efficient as a group (by considering its efficient semigroup presentation as a group presentation). Similarly, if G is efficient as a monoid, then it is clear that it is efficient as a group. Conversely one may ask whether an efficient group is efficient as a monoid or as a semigroup. In this section, we prove that efficient groups are efficient as monoids.

Before proving the main theorem of this section, we state and prove a technical lemma which we use throughout this and the next chapters.

Lemma 5.1 *Let $\mathcal{P} = \langle A \mid R \rangle$ be a semigroup presentation and let e be a word in A^+ .*

- (i) *If, for each $a \in A$, $ea = a$ (left identity) and there exists $u_a \in A^+$ such that $u_a a = e$ (left inverse), then \mathcal{P} defines a group with the identity e .*
- (ii) *If, for each $a \in A$, $ae = a$ (right identity) and there exists $v_a \in A^+$ such that $av_a = e$ (right inverse), then \mathcal{P} defines a group with the identity e .*

Proof (i) It is enough to show that, for each $a \in A$, $au_a = e$ and $ae = a$. Observe that from (i) we have that, for each $a \in A$,

$$(au_a)^2 \equiv a(u_a a)u_a = ae u_a \equiv a(ea')u'_a = aa'u'_a \equiv au_a$$

where $u_a \equiv a'u'_a$ and $a' \in A$, $u'_a \in A^*$. Let

$$u_a \equiv a_1 \cdots a_n$$

where $a_i \in A$ ($1 \leq i \leq n$). Then, by (ii), we have

$$u_1 a_1 = \cdots = u_n a_n = e$$

for some $u_i \in A^+$. Since

$$\begin{aligned} u_n \cdots u_1 (u_a a) a_1 \cdots a_n &= u_n \cdots u_1 (ea_1) a_2 \cdots a_n = u_n \cdots u_2 (u_1 a_1) a_2 \cdots a_n \\ &= u_n \cdots u_2 (ea_2) \cdots a_n = \cdots = u_n a_n = e, \end{aligned}$$

it follows from (i) that

$$\begin{aligned} au_a &= ea u_a = (u_n \cdots u_1 u_a a a_1 \cdots a_n) a u_a \equiv u_n \cdots u_1 u_a (au_a)^2 \\ &= u_n \cdots u_1 u_a a u_a \equiv u_n \cdots u_1 u_a a a_1 \cdots a_n = e. \end{aligned}$$

Moreover,

$$ae = a(u_a a) = (au_a)a = ea = a,$$

as required.

(ii) It is proved similarly. ■

For a similar proof, see [31, Proposition 1.3]. Notice that a similar result holds for monoid presentations.

Next we state and prove the main theorem of this section.

Theorem 5.2 *Let $\mathcal{P}_G = \langle A \mid R \rangle$ be a finite group presentation for a group G . Consider the monoid presentation*

$$\mathcal{P}_M = \langle A, A' \mid R', aa'a = 1 \ (a \in A) \rangle,$$

where $A' = \{ a' \mid a \in A \}$ is a copy of A and R' is obtained from R by replacing a^{-1} (if it occurs) by aa' in every relation in R . Then \mathcal{P}_M defines G as a monoid. Therefore G is efficient as a group if and only if it is efficient as a monoid.

Proof Since

$$aa' = aa'(aa'a) \equiv (aa'a)a'a = a'a,$$

it follows that

$$a^2a' = a'a^2 = aa'a = 1.$$

We deduce that a^2 is an inverse of a' and aa' is an inverse of a . Therefore, by Lemma 5.1, \mathcal{P}_M defines a group. It is clear that this group is isomorphic to the group G .

If G is efficient as a monoid, it is clear that it is efficient as a group by considering an efficient monoid presentation as a group presentation. Since $\text{def}(\mathcal{P}_G) = \text{def}(\mathcal{P}_M)$, it follows that if a group G is efficient as a group then it is efficient as a monoid. ■

Next we give another proof of the monoid efficiency of efficient groups. This proof is only for finite groups but the monoid presentation below is defined on fewer generators than the monoid presentation above.

Theorem 5.3 *Let $\mathcal{Q}_G = \langle A \mid R \rangle$ be a finite group presentation for a finite group G . Then G has a monoid presentation on $|A| + 1$ generators with the same deficiency as \mathcal{Q}_G . Moreover, G has a semigroup presentation on $|A| + 1$ generators with deficiency $|R| + 1$.*

Proof Let $\mathcal{Q}_G = \langle A \mid R \rangle$ be a finite group presentation on $A = \{ a_1, \dots, a_n \}$. Since G is finite, $|R| \geq n$ and we may assume n many of these relations to have the form

$$u_i a_1 \cdots a_i = 1 \quad (1 \leq i \leq n)$$

(since $r = s \Leftrightarrow rs^{-1} = 1 \Leftrightarrow (rs^{-1}(a_1 \cdots a_i)^{-1})a_1 \cdots a_i = 1$).

Consider the monoid presentation

$$\mathcal{Q}_M = \langle A, b \mid R', a_1 \cdots a_n b = 1 \rangle$$

where R' is obtained from R by replacing a_i^{-1} by $a_{i+1} \cdots a_n b a_1 \cdots a_{i-1}$ in each relation in which it occurs for $1 \leq i \leq n$. Since, for $1 \leq i \leq n$, we have

$$\begin{aligned} (a_{i+1} \cdots a_n b a_1 \cdots a_{i-1}) a_i &= (u_i a_1 \cdots a_i) a_{i+1} \cdots a_n b a_1 \cdots a_i \\ &\equiv u_i (a_1 \cdots a_n b) a_1 \cdots a_i = u_i a_1 \cdots a_i = 1 \end{aligned}$$

and $a_1 \cdots a_n b = 1$, it follows that every generator in \mathcal{P}_M has a left inverse. Therefore, by Lemma 5.1, \mathcal{Q}_M defines a group. It is clear that this group is the group G , and so the proof of the first part of the theorem is complete.

Next consider the semigroup presentation

$$\mathcal{P}_S = \langle A, b \mid R'', a_1 \cdots a_n b = a_1^m, a_1^m a_i = a_i \ (1 \leq i \leq n), a_1^m b = b \rangle$$

where m is the order of a_1 and R'' is obtained from R' by replacing 1 by a_1^m in each relation in which 1 appears.

It is clear that a_1^m is a left identity for the semigroup S defined by \mathcal{P}_S . Since $a_1^m a_{i+1} = a_{i+1}$ and $u_i a_1 \cdots a_i = a_1^m \in R''$, we have

$$\begin{aligned} (a_{i+1} \cdots a_n b a_1 \cdots a_{i-1}) a_i &= a_1^m a_{i+1} \cdots a_n b a_1 \cdots a_{i-1} \\ &= (u_i a_1 \cdots a_i) a_{i+1} \cdots a_n b a_1 \cdots a_i \\ &\equiv u_i (a_1 \cdots a_n b) a_1 \cdots a_i = u_i a_1^{m+1} \cdots a_i \\ &= u_i a_1 \cdots a_i = a_1^m. \end{aligned}$$

It follows from the relation $a_1 \cdots a_n b = a_1^m$ that each generator has a left inverse. Therefore, from Lemma 5.1, S is a group. It is clear that S is isomorphic to G , as required. ■

The above efficient monoid presentation is defined on only $(|A| + 1)$ generators rather than the $2|A|$ used in Theorem 5.2.

5.2 Efficiency of monoids as semigroups

However there is no result connecting monoid efficiency and semigroup efficiency of a finite monoid. To show this we give an example which shows that not all efficient monoids are efficient as semigroups.

Example 5.4 *Consider the monoid $Z = \{1, x, x^2\}$ with $x^3 = x^2$ which is obtained from the zero semigroup of order two by adding an identity 1. The monoid Z is efficient as a monoid but it is not efficient as a semigroup.*

Proof It is clear that the monoid presentation $\langle a \mid a^3 = a^2 \rangle$ defines Z as a monoid, and so Z is an efficient monoid (since the deficiency of this presentation is zero).

We show that Z is not an efficient semigroup, that is, if $\langle A \mid R \rangle$ is a semigroup presentation for Z , then $|R| > |A|$. We may assume that there is no trivial relation $w = w$ ($w \in A^+$) nor a relation of the form $(w = a) \in R$ such that $a \in A$ and $w \in (A \setminus \{a\})^+$ (otherwise we eliminate these kinds of relations or generators without increasing the deficiency).

Since every generating set of Z contains $\{1, x\}$, the sets

$$A_1 = \{a \in A \mid \pi(a) = 1\} \text{ and } A_2 = \{b \in A \mid \pi(b) = x\},$$

where π is the natural homomorphism from A^+ onto Z , are non-empty subsets of A . Then take

$$A_3 = A \setminus (A_1 \cup A_2) = \{c \in A \mid \pi(c) = x^2\}$$

which may be empty.

Consider $a \in A_1$ and $d \in A$ so that the relation $ad = d$ holds in Z . Therefore, there is a relation of the form $w_d = d$ with $|w_d| \geq 2$ (by the assumption on R). Then define the sets

$$R_1 = \{(w_a = a) \in R \mid a \in A_1\}, \quad R_2 = \{(w_b = b) \in R \mid b \in A_2\},$$

$$R_3 = \{ (w_c = c) \in R \mid c \in A_3 \}.$$

Notice that, by assumption, $|R_i| \geq |A_i|$ ($i = 1, 2, 3$) and they are disjoint subsets of R . Since $a \in A_1$ and $w_a = a$, we must have $w_a \in A_1^+$, and since $b \in A_2$ and $w_b = b$, we must have $w_b \in A_1^* b A_1^*$.

Finally, observe that the relation $b^3 = b^2$ ($b \in A_2$) holds in Z , and so there is a sequence

$$b^3 \equiv \alpha_1, \alpha_2, \dots, \alpha_n \equiv b^2$$

of words α_i ($1 \leq i \leq n$) in A^+ such that α_{i+1} is obtained from α_i ($1 \leq i \leq n-1$) by applying one relation from R . Notice that we cannot apply any relation from R_3 to any word $w \in (A_1 \cup A_2)^+$. Since applications of the relations from $R_1 \cup R_2$ to b^3 do not change the number of b 's and since they always yield a word w from $(A_1 \cup A_2)^+$, we have a relation $(r = s) \in R \setminus (R_1 \cup R_2 \cup R_3)$ so that $|R| > |A|$. Hence Z is an inefficient semigroup, as required. ■

It follows from the previous example that the presentation

$$\langle a, b \mid a^3 = a^2, bab = a, b^2 = b \rangle$$

is a minimal semigroup presentation for the monoid $\{1, x, x^2\}$ with $x^3 = x^2$.

Let S be a semigroup. If we consider a semigroup presentation of S as a monoid presentation, then it defines $S^1 = S \cup \{1\}$. The example above shows that S^1 may be an inefficient semigroup although S is an efficient semigroup. (Since the semigroup presentation $\langle a \mid a^3 = a^2 \rangle$ defines the zero semigroup \mathcal{Z}_2 of order 2, \mathcal{Z}_2 is an efficient semigroup.) However, this is not true in general. That is, there is an efficient semigroup S such that S^1 is also an efficient semigroup. For this consider the cyclic group C_n of order n given as a semigroup by

$$\langle a \mid a^{n+1} = a \rangle.$$

We claim that the semigroup C_n^1 with presentation

$$\langle a, b \mid a^{n+1} = a, ab = a, ba = a, b^2 = b \rangle,$$

which is obtained by adjoining an identity to C_n (note that a^n is not the identity anymore) is also efficient as a semigroup. In fact, the semigroup presentation

$$\langle a, b \mid ba^{n+1}b = a, b^2 = b \rangle$$

defines C_n^1 . Indeed,

$$ab = (ba^{n+1}b)b \equiv ba^{n+1}b^2 = ba^{n+1}b = a \text{ and}$$

$$ba = b(ba^{n+1}b) \equiv b^2a^{n+1}b = ba^{n+1}b = a.$$

It also follows that

$$a^{n+1} = (ba)a^{n-1}(ab) \equiv ba^{n+1}b = a,$$

and so the assertion is proved.

5.3 Semigroup efficiency of certain groups

If G is a group, then every semigroup presentation for G is also a group presentation for G . Therefore, if G is efficient as a semigroup, it is also efficient as a group. In general, we have not proved that an efficient group G is efficient as a semigroup. In this section we start to investigate the semigroup efficiency of certain efficient groups, namely finite abelian groups, dihedral groups D_{2n} (n even) of order $2n$ and generalised quaternion groups Q_n of order $4n$.

Theorem 5.5 *Finite abelian groups are efficient as semigroups. More precisely, a finite abelian group has a minimal semigroup presentation*

$$\mathcal{P}_S = \langle a_1, \dots, a_r \mid a_1^{q_1+1} = a_1, a_1^{q_1} = a_j^{q_j}, a_1 a_j a_1^{q_1-1} = a_j, a_k a_l = a_l a_k \\ (2 \leq j \leq r, 2 \leq k < l \leq r) \rangle,$$

where $r \geq 1$ and q_j divides q_{j+1} for all $j = 1, \dots, r-1$.

Proof A finite abelian group G of rank r can be expressed as a direct product of cyclic groups of order $q_j > 1$ ($1 \leq j \leq r$) with q_j dividing q_{j+1} for each j ($1 \leq j \leq r-1$) (see, for example, [53]). Moreover, G has finite second homology of rank $r(r-1)/2$ (see [40]). Therefore the standard group presentation for G , namely

$$\mathcal{P}_G = \langle x_1, \dots, x_r \mid x_j^{q_j} = 1 \ (1 \leq j \leq r) \ x_j x_k = x_k x_j \ (1 \leq j < k \leq r) \rangle$$

shows that G is efficient as a group.

We show that \mathcal{P}_S defines G as a semigroup. For this, it is enough to show that $a_1 a_j = a_j a_1$ ($2 \leq j \leq r$), that $a_1^{q_1}$ is an identity and that each $a_j^{q_j-1}$ ($1 \leq j \leq r$) is an inverse of a_j . By the first three group relations of \mathcal{P}_S we have

$$a_1 a_j = a_1^{q_1+1} a_j = a_1 a_j^{q_j} a_j \equiv a_1 a_j a_j^{q_j} = (a_1 a_j a_1^{q_1-1}) a_1 = a_j a_1.$$

Now by commutativity we have

$$a_j a_1^{q_1} = a_1 a_j a_1^{q_1-1} = a_j$$

and similarly $a_1^{q_1} a_j = a_j$ for $j = 1, \dots, r$. Therefore, $a_1^{q_1}$ is an identity and $a_j^{q_j-1}$ is an inverse of a_j for each $j = 1, \dots, r$, and so \mathcal{P}_S defines a group. It is clear that this group is the abelian group defined by \mathcal{P}_G . From $\text{def}(\mathcal{P}_S) = r(r-1)/2$, the result follows. \blacksquare

Theorem 5.6 *The dihedral group D_{2n} can be presented as a semigroup as follows:*

$$\mathcal{P} = \langle a, b \mid a^3 = a, \ a^2 = b^n, \ ab^{n-1}a = b \rangle.$$

When n is even, \mathcal{P} is a minimal presentation and D_{2n} is efficient as a semigroup.

Proof By the first relation, we have $a^2 a = a = a a^2$ and by the first and third relations, we have

$$a^2 b = a^2 (ab^{n-1}a) \equiv a^3 b^{n-1}a = ab^{n-1}a = b = ab^{n-1}a = ab^{n-1}a^3 = ba^2.$$

Therefore a^2 is an identity of the semigroup which is defined by the presentation \mathcal{P} . By the first and second relations, a is its own inverse and b^{n-1} is an inverse of b , and so \mathcal{P} defines a group, namely D_{2n} .

Since the second homology of D_{2n} with n even is the cyclic group of order 2 (see [40]), it follows that D_{2n} with n even is an efficient semigroup. ■

Since the second homology of D_{2n} with n odd is trivial (see [40]), \mathcal{P} is not necessarily a minimal presentation for D_{2n} with n odd. However we know that there is a minimal (deficiency zero) group presentation for D_{2n} with n odd. For example, the group presentation

$$\langle x, y \mid x^2 = y^n, x^{-1}y^{-(n+1)/2}xy^{(n+1)/2} = y \rangle$$

defines D_{2n} with n odd (see Proposition 1.42 to construct the above presentation).

Lemma 5.7 *The semigroup presentation*

$$\langle a, b \mid ababa = a, ab^{n-1}a^{n-2} = b \rangle$$

defines a group G_n and D_{2n} is a homomorphic image of G_n for odd n . Moreover, $G_n \cong D_{2n}$ for $n = 3, 5, 7, 9$, so that D_{2n} is efficient as a semigroup when $n = 3, 5, 7, 9$.

Proof Since

$$(ab)^2b = (ab)^2(ab^{n-1}a^{n-2}) \equiv (ababa)b^{n-1}a^{n-2} = ab^{n-1}a^{n-2} = b,$$

it follows from the first relation that $(ab)^2$ is a left identity. Since

$$(aba^2b^{n-1}a^{n-3})a \equiv aba(ab^{n-1}a^{n-2}) = (ab)^2,$$

$aba^2b^{n-1}a^{n-3}$ is a left inverse of a and it is clear that aba is a left inverse of b . Therefore, it follows from Lemma 5.1 that the presentation above defines a group, say G_n .

Next we show that the relations $ab^{n-1}a^{n-2} = b$ and $ababa = a$ hold in D_{2n} , that is they are consequences of the presentation in Theorem 5.6. Since $a^3 = a$ and n is odd, we have $ab^{n-1}a^{n-2} = ab^{n-1}a = b$. It follows from the relation $a^2 = b^n$ that

$$ababa = aba(ab^{n-1}a)a = aba^2b^{n-1}a^2 = abb^n b^{n-1}b^n \equiv ab^{3n} = a^7 = a.$$

Therefore we deduce that D_{2n} is a homomorphic image of G_n when n is odd.

A coset enumeration program shows that $|G_n| = |D_{2n}|$ for $n = 3, 5, 7, 9$. Therefore, D_{2n} is efficient as a semigroup when $n = 3, 5, 7, 9$, as required. ■

In general we do not know which group G_n is when $n > 9$. We do not even know the order of G_n for $n > 9$. The coset enumeration program we use fails to compute the order of G_n for $n > 9$.

Open problem. Does there exist a deficiency zero presentation for D_{2n} with n odd and $n > 9$?

Next we investigate the semigroup efficiency of the generalised quaternion group Q_n of order $4n$. A group presentation for Q_n is

$$\langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle$$

(see, for example, [35]).

Theorem 5.8 *The semigroup presentation*

$$\mathcal{P}_1 = \langle a, b \mid aba = b, ba^{n-1}b = a \rangle$$

defines the generalised quaternion group Q_n of order $4n$.

Proof Observe that, from the first relation, we have

$$a^{n-1}ba^{n-1} \equiv a^{n-2}(aba)a^{n-2} = a^{n-2}ba^{n-2} = \dots = aba = b. \quad (1)$$

It follows from the second relation and (1) that

$$a^n = (ba^{n-1}b)a^{n-1} \equiv b(a^{n-1}ba^{n-1}) = b^2. \quad (2)$$

From (2), the second relation and the first relation, we have

$$a^{2n}b \equiv aa^na^{n-1}b = ab^2a^{n-1}b \equiv ab(ba^{n-1}b) = aba = b$$

and also

$$a^{2n}a = b^4a \equiv b(b^2)ba = ba^nb a \equiv ba^{n-1}(aba) = ba^{n-1}b = a.$$

We conclude that a^{2n} is a left identity. Since b^3 is a left inverse of b (from (2)), and a^{2n-1} is a left inverse of a , it follows from Lemma 5.1 that \mathcal{P}_1 defines a group with identity a^{2n} . Since a^{2n} is an identity for the group and the relation $a^n = b^2$ holds in the group, this group may be given by the following group presentation:

$$\begin{aligned} & \langle a, b \mid aba = b, ba^{n-1}b = a, a^n = b^2, a^{2n} = 1 \rangle \\ & \cong \langle a, b \mid b^{-1}ab = a^{-1}, ba^{n-1}b = a, a^n = b^2, a^{2n} = 1 \rangle. \end{aligned}$$

Since the relation

$$ba^{n-1}b = ba^n(b^{-1}ab)b = bb^2b^{-1}ab^2 = b^2ab^2 = a^{2n+1} = a$$

holds in Q_n , this group is, in fact, the generalised quaternion group Q_n . Since the deficiency of \mathcal{P}_1 is zero, we conclude that Q_n is efficient as a semigroup. ■

Note that in this case, this efficient group presentation of Q_n is also an efficient semigroup presentation. In general (see for example [1] or [51]) a group presentation (without 1 and inverses) does not give a semigroup presentation for the same group.

5.4 Semigroup efficiency of direct powers of dihedral groups

In the previous section it was shown that the dihedral group D_{2n} with n even is efficient as a semigroup. In this section although we have not been able to prove that D_{2n} with $n > 9$ odd is efficient, we prove that the direct power D_{2n}^m is efficient as a semigroup for an arbitrary n and $m \geq 2$.

We begin with a technical lemma.

Lemma 5.9 *Let $\langle A \mid R \rangle$ and $\langle B \mid Q \rangle$ be two semigroup presentations for two groups G and H , respectively. Then the direct product $G \times H$ may be defined by the semigroup presentation*

$$\mathcal{B} = \langle A, B \mid R, Q, C, e = f \rangle$$

where $C = \{ab = ba \mid a \in A, b \in B\}$ and $e \in A^+$ and $f \in B^+$ are any two words representing the identity elements of G and H , respectively.

Proof Let S be the semigroup defined by \mathcal{B} . It is clear from the relation $e = f$ that e is the identity of S . For $a \in A$, there exists a word $w_a \in A^+ \subseteq (A \cup B)^+$ such that $w_a a = e$, and for $b \in B$ there exists a word $w_b \in B^+ \subseteq (A \cup B)^+$ such that $w_b b = f = e$. It follows from Lemma 5.1 that \mathcal{B} defines a group and it is clear that this group is the direct product $G \times H$, as required. ■

Notice that the last relation of \mathcal{B} above makes the semigroup efficiency of the direct product of efficient groups harder to prove than in the group case. It is proved in [21] that, for all m and n , the direct power D_{2n}^m of the dihedral group D_{2n} of order $2n$ is efficient as a group.

We apply the previous lemma to prove the following:

Theorem 5.10 *For any $m, n \geq 1$ with n even, the direct product D_{2n}^m is efficient as a semigroup.*

Proof By Theorem 5.6, D_{2n} has the following semigroup presentation:

$$\langle a, b \mid a^3 = a, a^2 = b^n, ab^{n-1}a = b \rangle.$$

From the previous lemma, it follows by induction that, for any positive integers m and n , D_{2n}^m may be presented by

$$\mathcal{P}_2 = \langle a_i, b_i \mid a_i^3 = a_i, a_i^2 = b_i^n, a_i b_i^{n-1} a_i = b_i, \quad (3)$$

$$a_k a_l = a_l a_k, a_k b_l = b_l a_k, b_k a_l = a_l b_k, b_k b_l = b_l b_k, \quad (4)$$

$$a_j^2 = a_1^2 \quad (1 \leq i \leq m, 2 \leq j \leq m, 1 \leq k < l \leq m) \rangle \quad (5)$$

Note that the deficiency of \mathcal{P}_2 is $(3m + 4m(m-1)/2 + (m-1)) - (2m) = 2m^2 - 1$. In [21], the rank of the Schur multiplier of D_{2n}^m was computed to be:

$$\text{rank}(H_2(D_{2n}^m)) = \begin{cases} \frac{1}{2}(m^2 - m) & \text{if } n \text{ is odd,} \\ 2m^2 - m & \text{if } n \text{ is even.} \end{cases}$$

Therefore \mathcal{P}_2 is not an efficient presentation for D_{2n}^m for any positive integers m and n except $m = 1$ and even n . However, we can construct an efficient presentation for even n using \mathcal{P}_2 .

For $2 \leq j \leq m$, observe that, from the relations (4), (5) and (3), we have

$$a_1 a_j a_1 = a_j \quad (2 \leq j \leq m) \quad (6)$$

since $a_1 a_j a_1 = a_j a_1^2 = a_j^3 = a_j$. Now we show that the relations $a_j^3 = a_j$ and $a_1 a_j = a_j a_1$ for $(2 \leq j \leq m)$ are consequences of the other relations and the relations (6). Indeed, from (5), (6) and (3), we have

$$a_j^3 = a_1^2(a_1 a_j a_1) \equiv a_1^3 a_j a_1 = a_1 a_j a_1 = a_j \quad (2 \leq j \leq m),$$

and, from (6), (5) and (3), we have

$$a_1 a_j = a_1(a_1 a_j a_1) \equiv a_1^2 a_j a_1 = a_j^3 a_1 = a_j a_1 \quad (2 \leq j \leq m).$$

Therefore, adding the $(m-1)$ relations (6) and removing the $(2m-2)$ redundant relations, we have the following presentation:

$$\begin{aligned} \mathcal{P}_3 = \langle a_i, b_i \mid & a_1^3 = a_1, a_1^2 = b_1^n, a_1 b_1^{n-1} a_1 = b_1, \\ & a_1 a_j a_1 = a_j, a_j^2 = b_j^n, a_j b_j^{n-1} a_j = b_j, a_j^2 = a_1^2 \\ & a_{k'} a_{l'} = a_{l'} a_{k'}, a_k b_l = b_l a_k, b_k a_l = a_l b_k, b_k b_l = b_l b_k \\ & (1 \leq i \leq m, 2 \leq j \leq m, 2 \leq k' < l' \leq m, 1 \leq k < l \leq m) \rangle \end{aligned}$$

for D_{2n}^m . Since the deficiency of \mathcal{P}_3 and the rank of $H_2(D_{2n}^m)$ are both $2m^2 - m$ when n is even, we conclude that D_{2n}^m is an efficient semigroup, as required. ■

Next we investigate the semigroup efficiency of D_{2n}^m for odd n . First we give the following result.

Proposition 5.11 *The presentation*

$$\mathcal{P}_4 = \langle x, y \mid x^{2n}y = y, (x^n y)^2 x = x, (xy)^2 x^{2n} = y^{2n} \rangle$$

is an efficient semigroup presentation for $D_{2n} \times D_{2n}$ with n odd.

Proof First we show that \mathcal{P}_4 defines a group. Since, from the second and first relations above,

$$x^{2n+1} = x^{2n}(x^n y)^2 x \equiv x^n(x^{2n}y)x^n y x = (x^n y)^2 x = x \quad (7)$$

and $x^{2n}y = y$, it follows that x^{2n} is a left identity.

Now we show that, for all positive integers k , $y^k x^n y^k = y x^n y$. Indeed, from the first and second relations of \mathcal{P}_4 , we have

$$y^2 x^n y^2 = y(x^{2n}y)x^n y^2 \equiv y x^n (x^n y)^2 y = y x^n (x^n y)^2 x^{2n} y = y x^n x^{2n} y = y x^n y.$$

Therefore $y^k x^n y^k = y x^n y$ follows by induction on k . For $k = 2n$, we have

$$y^{2n} x^n y^{2n} = y x^n y. \quad (8)$$

From the third relation and (7),

$$y^{2n} = (xy)^2 x^{2n} = (xy)^2 x^{4n} = y^{2n} x^{2n}.$$

From this, (8) and the second relation, it follows that

$$(x^n y^{2n})^2 = (x^n y^{2n})^2 x^{2n} \equiv x^n (y^{2n} x^n y^{2n}) x^{2n} = x^n (y x^n y) x^{2n} \equiv (x^n y)^2 x^{2n} = x^{2n}.$$

Therefore $x^n y^{2n} x^n y^{2n-1}$ is a left inverse of y . It follows from Lemma 5.1 that \mathcal{P}_4 defines a group with an identity x^{2n} . This group has the following group presentation

$$\langle x, y \mid x^{2n} = 1, (x^n y)^2 = 1, (xy)^2 = y^{2n} \rangle$$

which is a group presentation for D_{2n}^2 for odd n (see Lemma 1.45). Since the deficiency of \mathcal{P}_4 and $\text{rank}(M(D_{2n}^2))$ are both one, we conclude that D_{2n}^2 is efficient as a semigroup. ■

Now we give an efficient semigroup presentation for D_{2n}^3 with n odd.

Proposition 5.12 *The presentation*

$$\begin{aligned} \mathcal{P}_5 = \langle a, x, z \mid & \quad zazaz = z, (xz^n)^{2n} = a^2, x^{2n-1}ax = a, \\ & \quad x^{n-1}z = zx^{n+1}, z^{n-1}xz^{n+1} = x, x^{2n} = a^2 \rangle \end{aligned}$$

is an efficient semigroup presentation for D_{2n}^3 with n odd.

Proof First we prove that \mathcal{P}_5 defines a group with identity $(az)^2$. For this we apply Lemma 5.1. Indeed, from the fifth and first relations, we have

$$x(az)^2 = (z^{n-1}xz^{n+1})(az)^2 \equiv z^{n-1}xz^n z(az)^2 = z^{n-1}xz^{n+1} = x.$$

Moreover, from this and the third relation, we have

$$a(az)^2 = (x^{2n-1}ax)(az)^2 \equiv x^{2n-1}ax(az)^2 = x^{2n-1}ax = a.$$

Therefore, it follows from these and the first relation that $(az)^2$ is a right identity of the semigroup B_n defined by \mathcal{P}_5 .

Next we find right inverses for the generators. It is clear that zaz is a right inverse of a . We prove that $z^{n-2}xz^{n+1}x^{2n-2}axzaz$ is a right inverse of z and $x^{2n-2}axzaz$ is a right inverse of x . Indeed, from the fifth and third relations in \mathcal{P}_5 , we have

$$z(z^{n-2}xz^{n+1}x^{2n-2}axzaz) \equiv (z^{n-1}xz^{n+1})x^{2n-2}axzaz = (x^{2n-1}ax)zaz = (az)^2$$

and, from the third relation, we have

$$x(x^{2n-2}axzaz) \equiv (x^{2n-1}ax)zaz = (az)^2.$$

Therefore, we deduce that B_n is a group with the following group presentation:

$$\langle a, x, z \mid azaz = 1, (xz^n)^{2n} = a^2, x^{2n-1}ax = a, \\ x^{n-1}z = zx^{n+1}, z^{n-1}xz^{n+1} = x, x^{2n} = a^2 \rangle.$$

Next we show that the relations $a^4 = x^{4n} = 1$ hold in B_n . Indeed, from the third and sixth relations, we have

$$a^2 = (x^{2n-1}ax)(x^{2n-1}ax) \equiv x^{2n-1}ax^{2n}ax = x^{2n-1}a^4x = x^{6n} = a^6,$$

and so we have

$$a^4 = x^{4n} = 1. \tag{9}$$

From the fourth relation and (9), we have

$$x^{n-1}zx^{3n-1} \equiv (x^{n-1}z)x^{3n-1} = zx^{4n} = z,$$

and so

$$z = x^{n-1}zx^{3n-1} = x^{n-1}(x^{n-1}zx^{3n-1})x^{3n-1} \equiv x^{2n-2}zx^{6n-2} = \dots = x^{kn-k}zx^{3kn-k}$$

for any k . In particular, for $k = n$, we have

$$x^{n(n-1)}zx^{n(3n-1)} = z. \quad (10)$$

Since n is odd, either $n = 4m + 1$ or $n = 4m + 3$. If $n = 4m + 1$, then, from (9), we have

$$x^{n(n-1)} \equiv x^{n((4m+1)-1)} \equiv x^{4mn} = 1$$

and

$$x^{n(3n-1)} \equiv x^{n(3(4m+1)-1)} \equiv x^{12mn+2n} = x^{2n},$$

and so, from (10), we have $x^{2n} = 1$.

If $n = 4m + 3$, then from (9), we have

$$x^{n(3n-1)} \equiv x^{n(3(4m+3)-1)} \equiv x^{4(3m+2)n} = 1$$

and

$$x^{n(n-1)} \equiv x^{n((4m+3)-1)} \equiv x^{4mn+2n} = x^{2n},$$

and so, from (10), we have $x^{2n} = 1$.

Therefore, the relations $a^2 = x^{2n} = 1$ hold in B_n . Moreover, the relation $x^{2n-1}ax = a$ can be replaced by the relation $ax = xa$. Hence we obtain the following group presentation for B_n :

$$\langle a, x, z \mid (az)^2 = 1, (xz^n)^{2n} = 1, ax = xa, \\ x^{n-1}z = zx^{n+1}, z^{n-1}xz^{n+1} = x, x^{2n} = a^2 = 1 \rangle$$

which defines D_{2n}^3 for odd n (see Lemma 1.46). ■

Note that the group presentation above is not the efficient presentation given in [21]. It is shown in [21] that the relation $x^{2n} = 1$ is redundant. If we eliminate this redundant relation, we obtain the efficient group presentation for D_{2n}^3 given in [21].

We now consider the general case when n is odd.

Theorem 5.13 *For any integers $m \geq 2$ and odd n , the direct product D_{2n}^m is efficient as a semigroup.*

Proof First we prove that D_{2n}^m is efficient for m even. By Proposition 5.11, D_{2n}^2 has the efficient semigroup presentation \mathcal{P}_4 . It follows, from Lemma 5.9, that for any positive integer t , the semigroup presentation

$$\begin{aligned} \langle x_i, y_i \mid & x_i^{2n} y_i = y_i, (x_i^n y_i)^2 x_i = x_i, (x_i y_i)^2 x_i^{2n} = y_i^{2n}, \\ & x_k x_l = x_l x_k, x_k y_l = y_l x_k, y_k x_l = x_l y_k, y_k y_l = y_l y_k, \\ & x_j^{2n} = x_1^{2n} \ (1 \leq i \leq t, \ 1 \leq k < l \leq t, \ 2 \leq j \leq t) \rangle \end{aligned}$$

defines D_{2n}^{2t} . Since

$$x_1 y_j = y_j x_1, \ x_j^{2n} = x_1^{2n} \text{ and } x_j^{2n} y_j = y_j$$

($2 \leq j \leq t$) hold in D_{2n}^{2t} , we have

$$x_1 y_j x_1^{2n-1} = x_1^{2n} y_j = x_j^{2n} y_j = y_j. \quad (11)$$

Next we show that the relations $x_1 y_j = y_j x_1$ and $x_j^{2n} y_j = y_j$ ($2 \leq j \leq t$) are consequences of the other relations and the relations (11). First, from the second and first relations ($i = 1$), we have

$$x_1^{2n+1} = x_1^{2n} (x_1^n y_1)^2 x_1 \equiv x_1^n (x_1^{2n} y_1) x_1^n y_1 x_1 = (x_1^n y_1)^2 x_1 = x_1.$$

It follows from (11), the relation $x_j^{2n} = x_1^{2n}$ and the fact that $x_1^{2n+1} = x_1$, that we have

$$x_j^{2n} y_j = x_j^{2n} (x_1 y_j x_1^{2n-1}) = x_1^{2n+1} y_j x_1^{2n-1} = x_1 y_j x_1^{2n-1} = y_j$$

for $2 \leq j \leq t$. Moreover,

$$y_j x_j^{2n} = (x_1 y_j x_1^{2n-1}) x_1^{2n} = x_1 y_j x_1^{4n-1} = x_1 y_j x_1^{2n-1} = y_j.$$

From this fact, the relation $x_j^{2n} = x_1^{2n}$ and (11), we have

$$x_1 y_j = x_1 (y_j x_j^{2n}) = x_1 y_j x_1^{2n} \equiv (x_1 y_j x_1^{2n-1}) x_1 = y_j x_1.$$

Therefore, adding the relations $x_1 y_j x_1^{2n-1} = y_j$ ($2 \leq j \leq t$), and then removing the relations $x_j^{2n} y_j = y_j$ and $x_1 y_j = y_j x_1$ ($2 \leq j \leq t$), we have the following presentation

$$\begin{aligned} \langle x_i, y_i \mid & x_1^{2n} y_1 = y_1, (x_1^n y_1)^2 x_1 = x_1, (x_1 y_1)^2 x_1^{2n} = y_1^{2n}, \\ & x_1 y_j x_1^{2n-1} = y_j, (x_j^n y_j)^2 x_j = x_j, (x_j y_j)^2 x_j^{2n} = y_j^{2n}, x_j^{2n} = x_1^{2n} \\ & x_k x_l = x_l x_k, x_{k'} y_{l'} = y_{l'} x_{k'}, y_k x_l = x_l y_k, y_k y_l = y_l y_k \\ & (1 \leq i \leq t, 1 \leq k < l \leq t, 2 \leq k' < l' \leq t, 2 \leq j \leq t) \rangle \end{aligned}$$

for D_{2n}^{2t} with n odd. Since the deficiency of the presentation above and the rank of the Schur multiplier of D_{2n}^{2t} (with n odd) are the same number, $2t^2 - t$, it follows that D_{2n}^{2t} is efficient as a semigroup.

Next we prove that D_{2n}^m is efficient for $m \geq 3$ odd. Let $m = 2t + 3$ for some integer $t \geq 0$. If $t = 0$, we know from Proposition 5.12 that D_{2n}^3 is efficient as a semigroup. If $t \geq 1$, then it follows from Lemma 5.9, the presentation \mathcal{P}_5 for D_{2n}^3 and the presentation for D_{2n}^{2t} above, that the semigroup presentation

$$\begin{aligned} \langle a, x, z, x_i, y_i \mid & z a z a z = z, (x z^n)^{2n} = a^2, x^{2n-1} a x = a, \\ & x^{n-1} z = z x^{n+1}, z^{n-1} x z^{n+1} = x, x^{2n} = a^2, \\ & x_1^{2n} y_1 = y_1, (x_1^n y_1)^2 x_1 = x_1, (x_1 y_1)^2 x_1^{2n} = y_1^{2n}, \\ & x_1 y_j x_1^{2n-1} = y_j, (x_j^n y_j)^2 x_j = x_j, (x_j y_j)^2 x_j^{2n} = y_j^{2n}, \\ & x_k x_l = x_l x_k, x_{k'} y_{l'} = y_{l'} x_{k'}, y_k x_l = x_l y_k, y_k y_l = y_l y_k, \\ & a x_i = x_i a, a y_i = y_i a, x x_i = x_i x, x y_i = y_i x, \\ & z x_i = x_i z, z y_i = y_i z, x_i^{2n} = x^{2n} \\ & (1 \leq i \leq t, 1 \leq k < l \leq t, 2 \leq k' < l' \leq t, 2 \leq j \leq t) \rangle \end{aligned}$$

defines $D_{2n}^{2t+3} = D_{2n}^3 \times D_{2n}^{2t}$ as a semigroup. Notice that the deficiency of the above presentation is $2t^2 + 5t + 4$. However, the rank of the Schur multiplier of D_{2n}^{2t+3} is $(2t + 3)(2t + 2)/2 = 2t^2 + 5t + 3$ (see [21]).

To obtain an efficient presentation, first notice that the relation $x y_1 x^{2n-1} = y_1$ is a consequence of the relations $x y_1 = y_1 x$, $x_1^{2n} = x^{2n}$ and $x_1^{2n} y_1 = y_1$. Indeed,

observe that

$$xy_1x^{2n-1} = y_1x^{2n} = y_1x_1^{2n} = y_1.$$

Next notice that the relation $x^{2n+1} = x$ is a consequence of the first six relations of the presentation above (as we proved $x^{2n} = 1$ in the proof of Proposition 5.12). From this fact, the relation $x_1^{2n} = x^{2n}$ and the new relation $xy_1x^{2n-1} = y_1$, we show that the relations $x_1^{2n}y_1 = y_1$ and $xy_1 = y_1x$ are redundant. Indeed,

$$x_1^{2n}y_1 = x_1^{2n}(xy_1x^{2n-1}) = x^{2n+1}y_1x^{2n-1} = xy_1x^{2n-1} = y_1$$

and

$$y_1x_1^{2n} = (xy_1x^{2n-1})x_1^{2n} = xy_1x^{4n-1} = xy_1x^{2n-1} = y_1.$$

It also follows that

$$xy_1 = x(y_1x_1^{2n}) = (xy_1x^{2n-1})x = y_1x.$$

Therefore, the relations $x_1^{2n}y_1 = y_1$ and $xy_1 = y_1x$ can be replaced by the single new relation $xy_1x^{2n-1} = y_1$, and hence D_{2n}^{2t+3} is efficient as a semigroup, as required. ■

From Theorems 5.10 and 5.13, we may deduce that

Theorem 5.14 *The direct power D_{2n}^m is an efficient semigroup for $m \geq 2$.*

For odd n , we do not know whether D_{2n} is efficient as a semigroup. But we know that D_{2n}^m ($m \geq 2$) is efficient as a semigroup although the semigroup efficiency of direct product of groups is harder than the group case. This makes Open Problem 1 more interesting.

5.5 Semigroup efficiency of $PSL(2, p)$

The efficiency of $PSL(2, p)$ (as a group) has been studied in many papers (see for example [8], [63] and [71]). In this section, we prove that $PSL(2, p)$ is efficient as a semigroup for all primes p .

We begin with the (inefficient) group presentation of $PSL(2, p)$

$$\mathcal{P}(G, p) = \langle x, y \mid x^2 = 1, y^p = 1, (xy)^3 = 1, (xy^4xy^{\frac{p+1}{2}})^2 = 1 \rangle$$

for p an odd prime (which can be deduced from the presentation in Theorem 1.44).

Next we give an inefficient semigroup presentation for $PSL(2, p)$ which will be useful for Theorem 5.19.

Lemma 5.15 *If p is an odd prime, then*

$$\mathcal{G}_p = \langle x, y \mid x^3 = x, y^p = x^2, (xy)^3 = x^2, xy^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = y \rangle$$

is a semigroup presentation for $PSL(2, p)$.

Proof From the last and the first relations, we have

$$x^2y = x^3y^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = xy^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = y.$$

It follows from the second relation that x^2 is central, and so x^2 is the identity of the semigroup S_p defined by \mathcal{G}_p . Moreover, x and y have inverses, namely x and y^{p-1} , and so S_p is, in fact, a group. It is clear that the following

$$\mathcal{G}'_p = \langle x, y \mid x^2 = 1, y^p = 1, (xy)^3 = 1, y^{-1}xy^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = 1 \rangle$$

is a group presentation for S_p .

Since, from the first three relations of \mathcal{G}'_p ,

$$y^{-1}xy^{p-1}xy^{-1} = y^{-1}x^{-1}y^{-1}x^{-1}y^{-1} = x,$$

it follows from the last relation of \mathcal{G}'_p that

$$\begin{aligned} 1 &= y^{-1}xy^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} \\ &= (y^{-1}xy^{p-1}xy^{-1})y^4xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} \\ &= (xy^4xy^{\frac{p+1}{2}})^2, \end{aligned}$$

which is the last relation of $\mathcal{P}(G, p)$.

Now we prove that the last relation of \mathcal{G}'_p is a consequence of the relations of $\mathcal{P}(G, p)$. Since $y^{p-1}x^2y = y^{p-1}y = 1$, $(xy^4xy^{\frac{p+1}{2}})^2 = 1$ and $(xy)^3 = y^p = x^2 = 1$, it follows that

$$\begin{aligned} xy^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} &= xy^{p-1}x(y^{p-1}x^2y)y^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} \\ &\equiv xy^{p-1}xy^{p-1}x(xy^4xy^{\frac{p+1}{2}})^2 = xy^{p-1}xy^{p-1}x \\ &= xy^{p-1}x(xyxyx)x \equiv xy^{p-1}x^2yxyx^2 = y. \end{aligned}$$

Therefore, \mathcal{G}_p and $\mathcal{P}(G, p)$ define isomorphic groups, and so $S_p \cong PSL(2, p)$, as required. \blacksquare

Next we give a deficiency one semigroup presentation which defines a group and which will prove useful in obtaining an efficient presentation for $PSL(2, p)$.

Lemma 5.16 *The following semigroup presentation*

$$\mathcal{H}(p, k) = \langle x, y \mid y^p = x^2, yxyxy = x, xy^{kp-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = y \rangle$$

defines a group with the identity $x^2(xy^2)^p$ for each odd prime p and for each positive integer k .

Proof First notice that, from the first relation, x^2 and y^p are both central. Then we have

$$\begin{aligned} x^{2k+3}(xy^4xy^{\frac{p+1}{2}})^2 &= y^{(k+1)p}x(xy^4xy^{\frac{p+1}{2}})^2 && \text{(by } x^2 = y^p) \\ &= y^{kp+1}x^2y^{p-1}y^4xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} && \text{(since } x^2 \text{ is central)} \\ &= y^{kp+1}x^3xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} && \text{(by } x^2 = y^p) \\ &= yxy^{kp}x^2xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} && \text{(since } y^p \text{ is central)} \\ &= yxyx^2y^{kp-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} && \text{(since } x^2 \text{ is central).} \end{aligned}$$

Thus, from the last and second relations, we have

$$x^{2k+3}(xy^4xy^{\frac{p+1}{2}})^2 = yxyxy = x. \quad (12)$$

We also have

$$\begin{aligned}
x^4 y x y &= x^4 y x (x y^{kp-1} x y^3 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}}) \quad (\text{by the third relation}) \\
&= x^4 y^{kp} x x^2 y^3 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}} \quad (\text{since } x^2 \text{ is central}) \\
&= x^4 y^{kp} x y^{p-1} y^4 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}} \quad (\text{by } x^2 = y^p) \\
&= x y^{p-1} y^{kp} x^3 (x y^4 x y^{\frac{p+1}{2}})^2 \quad (\text{since } x^2 \text{ and } y^p \text{ are central}) \\
&= x y^{p-1} x^{2k+3} (x y^4 x y^{\frac{p+1}{2}})^2 \quad (\text{by } x^2 = y^p).
\end{aligned}$$

It follows from (12) that

$$y x y x^4 = x^4 y x y = x y^{p-1} x, \quad (13)$$

and, from the first relation,

$$x^{2k+2} y x y \equiv x^{2k-2} (x^4 y x y) = y^{(k-1)p} x y^{p-1} x = x y^{kp-1} x,$$

and so, from the last relation,

$$y x^{2k+2} (x y^4 x y^{\frac{p+1}{2}})^2 = x^{2k+2} y (x y^4 x y^{\frac{p+1}{2}})^2 = x y^{kp-1} x y^3 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}} = y. \quad (14)$$

From the second relation, for any $k \in \mathbb{N}$, we have $(y x y)^k x y^k = x$. In particular,

$$(y x y)^{p-1} x y^{p-1} = x. \quad (15)$$

It follows from the third and first relations that

$$\begin{aligned}
y (x y^2)^{p-1} &\equiv (y x y)^{p-1} y = ((y x y)^{p-1} x y^{p-1}) y^{(k-1)p} x y^3 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}} \\
&= x y^{(k-1)p} x y^3 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}} = x^{2k} y^3 x y^{\frac{p+1}{2}} x y^4 x y^{\frac{p+1}{2}},
\end{aligned}$$

and so, by multiplying by $x y$ on the left-hand side, we have

$$(x y^2)^p = x^{2k} (x y^4 x y^{\frac{p+1}{2}})^2. \quad (16)$$

Therefore, it follows from (12) and (14) that

$$x^3 (x y^2)^p = x \text{ and } y x^2 (x y^2)^p = y \quad (17)$$

so that $x^2(xy^2)^p$ is a right identity.

Next we prove that $y(xy^2)^p = (xy^2)^p y$. By the last and first relations, we have

$$\begin{aligned} yxy &= yx^2y^{kp-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = y^{kp}xy^{p-1}y^4xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} \\ &= xy^{p-1}x^{2k-1}(xy^4xy^{\frac{p+1}{2}})^2. \end{aligned}$$

It follows, by multiplying by $(yxy)^{p-1}$ on the left-hand side, that

$$y(xy^2)^{p-1}xy \equiv (yxy)^{p-1}yxy = (yxy)^{p-1}xy^{p-1}x^{2k-1}(xy^4xy^{\frac{p+1}{2}})^2,$$

and so, by (15), we have

$$y(xy^2)^{p-1}xy = x^{2k}(xy^4xy^{\frac{p+1}{2}})^2.$$

Therefore, from (16), we have that $y(xy^2)xy = (xy^2)^p$, and so we have

$$y(xy^2)^p = (xy^2)^p y. \quad (18)$$

It is clear that $x(xy^2)^p$ is a right inverse of x . Now we show that $x^2(xy^2)^{p-1}xy$ is a right inverse of y . From the first relation and (18),

$$y(x^2(xy^2)^{p-1}xy) = y(xy^2)^{p-1}xy^{p+1} = y(xy^2)^p y^{p-1} = y^p(xy^2)^p = x^2(xy^2)^p.$$

Therefore, from Lemma 5.1, we deduce that $\mathcal{H}(p, k)$ defines a group. ■

Let $H(p, k)$ denote the group defined by the semigroup presentation $\mathcal{H}(p, k)$.

Lemma 5.17 *For all odd k and all odd primes p , if the generator x satisfies $x^4 = 1$, then $x^2 = 1$ holds in $H(p, k)$.*

Proof First we show that

$$y^{p-1} = xyxyx \quad (19)$$

without assuming $x^4 = 1$. Observe that we have

$$\begin{aligned} y^{p-1} &= y^{p-1}x^{2k+2}(xy^4xy^{\frac{p+1}{2}})^2 && \text{(by (14))} \\ &= x(xy^{p-1}x)x^{2k-1}(xy^4xy^{\frac{p+1}{2}})^2 && \text{(since } x^2 \text{ is central)} \\ &= xyxyx^{2k+3}(xy^4xy^{\frac{p+1}{2}})^2 && \text{(by (13))} \\ &= xyxyx && \text{(by (12)),} \end{aligned}$$

as required. Next we have from the last and first relations of $\mathcal{H}(p, k)$ that

$$yxy = yx^2y^{kp-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = x^{2k+3}y^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}}.$$

Now assume that $x^4 = 1$. Then, since k is odd, we have $x^{2k+3} = x$, so that

$$yxy = xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}}.$$

It follows from (13) and $x^4 = 1$ that

$$xy^{p-1}x = xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}}.$$

Multiplying the above equation by $y^{2p-3}x^3$ on the left, we have

$$y^{p-4}x = xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}}$$

since $x^4 = y^{2p} = 1$. By multiplying the above equation by $x^2y^{p-1}x^3$ on the right, we have

$$y^{p-4}x^3y^{p-1}x^3 = xy^{\frac{p+1}{2}}xy^4x^3y^{\frac{3p-1}{2}}x^3. \quad (20)$$

Since

$$\begin{aligned} y^{p-4}x^3y^{p-1}x^3 &= y^{p-4}xy^{p-1}x && (\text{since } x^2 \text{ is central and } x^4 = 1) \\ &= y^{p-4}x^2yxyx^2 && (\text{by (19)}) \\ &= y^{2p-3}xyx^2 && (\text{since } x^2 = y^p) \\ &= y^{2p-2}xyxy^2x^2 && (\text{by the second relation}) \\ &= (y^{2p-2}x)y(x^3y^2), \end{aligned}$$

it follows from (20) that

$$(y^{2p-2}x)y(x^3y^2) = uy^4v$$

where $u = xy^{\frac{p+1}{2}}x$ and $v = x^3y^{\frac{3p-1}{2}}x^3$. Since $vu = x^3y^{2p}x = x^3x^4x = 1$, it follows from

$$((y^{2p-2}x)y(x^3y^2))^p = (uy^4u^{-1})^p$$

that $(y^{2p-2}x)y^p(x^3y^2) = uy^{4p}u^{-1}$, and so $x^2 = y^p = 1$, as required. \blacksquare

Before the main theorem of this section, we state a useful theorem from [13]. For any group G , we use G' and $Z(G)$ to denote the derived group of G and the centre of G , respectively. If $G = G'$ then G is said to be a *perfect group*. A *covering group* C of G is a group such that C has a subgroup A with $C/A \cong G$, $A \leq C' \cap Z(C)$ and $|A| = |M(G)|$, where $M(G)$ denotes the Schur multiplier of G . Thus it is clear that $SL(2, p)$ is a covering group of $PSL(2, p)$.

Theorem 5.18 *Let G be a finite perfect group. Suppose $G \cong H/B$ where $B \leq H' \cap Z(H)$. Then G has a unique covering group C and H is a homomorphic image of C .*

For a proof and more details, see [13].

Theorem 5.19 *For each prime p , $PSL(2, p)$ is efficient as a semigroup. Moreover, $PSL(2, p) \cong H(p, 11)$ for all odd primes p .*

Proof Since $PSL(2, 2) \cong D_6$ and D_6 is efficient (see Lemma 5.7), $PSL(2, 2)$ is efficient as a semigroup. For each odd prime p , we use the group $H(p, k)$. Note that, for all k and odd primes p , $PSL(2, p)$ is a homomorphic image of $H(p, k)$ since

$$yxyxy = (x^2y)xyxy = x(xy)^3 = x^3 = x$$

(see Lemma 5.15). For some k , if $x^3 = x$ (or $x^2 = 1$) holds in $H(p, k)$, then $PSL(2, p)$ is isomorphic to $H(p, k)$. Therefore, to prove $PSL(2, p)$ is efficient, it is enough to show that $x^3 = x$ holds in $H(p, k_0)$ for some k_0 .

First we show that $PSL(2, 3) \cong H(3, 1)$. Recall that

$$H(3, 1) \cong \langle x, y \mid y^3 = x^2, yxyxy = x, xy^2xy^3xy^2xy^4xy^2 = y \rangle.$$

and that $x^2(xy^2)^3$ is an identity. Observe that, from the first and last relations above, we have

$$x^3 = xy^3 = (xy^2)xy^2xy^3xy^2xy^4xy^2 \equiv (xy^2)^3yxy^2xy^4xy^2 = x^2(xy^2)^3yxy^2xyxy^2.$$

Since $x^2(xy^2)^3 = 1$, it follows from the second relation above that

$$x^3 = xyx^2xyxy^2 \equiv yxy(yxyxy)y = yxyxy = x,$$

as required. Notice that $H(3, 1) \cong H(3, 11)$ (we use a coset enumerate program for this).

Now consider the group $H = H(p, 11)$ for odd primes $p \geq 5$. Now we show that H is a perfect group, that is H/H' is trivial. Observe that we have

$$\begin{aligned} H/H' &\cong \langle x, y \mid y^p = x^2, y^3x = 1, y^{12p+6}x^5 = 1, xy = yx \rangle \\ &\cong \langle y \mid y^{p+6} = 1, y^{12p-9} = 1 \rangle \quad (\text{by eliminating } x = y^{-3}) \\ &\cong \langle y \mid y^{p+6} = 1, y^{81} = 1 \rangle \\ &\cong \{1\} \end{aligned}$$

since the highest common divisor of $p+6$ and 81 is one if $p \neq 3$. (For more details for the group G/G' see [35] or [36].)

Therefore, we have $x^2 \in H' = H$ and x^2 is central so that $B \leq H' \cap Z(H)$ where B is the subgroup generated by x^2 . Since

$$\begin{aligned} H/B &= \langle x, y \mid y^p = x^2 = 1, yxyxy = x, xy^{kp-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = y \rangle \\ &= \langle x, y \mid y^p = x^2 = 1, yxyxy = x, xy^{p-1}xy^3xy^{\frac{p+1}{2}}xy^4xy^{\frac{p+1}{2}} = y \rangle \\ &\cong PSL(2, p) \end{aligned}$$

by Lemma 5.15, it follows from Theorem 5.18 that H is either $PSL(2, p)$ or $SL(2, p)$. In both cases $x^4 = 1$ holds in H , and so, by Lemma 5.17, $x^2 = 1$ holds in H so that $PSL(2, p) \cong H/B \cong H$, as required. \blacksquare

Remark. The group $H(p, k)$ is perfect if and only if $2k + 5$ and $p + 6$ are coprime. The values of k for which this holds for all p are those for which $2k + 5$ is a power of 3.

It is not true, in general, that $PSL(2, p) \cong H(p, k)$ for all odd k . Even the orders of $PSL(2, p)$ and $H(p, k)$ may be different. For example, $|H(5, 3)| = 11 \times |PSL(2, 5)|$.

Finally, we give examples of efficient semigroups which lead us to consider the efficiency of direct products of the semigroups $PSL(2, p)$. It is known [20] that these are efficient as groups.

Example 5.20 *The semigroup presentation*

$$\langle a, b \mid a^7 = a, (a^3b)^2 = a^6, (a^2b)^2b^5 = b \rangle$$

defines $PSL(2, 2) \times PSL(2, 2)$. Therefore $PSL(2, 2) \times PSL(2, 2)$ is efficient as a semigroup.

Proof Since

$$a^6b = a^6(a^2b)^2b^5 \equiv a^7(aba^2b)b^5 = (a^2b)^2b^5 = b$$

from the last and first relations above, it follows that a^6 is a left identity. It is clear that a^5 is a left inverse of a and a^3ba^3 is a left inverse of b , and so, by Lemma 5.1, the above presentation defines a group with the identity a^6 . Thus this group may be given by the following group presentation:

$$\langle a, b \mid a^6 = 1, (a^3b)^2 = 1, (a^2b)^2b^4 = 1 \rangle.$$

It is known that this group presentation defines $PSL(2, 2) \times PSL(2, 2)$ efficiently (see [20]). Therefore $PSL(2, 2) \times PSL(2, 2)$ is efficient as a semigroup. ■

Since $PSL(2, 2)^2 \cong D_6^2$, the efficiency of $PSL(2, 2)^2$ also follows from Proposition 5.11.

Example 5.21 *The semigroup presentation*

$$\langle a, b \mid a^4 = a, b^3 = a^3, (ab)^6b = b, bab^2a^2ba = ab \rangle$$

defines $PSL(2, 3) \times PSL(2, 3)$. Therefore $PSL(2, 3) \times PSL(2, 3)$ is efficient as a semigroup.

Proof From the first and the third relations of the above presentation, we have

$$a^3b = a^3(ab)^6b \equiv a^4b(ab)^5b = ab(ab)^5b = b.$$

It follows from the second relation that a^3 is the identity, a^2 is the inverse of a and b^2 is the inverse of b . Therefore the above presentation defines a group with the identity a^3 . This group may be defined by the following group presentation

$$\langle a, b \mid a^3 = 1, b^3 = 1, (ab)^6 = 1, bab^2a^2ba = ab \rangle$$

which is given as an efficient group presentation of $PSL(2, 3) \times PSL(2, 3)$ (see [20]). ■

Open problem. Does there exist a deficiency two semigroup presentation for $PSL(2, p) \times PSL(2, p)$ with p prime, $p \geq 5$?

More generally, we have the following question:

Open problem. Is an efficient finite group G efficient as a semigroup?

Alternatively, are there any examples of finite efficient groups which are not efficient as semigroups?

Chapter 6

Minimal Presentations for Zero Semigroups, Free Semilattices and Rectangular Bands

In the previous chapter we considered groups as semigroups. Then we proved that certain classes of groups were efficient as semigroups. In this chapter we find certain infinite classes of both efficient and inefficient semigroups. For example, finite rectangular bands are efficient semigroups. By way of contrast we show that finite zero semigroups and free semilattices are never efficient. In the process, we calculate their second homology groups and find minimal presentations for them. Finally, we compare these results with some well-known results on the efficiency of groups.

Recall that we define the monoid S^1 by adjoining an identity 1 to a semigroup S . We now define the n th (left) integral homology of a semigroup S , $H_n(S)$, to be the n th (left) integral homology of the monoid S^1 , that is

$$H_n(S) = H_n(S^1).$$

The results of this chapter together with some of the results of the previous chapter will appear in Semigroup Forum (see [1]).

6.1 The n th homology groups of semigroups with a left or a right zero

In the following proposition, we compute the homology groups of a semigroup with a left or a right zero.

Theorem 6.1 *Let S be a semigroup with a left or right zero. Then, for $n \geq 1$, the n th homology of S is the trivial group.*

Proof We compute the homology groups of a semigroup S by using the bar resolution of \mathbb{Z} (see Chapter 1).

Recall that applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}S^1} -$, where \mathbb{Z} is a trivial right $\mathbb{Z}S^1$ -module, to the bar resolution of \mathbb{Z} , in effect, yields the following chain complex:

$$\bar{\mathbf{B}} = \cdots \longrightarrow \bar{B}_n \xrightarrow{\bar{\partial}_n} \bar{B}_{n-1} \xrightarrow{\bar{\partial}_{n-1}} \cdots \xrightarrow{\bar{\partial}_2} \bar{B}_1 \xrightarrow{\bar{\partial}_1} \bar{B}_0 \longrightarrow 0$$

where \bar{B}_n is the free abelian group on all $[x_1 | \dots | x_n]$ with $x_i \in S$ and, and the group homomorphism $\bar{\partial}_n$ is given by

$$\begin{aligned} \bar{\partial}_n([x_1 | \dots | x_n]) &= [x_2 | \dots | x_n] + \sum_{i=1}^{n-1} (-1)^i [x_1 | \dots | x_i x_{i+1} | \dots | x_n] \\ &\quad + (-1)^n [x_1 | \dots | x_{n-1}]. \end{aligned}$$

Now we apply Proposition 1.19 to show that the chain complex $\bar{\mathbf{B}}$ is exact for $n \geq 1$. For this, we only need to construct a contracting homotopy.

In the case when S has a left zero z , we construct a contracting homotopy for $\bar{\mathbf{B}}$, namely

$$\cdots \longleftarrow \bar{B}_{n+1} \xleftarrow{s_n} \bar{B}_n \xleftarrow{s_{n-1}} \cdots \xleftarrow{s_1} \bar{B}_1 \xleftarrow{s_0} \bar{B}_0 \longleftarrow 0$$

where each $s_n : \bar{B}_n \longrightarrow \bar{B}_{n+1}$ is defined by

$$s_n([x_1 | \dots | x_n]) = [z | x_1 | \dots | x_n].$$

Then observe that, for $[x_1|\dots|x_n] \in \bar{B}_n$, we have

$$\begin{aligned}
(\bar{\partial}_{n+1}s_n + s_{n-1}\bar{\partial}_n)[x_1|\dots|x_n] &= \bar{\partial}_{n+1}(s_n([x_1|\dots|x_n])) + s_{n-1}(\bar{\partial}_n([x_1|\dots|x_n])) \\
&= \bar{\partial}_{n+1}([z|x_1|\dots|x_n]) + s_{n-1}\left([x_2|\dots|x_n] + \left(\sum_{i=1}^{n-1}(-1)^i[x_1|\dots|x_ix_{i+1}|\dots|x_n]\right)\right. \\
&\quad \left.+ (-1)^n[x_1|\dots|x_{n-1}]\right) \\
&= \left([x_1|\dots|x_n] - [zx_1|x_2|\dots|x_n] + \left(\sum_{i=1}^{n-1}(-1)^{i+1}[z|x_1|\dots|x_ix_{i+1}|\dots|x_n]\right)\right. \\
&\quad \left.+ (-1)^{n+1}[z|x_1|\dots|x_{n-1}]\right) + \left([z|x_2|\dots|x_n] + \left(\sum_{i=1}^{n-1}(-1)^i[z|x_1|\dots|x_ix_{i+1}|\dots|x_n]\right)\right. \\
&\quad \left.+ (-1)^n[z|x_1|\dots|x_{n-1}]\right) = [x_1|\dots|x_n]
\end{aligned}$$

so that $\bar{\partial}_{n+1}s_n + s_{n-1}\bar{\partial}_n = I_{\bar{B}_n}$, where $I_{\bar{B}_n}$ is the identity homomorphism of \bar{B}_n . It follows that $\bar{\mathbf{B}}$ is exact at \bar{B}_n for $n \geq 1$, and hence the n th homology is trivial, as required.

In the case when S has a right zero z , a contracting homotopy can be defined by

$$t_n([x_1|\dots|x_n]) = (-1)^{n+1}[x_1|\dots|x_n|z]$$

and, similarly, we have

$$\begin{aligned}
(\bar{\partial}_{n+1}t_n + t_{n-1}\bar{\partial}_n)[x_1|\dots|x_n] &= \bar{\partial}_{n+1}(t_n([x_1|\dots|x_n])) + t_{n-1}(\bar{\partial}_n([x_1|\dots|x_n])) \\
&= \bar{\partial}_{n+1}((-1)^{n+1}[x_1|\dots|x_n|z]) + t_{n-1}\left([x_2|\dots|x_n] \right. \\
&\quad \left. + \left(\sum_{i=1}^{n-1}(-1)^i[x_1|\dots|x_ix_{i+1}|\dots|x_n]\right) + (-1)^n[x_1|\dots|x_{n-1}]\right) \\
&= \left((-1)^{n+1}[x_2|\dots|x_n|z] + \left(\sum_{i=1}^{n-1}(-1)^{n+1+i}[x_1|\dots|x_ix_{i+1}|\dots|x_n|z]\right)\right. \\
&\quad \left.+ (-1)^{2n+1}[x_1|\dots|x_{n-1}|x_nz] + (-1)^{2n+2}[x_1|\dots|x_n]\right) \\
&\quad + \left((-1)^n[x_2|\dots|x_n|z] + \left(\sum_{i=1}^{n-1}(-1)^{n+i}[x_1|\dots|x_ix_{i+1}|\dots|x_n|z]\right)\right. \\
&\quad \left.+ (-1)^{2n}[x_1|\dots|x_{n-1}|z]\right) \\
&= (-1)^{2n+2}[x_1|\dots|x_n] = [x_1|\dots|x_n]
\end{aligned}$$

so that $\bar{\partial}_{n+1}t_n + t_{n-1}\bar{\partial}_n = I_{\bar{B}_n}$, and so we conclude that the n th homology is trivial, as required. ■

It follows from Proposition 1.22 and Theorem 6.1 that we have the following corollary.

Corollary 6.2 *Let M be a monoid with a left or right zero. Then, for $n \geq 1$, the n th homology of M is the trivial group.*

6.2 Inefficiency of zero semigroups

Observe that from Theorem 6.1, a semigroup with zero has trivial second homology. In particular, the second homology of a zero semigroup is trivial. Next we investigate the efficiency of finite zero semigroups.

Theorem 6.3 *Let \mathcal{Z}_n be the zero semigroup of order n with $n \geq 2$. Then*

$$\text{def}_S(\mathcal{Z}_n) = (n-1)(n-2).$$

If $n \geq 3$, then \mathcal{Z}_n is inefficient.

Proof Let $\mathcal{Z}_n = \{z_0, z_1, \dots, z_{n-1}\}$ with $z_i z_j = z_0$, where z_0 is the zero element, and let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation for \mathcal{Z}_n . We may assume that there are no relations of the form $w = w$ ($w \in A^+$) nor of the form $w = a$ with $a \in A$, $w \in (A \setminus \{a\})^+$ (otherwise we eliminate the relations of these form without increasing the deficiency of \mathcal{P}).

Decompose A as $A = X \cup Y$, where X represents the non-zero elements of \mathcal{Z}_n and Y represents $\{z_0\}$. Since $z_i z_j = z_0$ for all $z_i, z_j \in \mathcal{Z}_n$, the elements z_1, \dots, z_{n-1} must belong to every generating set of \mathcal{Z}_n . It follows that $|X| \geq n-1$.

The set Y may be empty. We let

$$\begin{aligned} R_1 &= R \cap ((X^2 \times A^+) \cup (A^+ \times X^2)) = \{(u = v) \in R \mid u \in X^2 \text{ or } v \in X^2\}, \\ R_2 &= R \cap ((Y \times A^+) \cup (A^+ \times Y)) = \{(u = v) \in R \mid u \in Y \text{ or } v \in Y\}, \\ R_3 &= R \setminus (R_1 \cup R_2). \end{aligned}$$

Clearly, $R = R_1 \cup R_2 \cup R_3$, and both $R_1 \cap R_3$ and $R_2 \cap R_3$ are empty. We claim that $R_1 \cap R_2$ is also empty. Otherwise, we would have a relation of the form $x_i x_j = y$ ($x_i, x_j \in X, y \in Y$) which is a contradiction to the assumption on \mathcal{P} .

On the set X^2 define a binary relation

$$\rho = (X^2 \times X^2) \cap R_1 = \{(x_1 x_2 = x_3 x_4) \in R_1 \mid x_1, x_2, x_3, x_4 \in X\}$$

and let ρ^* be the equivalence relation generated by ρ . Then consider an arbitrary equivalence class C of ρ^* and let $x_1 x_2 \in C$. We must have

$$|(C \times C) \cap \rho| = |(C \times C) \cap R_1| \geq |C| - 1,$$

because any two elements of C must be connected by a chain of pairs from ρ . Also note that the relations $x_1 x_2 = x_1^3$ holds in \mathcal{Z}_n and that x_1^3 cannot be obtained from $x_1 x_2$ by applying relations from ρ . Hence there exists a relation of the form $(x_3 x_4 = u) \in R_1 \setminus (C \times C)$ with $x_3 x_4 \in C$ and $u \notin X^2$. Therefore,

$$|R_1| \geq \sum_{C \in X^2 / \rho^*} (|C \times C \cap R_1| + 1) \geq \sum_{C \in X^2 / \rho^*} |C| = |X^2|.$$

Noting that the relation $y^2 = y$ holds in \mathcal{Z}_n for each $y \in Y$, and hence there exists a relation $(w_y = y) \in R_2$ with $|w_y| \geq 2$ (otherwise we eliminate y which is a contradiction to the minimality of A). Therefore we have $|R_2| \geq |Y|$, and hence

$$\begin{aligned} \text{def}(\mathcal{P}) &= |R| - |A| = |R_1| + |R_2| + |R_3| - |X| - |Y| \\ &\geq |X^2| + |Y| - |X| - |Y| \geq (n-1)^2 - (n-1) = (n-1)(n-2). \end{aligned}$$

Finally, we show that \mathcal{Z}_n can be presented with a deficiency $(n-1)(n-2)$ presentation. For this purpose, we start with its Cayley table as a presentation, that is

$$\langle a_0, a_1, \dots, a_{n-1} \mid a_i a_j = a_0 \ (0 \leq i, j \leq n-1) \rangle.$$

By eliminating the generator $a_0 = a_1^2$, we obtain

$$\begin{aligned} \langle a_1, \dots, a_{n-1} \mid & a_1^4 = a_1^2, a_1^3 = a_1^2, a_i a_1^2 = a_1^2, a_1^2 a_i = a_1^2, a_k a_l = a_1^2 \\ & (1 < i < n; 1 \leq k, l < n) \rangle. \end{aligned}$$

We show that the relations $a_1^4 = a_1^2$, $a_i a_1^2 = a_1^2$ and $a_1^2 a_i = a_1^2$ for $1 < i < n$ are redundant. Indeed, from the relations $a_1^3 = a_1^2$, $a_i a_1 = a_1^2$ and $a_1 a_i = a_1^2$ ($1 < i < n$), we have

$$a_1^4 \equiv a_1^3 a_1 = a_1^3 = a_1^2,$$

and moreover,

$$a_i a_1^2 \equiv (a_i a_1) a_1 = a_1^3 = a_1^2 \quad \text{and} \quad a_1^2 a_i \equiv a_1 (a_1 a_i) = a_1^3 = a_1^2.$$

By eliminating these redundant relations and the trivial relation $a_1^2 = a_1^2$, we obtain the following presentation

$$\langle a_1, \dots, a_{n-1} \mid a_i a_j = a_1^3 \ (1 \leq i, j \leq n-1) \rangle.$$

of deficiency $(n-1)(n-2)$. Therefore the deficiency of \mathcal{Z}_n is $(n-1)(n-2)$. By Theorem 6.1, $H_2(\mathcal{Z}_n)$ is trivial and hence \mathcal{Z}_n is inefficient for $n \geq 3$. ■

Note that since

$$\mathcal{Z}_2 = \langle x \mid x^3 = x^2 \rangle,$$

\mathcal{Z}_2 is efficient.

6.3 Inefficiency of free semilattices

Next we investigate the free semilattice \mathcal{SL}_A over a finite set A . Recall that \mathcal{SL}_A is the set of all non-empty subsets of A with set-theoretic union as multiplication.

Theorem 6.4 *If A is a finite non-empty set of size n then*

$$\text{def}_S(\mathcal{SL}_A) = n(n-1)/2.$$

In particular, \mathcal{SL}_A is inefficient if $n \geq 2$.

Proof Let $\mathcal{P} = \langle X \mid R \rangle$ be a presentation for \mathcal{SL}_A . As in the previous theorem, we assume that there are no relations of the form $(w = w)$ ($w \in X^+$) nor $w = x$ where $x \in X$ and $w \in (X \setminus \{x\})^+$.

For each $a \in A$, let

$$X_a = \{ x \in X \mid x \text{ represents } \{a\} \}.$$

Note that $\{a\}$ belongs to every generating set of \mathcal{SL}_A so that X_a is non-empty. Let

$$R_1 = ((X \times X^+) \cup (X^+ \times X)) \cap R = \{(u = v) \in R \mid u \in X \text{ or } v \in X\}.$$

Since, for each $x \in X$, the relation $x^2 = x$ holds in \mathcal{SL}_A , it follows that R contains a relation of the form $x = w$ with $w \in X^+$. Note that $|w| \geq 2$ because of the minimality of \mathcal{P} and $|A|$. Therefore we have

$$|R_1| \geq |X|. \tag{1}$$

Next note that if a relation $w_1 = w_2$ holds in \mathcal{SL}_A and if $w_1 \in X_a^+$ then $w_2 \in X_a^+$ as well. Now let $a, b \in A$ ($a \neq b$) and $x \in X_a$, $y \in X_b$ be arbitrary. The relation $xy = yx$ holds in \mathcal{SL}_A and hence is a consequence of R . However, applying relations from R_1 to xy will always yield words from $X_a^+ X_b^+$, whereas yx does not have this form. We conclude that there is a relation $(u = v) \in R \setminus R_1$ such that both u and v represent the element $\{a, b\}$ of \mathcal{SL}_A . Hence

$$|R \setminus R_1| \geq n(n-1)/2. \tag{2}$$

By combining (1) and (2), we have

$$|R| - |X| = |R_1| + |R \setminus R_1| - |X| \geq n(n-1)/2.$$

If $A = \{a_1, \dots, a_n\}$, then it is clear that the presentation

$$\langle a_1, \dots, a_n \mid a_i^2 = a_i, a_j a_k = a_k a_j \ (1 \leq i \leq n, 1 \leq j < k \leq n) \rangle$$

defines \mathcal{SL}_A and has deficiency $n(n-1)/2$. We conclude that

$$\text{def}_S(\mathcal{SL}_A) = n(n-1)/2.$$

Since $A \in \mathcal{SL}_A$ is the zero element, by Theorem 6.1, the second homology of \mathcal{SL}_A is trivial. Therefore \mathcal{SL}_A is inefficient if $|A| = n \geq 2$. ■

Note that if $A = \{a\}$, then $\mathcal{SL}_A \cong \langle a \mid a^2 = a \rangle$, and hence \mathcal{SL}_A is efficient if $|A| = 1$.

6.4 The second homology of rectangular bands

In this section we compute the second homology group of finite rectangular bands.

Recall that the rectangular band $R_{m,n}$ is the set $I \times \Lambda$ where $I = \{1, \dots, m\}$ and $\Lambda = \{1, \dots, n\}$, with the multiplication given by $(i, \lambda)(j, \mu) = (i, \mu)$ for $(i, \lambda), (j, \mu) \in R_{m,n}$.

Theorem 6.5 *For any integers $m, n \geq 1$, we have*

$$H_2(R_{m,n}) = \mathbb{Z}^{(m-1)(n-1)}.$$

Proof Since $R_{1,n}$ is a right zero semigroup, it follows from Theorem 6.1 that $H_2(R_{1,n}) = \{1\}$. Similarly, $H_2(R_{m,1}) = \{1\}$. Thus we may assume that $m, n \geq 2$.

Now we consider the relevant part of the bar resolution of \mathbb{Z}

$$B_3 \xrightarrow{\partial_3} B_2 \xrightarrow{\partial_2} B_1,$$

where B_3 , B_2 and B_1 are the free $\mathbb{Z}R_{m,n}^1$ -modules on the set of formal symbols $[(i, \lambda)|(j, \mu)|(k, \nu)]$, $[(i, \lambda)|(j, \mu)]$ and $[(i, \lambda)]$ ($i, j, k \in I$; $\lambda, \mu, \nu \in \Lambda$) respectively,

and ∂_3 and ∂_2 are given by

$$\begin{aligned}\partial_3([(i, \lambda)|(j, \mu)|(k, \nu)]) &= (i, \lambda)[(j, \mu)|(k, \nu)] - [(i, \mu)|(k, \nu)] \\ &\quad + [(i, \lambda)|(j, \nu)] - [(i, \lambda)|(j, \mu)], \\ \partial_2([(i, \lambda)|(j, \mu)]) &= (i, \lambda)[(j, \mu)] - [(i, \mu)] + [(i, \lambda)].\end{aligned}$$

By applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}R_{m,n}^1} -$, where \mathbb{Z} is a trivial right $\mathbb{Z}R_{m,n}^1$ -module, we obtain the chain complex

$$\mathbb{Z} \otimes_{\mathbb{Z}R_{m,n}^1} B_3 \xrightarrow{1 \otimes \partial_3} \mathbb{Z} \otimes_{\mathbb{Z}R_{m,n}^1} B_2 \xrightarrow{1 \otimes \partial_2} \mathbb{Z} \otimes_{\mathbb{Z}R_{m,n}^1} B_1$$

of abelian groups. As before, this chain complex is isomorphic to

$$\bar{B}_3 \xrightarrow{\bar{\partial}_3} \bar{B}_2 \xrightarrow{\bar{\partial}_2} \bar{B}_1$$

where \bar{B}_3 , \bar{B}_2 and \bar{B}_1 are the free abelian groups on the set of all formal symbols $[(i, \lambda)|(j, \mu)|(k, \nu)]$, $[(i, \lambda)|(j, \mu)]$ and $[(i, \lambda)]$ ($i, j, k \in I$; $\lambda, \mu, \nu \in \Lambda$) respectively, and the group homomorphisms $\bar{\partial}_3$ and $\bar{\partial}_2$ are given by

$$\begin{aligned}\bar{\partial}_3([(i, \lambda)|(j, \mu)|(k, \nu)]) &= [(j, \mu)|(k, \nu)] - [(i, \mu)|(k, \nu)] + [(i, \lambda)|(j, \nu)] \\ &\quad - [(i, \lambda)|(j, \mu)], \\ \bar{\partial}_2([(i, \lambda)|(j, \mu)]) &= [(j, \mu)] - [(i, \mu)] + [(i, \lambda)].\end{aligned}$$

We find a basis set for $\ker(\bar{\partial}_2)$. Each $\alpha \in \bar{B}_2$ has the form;

$$\alpha = \sum_{i,j \in I} \sum_{\lambda, \mu \in \Lambda} \alpha(i, \lambda, j, \mu) [(i, \lambda)|(j, \mu)]$$

with $\alpha(i, \lambda, j, \mu) \in \mathbb{Z}$. It follows that $\alpha \in \ker(\bar{\partial}_2)$ if and only if

$$\begin{aligned}0 &= \bar{\partial}_2(\alpha) = \sum_{i,j \in I} \sum_{\lambda, \mu \in \Lambda} \alpha(i, \lambda, j, \mu) ([(j, \mu)] - [(i, \mu)] + [(i, \lambda)]) \\ &= \sum_{i \in I, \lambda \in \Lambda} \left(\alpha(i, \lambda, i, \lambda) + \sum_{\substack{j \neq i \text{ or} \\ \mu \neq \lambda}} (\alpha(j, \mu, i, \lambda) - \alpha(i, \mu, j, \lambda) \right. \\ &\quad \left. + \alpha(i, \lambda, j, \mu)) \right) [(i, \lambda)]\end{aligned}$$

or, equivalently, if and only if

$$\alpha(i, \lambda, i, \lambda) = \sum_{\substack{j \neq i \text{ or} \\ \mu \neq \lambda}} (-\alpha(j, \mu, i, \lambda) + \alpha(i, \mu, j, \lambda) - \alpha(i, \lambda, j, \mu)) \quad (3)$$

for each pair (i, λ) .

For fixed $\alpha(i_0, \lambda_0, j_0, \mu_0)$ with $i_0 \neq j_0$ or $\lambda_0 \neq \mu_0$, we set $\alpha(i_0, \lambda_0, j_0, \mu_0) = 1$ and all the other variables on the right-hand side in (3) to zero, and so we obtain

$$\alpha(i_0, \lambda_0, i_0, \lambda_0) = -1, \quad \alpha(i_0, \mu_0, i_0, \mu_0) = 1, \quad \text{and} \quad \alpha(j_0, \mu_0, j_0, \mu_0) = -1.$$

Therefore we obtain the following basis set for $\ker(\bar{\partial}_2)$:

$$\{ x(i, \lambda, j, \mu) \mid i, j \in I, \lambda, \mu \in \Lambda \text{ (} i \neq j \text{ or } \lambda \neq \mu \text{)} \}$$

where

$$x(i, \lambda, j, \mu) = [(i, \lambda)|(j, \mu)] - [(i, \lambda)|(i, \lambda)] + [(i, \mu)|(i, \mu)] - [(j, \mu)|(j, \mu)].$$

Now we find a generating set for $\text{im}(\bar{\partial}_3)$. If $y = [(i, \lambda)|(j, \mu)|(k, \nu)]$, then

$$\begin{aligned} \bar{\partial}_3(y) &= [(j, \mu)|(k, \nu)] - [(i, \mu)|(k, \nu)] + [(i, \lambda)|(j, \nu)] - [(i, \lambda)|(j, \mu)] \\ &= [(j, \mu)|(k, \nu)] - [(j, \mu)|(j, \mu)] + [(j, \nu)|(j, \nu)] - [(k, \nu)|(k, \nu)] \\ &\quad - [(i, \mu)|(k, \nu)] + [(i, \mu)|(i, \mu)] - [(i, \nu)|(i, \nu)] + [(k, \nu)|(k, \nu)] \\ &\quad + [(i, \lambda)|(j, \nu)] - [(i, \lambda)|(i, \lambda)] + [(i, \nu)|(i, \nu)] - [(j, \nu)|(j, \nu)] \\ &\quad - [(i, \lambda)|(j, \mu)] + [(i, \lambda)|(i, \lambda)] - [(i, \mu)|(i, \mu)] + [(j, \mu)|(j, \mu)], \end{aligned}$$

and so we have

$$\bar{\partial}_3([(i, \lambda)|(j, \mu)|(k, \nu)]) = x(j, \mu, k, \nu) - x(i, \mu, k, \nu) + x(i, \lambda, j, \nu) - x(i, \lambda, j, \mu).$$

Therefore an abelian group presentation of $H_2(R_{m,n})$ is

$$\begin{aligned} \langle &x(i, \lambda, j, \mu) \text{ (} i, j \in I, \lambda, \mu \in \Lambda \text{)} \mid x(i, \lambda, i, \lambda) = 0, \\ &x(j, \mu, k, \nu) - x(i, \mu, k, \nu) + x(i, \lambda, j, \nu) - x(i, \lambda, j, \mu) = 0, \\ &(i, j, k \in I; \lambda, \mu, \nu \in \Lambda) \rangle. \end{aligned} \quad (4)$$

By taking $k = \nu = 1$ and $\mu \neq 1$ in (4), we have

$$x(i, \lambda, j, \mu) = x(j, \mu, 1, 1) - x(i, \mu, 1, 1) + x(i, \lambda, j, 1).$$

It follows that, for $\lambda \neq 1$,

$$x(i, \lambda, i, \lambda) = x(i, \lambda, 1, 1) - x(i, \lambda, 1, 1) + x(i, \lambda, i, 1) = x(i, \lambda, i, 1).$$

Moreover, for $\nu, \mu \neq 1$,

$$\begin{aligned} & x(j, \mu, k, \nu) - x(i, \mu, k, \nu) + x(i, \lambda, j, \nu) - x(i, \lambda, j, \mu) \\ &= (x(j, \mu, k, 1) - x(j, \nu, 1, 1) + x(k, \nu, 1, 1)) - (x(i, \mu, k, 1) - x(i, \nu, 1, 1) \\ &\quad + x(k, \nu, 1, 1)) + (x(i, \lambda, j, 1) - x(i, \nu, 1, 1) + x(j, \nu, 1, 1)) - (x(i, \lambda, j, 1) \\ &\quad - x(i, \mu, 1, 1) + x(j, \mu, 1, 1)) \\ &= x(j, \mu, k, 1) - x(i, \mu, k, 1) + x(i, \mu, 1, 1) - x(j, \mu, 1, 1). \end{aligned}$$

For $\nu \neq 1$ and $\mu = 1$,

$$\begin{aligned} & x(j, 1, k, \nu) - x(i, 1, k, \nu) + x(i, \lambda, j, \nu) - x(i, \lambda, j, 1) \\ &= (x(j, 1, k, 1) - x(j, \nu, 1, 1) + x(k, \nu, 1, 1)) - (x(i, 1, k, 1) - x(i, \nu, 1, 1) \\ &\quad + x(k, \nu, 1, 1)) + (x(i, \lambda, j, 1) - x(i, \nu, 1, 1) + x(j, \nu, 1, 1)) - x(i, \lambda, j, 1) \\ &= x(j, 1, k, 1) - x(j, 1, k, 1). \end{aligned}$$

Therefore, eliminating the generators $x(i, \lambda, j, \mu)$ gives the presentation

$$\begin{aligned} & \langle x(i, \lambda, j, 1) \mid (i, j \in I, \lambda \in \Lambda) \mid x(i, 1, i, 1) = 0, x(i, \lambda, i, 1) = 0, \\ & x(j, \mu, k, 1) - x(i, \mu, k, 1) - x(j, \mu, 1, 1) + x(i, \mu, 1, 1) = 0, \end{aligned} \quad (5)$$

$$x(j, 1, k, 1) - x(i, 1, k, 1) = 0 \mid (i, j, k \in I, \lambda, \mu \in \Lambda, \lambda \neq 1, \mu \neq 1) \rangle. \quad (6)$$

Next set $k = j$ in (6) to obtain

$$x(i, 1, j, 1) = x(j, 1, j, 1) = 0$$

and in (5) to obtain

$$\begin{aligned} x(i, \mu, j, 1) &= x(j, \mu, j, 1) + x(i, \mu, 1, 1) - x(j, \mu, 1, 1) \\ &= x(i, \mu, 1, 1) - x(j, \mu, 1, 1) \end{aligned}$$

since $x(j, \mu, j, 1) = 0$ ($\mu \neq 1$). It follows, for $\lambda \neq 1$, that

$$x(i, \lambda, i, 1) = x(i, \lambda, 1, 1) - x(i, \lambda, 1, 1) = 0$$

and, for $\mu \neq 1$, that

$$\begin{aligned} &x(j, \mu, k, 1) - x(i, \mu, k, 1) - x(j, \mu, 1, 1) + x(i, \mu, 1, 1) \\ &= (x(j, \mu, 1, 1) - x(k, \mu, 1, 1)) - (x(i, \mu, 1, 1) - x(k, \mu, 1, 1)) \\ &\quad - (x(j, \mu, 1, 1) - x(1, \mu, 1, 1)) + (x(i, \mu, 1, 1) - x(1, \mu, 1, 1)) = 0. \end{aligned}$$

Therefore, by eliminating $x(i, 1, j, 1)$ and $x(i, \mu, j, 1)$ ($\mu \neq 1$), we obtain the presentation

$$\langle x(i, \mu, 1, 1) \ (i \in I, \mu \in \Lambda \setminus \{1\}) \mid x(1, \mu, 1, 1) = 0 \ (\mu \in \Lambda \setminus \{1\}) \rangle.$$

Finally, we eliminate $x(1, \mu, 1, 1)$ ($\mu \neq 1$) to obtain the free abelian group

$$\langle x(i, \mu, 1, 1) \ (i \in I \setminus \{1\}, \mu \in \Lambda \setminus \{1\}) \mid \rangle,$$

thus proving that $H_2(R_{m,n}^1) = \mathbb{Z}^{(m-1)(n-1)}$, as required. ■

6.5 Efficiency of rectangular bands

In this section we prove that finite rectangular bands are efficient. Therefore we obtain our first example of a family of efficient semigroups which are not groups.

It is easy to see that $R_{m,1}$ is isomorphic to the left zero semigroup of order m . Neumann showed in [47] that all finite left zero semigroups have deficiency zero presentations, namely

$$\langle a_1, \dots, a_m \mid a_1 a_2 = a_1, \dots, a_i a_{i+1} = a_i, \dots, a_n a_1 = a_n \rangle, \quad (7)$$

and so they are efficient. Similarly, one may show that the presentation

$$\langle b_1, \dots, b_n \mid b_1 b_2 = b_2, \dots, b_\lambda b_{\lambda+1} = b_{\lambda+1}, \dots, b_n b_1 = b_1 \rangle$$

defines the finite right zero semigroup $R_{1,n}$ of order n . Therefore all finite right zero semigroups are efficient. Notice that both left zero semigroups and right zero semigroups may only be generated by themselves, that is if $\langle X \rangle = R_{m,1} (R_{1,n})$, then $X = R_{m,1} (R_{1,n})$.

From now on we assume that $m, n \geq 2$. Before we show that rectangular bands are efficient, we give nice presentations for them. Note that since a rectangular band is, in fact, a Rees matrix semigroup over the trivial group, a presentation for $R_{m,n}$ can be deduced from a general presentation for Rees matrix semigroups given in [30], namely

$$\begin{aligned} \langle e, y_2, \dots, y_m, z_2, \dots, z_n \mid & e^2 = e, ey_i = e, y_i e = y_i, ez_\lambda = z_\lambda, \\ & z_\lambda e = e, z_\lambda y_i = e \ (2 \leq i \leq m, 2 \leq \lambda \leq n) \rangle. \end{aligned}$$

Proposition 6.6 *The rectangular band $R_{m,n}$ ($m, n \geq 2$) has a presentation*

$$\mathcal{P}_1 = \langle \quad a(i, 1), a(1, \lambda) \ (1 \leq i \leq m, 1 \leq \lambda \leq n) \mid$$

$$a(i, 1)a(i+1, 1) = a(i, 1) \ (1 \leq i \leq m), \tag{8}$$

$$a(1, \lambda)a(1, \lambda+1) = a(1, \lambda+1) \ (1 \leq \lambda \leq n), \tag{9}$$

$$a(1, \lambda)a(i, 1) = a(1, 1) \ (2 \leq i \leq m, 2 \leq \lambda \leq n) \rangle, \tag{10}$$

with the convention that $a(m+1, 1) = a(1, 1) = a(1, n+1)$ in terms of the generating set

$$X_{m,n} = \{ (i, 1) \mid 1 \leq i \leq m \} \cup \{ (1, \lambda) \mid 1 \leq \lambda \leq n \}.$$

Proof From $(i, \lambda) = (i, 1)(1, \lambda)$, we see immediately that $X_{m,n}$ generates $R_{m,n}$. A routine verification shows that $R_{m,n}$ satisfies all the relations (8), (9), (10). Hence $R_{m,n}$ is a homomorphic image of the semigroup S defined by \mathcal{P}_1 and, in

particular, $|S| \geq mn$. From (8) and (7), it follows that the semigroup S_1 of S generated by $\{ a(i, 1) \mid 1 \leq i \leq m \}$ is a semigroup of left zeros, and so $|S_1| \leq m$. Similarly, $|S_2| \leq n$, where S_2 is generated by $\{ a(1, \lambda) \mid 1 \leq \lambda \leq n \}$. Because of the relations (10), we have $S = S_1 S_2$, and so $|S| \leq mn$. We conclude that $|S| = mn$ and hence $S \cong R_{m,n}$. ■

Note that the deficiency of the presentation \mathcal{P}_1 is $(m-1)(n-1)+1$, which is just 1 greater than we require. We now give our efficient presentation for $R_{m,n}$.

Theorem 6.7 *The rectangular band $R_{m,n}$ ($m, n \geq 2$) has a presentation*

$$\begin{aligned} \mathcal{P}_2 = \langle & a(i, 1), a(1, \lambda) \ (2 \leq i \leq m; 2 \leq \lambda \leq n) \mid \\ & a(1, n)a(m, 1)a(2, 1) = a(1, n)a(m, 1), \\ & a(i, 1)a(i+1, 1) = a(i, 1) \ (2 \leq i \leq m-1), \\ & a(1, n)a(m, 1)a(1, 2) = a(1, 2), \\ & a(1, \lambda)a(1, \lambda+1) = a(1, \lambda+1) \ (2 \leq \lambda \leq n-1), \\ & a(m, 1)a(1, n)a(1, n)a(m, 1) = a(m, 1), \\ & a(1, \lambda)a(i, 1) = a(1, n)a(m, 1) \ (2 \leq i \leq m; 2 \leq \lambda \leq n; (i, \lambda) \neq (m, n)) \rangle \end{aligned} \quad (11)$$

in terms of the generating set $\{ (i, 1) \mid 2 \leq i \leq m \} \cup \{ (1, \lambda) \mid 2 \leq \lambda \leq n \}$.

Proof We apply Tietze transformations to \mathcal{P}_1 to obtain \mathcal{P}_2 . By (10), we have $a(1, 1) = a(1, n)a(m, 1)$ and so, by eliminating $a(1, 1)$ from \mathcal{P}_1 , we have the following presentation:

$$\begin{aligned} \langle & a(i, 1), a(1, \lambda) \ (2 \leq i \leq m; 2 \leq \lambda \leq n) \mid \\ & a(1, n)a(m, 1)a(2, 1) = a(1, n)a(m, 1), \\ & a(i, 1)a(i+1, 1) = a(i, 1) \ (2 \leq i \leq m-1), \\ & a(m, 1)a(1, n)a(m, 1) = a(m, 1), \end{aligned} \quad (12)$$

$$a(1, n)a(m, 1)a(1, 2) = a(1, 2), \quad (13)$$

$$a(1, \lambda)a(1, \lambda + 1) = a(1, \lambda + 1) \quad (2 \leq \lambda \leq n - 1), \quad (14)$$

$$a(1, n)a(1, n)a(m, 1) = a(1, n)a(m, 1), \quad (15)$$

$$a(1, \lambda)a(i, 1) = a(1, n)a(m, 1) \quad (2 \leq i \leq m; 2 \leq \lambda \leq n; (i, \lambda) \neq (m, n)).$$

Next we show that we may replace the relation (15) by the relation (11). First we show that (11) holds in $R_{m,n}$. Indeed, from (15) and (12), respectively, we have

$$a(m, 1)a(1, n)a(1, n)a(m, 1) = a(m, 1)a(1, n)a(m, 1) = a(m, 1).$$

A repeated application of (14) yields

$$a(1, n) = a(1, 2)a(1, 3) \cdots a(1, n) = a(1, 2)a(1, n), \quad (16)$$

and hence

$$\begin{aligned} a(1, n)a(m, 1) &= a(1, n)a(m, 1)a(1, n)a(1, n)a(m, 1) && \text{(by (11))} \\ &= a(1, n)a(m, 1)a(1, 2)a(1, n)a(1, n)a(m, 1) && \text{(by (16))} \\ &= a(1, 2)a(1, n)a(1, n)a(m, 1) && \text{(by (13))} \\ &= a(1, n)a(1, n)a(m, 1) && \text{(by (16))} \end{aligned}$$

and so we may replace the relation (15) by the relation (11). Next we prove that the relation (12) is redundant. Observe that we have

$$\begin{aligned} a(1, n)a(m, 1)a(1, n) &= a(1, n)a(m, 1)a(1, 2)a(1, n) && \text{(by (16))} \\ &= a(1, 2)a(1, n) && \text{(by (13))} \\ &= a(1, n) && \text{(by (16)).} \end{aligned}$$

Finally, from the previous relation and (11), it follows that

$$\begin{aligned} a(m, 1)a(1, n)a(m, 1) &= a(m, 1)(a(1, n)a(m, 1)a(1, n))a(m, 1) \\ &= a(m, 1)a(1, n)(a(m, 1)a(1, n)a(1, n)a(m, 1))a(1, n)a(m, 1) \\ &= a(m, 1)(a(1, n)a(m, 1)a(1, n))(a(1, n)a(m, 1)a(1, n))a(m, 1) \\ &= a(m, 1)a(1, n)a(1, n)a(m, 1) = a(m, 1). \end{aligned}$$

Therefore, by eliminating this redundant relation we obtain our presentation \mathcal{P}_2 , as required. \blacksquare

Corollary 6.8 *The rectangular band $R_{m,n}$ is efficient for arbitrary $m, n \geq 1$.*

Proof Recall that, by Theorem 6.5, $H_2(R_{m,n}) \cong \mathbb{Z}^{(m-1)(n-1)}$. If $n = 1$, then $R_{m,1}$ is a semigroup of left zeros, and is efficient by [47]. The case $m = 1$ is dual. If $m, n \geq 2$, then

$$\begin{aligned} \text{def}(\mathcal{P}_2) &= ((m-1)(n-1) + (n-2) + (m-2) + 3) - (m+n-1) \\ &= (m-1)(n-1) = \text{rank}(H_2(R_{m,n})) \end{aligned}$$

and hence \mathcal{P}_2 is an efficient presentation for $R_{m,n}$. \blacksquare

6.6 Remarks

Although finite left and right zero semigroups are efficient, we showed in the second section that zero semigroups of order at least 3 are not efficient. Moreover, $\text{def}_S(\mathcal{Z}_n) - \text{rank}(H_2(\mathcal{Z}_n))$ and $\text{def}_S(\mathcal{SL}_n) - \text{rank}(H_2(\mathcal{SL}_n))$ both increase with n . Note that \mathcal{Z}_n and \mathcal{SL}_n are both abelian semigroups. This contrasts with the case of finite abelian groups which are efficient even when considered as semigroups (as we proved in the previous chapter).

It is interesting to notice that although the second homology of a finite group is a finite abelian group, the second homology of a finite semigroup is not necessarily finite since $H_2(R_{m,n}^1) \cong \mathbb{Z}^{(m-1)(n-1)}$, which is free abelian group of rank $(m-1)(n-1)$.

If G and K are finite groups, then it is well-known (see, for example, [40]) that

$$H_2(G \times K) = H_2(G) \times H_2(K) \times (H_1(G) \otimes H_1(K)).$$

However, $R_{m,n}$ is a direct product of the left zero semigroup L_m of order m and the right zero semigroup R_n of order n (see [29, Theorem 1.13]). It follows, by Theorems 6.1 and 6.5 that

$$H_2(L_m \times R_n) \cong \mathbb{Z}^{(m-1)(n-1)} \neq 0 = H_2(L_m) \times H_2(R_n) \times (H_1(L_m) \otimes H_1(R_n))$$

for $m, n \geq 2$. Therefore, the above (Schur-Künneth) formula for the second homology of the direct product of groups is not valid for the direct product of semigroups.

Chapter 7

Efficiency of Finite Simple Semigroups

The purpose of this chapter is to investigate the efficiency of finite simple semigroups.

It is well-known that a finite semigroup S is simple if and only if it is isomorphic to a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ (see for example [48] or [29]). Here G is a group, I and Λ are non-empty sets, $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries from G and

$$\mathcal{M}[G; I, \Lambda; P] = \{ (i, g, \lambda) \mid i \in I, g \in G, \lambda \in \Lambda \}$$

with multiplication defined by

$$(i, g, \lambda)(j, h, \mu) = (i, gp_{\lambda j}h, \mu).$$

It is also well-known that the matrix P can be chosen to be *normal*, that is $p_{\lambda 1} = p_{1 i} = 1_G$ for all $\lambda \in \Lambda, i \in I$, where 1_G is the identity of G ; see for example [48] or [29, Theorem 3.4.2].

Let S be a finite simple semigroup, given as a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$ over a group G . We prove that the second homology of S is

$$H_2(S) = H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)}.$$

For a given semigroup presentation $\langle A_1 \mid R_1 \rangle$ of G , we find a presentation $\langle A \mid R \rangle$ for S such that $|R| - |A| = |R_1| - |A_1| + (|I| - 1)(|\Lambda| - 1) + 1$. We use this presentation to prove that S is efficient when $\langle A_1 \mid R_1 \rangle$ is an efficient semigroup presentation for G and R_1 contains a relation of a special form. In particular, we prove that finite Rees matrix semigroups $\mathcal{M}[G; I, \Lambda; P]$ over finite abelian groups, or direct powers D_{2n}^m ($m \geq 1$) of dihedral groups D_{2n} with even n , or generalised quaternion groups Q_r of order $4r$, or projective special linear groups $PSL(2, p)$ with p prime are efficient. Finally, we show that there exist non-simple efficient semigroups which have non-trivial second homology.

The most of the results of this chapter have been submitted for publication by H. Ayık, C. M. Campbell, J. J. O'Connor and N. Ruškuc (see [2]).

7.1 A rewriting system for Rees matrix semigroups

In the previous chapter, the bar resolution was used to compute the second integral homology of rectangular bands $R_{m,n}$ to be $\mathbb{Z}^{(m-1)(n-1)}$ and the n th ($n \geq 1$) homology of semigroups with a left or a right zero to be trivial. Here we use another resolution which is described by Squier in [61]. Since this resolution is defined by using a presentation in which the set of relations is a uniquely terminating rewriting system, we first find a presentation for a Rees matrix semigroup in which the set of relations is a uniquely terminating rewriting system. We begin by introducing some elementary concepts about rewriting systems.

Let A be a set and let A^* be the free monoid on A . A *rewriting system* R on A is a subset of $A^* \times A^*$. For $w_1, w_2 \in A^*$, we write $w_1 \equiv w_2$ if they are identical words. We say that w_1 *rewrites to* w_2 if there exist $b, c \in A^*$ and $(u, v) \in R$ such that $w_1 \equiv buc$ and $w_2 \equiv bvc$ and we write $w_1 \rightarrow w_2$. We denote by \rightarrow^* the reflexive transitive closure of \rightarrow and by \sim the equivalence relation generated

by \rightarrow .

For a word w we say that w is *reducible* if there is a word z such that $w \rightarrow z$; otherwise we call w *irreducible*. If $w \xrightarrow{*} y$ and y is irreducible, then we say that y is an *irreducible form* of w . A rewriting system R is said to be *terminating* if there is no infinite sequence (w_n) such that $w_n \rightarrow w_{n+1}$ for all $n \geq 1$. We denote by $|w|$ the length of the word w . We call R *length-reducing* if $|u| > |v|$ for all $(u, v) \in R$.

Let u and v be any words in A^* . Then we write $u \ll v$ if $|u| < |v|$ or if $|u| = |v|$ and u precedes v in the lexicographic order induced by some well-ordering on A . We call R a *lexicographic rewriting system* if $u \ll v$ for all $(v, u) \in R$. It is clear that if R is a lexicographic rewriting system, then R is a terminating rewriting system. We say that R is *confluent* if, for any $x, y, z \in A^*$ such that $x \xrightarrow{*} y$, $x \xrightarrow{*} z$, there exists $w \in A^*$ such that $y \xrightarrow{*} w$, $z \xrightarrow{*} w$. A rewriting system R is *complete* if it is both terminating and confluent. For a given R , define $R_1 \subseteq A^*$ to consist of all $r \in A^*$ such that there exists $(r, s) \in R$ for some $s \in A^*$. The system R is said to be *reduced* provided that, for each $(r, s) \in R$, we have $R_1 \cap A^*rA^* = \{r\}$ and s is R -irreducible. A reduced complete rewriting system $R \subseteq A^* \times A^*$ is called a *uniquely terminating rewriting system*.

Lemma 7.1 *Let R be a terminating rewriting system. Then the following are equivalent:*

- (i) R is confluent (and hence complete);
- (ii) for any $(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3}) \in R$, where r_2 is non-empty, there exists a word $w \in A^*$ such that $s_{1,2}r_3 \xrightarrow{*} w$, $r_1s_{2,3} \xrightarrow{*} w$; for any $(r_1r_2r_3, s_{1,2}), (r_2, s_{2,3}) \in R$, there exists a word $w \in A^*$ such that $s_{1,2} \xrightarrow{*} w$, $r_1s_{2,3}r_3 \xrightarrow{*} w$;
- (iii) any word $w \in A^*$ has exactly one irreducible form. Moreover $w \sim w'$ if and only if w and w' have the same irreducible form.

For a proof see, [27] or [61].

We define the *overlaps* to be the ordered pairs of the form

$$[(r_1 r_2, s_{1,2}), (r_2 r_3, s_{2,3})] \text{ and } [(r_4 r_5 r_6, s_{4,5}), (r_5, s_{5,6})]$$

where $(r_1 r_2, s_{1,2}), (r_2 r_3, s_{2,3}), (r_4 r_5 r_6, s_{4,5}), (r_5, s_{5,6}) \in R$, and r_2 and r_5 are non-empty words.

First we give a presentation for a Rees matrix semigroup with a normal matrix. For ease of notation we assume that I and Λ both contain a distinguished element denoted by 1.

Theorem 7.2 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a Rees matrix semigroup, where G is a group and $P = (p_{\lambda i})$ is a normal $\Lambda \times I$ matrix with entries from G . Let $\langle X \mid R \rangle$ be a semigroup presentation for G , let $e \in X^+$ be a non-empty word representing the identity of G , and let $Y = X \cup \{y_i \mid i \in I - \{1\}\} \cup \{z_\lambda \mid \lambda \in \Lambda - \{1\}\}$. Then the presentation*

$$\langle Y \mid R, y_i e = y_i, e y_i = e, z_\lambda e = e, e z_\lambda = z_\lambda, z_\lambda y_i = p_{\lambda i}$$

$$(i \in I - \{1\}, \lambda \in \Lambda - \{1\}) \rangle$$

defines S in terms of the generating set $\{(1, x, 1) \mid x \in X\} \cup \{(i, 1_G, 1) \mid i \in I - \{1\}\} \cup \{(1, 1_G, \lambda) \mid \lambda \in \Lambda - \{1\}\}$.

Proof The result is a special case of Theorem 6.2 in [30]. ■

In the previous presentation, there are some overlaps, for example $[y_i e = y_i, e y_{i'} = e]$, which show that the set of the relations is not a uniquely terminating rewriting system. Indeed, we have

$$y_i(e y_{i'}) \rightarrow y_i e \rightarrow y_i \text{ but } (y_i e) y_{i'} \rightarrow y_i y_{i'},$$

and so we need relations of the form $y_i y_{i'} = y_i$, which hold in S . Similarly we need relations of the form $z_\lambda z_{\lambda'} = z_{\lambda'}$, which also hold in S . Let $e \equiv x e'$ where $x \in X$, $e' \in X^*$. We assume that there is a relation $(r x = s) \in R$ so that we

have the overlap $[(rx, s), (xe'y_i, xe')]$. If we consider this overlap, then we need relations of the form $xy_i = x$ ($x \in X, i \in I - \{1\}$), which hold in S (since the relations $ey_i = e$ hold in S). Similarly, we need relations of the form $z_\lambda x = x$ ($x \in X, \lambda \in \Lambda - \{1\}$).

Now we construct a new presentation with a uniquely terminating rewriting system of relations.

We can take the presentation $\langle X \mid R \rangle$ to be the Cayley table of the group G , that is $X = G$ and $R = \{ (x_1x_2, x_3) \mid x_1, x_2, x_3 \in G, x_1x_2 = x_3 \text{ in } G \}$. It is clear that R is a uniquely terminating rewriting system on X . Let $x_0 \in X$ represent the identity of G . Then taking $e \equiv x_0$ and adding the new relations $xy_i = x$, $z_\lambda x = x$, $y_iy_{i'} = y_i$ and $z_\lambda z_{\lambda'} = z_{\lambda'}$ ($x \in X - \{x_0\}; i, i' \in I - \{1\}; \lambda, \lambda' \in \Lambda - \{1\}$) yields the presentation

$$\begin{aligned} \langle Y \mid R, & \ y_ix_0 = y_i, \ xy_i = x, \ y_iy_{i'} = y_i, \ z_\lambda x = x, \ x_0z_\lambda = z_\lambda \\ & z_\lambda z_{\lambda'} = z_{\lambda'}, \ z_\lambda y_i = p_{\lambda i} \ (i, i' \in I - \{1\}; \lambda, \lambda' \in \Lambda - \{1\}; x \in X) \rangle \end{aligned}$$

which defines $S = \mathcal{M}[G; I, \Lambda; P]$.

For ease of notation, we assume that G is finite and $X = \{x_0, x_1, \dots, x_k\}$ where x_0 is the representative of the identity of G . We further assume that the entries $p_{\lambda i}$ of the matrix P are represented by words of length one.

Theorem 7.3 *Let $\langle X \mid R \rangle$ be the Cayley table of the finite group G and let $x_0 \in X$ be the representative of the identity. With the above notation, the presentation*

$$\begin{aligned} \mathcal{P} = \langle Y \mid & \ R, \ y_ix_0 = y_i, \ x_ky_i = x_k, \ y_iy_{i'} = y_i, \ z_\lambda x_k = x_k, \\ & x_0z_\lambda = z_\lambda, \ z_\lambda z_{\lambda'} = z_{\lambda'}, \ z_\lambda y_i = p_{\lambda i} \\ & (0 \leq k \leq m; \ i, i' \in I - \{1\}; \ \lambda, \lambda' \in \Lambda - \{1\}) \rangle, \end{aligned}$$

which defines $S = \mathcal{M}[G; I, \Lambda; P]$, has a uniquely terminating rewriting system of relations on Y .

Proof Let Q denote the set of relations of \mathcal{P} . Recall that all rewriting rules in R have the form (x_1x_2, x_3) ($x_1, x_2, x_3 \in X$) so that all the rewriting rules in

Q are length reducing. Therefore Q is terminating. Moreover, observe that each word on the right hand-side of each rewriting rule is a generator, and each word on the left hand-side of each rewriting rule is not a subword of the words on the other left hand-sides of rewriting rules, and so Q is reduced. Therefore it remains to prove that Q is confluent. For this we first determine all overlaps. They are:

$$\begin{aligned}
U_{1,k,k',k''} &= [(x_k x_{k'}, x_l), (x_{k'} x_{k''}, x_{l'})], & U_{2,k,k',i} &= [(x_{k'} x_k, x_l), (x_k y_i, x_k)], \\
U_{3,k,\lambda} &= [(x_k x_0, x_k), (x_0 z_\lambda, z_\lambda)], & U_{4,k,i} &= [(y_i x_0, y_i), (x_0 x_k, x_k)], \\
U_{5,i,i'} &= [(y_i x_0, y_i), (x_0 y_{i'}, x_0)], & U_{6,i,\lambda} &= [(y_i x_0, y_i), (x_0 z_\lambda, z_\lambda)], \\
U_{7,k,i} &= [(x_k y_i, x_k), (y_i x_0, y_i)], & U_{8,k,i,i'} &= [(x_k y_i, x_k), (y_i y_{i'}, y_i)], \\
U_{9,i,i'} &= [(y_i y_{i'}, y_i), (y_{i'} x_0, y_{i'})], & U_{10,i,i',i''} &= [(y_i y_{i'}, y_i), (y_{i'} y_{i''}, y_{i'})], \\
U_{11,k,k',\lambda} &= [(z_\lambda x_k, x_k), (x_k x_{k'}, x_l)], & U_{12,k,i,\lambda} &= [(z_\lambda x_k, x_k), (x_k y_i, x_k)], \\
U_{13,\lambda,\lambda'} &= [(z_\lambda x_0, x_0), (x_0 z_{\lambda'}, z_{\lambda'})], & U_{14,k,\lambda} &= [(x_0 z_\lambda, z_\lambda), (z_\lambda x_k, x_k)], \\
U_{15,\lambda,\lambda'} &= [(x_0 z_\lambda, z_\lambda), (z_\lambda z_{\lambda'}, z_{\lambda'})], & U_{16,i,\lambda} &= [(x_0 z_\lambda, z_\lambda), (z_\lambda y_i, p_{\lambda i})], \\
U_{17,k,\lambda,\lambda'} &= [(z_\lambda z_{\lambda'}, z_{\lambda'}), (z_{\lambda'} x_k, x_k)], & U_{18,\lambda,\lambda',\lambda''} &= [(z_\lambda z_{\lambda'}, z_{\lambda'}), (z_{\lambda'} z_{\lambda''}, z_{\lambda''})], \\
U_{19,i,\lambda,\lambda'} &= [(z_\lambda z_{\lambda'}, z_{\lambda'}), (z_{\lambda'} y_i, p_{\lambda' i})], & U_{20,i,\lambda} &= [(z_\lambda y_i, p_{\lambda i}), (y_i x_0, y_i)], \\
U_{21,i,i',\lambda} &= [(z_\lambda y_i, p_{\lambda i}), (y_i y_{i'}, y_i)],
\end{aligned}$$

where $i, i', i'' \in I - \{1\}$; $\lambda, \lambda', \lambda'' \in \Lambda - \{1\}$; $1 \leq k, k', k'' \leq m$. Now we apply Lemma 7.1(ii).

For $U_{1,k,k',k''}$, we have some $x_{l'} \in Y^+$ such that

$$(x_k x_{k'}) x_{k''} \rightarrow x_l x_{k''} \rightarrow x_{l'} \text{ and } x_k (x_{k'} x_{k''}) \rightarrow x_k x_{l'} \rightarrow x_{l'}.$$

For $U_{2,k,k',i}$, we have some $x_l \in Y^+(X)$ such that

$$(x_{k'} x_k) y_i \rightarrow x_l y_i \rightarrow x_l \text{ and } x_{k'} (x_k y_i) \rightarrow x_{k'} x_k \rightarrow x_l.$$

For $U_{3,k,\lambda}$, we have some $x_k z_\lambda \in Y^+$ such that

$$(x_k x_0) z_\lambda \rightarrow x_k z_\lambda \text{ and } x_k (x_0 z_\lambda) \rightarrow x_k z_\lambda.$$

For $U_{4,k,i}$, we have some $y_i x_k \in Y^+$ such that

$$(y_i x_0) x_k \rightarrow y_i x_k \text{ and } y_i (x_0 x_k) \rightarrow y_i x_k.$$

For $U_{5,i,i'}$, we have some $y_i \in Y^+$ such that

$$(y_i x_0) y_{i'} \rightarrow y_i y_{i'} \rightarrow y_i \text{ and } y_i (x_0 y_{i'}) \rightarrow y_i x_0 \rightarrow y_i.$$

For $U_{6,i,\lambda}$, we have some $y_i z_\lambda \in Y^+$ such that

$$(y_i x_0) z_\lambda \rightarrow y_i z_\lambda \text{ and } y_i (x_0 z_\lambda) \rightarrow y_i z_\lambda.$$

For $U_{7,k,i}$, we have some $x_k \in Y^+$ such that

$$(x_k y_i) x_0 \rightarrow x_k x_0 \rightarrow x_k \text{ and } x_k (y_i x_0) \rightarrow x_k y_i \rightarrow x_k.$$

For $U_{8,k,i,i'}$, we have some $x_k \in Y^+$ such that

$$(x_k y_i) y_{i'} \rightarrow x_k y_{i'} \rightarrow x_k \text{ and } x_k (y_i y_{i'}) \rightarrow x_k y_i \rightarrow x_k.$$

For $U_{9,i,i'}$, we have some $y_i \in Y^+$ such that

$$(y_i y_{i'}) x_0 \rightarrow y_i x_0 \rightarrow y_i \text{ and } y_i (y_{i'} x_0) \rightarrow y_i y_{i'} \rightarrow y_i.$$

For $U_{10,i,i',i''}$, we have some $y_i \in Y^+$ such that

$$(y_i y_{i'}) y_{i''} \rightarrow y_i y_{i''} \rightarrow y_i \text{ and } y_i (y_{i'} y_{i''}) \rightarrow y_i y_{i'} \rightarrow y_i.$$

For $U_{11,k,k',\lambda}$, we have some $x_l \in Y^+$ such that

$$(z_\lambda x_k) x_{k'} \rightarrow x_k x_{k'} \rightarrow x_l \text{ and } z_\lambda (x_k x_{k'}) \rightarrow z_\lambda x_l \rightarrow x_l.$$

For $U_{12,k,i,\lambda}$, we have some $x_k \in Y^+$ such that

$$(z_\lambda x_k) y_i \rightarrow x_k y_i \rightarrow x_k \text{ and } z_\lambda (x_k y_i) \rightarrow z_\lambda x_k \rightarrow x_k.$$

For $U_{13,\lambda,\lambda'}$, we have some $z_{\lambda'} \in Y^+$ such that

$$(z_\lambda x_0) z_{\lambda'} \rightarrow x_0 z_{\lambda'} \rightarrow z_{\lambda'} \text{ and } z_\lambda (x_0 z_{\lambda'}) \rightarrow z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}.$$

For $U_{14,k,\lambda}$, we have some $x_k \in Y^+$ such that

$$(x_0 z_\lambda) x_k \rightarrow z_\lambda x_k \rightarrow x_k \text{ and } x_0(z_\lambda x_k) \rightarrow x_0 x_k \rightarrow x_k.$$

For $U_{15,\lambda,\lambda'}$, we have some $z_{\lambda'} \in Y^+$ such that

$$(x_0 z_\lambda) z_{\lambda'} \rightarrow z_\lambda z_{\lambda'} \rightarrow z_{\lambda'} \text{ and } x_0(z_\lambda z_{\lambda'}) \rightarrow x_0 z_{\lambda'} \rightarrow z_{\lambda'}.$$

For $U_{16,i,\lambda}$, we have some $p_{\lambda i} \in Y^+$ such that

$$(x_0 z_\lambda) y_i \rightarrow z_\lambda y_i \rightarrow p_{\lambda i} \text{ and } x_0(z_\lambda y_i) \rightarrow x_0 p_{\lambda i} \rightarrow p_{\lambda i}.$$

For $U_{17,k,\lambda,\lambda'}$, we have some $x_k \in Y^+$ such that

$$(z_\lambda z_{\lambda'}) x_k \rightarrow z_{\lambda'} x_k \rightarrow x_k \text{ and } z_\lambda(z_{\lambda'} x_k) \rightarrow z_\lambda x_k \rightarrow x_k.$$

For $U_{18,\lambda,\lambda',\lambda''}$, we have some $z_{\lambda''} \in Y^+$ such that

$$(z_\lambda z_{\lambda'}) z_{\lambda''} \rightarrow z_{\lambda'} z_{\lambda''} \rightarrow z_{\lambda''} \text{ and } z_\lambda(z_{\lambda'} z_{\lambda''}) \rightarrow z_\lambda z_{\lambda''} \rightarrow z_{\lambda''}.$$

For $U_{19,i,\lambda,\lambda'}$, we have some $p_{\lambda' i} \in Y^+$ such that

$$(z_\lambda z_{\lambda'}) y_i \rightarrow z_{\lambda'} y_i \rightarrow p_{\lambda' i} \text{ and } z_\lambda(z_{\lambda'} y_i) \rightarrow z_{\lambda'} p_{\lambda' i} \rightarrow p_{\lambda' i}.$$

For $U_{20,i,\lambda}$, we have some $p_{\lambda i} \in Y^+$ such that

$$(z_\lambda y_i) x_0 \rightarrow p_{\lambda i} x_0 \rightarrow p_{\lambda i} \text{ and } z_\lambda(y_i x_0) \rightarrow z_\lambda y_i \rightarrow p_{\lambda i}.$$

Finally, for $U_{21,i,i',\lambda}$, we have some $p_{\lambda i} \in Y^+$ such that

$$(z_\lambda y_i) y_{i'} \rightarrow p_{\lambda i} y_{i'} \rightarrow p_{\lambda i} \text{ and } z_\lambda(y_i y_{i'}) \rightarrow z_\lambda y_i \rightarrow p_{\lambda i}.$$

Therefore we have proved that Q is confluent, and so it is a uniquely terminating rewriting system on Y . ■

7.2 The second homology of Rees matrix semigroups

Now we describe the resolution of \mathbb{Z} given by Squier in [61], which we use to compute the second homology of a finite Rees matrix semigroup.

Let S be a monoid and let $\langle A \mid R \rangle$ be a presentation for S in which R is a uniquely terminating rewriting system. Then Squier defined a free resolution of the trivial $\mathbb{Z}S$ -module \mathbb{Z} as follows:

$$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where P_0 is the free $\mathbb{Z}S$ -module on a single formal symbol $[\]$, the augmentation map $\varepsilon : P_0 \longrightarrow \mathbb{Z}$ is defined by $\varepsilon([\]) = 1$, P_1 is the free $\mathbb{Z}S$ -module on the set of formal symbols $[x]$ for all $x \in A$ and $\partial_1 : P_1 \longrightarrow P_0$ is defined by

$$\partial_1([x]) = (x - 1)[\]$$

where $x \in A$. Further P_2 is the free $\mathbb{Z}S$ -module on the set of formal symbols $[r, s]$, one for each $(r, s) \in R$. For $x \in A$, we define a function $\partial/\partial_x : A^* \longrightarrow \mathbb{Z}A^*$ inductively by

$$\begin{aligned} \partial/\partial_x(1) &= 0 \\ \partial/\partial_x(wx) &= \partial/\partial_x(w) + w \quad (w \in A^*) \\ \partial/\partial_x(wy) &= \partial/\partial_x(w) \quad (w \in A^* \text{ and } y \neq x). \end{aligned}$$

This function, ∂/∂_x , is called a *derivation*.

Now we define $\partial_2 : P_2 \longrightarrow P_1$ by

$$\partial_2([r, s]) = \sum_{x \in A} \phi(\partial/\partial_x(r) - \partial/\partial_x(s))[x]$$

where $\phi : \mathbb{Z}A^* \longrightarrow \mathbb{Z}S$ is induced by the natural homomorphism from A^* to S .

Next, P_3 is the free $\mathbb{Z}S$ -module on the set of overlaps $[(r_1r_2, s_{1,2}), (r_2r_3, s_{2,3})]$ from R . Let w be in A^* and let u be the irreducible form of w . Then we have a

sequence

$$w \equiv b_1 r_1 c_1, b_1 s_1 c_1 \equiv b_2 r_2 c_2, \dots, b_q s_q c_q \equiv u$$

where $b_i, c_i \in A^*$ and $(r_i, s_i) \in R$ for all $i = 1, \dots, q$. Define $\Phi : A^* \longrightarrow P_2$ by

$$\Phi(w) = \sum_{i=1}^q \phi(b_i)[r_i, s_i].$$

Now we define $\partial_3 : P_3 \longrightarrow P_2$ by

$$\partial_3([(r_1 r_2, s_{1,2}), (r_2 r_3, s_{2,3})]) = r_1[r_2 r_3, s_{2,3}] - [r_1 r_2, s_{1,2}] + \Phi(r_1 s_{2,3}) - \Phi(s_{1,2} r_3).$$

Squier [61] showed that $P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$ is an exact sequence when R is a uniquely terminating rewriting system.

We now use this resolution to compute the second homology of a finite Rees matrix semigroup $\mathcal{M}[G; I, \Lambda; P]$.

Theorem 7.4 *Let $S = \mathcal{M}[G; I, \Lambda; P]$ be a finite Rees matrix semigroup. Then the second integral homology of S is*

$$H_2(S) = H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)}.$$

Proof Without loss of generality we may assume that P is normal. We consider the uniquely terminating rewriting system Q on Y given in Theorem 7.3 and the resolution of \mathbb{Z} arising from it. By applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}S^1} -$ to this resolution, we obtain the chain complex of abelian groups

$$\mathbb{Z} \otimes P_3 \xrightarrow{1 \otimes \partial_3} \mathbb{Z} \otimes P_2 \xrightarrow{1 \otimes \partial_2} \mathbb{Z} \otimes P_1 \xrightarrow{1 \otimes \partial_1} \mathbb{Z} \otimes P_0 \xrightarrow{1 \otimes \epsilon} \mathbb{Z} \otimes \mathbb{Z} \longrightarrow 0,$$

or simply

$$\bar{P}_3 \xrightarrow{\bar{\partial}_3} \bar{P}_2 \xrightarrow{\bar{\partial}_2} \bar{P}_1 \xrightarrow{\bar{\partial}_1} \mathbb{Z} \longrightarrow 0,$$

where \bar{P}_1 , \bar{P}_2 and \bar{P}_3 are the free abelian groups on the sets of formal symbols $[x]$ ($x \in Y$), $[r, s]$ ($(r, s) \in Q$) and $[(r_1 r_2, s_{1,2}), (r_2 r_3, s_{2,3})]$, one for each overlap

from Q , respectively. The mappings $\bar{\partial}_2 : \bar{P}_2 \rightarrow \bar{P}_1$ and $\bar{\partial}_3 : \bar{P}_3 \rightarrow \bar{P}_2$ are defined respectively by

$$\bar{\partial}_2([r, s]) = \sum_{x \in Y} ((\text{the number of } x\text{'s in } r) - (\text{the number of } x\text{'s in } s))[x]$$

and

$$\bar{\partial}_3([(r_1 r_2, s_{1,2}), (r_2 r_3, s_{2,3})]) = [r_2 r_3, s_{2,3}] - [r_1 r_2, s_{1,2}] + \bar{\Phi}(r_1 s_{2,3}) - \bar{\Phi}(s_{1,2} r_3),$$

where $\bar{\Phi}$ is defined by

$$\bar{\Phi}(w) = \sum_{i=1}^q [r_i, s_i]$$

if $\Phi(w) = \sum_{i=1}^q \phi(b_i)[r_i, s_i]$.

Before we compute the second homology of S , $H_2(S) \cong \ker \bar{\partial}_2 / \text{im } \bar{\partial}_3$, we note that $H_2(G) \cong \ker \bar{\partial}_2^G / \text{im } \bar{\partial}_3^G$ where $\ker \bar{\partial}_2^G$ is the free abelian group on $\{W_j \mid j \in J\}$ and $\text{im } \bar{\partial}_3^G$ is the free abelian group on $\{V_l \mid l \in L\}$ which are found by using the Squier resolution on the Cayley table $\langle X \mid R \rangle$ of G . (Recall that R is a uniquely terminating system on X .) Notice that since G is a finite group, $H_2(G)$ is finite, and so $|J| = |L|$. Moreover, since

$$\bar{\partial}_2^G([x^2, u_1] + [u_1 x, u_2] + \dots + [u_{n_x-1} x, x]) = n_x [x] \quad (1)$$

where $x \in X$, $u_i = x^{i+1}$ and n_x is the order of x , we have $\text{rank}(\text{im } \bar{\partial}_2^G) = |X| = |G|$, and so $|J| = |L| = |G|^2 - |G|$.

Now we find a generating set for $\text{im } \bar{\partial}_3$ by using the overlaps from the proof of Theorem 7.3. First observe that $\bar{\partial}_3(U_{1,k,k',k''})$ gives a generating set which may be reduced to the basis $\{V_l \mid l \in L\}$ for $\text{im } \bar{\partial}_3^G$. Next we have

$$\begin{aligned} \bar{\partial}_3(U_{2,k,k',i}) &= [x_k y_i \rightarrow x_k] - [x_{k'} x_k \rightarrow x_l] + \bar{\Phi}(x_{k'} x_k) - \bar{\Phi}(x_l y_i) \\ &= [x_k y_i \rightarrow x_k] - [x_l y_i \rightarrow x_l] \end{aligned}$$

since $\bar{\Phi}(x_{k'} x_k) = [x_{k'} x_k \rightarrow x_l]$ and $\bar{\Phi}(x_l y_i) = [x_l y_i \rightarrow x_l]$. Similarly, we compute that

$$\begin{aligned} \bar{\partial}_3(U_{3,k,\lambda}) &= [x_0 z_\lambda \rightarrow z_\lambda] - [x_k x_0 \rightarrow x_k] + \bar{\Phi}(x_k z_\lambda) - \bar{\Phi}(x_k z_\lambda) \\ &= [x_0 z_\lambda \rightarrow z_\lambda] - [x_k x_0 \rightarrow x_k] \end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{4,k,i}) &= [x_0x_k \rightarrow x_k] - [y_ix_0 \rightarrow y_i] + \bar{\Phi}(y_ix_k) - \bar{\Phi}(y_ix_k) \\
&= [x_0x_k \rightarrow x_k] - [y_ix_0 \rightarrow y_i]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{5,i,i'}) &= [x_0y_{i'} \rightarrow x_0] - [y_ix_0 \rightarrow y_i] + \bar{\Phi}(y_ix_0) - \bar{\Phi}(y_iy_{i'}) \\
&= [x_0y_{i'} \rightarrow x_0] - [y_iy_{i'} \rightarrow y_i]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{6,i,\lambda}) &= [x_0z_\lambda \rightarrow z_\lambda] - [y_ix_0 \rightarrow y_i] + \bar{\Phi}(y_iz_\lambda) - \bar{\Phi}(y_iz_\lambda) \\
&= [x_0z_\lambda \rightarrow z_\lambda] - [y_ix_0 \rightarrow y_i]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{7,k,i}) &= [y_ix_0 \rightarrow y_i] - [x_ky_i \rightarrow x_k] + \bar{\Phi}(x_ky_i) - \bar{\Phi}(x_kx_0) \\
&= [y_ix_0 \rightarrow y_i] - [x_kx_0 \rightarrow x_k]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{8,k,i,i'}) &= [y_iy_{i'} \rightarrow y_i] - [x_ky_i \rightarrow x_k] + \bar{\Phi}(x_ky_i) - \bar{\Phi}(x_ky_{i'}) \\
&= [y_iy_{i'} \rightarrow y_i] - [x_ky_{i'} \rightarrow x_k]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{9,i,i'}) &= [y_{i'}x_0 \rightarrow y_{i'}] - [y_iy_{i'} \rightarrow y_i] + \bar{\Phi}(y_iy_{i'}) - \bar{\Phi}(y_ix_0) \\
&= [y_{i'}x_0 \rightarrow y_{i'}] - [y_ix_0 \rightarrow y_i]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{10,i,i',i''}) &= [y_{i'}y_{i''} \rightarrow y_{i'}] - [y_iy_{i'} \rightarrow y_i] + \bar{\Phi}(y_iy_{i'}) - \bar{\Phi}(y_iy_{i''}) \\
&= [y_{i'}y_{i''} \rightarrow y_{i'}] - [y_iy_{i''} \rightarrow y_i]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{11,k,k',\lambda}) &= [x_kx_{k'} \rightarrow x_l] - [z_\lambda x_k \rightarrow x_k] + \bar{\Phi}(z_\lambda x_l) - \bar{\Phi}(x_kx_{k'}) \\
&= [z_\lambda x_l \rightarrow x_l] - [z_\lambda x_k \rightarrow x_k]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{12,k,i,\lambda}) &= [x_ky_i \rightarrow x_k] - [z_\lambda x_k \rightarrow x_k] + \bar{\Phi}(z_\lambda x_k) - \bar{\Phi}(x_ky_i) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{13,\lambda,\lambda'}) &= [x_0z_{\lambda'} \rightarrow z_{\lambda'}] - [z_\lambda x_0 \rightarrow x_0] + \bar{\Phi}(z_\lambda z_{\lambda'}) - \bar{\Phi}(x_0z_{\lambda'}) \\
&= [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}] - [z_\lambda x_0 \rightarrow x_0]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{14,k,\lambda}) &= [z_\lambda x_k \rightarrow x_k] - [x_0z_\lambda \rightarrow z_\lambda] + \bar{\Phi}(x_0x_k) - \bar{\Phi}(z_\lambda x_k) \\
&= [x_0x_k \rightarrow x_k] - [x_0z_\lambda \rightarrow z_\lambda]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial}_3(U_{15,\lambda,\lambda'}) &= [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}] - [x_0z_\lambda \rightarrow z_\lambda] + \bar{\Phi}(x_0z_{\lambda'}) - \bar{\Phi}(z_\lambda z_{\lambda'}) \\
&= [x_0z_{\lambda'} \rightarrow z_{\lambda'}] - [x_0z_\lambda \rightarrow z_\lambda]
\end{aligned}$$

$$\begin{aligned}\bar{\partial}_3(U_{16,i,\lambda}) &= [z_\lambda y_i \rightarrow p_{\lambda i}] - [x_0 z_\lambda \rightarrow z_\lambda] + \bar{\Phi}(x_0 p_{\lambda i}) - \bar{\Phi}(z_\lambda y_i) \\ &= [x_0 p_{\lambda i} \rightarrow p_{\lambda i}] - [x_0 z_\lambda \rightarrow z_\lambda]\end{aligned}$$

$$\begin{aligned}\bar{\partial}_3(U_{17,k,\lambda,\lambda'}) &= [z_{\lambda'} x_k \rightarrow x_k] - [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}] + \bar{\Phi}(z_\lambda x_k) - \bar{\Phi}(z_{\lambda'} x_k) \\ &= [z_\lambda x_k \rightarrow x_k] - [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}]\end{aligned}$$

$$\begin{aligned}\bar{\partial}_3(U_{18,\lambda,\lambda',\lambda''}) &= [z_{\lambda'} z_{\lambda''} \rightarrow z_{\lambda''}] - [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}] + \bar{\Phi}(z_\lambda z_{\lambda''}) - \bar{\Phi}(z_{\lambda'} z_{\lambda''}) \\ &= [z_\lambda z_{\lambda''} \rightarrow z_{\lambda''}] - [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}]\end{aligned}$$

$$\begin{aligned}\bar{\partial}_3(U_{19,i,\lambda,\lambda'}) &= [z_{\lambda'} y_i \rightarrow p_{\lambda' i}] - [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}] + \bar{\Phi}(z_\lambda p_{\lambda' i}) - \bar{\Phi}(z_{\lambda'} y_i) \\ &= [z_\lambda p_{\lambda' i} \rightarrow p_{\lambda' i}] - [z_\lambda z_{\lambda'} \rightarrow z_{\lambda'}]\end{aligned}$$

$$\begin{aligned}\bar{\partial}_3(U_{20,i,\lambda}) &= [y_i x_0 \rightarrow y_i] - [z_\lambda y_i \rightarrow p_{\lambda i}] + \bar{\Phi}(z_\lambda y_i) - \bar{\Phi}(p_{\lambda i} x_0) \\ &= [y_i x_0 \rightarrow y_i] - [p_{\lambda i} x_0 \rightarrow p_{\lambda i}]\end{aligned}$$

$$\begin{aligned}\bar{\partial}_3(U_{21,i,i',\lambda}) &= [y_i y_{i'} \rightarrow y_i] - [z_\lambda y_i \rightarrow p_{\lambda i}] + \bar{\Phi}(z_\lambda y_i) - \bar{\Phi}(p_{\lambda i} y_{i'}) \\ &= [y_i y_{i'} \rightarrow y_i] - [p_{\lambda i} y_{i'} \rightarrow p_{\lambda i}].\end{aligned}$$

Let

$$\begin{aligned}V_k &= \bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)]) + \bar{\partial}_3([(x_0 x_0, x_0), (x_0 x_k, x_k)]) \\ &= [x_k x_0, x_k] - [x_0 x_k, x_k] \in \text{im } \bar{\partial}_3^G,\end{aligned}$$

and let

$$V_{k,i} = [y_i x_0, y_i] - [x_k x_0, x_k], \quad V_{k,i,i'} = [y_i y_{i'}, y_i] - [x_k y_{i'}, x_k],$$

$$V_{k,\lambda} = [x_0 z_\lambda, z_\lambda] - [x_0 x_k, x_k], \quad V_{k,\lambda,\lambda'} = [z_\lambda z_{\lambda'}, z_{\lambda'}] - [z_\lambda x_k, x_k]$$

$$(0 \leq k \leq m, i, i' \in I - \{1\}, \lambda, \lambda' \in \Lambda - \{1\}).$$

Now observe that we have

$$\begin{aligned}
\bar{\partial}_3(U_{2,k,k',i}) &= -V_{k,i',i} + V_{l,i',i}, & \bar{\partial}_3(U_{3,k,\lambda}) &= V_{k,\lambda} + V_k, \\
\bar{\partial}_3(U_{4,k,i}) &= -V_{k,i} + V_k, & \bar{\partial}_3(U_{5,i,i'}) &= -V_{0,i,i'}, \\
\bar{\partial}_3(U_{6,i,\lambda}) &= V_{k,\lambda} - V_{k,i} + V_k, & \bar{\partial}_3(U_{7,k,i}) &= V_{k,i}, \\
\bar{\partial}_3(U_{8,k,i,i'}) &= V_{k,i,i'}, & \bar{\partial}_3(U_{9,i,i'}) &= -V_{k,i'} - V_{k,i}, \\
\bar{\partial}_3(U_{10,i,i',i''}) &= V_{k,i',i''} - V_{k,i,i''}, & \bar{\partial}_3(U_{11,k,k',\lambda}) &= V_{k,\lambda,\lambda'} - V_{l,\lambda,\lambda'}, \\
\bar{\partial}_3(U_{12,k,i,\lambda}) &= 0, & \bar{\partial}_3(U_{13,\lambda,\lambda'}) &= V_{0,\lambda,\lambda'}, \\
\bar{\partial}_3(U_{14,k,\lambda}) &= -V_{k,\lambda}, & \bar{\partial}_3(U_{15,\lambda,\lambda'}) &= V_{k,\lambda'} - V_{k,\lambda}, \\
\bar{\partial}_3(U_{16,i,\lambda}) &= -V_{k_0,\lambda} \ (x_{k_0} \equiv p_{\lambda i}), & \bar{\partial}_3(U_{17,k,\lambda,\lambda'}) &= -V_{k,\lambda,\lambda'}, \\
\bar{\partial}_3(U_{18,\lambda,\lambda',\lambda''}) &= V_{k,\lambda,\lambda''} - V_{k,\lambda,\lambda'}, & \bar{\partial}_3(U_{19,i,\lambda,\lambda'}) &= -V_{k_1,\lambda,\lambda'} \ (x_{k_1} \equiv p_{\lambda' i}), \\
\bar{\partial}_3(U_{20,i,\lambda}) &= V_{k_0,i} \ (x_{k_0} \equiv p_{\lambda i}), & \bar{\partial}_3(U_{21,i,i',\lambda}) &= V_{k_0,i,i'} \ (x_{k_0} \equiv p_{\lambda i}).
\end{aligned}$$

Therefore

$$B = \{V_l, V_{k,i}, V_{k,i,i'}, V_{k,\lambda}, V_{k,\lambda,\lambda'} \mid l \in L; 0 \leq k \leq m; i, i' \in I - \{1\}; \lambda, \lambda' \in \Lambda - \{1\}\}$$

generates $\text{im } \bar{\partial}_3$.

Next we find a basis for $\ker \bar{\partial}_2$. First notice that since $\bar{\partial}_2([y_{i'}y_i, y_{i'}]) = [y_i]$ and $\bar{\partial}_2([z_{\lambda}z_{\lambda'}, z_{\lambda'}]) = [z_{\lambda}]$, it follows from (1) that

$$\text{rank}(\text{im } \bar{\partial}_2) = \text{rank}(\bar{P}_1) = |G| + (|\Lambda| - 1) + (|I| - 1).$$

Therefore

$$\begin{aligned}
\text{rank}(\ker \bar{\partial}_2) &= \text{rank}(\bar{P}_2) - \text{rank}(\bar{P}_1) = (|G|^2 - |G|) + |G|((|\Lambda| - 1) \\
&\quad + (|I| - 1)) + (|\Lambda| - 1)^2 + (|I| - 1)^2 + (|\Lambda| - 1)(|I| - 1).
\end{aligned}$$

Since each $\alpha \in \bar{P}_2$ has the form

$$\begin{aligned}
\alpha &= \sum_{k,k'=0}^m \alpha_{x_k, x_{k'}} [x_k x_{k'}, x_l] + \sum_{i \in I - \{1\}} \left(\alpha_{1,i} [y_i x_0, y_i] + \sum_{i' \in I - \{1\}} \alpha_{2,i,i'} [y_{i'} y_i, y_{i'}] \right. \\
&\quad \left. + \sum_{k=0}^m \alpha_{3,k,i} [x_k y_i, x_k] \right) + \sum_{\lambda \in \Lambda - \{1\}} \left(\beta_{1,\lambda} [x_0 z_{\lambda}, z_{\lambda}] + \sum_{\lambda' \in \Lambda - \{1\}} \beta_{2,\lambda,\lambda'} [z_{\lambda} z_{\lambda'}, z_{\lambda'}] \right. \\
&\quad \left. + \sum_{k=0}^m \beta_{3,k,\lambda} [z_{\lambda} x_k, x_k] + \sum_{i \in I - \{1\}} \gamma_{\lambda,i} [z_{\lambda} y_i, p_{\lambda i}] \right)
\end{aligned}$$

where all the coefficients are integers, $\alpha \in \ker \bar{\partial}_2$ if and only if

$$\begin{aligned}
0 = \bar{\partial}_2(\alpha) &= \sum_{k,k'=0}^m \alpha_{x_k, x_{k'}} ([x_k] + [x_{k'}] - [x_l]) \\
&+ \sum_{i \in I - \{1\}} \left(\alpha_{1,i} [x_0] + \sum_{i' \in I - \{1\}} \alpha_{2,i,i'} [y_i] + \sum_{k=0}^m \alpha_{3,k,i} [y_i] \right) \\
&+ \sum_{\lambda \in \Lambda - \{1\}} \left(\beta_{1,\lambda} [x_0] + \sum_{\lambda' \in \Lambda - \{1\}} \beta_{2,\lambda,\lambda'} [z_\lambda] + \sum_{k=0}^m \beta_{3,k,\lambda} [z_\lambda] \right) \\
&+ \sum_{i \in I - \{1\}} \gamma_{\lambda,i} ([z_\lambda] + [y_i] - [p_{\lambda i}]).
\end{aligned}$$

Equivalently, $\alpha \in \ker \bar{\partial}_2$ if and only if

$$\alpha_{x_0, x_0} = - \sum_{k=1}^m (\alpha_{x_k, x_0} + \alpha_{x_0, x_k} - \alpha_{x_k, x_k^{-1}}) - \sum_{i \in I - \{1\}} \alpha_{1,i} - \sum_{\lambda \in \Lambda - \{1\}} \beta_{1,\lambda} \quad (2)$$

$$\begin{aligned}
&+ \sum_{\substack{\lambda \in \Lambda - \{1\}, i \in I - \{1\} \\ p_{\lambda i} \equiv x_0}} \gamma_{\lambda,i}, \\
0 &= 2\alpha_{x_k, x_k} + \sum_{\substack{k'=1 \\ k' \neq k}}^m (\alpha_{x_k, x_{k'}} + \alpha_{x_{k'}, x_k} - \alpha_{x_{k'}, x_{k'}^{-1} x_k}) \\
&- \sum_{\substack{\lambda \in \Lambda - \{1\}, i \in I - \{1\} \\ p_{\lambda i} \equiv x_k}} \gamma_{\lambda,i} \quad (1 \leq k \leq m),
\end{aligned} \quad (3)$$

$$\alpha_{2,i,2} = - \left(\sum_{i' \in I - \{1,2\}} \alpha_{2,i,i'} + \sum_{k=0}^m \alpha_{3,k,i} + \sum_{\lambda \in \Lambda - \{1\}} \gamma_{\lambda,i} \right) \quad (i \in I - \{1\}), \quad (4)$$

$$\beta_{2,\lambda,2} = - \left(\sum_{\lambda' \in \Lambda - \{1,2\}} \beta_{2,\lambda,\lambda'} + \sum_{k=0}^m \beta_{3,k,\lambda} + \sum_{i \in I - \{1\}} \gamma_{\lambda,i} \right) \quad (\lambda \in \Lambda - \{1\}). \quad (5)$$

We have assumed that $|I|, |\Lambda| \geq 2$ and that 2 is a common element. The cases $|I| = 1$ or $|\Lambda| = 1$ are considered later. By using the system of equations above, we find a basis for $\ker \bar{\partial}_2$. First, if we take all $\alpha_{1,i}$, $\alpha_{2,i,i'}$, $\alpha_{3,k,i}$, $\beta_{1,\lambda}$, $\beta_{2,\lambda,\lambda'}$, $\beta_{3,k,\lambda}$ and $\gamma_{\lambda,i}$ to be zero, we have

$$\sum_{x_k, x_{k'} \in X} \alpha_{x_k, x_{k'}} ([x_k] + [x_{k'}] - [x_l]) = 0,$$

which gives the basis $\{W_j \mid j \in J\}$ of $\ker \bar{\partial}_2^G$ where $H_2(G) = \ker \bar{\partial}_2^G / \text{im } \bar{\partial}_3^G$.

Now if we fix $\alpha_{1,i} = 1$ and all the other variables on the right-hand side in (2)–(5) to be zero, then we obtain $\alpha_{x_0, x_0} = -1$. Therefore we obtain the following generators:

$$W_i = [y_i x_0, y_i] - [x_0^2, x_0] \quad (i \in I - \{1\}).$$

By using similar arguments, we obtain certain other generators:

$$W_\lambda = [x_0 z_\lambda, z_\lambda] - [x_0^2, x_0] \quad (\lambda \in \Lambda - \{1\}),$$

$$W_{i,k} = [y_2 y_i, y_2] - [x_k y_i, x_k] \quad (0 \leq k \leq m, i \in I - \{1\}),$$

$$W_{\lambda,k} = [z_\lambda z_2, z_2] - [z_\lambda x_k, x_k] \quad (0 \leq k \leq m, \lambda \in \Lambda - \{1\}),$$

$$W_{i,i'} = [y_{i'} y_i, y_{i'}] - [y_2 y_i, y_2] \quad (i, i' \in I - \{1\}, i' \neq 2),$$

$$W_{\lambda,\lambda'} = [z_\lambda z_{\lambda'}, z_{\lambda'}] - [z_\lambda z_2, z_2] \quad (\lambda, \lambda' \in \Lambda - \{1\}, \lambda' \neq 2).$$

We note that to construct a basis for $\ker \bar{\partial}_2$ we need a further $(|\Lambda| - 1)(|I| - 1)$ independent elements. We will see that we do not need to identify these remaining elements $W_{\lambda,i}$ ($\lambda \in \Lambda - \{1\}, i \in I - \{1\}$) of the basis:

$$Z = \{ W_j, W_i, W_\lambda, W_{i,k}, W_{\lambda,k}, W_{i,i'}, W_{\lambda,\lambda'}, W_{\lambda,i} \mid j \in J; 0 \leq k \leq m;$$

$$i, i' \in I - \{1\} (i' \neq 2); \lambda, \lambda' \in \Lambda - \{1\} (\lambda' \neq 2) \}.$$

Now we express the V 's in B in terms of the W 's in Z . First, for each $l \in L$, write $V_l(W)$ for the expression of V_l in terms of the W_j ($j \in J$) as in the calculation of $H_2(G)$. Now observe that

$$V_{0,i} = W_i \quad V_{k,i} = W_i + \bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)]) \quad (k \neq 0),$$

$$V_{0,\lambda} = W_\lambda \quad V_{k,\lambda} = W_\lambda - \bar{\partial}_3([(x_0 x_0, x_0), (x_0 x_k, x_k)]) \quad (k \neq 0),$$

$$V_{k,2,i} = W_{i,k} \quad V_{k,i',i} = W_{i,i'} + W_{i,k} \quad (i' \neq 2),$$

$$V_{k,\lambda,2} = W_{\lambda,k} \quad V_{k,\lambda,\lambda'} = W_{\lambda,\lambda'} + W_{\lambda,k} \quad (\lambda' \neq 2).$$

We obtain the following abelian group presentation for $H_2(S)$:

$$\begin{aligned} \langle Z \mid & V_l(W) = 0, W_i = 0, W_i + V_k(W) = 0 \ (k \neq 0), W_\lambda = 0, \\ & W_\lambda + V'_k(W) = 0 \ (k \neq 0), W_{i,k} = 0, W_{i,i'} + W_{i,k} = 0 \ (i' \neq 2), \\ & W_{\lambda,k} = 0, W_{\lambda,\lambda'} + W_{\lambda,k} = 0 \ (\lambda' \neq 2) \\ & (l \in L; 0 \leq k \leq m; \lambda, \lambda' \in \Lambda - \{1\}; i, i' \in I - \{1\}) \rangle \end{aligned}$$

where $V_k(W)$ expresses $\bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)])$ in terms of the W_j and $V'_k(W)$ expresses $\bar{\partial}_3([(x_0 x_0, x_0), (x_0 x_k, x_k)])$ in terms of the W_j . It is clear that some of the generators in the above presentation are redundant. By eliminating these redundant generators, we obtain the abelian group presentation:

$$\langle V_j, W_{\lambda,i} \ (j \in J; \lambda \in \Lambda - \{1\}; i \in I - \{1\}) \mid V_l(W) = 0 \ (l \in L) \rangle$$

which defines the abelian group

$$H_2(G) \times \mathbb{Z}^{(|I|-1)(|\Lambda|-1)},$$

as required.

Now we assume that $|\Lambda| = 1$, $|I| > 1$ and we prove that $H_2(S) = H_2(G)$. In this case, first observe that we have the following overlaps in the proof of Theorem 7.3:

$$\begin{aligned} U'_{1,k,k',k''} &= [(x_k x_{k'}, x_l), (x_{k'} x_{k''}, x_l)], & U'_{2,k,k',i} &= [(x_{k'} x_k, x_l), (x_k y_i, x_k)], \\ U'_{3,k,i} &= [(y_i x_0, y_i), (x_0 x_k, x_k)], & U'_{4,i,i'} &= [(y_i x_0, y_i), (x_0 y_{i'}, x_0)], \\ U'_{5,k,i} &= [(x_k y_i, x_k), (y_i x_0, y_i)], & U'_{6,k,i,i'} &= [(x_k y_i, x_k), (y_i y_{i'}, y_i)], \\ U'_{7,i,i'} &= [(y_i y_{i'}, y_i), (y_{i'} x_0, y_{i'})], & U'_{8,i,i',i''} &= [(y_i y_{i'}, y_i), (y_{i'} y_{i''}, y_{i'})], \end{aligned}$$

where $i, i', i'' \in I - \{1\}$ and $1 \leq k, k', k'' \leq m$.

Now we find a generating set for $\text{im } \bar{\partial}_3$ by using the overlaps above. First observe that, as before, $\bar{\partial}_3(U'_{1,k,k',k''})$ gives a generating set which may be reduced to the basis $\{V_l \mid l \in L\}$ for $\text{im } \bar{\partial}_3^G$ as before. Similarly, next we have

$$\begin{aligned} \bar{\partial}_3(U'_{2,k,k',i}) &= [x_k y_i \rightarrow x_k] - [x_{k'} x_k \rightarrow x_l] + \bar{\Phi}(x_{k'} x_k) - \bar{\Phi}(x_l y_i) \\ &= [x_k y_i \rightarrow x_k] - [x_l y_i \rightarrow x_l] \end{aligned}$$

since $\bar{\Phi}(x_k'x_k) = [x_k'x_k \rightarrow x_l]$ and $\bar{\Phi}(x_ly_i) = [x_ly_i \rightarrow x_l]$. Similarly, we compute that

$$\begin{aligned}
\bar{\partial}_3(U'_{3,k,i}) &= [x_0x_k \rightarrow x_k] - [y_ix_0 \rightarrow y_i] + \bar{\Phi}(y_ix_k) - \bar{\Phi}(y_ix_k) \\
&= [x_0x_k \rightarrow x_k] - [y_ix_0 \rightarrow y_i] \\
\bar{\partial}_3(U'_{4,i,i'}) &= [x_0y_{i'} \rightarrow x_0] - [y_ix_0 \rightarrow y_i] + \bar{\Phi}(y_ix_0) - \bar{\Phi}(y_iy_{i'}) \\
&= [x_0y_{i'} \rightarrow x_0] - [y_iy_{i'} \rightarrow y_i] \\
\bar{\partial}_3(U'_{5,k,i}) &= [y_ix_0 \rightarrow y_i] - [x_ky_i \rightarrow x_k] + \bar{\Phi}(x_ky_i) - \bar{\Phi}(x_kx_0) \\
&= [y_ix_0 \rightarrow y_i] - [x_kx_0 \rightarrow x_k] \\
\bar{\partial}_3(U'_{6,k,i,i'}) &= [y_iy_{i'} \rightarrow y_i] - [x_ky_i \rightarrow x_k] + \bar{\Phi}(x_ky_i) - \bar{\Phi}(x_ky_{i'}) \\
&= [y_iy_{i'} \rightarrow y_i] - [x_ky_{i'} \rightarrow x_k] \\
\bar{\partial}_3(U'_{7,i,i'}) &= [y_{i'}x_0 \rightarrow y_{i'}] - [y_iy_{i'} \rightarrow y_i] + \bar{\Phi}(y_iy_{i'}) - \bar{\Phi}(y_ix_0) \\
&= [y_{i'}x_0 \rightarrow y_{i'}] - [y_ix_0 \rightarrow y_i] \\
\bar{\partial}_3(U'_{8,i,i',i''}) &= [y_{i'}y_{i''} \rightarrow y_{i'}] - [y_iy_{i'} \rightarrow y_i] + \bar{\Phi}(y_iy_{i'}) - \bar{\Phi}(y_iy_{i''}) \\
&= [y_{i'}y_{i''} \rightarrow y_{i'}] - [y_iy_{i''} \rightarrow y_i].
\end{aligned}$$

Let

$$\begin{aligned}
V_k &= \bar{\partial}_3([(x_kx_0, x_k), (x_0x_0, x_0)]) + \bar{\partial}_3([(x_0x_0, x_0), (x_0x_k, x_k)]) \\
&= [x_kx_0, x_k] - [x_0x_k, x_k] \in \text{im } \bar{\partial}_3^G,
\end{aligned}$$

and let

$$V_{k,i} = [y_ix_0, y_i] - [x_kx_0, x_k], \quad V_{k,i,i'} = [y_iy_{i'}, y_i] - [x_ky_{i'}, x_k]$$

where $0 \leq k \leq m$ and $i, i' \in I - \{1\}$. Then we have

$$\begin{aligned}
\bar{\partial}_3(U'_{2,k,k',i}) &= -V_{k,i',i} + V_{l,i',i}, & \bar{\partial}_3(U'_{3,k,i}) &= -V_{k,i} + V_k, \\
\bar{\partial}_3(U'_{4,i,i'}) &= -V_{0,i,i'}, & \bar{\partial}_3(U'_{5,k,i}) &= V_{k,i}, \\
\bar{\partial}_3(U'_{6,k,i,i'}) &= V_{k,i,i'}, & \bar{\partial}_3(U'_{7,i,i'}) &= -V_{k,i'} - V_{k,i}, \\
\bar{\partial}_3(U'_{8,i,i',i''}) &= V_{k,i',i''} - V_{k,i,i''}.
\end{aligned}$$

Therefore

$$B' = \{ V_l, V_{k,i}, V_{k,i,i'} \mid l \in L; 0 \leq k \leq m; i, i' \in I - \{1\} \}$$

generates $\text{im } \bar{\partial}_3$.

Next we find a basis for $\ker \bar{\partial}_2$. Since each $\alpha \in \bar{P}_2$ has the form

$$\begin{aligned} \alpha = & \sum_{k,k'=0}^m \alpha_{x_k, x_{k'}} [x_k x_{k'}, x_l] + \sum_{i \in I - \{1\}} \left(\alpha_{1,i} [y_i x_0, y_i] + \sum_{i' \in I - \{1\}} \alpha_{2,i,i'} [y_{i'} y_i, y_{i'}] \right. \\ & \left. + \sum_{k=0}^m \alpha_{3,k,i} [x_k y_i, x_k] \right) \end{aligned}$$

where all the coefficients are integers, $\alpha \in \ker \bar{\partial}_2$ if and only if

$$\begin{aligned} 0 = \bar{\partial}_2(\alpha) = & \sum_{k,k'=0}^m \alpha_{x_k, x_{k'}} ([x_k] + [x_{k'}] - [x_l]) \\ & + \sum_{i \in I - \{1\}} \left(\alpha_{1,i} [x_0] + \sum_{i' \in I - \{1\}} \alpha_{2,i,i'} [y_i] + \sum_{k=0}^m \alpha_{3,k,i} [y_i] \right). \end{aligned}$$

Equivalently, $\alpha \in \ker \bar{\partial}_2$ if and only if

$$\begin{aligned} \alpha_{x_0, x_0} = & - \sum_{k=1}^m (\alpha_{x_k, x_0} + \alpha_{x_0, x_k} - \alpha_{x_k, x_k^{-1}}) - \sum_{i \in I - \{1\}} \alpha_{1,i} \\ 0 = & 2\alpha_{x_k, x_k} + \sum_{\substack{k'=1 \\ k' \neq k}}^m (\alpha_{x_k, x_{k'}} + \alpha_{x_{k'}, x_k} - \alpha_{x_{k'}, x_k^{-1} x_k}) \quad (1 \leq k \leq m) \\ \alpha_{2,i,2} = & - \left(\sum_{i' \in I - \{1,2\}} \alpha_{2,i,i'} + \sum_{k=0}^m \alpha_{3,k,i} + \sum_{\lambda \in \Lambda - \{1\}} \gamma_{\lambda,i} \right) \quad (i \in I - \{1\}). \end{aligned}$$

As before, by using the above system of equations, we obtain the following basis for $\ker \bar{\partial}_2$:

$$Z' = \{ W_j, W_i, W_{i,k}, W_{i,i'} \mid j \in J; 0 \leq k \leq m; i, i' \in I - \{1\} (i' \neq 2) \}$$

where $\{ W_j \mid j \in J \}$ is the basis for $\ker \bar{\partial}_2^G$ ($H_2(G) = \ker \bar{\partial}_2^G / \text{im } \bar{\partial}_3^G$) and

$$\begin{aligned} W_i = & [y_i x_0, y_i] - [x_0^2, x_0] \quad (i \in I - \{1\}), \\ W_{i,k} = & [y_2 y_i, y_2] - [x_k y_i, x_k] \quad (0 \leq k \leq m; i \in I - \{1\}), \\ W_{i,i'} = & [y_{i'} y_i, y_{i'}] - [y_2 y_i, y_2] \quad (i, i' \in I - \{1\}, i' \neq 2). \end{aligned}$$

Next we express the V 's in B' in terms of the W 's in Z' . First, for each $l \in L$, write $V_l(W)$ for the expression of V_l in terms of the W_j ($j \in J$) as in the calculation of $H_2(G)$. Now observe that we have

$$\begin{aligned} V_{0,i} &= W_i, & V_{k,i} &= W_i + \bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)]) \quad (k \neq 0), \\ V_{k,2,i} &= W_{i,k}, & V_{k,i',i} &= W_{i,i'} + W_{i,k} \quad (i' \neq 2). \end{aligned}$$

Therefore, we obtain the following abelian group presentation for $H_2(S)$:

$$\begin{aligned} \langle Z' \mid & V_l(W) = 0, \quad W_i = 0, \quad W_i + V_k(W) = 0 \quad (k \neq 0), \quad W_{i,k} = 0, \\ & W_{i,i'} + W_{i,k} = 0 \quad (l \in L; \quad 0 \leq k \leq m; \quad i, i' \in I - \{1\}, \quad (i' \neq 2)) \rangle \end{aligned}$$

where $V_k(W)$ expresses $\bar{\partial}_3([(x_k x_0, x_k), (x_0 x_0, x_0)])$ in terms of the W_j . It is clear that some of the generators in the above presentation are redundant. By eliminating these redundant generators, we obtain the abelian group presentation:

$$\langle V_j \mid V_l(W) = 0 \quad (l \in L) \rangle$$

which defines the abelian group $H_2(G)$, as required.

Since $\mathcal{M}[G; I, \{1\}; P] \cong \mathcal{M}[G; \{1\}, I; P]$ and $\mathcal{M}[G; , \{1\}, \{1\}; P] \cong G$, the proof is now complete. \blacksquare

7.3 A small presentation for Rees matrix semigroups

Consider the presentation for $S = \mathcal{M}[G; I, \Lambda; P]$, a Rees matrix semigroup with P normal, which is given in Theorem 7.2 by

$$\mathcal{P}_1 = \langle Y \mid R, \quad y_i e = y_i, \quad e y_i = e \quad (2 \leq i \leq m), \quad (6)$$

$$z_\lambda e = e, \quad e z_\lambda = z_\lambda \quad (2 \leq \lambda \leq n), \quad (7)$$

$$z_\lambda y_i = p_{\lambda i} \quad (2 \leq i \leq m, \quad 2 \leq \lambda \leq n) \rangle$$

where e is a non-empty representative of the identity of G , and where $I = \{1, \dots, m\}$ and $\Lambda = \{1, \dots, n\}$. From now on, we write $S = \mathcal{M}[G; m, n; P]$ instead of $S = \mathcal{M}[G; I, \Lambda; P]$.

The deficiency of \mathcal{P}_1 is given by $\text{def}(\mathcal{P}_1) = \text{def}(\mathcal{P}_G) + (m-1)(n-1) + (m-1) + (n-1)$, where $\mathcal{P}_G = \langle X \mid R \rangle$ is a semigroup presentation for G . With the above notation, we give a presentation for S with deficiency $\text{def}(\mathcal{P}_G) + (m-1)(n-1) + 1$, which is one higher than the rank of $H_2(S)$ (see Theorem 7.4) if \mathcal{P}_G is an efficient semigroup presentation for G .

Proposition 7.5 *The presentation*

$$\mathcal{P}_2 = \langle Y \mid R, \quad ey_2 = e, \quad y_i y_{i+1} = y_i \quad (2 \leq i \leq m-1), \quad (8)$$

$$ez_2 = z_2, \quad z_\lambda z_{\lambda+1} = z_{\lambda+1} \quad (2 \leq \lambda \leq n-1), \quad (9)$$

$$y_m z_n e = y_m, \quad (10)$$

$$z_\lambda y_i = p_{\lambda i} \quad (2 \leq i \leq m, \quad 2 \leq \lambda \leq n) \rangle$$

defines the Rees matrix semigroup $S = \mathcal{M}[G; m, n; P]$ with $m, n > 1$.

Proof From (6), we have

$$y_i y_{i+1} = (y_i e) y_{i+1} \equiv y_i (e y_{i+1}) = y_i e = y_i \quad (2 \leq i \leq m-1).$$

Similarly, from (7), we have

$$z_\lambda z_{\lambda+1} = z_\lambda (e z_{\lambda+1}) \equiv (z_\lambda e) z_{\lambda+1} = e z_{\lambda+1} = z_{\lambda+1} \quad (2 \leq \lambda \leq n-1).$$

Moreover, from (7) and (6), we have

$$y_m z_n e = y_m e = y_m.$$

Therefore, every relation in \mathcal{P}_2 holds in S . Now we show that every relation in \mathcal{P}_1 is a consequence of the relations in \mathcal{P}_2 .

By induction, it follows from (8) that $y_i y_{i'} = y_i$ ($2 \leq i < i' \leq m$). In particular,

$$y_i y_m = y_i \text{ and } y_2 y_i = y_2 \quad (2 \leq i \leq m). \quad (11)$$

Similarly, from (9),

$$z_\lambda z_n = z_n \text{ and } z_2 z_\lambda = z_\lambda \quad (2 \leq \lambda \leq n). \quad (12)$$

Since G is finite and e is a representative of the identity of G , there exists $k \in \mathbb{N}$ such that the relation $p_{nm}^k = e$ holds in G , and so $(z_n y_m)^k = e$ is a consequence of the relations from $R \cup \{z_n y_m = p_{nm}\}$. It follows from (12), (9) and (10) that

$$z_n e = (z_2 z_n) e = (e z_2) z_n e = e z_n e = (z_n y_m)^{k-1} z_n (y_m z_n e) = (z_n y_m)^k = e. \quad (13)$$

Moreover, since $e^2 = e$ is a consequence of the relations from R , it follows from (10) that

$$y_m e = (y_m z_n e) e = y_m z_n e = y_m. \quad (14)$$

Next we show that the remaining relations of \mathcal{P}_1 hold. From (11) and (14), we have

$$y_i e = (y_i y_m) e \equiv y_i (y_m e) = y_i y_m = y_i \quad (2 \leq i \leq m-1)$$

and, from (8) and (11), we have

$$e y_i = (e y_2) y_i \equiv e (y_2 y_i) = e y_2 = e \quad (3 \leq i \leq m).$$

and from (12), (9) and (13), we have

$$e z_\lambda = e (z_2 z_\lambda) \equiv (e z_2) z_\lambda = z_2 z_\lambda = z_\lambda \quad (3 \leq \lambda \leq n)$$

and

$$z_\lambda e = z_\lambda (z_n e) \equiv (z_\lambda z_n) e = z_n e = e \quad (2 \leq \lambda \leq n-1),$$

as required. ■

7.4 Efficiency of Rees matrix semigroups

The presentation \mathcal{P}_2 is not efficient, but it proves useful in the following results. In Chapter 5 we proved that finite abelian groups and direct powers of the dihedral groups D_{2r}^m with r even are efficient as semigroups. In particular, we found efficient semigroup presentations of the form $\langle X \mid R_1, xux = x \rangle$ with identity xu .

Theorem 7.6 *Let $S = \mathcal{M}[G; m, n; P]$ be a finite Rees matrix semigroup with P normal. If G has a semigroup presentation of the form $\mathcal{P}_G = \langle X \mid R_1, xux = x \rangle$ with identity xu ($x \in X, u \in X^+$), then S has a semigroup presentation whose deficiency is $\text{def}(\mathcal{P}_G) + (m - 1)(n - 1)$.*

Proof First assume that $m, n > 1$ and consider the presentation \mathcal{P}_2 for S . Take $e \equiv xu$. Since, from (8) and (13), the relations $xuy_2 = xu$, $z_nx = x$ and $xux = x$ hold in S , we have

$$xuy_2z_nx \equiv (xuy_2)z_nx = xu(z_nx) = xux = x.$$

Therefore, S is a homomorphic image of the semigroup T defined by the presentation obtained from \mathcal{P}_2 by adding the relation $xuy_2z_nx = x$ and removing the relations $xuy_2 = xu$ and $xux = x$:

$$\mathcal{P}_3 = \langle Y \mid R_1, \quad xuy_2z_nx = x, \tag{15}$$

$$y_iy_{i+1} = y_i \quad (2 \leq i \leq m - 1), \tag{16}$$

$$xuz_2 = z_2, \tag{17}$$

$$z_\lambda z_{\lambda+1} = z_{\lambda+1} \quad (2 \leq \lambda \leq n - 1), \tag{18}$$

$$y_m z_n x u = y_m, \tag{19}$$

$$z_\lambda y_i = p_{\lambda i} \quad (2 \leq i \leq m, 2 \leq \lambda \leq n) \rangle.$$

Note that if $m = 2$, then (16) is absent and if $n = 2$, then (18) is absent. Now we show that the relations $xuy_2 = xu$ and $xux = x$ hold in T so that $S \cong T$.

As before, from (16), (18) and (17), we have

$$y_2 y_m = y_2, \quad z_2 z_n = z_n \text{ and } x u z_n = z_n. \quad (20)$$

It follows from (20), (19) and (15) that

$$x u y_2 = x u (y_2 y_m) = x u y_2 (y_m z_n x u) = (x u y_2 z_n x) u = x u \quad (21)$$

and also that

$$z_n x = (z_2 z_n) x = (x u z_2) z_n x = x u z_n x = (x u y_2) z_n x = x. \quad (22)$$

Therefore, from (15), (21), (20) and (22), we have

$$x u x = x u (x u y_2 z_n x) = x u (x u z_n) x = (x u z_n) x = z_n x = x$$

and hence S is efficient, as required.

If $m = 1$, then

$$\mathcal{P}'_3 = \langle X, z_2, \dots, z_n \mid R_1, \quad u x z_2 = z_2, \quad z_\lambda z_{\lambda+1} = z_{\lambda+1} \quad (2 \leq \lambda \leq n-1), \quad x z_n u x = x \rangle$$

is an efficient presentation for S . Indeed, from the relations $z_\lambda z_{\lambda+1} = z_{\lambda+1}$ ($2 \leq \lambda \leq n-1$), we have $z_2 z_n = z_n$. It follows that

$$z_n u x = (z_2 z_n) u x = (u x z_2) z_n u x = u (x z_n u x) = u x,$$

and so

$$x u x = x u (x z_n u x) = x u x (z_2 z_n) u x \equiv x (u x z_2) z_n u x = x (z_2 z_n) u x = x z_n u x = x.$$

Therefore, since $x z_n u x = x u x = x$, we may replace the relation $x z_n u x = x$ by the relations $z_n u x = u x$ and $x u x = x$ to obtain the presentation

$$\langle X, z_\lambda \mid R_1, \quad x u x = x, \quad u x z_2 = z_2, \quad z_\lambda z_{\lambda+1} = z_{\lambda+1} \quad (2 \leq \lambda \leq n-1), \quad z_n u x = u x \rangle$$

which is a presentation for S by Theorem 7.5, as required.

Similarly, if $n = 1$, then

$$\mathcal{P}_3'' = \langle X, y_2, \dots, y_m \mid R_1, \quad xy_2x = x, \quad y_iy_{i+1} = y_i \ (2 \leq i \leq m-1), \quad y_mxu = y_m \rangle$$

is an efficient presentation for S . The proof is now complete. \blacksquare

As we mentioned at the beginning of this section, finite abelian groups and dihedral groups D_{2r} with r even, have efficient semigroup presentations of the required form. Therefore we have the following results:

Corollary 7.7 *Finite Rees matrix semigroups over finite abelian groups or dihedral groups with even degree are efficient.*

Proof Since $\text{rank } H_2(D_{2r}) = 1$ with r even and

$$\langle a, b \mid a^3 = a, \quad a^2 = b^n, \quad ab^{n-1}a = b \rangle$$

is an efficient presentation for D_{2r} with r even (see Theorem 5.6), it follows by Theorem 7.6 that finite Rees matrix semigroups over dihedral groups with even degree are efficient.

Since $\text{rank } H_2(A) = t(t-1)/2$ where A is a finite abelian group of rank t and

$$\begin{aligned} \langle a_1, \dots, a_t \mid & \quad a_1^{q_1+1} = a_1, \quad a_1^{q_1} = a_j^{q_j}, \quad a_1a_ja_1^{q_1-1} = a_j, \quad a_ka_l = a_la_k \\ & \quad (2 \leq j \leq t, \quad 2 \leq k < l \leq t) \rangle, \end{aligned}$$

where $q_1 > 1$ and q_j divides q_{j+1} for all $j = 1, \dots, t-1$, is an efficient presentation for A (see Theorem 5.5), it follows by Theorem 7.6 that finite Rees matrix semigroups over abelian groups are efficient. \blacksquare

In Chapter 5, we found an efficient semigroup presentations for the direct power D_{2r}^t with r even of the form $\langle a_i, b_i \mid a_i^3 = a_i, \quad R_1 \rangle$ with identity a_1^2 (see Theorem 5.10). By Theorem 7.6, we have the following result.

Corollary 7.8 *Finite Rees matrix semigroups over the direct power D_{2r}^k with r even are efficient.*

Next we give another efficient semigroup presentation for the generalised quaternion groups. This presentation will be useful for the efficiency of finite Rees matrix semigroups over generalised quaternion groups.

Proposition 7.9 *The semigroup presentation*

$$\langle a, b \mid ba^{r-1}b = a, ba^rb^2 = b \rangle$$

defines the generalised quaternion group Q_r of order $4r$ ($r \geq 2$).

Proof Since $\langle a, b \mid ba^{r-1}b = a, aba = b \rangle$ defines Q_r with identity a^{2r} and since the relation $a^r = b^2$ holds in Q_r (see (2) in Chapter 5), it is enough to show that the relation $aba = a$ is a consequence of the relations $ba^{r-1}b = a$ and $ba^rb^2 = b$. First observe that we have

$$a^{r+1}b^2 = (ba^{r-1}b)a^rb^2 \equiv ba^{r-1}(ba^rb^2) = ba^{r-1}b = a$$

and

$$\begin{aligned} b^2a^{r+1} &= b^2a^r(ba^{r-1}b) = b^2a^rb(ba^{r-1}b)a^{r-2}b \equiv b(ba^rb^2)a^{r-1}b)a^{r-2}b \\ &= b(ba^{r-1}b)a^{r-2}b = ba^{r-1}b = a. \end{aligned}$$

It follows that

$$aba = ab(a^{r+1}b^2) = (ba^{r-1}b)ba^{r+1}b^2 \equiv ba^{r-1}(b^2a^{r+1})b^2 = ba^rb^2 = b,$$

as required. ■

Therefore we have another efficient semigroup presentation for Q_r . Moreover, since the relation $ba^rb = a^{2r}$ holds in Q_r , this presentation of the form $\langle X \mid R_1, xux = x \rangle$ with identity xu .

Corollary 7.10 *Finite Rees matrix semigroups over the generalised quaternion groups Q_r of order $4r$ are efficient.*

Proof The result follows from Proposition 7.9 and Theorem 7.6. ■

Next we consider the finite Rees matrix semigroups over the projective special linear groups $PSL(2, p)$ with p prime.

Corollary 7.11 *Let $S = \mathcal{M}[G; m, n; P]$ be a finite Rees matrix semigroup (with P normal). If G is the group $PSL(2, p)$ with p prime, then S is efficient.*

Proof Since $PSL(2, 2) \cong D_6$ and $D_6 \cong \langle a, b \mid ababa = a, ab^2a = b \rangle$ (see Lemma 5.7), it follows from Theorem 7.6 that S is efficient when $p = 2$.

If p is an odd prime, then, from Theorem 5.19, the presentation:

$$\begin{aligned} PSL(2, p) &\cong \langle a, b \mid b^p = a^2, babab = a, ab^{11p-1}ab^3ab^{(p+1)/2}ab^4ab^{(p+1)/2} = b \rangle \\ &\cong \langle a, b \mid b^p = a^2, babab = a, b(abab^{11p}ab^3ab^{(p+1)/2}ab^4ab^{(p-1)/2})b = b \rangle \end{aligned}$$

is an efficient presentation for $PSL(2, p)$.

Since $b(abab^{11p}ab^3ab^{(p+1)/2}ab^4ab^{(p-1)/2})$ is a representative of the identity of $PSL(2, p)$, it follows from Theorem 7.6 that S is efficient, as well. ■

7.5 Efficient non-simple semigroups

All the efficient semigroups in the previous chapters and in this chapter so far are simple semigroups. In this section, we give two families of efficient non-simple semigroups which have non-trivial second homology.

Consider the following presentation:

$$\langle a_1, \dots, a_r \mid a_i^{n_i+1} = a_i \ (1 \leq i \leq r), \ a_j a_i = a_i a_j \ (1 \leq i < j \leq r) \rangle$$

where $n_1 > 1$ and n_i divides n_{i+1} for $i = 1, \dots, r-1$.

This semigroup presentation is related to the standard group presentation of the abelian group $C_{n_1} \times \dots \times C_{n_r}$, where C_{n_i} is the cyclic group of order n_i .

For $r \geq 2$, it is clear that this semigroup presentation defines a commutative semigroup S but not an abelian group. For $r \geq 2$, the subset

$$I = \{a_1^{m_1} \cdots a_r^{m_r} \mid 1 \leq m_i \leq n_i \text{ for } i = 1, \dots, r\}$$

is a proper ideal of S , so that S is not simple.

Theorem 7.12 *Let S be the semigroup defined by the following presentation*

$$\langle a_1, \dots, a_r \mid a_i^{n_i+1} = a_i \ (1 \leq i \leq r), \ a_j a_i = a_i a_j \ (1 \leq i < j \leq r) \rangle$$

where $n_1 > 1$ and n_i divides n_{i+1} for $i = 1, \dots, r-1$. Then the second homology of S is

$$C_{n_1}^{(r-1)} \times C_{n_2}^{(r-2)} \times \cdots \times C_{n_{r-1}}.$$

In particular, S is an efficient semigroup.

Proof First we determine all overlaps. They are:

$$U_{0,i,k} = [(a_i^{n_i+1-k} a_i^k, a_i), (a_i^k a_i^{n_i+1-k}, a_i)] \ (1 \leq i \leq r, \ 1 \leq k \leq n_i + 1),$$

$$U_{1,i,j} = [(a_j^{n_j+1}, a_j), (a_j a_i, a_i a_j)] \ (1 \leq i < j \leq r),$$

$$U_{2,i,j} = [(a_j a_i, a_i a_j), (a_i^{n_i+1}, a_i)] \ (1 \leq i < j \leq r),$$

$$U_{3,i,j,k} = [(a_k a_j, a_j a_k), (a_j a_i, a_i a_j)] \ (1 \leq i < j < k \leq r).$$

Now we apply Lemma 7.1. For $U_{0,i,k}$, there exists $a_i^{n_i+2-k}$ such that

$$(a_i^{n_i+1-k} a_i^k) a_i^{n_i+1-k} \rightarrow a_i^{n_i+2-k} \text{ and } a_i^{n_i+1-k} (a_i^k a_i^{n_i+1-k}) \rightarrow a_i^{n_i+2-k}.$$

For $U_{1,i,j}$, there exists $a_i a_j$ such that

$$(a_j^{n_j+1}) a_i \rightarrow a_j a_i \rightarrow a_i a_j \text{ and}$$

$$a_j^{n_j} (a_j a_i) \rightarrow a_j^{n_j-1} (a_j a_i) a_j \rightarrow \cdots \rightarrow a_i a_j^{n_j+1} \rightarrow a_i a_j.$$

For $U_{2,i,j}$, there exists $a_i a_j$ such that

$$(a_j a_i) a_i^{n_i} \rightarrow a_i (a_j a_i) a_j^{n_i-1} \rightarrow \cdots \rightarrow a_i^{n_i+1} a_j \rightarrow a_i a_j \text{ and}$$

$$a_j(a_i^{n_i+1}) \rightarrow a_j a_i \rightarrow a_i a_j.$$

For $U_{3,i,j,k}$, there exists $a_i a_j a_k$ such that

$$(a_k a_j) a_i \rightarrow a_j (a_k a_i) \rightarrow (a_j a_i) a_k \rightarrow a_i a_j a_k \text{ and}$$

$$a_k (a_j a_i) \rightarrow (a_k a_i) a_j \rightarrow a_i (a_k a_j) \rightarrow a_i a_j a_k.$$

Therefore the rewriting system is confluent. If we consider length-lexicographic ordering ($a_1 < a_2 < \dots < a_r$), then it is clear that the rewriting rule is terminating and reduced and hence it is a uniquely terminating rewriting system.

Next we compute the second homology of S by using the Squier resolution. By applying the functor $\mathbb{Z} \otimes_{\mathbb{Z}S^1} -$ to that resolution, we obtain the chain complex of abelian groups

$$\bar{P}_3 \xrightarrow{\bar{\partial}_3} \bar{P}_2 \xrightarrow{\bar{\partial}_2} \bar{P}_1 \xrightarrow{\bar{\partial}_1} \mathbb{Z} \longrightarrow 0$$

where \bar{P}_1 , \bar{P}_2 and \bar{P}_3 are the free abelian groups on the sets of generators A , relations (rewriting rules) R and overlaps, respectively. The mappings

$$\bar{\partial}_2 : \bar{P}_2 \rightarrow \bar{P}_1 \text{ and } \bar{\partial}_3 : \bar{P}_3 \rightarrow \bar{P}_2$$

are defined respectively by

$$\bar{\partial}_2([r, s]) = \sum_{a \in A} ((\text{the number of } a\text{'s in } r) - (\text{the number of } a\text{'s in } s)) [a]$$

and

$$\bar{\partial}_3([(r_1 r_2, s_{1,2}), (r_2 r_3, s_{2,3})]) = [r_2 r_3, s_{2,3}] - [r_1 r_2, s_{1,2}] + \bar{\Phi}(r_1 s_{2,3}) - \bar{\Phi}(s_{1,2} r_3),$$

where $\bar{\Phi}$ is defined by $\bar{\Phi}(w) = \sum_{i=1}^n [r_i, s_i]$ if $\Phi(w) = \sum_{i=1}^q \phi(b_i) [r_i, s_i]$.

First we find a basis for $\ker \bar{\partial}_2$. Each $\alpha \in \bar{P}_2$ has the form

$$\alpha = \sum_{i=1}^r \alpha_i [a_i^{n_i+1}, a_i] + \sum_{1 \leq i < j \leq r} \alpha_{i,j} [a_j a_i, a_i a_j].$$

where all the coefficients are integers. Thus $\alpha \in \ker \bar{\partial}_2$ if and only if

$$0 = \bar{\partial}_2(\alpha) = \sum_{i=1}^r \alpha_i n_i [a_i].$$

It follows that $\alpha \in \ker \bar{\partial}_2$ if and only if $\alpha_i = 0$ ($1 \leq i \leq r$). Therefore the set

$$Z = \{ V_{i,j} = [a_j a_i, a_i a_j] \mid 1 \leq i < j \leq r \}$$

is a basis for $\ker \bar{\partial}_2$.

Next we find a basis for $\text{im } \bar{\partial}_3$. For this we first find the image of the overlaps above under $\bar{\partial}_3$.

$$\begin{aligned} \bar{\partial}_3(U_{0,i,k}) &= [a_i^{n_i+1}, a_i] - [a_i^{n_i+1}, a_i] + \bar{\Phi}(a_i^{n_i+1-k} a_i) - \bar{\Phi}(a_i a_i^{n_i+1-k} a_i) = 0 \\ \bar{\partial}_3(U_{1,i,j}) &= [a_j a_i, a_i a_j] - [a_j^{n_j+1}, a_j] + \bar{\Phi}(a_j^{n_j} a_i a_j) - \bar{\Phi}(a_j a_i) \\ &= -[a_j^{n_j+1}, a_j] + \bar{\Phi}(a_j^{n_j} a_i a_j) \\ &= -[a_j^{n_j+1}, a_j] + n_j [a_j a_i, a_i a_j] + [a_j^{n_j+1}, a_j] \\ &= n_j [a_j a_i, a_i a_j] = n_j V_{i,j} \\ \bar{\partial}_3(U_{2,i,j}) &= [a_i^{n_i+1}, a_i] - [a_j a_i, a_i a_j] + \bar{\Phi}(a_j a_i) - \bar{\Phi}(a_i a_j a_i^{n_i}) \\ &= [a_i^{n_i+1}, a_i] - \bar{\Phi}(a_i a_j a_i^{n_i}) \\ &= [a_i^{n_i+1}, a_i] - n_i [a_j a_i, a_i a_j] - [a_i^{n_i+1}, a_i] \\ &= -n_i [a_j a_i, a_i a_j] = -n_i V_{i,j} \\ \bar{\partial}_3(U_{3,i,j,k}) &= [a_j a_i, a_i a_j] - [a_k a_j, a_j a_k] + \bar{\Phi}(a_k a_i a_j) - \bar{\Phi}(a_j a_k a_i) \\ &= [a_j a_i, a_i a_j] - [a_k a_j, a_j a_k] + [a_k a_i, a_i a_k] \\ &\quad + [a_k a_j, a_j a_k] - [a_k a_i, a_i a_k] - [a_j a_i, a_i a_j] = 0. \end{aligned}$$

Since n_i divides n_j for $1 \leq i < j \leq r$, it follows that, for $1 \leq i < j \leq r$, the generator $n_j V_{i,j}$ is redundant. Hence

$$B = \{ n_i [a_j a_i, a_i a_j] \mid 1 \leq i < j \leq r \}$$

is a generating set (basis) for $\text{im } \bar{\partial}_3$. Therefore the second homology of S may be given by the following abelian group presentation:

$$\langle V_{i,j} \mid n_i V_{i,j} = 0 \ (1 \leq i < j \leq r) \rangle,$$

where $n_1 > 1$ and $n_i | n_{i+1}$ ($i = 1, \dots, r-1$), which defines the abelian group

$$C_{n_1}^{(r-1)} \times C_{n_2}^{(r-2)} \times \dots \times C_{n_{r-1}}.$$

where $n_1 > 1$ and $n_i | n_{i+1}$ ($i = 1, \dots, r-1$).

Since the deficiency of \mathcal{P} and the rank of $H_2(S)$ are the same number $r(r-1)/2$, it follows that S is efficient, as required. ■

We now give another example of an efficient non-simple semigroup. This time we give a non-commutative semigroup whose second homology is infinite.

Consider the semigroup T defined by the following semigroup presentation:

$$\langle a, b \mid a^2 = a, b^2 = b, (ab)^2 = ab \rangle.$$

Let w be a word in $\{a, b\}^+$. Then apply the first two relations so that we have a word $w' \in \{a, b\}^+$ such that $w = w'$ holds in T and, a^2 and b^2 are not subwords of w' . Moreover apply the last relation to obtain a word $w'' \in \{a, b\}^+$ such that $|w''| \leq 4$ and if $|w''| = 4$ then $w'' = baba$. Therefore we have

$$T = \{ a, b, ab, ba, aba, bab, baba \}$$

since the set contains one and only one representative for each element of T .

Next consider the subset $J = \{ ab, aba, bab, baba \}$ of T . It is clear that J is a proper (minimal) ideal of T . Therefore T is non-simple, and clearly is non-commutative.

Theorem 7.13 *Let T be the semigroup defined by the presentation*

$$\langle a, b \mid a^2 = a, b^2 = b, (ab)^2 = ab \rangle.$$

Then $H_2(S) = \mathbb{Z}$, and so T is efficient.

Proof It is clear that the rewriting system of the relations is terminating and reduced. The overlaps are:

$$U_1 = [(a^2, a), (a^2, a)], \quad U_2 = [(aa, a), (aa, a)], \quad U_3 = [(a^2, a), ((ab)^2, ab)],$$

$$U_4 = [(b^2, b), (b^2, b)], \quad U_5 = [(bb, b), (b, b)], \quad U_6 = [((ab)^2, ab), (b^2, b)].$$

Now we apply Lemma 7.1 for U_3 and U_6 . (The others are clear.) For U_3 , there exists ab such that

$$(a^2)bab \rightarrow abab \rightarrow ab \text{ and } a(ab)^2 \rightarrow a^2b \rightarrow ab.$$

For U_6 , there exists ab such that

$$(ab)^2b \rightarrow ab^2 \rightarrow ab \text{ and } aba(b^2) \rightarrow (ab)^2 \rightarrow ab.$$

We now use the Squier resolution. As before, $\alpha \in \bar{P}_2$ has the form

$$\alpha = \alpha_1[a^2, a] + \alpha_2[b^2, b] + \alpha_3[(ab)^2, ab]$$

where α_1 , α_2 , and α_3 are all integers. Thus $\alpha \in \ker \bar{\partial}_2$ if and only if

$$0 = \bar{\partial}_2(\alpha) = \alpha_1[a] + \alpha_2[b] + \alpha_3([a] + [b]).$$

Thus $\alpha \in \ker \bar{\partial}_2$ if and only if $\alpha_1 = \alpha_2 = -\alpha_3$, and so $\ker \bar{\partial}_2$ is the free abelian group on the unique symbol

$$V = [(ab)^2, ab] - [a^2, a] - [b^2, b].$$

Next we find the image of the above overlaps under $\bar{\partial}_3$. For this, we first find the image of the overlaps above under $\bar{\partial}_3$.

$$\begin{aligned} \bar{\partial}_3(U_1) &= [a^2, a] - [a^2, a] + \bar{\Phi}(a) - \bar{\Phi}(a) = 0 \\ \bar{\partial}_3(U_2) &= [a^2, a] - [a^2, a] + \bar{\Phi}(a^2) - \bar{\Phi}(a^2) = 0 \\ \bar{\partial}_3(U_3) &= [(ab)^2, ab] - [a^2, a] + \bar{\Phi}(a^2b) - \bar{\Phi}((ab)^2) = 0 \\ \bar{\partial}_3(U_4) &= [b^2, b] - [b^2, b] + \bar{\Phi}(b) - \bar{\Phi}(b) = 0 \\ \bar{\partial}_3(U_5) &= [b^2, b] - [b^2, b] + \bar{\Phi}(b^2) - \bar{\Phi}(b^2) = 0 \\ \bar{\partial}_3(U_6) &= [b^2, b] - [(ab)^2, ab] + \bar{\Phi}((ab)^2) - \bar{\Phi}(ab^2) = 0. \end{aligned}$$

Hence $\text{im } \bar{\partial}_3$ is the trivial group. It follows that $H_2(T) \cong \ker \bar{\partial}_2$, that is

$$H_2(T) = \mathbb{Z}.$$

Since the rank of $H_2(T)$ and the deficiency of the above presentation are both one, it follows that T is efficient, as required. ■

Chapter 8

Efficiency of Wreath Products of Finite Groups

The aim of this chapter is to investigate the (semi)group efficiency of wreath products of finite groups.

The results of this chapter have been submitted for publication by H. Ayık, C. M. Campbell, J. J. O'Connor and N. Ruškuc (see [4]).

8.1 Wreath products of finite groups

In this section, we investigate the group efficiency of wreath products of finite groups. The main result of this section is to extend a result of Jamali in [33] from C_2 to certain finite efficient groups.

Let G and H be finite groups. The (standard) *wreath product* of G and H , denoted by $G \wr H$, is a split extension of the direct product of $|H|$ copies of G by H . Let $I = I(H)$ denote the set of all involutions of H and let $J = J(H)$ be a minimal set of non-involutions of H such that $J \cup J^{-1}$ contains all non-involutions of $H \setminus \{1\}$. Observe that we have $H \setminus \{1\} = I \cup J \cup J^{-1}$ and $|J| = (|H| - |I| - 1)/2$.

Let $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$ be finite group presentations for the finite groups G

and H , respectively. Then it is a well-known fact that

$$\langle X, Y \mid R, S, [x_1, x_2^h] = 1 \ (x_1, x_2 \in X, h \in I(H) \cup J(H)) \rangle, \quad (1)$$

is a group presentation for the wreath product $G \wr H$ (see, for example, [34] or [35]).

For a group G , we denote the Schur multiplier of G by $M(G)$, and the tensor product of G by itself by $G \otimes G$. Then the Schur multiplier of the wreath product of G and H , by [40, Theorem 6.3.3] (see also [10]), is

$$M(G \wr H) = M(G) \times M(H) \times (G \otimes G)^{|J|} \times (G \# G)^{|I|}, \quad (2)$$

where $J = J(H)$, $I = I(H)$, and if

$$G/G' = \prod_{i=1}^t C_{m_i}$$

with C_{m_i} the cyclic group of order $m_i > 1$ and $m_i \mid m_{i+1}$ for $i = 1, \dots, t-1$, then

$$G \# G = \left(\prod_{i=1}^{t-1} C_{m_i}^{t-i} \right) \times C_2^s \quad (3)$$

where s is the number of even m_i ($1 \leq i \leq t$) (see [40, Lemma 6.3.4]). Moreover, we have

$$G \otimes G = \prod_{i=1}^t C_{m_i}^{(2t-2i+1)}. \quad (4)$$

Note that the rank of $G \otimes G$ is given by

$$\text{rank}(G \otimes G) = \sum_{i=1}^t (2t - 2i + 1) = t^2$$

and the rank of $G \# G$ is given by

$$\text{rank}(G \# G) = \left(\sum_{i=1}^{t-1} (t - i) \right) + s = \frac{1}{2}(t^2 - t) + s.$$

The numbers m_1, \dots, m_t where $G/G' = \prod_{i=1}^t C_{m_i}$ with C_{m_i} the cyclic group of order $m_i > 1$ and $m_i \mid m_{i+1}$ for $i = 1, \dots, t-1$ are called the *invariant factors* of the abelian group G/G' .

With the above notation, we have the following result:

Theorem 8.1 *Let G and H be finite efficient groups and let G have an efficient presentation on $\text{rank}(G/G') \geq 1$ generators. If each abelian group G/G' , $M(G)$ and $M(H)$ is trivial or has even invariant factors, then $G \wr H$ is an efficient group.*

Proof Let $J = J(H)$, $I = I(H)$ and $G/G' = \prod_{i=1}^t C_{m_i}$ of rank $t \geq 1$. We suppose that all m_i ($1 \leq i \leq t$) are even and $m_i | m_{i+1}$ for $i = 1, \dots, t-1$. From the hypotheses of the theorem, if $\text{rank}(M(G)) = r_1$ and $\text{rank}(M(H)) = r_2$ (r_1 or r_2 may be zero), then $C_2^{r_1}$ and $C_2^{r_2}$ are subgroups of $M(G)$ and $M(H)$, respectively. Since all the invariant factors of G/G' are even, it follows from (3) and (4) that $C_2^{t^2|J|}$ is a subgroup of $(G \otimes G)^{|J|}$ and that $C_2^{(\frac{t^2+t}{2})|I|}$ is a subgroup of $(G \# G)^{|I|}$. Therefore, it follows from (2) that C_2^r , where $r = r_1 + r_2 + t^2|J| + (\frac{t^2+t}{2})|I|$ is a subgroup of $M(G \wr H)$ so that

$$\text{rank}(M(G \wr H)) \geq r_1 + r_2 + t^2|J| + (\frac{t^2+t}{2})|I|.$$

From (2),

$$\begin{aligned} \text{rank}(M(G \wr H)) &\leq \text{rank}(M(G)) + \text{rank}(M(H)) + |J|(\text{rank}(G \otimes G)) \\ &\quad + |I|(\text{rank}(G \# G)) = r_1 + r_2 + |J|t^2 + |I|(t^2 + t)/2, \end{aligned}$$

and so we have equality.

Suppose that G has an efficient presentation $\langle X \mid R \rangle$ where $X = \{x_1, \dots, x_t\}$, and H has an efficient presentation $\langle Y \mid S \rangle$. It follows from (1) that $G \wr H$ has the following group presentation:

$$\langle X, Y \mid R, S, [x_i, x_j^h] = 1 \ (1 \leq i, j \leq t, h \in I \cup J) \rangle.$$

If I is empty, then it is clear that the above presentation is an efficient presentation for $G \wr H$.

Now assume that I is not empty. Then we show that the relations of the form $[x_i, x_j^h] = 1$ with $i > j$ and $h \in I$ are redundant. Indeed, from the relation

$[x_j, x_i^h] = 1$, first we have $[x_j^{-1}, x_i^h] = 1$. Since $h^{-1} = h$, it follows that

$$\begin{aligned} [x_i, x_j^h] &= x_i^{-1}(hx_jh)^{-1}x_i(hx_jh) = x_i^{-1}hx_j^{-1}hx_ihx_jh \\ &= (hx_j^{-1}x_jh)x_i^{-1}hx_j^{-1}hx_ihx_jh \equiv hx_j^{-1}(x_jhx_i^{-1}hx_j^{-1}hx_ih)x_jh \\ &\equiv hx_j^{-1}[x_j^{-1}, x_i^h]x_jh = hx_j^{-1}x_jh = 1, \end{aligned}$$

as required. Therefore, by eliminating these redundant relations, we have the following presentation

$$\begin{aligned} \langle X, Y \mid R, S, [x_i, x_j^h] = 1 \quad (1 \leq i, j \leq t, h \in J), \\ [x_i, x_j^h] = 1 \quad (1 \leq i \leq j \leq t, h \in I) \rangle \end{aligned} \quad (5)$$

for $G \wr H$. Since the deficiency of the above presentation and the rank of $G \wr H$ are the same number $r_1 + r_2 + |J|t^2 + |I|(t^2 + t)/2$, the proof is now complete. ■

Since the abelian groups $C_{n_1} \times \cdots \times C_{n_t}$ with n_i even ($i = 1, \dots, t$), the dihedral groups D_{2n} of order $2n$ with n even and the generalised quaternion groups Q_n of order $4n$ ($\cong \langle a, b \mid aba = b, ba^{n-1}b = a \rangle$) with n even satisfy the conditions in the previous theorem, we have the following corollary:

Corollary 8.2 *Let H be an efficient group and let n, n_1, \dots, n_t be positive even integers such that n_i divides n_{i+1} ($i = 1, \dots, t-1$). If the Schur multiplier $M(H)$ of H is trivial or if all the invariant factors of $M(H)$ are even, then*

- (i) $(C_{n_1} \times \cdots \times C_{n_t}) \wr H$ is efficient,
- (ii) $D_{2n} \wr H$ is efficient and
- (iii) $Q_n \wr H$ is efficient.

Proof (i) Let $G = C_{n_1} \times \cdots \times C_{n_t}$ with n_1 even and let n_i divide n_{i+1} ($i = 1, \dots, t-1$). It is a well-known fact that

$$H_2(G) = C_{n_1}^{t-1} \times \cdots \times C_{n_{t-2}}^2 \times C_{n_{t-1}}$$

(see [40, Corollary 2.2.12] or [60]). Therefore the following presentation

$$\mathcal{P}_G = \langle x_1, \dots, x_t \mid x_j^{n_j} = 1 \quad (1 \leq j \leq t) \quad x_j x_k = x_k x_j \quad (1 \leq j < k \leq t) \rangle$$

is an efficient group presentation on $\text{rank}(G/G') = t$ generators for G (since $G/G' \cong G$). Now the efficiency of $G \wr H$ follows from Theorem 8.1.

(ii) Let n be even. Since $H_2(D_{2n}) = C_2$ (see [40, Proposition 2.11.4]),

$$D_{2n} \cong \langle a, b \mid a^2 = 1, b^n = 1, ab^{n-1}a = b \rangle$$

and since

$$\begin{aligned} D_{2n}/D'_{2n} &\cong \langle a, b \mid a^2 = 1, b^n = 1, a^2b^{n-2} = 1, ba = ab \rangle \\ &\cong \langle a, b \mid a^2 = 1, b^n = 1, b^{n-2} = 1, ba = ab \rangle \\ &\cong \langle a, b \mid a^2 = 1, b^2 = 1, ba = ab \rangle \\ &\cong C_2 \times C_2, \end{aligned}$$

the efficiency of $D_{2n} \wr H$ follows from Theorem 8.1.

(iii) Let n be even. Since $H_2(Q_n)$ is trivial (see [40, Example 2.4.8]),

$$Q_n \cong \langle a, b \mid aba = b, ba^{n-1}b = a \rangle$$

and since

$$\begin{aligned} Q_n/Q'_n &\cong \langle a, b \mid a^2 = 1, b^2a^{n-2} = 1, ba = ab \rangle \\ &\cong \langle a, b \mid a^2 = 1, b^2 = 1, ba = ab \rangle \\ &\cong C_2 \times C_2, \end{aligned}$$

the efficiency of $Q_n \wr H$ follows from Theorem 8.1. ■

Since, for any integer m and prime p , the Schur multipliers of the cyclic group C_m , the dihedral group D_{2m} , the generalized quaternion group Q_m , the alternating group A_m , the special linear group $SL(2, p)$ and the projective special linear group $PSL(2, p)$ are either trivial or a cyclic group of even order (see [40]), it follows from Corollary 8.2 that, for n even, the groups:

$$C_n \wr C_m, C_n \wr D_{2m}, C_n \wr Q_m, C_n \wr SL(2, p), C_n \wr PSL(2, p), D_{2n} \wr C_m, \text{ etc.}$$

are all efficient.

Next we prove that a wreath product of any two cyclic groups is efficient. This will give an example of an efficient wreath product $G \wr H$ such that not all the invariants of G/G' are necessarily even.

Proposition 8.3 *For any n and m , $C_n \wr C_m$ is an efficient group.*

Proof From the previous corollary, it is enough to prove the result for n odd. First note that $C_n \otimes C_n = C_n$ and $C_n \# C_n$ is the trivial group when n is odd. It follows from (2)–(4) that

$$M(C_n \wr C_m) = \begin{cases} (C_n \otimes C_n)^{|J|} \times (C_n \# C_n)^{|I|} = C_n^{(m-1)/2} & \text{if } m \text{ is odd,} \\ (C_n \otimes C_n)^{|J|} \times (C_n \# C_n)^{|I|} = C_n^{(m-2)/2} & \text{if } m \text{ is even} \end{cases}$$

where $I = I(C_m)$ and $J = J(C_m)$. Therefore, if m is odd, it follows from (1) that the presentation

$$\langle a, b \mid a^n = 1, b^m = 1, [a, b^{-i}ab^i] = 1 \ (1 \leq i \leq (m-1)/2) \rangle$$

is an efficient presentation for $C_n \wr C_m$.

Now we assume that m is even. We prove that the presentation

$$\mathcal{P} = \langle a, b \mid a^n = b^m, (ab^{m/2})^2 = b^{m/2}ab^{-m/2}a, [a, b^{-i}ab^i] = 1 \ (1 \leq i \leq (m-2)/2) \rangle$$

defines $C_n \wr C_m$. For this, it is enough to prove that $b^m = 1$.

Indeed, from the second relation of \mathcal{P} , we have

$$ab^{m/2}a^{-1} = b^{-m/2}a^{-1}b^{m/2}ab^{-m/2}.$$

Since b^m is a central element, by squaring both sides of the above equation, we have

$$ab^ma^{-1} = b^{-m/2}a^{-1}b^{m/2}ab^{-m}a^{-1}b^{m/2}ab^{-m/2} = b^{-m},$$

and so we obtain $b^{2m} = 1$.

Again from the second relation, we have

$$ab^{m/2}ab^{m/2}a^{-1} = b^{m/2}ab^{-m/2}.$$

This time, taking the n th power, we have

$$a(b^{m/2}ab^{m/2})^na^{-1} = b^{m/2}a^nb^{-m/2} = a^n$$

since a^n is central. It follows from the first relation that $(b^{m/2}ab^{m/2})^n = b^m$, and so $(ab^m)^n = b^m$. Since $a^n = b^m$, $b^{2m} = 1$ and n is odd, it follows that

$$b^m = (ab^m)^n = a^n b^{nm} = b^{(n+1)m} = 1,$$

as required. ■

Let G and H be any two finite groups, and let $W = G \wr H$. Then it follows from (1) that

$$W/W' = (G/G') \times (H/H'). \quad (6)$$

We denote the direct power of k copies of G by G^k , that is $G^k = G \times \cdots \times G$. However, since wreath products of groups are not associative, we introduce the convention for the wreath product of k copies of G as below:

$$G^{\text{wr}(k)} = ((\cdots (((G \wr G) \wr G) \wr G) \wr \cdots \wr G) \wr G) \wr G.$$

With the above notation, we have the following results:

Lemma 8.4 *Let G be a finite group and let k be a positive integer. Then*

- (i) $(G^{\text{wr}(k)})/(G^{\text{wr}(k)})' = (G/G')^k$;
- (ii) $G^{\text{wr}(k)} \sharp G^{\text{wr}(k)} = (G \sharp G)^k$;
- (iii) $G^{\text{wr}(k)} \otimes G^{\text{wr}(k)} = (G \otimes G)^k$; and
- (iv) $M(G^{\text{wr}(k)}) = M(G)^k \times (G \sharp G)^{(k(k-1)/2)|J|} \times (G \otimes G)^{(k(k-1)/2)|I|}$

where $I = I(G)$ and $J = J(G)$.

Proof (i) Inductively, the result follows from (6).

(ii) The result follows from (3) and (i).

(iii) The result follows from (4) and (i).

(iv) For $k = 2$, the result follows from (2). Now we use the inductive hypothesis to prove it for any positive integer $k + 1$.

First assume that the equation holds for k . Then, for $k + 1$, it follows from (2), (ii) and (iii) that

$$\begin{aligned}
 M(G^{\text{wr}(k+1)}) &= M(G^{\text{wr}(k)} \wr G) \\
 &= M(G^{\text{wr}(k)}) \times M(G) \times (G \otimes G)^{k|J|} \times (G \# G)^{k|I|} \\
 &= M(G)^k \times (G \# G)^{(k(k-1)/2)|J|} \times (G \otimes G)^{(k(k-1)/2)|I|} \times M(G) \\
 &\quad \times (G \otimes G)^{k|J|} \times (G \# G)^{k|I|} \\
 &= M(G)^{k+1} \times (G \# G)^{(k(k+1)/2)|J|} \times (G \otimes G)^{(k(k+1)/2)|I|}
 \end{aligned}$$

where $I = I(G)$ and $J = J(G)$, as required. ■

Next we give two families of efficient groups which are wreath powers of the groups C_n and D_{2n} .

Theorem 8.5 *Let n be an even integer. Then, for any integer k , the groups $C_n^{\text{wr}(k)}$ and $D_{2n}^{\text{wr}(k)}$ are efficient.*

Proof It is well-known that the groups C_n and D_{2n} are efficient (even as semigroups (see Theorems 5.5 and 5.6)). When $k = 2$, the result is a consequence of Corollary 8.2. Now we use the inductive hypothesis for $k \geq 3$. Since the invariant factors of $C_n/C'_n = C_n$ and $D_{2n}/D'_{2n} = C_2 \times C_2$ are even, and since $M(C_n)$ is trivial and $M(D_{2n}) = C_2$, it follows from Lemma 8.4 that all the invariant factors of the following abelian groups

$$(C_n^{\text{wr}(k)})/(C_n^{\text{wr}(k)})', \quad (D_{2n}^{\text{wr}(k)})/(D_{2n}^{\text{wr}(k)})', \quad M(C_n^{\text{wr}(k)}) \quad \text{and} \quad M(D_{2n}^{\text{wr}(k)})$$

are even. From Lemma 8.4(i), we deduce that the rank of $(C_n^{\text{wr}(k)})/(C_n^{\text{wr}(k)})'$ is k and the rank of $(D_{2n}^{\text{wr}(k)})/(D_{2n}^{\text{wr}(k)})'$ is $2k$. It follows from (1) that $C_n^{\text{wr}(k)}$ has an

efficient presentation on k generators and $D_{2n}^{\text{wr}(k)}$ has an efficient presentation on $2k$ generators. Since $M(C_n)$ is trivial and $M(D_{2n}) = C_2$, the result follows from Theorem 8.1. ■

8.2 The semigroup efficiency of wreath products of groups

Now we investigate the semigroup efficiency of wreath products of finite groups. From (5) and a similar argument to Lemma 5.9, we have the following immediate result for wreath products of groups:

Lemma 8.6 *Let $\langle X \mid R \rangle$ and $\langle Y \mid S \rangle$ be semigroup presentations for finite groups G and H , and let $e \in X^+$ and $f \in Y^+$ be any representatives of the identity elements of G and H . Then the semigroup presentation*

$$\begin{aligned} \langle X, Y \mid R, S, e = f, \quad & x_i h' x_j h = h' x_j h x_i \quad (1 \leq i, j \leq t, h \in J), \\ & x_i h x_j h = h x_j h x_i \quad (1 \leq i \leq j \leq t, h \in I) \rangle, \end{aligned}$$

where $I = I(H)$, $J = J(H)$ as in the previous section and $h' \in Y^+$ represents an inverse of $h \in I \cup J$, defines $G \wr H$ as a semigroup.

We first give a semigroup presentation for the wreath product of two cyclic groups.

Lemma 8.7 *The semigroup presentation*

$$\begin{aligned} \mathcal{P}_{m,n} = \langle a, b \mid & a^{n+1} = a, a^n = b^m, ba^{n-1}b^{m-1}a^{n-1}bab^{m-1}ab = b, \\ & a^{n-1}b^{m-i}a^{n-1}b^i ab^{m-i}ab^i = a^n \quad (2 \leq i \leq [m/2]) \rangle, \end{aligned}$$

where $[m/2]$ denotes the integer part of $m/2$, defines $C_n \wr C_m$ as a semigroup.

Proof First we prove that $b^{m+1} = b$. Indeed, from the first three relations of $\mathcal{P}_{m,n}$, we have

$$\begin{aligned} b^{m+1} &= a^n b = a^n (ba^{n-1} b^{m-1} a^{n-1} b a b^{m-1} a b) = ba^{2n-1} b^{m-1} a^{n-1} b a b^{m-1} a b \\ &= ba^{n-1} b^{m-1} a^{n-1} b a b^{m-1} a b = b. \end{aligned}$$

It follows that $a^n b = ba^n = b^{m+1} = b$ so that a^n is an identity of the semigroup S defined by $\mathcal{P}_{m,n}$. Since both a and b have inverses, $\mathcal{P}_{m,n}$ defines a group. It follows from Lemma 8.6 that $\mathcal{P}_{m,n}$, in fact, defines $C_n \wr C_m$ as a semigroup. ■

Let n and m be even positive integers. Then, from (2)–(4), we have

$$M(C_n \wr C_m) = (C_n \otimes C_n)^{|J(C_m)|} \times (C_n \# C_n)^{|I(C_m)|} = C_n^{|J(C_m)|} \times C_2 = C_n^{(m-2)/2} \times C_2.$$

Similarly, from (2)–(4), we have

$$M(C_n \wr C_m) = \begin{cases} C_n^{(m-1)/2} & \text{if } m \text{ is odd} \\ C_n^{(m-2)/2} & \text{if } m \text{ is even and } n \text{ is odd} \\ C_n^{(m-2)/2} \times C_2 & \text{if both } m \text{ and } n \text{ are even,} \end{cases}$$

and so we have

$$\text{rank}(M(C_n \wr C_m)) = \begin{cases} (m-2)/2 & \text{if } m \text{ is even and } n \text{ is odd} \\ \lfloor m/2 \rfloor & \text{otherwise.} \end{cases}$$

Therefore, since the deficiency of $\mathcal{P}_{m,n}$ is $\lfloor m/2 \rfloor$, we have the following result.

Theorem 8.8 *If m is odd or if both m and n are even, then $C_n \wr C_m$ is efficient as a semigroup.* ■

It is an open problem whether a finite group which is efficient as a group is necessarily also efficient as a semigroup. We do not know an efficient semigroup presentation for $C_n \wr C_m$ with m even and n odd, and if none existed it would give a counterexample for the above problem.

Next we investigate the semigroup efficiency of the wreath product of two dihedral groups. Since

$$D_{2n}/D'_{2n} = \begin{cases} C_2 & \text{if } n \text{ is odd,} \\ C_2^2 & \text{if } n \text{ is even,} \end{cases}$$

it follows from (3) and (4), respectively that

$$D_{2n} \sharp D_{2n} = \begin{cases} C_2 & \text{if } n \text{ is odd,} \\ C_2^3 & \text{if } n \text{ is even} \end{cases}$$

and

$$D_{2n} \otimes D_{2n} = \begin{cases} C_2 & \text{if } n \text{ is odd,} \\ C_2^4 & \text{if } n \text{ is even.} \end{cases}$$

Suppose that $\langle c, d \mid c^2 = 1, d^m = 1, cd^{m-1}c = d \rangle$, is a group presentation for D_{2m} . Then $I(D_{2m}) = \{c, cd, \dots, cd^{m-1}\}$ for odd m and $I(D_{2m}) = \{c, cd, \dots, cd^{m-1}, d^{m/2}\}$ for even m . Since

$$M(D_{2n}) = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ C_2 & \text{if } n \text{ is even,} \end{cases}$$

it follows from (2) that, if m and n are odd, then

$$M(D_{2n} \wr D_{2m}) = C_2^{(m-1)/2} \times C_2^m = C_2^{(3m-1)/2}.$$

If m is even and if n is odd, then

$$M(D_{2n} \wr D_{2m}) = C_2 \times C_2^{(m-2)/2} \times C_2^{m+1} = C_2^{(3m+2)/2}.$$

If m is odd and if n is even, then

$$M(D_{2n} \wr D_{2m}) = C_2 \times C_2^{4(m-1)/2} \times C_2^{3m} = C_2^{5m-1}.$$

Finally, if m and n are even, then

$$M(D_{2n} \wr D_{2m}) = C_2 \times C_2 \times C_2^{4(m-2)/2} \times C_2^{3(m+1)} = C_2^{5m+1}.$$

Therefore we have

$$\text{rank}(M(D_{2n} \wr D_{2m})) = \begin{cases} (3m-1)/2 & \text{if both } m \text{ and } n \text{ are odd,} \\ (3m+2)/2 & \text{if } m \text{ is even and } n \text{ is odd,} \\ 5m-1 & \text{if } m \text{ is odd and } n \text{ is even,} \\ 5m+1 & \text{if both } m \text{ and } n \text{ are even.} \end{cases} \quad (7)$$

Next we use the semigroup presentation for D_{2n} given in Theorem 5.6 and Lemma 8.6 to obtain a semigroup presentation for $D_{2n} \wr D_{2m}$. If m is even, then the semigroup presentation

$$\begin{aligned} \mathcal{B}_{m,n} = \langle & a, b, c, d \mid a^3 = a, b^n = a^2, ab^{n-1}a = b, \\ & c^3 = c, d^m = c^2, cd^{m-1}c = d, a^2 = c^2, \\ & ad^{m-i}ad^i = d^{m-i}ad^ia, ad^{m-i}bd^i = d^{m-i}bd^ia, \quad (1 \leq i \leq (m-2)/2) \\ & bd^{m-i}ad^i = d^{m-i}ad^ib, bd^{m-i}bd^i = d^{m-i}bd^ib, \quad (1 \leq i \leq (m-2)/2) \\ & (ac)^2 = (ca)^2, (acd^j)^2 = (cd^ja)^2, \quad (1 \leq j \leq m-1) \\ & acd^j bcd^j = cd^j bcd^j a, (bcd^j)^2 = (cd^jb)^2, \quad (0 \leq j \leq m-1) \\ & (ad^{m/2})^4 = a^2, ad^{m/2}bd^{m/2} = d^{m/2}bd^{m/2}a, bd^{m/2}bd^{m/2} = d^{m/2}bd^{m/2}b \rangle \end{aligned}$$

defines $D_{2n} \wr D_{2m}$. If m is odd, then the semigroup presentation

$$\begin{aligned} \mathcal{C}_{m,n} = \langle & a, b, c, d \mid a^3 = a, b^n = a^2, ab^{n-1}a = b, \\ & c^3 = c, d^m = c^2, cd^{m-1}c = d, a^2 = c^2, \\ & ad^{m-i}ad^i = d^{m-i}ad^ia, ad^{m-i}bd^i = d^{m-i}bd^ia, \quad (1 \leq i \leq (m-1)/2) \\ & bd^{m-i}ad^i = d^{m-i}ad^ib, bd^{m-i}bd^i = d^{m-i}bd^ib, \quad (1 \leq i \leq (m-1)/2) \\ & (ac)^2 = (ca)^2, (acd^j)^2 = (cd^ja)^2, \quad (1 \leq j \leq m-1) \\ & acd^j bcd^j = cd^j bcd^j a, (bcd^j)^2 = (cd^jb)^2, \quad (0 \leq j \leq m-1) \rangle \end{aligned}$$

defines $D_{2n} \wr D_{2m}$.

Lemma 8.9 *The relation $(ac)^4c = c$ holds in $D_{2n} \wr D_{2m}$. Moreover, if we add this relation to the presentations $\mathcal{B}_{m,n}$ and $\mathcal{C}_{m,n}$, then the relations $c^3 = c$ and $(ac)^2 = (ca)^2$ become redundant.*

Proof First we show that the relation $(ac)^4c = c$ is a consequence of the relations: $c^3 = c$, $a^3 = a$, $a^2 = c^2$ and $(ac)^2 = (ca)^2$. Indeed, observe that

$$(ac)^4 = (ac)^2(ca)^2 \equiv acac^2aca = aca^4ca = a^8 = a^2,$$

and so

$$(ac)^4c = a^2c = c^3 = c.$$

Now we show that the relations $c^3 = c$ and $(ac)^2 = (ca)^2$ are consequences of the relations: $a^3 = a$, $a^2 = c^2$ and $(ac)^4c = c$. Indeed, observe that

$$c^3 = a^2c = a^2(ac)^4c \equiv a^3c(ac)^3c = (ac)^4c = c.$$

Since $a(caca^2cac) = a^3(cac^2ac) = a^9 = a$, we have

$$\begin{aligned} (ca)^2 &= (ca)^2(caca^2cac) \equiv (ca)^4(ac)^2 = (ca)^4a^3cac = (ca)^4c^2(ac)^2 \\ &\equiv c((ac)^4c)(ac)^2 = c^2(ac)^2 = a^3cac = (ac)^2, \end{aligned}$$

as required. ■

Let $\mathcal{E}_{m,n}$ denote the semigroup presentation which is obtained from $\mathcal{B}_{m,n}$ by adding the relation $(ac)^4c = c$ and then removing the relations $c^3 = c$ and $(ac)^2 = (ca)^2$.

Theorem 8.10 *If both m and n are even, $D_{2n} \wr D_{2m}$ is efficient as a semigroup.*

Proof Since

$$\text{def}(\mathcal{E}_{m,n}) = (6 + 4((m-2)/2) + 3m + 3) - 4 = 5m + 1 = \text{rank}(M(D_{2n} \wr D_{2m})),$$

it follows that $D_{2n} \wr D_{2m}$ is efficient as a semigroup. ■

Next consider the following semigroup presentation

$$\mathcal{G}_m = \langle c, d \mid cdcdc = c, cd^{m-1}c^{m-2} = d \rangle$$

which defines a group G_m (see Lemma 5.7). Moreover, it is known that G_m is isomorphic to D_{2m} for $m = 3, 5, 7, 9$.

Theorem 8.11 *Let n be any even integer. Then the groups $D_{2n} \wr D_{2m}$ for $m = 3, 5, 7, 9$ are efficient as semigroups.*

Proof Let n be a positive even integer and let m be one of the odd numbers: 3, 5, 7 and 9. Then consider the semigroup presentation $\mathcal{C}'_{m,n}$ which is obtained from $\mathcal{C}_{m,n}$ by replacing the relations $c^3 = c$, $a^2 = c^2$, $d^m = c^2$, $(ac)^2 = (ca)^2$ and $cd^{m-1}c = d$ by the relations $cdcdc = c$, $cd^{m-1}c^{m-2} = d$, $(ac)^4c = c$ and $a^2 = (cd)^2$. Since $(cd)^2$ is a representative of the identity the group defined by the semigroup presentation \mathcal{G}_m , it follows from Lemmas 8.6 and 8.9 that $\mathcal{C}'_{m,n}$ defines $D_{2n} \wr D_{2m}$ as a semigroup for $m = 3, 5, 7, 9$. Now we prove that the relation $cdcdc = c$ in $\mathcal{C}'_{m,n}$ is redundant, and so $D_{2n} \wr D_{2m}$ is efficient as a semigroup. Indeed, we have

$$cdcdc = a^2c = a^2((ac)^4c) = a^3c(ac)^3c = (ac)^4c = c,$$

as required. ■

8.3 Some inefficient groups

Notice that all inefficient groups are also inefficient as semigroups. Examples of inefficient semigroups which are semigroups but not groups are given in Chapter 6. Examples of inefficient groups were first given by Swan in [64], and more examples can be found in [70], [41], [50] and [33]. In this section we give further examples of inefficient groups.

Theorem 8.12 *Let G and H be finite groups and let the Schur multiplier of H be trivial. Suppose that the first invariant factor of G/G' is odd and not coprime to the first invariant factor of the Schur multiplier of G (provided this is non-trivial). Denote the number of involutions in H by m and the rank of G/G' by t . If $mt \geq 2$, then $G \wr H$ is an inefficient group.*

Proof Let s be the number of even $n_i > 1$ ($1 \leq i \leq t$) where $G/G' = \prod_{i=1}^t C_{n_i}$ with $n_i | n_{i+1}$, $i = 1, \dots, t-1$. Since n_1 is odd and $t-1 \geq s$, it follows from (3) that

$$G \sharp G = \left(\prod_{i=1}^{t-1} C_{n_i}^{t-i} \right) \times C_2^s = C_{2n_1}^s \times C_{n_1}^{t-s-1} \times \prod_{i=2}^{t-1} C_{n_i}^{t-i}.$$

Hence, from (2) and (3),

$$M(G \wr H) = M(G) \times \left(\prod_{i=1}^t C_{n_i}^{2t-2i+1} \right)^{(n-m-1)/2} \times (C_{2n_1}^s \times C_{n_1}^{t-s-1} \times \prod_{i=2}^{t-1} C_{n_i}^{t-i})^m$$

where $|H| = n$ and m is the number of involutions of H . From the fact that, for any abelian groups A and B ,

$$\text{rank}(A \times B) \leq \text{rank}(A) + \text{rank}(B),$$

arguing as in the previous section, we deduce

$$\begin{aligned} \text{rank}(M(G \wr H)) &= r + t^2(n-m-1)/2 + (t^2-t)m/2 \\ &= r + t^2(n-1)/2 - mt/2 \leq r + t^2((n-1)/2) - 1, \end{aligned} \quad (8)$$

where $\text{rank}(M(G)) = r$, since $mt \geq 2$.

Next consider the direct product of n copies of G , G^n , which is a subgroup of $G \wr H$. By using the Schur-Künneth formula inductively, we have

$$M(G^n) = M(G)^n \times (G \otimes G)^{n(n-1)/2}$$

(for more details see [22]), and so, from the hypotheses of the theorem, we have

$$\text{rank}(M(G^n)) = nr + t^2(n(n-1)/2).$$

Since the index of G^n in $G \wr H$ is n , from Lemma 1.38, we have that

$$\text{def}(G \wr H) \geq \frac{nr + t^2(n(n-1)/2) + 1}{n} - 1 > r + t^2(n-1)/2 - 1.$$

It follows from (8) that $\text{def}(G \wr H) > \text{rank}(M(G \wr H))$, and so $G \wr H$ is inefficient, as required. ■

Let $A = C_{n_1} \times \cdots \times C_{n_t}$ be an abelian group of rank $t \geq 2$, n_1 odd and $n_i | n_{i+1}$ ($i = 1, \dots, t-1$). Since the groups C_{2n} , D_{2n} and $SL(2, p)$ with n odd and p prime have trivial Schur multipliers and contain at least one involution, it follows from the previous theorem that the groups:

$$A \wr C_{2n}, \quad A \wr D_{2n} \quad \text{and} \quad A \wr SL(2, p)$$

are inefficient groups. Moreover, for odd m and n , the group $C_m \wr D_{2n}$ is inefficient if $n \geq 3$.

Let

$$G_1 = \langle a, b \mid a^3 = 1, ba = ab^2 \rangle,$$

$$G_2 = \langle a, b \mid a^3 = b^3 = (ab)^2 \rangle,$$

$$G_3 = \langle a, b \mid a^3 = 1, bab^2 = a \rangle$$

be groups of order 21, 24 and 27, respectively. Since they have trivial Schur multipliers (see [40]) and since

$$(G_1/G'_1) = (G_2/G'_2) = C_3 \quad \text{and} \quad (G_3/G'_3) = C_3 \times C_3,$$

it follows that $G_i \wr D_{2n}$ ($i = 1, 2, 3$) are inefficient for odd $n \geq 3$.

Let $G = C_q \times SL(2, p)$ with q odd and p prime. Since

$$C_q \otimes SL(2, p) = (C_q/C'_q) \otimes (SL(2, p)/SL(2, p)') = C_q \otimes \{1\} = \{1\},$$

it follows from the Schur-Künneth formula that

$$M(G) = M(C_q \times SL(2, p)) = M(C_q) \times M(SL(2, p)) \times (C_q \otimes SL(2, p)) = \{1\}.$$

Since $G/G' = C_q$, it follows from the previous theorem that for any group H with trivial Schur multiplier and at least 2 involutions, $(C_q \times SL(2, p)) \wr H$ is inefficient. In particular, for odd $n \geq 3$, $(C_q \times SL(2, p)) \wr D_{2n}$ is inefficient.

We can generalise this last example.

Corollary 8.13 *Let G and H be finite groups, let the Schur multiplier of H be trivial, and let q be any odd integer. If G is a perfect group, then the group $(C_q \times G) \wr H$ is inefficient.*

Proof The proof is as in the last example. ■

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