On Some Properties of Vector Space based Graphs

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Abstract

In this paper, we study some problems related to subspace inclusion graph $\mathcal{I}n(\mathbb{V})$ and subspace sum graph $\mathcal{G}(\mathbb{V})$ of a finite-dimensional vector space $\mathbb{V}$. Namely, we prove that $\mathcal{I}n(\mathbb{V})$ is a Cayley graph as well as Hamiltonian when the dimension of $\mathbb{V}$ is 3. We also find the exact value of independence number of $\mathcal{G}(\mathbb{V})$ when the dimension of $\mathbb{V}$ is odd. The above two problems were left open in previous works in literature. Moreover, we prove that the determining numbers of $\mathcal{I}n(\mathbb{V})$ and $\mathcal{G}(\mathbb{V})$ are bounded above by 6. Finally, we study some forbidden subgraphs of these two graphs.

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1. Introduction

Graphs defined on vector spaces have been studied extensively in the last few years [2, 3, 4, 5, 6, 7, 12], and as a result various types of graphs on finite-dimensional vector spaces and numerous problems related to them have surfaced recently. As these graphs inherit the rich structure possessed by the finite-dimensional vector spaces, their combinatorial properties are also worth studying. In this paper, we focus on two such graphs, namely *subspace inclusion graph* [5] and *subspace sum graph* [7], and study some problems related to these two graphs.

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Definition 1.1. *Let $\mathbb{V}$ be a finite-dimensional vector space over a field F of dimension greater than 1 and V be the collection of non-trivial proper subspaces of $\mathbb{V}$. The subspace inclusion graph $\mathcal{I}n(\mathbb{V})$ is a graph on V as the set of vertices and two distinct vertices W_1 and W_2 are adjacent if $W_1 \subset W_2$ or $W_2 \subset W_1$. The subspace sum graph $\mathcal{G}(\mathbb{V})$ is a graph on V as the set of vertices and two distinct vertices W_1 and W_2 are adjacent if $W_1 + W_2 = \mathbb{V}$.*

The subspace inclusion graph $\mathcal{I}n(\mathbb{V})$ was introduced in [5]. The automorphism group and independence number of $\mathcal{I}n(\mathbb{V})$ were determined in [18] and [13], respectively. In [6], it was observed that $\mathcal{I}n(\mathbb{V})$ possesses special properties when $\dim(\mathbb{V}) = 3$. In particular, the authors posed four conjectures on $\mathcal{I}n(\mathbb{V})$ when $\dim(\mathbb{V}) = 3$. Two of those conjectures, namely $\mathcal{I}n(\mathbb{V})$ is distance transitive and $\gamma(\mathcal{I}n(\mathbb{V})) = 2q$ where $|F| = q$ and $\dim(\mathbb{V}) = 3$, has been proved in [21]. The other two conjectures, namely $\mathcal{I}n(\mathbb{V})$ is a Cayley graph and $\mathcal{I}n(\mathbb{V})$ is Hamiltonian when $\dim(\mathbb{V}) = 3$ were left open. In this paper, we resolve these two conjectures affirmatively.

The subspace sum graph $\mathcal{G}(\mathbb{V})$ was introduced in [7] and its automorphism group and independence number was determined in [16] and [13], respectively. In particular, authors in [13] found the exact value of independence number of $\mathcal{G}(\mathbb{V})$ when $\dim(\mathbb{V})$ is odd and provided an upper bound on the independence when $\dim(\mathbb{V})$ is even. In this paper, we determine the exact value of $\alpha(\mathcal{G}(\mathbb{V}))$, when $\dim(\mathbb{V})$ is even.

We next focus our attention to finding the determining number of $\mathcal{I}n(\mathbb{V})$ and $\mathcal{G}(\mathbb{V})$. We found out to our surprise that the determining number of $\mathcal{I}n(\mathbb{V})$ and $\mathcal{G}(\mathbb{V})$ of any finite-dimensional vector space of dimension ≥ 3 is at most 6.

We also characterize when these graphs forbid certain induced subgraphs. In particular, we study when the graphs $\mathcal{I}n(\mathbb{V})$ and $\mathcal{G}(\mathbb{V})$ are cographs, chordal graphs, split graphs and threshold graphs.

1.1. Preliminaries

We first recall some definitions and results on graph theory. For undefined terms and results, please refer to [20]. A graph $G = (V, E)$ is said to be *Hamiltonian* if there exists a cycle which passes through each vertex of the graph. A subset S of V is said to be independent set if no two vertices of S are adjacent in G. The cardinality of the maximum independent set is called the *independence number* of G and is denoted by $\alpha(G)$. The group of automorphisms of G is denoted by $\mathsf{Aut}(G)$. A *matching* in a graph is a set of edges that do not have a set of common vertices. A *perfect matching* is a matching that matches all the vertices of the graph. A *maximum matching* is a matching that contains the largest possible number of edges and the *matching number* $\mu(G)$ of G is the size of a maximum matching. A subset S of V is said to be a determining set [1] of G, if identity map is the only automorphism which fixes S elementwise. The size of the smallest determining set is called the *determining number* or fixing number of G and is denoted by $Det(G)$. Let $\mathbb{G}$ be a group and S be an inverse-symmetric subset of $\mathbb{G}$ (i.e., $x \in S \Rightarrow x^{-1} \in S$) such that identity of the group is not in S. The Cayley graph of $\mathbb{G}$ with respect to S is a graph with vertex set $\mathbb{G}$ and two distinct group elements x and y are adjacent if $xy^{-1} \in S$. For any graph G, the automorphism group of G is denoted by $\mathsf{Aut}(G)$. A subgroup H of $\mathsf{Aut}(G)$

is said to be a regular subgroup if for any two vertices x and y of G, there exists a unique automorphism $\varphi \in H$ such that $\varphi(x) = y$. A graph G is a Cayley graph if and only if $\mathsf{Aut}(G)$ possesses a regular subgroup. (See Lemma 3.7.1 and 3.7.2 in [10])

2. Subspace Inclusion Graph

It was proved in [6] and [21] that if $\mathbb{V}$ is a 3-dimensional vector space over a finite field F with q elements, then $\mathcal{I}n(\mathbb{V})$ is a bipartite distance transitive graphs with one- and two-dimensional subspaces forming the partite sets each with q^2+q+1 vertices. In the next two theorems, we prove that for $\dim(\mathbb{V}) = 3$, $\mathcal{I}n(\mathbb{V})$ is a Cayley graph and it is Hamiltonian.

Theorem 2.1. *Let $\mathbb{V}$ be a 3-dimensional vector space over a finite field F with q elements. Then $\mathcal{I}n(\mathbb{V})$ is a Cayley graph.*

Proof. Let K be a finite field of order q^3. Then K/F is a field extension, K is a 3-dimensional vector space over F and K is isomorphic to $\mathbb{V}$ as F-vector space.

As K is a finite field, its multiplicative group K^* is a cyclic group of order q^3-1. Let $a \in K^*$ be an element of order q^3-1. Consider the left multiplication map $T_a : K \to K$ by $T_a(x) = ax$. Clearly, T_a is a linear isomorphism on K which permutes the non-zero vectors in a single cycle of length q^3-1. Now, as $b = a^{q^2+q+1}$ is a $(q-1)$-th root of unity, $b \in F$. Thus T_b fixes all 1-dimensional subspaces of K. Thus T_a induces a cyclic permutation of order q^2+q+1 on the 1-dimensional subspaces, permuting them in a single cycle. Thus the group H generated by T_a acts as automorphisms of the graph, having one orbit on the 1-dimensional subspaces and one orbit on the 2-dimensional subspaces.

Let U be a 1-dimensional subspace and W be a 2-dimensional subspace such that $U \subset W$. Then ${T_a}^i(U) \subset {T_a}^i(W)$ for all values of i. Note that as i varies from 0 to q^2+q+1, ${T_a}^i(U)$ and ${T_a}^i(W)$ vary over the set of all 1-dimensional subspaces and set of all 2-dimensional subspaces respectively. Define a map S on the set of all 1 and 2-dimensional subspaces such that S interchanges ${T_a}^i(U)$ and ${T_a}^{-i}(W)$. Clearly, S is a bijection on $\mathcal{I}n(\mathbb{V})$. We prove that S is a graph automorphism.

If ${T_a}^j(U) \sim {T_a}^k(W)$, i.e., ${T_a}^j(U) \subset {T_a}^k(W)$, then applying ${T_a}^{-j-k}$ on both sides, we get ${T_a}^{-k}(U) \subset {T_a}^{-j}(W)$. So S maps the edge $\{{T_a}^j(U), {T_a}^k(W)\}$ to the edge $\{{T_a}^{-k}(U), {T_a}^{-j}(W)\}$. Thus S is an automorphism. One can easily check that S is an involution and $ST_aS = {T_a}^{-1}$. Hence, the dihedral group $\langle T_a, S\rangle$ of order $2(q^2+q+1)$ acts regularly on $\mathcal{I}n(\mathbb{V})$ and $\mathcal{I}n(\mathbb{V})$ is a Cayley graph. □

Theorem 2.2. *Let $\mathbb{V}$ be a 3-dimensional vector space over a finite field F. Then $\mathcal{I}n(\mathbb{V})$ is Hamiltonian.*

Proof. Let a, K and T_a be as in the proof of Theorem 2.1. Consider $U = \langle 1\rangle$ and $W = \langle 1, a\rangle$ as subspaces of K. Clearly U and $T_a(U)$ are subspaces of W. So $\{U, W\}$ and $\{T_a(U), W\}$ are edges in $\mathcal{I}n(\mathbb{V})$. So, we can construct a Hamiltonian cycle in $\mathcal{I}n(\mathbb{V})$ as $(U, W, T_a(U), T_a(W), {T_a}^2(U), \ldots, {T_a}^{k-1}(U), {T_a}^{k-1}(W), U)$ where $k = q^2+q+1$. □

3. Independence Number of Subspace Sum Graph

Ma and Wang [13, Theorem 2.5] determined the exact value of the independence number of the graph $\mathcal{I}n(\mathbb{V})$ where $\mathbb{V}$ is a finite-dimensional vector space over finite field. Moreover, they were also able to find explicitly the exact value of the independence number of the subspace sum graph when the dimension of the vector space is odd. In this section, we are interested in finding the exact value of the independence number of the subspace sum graph when the dimension of the vector space is even. We begin with some known results about the number of k-dimensional subspaces of an n-dimensional vector space. Throughout this section, F is a finite field of order q.

Lemma 3.1 ([11]). *The number of k-dimensional subspaces of an n-dimensional vector space over F is the following q-binomial coefficient*

$$\binom{n}{k}_q = \frac{(q^n-1)(q^{n-1}-1)\dots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\dots(q-1)}.$$

Lemma 3.2 ([14]). *For a fixed positive integer n, the q-binomial coefficients $\binom{n}{k}_q$ is a polynomial in the variable q and the coefficients satisfy the following symmetry:*

$$\binom{n}{k}_q = \binom{n}{n-k}_q, \textit{ for } 1 \le k \le n-1.$$

We are now in a position to state the result of Ma and Wang [13, Theorem 3.7] on the independence number of the subspace sum graph when the dimension of the vector space is odd.

Theorem 3.1. *If $n = 2m-1$ is odd, then*

$$\alpha(\mathcal{G}(\mathbb{V})) = \frac{1}{2}\Big(\sum_{k=0}^{n}\binom{n}{k}_q - 2\Big).$$

In course of their proof, they constructed an independent set of the above size and also proved that when $n = 2m-1$, the graph $\mathcal{G}(\mathbb{V})$ has a perfect matching and hence the matching number is $\frac{1}{2}\Big(\sum_{k=0}^{n}\binom{n}{k}_q - 2\Big)$.

But when $n = 2m$, the maximum size of the independent set they could construct, is not the same as the size of the maximum matching and hence they left finding the exact independence number as a topic of further research. Here we prove the following result to answer their question.

Theorem 3.2. *Let $n = 2m$ be a positive integer. Then,*

$$\alpha(\mathcal{G}(\mathbb{V})) = \binom{2m}{1}_q + \dots + \binom{2m}{m-1}_q + \binom{2m-1}{m-1}_q.$$

To prove Theorem 3.2, we need some more results. First we state the following version of the Erdős–Ko–Rado Theorem [9] for t-intersecting families of k-dimensional subspaces of a n-dimensional vector space $\mathbb{V}$ over F. Let $V(n,k)$ denote the set of k-dimensional subspaces of a n-dimensional vector space V over F. A family $\mathcal{F} \subseteq V(n,k)$ is said to be t-intersecting if for any $W_1, W_2 \in \mathcal{F}$. we have $\dim(W_1 \cap W_2) \geq t$.

Theorem 3.3 (Frankl–Wilson)**.** *Suppose $n \geq 2k-t$ and a family $\mathcal{F} \subset V(n,k)$ is t-intersecting. Then,*

$$|\mathcal{F}| \leq \max\left\{ \binom{n-t}{k-t}_q, \binom{2k-t}{k}_q \right\}.$$

Let $n = 2m$. For $1 \leq k \leq m$, we define Γ_k to be a bipartite graph with vertex set $V(n,k) \cup V(n,n-k)$ and there is an edge between a vertex $U \in V(n,k)$ and a vertex $W \in V(n,n-k)$ if and only if $U + W = V$. Note that Γ_k can be obtained from the induced subgraph of the subspace sum graph with vertex set $V(n,k) \cup V(n,n-k)$ by deleting all possible edges between vertices of $V(n,n-k)$. Ma and Wang [13, Lemma 3.4] proved the following Lemma on the existence of perfect matching of Γ_k.

Lemma 3.3. *For positive integers $n = 2m$ and $1 \leq k \leq m-1$, Γ_k has a perfect matching.*

We are now in a position to prove Theorem 3.2.

Proof of Theorem 3.2. We fix a particular nonzero vector α of V. Let Y denote the set of all m-dimensional subspaces of V containing α. Let

$$S = Y \cup V(2m,1) \cup V(2m,2) \cup \cdots \cup V(2m,m-1).$$

We first prove that S is indeed an independent set. Let $W_1, W_2 \in S$. If both $W_1 \in Y$ and $W_2 \in Y$, we have $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) \leq 2m-1 < n$ and therefore $W_1 \nsim W_2$. If atleast one of W_1 and W_2 is not from Y, we have $\dim(W_1 + W_2) \leq \dim(W_1) + \dim(W_2) \leq 2m-1$. Hence, S is an independent set and it can be checked that $|S| = \binom{2m}{1}_q + \cdots + \binom{2m}{m-1}_q + \binom{2m-1}{m-1}_q$.

We now prove that this is the maximum possible cardinality of an independent set.

Let T be an independent set of $\mathcal{G}(\mathbb{V})$. Thus, for $1 \leq k \leq m-1$, $T \cap \Gamma_k$ is also an independent set. Now, by Lemma 3.3, Γ_k has a perfect matching and by Lemma 3.1 and 3.2, we have

$$|T \cap \Gamma_k| \leq \frac{1}{2}|\Gamma_k| = \frac{1}{2}\left(\binom{2m}{k}_q + \binom{2m}{2m-k}_q\right) = \binom{2m}{k}_q. \tag{1}$$

We now consider $T \cap \Gamma_m$ which is again an independent set. Thus, for any $W_1, W_2 \in T \cap \Gamma_m$, we have $W_1 \nsim W_2$ and $\dim(W_1) = m = \dim(W_2)$. This forces us to have

$\dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 + W_2) \geq 1$. Hence the family $T \cap \Gamma_m$ is a 1-intersecting family of m-dimensional subspaces. Thus by setting $n = 2m$, $k = m$ and $t = 1$ in Theorem 3.3, we have

$$|T \cap \Gamma_m| \leq \max\{\binom{2m-1}{m-1}_q, \binom{2m-1}{m}_q\} = \binom{2m-1}{m-1}_q. \tag{2}$$

From (1) and (2), we get

$$|T| = \sum_{k=1}^{m-1} |T \cap \Gamma_k| + |T \cap \Gamma_m| \leq \binom{2m}{1}_q + \cdots + \binom{2m}{m-1}_q + \binom{2m-1}{m-1}_q.$$

This completes the proof of Theorem 3.2. □

4. Determining Number of Subspace Inclusion Graph and Subspace Sum Graph

In this section, we provide bounds for the determining number of subspace inclusion graph and subspace sum graph. We begin with a definition.

A *base* for a permutation group is a sequence of points in the domain of the group whose pointwise stabiliser is the identity. The *base size* is the minimum cardinality of a base. The *determining number* of a graph is the base size of its automorphism group.

Theorem 4.1. *The determining number of $\mathcal{I}n(\mathbb{V})$ of a finite dimensional vector space of dimension 3 or higher is at most 6.*

Proof. Wang and Wong [18] proved that the automorphism group of $\mathcal{I}n(\mathbb{V})$ of the n-dimensional vector space $\mathbb{V}$ over $\mathrm{GF}(q)$ is $\mathrm{P\Gamma L}(n,q) : C_2$, where $\mathrm{P\Gamma L}(n,q)$ is the group generated by invertible linear maps on $\mathbb{V}$ and field automorphisms acting coordinatewise, modulo the normal subgroup of scalar maps, and C_2 is the inverse-transpose automorphism which acts on the graph as a duality map (exchanging subspaces of dimensions k and $n-k$). It is also known that the general linear group $\mathrm{GL}(n,q)$ is generated by two elements. (See [19] for explicit generators.)

The proof will separate the cases of even and odd dimension. We begin with the even-dimensional case.

First we show that the subgroup $\mathrm{PGL}(n,q)$, omitting the field automorphisms and inverse-transpose automorphism, has a base of size 5, which we give explicitly. Let $n = 2m$, and let S, T be two matrices which generate $\mathrm{GL}(m,q)$. We take the vector space $\mathbb{V}$ to have the form $\mathbb{V} = E \oplus F$, where E has basis $\{e_1, \ldots, e_m\}$ and F has basis $\{f_1, \ldots, f_m\}$. Our five subspaces are

- $W_1 = E$;
- $W_2 = F$;
- $W_3 = \langle e_1 + f_1, \ldots, e_m + f_m \rangle$;

- $W_4 = \langle e_1 + S(e_1), \ldots, e_m + S(e_m) \rangle$;
- $W_5 = \langle e_1 + T(e_1), \ldots, e_m + T(e_m) \rangle$.

For the last two, we re-interpret S and T as maps from E to F whose matrices (relative to the bases for E and F) are equal to the matrices on the m-dimensional space in their definition. Now we have to show that the only invertible linear maps on V which fix all five subspaces are the scalars.

The stabiliser of W_1 and W_2 in $\mathrm{GL}(n,q)$ is $\mathrm{GL}(m,q) \times \mathrm{GL}(m,q)$. Now W_3 induces a linear bijection from E to F, defined by $e \mapsto f$ if and only if $e + f \in W_3$. This bijection must be preserved by the stabilizer of these three subspaces, which is thus $\mathrm{GL}(m,q)$, acting on the same way on E and F.

In a similar way, W_4 and W_5 also induce bijections from E to F. Now applying the W_4 map and the inverse of the W_3 map gives a linear map on E which acts as the transformation S. Similarly the W_5 map and the inverse of the W_3 map gives the transformation T on E. As before, the stabiliser of the five subspaces must commute with S and T. However, since S and T generate $\mathrm{GL}(m,q)$, this means that the stabiliser of the five subspaces is contained in the centre of $\mathrm{GL}(n,q)$, which consists of scalar matrices only. So the stabiliser is the identity in $\mathrm{PGL}(n,q)$, that is, acts trivially on the graph as required. So these five matrices form a basis for $\mathrm{PGL}(n,q)$.

From this, we see that the group induced on the subspace graph by the stabiliser of $W_1, \ldots, W_5$ in its automorphism group is contained in $C_r : C_2$, where $q = p^r$ with p prime, so that C_r is the automorphism group of the field of order q. (We cannot say exactly which subgroup, or how it acts, since this may depend on the choice of S and T.) We will choose the sixth subspace W_6 to have dimension 1. It follows that the stabiliser of $W_1, \ldots, W_6$ cannot induce a duality map, and so is contained in C_r. The group of field automorphisms, in its coordinatewise action, has an orbit of length r on V (containing a vector $(1, a, \ldots)$, for example, where a is a generator of $\mathrm{GF}(q)$ over $\mathrm{GF}(p)$); the stabiliser of this vector is the identity. So if we choose W_6 to lie in an orbit of maximum length, then certainly its stabiliser in the group of field and duality automorphisms will be trivial. Thus $\{W_1, \ldots, W_6\}$ is the required base.

Now we turn to the odd-dimensional case, with $n = 2m + 1$. We assume first that $m > 1$. Let $V = V(n,q)$, and let U be a subspace of dimension $n - 1 = 2m$. We begin by choosing six subspaces $W_1, \ldots, W_6$ forming a basis for $\mathrm{P\Gamma L}(2m,q)$ as above. The stabiliser of these six subspaces in $\mathrm{P\Gamma L}(2m+1,q)$ fixes U (which is spanned by the first two of them) and acts trivially on U. The subspaces of V not contained in U can be regarded as the elements of the affine geometry of dimension $2m$ over $\mathrm{GF}(q)$, and the stabiliser of all subspaces contained in U acts on it as a group of affine transformations fixing the hyperplane at infinity pointwise; these are just the translations and dilations of the affine space, that is, the maps $v \mapsto cv + w$ where $w \in \mathrm{GF}(q)^{2m}$ and $c \in \mathrm{GF}(q)$, $c \neq 0$.

We claim that the stabiliser of two skew affine subspaces whose dimensions are not complementary (in the group of translations and dilations) is trivial. For this group can contain no non-trivial translations, since the translation vector would have to fix both subspaces. Also, it cannot contain any dilation centre, since the dilation centre would have to lie in both subspaces.

So we let W_i be one of $W_1, \ldots, W_5$ not containing W_6. Choose $x \in V \setminus U$; and let $W_i^+ = \langle W_i, x\rangle$, and $A = W_i^+ \setminus U$, an affine subspace of $V \setminus U$ with affine dimension $m+1$. For any $y \in V \setminus U$, if $W_6^+ = \langle W_6, y\rangle$, then $A_2 = W_6^+ \setminus U$ is a 2-dimensional affine subspace disjoint from U. Now replace W_i and W_6 in our basis by W_i^+ and W_6^+, and set $W_j^+ = W_j$ for $j \leq 5$, $j \neq i$.

We claim that $\{W_1^+, \ldots, W_6^+\}$ is a basis. For the stabiliser of these six subspaces fixes U, which is spanned by some two of $W_1^+, \ldots, W_5^+$ except W_i^+. (For example, any two of W_1, W_2 and W_3 span U.) Thus it fixes $W_i = W_i^+ \cap U$ for $i = 1, \ldots, 5$; these five form a basis for the group induced on subspaces of U. It also fixes the two affine spaces A_1 and A_2. It follows from our argument that this stabiliser is trivial.

Finally we have to deal with the case $m = 1$, when the vector space V has dimension 3. Now it is familiar in the theory of projective planes that the five points

$$\langle(1,0,0)\rangle, \langle(0,1,0)\rangle, \langle(0,0,1)\rangle, \langle(1,1,1)\rangle, \langle(1,a,a^2)\rangle,$$

where a is a primitive element of $\mathrm{GF}(q)$, form a base for the collineation group of the Desarguesian projective plane. So there is a determining set of size 5 in this case. $\square$

Now, we present a lower bound on the determining number of $\mathcal{I}n(\mathbb{V})$.

Theorem 4.2. *It* $\dim(\mathcal{V}) \geq 3$*, then the determining number of* $\mathcal{I}n(\mathbb{V})$ *is at least* 4 *for infinitely many choices of* n *and* q*, where* q *is the order of the base field.*

Proof. The order of the automorphism group of $\mathcal{I}n(\mathbb{V})$ is $2q^{n^2-1}\log q(1-O(q^{-1}))$. On the other hand, the number of vertices of the graph is at least $q^{n^2/4}(1-O(q^{-1}))$; so the number of choices of three subspaces is smaller than the order of the automorphism group, for all but finitely many q (given n). If there is a base of size 3, then the number of its distinct images under the automorphism group G would be equal to $|G|$. $\square$

Analogous results to that of Theorem 4.1 and Theorem 4.2 for upper bounds and lower bounds of determining number of the subspace sum graph also hold:

Theorem 4.3. *Let* $\mathcal{V}$ *be a finite vector space with dimension* $n \geq 3$ *over the field with* q *elements.*

(a) The determining number of $\mathcal{G}(\mathbb{V})$ *is at most* 6*, and is at most* 5 *if* q *is prime.*

(b) There are infinitely many choices of n *and* q *such that the determining number of* $\mathcal{G}(\mathbb{V})$ *is at least* 4.

Proof. For $n = \dim(\mathbb{V}) > 2$, the automorphism group of the subspace sum graph $\mathcal{G}(\mathbb{V})$ is $\mathrm{P\Gamma L}(n,q)$, a subgroup of index 2 in the automorphism group of the subspace inclusion graph not containing the duality map [16].

Since $\mathsf{Aut}(\mathcal{G}(\mathbb{V}))$ is a subgroup of the automorphism group of the subspace inclusion graph, its determining number is bounded above by that of the latter group, and hence is at most 6. Indeed, if the field has prime order, then its automorphism group is $\mathrm{PGL}(n, q)$, and as we saw earlier, five subspaces suffice to form a base in this case.

The proof of the lower bound is almost exactly as in Theorem 4.2. □

5. Forbidden subgraphs of Subspace Inclusion graph and Subspace Sum graph

A graph G is a cograph if it has no induced subgraph which is isomorphic to the four-vertex path P_4. That is, a graph is a cograph if and only if for any four vertices $\{v_1, v_2, v_3, v_4\}$, if $\{v_1, v_2\}, \{v_2, v_3\}$ and $\{v_3, v_4\}$ are edges of the graph then at least one of $\{v_1, v_3\}, \{v_1, v_4\}$ or $\{v_2, v_4\}$ is also an edge. A threshold graph is a graph containing no induced subgraph isomorphic to P_4, C_4 or $2K_2$. Therefore every threshold graph is a cograph.

A graph G is split if the vertex set is the disjoint union of two subsets A and B such that A induces a complete graph and B a null graph. An equivalent definition is that a graph is split if and only if it contains no induced subgraph isomorphic to C_4, C_5 or $2K_2$.

A graph G is chordal if it contains no induced cycles of length greater than 3; in other words, every cycle on more than 3 vertices has a chord.

We now restrict our attention on the dimension for the graphs $\mathcal{I}n(\mathbb{V})$ and $\mathcal{G}(\mathbb{V})$ to be cographs, threshold, split and chordal graphs. At first we observe the following when $\dim(\mathbb{V}) = 2$.

1. The vertices of $\mathcal{I}n(\mathbb{V})$ are only the one-dimensional subspaces and therefore, it is edgeless.

2. The vertices of the graph $\mathcal{G}(\mathbb{V})$ are the one-dimensional subspaces of $\mathbb{V}$. Now, the sum of two distinct 1-dimensional subspaces in a 2-dimensional vector space is 2-dimensional and hence equal to $\mathbb{V}$. So $\mathcal{G}(\mathbb{V})$ is complete.

Hence, throughout this section we assume $\dim(\mathbb{V}) \geq 3$ and we at first prove the following result on the subspace inclusion graph $\mathcal{I}n(\mathbb{V})$.

Theorem 5.1. *For any vector space* $\mathbb{V}$ *with* $\dim(\mathbb{V}) \geq 3$, *the graph* $\mathcal{I}n(\mathbb{V})$ *is*

1. *never a cograph and hence never a threshold graph,*

2. *never a chordal graph and hence never a split graph.*

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V. Suppose $n \geq 3$.

1. Consider the following vertices: $W_1 = \langle \mathcal{B} \setminus \{\alpha_2, \alpha_3\}\rangle$, $W_2 = \langle \mathcal{B} \setminus \{\alpha_2\}\rangle$, $W_3 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_2\}\rangle$, and $W_4 = \langle \mathcal{B} \setminus \{\alpha_1\}\rangle$. It can be seen that $W_1 \subseteq W_2, W_3 \subseteq W_2$ and $W_3 \subseteq W_4$ but $W_1 \nsim W_3, W_1 \nsim W_4$ and $W_2 \nsim W_4$. Thus, the graph contains a P_4 and hence it is not a cograph.

2. We now show that this is not a chordal graph: $W_1 = \langle \mathcal{B} \setminus \{\alpha_2, \alpha_3\}\rangle$, $W_2 = \langle \mathcal{B} \setminus \{\alpha_3\}\rangle$, $W_3 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_3\}\rangle$, $W_4 = \langle \mathcal{B} \setminus \{\alpha_1\}\rangle$, $W_5 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_2\}\rangle$, and $W_6 = \langle \mathcal{B} \setminus \{\alpha_2\}\rangle$. Thus the following is a 6-cycle in the graph $\mathcal{I}n(\mathbb{V})$:

$$W_1 \sim W_2 \sim W_3 \sim W_4 \sim W_5 \sim W_6 \sim W_1.$$

One can check that there is no chord in this 6-cycle and hence it is not a chordal graph.

This completes the proof. $\square$

We now move on to the subspace sum graph and here we have the following result.

Theorem 5.2. *For any vector space* $\mathbb{V}$ *with* $\dim(\mathbb{V}) \geq 3$, *the graph* $\mathcal{G}(\mathbb{V})$ *is*

1. *never a cograph and hence not a threshold graph;*

2. *a chordal graph if and only if* $\dim(\mathbb{V}) = 3$*;*

3. *a split graph if and only if* $\dim(\mathbb{V}) = 3$.

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V. Suppose $n \geq 3$.

1. Consider the following vertices: $W_1 = \langle \mathcal{B} \setminus \{\alpha_2, \alpha_3\}\rangle$, $W_2 = \langle \mathcal{B} \setminus \{\alpha_1\}\rangle$, $W_3 = \langle \mathcal{B} \setminus \{\alpha_3\}\rangle$, and $W_4 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_2\}\rangle$. It can be easily checked that $W_1 + W_2 = \mathbb{V}$, $W_2 + W_3 = \mathbb{V}$ and $W_3 + W_4 = \mathbb{V}$ but $W_1 \nsim W_3, W_1 \nsim W_4$ and $W_2 \nsim W_4$. Therefore, the graph contains a P_4 and hence it is not a cograph.

2. Suppose $\dim(\mathbb{V}) = 3$ and let the following be a k-cycle with $k \geq 4$:

$$W_1 \sim W_2 \sim W_3 \sim W_4 \sim W_5 \sim \ldots W_k \sim W_1.$$

If none of the W_is have dimension 1, then of course we can find many chords. If for any $2 \leq i \leq k-1$, W_i has dimension 1 then both W_{i-1} and W_{i+1} has dimension 2 and as $k \geq 4$, the edge $W_{i-1} \sim W_{i+1}$ is a chord of the k-cycle. If W_1 has dimension 1 then $W_k \sim W_2$ is a chord and if W_k has dimension 1 then $W_{k-1} \sim W_1$ is a chord. Thus for $n = 3$, the graph is chordal.

For the other direction, let $\dim(\mathbb{V}) \geq 4$ and let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V. Suppose $n \geq 4$. Consider the following vertices: $W_1 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_2\}\rangle$, $W_2 = \langle \mathcal{B} \setminus \{\alpha_3, \alpha_4\}\rangle$, $W_3 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_2, \alpha_4\}\rangle$, and $W_4 = \langle \mathcal{B} \setminus \{\alpha_1, \alpha_3, \alpha_4\}\rangle$. It can be seen that $W_1 \sim W_2 \sim W_3 \sim W_4 \sim W_1$ but $W_1 \nsim W_3$ and $W_2 \nsim W_4$. Thus, the graph contains a C_4 which does not contain any chord and therefore the graph is not chordal.

3. This was proved by Venkatasalam and Chelliah [17, Theorem 4.2].

This completes the proof. $\square$

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